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AN AXIOMATIZATION OF THE PAZNER-SCHMEIDLER RULES  
IN LARGE FAIR DIVISION PROBLEMS

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## RÉSUMÉ

Nous étudions une version du problème de partage équitable à population et préférences variables. Nous montrons que les axiomes d'efficacité et d'anonymat, la condition que chaque agent préfère sa part à la part moyenne et une condition de solidarité par rapport aux changements dans la population et les préférences caractérisent la solution de Pazner-Schmeidler.

Mots clés: partage équitable, équivalence égalitaire, solidarité.

## ABSTRACT

We consider a version of the fair division problem where the population of individuals is allowed to vary in the set of (Borel) subsets of the unit interval. Preferences are variable as well, while the aggregate bundle to be divided is fixed. A *Pazner-Schmeidler rule* always selects an efficient allocation that equalizes utilities calibrated along the ray through the aggregate bundle. Efficiency, anonymity, the equal split lower bound on utilities, and a property of solidarity with respect to changes in population and preferences characterize the Pazner-Schmeidler rules.

Key words: fair division, egalitarian equivalence, solidarity.



## 1. INTRODUCTION

A central puzzle in normative economics is the so-called fair division problem. An aggregate bundle of goods is to be divided among a group of individuals who are collectively entitled to it. The individuals are characterized by their ordinal preferences; they have no private endowments. If we insist on allocating goods efficiently, equal split is generally ruled out. This poses the challenge of defining a notion of fairness that would be compatible with efficiency.

There is on this issue a sizeable literature which we will not systematically survey here, referring the reader to Moulin (1990). Without too much caricaturing, however, one may distinguish two general approaches. One is centered around the idea of equalizing opportunities. Perhaps the most natural way to achieve this ideal is to divide the aggregate bundle equally and "let people trade". Competitive equilibrium allocations from equal split have several nice properties: (i) they are efficient, (ii) no one ever prefers anyone else's bundle over his own, and (iii) everyone finds his own bundle at least as good as the mean bundle. Condition (ii) is known as the no-envy property and condition (iii) is often called the equal split lower bound. In large economies, efficiency and no-envy are essentially characteristic of competitive allocations from equal split [Varian (1974), Champsaur-Laroque (1981), Thomson-Zhou (1993)].

The second approach to the fair division problem tries to equalize utilities rather than opportunities. The suggestion, here, is to represent preferences by means of utility functions and to divide the aggregate bundle so as to equalize the individual utility levels. One possible way to construct the utility functions is as follows. First, pick a strictly positive numeraire (bundle)  $\omega_0$ , possibly different from the aggregate endowment. Given a continuous and increasing preference preordering, measure the utility of any particular consumption bundle by the number  $\theta$  which would make  $\theta\omega_0$  exactly as good as that bundle. An allocation equalizing such utilities may be called *egalitarian-equivalent relative to the numeraire*  $\omega_0$ ; indeed, all individuals are indifferent between the recommended allocation and some (typically infeasible) egalitarian one at which everyone receives a certain common multiple of the numeraire bundle. An allocation is called egalitarian-equivalent if it is egalitarian-equivalent relative to some numeraire. Pazner-Schmeidler (1978), who propose the concept, prove its compatibility with efficiency.

Choosing an efficient allocation which is egalitarian-equivalent relative to a *fixed* reference bundle has several advantages. It guarantees a threefold form of

solidarity among the individuals which, we feel, is no less appealing than the no-envy property. First, no one ever suffers from an increase in the aggregate bundle; this is the resource monotonicity condition of Moulin-Thomson (1988). In fact, any change in the aggregate bundle necessarily affects all individuals in the same direction, a *resource solidarity* condition à la Keiding-Moulin (1991). Secondly, the *population monotonicity* axiom [Thomson (1987, 1994)] is satisfied: nobody ever benefits from the arrival of newcomers. Thirdly, any change in the preferences of some individuals affects the others in a common direction, a *preference solidarity* property discussed in other contexts by Moulin (1987), Sprumont (1994) and Thomson (1992, 1993).

Among the drawbacks of efficient fixed-numeraire egalitarian-equivalent rules, two are of particular relevance to our discussion. Let us first record that these rules must violate the equal split lower bound since all efficient resource monotonic rules do [Moulin-Thomson (1988)]. Another, perhaps more fundamental difficulty, lies in the arbitrary character of the numeraire bundle. This point is stressed by Crawford (1979) and is well illustrated by the work of Dutta-Vohra (1993) who axiomatize (in a slightly different setting) the subclass of rules where the numeraire bundle contains a single good only, but must remain silent on the choice of this good.

An obvious solution to the problem is to choose the aggregate bundle itself as the numeraire. This is advocated by Pazner-Schmeidler in their original paper on the basis that only this choice would meet the no-envy test in two-individual problems. We will therefore call the resulting rules the *Pazner-Schmeidler rules*.<sup>1</sup>

As noted by Thomson (1987) and Moulin (1990), Pazner-Schmeidler's choice of the numeraire restores both the equal split lower bound and the neutrality property with respect to goods [formally defined in Moulin (1991)]. These properties are recovered at the severe cost of losing resource monotonicity: what made the fixed-numeraire rules resource monotonic was precisely that the numeraire bundle remained constant for all possible values of the aggregate endowment. Yet, population monotonicity and preference solidarity continue to hold for the Pazner-Schmeidler rules since the numeraire remains independent of both the population and the preference profile. In fact, solidarity is guaranteed even under joint changes in preferences and population.

Our purpose here is to present an axiomatization of the Pazner-Schmeidler rules. The main axioms are the solidarity property with respect to joint changes in preferences

<sup>1</sup> In any given fair division problem, there may be several efficient allocations which are egalitarian-equivalent with respect to the aggregate bundle. There are therefore several Pazner-Schmeidler rules, but all are Pareto-equivalent.

and population that we just mentioned, and the equal split lower bound. In addition, we only impose the standard conditions of efficiency and anonymity. Neutrality with respect to the goods is not required; it is implied by our four axioms.

A proviso is in order. Our result is cast in the framework of large fair division problems modeled à la Aumann (1964). In this respect, it is comparable to the characterizations of the competitive rule from equal split described earlier. Our theorem is unlikely to hold, at least in the exact sense, in the finite case. Resorting to a continuum of individuals strengthens the (otherwise discrete and less tractable) axiom of population monotonicity included in the joint solidarity condition.

The paper is organized as follows. Section 2 sets up the model. Section 3 defines the Pazner-Schmeidler rules, spells out the axioms, and states the theorem. The following three sections are devoted to the proof. Because of the continuum assumption, the anonymity axiom is of a slightly more delicate use than in finite models; section 4 prepares the way for its subsequent application. Section 5 derives various continuity properties implied by efficiency and our solidarity axiom, especially when they are used along with anonymity. Here, particularly in Lemma 6, the continuum formulation is essential. Section 6 completes the proof in two stages. We first demonstrate, under a technical range condition, that an efficient, anonymous rule satisfying solidarity must equalize some measure of the individuals' utilities: this is Lemma 8, which we find of independent interest. Imposing the equal split lower bound pinpoints the Pazner-Schmeidler rules. We conclude in section 7 by showing that neither population monotonicity nor preference solidarity may replace the solidarity axiom in our characterization. Two open questions are formulated.

## 2. LARGE FAIR DIVISION PROBLEMS

The population under consideration is represented by the measure space  $(I, \mathcal{B}, \lambda)$ , where  $I = [0, 1)$ ,  $\mathcal{B}$  is the  $\sigma$ -field of Borel subsets of  $I$ , and  $\lambda$  is Lebesgue measure. Elements of  $I$  are called *individuals* and are denoted by  $i, j$ , and so on; elements of  $\mathcal{B}$  are called *societies* and are denoted by  $S, T$ , and so on. A subsociety of society  $T$  is any society  $S \subset T$ . We let  $\mathcal{B}^0 = \{S \in \mathcal{B} \mid \lambda(S) = 0\}$ ,  $\mathcal{B}^+ = \{S \in \mathcal{B} \mid 0 < \lambda(S)\}$ ,  $\mathcal{B}^- = \{S \in \mathcal{B} \mid \lambda(S) < 1\}$ , and  $\mathcal{B}^{+-} = \mathcal{B}^+ \cap \mathcal{B}^-$ . Societies in  $\mathcal{B}^0$  are called *null*, societies in  $\mathcal{B}^+$  are called *positive*, and societies in  $\mathcal{B}^-$  are called *proper*. A statement holds for *almost all* individuals in a society  $S$  if it holds for all individuals in  $S$ , except for those in some null subsociety of  $S$ . Notice that if  $S$  is null, the statement is vacuously true.

A fixed commodity bundle  $\omega \in \mathbb{R}_{++}^{\ell}$  is given. The positive integer  $\ell$  represents the number of commodities entering the bundle  $\omega$ . Let  $D^0$  be the set of continuous, quasiconcave, strictly increasing real-valued functions on  $\mathbb{R}_{+}^{\ell}$ . The *utility domain*, which is fixed, is assumed to be a countable subset  $D$  of  $D^0$ . A *utility function* is any element of  $D$ . A (utility) *profile* (for society  $S$ ) is a mapping  $U_S : S \rightarrow D$  such that  $U_S^{-1}(u) \in \mathcal{S}$  for every  $u \in D$ . When the context clearly indicates which utility profile  $U_S$  is being considered and when individual  $i$  belongs to  $S$ , we often denote his utility function  $U_S(i)$  by  $u_i$ . If  $S$  is a subsociety of  $T$  and  $U_T$  is a profile for  $T$ ,  $U_T|_S$  denotes the restriction of the mapping  $U_T$  to  $S$ . A society endowed with a utility profile is called an *economy*. An economy  $(S, U_S)$  is called null, positive or proper if  $S$  is a null, positive or proper society. The set of all economies is denoted by  $\mathcal{E}$  and the set of positive economies is denoted by  $\mathcal{E}^+$ .

A consumption bundle for individual  $i$  is a vector  $x(i) = (x^1(i), \dots, x^{\ell}(i))$  in  $\mathbb{R}_{+}^{\ell}$ . An *allocation* [for the economy  $(S, U_S)$ ] is an integrable mapping  $x_{S, U_S} : S \rightarrow \mathbb{R}_{+}^{\ell}$  specifying a consumption bundle for each member of society  $S$  in a way that satisfies the feasibility constraint  $\int_S x_{S, U_S}^h(i) d\lambda(i) \leq \omega^h$  for  $h = 1, \dots, \ell$ . If  $T$  is a subsociety of  $S$ , we let  $\int_T x_{S, U_S}$  denote the  $\ell$ -dimensional vector whose  $h$ th coordinate is  $\int_T x_{S, U_S}^h(i) d\lambda(i)$ . The set of allocations for the economy  $(S, U_S)$  is denoted by  $X(S, U_S)$  and we let  $X = \cup_{(S, U_S) \in \mathcal{E}} X(S, U_S)$ . An (allocation) *rule* is a mapping  $F : \mathcal{E} \rightarrow X$  which assigns to each economy  $(S, U_S)$  an allocation  $f_{S, U_S} = F(S, U_S) \in X(S, U_S)$ . If  $(S, U_S) \in \mathcal{E}$  and  $i \in S$ , we write  $F_i(S, U_S) = f_{S, U_S}(i)$ .

### 3. AN AXIOMATIZATION OF THE PAZNER-SCHMEIDLER ALLOCATION RULES

Given an economy  $(S, U_S)$ , we say that the allocation  $y_{S, U_S}$  *Pareto-dominates* the allocation  $x_{S, U_S}$  if  $u_i(y_{S, U_S}(i)) \geq u_i(x_{S, U_S}(i))$  for almost all  $i \in S$  and  $u_i(y_{S, U_S}(i)) > u_i(x_{S, U_S}(i))$  for all  $i$  in some positive subsociety of  $S$ . The allocations are *Pareto-equivalent* if  $u_i(y_{S, U_S}(i)) = u_i(x_{S, U_S}(i))$  for almost all  $i \in S$ . An allocation is *efficient* if



no allocation Pareto-dominates it. An allocation  $x_{S,U_S}$  is called *egalitarian-equivalent* (relative to  $\omega$ ) if there exists some  $\theta \in \mathbb{R}$  such that  $u_i(x_{S,U_S}(i)) = u_i(\theta\omega)$  for almost all  $i \in S$ . An allocation which is efficient and egalitarian-equivalent is called (a) *Pazner-Schmeidler* (allocation). If  $S$  is a null society, all allocations are (trivially) Pazner-Schmeidler. If  $S$  is positive, there may exist several Pazner-Schmeidler allocations, but all are Pareto-equivalent. A rule  $F$  is (a) Pazner-Schmeidler (rule) if for every economy  $(S, U_S)$ ,  $F(S, U_S)$  is a Pazner-Schmeidler allocation.

We now present the four axioms that will be used to characterize the Pazner-Schmeidler allocation rules. Note that none of them restricts the behavior of  $F$  on null economies. Our first two axioms are standard and easily stated:

*Efficiency.* For every  $(S, U_S) \in \mathcal{E}^+$ , no  $x_{S,U_S} \in X(S, U_S)$  Pareto-dominates  $F(S, U_S)$ .

*Equal Split Lower Bound.* For every  $(S, U_S) \in \mathcal{E}^+$ ,  $u_i(F_i(S, U_S)) \geq u_i(\omega / \lambda(S))$  for almost all  $i \in S$ .

Our third axiom is the familiar property of Anonymity. For any society  $S$ , let us denote by  $\mathcal{A}_S$  the set of subsocieties of  $S$ ;  $\mathcal{A}_S$  is just the  $\sigma$ -field of Borel subsets of  $S$ . An *automorphism on  $S$*  is a bijective mapping  $\pi : S \rightarrow S$  for which the image and inverse image of every measurable set are measurable:  $\pi(T), \pi^{-1}(T) \in \mathcal{A}_S$  if  $T \in \mathcal{A}_S$ . An automorphism  $\pi$  on  $S$  is *measure-preserving* if  $\lambda(\pi(T)) = \lambda(T)$  for every  $T \in \mathcal{A}_S$ . Whenever we do not specify on which set an automorphism is defined, it is understood that this automorphism is on  $I$ . The *image* of an economy  $(S, U_S)$  under an automorphism  $\pi$  is the economy  $(\pi(S), U_{\pi(S)}^\pi)$ , where  $\pi(S) := \{\pi(i) \mid i \in S\}$  and  $U_{\pi(S)}^\pi(i) = U_S(\pi^{-1}(i))$  for all  $i \in \pi(S)$ .

*Anonymity.* For every measure-preserving automorphism  $\pi$  on  $I$  and every  $(S, U_S) \in \mathcal{E}^+$ ,  $u_i(F_{\pi(i)}(\pi(S), U_{\pi(S)}^\pi)) = u_i(F_i(S, U_S))$  for almost all  $i \in S$ .

Our last axiom is the only original one, even though it is related to solidarity properties previously studied in the literature:

*Solidarity.* For all  $(S, U_S), (T, V_T), (R, W_R) \in \mathcal{E}^+$  such that  $S \subset T \cap R$  and  $U_S = V_T|_S = W_R|_S$ , one of the following statements holds: (i)  $u_i(F_i(T, V_T)) = u_i(F_i(R, W_R))$  for almost all  $i \in S$ , (ii)  $u_i(F_i(T, V_T)) < u_i(F_i(R, W_R))$  for almost all  $i \in S$ , (iii)  $u_i(F_i(T, V_T)) > u_i(F_i(R, W_R))$  for almost all  $i \in S$ .

This axiom implies the classical axiom of Population Solidarity (by taking  $T = S$ ) and the axiom of Preference Solidarity (by taking  $T = R$ ).

We will prove:

*Theorem.* A rule satisfies Efficiency, the Equal Split Lower Bound, Anonymity, and Solidarity if and only if it is a Pazner-Schmeidler rule.

#### 4. USING THE ANONYMITY AXIOM

This section prepares the way for subsequent applications of the Anonymity axiom by establishing the existence of some measure-preserving automorphisms of special interest. We also prove that Anonymity implies the well-known property of Equal Treatment of Equals.

For  $0 \leq a \leq b \leq 1$ , let  $[a, b) := \{x \mid a \leq x < b\}$ . Any set of this type will be called an interval society. Our notation is slightly unconventional since it allows the case where  $a = b$ ; by definition,  $[a, a) = \emptyset$  for all  $a \in [0, 1]$ . We call *simple* any society which is the union of finitely many disjoint interval societies. We will use the term for economies as well. Notice that every simple society which is nonempty is necessarily positive. We let  $\mathcal{B}^*$  denote the set of all simple societies and  $\mathcal{B}^{*+} = \mathcal{B}^* \cap \mathcal{B}^+$ . The obvious proof of the following fact is left to the reader:

*Lemma 1.* For all  $S, T \in \mathcal{B}^*$  such that  $S \subset T$ , there exists a measure-preserving automorphism  $\pi$  on  $I$  such that  $\pi(S) = [0, \lambda(S)) \subset [0, \lambda(T)) = \pi(T)$ .

So much for simple societies. Given now an arbitrary  $S \in \mathcal{B}^+$ , define a *mixing sequence* for  $S$  to be a sequence  $\{\pi_n\}_{n \in \mathbb{N}}$  of measure-preserving automorphisms on  $S$  such that, for all  $T, T' \in \mathcal{B}_S$ ,  $\lambda(\pi_n(T) \cap T') \rightarrow \lambda(T)\lambda(T')$ . Because  $S$  is uncountable,  $(S, \mathcal{B}_S)$  is isomorphic to the measurable space formed by the closed unit interval and its Borel subsets [see, e.g., Proposition 1.1. in Aumann-Shapley (1974)]. Proposition 14.3, in Aumann-Shapley (1974), therefore applies: a mixing sequence for  $S$  exists. We use this fact in our next two lemmata.

**Lemma 2.** For all  $S, S', T, T' \in \mathcal{B}^+$ , there exists a measure-preserving automorphism  $\pi$  on  $I$  such that  $\pi(S) \cap S' \in \mathcal{B}^+$  and  $\pi(T) \cap T' \in \mathcal{B}^+$ .

*Proof.* Let  $S, S', T, T' \in \mathcal{B}^+$ . Let  $\{\pi_n\}$  be a mixing sequence for  $I$ . By definition, there exists  $n(S, S')$  and  $n(T, T')$  such that, for all  $n \geq \max\{n(S, S'), n(T, T')\}$ , we have  $\lambda(\pi_n(S) \cap S') > 0$  and  $\lambda(\pi_n(T) \cap T') > 0$   $\square$

Let us say that a rule  $F$  satisfies the property of *Equal Treatment of Equals* if for every  $(S, U_S) \in \mathcal{E}$  and every  $u \in D$  there is a number  $\alpha(u, S, U_S)$  such that  $u(F_i(S, U_S)) = \alpha(u, S, U_S)$  for almost all  $i \in U_S^{-1}(u)$ . If such a number exists, it is obviously unique when  $S$  is positive.

**Lemma 3.** If  $F$  satisfies Anonymity, then it satisfies *Equal Treatment of Equals*.

*Proof.* Suppose that  $F$  satisfies Anonymity but violates *Equal Treatment of Equals*. There must then exist  $(S, U_S) \in \mathcal{E}$ ,  $u \in D$ ,  $\alpha \in \mathbb{R}$ , and  $T, T' \in \mathcal{B}^+$  such that  $U_S(i) = u$  for all  $i \in T' \cup T^+$ , but  $u(F_i(S, U_S)) < \alpha < u(F_j(S, U_S))$  for all  $i \in T'$  and all  $j \in T^+$ . Set  $T = T' \cup T^+$ . Since  $T \in \mathcal{B}^+$ , there exists a mixing sequence  $\{\tau_n\}$  for  $T$ . For every  $n$ , define the mapping  $\pi_n$  on  $I$  by  $\pi_n(i) = \tau_n(i)$  if  $i \in T$  and  $\pi_n(i) = i$  if  $i \in I \setminus T$ . Since  $\tau_n$  is a measure-preserving automorphism on  $T$ ,  $\pi_n$  is a measure-preserving automorphism on  $I$ . For notational convenience, let  $(S_n, U_{S_n}^n)$  denote the image of  $(S, U_S)$  under  $\pi_n$ . Observe that  $S_n = S$  and  $U_{S_n}^n(i) = U_S(i)$  for all  $i \in S$ . Therefore,  $F(S_n, U_{S_n}^n) = F(S, U_S)$  and consequently  $u(F_i(S_n, U_{S_n}^n)) < \alpha$  for all  $i \in T'$ . By Anonymity, however,  $u(F_i(S_n, U_{S_n}^n)) > \alpha$  for almost all  $i \in \pi_n(T^+) = \tau_n(T^+)$ . Since  $\lambda(\tau_n(T^+) \cap T') > 0$  for all  $n$  large enough, we have a contradiction  $\square$

## 5. SOME CONTINUITY LEMMATA

Efficiency and Solidarity, especially when they are used in conjunction with *Equal Treatment of Equals* or *Anonymity*, imply some nice continuity properties. This section collects them in a series of four lemmata. Lemma 4, which relies solely on *Efficiency and Solidarity*, is a very basic fact. It will be used in the next section. Lemma 5, which assumes *Equal Treatment of Equals*, is mainly used in the proofs of

Lemmata 6 and 7; it also plays a direct but minor role in the next section. Lemma 6, which requires Anonymity, is the central result of this section and will be crucial in proving our theorem. In a sense to be made precise shortly, it asserts that there is no loss of generality in assuming that a positive society of individuals holding an a priori fixed utility function  $u_0$  is present in all positive economies. Lemma 7 establishes a useful regularity property of the range of a rule satisfying Efficiency, Anonymity, and Solidarity.

If  $S, T \in \mathcal{B}$ ,  $S \Delta T$  denotes the symmetric difference between  $S$  and  $T$ , i.e.,  $S \Delta T = (S \setminus T) \cup (T \setminus S)$ . In order to alleviate the text, we adopt the following convention:

*Notation 1.* Whenever a fixed profile  $U_1$  is being considered for society 1, economies are identified with societies:  $S$  stands for  $(S, U_1|_S)$  if  $S \in \mathcal{B}$ . Moreover, if  $u \in D$ , we write  $S_u$  instead of  $(U_1|_S)^{-1}(u)$  and let  $\mathcal{B}_u^+ := \{S \in \mathcal{B} \mid S_u \in \mathcal{B}^+\}$ .

*Lemma 4.* Let  $F$  satisfy Efficiency and Solidarity, let  $U_1 \in D^I$  and let  $S, T \in \mathcal{B}$ . If  $\lambda(S \Delta T) = 0$ , then  $u_i(F_i(S)) = u_i(F_i(T))$  for almost all  $i \in S \cap T$ .

*Proof.* Let  $F, U_1, S$  and  $T$  satisfy the assumptions of the lemma. Contrary to the claim, suppose that, say,  $u_i(F_i(S)) > u_i(F_i(T))$  for all  $i$  in some positive subsociety of  $S \cap T$ . By Solidarity,  $u_i(F_i(S)) > u_i(F_i(T))$  for almost all  $i \in S \cap T$ . Define the mapping  $x$  on  $T$  by  $x(i) = f_S(i)$  if  $i \in T \cap S$  and  $x(i) = f_T(i)$  if  $i \in T \setminus S$ . Since  $\lambda(S \Delta T) = 0$ ,  $\int_T x = \int_S f_S = \omega$ , i.e.,  $x \in X(T)$ . This contradicts the assumption that  $f_T$  is an efficient allocation in  $X(T)$   $\square$

Consider again a fixed profile  $U_1$ . Recall that if a rule  $F$  satisfies Equal Treatment of Equals, then for each  $u \in D$  and  $S \in \mathcal{B}_u^+$ , there is a unique number  $\alpha(u, S)$  such that  $u(F_i(S)) = \alpha(u, S)$  for almost all  $i \in S$ .

*Lemma 5.* Let  $F$  satisfy Efficiency, Equal Treatment of Equals, and Solidarity. Let  $U_1 \in D^I$ ,  $u \in D$  and  $S \in \mathcal{B}_u^+$ . Then, for each  $\varepsilon > 0$ , there is some  $\delta > 0$  such that, for any  $S' \in \mathcal{B}_u^+$ ,  $|\alpha(u, S) - \alpha(u, S')| < \varepsilon$  whenever  $\lambda(S \Delta S') < \delta$ .

*Proof.* Let  $F, U, u$  and  $S$  be as required. Suppose, by way of contradiction, that there exists some  $\varepsilon > 0$  and a sequence  $\{S_n\}_{n \in \mathbb{N}}$  in  $\mathcal{X}_u^+$  such that  $\lambda(S \Delta S_n) \rightarrow 0$  but  $|\alpha(u, S) - \alpha(u, S_n)| \geq \varepsilon$  for all  $n$ . Extracting a subsequence if necessary, we may assume that one of the following statements holds:

$$\text{for all } n, \alpha(u, S_n) \leq \alpha(u, S) - \varepsilon, \quad (1)$$

$$\text{for all } n, \alpha(u, S) \leq \alpha(u, S_n) - \varepsilon. \quad (2)$$

We will assume (1), but the proof goes through, *mutatis mutandis*, when (2) holds. For each  $n$ , let  $T_n := S_n \cup S$ . Since  $\lambda(T_n \setminus S) = \lambda(T_n \Delta S) = \lambda(S \Delta S_n)$  for every  $n$ , we have  $\lambda(T_n \setminus S) \rightarrow 0$ . Since  $S_n \subset T_n$  for each  $n$ , (1), Efficiency and Solidarity imply:

$$\text{for all } n, u(F_i(T_n)) \leq u(F_i(S)) - \varepsilon \text{ for almost all } i \in S_u, \quad (3)$$

and

$$\text{for all } n, u_i(F_i(T_n)) < u_i(F_i(S)) \text{ for almost all } i \in S. \quad (4)$$

Recall that the utility function  $u$  is continuous. Therefore (3) implies that for each  $i \in S_u$ , we can find some  $z(i) \in \mathbb{R}_{++}^\ell$  such that

$$\text{for all } n, u(F_i(T_n)) < u(F_i(z) - z(i)). \quad (5)$$

Fix a subsociety  $R_u$  of  $S_u$  such that  $\lambda(R_u) < \lambda(S)$  (if  $\lambda(S_u) < \lambda(S)$ , we may simply take  $R_u = S_u$ ). For each  $n$ , define the mapping  $x_n$  on  $T_n$  as follows:

$$x_n(i) = \begin{cases} f_{T_n}(i) & \text{if } i \in T_n \setminus S, \\ f_S(i) - z(i) & \text{if } i \in R_u, \\ f_S(i) + \frac{1}{\lambda(S \setminus R_u)} \left( \int_{R_u} z - \int_{T_n \setminus S} f_{T_n} \right) & \text{if } i \in S \setminus R_u. \end{cases}$$

Observe that  $\int_{T_n} x_n = \int_S f_S = \omega$ , i.e.,  $x_n \in X(T_n)$  for each  $n$ . Since  $\lambda(T_n \setminus S) \rightarrow 0$ , we know that  $\int_{T_n \setminus S} f_{T_n} \rightarrow 0$ . Since  $\int_{R_u} z \in \mathbb{R}_{++}^\ell$ , there exists some  $n^* \in \mathbb{N}$  such that  $\int_{R_u} z - \int_{T_n \setminus S} f_{T_n} \in \mathbb{R}_{++}^\ell$  for all  $n \geq n^*$ . Since utility functions are strictly increasing,  $u_i(x_n(i)) >$

$u_i(f_S(i))$  for all  $i \in S \setminus R_u$  and  $n \geq n^*$ . In view of (4) and (5), this means that  $x_n$  Pareto-dominates  $f_T$  for all  $n \geq n^*$ , contradicting Efficiency  $\square$

With Lemma 5 in hand, we may now turn to the most important result of this section. Fix an arbitrary utility function  $u_0$ . Suppose that a (positive, proper, and simple) society of individuals holding some common utility function  $u$  reaches a given utility level in some (positive, proper, and simple) economy. Then, the same utility level is also reached (by a positive society with the same utility function  $u$ ) in a (proper) economy where a positive society holds the utility function  $u_0$ . Let us be more precise:

**Lemma 6.** *Let  $F$  satisfy Efficiency, Anonymity, and Solidarity. Let  $u, u_0 \in D$  and let the economy  $(S, U_S)$  be such that  $S \in \mathcal{B}^{**}$  and  $U_S^{-1}(u) \in \mathcal{B}^{**}$ . Then, there exists an economy  $(T, V_T)$  such that  $T \in \mathcal{B}^{**}$ ,  $V_T^{-1}(u) \in \mathcal{B}^{**}$ ,  $V_T^{-1}(u_0) \in \mathcal{B}^{**}$ , and  $u(F_i(S, U_S)) = u(F_j(T, V_T))$  for almost all  $i \in U_S^{-1}(u)$  and almost all  $j \in V_T^{-1}(u)$ .*

*Proof.* Let  $F, u, u_0$  and  $(S, U_S)$  be as required. By Anonymity and Lemma 1, there is no loss of generality in assuming that  $U_S^{-1}(u) = ]0, b_u) \subset ]0, b) = S$ , where  $0 < b_u \leq b < 1$ . Define  $U_1 \in D^1$  by  $U_1(i) = U_S(i)$  if  $i \in ]0, b)$  and  $U_1(i) = u_0$  if  $i \in ]b, 1)$ . This profile being fixed throughout the proof, we will again use Notation 1. For each  $T \in \mathcal{B}_u^*$ , there exists, by Equal Treatment of Equals, a unique number  $\alpha(u, T)$  such that  $u(F_i(T)) = \alpha(u, T)$  for almost all  $i \in T$ . Choose  $c \in (0, b_u)$ . By Efficiency and Solidarity,

$$\alpha(u, 1) < \alpha(u, ]0, b)) < \alpha(u, ]0, c)). \quad (6)$$

For each  $a \in [c, 1]$ , define  $\alpha_c(a) := \alpha(u, ]0, c) \cup ]a, 1)$  (recall that  $]1, 1)$  is well defined and equal to the empty set). Rewrite (6) as follows:

$$\alpha_c(c) < \alpha(u, ]0, b)) < \alpha_c(1).$$

By Lemma 5, the mapping  $\alpha_c : [c, 1] \rightarrow \mathbb{R}$  is continuous. By the intermediate value theorem, there exists some  $a^* \in (c, 1)$  such that  $\alpha_c(a^*) = \alpha(u, ]0, b))$ . Defining  $T^* = ]0, c) \cup ]a^*, 1)$ , we have  $u(F_j(T^*)) = u(F_i(S))$  for almost all  $j \in T^*$  and almost all  $i \in S$ . Since  $T^*, T_u^*, T_{u_0}^* \in \mathcal{B}^{**}$ , we are done  $\square$

For any  $u \in D$ , let  $\bar{\alpha}(u) := \sup\{u(x) \mid x \in \mathbb{R}_+^I\}$ , with the convention that  $\bar{\alpha}(u) = +\infty$  if  $u$  has no upper bound. Given  $u \in D$  and a rule  $F$ , define  $A^{*+}(u, F) := \{\alpha \in \mathbb{R} \mid \exists (S, U_S) \in \mathcal{E} : \mathcal{S} \in \mathcal{S}^{*+}, U_S^{-1}(u) \in \mathcal{S}^{*+}, \text{ and } u(F_1(S, U_S)) = \alpha \text{ for almost all } i \in U_S^{-1}(u)\}$ .

**Lemma 7.** *Let  $F$  satisfy Efficiency, Anonymity, and Solidarity. Then, for each  $u \in D$ , there exists some  $\underline{\alpha}(u, F) \in [-\infty, \bar{\alpha}(u)]$  such that  $A^{*+}(u, F) = (\underline{\alpha}(u, F), \bar{\alpha}(u))$ .*

*Proof.* Let  $F$  be as required and fix  $u \in D$ . We start off with the following observation:

$$\forall (S, U_S) \in \mathcal{E}^+ \text{ such that } U_S(i) = u \text{ for all } i \in S, u(F_1(S, U_S)) = u(\omega / \lambda(S)) \text{ for almost all } i \in S. \quad (7)$$

The reason is the following. By Equal Treatment of Equals,  $u(F_1(S, U_S)) = \alpha$  for almost all  $i \in S$  and some  $\alpha \in \mathbb{R}$ . Clearly  $\alpha \geq u(\omega / \lambda(S))$  for otherwise  $F(S, U_S)$  would be Pareto-dominated by the equal split allocation in which  $x_{S, U_S}(i) = \omega / \lambda(S)$  for each  $i \in S$ . But if  $\alpha > u(\omega / \lambda(S))$ , we get  $\alpha > u\left[\frac{1}{\lambda(S)} \int_S f_{S, U_S}\right] \geq \alpha$ , where the last inequality holds by quasiconcavity of  $u$ . This is absurd, hence  $\alpha = u(\omega / \lambda(S))$ .

Define  $\underline{\alpha}(u, F) = \inf A^{*+}(u, F)$ , with the convention that  $\underline{\alpha}(u, F) = -\infty$  if  $A^{*+}(u, F)$  has no lower bound. We claim that

$$(\underline{\alpha}(u, F), \bar{\alpha}(u)) \subset A^{*+}(u, F). \quad (8)$$

Fix  $\alpha \in (\underline{\alpha}(u, F), \bar{\alpha}(u))$ . By definition of  $\underline{\alpha}(u, F)$ , there exists  $\beta \in [\underline{\alpha}(u, F), \alpha) \cap A^{*+}(u, F)$ . Let  $(S, U_S) \in \mathcal{E}$  be such that  $S \in \mathcal{S}^{*+}$ ,  $U_S^{-1}(u) \in \mathcal{S}^{*+}$ , and  $u(F_1(S, U_S)) = \beta$  for almost all  $i \in U_S^{-1}(u)$ . By Anonymity and Lemma 1, we may suppose that  $U_S^{-1}(u) = [0, b_u] \subset [0, b] = S$ . Let  $U_1$  be any profile in  $D^I$  such that  $U_1|_S = U_S$ . Keeping this profile fixed, we may again resort to Notation 1. Choose  $\gamma \in (\max\{\alpha, u(\omega / b_u)\}, \bar{\alpha}(u))$ . By continuity of  $u$ , there is some  $c \in (0, b_u)$  such that  $\gamma = u(\omega / c)$ . By (7),  $u(F_1([0, c])) = \gamma$  for almost all  $i \in [0, c]$ . For every  $a \in [c, b)$ ,

there exists by Equal Treatment of Equals a unique number  $\alpha_c(a)$  such that  $u(F_i(\{0, a\})) = \alpha_c(a)$  for almost all  $i \in [0, c)$ . By Lemma 5, the mapping  $\alpha_c : [c, b] \rightarrow \mathbb{R}$  is continuous. Since  $\alpha_c(b) = \beta < \alpha < \gamma = \alpha_c(c)$ , the intermediate value theorem guarantees the existence of a number  $a^* \in (c, b)$  such that  $\alpha_c(a^*) = \alpha$ . This means that  $u(F_i(\{0, a^*\})) = \alpha$  for almost all  $i \in [0, c)$ , hence, by Equal Treatment of Equals, for all  $i \in [0, a^*]_U$ . Thus,  $\alpha \in A^{*+}(u, F)$ , proving (8).

To complete the proof, we need to check that  $\underline{\alpha}(u, F) \in A^{*+}(u, F)$  and  $\bar{\alpha}(u) \in A^{*+}(u, F)$ . The latter statement being obvious, let us concentrate on the former. Suppose  $\underline{\alpha}(u, F) \in A^{*+}(u, F)$ . Let  $(S, U_S)$  be an economy such that  $S \in \mathcal{E}^{*+}$ ,  $U_S^{-1}(u) \in \mathcal{E}^{*+}$ , and  $u(F_i(S, U_S)) = \underline{\alpha}(u, F)$  for almost all  $i \in U_S^{-1}(u)$ . Since  $S \in \mathcal{E}^{*+}$ , there exists an economy  $(T, V_T)$  such that  $T \in \mathcal{E}^{*+}$ ,  $V_T^{-1}(u) \in \mathcal{E}^{*+}$ ,  $S \subset T$ ,  $\lambda(S) < \lambda(T)$ , and  $V_T|_S = U_S$ . By Efficiency, Equal Treatment of Equals, and Solidarity, there exists some  $\alpha < \underline{\alpha}(u, F)$  such that  $u(F_i(T, V_T)) = \alpha$  for almost all  $i \in V_T^{-1}(u)$ , contradicting the definition of  $\underline{\alpha}(u, F)$   $\square$

## 6. PROOF OF THE THEOREM

We proceed in two steps. We first demonstrate, under a range regularity condition, that every rule satisfying Efficiency, Anonymity, and Solidarity, must rescale the individuals' utility functions and allocate the aggregate bundle so as to equalize the rescaled utility levels. This is Lemma 8 below.<sup>2</sup> We then show how the Equal Split Lower Bound pinpoints an essentially unique procedure leading to the Pazner-Schmeidler rules.

Given  $u \in D$  and a rule  $F$ , define  $A^+(u, F) := \{\alpha \in \mathbb{R} \mid \exists (S, U_S) \in \mathcal{E} : S \in \mathcal{E}^{*+}, U_S^{-1}(u) \in \mathcal{E}^{*+}, \text{ and } u(F_i(S, U_S)) = \alpha \text{ for almost all } i \in U_S^{-1}(u)\}$ . Obviously,  $A^{*+}(u, F) \subset A^+(u, F)$ . We say that  $F$  has *nice range* if  $A^{*+}(u, F) = A^+(u, F)$  for each  $u \in D$ . Let us formally define a mapping  $\hat{u}$  on  $\mathbb{R}_+^I$  to be a *rescaling* of  $u \in D$  if for all  $x, y \in \mathbb{R}_+^I$ ,  $\hat{u}(x) \leq \hat{u}(y)$  if and only if  $u(x) \leq u(y)$ . We call the mapping  $\wedge : u \mapsto \hat{u}$  a (*rescaling*) *procedure*.

<sup>2</sup> Although we conjecture that the range condition is superfluous, we do not attempt to dispense with it at this stage. It will be easier to drop it later on.



**Lemma 8.** *Suppose that  $F$  satisfies Efficiency, Anonymity, Solidarity, and has nice range. Then, there exists a rescaling procedure  $\hat{\cdot}$  such that*

$$\forall (S, U_S) \in \mathcal{E}, \exists \beta(S, U_S) \in \mathbb{R} : \hat{u}_i(F_i(S, U_S)) = \beta(S, U_S) \text{ for almost all } i \in S. \quad (9)$$

*Proof.* Let  $F$  satisfy the assumptions of the lemma. Choose  $u_0$  arbitrarily in  $D$ . For each  $u \in D$ , we know from the nice range assumption and Lemma 7 that  $A^{++}(u, F) = (\underline{\alpha}(u, F), \bar{\alpha}(u))$  for some  $\underline{\alpha}(u, F) \in ]-\infty, \bar{\alpha}(u))$ , where  $\bar{\alpha}(u) := \sup\{u(x) \mid x \in \mathbb{R}_+^{\ell}\}$ . Construct the mapping  $\hat{u} : \mathbb{R}_+^{\ell} \rightarrow \mathbb{R}$  in two pieces. If  $u(x) > \underline{\alpha}(u, F)$ , define  $\hat{u}(x)$  by the condition:

$$\begin{aligned} \exists (S, U_S) \in \mathcal{E} : U_S^{-1}(u), U_S^{-1}(u_0) \in \mathcal{E}^{++}, u(F_i(S, U_S)) = u(x) \text{ for almost all } \\ i \in U_S^{-1}(u), \text{ and } u_0(F_i(S, U_S)) = \hat{u}(x) \text{ for almost all } i \in U_S^{-1}(u_0). \end{aligned} \quad (10)$$

If  $u(x) \leq \underline{\alpha}(u, F)$ , simply let

$$\hat{u}(x) = u(x) + \beta(u, F) - \underline{\alpha}(u, F), \quad (11)$$

where  $\beta(u, F) := \inf\{\hat{u}(x) \mid u(x) > \underline{\alpha}(u, F)\}$ . We claim that  $\hat{u}$  is well defined (Step 1), that it is a rescaling of  $u$  (Step 2), and that  $F$  equalizes the utilities rescaled according to the procedure  $\hat{\cdot}$  (Step 3).

**Step 1:** for each  $u \in D$ ,  $\hat{u}$  is a well-defined mapping.

We need only show that whenever  $u(x) > \underline{\alpha}(u, F)$ , there exists a unique number  $\hat{u}(x)$  satisfying (10). When  $u(x) \leq \underline{\alpha}(u, F)$ , existence and uniqueness of  $\hat{u}(x)$  immediately follow from (11). Let us thus assume that  $u(x) > \underline{\alpha}(u, F)$ . Then,  $u(x) \in A^{++}(u, F)$ , i.e., there exists an economy  $(S, U_S)$  such that  $S \in \mathcal{E}^{++}$ ,  $U_S^{-1}(u) \in \mathcal{E}^{++}$ , and  $u(F_i(S, U_S)) = u(x)$  for almost all  $i \in U_S^{-1}(u)$ . By Lemma 6, there exists an economy  $(T, V_T)$  such that  $V_T^{-1}(u), V_T^{-1}(u_0) \in \mathcal{E}^{++}$  and  $u(F_i(T, V_T)) = u(x)$  for almost all  $i \in V_T^{-1}(u)$ . By Equal Treatment of Equals, there exists a number  $\hat{u}(x)$  such that  $u_0(F_i(T, V_T)) = \hat{u}(x)$  for almost all  $i \in V_T^{-1}(u_0)$ . By construction, this number satisfies (10).

Having proved existence, let us now establish uniqueness. Suppose, by way of contradiction, that there exist  $(S, U_S), (T, V_T) \in \mathcal{Z}$  such that  $U_S^{-1}(u), U_S^{-1}(u_0), V_T^{-1}(u), V_T^{-1}(u_0) \in \mathcal{X}^{+-}$  and

$$u(F_i(S, U_S)) = u(x) = u(F_j(T, V_T)) \text{ for almost all } i \in U_S^{-1}(u) \text{ and almost all } j \in V_T^{-1}(u), \quad (12)$$

while

$$u_0(F_i(S, U_S)) = \beta < \beta = u_0(F_j(T, V_T)) \text{ for almost all } i \in U_S^{-1}(u_0) \text{ and almost all } j \in V_T^{-1}(u_0). \quad (13)$$

By Anonymity and Lemma 2, we may assume that  $\lambda(U_S^{-1}(u) \cap V_T^{-1}(u)) > 0$  and  $\lambda(U_S^{-1}(u_0) \cap V_T^{-1}(u_0)) > 0$ . But then, (12) and (13) contradict Solidarity.

*Step 2: for each  $u \in D$ ,  $\hat{u}$  is a rescaling of  $u$ .*

We fix  $u \in D$ , assume that  $u(x) < u(y)$ , and prove that  $\hat{u}(x) < \hat{u}(y)$ . (A similar argument would prove that  $\hat{u}(x) = \hat{u}(y)$  if  $u(x) = u(y)$ .) There are three cases.

*Case (i):  $u(y) \leq \underline{\alpha}(u, F)$ .*

Then, it is obvious from (11) that  $\hat{u}(x) < \hat{u}(y)$ .

*Case (ii):  $\underline{\alpha}(u, F) \leq u(x)$ .*

By definition of  $\wedge$ , there exist  $(S, U_S), (T, V_T) \in \mathcal{Z}$  such that  $U_S^{-1}(u), U_S^{-1}(u_0), V_T^{-1}(u), V_T^{-1}(u_0) \in \mathcal{X}^{+-}$  and

$$u(F_i(S, U_S)) = u(x) \text{ for almost all } i \in U_S^{-1}(u),$$

$$u_0(F_i(S, U_S)) = \hat{u}(x) \text{ for almost all } i \in U_S^{-1}(u_0),$$

$$u(F_i(T, V_T)) = u(y) \text{ for almost all } i \in V_T^{-1}(u),$$

$$u_0(F_i(T, V_T)) = \hat{u}(y) \text{ for almost all } i \in V_T^{-1}(u_0).$$

Using Anonymity and Lemma 2, we may once again assume that  $\lambda(U_S^{-1}(u) \cap V_T^{-1}(u)) > 0$  and  $\lambda(U_S^{-1}(u_0) \cap V_T^{-1}(u_0)) > 0$ . Since  $u(x) < u(y)$ ,  $u(F_i(S, U_S)) < u(F_i(T, V_T))$  for almost all  $i \in U_S^{-1}(u) \cap V_T^{-1}(u)$ . By Solidarity,  $u_0(F_i(S, U_S)) < u_0(F_i(T, V_T))$  for almost all  $i \in U_S^{-1}(u_0) \cap V_T^{-1}(u_0)$ , implying that  $\hat{u}(x) < \hat{u}(y)$ .

*Case (iii):*  $u(x) \leq \underline{\alpha}(u, F) < u(y)$ .

By (11),  $\hat{u}(x) \leq \underline{\beta}(u, F)$ . By definition,  $\underline{\beta}(u, F) \leq \hat{u}(y)$ . We claim that this inequality is strict. Indeed, suppose  $\hat{u}(y) = \underline{\beta}(u, F)$ . As  $u$  is continuous, there exists  $z \in \mathbb{R}_+^L$  such that  $\underline{\alpha}(u, F) < u(z) < u(y)$ . From case (ii),  $\hat{u}(z) < \hat{u}(y)$ . Hence,  $\hat{u}(z) < \underline{\beta}(u, F)$ , contradicting the definition of  $\underline{\beta}(u, F)$ . It follows that  $\hat{u}(x) < \hat{u}(y)$ .

*Step 3: (9) holds for the rescaling procedure  $\wedge$  defined by (10)–(11).*

Since  $F$  satisfies Equal Treatment of Equals, there is for each utility function  $u$  and each economy  $(S, U_S)$  such that  $U_S^{-1}(u) \in \mathcal{E}^+$  a unique number  $\alpha(u, S, U_S)$  such that  $u(F_i(S, U_S)) = \alpha(u, S, U_S)$  for almost all  $i \in U_S^{-1}(u)$ . Since  $\wedge$  is a rescaling procedure, there also exists a unique number  $\beta(u, S, U_S)$  such that  $\hat{u}(F_i(S, U_S)) = \beta(u, S, U_S)$  for almost all  $i \in U_S^{-1}(u)$ . For any two utility functions  $u$  and  $v$ , it is obvious that  $\beta(u, S, U_S) = \beta(v, S, U_S)$  in those economies  $(S, U_S)$  where  $U_S^{-1}(u)$ ,  $U_S^{-1}(v)$ , and  $U_S^{-1}(u_0)$  are positive. In order to establish that  $F$  equalizes the rescaled utilities in *all* economies in  $\mathcal{E}$ , however, we need a continuity property of the rescaled utilities. The property is the analogue of Lemma 5. Since it holds for a fixed profile  $U_1$ , we use Notation 1:

Let  $U_1 \in D^1$ ,  $u \in D$  and  $S \in \mathcal{E}_u^+$ . Then, for each  $\varepsilon > 0$ , there is some  $\delta > 0$  such that, for any  $S' \in \mathcal{E}_u^+$ ,  $|\beta(u, S) - \beta(u, S')| < \varepsilon$  whenever  $\lambda(S \Delta S') < \delta$ . (14)

The proof of (14) goes as follows. Fix  $U_1$ ,  $u$  and  $S$  as required. Distinguish two cases.

*Case (i):*  $\alpha(u, S) > \underline{\alpha}(u, F)$ .

Then (10) applies: there exists  $(T, V_T) \in \mathcal{E}$  such that  $V_T^{-1}(u) \in \mathcal{E}^+$ ,  $V_T^{-1}(u_0) \in \mathcal{E}^+$ , and

$$\alpha(u, S) = \alpha(u, T, V_T) \text{ and } \beta(u, S) = \alpha(u_0, T, V_T). \quad (15)$$

Suppose, contrary to the claim, that there exists some  $\varepsilon > 0$  and a sequence  $\{S_n\}_{n \in \mathbb{N}}$  in  $\mathcal{E}_u^+$  such that  $\lambda(S_n \Delta S) \rightarrow 0$  but  $|\beta(u, S_n) - \beta(u, S)| \geq \varepsilon$  for all  $n$ . Extracting a subsequence if necessary, we may assume that either  $\beta(u, S_n) \leq \beta(u, S) - \varepsilon$  for all  $n$ , or

$$\beta(u, S) \leq \beta(u, S_n) - \varepsilon \text{ for all } n. \quad (16)$$

We will assume (16); the argument is easily adapted to the other case. By Lemma 5,

$$\alpha(u, S_n) \rightarrow \alpha(u, S). \quad (17)$$

We may therefore assume, taking again a subsequence if needed, that  $\alpha(u, S_n) > \alpha(u, S)$  for all  $n$ . Applying (10) once more, we can find for each  $n$  an economy  $(T_n, V_{T_n}^n)$  such that  $(V_{T_n}^n)^{-1}(u) \in \mathcal{E}^+$ ,  $(V_{T_n}^n)^{-1}(u_0) \in \mathcal{E}^+$ , and

$$\alpha(u, S_n) = \alpha(u, T_n, V_{T_n}^n) \text{ and } \beta(u, S_n) = \alpha(u_0, T_n, V_{T_n}^n). \quad (18)$$

From (15), (16) and (18),  $\alpha(u_0, T, V_T) \leq \alpha(u_0, T_n, V_{T_n}^n) - \varepsilon$  for all  $n$ . We can therefore find some  $\beta^*$  such that

$$\alpha(u_0, T, V_T) < \beta^* < \alpha(u_0, T_n, V_{T_n}^n) \text{ for all } n. \quad (19)$$

Since  $\alpha(u_0, T, V_T)$  and each  $\alpha(u_0, T_n, V_{T_n}^n)$  belong to  $A^{*+}(u_0, F) = A^{*+}(u_0, F)$ , Lemma 7 ensures that  $\beta^* \in A^{*+}(u_0, F)$ . That is, there exists an economy  $(R_0, W_{R_0}^0)$  such that  $R_0 \cdot (W_{R_0}^0)^{-1}(u_0) \in \mathcal{E}^{*+}$  and  $\alpha(u_0, R_0, W_{R_0}^0) = \beta^*$ . Applying Lemma 6 (with  $u_0$  playing the role of  $u$  and vice-versa), there exists an economy  $(R, W_R)$  such that  $R \cdot W_R^{-1}(u_0), W_R^{-1}(u) \in \mathcal{E}^{*+}$  and  $\alpha(u_0, R, W_R) = \beta^*$ . Rewriting (19), we get

$$\alpha(u_0, T, V_T) < \alpha(u_0, R, W_R), \quad (20)$$

and, for every  $n$ ,

$$\alpha(u_0, R, W_R) < \alpha(u_0, T_n, V_{T_n}^n). \quad (21)$$

Consider inequality (20). By Lemma 2 and Anonymity, we may assume that  $V_T^{-1}(u) \cap W_R^{-1}(u) \in \mathcal{X}^+$  and  $V_T^{-1}(u_0) \cap W_R^{-1}(u_0) \in \mathcal{X}^+$ . Applying Solidarity yields

$$\alpha(u, T, V_T) < \alpha(u, R, W_R). \quad (22)$$

Likewise, fix  $n$  and consider (21). Resorting again to Lemma 2 and Anonymity, let us assume that  $W_R^{-1}(u) \cap (V_{T_n}^n)^{-1}(u) \in \mathcal{X}^+$  and  $W_R^{-1}(u_0) \cap (V_{T_n}^n)^{-1}(u_0) \in \mathcal{X}^+$ . By Solidarity,

$$\alpha(u, R, W_R) < \alpha(u, T_n, V_{T_n}^n). \quad (23)$$

By (15), (17) and (18), however,  $\alpha(u, T_n, V_{T_n}^n) \rightarrow \alpha(u, T, V_T)$ . Because of (22), there must exist some  $n^* \in \mathbb{N}$  such that  $\alpha(u, T_n, V_{T_n}^n) < \alpha(u, R, W_R)$  for all  $n \geq n^*$ . Since (23) holds for all  $n$ , we have a contradiction.

*Case (ii):*  $\alpha(u, S) \leq \underline{\alpha}(u, F)$ .

By definition of  $\underline{\alpha}(u, F)$ , we must have  $\lambda(S) = 1$ . Moreover,  $\alpha(u, S) = \underline{\alpha}(u, F)$ . (Indeed, if  $\alpha(u, S) < \underline{\alpha}(u, F)$ , take a sequence  $\{S_n\}_{n \in \mathbb{N}}$  in  $\mathcal{X}_u^+$  such that  $\lambda(S_n) < 1$  for each  $n$  and  $\lambda(S_n) \rightarrow 1$ . By Lemma 5,  $\alpha(u, S_n) \rightarrow \alpha(u, S)$ , hence  $\alpha(u, S_n) < \underline{\alpha}(u, F)$  for  $n$  large enough, a contradiction.) Now (11) applies:

$$\beta(u, S) = \alpha(u, S) + \underline{\beta}(u, F) - \underline{\alpha}(u, F) = \underline{\beta}(u, F). \quad (24)$$

Suppose again, contrary to the claim, that there exists  $\varepsilon > 0$  and a sequence  $\{S_n\}_{n \in \mathbb{N}}$  in  $\mathcal{X}_u^+$  such that  $\lambda(S_n \Delta S) \rightarrow 0$  but  $|\beta(u, S_n) - \beta(u, S)| \geq \varepsilon$  for all  $n$ . Since  $\lambda(S) = 1$ , Lemma 4 implies that  $\lambda(S_n) < 1$  for each  $n$ . By Efficiency and Solidarity, then,

$$\beta(u, S) \leq \beta(u, S_n) - \varepsilon \text{ for all } n. \quad (25)$$

On the other hand,  $\alpha(u, S_n) \in A^+(u, F)$ , hence  $\alpha(u, S_n) > \underline{\alpha}(u, F)$  for each  $n$ . Therefore, (10) applies here: for each  $n$ , there is an economy  $(T_n, V_{T_n}^n)$  such that  $(V_{T_n}^n)^{-1}(u) \in \mathcal{E}^+$ ,  $(V_{T_n}^n)^{-1}(u_0) \in \mathcal{E}^+$ , and

$$\alpha(u, S_n) = \alpha(u, T_n, V_{T_n}^n) \text{ and } \beta(u, S_n) = \alpha(u_0, T_n, V_{T_n}^n). \quad (26)$$

Gathering (24), (25) and (26), we can find some  $\beta^*$  such that

$$\beta(u, F) < \beta^* < \alpha(u_0, T_n, V_{T_n}^n) \text{ for all } n. \quad (27)$$

Notice that  $\beta(u, F) \geq \underline{\alpha}(u_0, F)$ . (Indeed,  $\beta(u, F) = \inf\{\hat{u}(x) \mid u(x) > \underline{\alpha}(u, F)\} = \inf\{\alpha(u_0, T, V_T) \mid (T, V_T) \in \mathcal{E}, V_T^{-1}(u) \in \mathcal{E}^+, V_T^{-1}(u_0) \in \mathcal{E}^+\}$ . The latter set being included in  $A^+(u_0, F)$ , it follows that  $\beta(u, F) \geq \inf A^+(u_0, F) = \underline{\alpha}(u_0, F)$ ). Therefore, by Lemma 7,  $\beta^* \in A^{*+}(u_0, F)$ . As in case (i), we may apply Lemma 6 to find an economy  $(R, W_R)$  such that  $R, W_R^{-1}(u_0), W_R^{-1}(u) \in \mathcal{E}^+$  and  $\alpha(u_0, R, W_R) = \beta^*$ . From (27), then,

$$\alpha(u_0, R, W_R) < \alpha(u_0, T_n, V_{T_n}^n) \text{ for every } n. \quad (28)$$

By Lemma 5, however,  $\alpha(u, S_n) - \alpha(u, S)$ . By (26) and since  $\alpha(u, S) = \underline{\alpha}(u, F)$ ,  $\alpha(u, T_n, V_{T_n}^n) - \underline{\alpha}(u, F)$ . By definition,  $\alpha(u, R, W_R) \in A^+(u, F)$ , hence  $\underline{\alpha}(u, F) < \alpha(u, R, W_R)$ . It follows that

$$\alpha(u, T_n, V_{T_n}^n) < \alpha(u, R, W_R) \text{ for every } n \text{ large enough.} \quad (29)$$

Invoking Lemma 2 and Anonymity, we may assume that  $W_R^{-1}(u) \cap (V_{T_n}^n)^{-1}(u)$  and  $W_R^{-1}(u_0) \cap (V_{T_n}^n)^{-1}(u_0)$  are positive: (28) and (29) then violate the Solidarity axiom.

Having proved (14), we are now in a position to show that  $F$  equalizes the utilities rescaled according to  $\wedge$ . Suppose on the contrary that (9) fails. Since  $D$  is countable, there must exist two utility functions  $u, v$  and some economy  $(S, U_S)$  such that  $U_S^{-1}(u), U_S^{-1}(v) \in \mathcal{X}^+$  and

$$\beta(u, S, U_S) < \beta(v, S, U_S). \quad (30)$$

Distinguish three cases.

Case (i):  $U_S^{-1}(u_0) \in \mathcal{X}^+$ .

Then, by definition of  $\wedge$ , there is a unique number  $\alpha(u_0, S, U_S)$  such that  $\beta(u, S, U_S) = \alpha(u_0, S, U_S) = \beta(v, S, U_S)$ , contradicting (30).

Case (ii):  $U_S^{-1}(u_0) \in \mathcal{X}^0$  but  $S \in \mathcal{X}$ .

Then, we can find a sequence  $\{S_n\}_{n \in \mathbb{N}}$  in  $\mathcal{X}$  such that  $S \subset S_n$  and  $\lambda(S) < \lambda(S_n)$  for each  $n$ , with  $\lambda(S_n \setminus S) \rightarrow 0$ . Defining  $U_1$  by  $U_1(i) = U_S(i)$  if  $i \in S$  and  $U_1(i) = u_0$  if  $i \in I \setminus S$ , we know from (14) that  $\beta(u, S_n, U_1|_{S_n}) \rightarrow \beta(u, S, U_S)$  and  $\beta(v, S_n, U_1|_{S_n}) \rightarrow \beta(v, S, U_S)$ . Because of (30),  $\beta(u, S_n, U_1|_{S_n}) < \beta(v, S_n, U_1|_{S_n})$  for all large enough  $n$ .

By construction, however,  $(U_1|_{S_n})^{-1}(u_0) \in \mathcal{X}^+$  for every  $n$ . We know from case (i) that this is impossible.

Case (iii):  $U_S^{-1}(u_0) \in \mathcal{X}^0$  and  $\lambda(S) = 1$ .

Let  $\{S_n\}_{n \in \mathbb{N}}$  be a sequence in  $\mathcal{X}$  such that  $S_n \subset S$  and  $\lambda(S_n) < \lambda(S)$  for each  $n$ , with  $\lambda(S \setminus S_n) \rightarrow 0$ . For all values of  $n$  sufficiently large,  $(U_S|_{S_n})^{-1}(u) \in \mathcal{X}^+$  and  $(U_S|_{S_n})^{-1}(v) \in \mathcal{X}^+$ . From (14),  $\beta(u, S_n, U_S|_{S_n}) \rightarrow \beta(u, S, U_S)$  and  $\beta(v, S_n, U_S|_{S_n}) \rightarrow \beta(v, S, U_S)$ . Because of (30),  $\beta(u, S_n, U_1|_{S_n}) < \beta(v, S_n, U_1|_{S_n})$  for every  $n$  large enough. We know from case (ii) that this is impossible  $\square$ .

We are now in a position to prove our theorem.

*Proof of the Theorem.*

It is easy to check that the Pazner-Schmeidler rules satisfy Efficiency, the Equal Split Lower Bound, Anonymity and Solidarity. Conversely, fix a rule  $F$  satisfying all four axioms. First, we check that  $F$  has nice range, i.e., that  $A^+(u, F) \subset A^{**}(u, F)$  for all  $u \in D$ . Let thus  $u \in D$  and  $\alpha \in A^+(u, F)$ . By the Equal Split Lower Bound,  $\alpha > u(\omega)$ . Pick  $s \in (0, 1)$  such that  $u(\omega / s) = \alpha$ . Define  $S = [0, s)$  and let  $U_S(i) = u$  for all  $i \in S$ . By (7),  $u(F_i(S, U_S)) = \alpha$  for almost all  $i \in S$ , hence,  $\alpha \in A^{**}(u)$ .

Suppose now that  $F$  is not a Pazner-Schmeidler rule. Since  $D$  is countable, there must exist  $\eta, \theta \in \mathbb{R}_+$  such that

$$\eta < \theta;$$

and  $u, v \in D$  and  $(S, U_S) \in \mathcal{S}$  such that  $U_S^{-1}(u), U_S^{-1}(v) \in \mathcal{S}^+$ ,

$$u(F_i(S, U_S)) = u(\eta\omega) \text{ for almost all } i \in U_S^{-1}(u),$$

and

$$v(F_i(S, U_S)) = v(\theta\omega) \text{ for almost all } i \in U_S^{-1}(v).$$

By Lemma 8, there exists a rescaling procedure  $\hat{\cdot}$  for which (9) holds. Hence,

$$\hat{u}(\eta\omega) = \hat{v}(\theta\omega). \quad (31)$$

By the Equal Split Lower Bound,  $1 \leq 1 / \lambda(S) \leq \eta$ . We may therefore find some  $t \in (0, 1)$  such that

$$\eta < 1 / t < \theta. \quad (32)$$

Let  $T = [0, t)$ . Define the profile  $V_T \in D^I$  by  $V_T(i) = v$  if  $0 \leq i < t$  and  $V_T(i) = u$  if  $t \leq i < 1$ . We keep this profile fixed and use Notation 1. From (7) and (32),

$$\hat{v}(F_i(T)) = \hat{v}(\omega / t) < \hat{v}(\theta\omega) \text{ for almost all } i \in T.$$



Choose any  $n > 1 / 1 - t$  and set  $T_n := [0, t + \frac{1}{n}]$ . By Efficiency and Solidarity,  $\hat{v}(F_i(T_n)) < \hat{v}(F_i(T))$  for almost all  $i \in T$ . Therefore,

$$\hat{v}(F_i(T_n)) < \hat{v}(\theta\omega) \text{ for almost all } i \in T. \quad (33)$$

By (9), however,  $\hat{v}(F_i(T_n)) = \hat{u}(F_j(T_n))$  for almost all  $i \in [0, t)$  and almost all  $j \in [t, t + \frac{1}{n}]$ . From (31) and (33), then,

$$\hat{u}(F_j(T_n)) < \hat{u}(\eta\omega) \text{ for almost all } j \in [t, t + \frac{1}{n}].$$

But for  $n$  large enough, it follows from (32) that  $\eta < 1 / (t + \frac{1}{n})$  and, therefore,

$$\hat{u}(F_j(T_n)) < \hat{u}(\eta\omega) < \hat{u}(\omega / (t + \frac{1}{n})) \text{ for almost all } j \in [t, t + \frac{1}{n}].$$

Since  $\hat{u}$  is just a rescaling of  $u$ , this violates the Equal Split Lower Bound  $\square$

## 7. CONCLUSION

(1). As Solidarity is a twofold, hence quite strong, condition, it is natural to try and weaken it. As mentioned in Section 3, a weaker and better known property is the Population Solidarity axiom:

*Population Solidarity.* For all  $(S, U_S), (R, W_R) \in \mathcal{G}^+$  such that  $S \subset R$  and  $U_S = W_R|_S$ , one of the following statements holds: (i)  $u_i(F_i(S, U_S)) = u_i(F_i(R, W_R))$  for almost all  $i \in S$ , (ii)  $u_i(F_i(S, U_S)) < u_i(F_i(R, W_R))$  for almost all  $i \in S$ , (iii)  $u_i(F_i(S, U_S)) > u_i(F_i(R, W_R))$  for almost all  $i \in S$ .

When used with Efficiency, this property implies the standard Population Monotonicity condition: no one benefits from a population increase. We will show by means of a simple example that our characterization theorem no longer holds when Solidarity is replaced with Population Solidarity. Recall, however, that our theorem holds for any fixed countable domain of (continuous, quasiconcave and strictly increasing) utility functions. What we provide below is merely a particular domain on which an allocation rule satisfying Efficiency, the Equal Split Lower Bound, Anonymity and Population Solidarity need not be Pazner-Schmeidler.

Let there be two goods, i.e.,  $\ell = 2$ , let  $\omega = (1, 1)$  and let  $D$  consist of the utility functions  $u$  and  $v$  given by

$$u(x^1, x^2) = 2x^1 + x^2, v(x^1, x^2) = x^1 + 2x^2 \text{ for all } (x^1, x^2) \in \mathbb{R}_+^2.$$

Since the behavior of an allocation rule on null economies is irrelevant, let us define our rule  $F$  on the positive economies only. For any positive economy  $(S, U_S)$ , let  $F_i(S, U_S) = \phi(u, S, U_S)$  for all  $i \in U_S^{-1}(u)$  and  $F_j(S, U_S) = \phi(v, S, U_S)$  for all  $j \in U_S^{-1}(v)$ , where

$$\phi(u, S, U_S) = \begin{cases} \left[ \frac{3}{2\lambda(S)}, 0 \right] & \text{if } \lambda(U_S^{-1}(u)) \leq 2\lambda(U_S^{-1}(v)), \\ \left[ \frac{1}{\lambda(U_S^{-1}(u))}, \frac{\lambda(U_S^{-1}(u)) - 2\lambda(U_S^{-1}(v))}{\lambda(U_S^{-1}(u)) \lambda(S)} \right] & \text{otherwise,} \end{cases}$$

and

$$\phi(v, S, U_S) = \begin{cases} \left[ \frac{2\lambda(U_S^{-1}(v)) - \lambda(U_S^{-1}(u))}{2\lambda(U_S^{-1}(v)) \lambda(S)}, \frac{1}{\lambda(U_S^{-1}(v))} \right] & \text{if } \lambda(U_S^{-1}(u)) \leq 2\lambda(U_S^{-1}(v)), \\ \left[ 0, \frac{3}{\lambda(S)} \right] & \text{otherwise.} \end{cases}$$

In each economy, this rule picks the efficient allocation which treats equals equally and gives the individuals with utility function  $u$  just enough to bring them to their equal split utility level. We let the reader check that Efficiency, the Equal Split Lower Bound, Anonymity and Population Solidarity are indeed satisfied. The rule is not a Pazner-Schmeidler rule; it violates Solidarity.

(2). The other possible weakening of Solidarity that naturally comes to mind is the Preference Solidarity axiom:

**Preference Solidarity.** If  $(S, U_S), (R, V_R), (R, W_R) \in \mathcal{E}^+$ ,  $S \subset R$  and  $U_S = V_R|_S = W_R|_S$ , then one of the following statements holds: (i)  $u_i(F_i(R, V_R)) = u_i(F_i(R, W_R))$  for almost all  $i \in S$ , (ii)  $u_i(F_i(R, V_R)) < u_i(F_i(R, W_R))$  for almost all  $i \in S$ , (iii)  $u_i(F_i(R, V_R)) > u_i(F_i(R, W_R))$  for almost all  $i \in R$ .

Substituting this axiom for Solidarity again opens the way to allocation rules which are not Pazner-Schmeidler. An interesting class consists of the "relative egalitarian" rules constructed as follows. Choose any rescaling procedure that makes all utility functions unbounded and keeps them continuous. For each economy, select an efficient allocation that equalizes the rescaled utility *gains* above equal split. All such rules are efficient, anonymous, satisfy the Equal Split Lower Bound and Preference Solidarity. They may, however, violate the Population Solidarity requirement. To see this, consider again the two-utility-function example discussed earlier in this section. Define the mapping  $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}$  by

$$\rho(x) = \begin{cases} x & \text{if } x \leq 3, \\ 3 + 3(x - 3) & \text{if } 3 < x \leq \frac{36}{11}, \\ \frac{42}{11} + \frac{1}{4}(x - \frac{36}{11}) & \text{if } \frac{36}{11} < x. \end{cases}$$

Since  $\rho$  is continuous and strictly increasing,  $\hat{u} := \rho \circ u$  is a rescaling of  $u$ . Define  $\hat{v} := v$  and consider now the efficient rule  $F$  which equalizes the rescaled utility gains above equal split. Let  $U_1(i) = u$  for all  $i \in [0, \frac{1}{2}]$  and  $U_1(i) = v$  for all  $i \in [\frac{1}{2}, 1)$ . Let  $S = [0, \frac{11}{24}] \cup [\frac{1}{2}, \frac{23}{24}]$ . We get

$$F_i(I, U_1) = \begin{cases} (2, 0) & \text{if } i \in [0, \frac{1}{2}), \\ (0, 2) & \text{if } i \in [\frac{1}{2}, 1), \end{cases}$$

$$F_i(S, U_1|_S) = \begin{cases} (\frac{24}{11}, \frac{4}{11}) & \text{if } i \in [0, \frac{11}{24}), \\ (0, \frac{20}{11}) & \text{if } i \in [\frac{1}{2}, \frac{23}{24}). \end{cases}$$

This constitutes a clear violation of Population Solidarity since the agents with utility function  $u$  get a physically bigger share in the smaller economy while the others get a smaller share.

The above examples indicate what is perhaps the most annoying limitation of the paper, namely, the assumption of a countable domain. They leave open two interesting questions: (i) is our characterization valid on the domain of all continuous, quasiconcave, strictly increasing utility functions, and (ii) if the result does extend to the uncountable case, can Solidarity be weakened in that context?

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