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L'équivalence entre le local-réalisme et le principe de non-signalement

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RÉSUMÉ

Cette thèse par articles réfute une position largement répandue en physique selon laquelle la mécanique quantique est une théorie qui ne peut pas être simultanément locale et réaliste. Pour ceci, nous démontrons l'équivalence entre le local-réalisme et l'impossibilité de communiquer instantanément.

Le premier article concerne la boîte de Popescu-Rorhlich. Celle-ci est une théorie jouet violant maximalement une inégalité de Bell. Une interprétation locale-réaliste de la boîte de Popescu-Rorhlich est présentée. Cette interprétation est basée sur la théorie des mondes multiples d'Everett. Il s'agit de la preuve la plus simple possible qu'une théorie peut ne pas être décrite par des variables cachées locales et peut pourtant être locale-réaliste. Une réponse à l'argument Einstein-Podolsky-Rosen est également fournie.

Dans le second article, une définition formelle de la notion de théorie locale-réaliste est présentée, ainsi que de théorie opérationnelle non-signalante. La thèse philosophique selon laquelle la version formelle de local-réalisme et de théorie non-signalante correspondent aux notions intuitives est avancée. On prouve que toute théorie locale-réaliste est une théorie non-signalante. Sous l'hypothèse d'une dynamique réversible, on prouve également que toute théorie non-signalante possède un modèle local-réaliste. Un corrolaire est l'existence d'un modèle local-réaliste pour la mécanique quantique unitaire.

Finalement, dans le troisième article, on prouve que la fonction d'onde universelle de la mécanique quantique ne peut pas être une description complète d'une réalité locale. Ceci amène à la nécessité de compléter la mécanique quantique, ainsi qu'en rêvait Einstein. Pour ce faire, un modèle local-réaliste pour la mécanique quantique basé sur le calcul matriciel est présenté.

Mots clés : Argument Einstein-Podolsky-Rosen, Localité, Mécanique quantique, Non-signalement, Réalisme, Théorème de Bell, Théorie d'Everett.

ABSTRACT

This thesis by articles refutes the largely held belief among physicists that quantum physics cannot be local-realistic. We do this by showing the equivalence between local realism and the impossibility of communicating instantaneously.

The first article concerns the Popescu-Rorhlich box. This box is a toy model maximally breaking a Bell inequality. A local-realistic interpretation of the Popescu-Rorhlich box is presented. This interpretation is based on Everett's many-world interpretation of quantum mechanics. It is the simplest possible proof that a theory can be local-realistic, and yet impossible to describe by local hidden variables. A response to the Einstein-Podolsky-Rosen argument is also given.

In the second article, a formal definition of local realism is given, as well as a formal definition of a no-signalling operational theory. The philosophical thesis that these formal definitions correspond to the intuitive notions is presented. We prove that every local-realistic theory is no-signalling. Under the hypothesis of reversible dynamics, we also prove that every no-signalling theory has a local-realistic model. A corollary is the existence of a local-realistic model for quantum mechanics.

Lastly, in the third article, we prove that the universal wavefunction of quantum theory cannot be a complete description of local reality. This leads to the necessity of completing quantum theory, which Einstein dreamt of. To do this, we develop a local-realistic model for quantum mechanics built around matrix calculus.

Keywords: Bell's theorem, Einstein-Podolsky-Rosen argument, Everett's theory, Locality, No-signalling, Quantum theory, Realism.

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CHAPITRE 1

INTRODUCTION

La physique quantique est la théorie des systèmes à l'échelle atomique. Le formalisme a été développé par Niels Bohr, Max Born, Werner Heisenberg et Erwin Schrödinger dans les années 1920. Les postulats de la mécanique quantique défient la conception classique de la physique. Un aspect parmi les plus frappants de l'interprétation orthodoxe de l'époque, celle de Copenhague, est que l'observation d'un système a un effet aléatoire, discontinu et irréversible, alors que tant qu'un système n'est pas observé, il a une évolution déterministe, continue et réversible. Dans cette vision standard de la mécanique quantique, le caractère d'une observation est inhéremment et irréductiblement stochastique.

En 1935, peu de temps après le développement des postulats de la mécanique quantique, Albert Einstein, Boris Podolsky et Nathan Rosen s'attaquent à l'interprétation de Copenhague de la mécanique quantique et déclarent que le hasard d'un système n'est pas inhérent, que certaines des quantités physiques qui semblent stochastiques sont en réalité prédéterminées. Leur conclusion est que la mécanique quantique est une théorie incomplète : un système quantique ne serait pas complètement décrit par sa fonction d'onde.

L'argumentation d'Einstein, Podolsky et Rosen est basée sur deux principes :

Principe du Réalisme : Il existe un monde extérieur à nos observations et celui-ci détermine nos observations.

Principe de Localité : Aucune action effectuée à un point A ne peut avoir un effet à un point B à une vitesse plus rapide que celle de la lumière.

Une théorie qui obéit à ces deux principes est dite *locale-réaliste*. Une façon locale-réaliste de compléter la mécanique quantique serait de rajouter des variables cachées locales qui dictent de quelle façon un système quantique se comporte lorsqu'il est observé. Une telle façon d'interpréter la mécanique quantique nous ramènerait à une physique plus classique, et mettrait fin à cette hérésie Copenhaguienne du monde dans laquelle Dieu joue aux dés avec l'univers.

En 1960, John Bell démontre qu'aucune théorie à variables cachées locales ne peut décrire la mécanique quantique. Nombreux sont ceux qui concluent que la mécanique quantique ne peut pas avoir d'interprétation locale-réaliste. Depuis ce temps, ces mêmes personnes proclament qu'Einstein avait tort dans son désir d'obtenir une théorie plus complète.

Par contre, cette conclusion est basée sur l'interprétation Copenhaguienne de la mécanique quantique, dans laquelle un observateur n'est pas considéré comme un système quantique ordinaire obéissant aux lois de la mécanique quantique. Bell ne tient aucunement compte du fait que dès 1950, Hugh Everett III complétait sa thèse doctorale sur la théorie de la fonction d'onde universelle, dans laquelle il généralise la théorie quantique et permet aux systèmes observateurs d'être des systèmes quantiques. Il élimine la distinction artificielle et floue entre le macroscopique et le microscopique, entre l'observateur et l'observé, le classique et le quantique. Selon Everett, l'acte d'observer un système n'est rien d'autre qu'une interaction ordinaire entre le système qui observe et le système observé. Avec cette synthèse vient une conséquence qui peut sembler aussi contre-intuitive que le modèle héliocentrique du système solaire : Un système observateur peut être en superposition de plusieurs états distints. On vivrait dans un multivers dans lequel des myriades de copies de nous existent. À chaque observation, un observateur se sépare en plusieurs copies.

En 2000, David Deutsch et Patrick Hayden, en se servant des idées d'Everett, démontrent que la mécanique quantique possède malgré tout une interprétation locale-réaliste en présentant un formalisme mathématique dans lequel les systèmes quantiques sont décrits de façon parfaitement locale. Par contre, leur modèle n'est pas très rigoureux, les arguments sont surtout intuitifs. Aucune définition mathématique claire de la notion de réalisme local n'est donnée. Et bien qu'ils aient ultimement réussi à prouver la localité de la mécanique quantique, on pourrait encore se poser la question : "Pour quelle raison la mécanique quantique est-elle une théorie locale ?"

Ceci nous amène à cette thèse. Son objectif est de répondre à cette question à caractère métaphysique par une réponse métaphysique, plutôt que par un formalisme mathématique. Pour cela, deux tâches sont accomplies. D'abord une définition rigoureuse et formelle du réalisme local est donnée. Puis, une preuve que toute théorie physique dont la dynamique est réversible et qui ne permet pas de signaler de l'information à une vitesse plus rapide que celle de la lumière possède une interprétation locale-réaliste. La réponse profonde pour laquelle la mécanique quantique est une théorie dans laquelle aucune action sur un système A n'a un effet (observable ou non) plus rapide que la vitesse de la lumière sur un système B est qu'aucune action n'a un effet *observable* à une vitesse supérieure à celle de la lumière.

Ceci est une thèse par articles, ces articles demandent très peu de préalables pour être lus. Le premier et le second n'en demandent aucun, alors que le troisième demande de connaître les postulats de la mécanique quantique.

Eppur si muove.

1.1 Les articles

1.1.1 Une interprétation locale-réaliste des boîtes "nonlocales"

Dans le premier article, nous visiterons un monde imaginaire, celui de la boîte "nonlocale" de Sandu Popescu et Daniel Rorhlich. Il s'agit d'une théorie jouet qui contient les mêmes problèmes conceptuels et métaphysiques que la mécanique quantique, tout en étant d'une immense simplicité mathématique. Nous nous servirons d'abord de ce monde imaginaire pour introduire l'argument Einstein-Podolsky-Rosen. Ensuite, nous verrons le théorème de Bell, ce qu'est une théorie à variables cachées locales, et nous apprendrons pourquoi ni la mécanique quantique, ni la boîte non-locale ne peuvent être décrites par une théorie de ce type. Néanmoins, nous découvrirons que la boîte nonlocale possède une interprétation parfaitement locale ! Ceci permet de prouver que la notion de réalisme local n'est pas restreinte par la notion de théorie à variables cachées locales.

Ce premier article a d'abord été présenté sous la forme d'une affiche illustrée, qui est disponible en annexe.

1.1.2 L'équivalence du réalisme local et du principe de non-signalement

Ensuite, dans cet article qui constitue le cœur de cette thèse, notre premier but est de développer une notion formelle de théorie locale-réaliste et de théorie opérationelle non-signalante qui correspondent aux notions intuitives. En tant que tel, une réponse à une telle question aura un aspect formel, qui peut être traitée par les mathématiques pures avec rigueur, et également un aspect nécessairement informel, qui sera développé dans un language naturel, et qui héritera et de sa richesse et de son ambiguïté. Il s'agit d'une thèse philosophique, au même titre que la thèse de Church-Turing selon laquelle les notions formelles de calculabilité présentées par Alonzo Church et Alan Turing correspondent à la notion de calculabilité intuitive.

Une définition claire de théorie locale-réaliste et de théorie opérationnelle non-signalante permet de prouver le théorème principal de cette thèse, qui dit que toute théorie non-signalante dont la dynamique est réversible possède une interprétation locale-réaliste. Ceci inclut comme cas particuler la mécanique quantique unitaire, et généralise le résultat déjà prouvé par Deutsch et Hayden.

1.1.3 Un formalisme local-réaliste pour la mécanique quantique

Finalement, dans le dernier article, nous introduisons un formalisme matriciel local-réaliste pour la mécanique quantique. Celui-ci diffère du formalisme présenté dans le second article. Notons que Deutsch et Hayden ont déjà prouvé la même chose, mais sans être aussi rigoureux ou encore aller dans les détails. Également, une preuve est donnée que la fonction d'onde universelle telle qu'introduite par Everett ne peut *pas* être une description complète de la réalité, et qu'il faut donc compléter la mécanique quantique si l'on veut rétablir une vision locale-réaliste de la mécanique quantique. En conclusion, Einstein avait raison une fois de plus !

CHAPITRE 2

PARALLEL LIVES : A LOCAL-REALISTIC INTERPRETATION OF "NONLOCAL" BOXES

By Gilles Brassard and Paul Raymond-Robichaud

ABSTRACT

We carry out a thought experiment in an imaginary world. Our world is both local and realistic, yet it violates a Bell inequality more than does quantum theory. This serves to debunk the myth that equates local realism with local hidden variables in the simplest possible manner. Along the way, we reinterpret the celebrated 1935 argument of Einstein, Podolsky and Rosen, and come to the conclusion that they were right in their questioning the completeness of quantum theory, provided one believes in a local-realistic universe. Throughout our journey, we strive to explain our views from first principles, without expecting mathematical sophistication nor specialized prior knowledge from the reader.

2.1 Introduction

Quantum theory is often claimed to be nonlocal, or more precisely that it cannot satisfy simultaneously the principles of locality and realism. These principles can be informally stated as follows.

Principle of realism: There is a real world and observations are determined by the state of the real world.

Principle of locality: No action taken at some point can have any effect at some remote point at a speed faster than light.

We give a formal definition of local realism in a companion paper [8]; here we strive to remain at the intuitive level and explain all our concepts, results and reasonings without expecting mathematical sophistication nor specialized prior knowledge from the reader.

The belief that quantum theory is nonlocal stems from the correct fact proved by John Bell [2] that it cannot be described by a *local hidden variable theory*, as we shall explain later. However, the claim of nonlocality for quantum theory is also based on the incorrect equivocation of local hidden variable theories with local realism, leading to the following fallacious argument:

1. Any local-realistic world must be described by local hidden variables.

2. Quantum theory cannot be described by local hidden variables.

3. *Ergo*, quantum theory cannot be both local and realistic.

The first statement is false, as we explain at length in this paper; the second is true; the third is a legitimate application of *modus tollens*,[1] but the argument is unsound since it is based on an incorrect premise. As such, our reasoning does not imply that quantum theory can be both local and realistic, but it establishes decisively that the usual reasoning against the local realism of quantum theory is fundamentally flawed.

In a companion paper, we go further and explicitly derive a full and complete local-realistic interpretation for finite-dimensional unitary quantum theory [7], which had already been discovered by Patrick Hayden and David Deutsch [16]. See also Ref. [36]. Going further, we show in another companion paper [8] that the local realism of quantum

[1] According to *modus tollens*, if p implies q but q is false, then p must be false as well.

theory is but a particular case of the following more general statement: *Any* reversible-dynamics theory that does not allow instantaneous signalling admits a local-realistic interpretation.

In order to invalidate statement (1) above, we exhibit an imaginary world that is both local and realistic, yet that cannot be described by local hidden variables. Our world is based on the so-called *nonlocal box*, introduced by Sandu Popescu and Daniel Rorhlich [32], which is already known to violate a Bell inequality even more than quantum theory (more on this later), which indeed implies that it cannot be explained by local hidden variables (more on this later also). Nevertheless, we provide a full local-realistic explanation for our imaginary world. Even though this world is not the one in which we live, its mathematical consistency suffices to debunk the myth that equates local realism with local hidden variables. In conclusion, the correct implication of Bell's theorem is that quantum theory cannot be described by local hidden variables, *not* that it is not local-realistic. *That's different!*

Given that quantum theory has a local-realistic interpretation, why bother with nonlocal boxes, which only exist in a fantasy world? The main virtue of the current paper, compared to Refs. [7, 8, 16, 36], is to invalidate the fallacious, yet ubiquitous, argument sketched above in the simplest and easiest possible way, with no needs to resort to sophisticated mathematics. The benefit of working with nonlocal boxes, rather than dealing with all the intricacies of quantum theory, was best said by Jeffrey Bub in his book on *Quantum Mechanics for Primates*: "The conceptual puzzles of quantum correlations arise without the distractions of the mathematical formalism of quantum mechanics, and you can see what is at stake—where the clash lies with the usual presuppositions about the physical world" [9].

The current paper is an expansion of an informal self-contained 2012 poster reproduced in the Appendix, which explains our key ideas in the style of a graphic novel, as well as of a brief account in a subsequent paper [5]. A very similar concept had already been formulated by Colin Bruce in his popular science book on *Schrödinger's*

Rabbits [10, pp. 130–132]. To the best of our knowledge, that was the first local-realistic description of an imaginary world that cannot be described by local hidden variables.

After this introduction, we describe the Popescu-Rorhlich nonlocal boxes, perfect as well as imperfect, in Section 2.2. We elaborate on no-signalling and local-realistic theories in Section 2.3, as well as introducing the notion of local hidden variables, which is illustrated with an application of the Einstein-Podolsky-Rosen argument [17]. Bell's Theorem is reviewed in Section 2.4 in the context of nonlocal boxes, and we explain why they cannot be described by local hidden variables. The paper culminates with Section 2.5, in which we expound our theory of *parallel lives* and how it allows us to show that "nonlocal" boxes are perfectly compatible with both locality and realism. Having provided a solution to our conundrum, we revisit Bell's Theorem and the Einstein-Podolsky-Rosen argument in Section 2.6 in order to understand how they relate to our imaginary world. There, we argue that our theory of parallel lives is an unavoidable consequence of postulating that the so-called nonlocal boxes are in fact local and realistic. We conclude with a discussion of our results in Section 2.7. Finally, we reproduce in the Appendix our 2012 graphic-novel-like poster that illustrates our main concepts. Throughout our journey, we strive to illustrate how the arguments formulated in terms of nonlocal boxes and the more complex quantum theory are interlinked.

2.2 The imaginary world

We now proceed to describe how our imaginary world is perceived by its two inhabitants, Alice and Bob. We postpone to Section 2.5 a description of what is *really* going on in that world. The main ingredient that makes our world interesting is the presence of perfect nonlocal boxes, a theoretical idea invented by Sandu Popescu and Daniel Rorhlich [32].

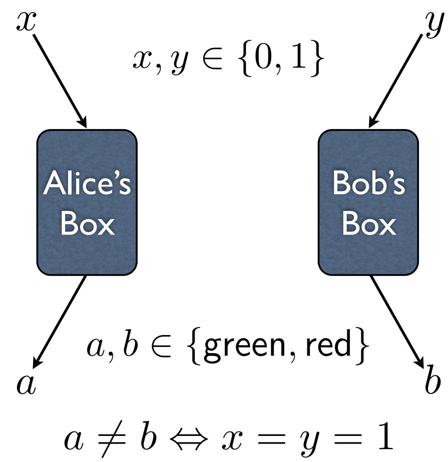

Figure 2.1: Nonlocal boxes.

2.2.1 The nonlocal box

Nonlocal boxes always come in pairs, one box is given to Alice and the other to Bob. [2] One can think of a nonlocal box as an ordinary-looking box with two buttons labelled 0 and 1. Whenever a button is pushed, the box instantaneously flashes either a red or a green light, both outcomes being equally likely. This concept is illustrated in Figure 2.1.

If Alice and Bob meet to compare their results after they have pushed buttons, they will find that each pair of boxes has produced outputs that are correlated in the following way: Whenever they have both pushed the 1 input button, their boxes have flashed different colours, but if at least one of them has pushed the 0 input button, their boxes have flashed the same colour. See Table 2.I.

For example, if Alice pushes 1 and sees green, whereas Bob pushes 0, she will discover when she meets Bob that he has also seen green. However, if Alice pushes 1 and sees green (as before), whereas Bob pushes 1 instead, she will discover when they meet that he has seen red.

[2] Some people prefer to think of the nonlocal box as consisting of both boxes, so that the pair of boxes that we describe here constitutes a single nonlocal box. It's a matter of taste.

Table 2.I: Behaviour of nonlocal boxes.

Alice's Input	Bob's Input	Output colours
0	0	Identical
0	1	Identical
1	0	Identical
1	1	Different

A nonlocal box is one-time use: once a button has been pushed and a colour has flashed, the box will forever flash that colour and is no longer responsive to new inputs. However, Alice and Bob have an unlimited supply of such pairs of disposable nonlocal boxes.

2.2.2 Testing the boxes

Our two inhabitants, Alice and Bob, would like to verify that their nonlocal boxes behave according to Table 2.I indeed. Here is how they proceed.

1. Alice and Bob travel far apart from each other with a large supply of numbered unused boxes, so that Alice's box number i is the one that is paired with Bob's box number i.

2. They flip independent unbiased coins labelled 0 and 1 and push the corresponding input buttons on their nonlocal boxes. They record for each box number the randomly-chosen input and the observed resulting colour. Because they are sufficiently far apart, the experiment can be performed with sufficient simultaneity that Alice's box cannot know the result of Bob's coin flip (hence the input to Bob's box) before it has to flash its own light, and vice versa.

3. After many trials, Alice and Bob come back together and verify that the boxes work perfectly: no matter how far they were from each other and how simulta-

neously the experiment is conducted, the correlations promised in Table 2.I are realized for each and every pair of boxes.

Note that Alice or Bob cannot confirm that the promised correlations are established until they have met, or at least sent a signal to each other. In other words, data collected locally at Alice's and at Bob's need to be brought together before any conclusion can be drawn. This detail may seem insignificant at first, but it will turn out to be crucial in order to give a local-realistic explanation for "nonlocal" boxes.

2.2.3 Imperfect nonlocal boxes

So far, we have talked about perfect nonlocal boxes, but we could consider nonlocal boxes that are sometimes allowed to give incorrectly correlated outputs. We say that a nonlocal box *works with probability p* if it behaves according to Table 2.I with probability p. With complementary probability $1-p$, the opposite correlation is obtained.

2.2.3.1 Quantum theory and nonlocal boxes

Although we shall concentrate on *perfect* nonlocal boxes in this paper, quantum theory makes it possible to implement nonlocal boxes that work with probability

$$p_{\text{quant}} = \cos^2\left(\tfrac{\pi}{8}\right) = \tfrac{2+\sqrt{2}}{4} \approx 85\%$$

but no better according to Cirel'son's theorem [12]. It follows that our imaginary world is distinct from the world in which we live since perfect nonlocal boxes cannot exist according to quantum theory.

For our purposes, The precise mathematics and physics that is needed to understand how it is possible for quantum theory to implement nonlocal boxes with probability p_{quant} does not matter. Let us simply say that it is made possible by harnessing *entanglement* in a clever way. Entanglement, which is the most nonclassical of all quantum resources, is at the heart of quantum information science. It was discovered by Einstein, Podolsky and Rosen in 1935 in Einstein's most cited paper [17], although there is some evidence that Erwin Schrödinger had discovered it earlier. It is also because of entanglement that the quantum world in which we live is often thought to be nonlocal.

2.3 The many faces of locality

Recall that the Principle of locality claims that no action taken at some point can have any effect at some remote point at a speed faster than light. An apparently weaker principle would allow such effects provided they cannot be observed at the remote point. This is the Principle of no-signalling, which we now explain.

2.3.1 No-signalling

It is important to realize that nonlocal boxes do not enable instantaneous communication between Alice and Bob. Indeed, no matter which button Alice pushes (or if she does not push a button at all), Bob has an equal chance of seeing red or green flashing from his box whenever he pushes either of his buttons. Said otherwise, no action that Alice can take has any effect whatsoever on the probabilities of events that Bob can observe.

It follows that our imaginary world shares an important property with the quantum world: it obeys the principle of no-signalling.

Principle of no-signalling: No action taken at some point can have any *observable* effect at some remote point at a speed faster than light.

Among observable effects, we include anything that would affect the probability distribution of outputs from any device. The principle of no-signalling implies in general that for any pair of devices shared by Alice and Bob (not only Popescu-Rorhlich nonlocal boxes), Bob's output distribution depends only on Bob's input, and not on Alice's input, provided they are sufficiently far from each other and Alice does not provide her input to her device too much before Bob's device must produce its output.

2.3.2 Local realism implies no-signalling

The principle of no-signalling follows from the principles of locality and realism: any local-realistic world is automatically no-signalling, as shown by the following informal argument.

1. By the principle of locality, no action taken at point A can have any effect on the state of the world at point B faster than at the speed of light.
2. By the principle of realism, anything observable at point B is a function of the state of the world at that point.
3. It follows that no action at point A can have an observable effect at point B faster than at the speed of light.

Here, we have relied on the tacit assumption that in a local-realistic world, what is a observable at some point is a function of the state of the world at that same point. The above argument is fully formalized, with all hypotheses made explicit, in Ref. [8].

2.3.3 Local hidden variable theories

The most usual type of local-realistic theories, which was studied in particular by John Bell [2], is based on local hidden variables (explained below). The misconception according to which all local-realistic theories have to be of that type has led to the widespread incorrect belief that quantum theory cannot be local-realistic because it cannot be based on local hidden variables according to Bell's theorem.

In the idealized context of nonlocal boxes, a local hidden variable theory would consider arbitrarily sophisticated pairs of devices that are allowed to share randomness for the purpose of explaining the observed behaviour. The individual boxes would also be allowed independent sources of internal randomness. The initial shared randomness, along with the internal randomness and the inputs provided by Alice and Bob would be used to determine which colours to flash. However, what is *not* allowed is for the output of one of Alice's boxes to depend on the input of Bob's corresponding box, or vice versa. This can be enforced by the principle of locality provided both input buttons are pushed with sufficient simultaneity to prevent a signal from one box to reach the other in time, even at the speed of light, to allow "cheating".

In any such theory (not just those pertaining to nonlocal boxes), it is always possible to remove internal sources of randomness and replace them by parts of the initial source of shared randomness that would be used by one side only (provided we allow a continuously infinite amount of shared randomness). However, the following section shows that, in the case of *perfect* nonlocal boxes, internal randomness should *never* be used to influence the behaviour of boxes.

2.3.4 The Einstein-Podolsky-Rosen argument

Even though they were obviously not talking about Popescu-Rorhlich nonlocal boxes, the original 1935 argument of Einstein, Podolsky and Rosen applies *mutatis mutandis* to prove that, in the context of local hidden variable theories, the output of Bob's nonlocal box should be completely determined by the initial randomness shared between Alice's and Bob's boxes and by Bob's input (and vice versa, with Alice and Bob interchanged).

1. Suppose Alice pushes her input button first.
2. When she pushes her button, this cannot have any instantaneous effect on Bob's box, by the principle of locality.

3. After Alice has seen her output, she can know with certainty what colour Bob will see as a function of his input (even though she does not know which input he will choose). For example, if Alice has pushed 1 and has seen green, she knows that if Bob chooses to push 0 he will also see green, whereas if he chooses to push 1 he will see red.

4. Since it is possible for Alice to know with certainty what colour Bob will see when he pushes either button, and she can obtain this knowledge without influencing his system, it must be that his colour was *predetermined* as a function of which button he would push. This predetermination can only come from the initial source of shared randomness, and errors could occur if it were influenced by local randomness at Bob's.

This argument was used in the original 1935 Einstein-Podolsky-Rosen paper [17] to prove, under the implicit assumption of local hidden variables, that there are instances in quantum theory in which both the position and momentum of a particle must be simultaneously defined. This clashed with the Copenhagen vision of quantum theory, according to which Heisenberg's uncertainty principle is not due merely to the fact that measuring one of those properties necessarily disturbs the other, but that they can never be fully defined simultaneously. The conclusion of Einstein, Podolsky and Rosen was that (Copenhagen) quantum-mechanical description of physical reality cannot be considered complete. After Niels Bohr's response [3], the physics community consensus was largely in his favour, asserting that the EPR argument was unsound and that the Copenhagen interpretation is indeed complete. In a companion paper [7], we prove that, under the metaphysical principle of local realism, it is Einstein who was correct after all in arguing that the usual description of quantum physics cannot be a complete description of reality, and furthermore we provide a completion of quantum theory that makes it possible.

But let us come back to our imaginary world of nonlocal boxes...

2.3.5 Local hidden variable theory for nonlocal boxes

In a local hidden variable theory for nonlocal boxes, we have seen that all correlations should be explained by the initial shared randomness. Since each box implements a simple one-bit to one-colour-out-of-two function, it needs to use only two bits of the randomness shared with its twin box to do so. It is natural to call those bits A_0 and A_1 for Alice, and B_0 and B_1 for Bob. If we define function

$$\mathsf{c} : \{0,1\} \rightarrow \{\mathsf{green}, \mathsf{red}\}$$

by $\mathsf{c}(0) = \mathsf{green}$ and $\mathsf{c}(1) = \mathsf{red}$, then Alice's box would flash colour $a = \mathsf{c}(A_x)$ when input button x is pushed by Alice, whereas Bob's box would flash colour $b = \mathsf{c}(B_y)$ when input button y is pushed by Bob. See Figure 2.1 again.

In order to fulfil the requirements of nonlocal boxes given by Table 2.I, it is easy to verify that the four local hidden variables must satisfy the condition

$$A_x \oplus B_y = x \cdot y \tag{2.1}$$

for all $x, y \in \{0,1\}$ simultaneously, where "$\oplus$" and "$\cdot$" denote the sum and the product modulo 2, respectively. For example, if Alice selects $x = 0$ and Bob $y = 1$, their boxes must flash the same colour $a = \mathsf{c}(A_0) = \mathsf{c}(B_1) = b$ according to Table 2.I, and therefore the hidden variables A_0 and B_1 must be equal since function c is one-to-one. In symbols, $A_0 = B_1$, which is equivalent to $A_0 \oplus B_1 = 0$, which indeed is equal to $x \cdot y$ in this case.

Is this possible?

2.4 Bell's Theorem

Theorem 2.4.1 (Bell's Theorem)**.** No local hidden variable theory can explain a nonlocal box that would work with a probability better than 75%. In particular, no local hidden variable theory can explain perfect nonlocal boxes.

Proof. We have just seen that any local hidden variable theory that enables perfect non-local boxes would have to satisfy Equation (2.1) for all $x, y \in \{0,1\}$. This gives rise to the following four explicit equations.

$$
\begin{aligned}
A_0 \oplus B_0 &= 0 \\
A_0 \oplus B_1 &= 0 \\
A_1 \oplus B_0 &= 0 \\
A_1 \oplus B_1 &= 1
\end{aligned}
$$

If we sum modulo 2 the equations on both sides and rearrange the terms using the associativity and commutativity of addition modulo 2, as well as the fact that any bit added modulo 2 to itself gives 0, we get:

$$
\begin{aligned}
(A_0 \oplus B_0) \oplus (A_0 \oplus B_1) \oplus (A_1 \oplus B_0) \oplus (A_1 \oplus B_1) &= 0 \oplus 0 \oplus 0 \oplus 1 \\
(A_0 \oplus A_0) \oplus (A_1 \oplus A_1) \oplus (B_0 \oplus B_0) \oplus (B_1 \oplus B_1) &= 1 \\
0 \oplus 0 \oplus 0 \oplus 0 &= 1 \\
0 &= 1,
\end{aligned}
$$

which is a contradiction. Therefore, it is not possible for all four equations to hold simultaneously. At least one of the four possible choices of buttons pushed by Alice and Bob is bound to give incorrect results. It follows that any attempt at creating a nonlocal box that works with probability better than $^3/_4 = 75\%$ is doomed to fail in any theory based on local hidden variables. □

The reader can easily verify from the proof of Theorem 2.4.1 that any three of the four equations can be satisfied by a proper choice of local hidden variables. For example, setting $A_0 = B_0 = A_1 = B_1 = 0$ results in the first three equations being satisfied, but not the fourth. A more interesting strategy would be for Alice's box to produce $A_x = x$ and for Bob's box to produce $B_y = 1 - y$. In this case, the last three equations are satisfied but not the first. For each equation, there is a simple strategy that satisfies the other three but not that one (more than one such strategy in fact). More generally, it is possible to create a pair of nonlocal boxes that work with probability 75% *regardless of the input provided to them* if they share three bits of randomness. The first two bits determine which one of the four equations is jettisoned, thus defining an arbitrary pre-agreed strategy that fulfils the other three. If the third random bit is 1, both boxes will in fact produce the complement of the output specified in their strategy (which has no effect on which equations are satisfied). The purpose of this third shared random bit is that a properly functioning pair of Popescu-Rorhlich boxes should produce a random output on each side if we only consider marginal probabilities.

We say of any world in which nonlocal boxes exist that work with a probability better than 75% that it *violates a Bell inequality* in honour of John Bell, who established the first result along the lines of Theorem 2.4.1, albeit not explicitly the one described here [2].

2.4.1 Quantum theory and Bell's theorem

The usual conclusion from Theorem 2.4.1 is that any world containing nonlocal boxes that work with a probability better than 75% cannot be both local and realistic. Since quantum theory enables boxes that work $\approx 85\%$ of the time, as we have seen in Section 2.2.3.1, it seems inescapable that the quantum world cannot be local-realistic.

Similarly, it is tempting to assert that the more a Bell inequality is violated by a theory, the more nonlocal it is. In particular, our imaginary world would be more non-local than quantum theory itself. As we shall now see—and this is the main point of this paper—all these conclusions are unsound because local realism and local hidden variables should not be equated.

2.5 A local realistic solution—Parallel lives

Here is how the seemingly impossible is accomplished. Let us assume for simplicity that Alice and Bob have a single pair of "nonlocal" boxes at their disposal, which is sufficient to rule out local hidden variable explanations. When Alice pushes a button on her box, she splits in two, together with her box. One Alice sees the red light flash on her box, whereas the other sees the green light flash. Both Alices are equally real. However, they are now living *parallel lives*: they will never be able to see each other or interact with each other. In fact, neither Alice is aware of the existence of the other, unless they infer it by pure thought as the only reasonable explanation for what they will experience when they test their boxes according to Section 2.2.2. From now on, any unsplit object (or person) touched by either Alice or her box splits. This does not have to be physical touching: a message sent by Alice has the same splitting effect on any object it encounters. Hence, Alice's splitting ripples through space, but at a speed that cannot exceed that of light. It is crucial to understand that it is *not* the entire universe that splits instantaneously when Alice pushes her button, as this would be a highly nonlocal effect.

The same thing happens to Bob when he pushes a button on his box. He splits in two and neither copy is aware of the other Bob. One sees a red light flash and the other sees a green light flash. If both Alice and Bob push a button at about the same time, we have two independent Alices and two independent Bobs, and for now the Alices and the Bobs are also independent of one another.

It is only when Alice and Bob interact that correlations are established. Let us assume for the moment that both Alice and Bob always push their buttons before interacting. The magical rule is that an Alice is allowed to interact with a Bob if and only if they jointly satisfy the conditions of the nonlocal box set out in Table 2.I.

For example, if Alice pushes button 1, she splits. Consider the Alice who has seen green. Her system can be imagined to carry the following rule: You are allowed to interact with Bob if either he has pushed the 0 button on his box and seen green, or he has pushed the 1 button and seen red. Should this Alice ever come in presence of a Bob who has pushed the 1 button and seen green, she would simply not become aware of his presence and could walk right through him without either one of them noticing anything. Of course, the other Alice, the one who has seen red after having pushed her 1 button, would be free to shake hands with that Bob.

If Bob has pushed button 0 and seen green, his system can likewise be imagined to carry the following rule: You are allowed to interact with Alice if and only if she has seen green, regardless of which button she has pushed. It is easy to generalize this idea to all cases covered by Table 2.I because there will always be one green Alice and one red Alice, one green Bob and one red Bob, and whenever green Alice is allowed to interact with one Bob, red Alice is allowed to interact with the other Bob. From their perspective, each Alice and each Bob will observe correlations that seem to come "from outside space-time" [23]. However, this interpretation is but an illusion due to their intrinsic inability to perceive some of the actors in the world in which they live.

Our imaginary world is fully local because Alice's state is allowed to depend only on her own input and output at the moment she pushes a button. It is true that the mysterious correlations given in Table 2.I would be impossible for any local hidden variable theory. However, Alice and Bob cannot *experience* those correlations before they actually meet (or at least before they share their data), *and these encounters cannot take place faster than at the speed of light.* When they meet, the correlations they experience are simply due to the matching rule that determines which Alices are allowed to interact with which Bobs, and *not* to a magical (because nonlocal) spukhafte Fernwirkungen ("Spooky action at a distance"), which was so abhorrent to Einstein, and rightly so.

What if Alice pushes her button, but Bob does not? In the discussion above, we had assumed for simplicity that both Alice and Bob had pushed buttons on their boxes before interacting. A full story should include various other scenarios. It could be that Alice pushes a button on her box and travels to interact with a Bob who has not yet touched his box. Or it could be that after pushing a button on her box, only the Alice who has seen green travels to interact with Bob, whereas the Alice who has seen red stays where she is.

For instance, consider the case in which Alice has pushed the 1 button on her box, split, and only the Alice who has seen green travels to meet unsplit Bob. At the moment they meet, Bob and his box will automatically split. One Bob will now own a box programmed as follows: "if button 0 is pushed, flash green, but if button 1 is pushed, flash red"; the other Bob will own a box containing the complementary program, with "green" and "red" interchanged. As for our travelling Alice, she will see the first one of those Bobs and be completely oblivious of the other, who will not even be aware that an Alice has just made the trip to meet him.

It would be tedious, albeit elementary, to go through an exhaustive list of all possible scenarios. We challenge the interested reader to figure out how to make our imaginary world behaves according to Table 2.I in all cases. But rather than getting bored at this exercise, why not contemplate the Appendix, which illustrates the concept of parallel lives in the form of a graphic novel?

2.5.1 Quantum theory and parallel lives

We coined the term "parallel lives" for the idea that a system is allowed to be in a superposition of several states, but that all splittings occur locally. This is distinct from the concept known as "many world", according to which the entire universe would split whenever Alice pushes a button on her box (or makes a quantum measurement that has more than one possible outcome according to standard theory). Nevertheless, our idea of parallel lives was directly inspired by the many-world interpretation, whose pioneer was Hugh Everett [18] more than six decades ago. Later, David Deutsch and Patrick Hayden provided the first local interpretation of quantum mechanics [16]. Even though they did not use the term "parallel lives", their approach was akin to ours. In their solution, the evolution of the quantum world is fully local, and individual systems, including observers, are allowed to be in superposition. In a companion paper [7], we offer our own local formalism for quantum theory along the lines of this paper, complete with full proofs of our assertions.

2.6 Revisiting Bell's theorem and the Einstein-Podolsky-Rosen argument

Having provided a solution to our conundrum with the explicit construction of a local-realistic imaginary world in which perfect Popescu-Rorhlich "nonlocal" boxes are possible, we revisit the Einstein-Podolsky-Rosen argument in order to understand how it relates to our imaginary world. This leads us to conclude that our theory of parallel lives is an unavoidable consequence of postulating that those boxes are compatible with local realism.

2.6.1 Parallel lives versus hidden variable theories

To understand the main difference between parallel lives and local hidden variable theories, consider again the scenario according to which Alice has pushed the 1 button and her box has flashed a green colour. According to local hidden variable theories, she would know with certainty what colour Bob will see as a function of his choice of input: he will also see green if he pushes the 0 button, but he will see red if he pushes the 1 button. This was at the heart of the Einstein-Podolsky-Rosen argument of Section 2.3.4 to the effect that the colours flashed by Bob's box had to be predetermined as a function of which button he would push since Alice could know this information without interacting with Bob's box. To quote the original argument, "If, without in any way disturbing a system, we can predict with certainty the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity" [17]. The "element of physical reality" in question is what we now call local hidden variables and the "physical quantity" is the mapping between input buttons and output colours.

The parallel-lives interpretation is fundamentally different. Whenever Alice pushes a button on her box, she cannot infer anything about Bob's box. Instead, she can predict how her various lives will interact with Bob's in the future, when (and if) they meet. Consider for example the case in which both Alice and Bob push their input buttons, whose immediate effect is the creation of two Alices and two Bobs. Let us call them Green-Alice, Red-Alice, Green-Bob and Red-Bob, depending on which colour they have seen. If the original Alice had pushed her 1 button, Green-Alice may now infer that she will interact with Red-Bob if he has also pushed his 1 button, whereas she will interact with Green-Bob if he has pushed his 0 button. The opposite statement is true of Red-Alice. As we can see, this is a purely local process since this instantaneous knowledge of both Alices has no influence on whatever the faraway Bobs may observe, which is both colours!

2.6.2 How an apparent contradiction leads to parallel lives

Consider the following argument concerning nonlocal boxes, and pretend that you have never heard of parallel lives (nor many-world), yet you believe in locality.

1. Let us say that Alice pushes button 1 on her box. Without loss of generality, say that her box flashes the green colour.

2. Now, we know that Bob will see green if he pushes his 0 button, whereas he will see red if he pushes his 1 button, according to Table 2.I. By the principle of locality, this conclusion holds regardless of Alice's previous action since she was too far for her choice of button to influence Bob's box.

3. What would have happened had Alice pushed her 0 button instead at step 1? She must see the same colour as Bob, regardless of Bob's choice of button, since her

pushing button 0 precludes the possibility that both Alice and Bob will press their 1 button, which is the only case yielding different colours, again according to Table 2.I.

4. Statements (2) and (3) imply together that when Alice pushes her 0 button, she must see both red and green!

Despite appearances, statement 4 is not a contradiction, and indeed it can be resolved. Both results seen by Alice must be equally real by logical necessity. The only way for her to see both colours, *yet be convinced she saw only one*, is that there are in fact two Alices unaware of one another. In other words, the postulated locality of Popescu-Rorhlich "nonlocal" boxes *forces us* into a parallel-lives theory, which, far from being a postulate, is in fact an ineluctability.

If both Alices are indeed mathematically necessary to describe a local-realistic world, then both Alices are real in that world, inasmuch as any mathematical quantity that is necessary to describe reality corresponds to something that is real. Here, we accepted as a philosophical axiom the claim that whenever a mathematical quantity is necessary to describe reality, that quantity corresponds to something that is real, and is not a mere artifact of the theory.

The same conclusion applies whenever any theory is shown to be inconsistent with all possible local hidden variable theories. Indeed, such theories carry the rarely-mentioned assumption that once concluded, any experiment has a single outcome. Other outcomes that could have been possible simply did not occur. The obvious resolution of any such inconsistency is to accept the fact that all possible outcomes occur within parallel lives of the experimenter.

2.7 Conclusions

We have exhibited a local-realistic imaginary world that violates a Bell inequality. For this purpose, we introduced the concept of *parallel lives*, but argued subsequently that this was an unavoidable consequence of postulating that the so-called nonlocal boxes are in fact local and realistic. The main virtue of our work is to demonstrate in an exceeding simple way that local reality can produce correlations that are impossible in any theory based on local hidden variables. In particular, it is fallacious to conclude that quantum theory is nonlocal simply because it violates Bell's inequality.

In quantum theory, ideas analogous to ours can be traced back at least to Hugh Everett [18]. They were developed further by David Deutsch and Patrick Hayden [16], and subsequently by Colin Bruce [10, pp. 130–132]. The latter gave the first local-realistic explanation for a theory that is neither quantum nor classical. In companion papers, we have proven that unitary quantum mechanics is local-realistic [7] (which had already been shown by Deutsch and Hayden [16]) and, more generally, that this is true for any reversible-dynamics no-signalling operational theory [8]. The latter paper provides a host of suggestions in its final section for a reader eager to pursue this line of work in yet unexplored directions.

Throughout our journey, we have revisited several times the 1935 Einstein-Podolsky-Rosen argument and came to the conclusion that they were right in questioning the completeness of Bohr's Copenhagen quantum theory. Perhaps Einstein was right in his belief of a local-realistic universe after all and in wishing for quantum theory to be completed? Perhaps we live parallel lives…

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CHAPITRE 3

THE EQUIVALENCE OF LOCAL-REALISTIC AND NO-SIGNALLING THEORIES

By Gilles Brassard and Paul Raymond-Robichaud

ABSTRACT

We provide a framework to describe all local-realistic theories and all no-signalling operational theories. We show that when the dynamics is reversible, these two concepts are equivalent. In particular, this implies that unitary quantum theory can be given a local-realistic model.

3.1 Introduction

On 21st October 2015, the *New York Times* touted "Sorry, Einstein. Quantum Study Suggests 'Spooky Action' Is Real" [31]. Indeed, as the daily continued, "objects separated by great distance can instantaneously affect each other's behavior". This dramatic headline was prompted by the successful completion of an ambitious experiment in Delft, the Netherlands, in which the world's first so-called "loophole-free Bell test" had been realized, whose objective was to verify that the predictions of quantum theory continue to rule even when they could not be given a classical explanation by no-faster-than-light signalling (locality loophole) nor by exploiting falsely inefficient detectors (detection loophole) [24]. Based on the work of John Bell [2], the Dutch paper concluded in its abstract that their "data hence imply statistically significant rejection of the local-realist null hypothesis".[1]

[1] To be fair, the Dutch team admitted towards the end of their paper that their "observation of a statistically significant loophole-free Bell inequality violation thus indicates rejection of all local-realist theories

Has the Dutch experiment definitely established the nonlocality of our quantum universe, barring a statistical fluke? Obviously not since David Deutsch and Patrick Hayden had already shown fifteen years earlier that quantum theory, with all its seemingly nonlocal predictions, can be given a fully local-realistic interpretation [16]! Subsequently, we had shown [4, 5] how easy it is to give a local-realistic interpretation for the Popescu-Rohrlich so-called *nonlocal* boxes [32], even though they seem to violate locality even more than quantum theory, thus establishing in the clearest and simplest possible way the fact that a mere violation of Bell's inequalities is no proof of nonlocality. The explanation for this conundrum is that there are more general ways for a world to be local-realistic than having to be ruled by local hidden variables, which was the only form of local realism considered by Bell in his paper [2]. We expound on the local construction of "nonlocal" boxes in a companion paper [6].

In the current paper, we give original formal definitions for the concepts of local-realistic theories and no-signalling operational theories. We argue in favour of our thesis, according to which our definitions capture the intuitive notions in the most general way possible. Those definitions are among our main contributions because they are required in order to give a complete and formal proof of our main result to the effect that *any* reversible-dynamics physical theory according to which it is impossible to signal information instantaneously can be given a local-realistic interpretation. As one specific example, this applies to unitary quantum theory, which provides an alternative proof of the Deutsch-Hayden result mentioned above [16]. Actually, the specific case of quantum theory is of such importance that we devote an entire companion paper to it [7], in which we prove in particular that the universal wavefunction cannot be the complete description of a *local* universe: it merely describes what can be observed from within. In other words, the universal wavefunction is but a *shadow* of the real world. It follows that if we believe in local realism—as Albert Einstein arguably did—the answer to the question

that accept that the number generators produce a free random bit in a timely manner *and that the outputs are final once recorded in the electronics*" [24] (our emphasis). In other words, additional loopholes exist, which their experiment admittedly did not take into account, including the possibility that measurements have no definite outcomes.

asked in the title of the celebrated 1935 Einstein-Podolsky-Rosen (EPR) paper [17] is a resounding no: the (Copenhagen) quantum-mechanical description of physical reality can*not* be considered complete, and furthermore it *can* be completed, as shown in [7, 16]. Said otherwise, the *New York Times* headline notwithstanding, Einstein does not have to be sorry: *he was right!*

This paper is organized as follows. After this introduction, we set the stage in Section 3.2 by laying the conceptual foundations of realism in an informal and intuitive manner. There, inspired by Immanuel Kant [19, 26], we introduce the essential notions of noumenal and phenomenal worlds. This is followed in Section 3.3 by a formal mathematical definition of what we call the structure of realism, and then in Section 3.4 the structure of *local* realism. Section 3.5 defines the notion of no-signalling operational theories without recourse to probabilities, which is more general than the usual approach. The paper culminates on a complete and formal proof in Section 3.6 of our main result to the effect that *all* reversible-dynamics no-signalling operational theories can be explained by a local-realistic model. We conclude in Section 3.7 with a final discussion and suggestions for further research.

3.2 Conceptual foundations of realism

Do you believe that when you place a delicious apple pie in your refrigerator and close the door, the pie continues to exist even though you are no longer looking at it? If so, congratulations, you are a realist, an adherent of the philosophical position called *realism*! What is realism? It is the principle that posits the existence of a world existing outside of our immediate subjective experience and that this world determines our subjective experience.

3.2.1 Appearance vs reality

This outside world can be called the *real world*, the *external* world, the *objective* world, or in Kantian terminology: the *noumenal* world [19, 26]. It describes the world as it is rather than the world as it can observed, or known through sensory experience.

According to realism, our subjective experience, our perceptions, our sense-data, are determined by the state of the external world. The portion of the real world that is observable or perceptible is called the *perceptive world*, or alternatively the *observable* world, or in Kantian terminology: the *phenomenal* world.[2]

To be perceptible does not mean to be perceived directly. If we scan a molecule with an atomic force microscope, the properties thus observed are perceptible even though we are not observing the molecule directly with our naked eyes. The only limit to what kind of measurement device may be used to determine what is perceptible are the laws of Nature, not merely the currently available technology. Also, to be perceptible does not imply to be perceived right now by some observer. Even if no one is looking right now at your delicious apple pie, it is still possible to look at it, inasmuch as no laws of Nature prevent us from opening the fridge door. More to the point, the far side of the Moon existed even before we had the technology that allowed us to observe it. Thus, we include as part of the perceptible world, not what is perceived now, but rather all potential perceptions, present and future.

After drawing this distinction between the noumenal world and the phenomenal world, an astute reader might ask: "Why do you need the noumenal world at all? Couldn't you get rid of it? After all, if something cannot be observed, it cannot be inferred as real!" Our first answer would be that the reader is not even able to infer the existence of *anything* outside her own senses, and might as well be a solipsist whose whole existence is constrained by her sense data, out of which nothing can be logically inferred. Any model able to make predictions needs to include various concepts outside

[2] For a good discussion on the relation between the observable world and the real world, see the work of Bertrand Russell [33].

of our immediate perceptions, in such a way that the consequences of these concepts give rise to our immediate perceptions, present as well as future. Thus, any such model needs at the very least to include the observable, rather than merely the observed.

Furthermore, we shall see that additional metaphysical principles, such as locality, in a world that follows the laws of quantum theory at the phenomenal level, will force us to make the noumenal world richer and deeper than its immediate phenomenal counterpart.

What is the relation between the noumenal world and the phenomenal world? What is perceptible must follow a process parallel to what exists. As the noumenal world evolves, so does the phenomenal world. Any property that exists in the phenomenal world arises from a property in the noumenal world.

We can represent the relation between the noumenal world and the phenomenal world with the following diagram.

$$\begin{array}{ccc} \mathsf{Noumenal}_1 & \xrightarrow{\;U\;} & \mathsf{Noumenal}_2 \\ \varphi \big\downarrow & & \big\downarrow \varphi \\ \mathsf{Phenomenal}_1 & & \mathsf{Phenomenal}_2 \end{array}$$

Here, φ is a mapping that determines the state of the phenomenal world in function of the state of the noumenal world. We refer to a state of the noumenal world as a *noumenal state* and to a state of the phenomenal world as a *phenomenal state*. Any phenomenal state arises from at least one noumenal state. Thus φ is *surjective*, as a mathematician would say.

The left part of the picture illustrates the following idea: when the noumenal world is in state $\mathsf{Noumenal}_1$, it has a corresponding phenomenal state $\mathsf{Phenomenal}_1$, which is determined by applying φ to $\mathsf{Noumenal}_1$:

$$\mathsf{Phenomenal}_1 = \varphi(\mathsf{Noumenal}_1)\,.$$

The Law of Nature that determines the evolution of the noumenal world is represented by U in this diagram. We can think of U as an *operation* that takes as input a noumenal state and outputs a new noumenal state. If we were considering only an *isolated subsystem* of the universe, the choice of operation would be determined by the laws of Nature and also by the state of the *environment*, which is the part of the universe external to the isolated system.

The upper part of the picture illustrates the following fact: if we apply an operation U to an isolated system that was in state $\mathsf{Noumenal}_1$, the new state of the system, $\mathsf{Noumenal}_2$, is determined only by its previous state and the operation. This can be summarized in the following equation:

$$\mathsf{Noumenal}_2 = U \cdot \mathsf{Noumenal}_1 \,.$$

Note that we wrote "$U \cdot \mathsf{Noumenal}$" above, rather than the more familiar form "$U(\mathsf{Noumenal})$", because we should not think here of U as a *function*, but rather "$\cdot$" is an *action* and U *acts* on the noumenal state according to that action. This allows us to use the same U to act differently on noumenal and phenomenal states by invoking different actions. Nevertheless, for ease of notation, we shall revert to writing $U(\bullet)$ once the concepts are rigorously established.

Finally, in the right part of the picture, we see that from the new noumenal state, $\mathsf{Noumenal}_2$, corresponds a phenomenal state, $\mathsf{Phenomenal}_2$. Mathematically:

$$\mathsf{Phenomenal}_2 = \varphi(\mathsf{Noumenal}_2) \,.$$

3.2.2 Parallel process between noumenal and phenomenal worlds

A question arises naturally when considering an isolated system: is it possible to describe its phenomenal evolution without having recourse to the noumenal world? Could we explain the evolution of phenomenal states only in terms of phenomenal states and operations applied on them? Could we explain the evolution of state $\mathsf{Phenomenal}_1$ to state $\mathsf{Phenomenal}_2$ through operation U, without having recourse to the underling state $\mathsf{Noumenal}_1$ giving rise to state $\mathsf{Noumenal}_2$?

Mathematically, can the following equation be verified:

$$\mathsf{Phenomenal}_2 = U \star \mathsf{Phenomenal}_1 \,,$$

where we have used "$\star$" to distinguish this action from the one on noumenal states, which was denoted "$\cdot$" above? We now argue that the answer is yes.

Given a state $\mathsf{Phenomenal}_1$, how can we determine its evolution according to operation U? Certainly, if we knew the underlying state $\mathsf{Noumenal}_1$ that led to state $\mathsf{Phenomenal}_1$, we could apply U on $\mathsf{Noumenal}_1$ and this would determine the new evolved state $\mathsf{Noumenal}_2$, from which we could determine the corresponding state $\mathsf{Phenomenal}_2$.

However, there is a potential difficulty with this reasoning: this works directly only if to a phenomenal state corresponds a unique underlying noumenal state. Could there be two distinct noumenal states underlying the same phenomenal state? If so, this would run against a principle attributed to Gottfried Wilhelm Leibniz, which we shall discuss in Section 3.2.3. For now, let us consider this possibility and see how it can be a problem for our argument.

Suppose we have two distinct noumenal states, $\mathsf{Noumenal}_1$ and $\mathsf{Noumenal}_1^*$, which correspond to the same phenomenal state $\mathsf{Phenomenal}_1$, meaning that $\varphi(\mathsf{Noumenal}_1) = \varphi(\mathsf{Noumenal}_1^*) = \mathsf{Phenomenal}_1$. These noumenal states will evolve according to operation U and give rise to states $\mathsf{Noumenal}_2$ and $\mathsf{Noumenal}_2^*$, respectively. To these evolved noumenal states correspond phenomenal states $\mathsf{Phenomenal}_2 = \varphi(\mathsf{Noumenal}_2)$ and $\mathsf{Phenomenal}_2^* = \varphi(\mathsf{Noumenal}_2^*)$. In order to be able to determine the evolution of state $\mathsf{Phenomenal}_1$ as a function of operation U *without needing any recourse to noumenal states*, it must be that $\mathsf{Phenomenal}_2 =$

$\mathsf{Phenomenal}_2^*$, so that it makes no difference which underlying noumenal state determined $\mathsf{Phenomenal}_1$. In this case, mathematicians would say that the phenomenal evolution is *well-defined*.

To find out if the evolution of phenomenal states is indeed well-defined, all we have to do is consider this auxiliary question: If we have two distinct underlying noumenal states $\mathsf{Noumenal}_1$ and $\mathsf{Noumenal}_1^*$ giving rise to the same state $\mathsf{Phenomenal}_1$, do they necessarily give rise to the same phenomenal state after evolution through some operation U?

We now argue that the answer to this conundrum is yes. Suppose we have two states $\mathsf{Noumenal}_1$ and $\mathsf{Noumenal}_1^*$ corresponding to the same state $\mathsf{Phenomenal}_1$. Remember that we include in our definition of the phenomenal world, not only the immediate subjective reality, the phenomenal experience that exists now, but any potential subjective reality, any potential phenomenal experience. If there is no *potential* difference now in the subjective reality between two noumenal states, there can be no difference in the future.

Thus, a state $\mathsf{Phenomenal}_1$, on which we apply an operation U, will evolve to a well-defined state $\mathsf{Phenomenal}_2$. Hence, we can indeed write:

$$\mathsf{Phenomenal}_2 = U \star \mathsf{Phenomenal}_1\,.$$

This allows us to update our picture:

$$\begin{array}{ccc} \mathsf{Noumenal}_1 & \xrightarrow{\;U\;} & \mathsf{Noumenal}_2 \\ \varphi\big\downarrow & & \big\downarrow\varphi \\ \mathsf{Phenomenal}_1 & \xrightarrow[\;U\;]{} & \mathsf{Phenomenal}_2 \end{array}$$

This new diagram illustrates the fact that there are two ways in which the same state $\mathsf{Phenomenal}_2$ can be determined from $\mathsf{Noumenal}_1$.

- We can apply first operation U to $\mathsf{Noumenal}_1$ to obtain $\mathsf{Noumenal}_2$ and then apply function φ to $\mathsf{Noumenal}_2$ and determine $\mathsf{Phenomenal}_2$;
- or we could apply first φ to $\mathsf{Noumenal}_1$ to determine $\mathsf{Phenomenal}_1$ and then we apply U to $\mathsf{Phenomenal}_1$ to obtain $\mathsf{Phenomenal}_2$.

A diagram with this property is called a *commuting* diagram. This diagram illustrates the fact that that there is a parallel process between the evolution of the noumenal world and the phenomenal world. It states that the evolution of the phenomenal consequences of the noumenal world are the phenomenal consequences of the evolution of the noumenal world. This concept is reminiscent of a remarkable nineteenth-century principle due to Heinrich Hertz.[3]

Mathematically:

$$\mathsf{Phenomenal}_2 = \varphi(U \cdot \mathsf{Noumenal}_1) = U \star \varphi(\mathsf{Noumenal}_1)\,.$$

A mathematical relation representing such a parallel process is called a *homomorphism*.

A philosopher would say that the evolution of the phenomenal world according to the Laws of Nature is an *epiphenomenon*: Understanding how the noumenal world evolves and the relation between the noumenal world and the phenomenal world is sufficient to describe how the Laws of Nature lead to the evolution of the phenomenal world. Causality should be understood strictly in terms of the evolution of the noumenal world.

3.2.3 Leibniz's Principle

Our previous discussion was made necessary by the possibility of two different noumenal states that can give rise to the same phenomenal state. This possibility runs

[3] "We form for ourselves images or symbols of external objects; and the form which we give them is such that the necessary consequents of the images in thought are always the images of the necessary consequents in nature of the things pictured. In order that this requirement may be satisfied, there must be a certain conformity between nature and our thought." [25].

against a principle attributed to Leibniz [27], which claims that if there is no possible perceptible difference between two objects, then these objects are the same, not superficially, but fundamentally. A complete discussion of Leibniz's principle is given by Peter Forrest [20].

According to Leibniz's principle, if two phenomenal states are equal, then they must arise from the same noumenal state. Mathematically, Leibniz's principle posits that φ is *injective*. However, we prove in companion paper [7] that Leibniz's principle is actually false under additional metaphysical principles: locality and the phenomenal validity of quantum theory inevitably lead to a deeper reality beyond what can be observed locally in a system. This necessity for a noumenal world implies the falsification of Leibniz's principle. Let us note that were Leibniz's principle correct, there would be a *bijective* correspondence between the noumenal world and the phenomenal world since φ would be both injective (by Leibniz's principle) and surjective (by definition). It would follow that to any noumenal state corresponds one and exactly one phenomenal state and vice versa. Furthermore, homomorphism φ would in fact be an isomorphism. Knowing the structure of the phenomenal world would be more than sufficient to know the structure of the noumenal world. As such, postulating any reality beyond the observable, while philosophically interesting, would be mathematically futile. It would lead to no supplementary explanatory power. After all, any isomorphism is a mere rebranding of terms.

According to the Encycloædia Britannica, "Kant claimed that man's speculative reason can only know phenomena and can never penetrate to the noumenon" [19]. Nevertheless, we demonstrate here and in our companion article on quantum theory [7] that the noumenal world in which we live can be apprehended by pure reason.

3.3 The structure of realism

From conceptual considerations, we are now moving towards developing a mathematical framework. We want to formalize mathematically the notions introduced earlier.

Let us begin by defining the *structure of realism*. The structure of realism is the list of mathematical axioms that characterize a realist theory. We shall call these axioms "requirements". This structure can be satisfied by many different theories. A concrete theory that satisfies these axioms is called a *realist model*.

The words "structure" and "model" are borrowed from universal algebra. For instance, the *structure* of a group is defined by the list of axioms that characterize an object as being a group. However, a particular group is a *model* for the structure of a group. For example, the integers with ordinary addition, $(\mathbb{Z}, +)$, provide a model of a group. There can be non-isomorphic models that satisfy the same structure. For example, $(\mathbb{R}, +)$ is not isomorphic to $(\mathbb{Z}, +)$ but is a group nevertheless. A more formal treatment of structures and models is given in Ref. [22].

First, we introduce realism in a theory consisting of a single *system*. A system is a part of the universe, or possibly the entire universe. Once we have built the machinery for a theory composed of a single system, we shall investigate in Section 3.4 the structure of *local realism*, in which there can be several systems that can evolve independently and interact with each others.

We follow Kant's terminology [19, 26], and as such we distinguish two kinds of states in a system, as mentioned informally in Section 3.2.

Noumenal State: The noumenal state of a system is its complete description. It describes the system as it is, rather than what can be observed about it, or known through sensory experience. It describes not only what can be observed from a system, but also how the system can interact with other systems. It is a state of being. It describes the system in itself, including parts that are not accessible locally or at all. Another term used in quantum foundations literature to describe the noumenal state would be the *ontic* state [35].

Phenomenal State: The phenomenal state of a system is a complete description of what is *locally* observable in that system. The phenomenal state is a complete description of all the observable properties potentially accessible in a system. It is what is observable in a system; not what is actually observed. The phenomenal state contains everything that can be observed through arbitrarily powerful technology. The only restriction on the technology is that it must abide by the laws of Nature.

Our terminology reflects the difference between appearance and reality. An alternative distinction, which is somewhat orthogonal, concerns the difference between existence and knowledge. The theories of existence and of knowledge are dealt with in the respective branches of philosophy called *ontology* and *epistemology*. Following that path would have led to the distinction between the *ontic* state of a system and its *epistemic* state [35]. The ontic state corresponds to what we have called the noumenal state. However, the epistemic state corresponds to what is known about a system by some observer [28], which might be subjective and vary from one observer to another [21, 29]. It should be emphasized that our phenomenal states are *not* states of knowledge, neither are they relative to an observer. Hence, epistemic and phenomenal states are two fundamentally different notions.

Now that we have explained our choice of terms, we shall describe the mathematical objects that are associated to a system.

Noumenal state space. Associated to a system is a *noumenal state space*, which is a set of *noumenal states*.

Phenomenal state space. Associated to a system is a *phenomenal state space*, which is a set of *phenomenal states*.

Operations. Associated to a system is a set of *operations*, which comes with a composition operator denoted "$\circ$". We require that:

1. If U and V are operations, $U \circ V$ is an operation;

2. If U, V and W are operations, $U \circ (V \circ W) = (U \circ V) \circ W$;

3. There exists an identity operation I such that, for all operations U,

$$I \circ U = U \circ I = U\,.$$

When there is no ambiguity, we shall omit the composition operator and write UV instead of $U \circ V$. Mathematicians would refer to the structure defining these operations as a *monoid*.

Reversibility of operations. An important natural principle is that the laws of physics are reversible. This principle is not necessary to characterize either realism nor local realism. However, the reversibility of the laws of physics can be easily expressed within our framework. It means that to each operation U, there corresponds an operation U^{-1} such that $U\,U^{-1} = U^{-1}U = I$. In other words, a reversible realistic structure is a realistic structure in which the operations form a *group*, which is a particular type of monoid.

Definition 3.3.1 (Operation action)**.** Let $\mathsf{Operations}$ be a set of operations and S be a set. An *operation action* on set S is a binary operation $\star : \mathsf{Operations} \times S \to S$ that satisfies, for all operations U and V and for all element s of the set S,

1. $U \star (V \star s) = (UV) \star s\,;$
2. if I is the identity operation, then $I \star s = s$.

Mathematicians refer to the operation $\star$ as a *monoid action*. A *group action* is the special case of a monoid action when the monoid is a group. Sometimes, an action can be characterized precisely in terms of how it acts on a given set. This leads to the concept of a *faithful* action.

Definition 3.3.2 (Faithful action)**.** Let $\star$ be an operation action on a set S. The action is *faithful* if whenever $U \star s = V \star s$ for all $s \in S$, then $U = V$.

Associated to a system, we require the existence of a faithful operation action on the noumenal state space, and an operation action on the phenomenal state space. Note that the phenomenal operation action is not required to be faithful. Thus, any operation is fully characterized by how it acts on noumenal states. The faithfulness of the noumenal action is not fundamental because any noumenal action can be made faithful by replacing operations by equivalence classes of operations, in effect equating any two operations that act identically on all possible noumenal states. However, it is algebraically very useful and natural to impose noumenal faithfulness.

Definition 3.3.3 (Noumenal-phenomenal homomorphism)**.** Let "$\cdot$" be the action on noumenal states and "$\star$" be the action on phenomenal states, and let ϕ be a mapping whose domain is the noumenal state space and whose range is the phenomenal state space. We say that ϕ is a *noumenal-phenomenal homomorphism*[4] if, for all operation U and all noumenal state N,

$$\phi(U \cdot N) = U \star \phi(N)\,.$$

When no ambiguity can arise, we omit writing the operation action, and instead we use the more familiar notation in which the object on which the operation acts is written in parenthesis, as if the operation were a function. For example, the equation above can be written equivalently as

$$\phi(U(N)) = U(\phi(N))\,.$$

The noumenal-phenomenal epimorphism. Associated to a system, we require the existence of a specific noumenal-phenomenal homomorphism, which has to be *surjective*: we call it *the noumenal-phenomenal epimorphism*[5] and denote it φ.

The operations act in a way that leads to the parallel evolution of the noumenal world and the phenomenal world, as explained intuitively in Section 3.2.2. This is best illustrated by the commuting diagram that we had seen previously:

[4] A *homomorphism* is a function that preserves relations between operations.
[5] An *epimorphism* is a surjective homomorphism.

$$\begin{array}{ccc} N_1 & \xrightarrow{U} & N_2 \\ \varphi\big\downarrow & & \big\downarrow\varphi \\ \rho_1 & \xrightarrow[U]{} & \rho_2 \end{array}$$

Even though we do not require the phenomenal action to be faithful in general, it will be useful to know that any noumenal action is automatically faithful whenever the corresponding phenomenal action is faithful (but not vice versa).

Theorem 3.3.1. Whenever the phenomenal action is faithful, the faithfulness requirement of the noumenal action is automatically verified.

Proof. Let "$\star$" and "$\cdot$" be the phenomenal and noumenal actions, respectively, and let φ be the noumenal-phenomenal epimorphism. Consider any two operations U and V for which $U \cdot N = V \cdot N$ for all noumenal state N. Our task is to prove that $U = V$. For this purpose, consider now any phenomenal state ρ and let N be any noumenal state such that $\rho = \varphi(N)$, whose existence is guaranteed by the surjectivity of the noumenal-phenomenal epimorphism.

$$\begin{aligned} U \cdot N = V \cdot N \quad &\Longrightarrow \quad \varphi(U \cdot N) = \varphi(V \cdot N) \\ &\Longrightarrow \quad U \star \varphi(N) = V \star \varphi(N) \\ &\Longrightarrow \quad U \star \rho = V \star \rho \end{aligned}$$

We have thus established that $U \star \rho = V \star \rho$ for all phenomenal state ρ, which implies that $U = V$ by faithfulness of the phenomenal action. This proves that the noumenal action is faithful as well. □

3.4 The structure of local realism

Now that we have defined the structure of realism, the idea of a world outside of our perceptions and how it relates to our perceptions, we can introduce an additional concept: locality.

3.4.1 Locality, informally

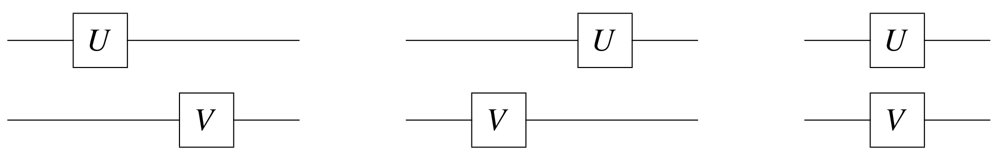
Informally, locality is a principle according to which the world is not an amorphous and indivisible blob: it can be divided into separate smaller parts, called *systems*, which interact with one another. Furthermore, if systems are disjoint and non-interacting, their states cannot influence one another. More generally, systems can influence one another provided they are sufficiently close. In contrast, they cannot influence one another when they are sufficiently far apart.

Relativity theory as a special case of locality. In relativity theory, the speed of light determines which systems are sufficiently far away that they cannot influence one another. Relativity is a theory about causality: It tells us whether or not a system can influence another. For our abstract purposes, any other theory that provides this sort of information is just as legitimate as relativity theory. We do not wish to give a precise meaning to "sufficiently close" and "sufficiently far".

Relativity theory can be contrasted with quantum theory. While relativity describes causality, and tells us which systems can and cannot interact with another, quantum theory describes what are the (phenomenal) states of systems, what kind of operations can be done on these states and what can be observed in a system.

If anything, the main lesson of the shift from Newtonian space-time to relativity is this: events are not related through a total order, but through a partial order. There exist events that are not causally related to one another in either order. In relativity, it does not matter whether Alice measures before Bob or Bob measures before Alice when they are spacelike separated. This is true because in reality, neither is measuring before the other! The idea of a total ordering between events occurring in spacelike separated systems is a myth.

Example of locality in our framework. Provided systems A and B are sufficiently far apart, it does not matter if we perform operation U first on system A and nothing on system B, followed by nothing on system A and V on system B, or if first we do nothing on system A and V on system B, followed by U on system A and nothing on system B. In either case, this simply corresponds to performing U on A and V on B. This is illustrated by the following three circuits, inspired by quantum computational networks [13], whose effect is identical.



Simply put, it is not meaningful to say that U was done before V or vice versa.

3.4.2 Systems

What is a system? A system is a part of the universe. The rest of the universe is called the *environment* of the system. The universe itself is a system. A system can be in one out of several possible noumenal states.

The state of a system evolves over time, according to the laws of Nature, and so does the state of the environment. However, a computer scientist would refer to this as *operations* applied on a system by the laws of Nature and the environment. The precise operation being applied, which is a function of the laws of Nature and the state of the environment, does not concern us.

A system is *open* when it interacts with the rest of the universe, and *closed*, equivalently called *isolated*, when it does not. When it is closed, nothing from the system can escape to the environment. Except for the universe itself, a closed system is a bit of an idealization.

We want to investigate how various systems relate to each others. For example, if we have a system A and a system B, we might be interested in the part of the universe that is common to both; this is another system, denoted $A \sqcap B$. In order to formalize this notion, we introduce a mathematical framework that describes all the parts of the universe we wish to consider, i.e. all systems and how they relate to each others.

Definition 3.4.1 (Lattice of systems)**.** A lattice of systems is a 6-tuple $(\mathscr{S}, \sqcup, \sqcap, \bar{\cdot}, S, \emptyset)$, where $\mathscr{S}$ is a set of elements called *systems*.

There are two special systems:

1. S, which is the *whole system* being considered, hereinafter called the *global system*. It could be the entire universe. Alternatively, it could be something much smaller, like a quantum computer or a single photon.
2. The *empty system* $\emptyset$, which contains no parts at all.

Let A and B be systems, then:

1. There exists a system $A \sqcup B$, the *union* of A and B.
2. There exists a system $A \sqcap B$ the *intersection* of A and B.
3. There exists a system $\overline{A}$, the *complement* of A, which is defined so that $A \sqcap \overline{A} = \emptyset$ and $A \sqcup \overline{A} = S$. Intuitively, it is composed of all the parts of S that are not in A.

The operations ($\sqcup$, $\sqcap$, $\bar{\cdot}$) and distinguished elements (S, $\emptyset$) behave like their usual set-theoretic counterparts. We use the slightly different notation of $\sqcup$, $\sqcap$, rather than $\cup$, $\cap$, to emphasize the fact that the operations $\sqcup$ and $\sqcap$ are purely algebraic in nature. A structure $(\mathscr{S}, \sqcup, \sqcap, \bar{\cdot}, S, \emptyset)$ that respects the usual set theoretic identities is called a *boolean lattice*.

Note that a more general theory might be possible if we did not impose that $A \sqcup B$ and $A \sqcap B$ be systems. We leave for future work this potential generalization in which we would no longer be able to use the fact that systems form a boolean lattice.

Definition 3.4.2 (Subsystem)**.** System A is a *subsystem* of a system B, written $A \sqsubseteq B$, if $A \sqcap B = A$.

Definition 3.4.3 (Disjoint systems)**.** Systems A and B are *disjoint* if they have no parts in common, i.e. $A \sqcap B = \emptyset$.

Note that the empty system is a subsystem of all systems, including itself, and that it is disjoint from all systems, again including itself.

Definition 3.4.4 (Composite system)**.** Let A and B be disjoint. The system $A \sqcup B$ is a *composite system*, composed of systems A and B. For convenience, we denote it by AB, rather than $A \sqcup B$.

Since $\sqcup$ is commutative, we have that $AB = A \sqcup B = B \sqcup A = BA$. It follows that $N^{AB} = N^{BA}$ and $\rho^{AB} = \rho^{BA}$ for any disjoint systems A and B.

Since $\sqcup$ is also associative, we have $A(BC) = (AB)C$ for any three mutually disjoint systems A, B and C. Thus, we shall simply write ABC to denote the composite system consisting of A, B and C.

Generalized union. In the most general study of local-realistic structures, we could be interested in cases in which a given system X can be represented as the union of many (possibly infinitely many, even possibly *uncountably* many) systems. Let $\mathscr{J}$ be a set of systems. A system X is the *union* of all systems in $\mathscr{J}$, denoted $X = \bigsqcup_{A \in \mathscr{J}} A$, if

$$(\forall A \in \mathscr{J}\ A \sqsubseteq X) \wedge (\forall B \in \mathscr{S}\ (\forall A \in \mathscr{J}\ A \sqsubseteq B) \implies X \sqsubseteq B).$$

Such an X may not exist in case $\mathscr{J}$ is infinite, in which case we are not allowed to write $\bigsqcup_{A \in \mathscr{J}} A$, but it is unique if it exists. Note that in set theory $\bigsqcup_{A \in \mathscr{J}} A$ is the usual $\bigcup_{A \in \mathscr{J}} A$.

Even though we shall not make use of the notion of generalized union to prove the main result of this paper, we have defined it in order to lay the groundwork for future research on local-realistic structures.

We now provide more details on the state spaces and operations on the various systems, in accordance with Section 3.3. Associated to each system A, we have:

Noumenal States: A noumenal state space denoted $\mathsf{Noumenal\text{-}Space}^A$; particular noumenal states of A are denoted N^A, N_i^A, N_1^A, etc.

Phenomenal States: A phenomenal state space denoted $\mathsf{Phenomenal\text{-}Space}^A$; particular phenomenal states of A are denoted ρ^A, ρ_i^A, ρ_1^A, etc.

Operations: A set of operations denoted $\mathsf{Operations}^A$; particular operations are denoted U^A, V^A, etc.; among them I^A denotes the identity operation on system A. When, there is no ambiguity, we drop the superscript and write simply U, V and I.

Noumenal-Phenomenal Epimorphism: A noumenal-phenomenal epimorphism on system A, denoted φ_A. When there is no ambiguity, we write φ instead of φ_A. For example, instead of writing $\varphi_A\left(N^A\right) = \rho^A$ we may write $\varphi\left(N^A\right) = \rho^A$, and we refer to φ as *the* noumenal-phenomenal epimorphism.

Now that we have established a notation for the various objects associated with systems, let us see how objects belonging to different systems relate with one another.

3.4.3 Splitting and merging

As we introduced informally in Section 3.4.1, the world can be decomposed into several parts according to a local-realistic theory. These parts exist in such a way that the state of the whole determines the state of the parts, and conversely the state of the whole is fully determined by the state of the parts. Note that the latter is *not* the case with standard quantum theory since entangled states cannot be recovered from the state of their subsystems. This is the reason why the usual formalism does not provide a local-realistic model of quantum theory.

3.4.3.1 Splitting and merging, intuitively

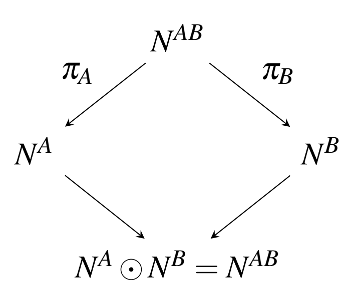
Given a composite system AB, its noumenal state N^{AB} can be decomposed in two states: A noumenal state N^A, in the state space of A, and a noumenal state N^B, in the state space of B. Informally, the state of the parts is determined by the state of the whole. For this purpose, we shall introduce formally in Section 3.4.3.2 two *projectors*, π_A and π_B, which split a system in the following way:

$$N^A = \pi_A\left(N^{AB}\right) \text{ and } N^B = \pi_B\left(N^{AB}\right).$$

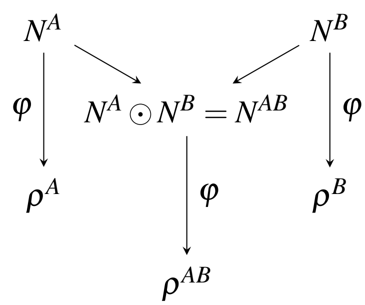
Furthermore, the two noumenal states N^A and N^B determine completely the noumenal state N^{AB}. Informally, the state of the whole is determined by the state of the parts. For this purpose, we shall introduce formally in Section 3.4.3.7 a *join product* "$\odot$", which merges the noumenal states of systems A and B as follows:

$$N^{AB} = N^A \odot N^B.$$

This is illustrated by the following diagram.



Note that such a diagram would not be possible at the phenomenal level in quantum theory, if we replaced N by ρ throughout, which is the main motivation for our introduction of the noumenal world. Nevertheless, even though the phenomenal state ρ^{AB} of joint system AB cannot be determined from the phenomenal states ρ^A and ρ^B of subsystems A and B, it *can* be determined (as well as ρ^A and ρ^B) from the *noumenal* states N^A and N^B of A and B, as illustrated by the following diagram.



Let us now proceed formally.

3.4.3.2 Noumenal projectors

For all systems A and B such that A is a subsystem of B, we require the existence of a function denoted π_A^B, which is called the *noumenal projector* from system B onto system A. Projector π_A^B is a surjective function from the noumenal space of system B to the noumenal space of system A. The intuitive reason for which we require that π_A^B be surjective is that each state of system A must arise from at least one state of any of its supersystems, such as B.

Furthermore, we require that if we have any noumenal state N^C belonging to a system C such that A is a subsystem of B, which is itself a subsystem of C, then the following relation must hold between projectors:

$$\left(\pi_A^B \circ \pi_B^C\right)\left(N^C\right) = \pi_A^C\left(N^C\right).$$

To say it more simply, if $A \sqsubseteq B \sqsubseteq C$, then

$$\pi_A^B \circ \pi_B^C = \pi_A^C \,.$$

Since there will be no ambiguity, we shall omit the superscript and we shall refer to π_A as *the* noumenal projector onto system A, regardless of the supersystem from which we project. For example, the equation above becomes

$$\pi_A \circ \pi_B = \pi_A \,.$$

This equation implies that projectors are idempotent,[6] which is the usual definition of projectors:

$$\pi_A \circ \pi_A = \pi_A \,.$$

Theorem 3.4.1. For any noumenal state N^A of system A,

$$\pi_A \left(N^A \right) = N^A \,.$$

Proof. Our surjectivity requirement on π_A^A imposes that there must exist a state N_α^A in the noumenal space of system A such that $\pi_A \left(N_\alpha^A \right) = N^A$. Therefore,

$$\pi_A \left(N^A \right) = \pi_A \circ \pi_A \left(N_\alpha^A \right) = \pi_A \left(N_\alpha^A \right) = N^A \,.$$

Of course, this N_α^A is none other than the original N^A since $\pi_A \left(N_\alpha^A \right) = N^A$ by definition of N_α^A, but also $\pi_A \left(N_\alpha^A \right) = N_\alpha^A$ by the theorem itself. □

3.4.3.3 Phenomenal projector

Moving now from noumenal to phenomenal states, we wish to express the following idea: If A is a subsystem of B, anything that can be observed about system A is fully

[6] By definition, x is idempotent under operation "$\cdot$" when $x \cdot x = x$.

determined by what can be observed about system B because any observation of A is also an observation of B. This leads to the following requirement. For all systems A, we require the existence of a phenomenal projector. These phenomenal projectors follow the same requirements as noumenal projectors, as stated in Section 3.4.3.2, *mutatis mutandis*. As an abuse of notation, we also denote the phenomenal projector onto system A by π_A, since no ambiguity will be possible with the corresponding noumenal projector π_A.

3.4.3.4 Relation between noumenal and phenomenal projectors

We require that for all systems A and B such that A is a subsystem of B, and all noumenal states N^B of B, the noumenal and phenomenal projections onto the system A are related by the following commutative relation:

$$\pi_A\left(\varphi\left(N^B\right)\right) = \varphi\left(\pi_A\left(N^B\right)\right).$$

Note that the π_A on the left is a phenomenal projector, whereas the π_A on the right is a noumenal projector. Note also that the $\varphi(\cdot)$ on the left is shorthand for $\varphi_B(\cdot)$, whereas the $\varphi(\cdot)$ on the right is shorthand for $\varphi_A(\cdot)$.

The relation between the noumenal and phenomenal projectors is best visualized by the fact that the following diagram commutes.

$$\begin{array}{ccc} N^B & \xrightarrow{\pi_A} & N^A \\ \varphi\big\downarrow & & \big\downarrow\varphi \\ \rho^B & \xrightarrow[\pi_A]{} & \rho^A \end{array}$$

This relation leads to the following natural definition for a family [7] of homomorphisms.

Definition 3.4.5 (Consistent family of noumenal-phenomenal homomorphisms)**.** Recall that $\mathscr{S}$ is the set of all systems. For any system A, let ϕ_A be a noumenal-phenomenal homomorphism for system A. We say that $(\phi_A)_{A\in\mathscr{S}}$ is a *consistent family of noumenal-phenomenal homomorphisms* if, for all systems A and B such that A is a subsystem of B, and for all noumenal states N^B of system B, the following relation holds.

$$\pi_A\big(\phi_B\big(N^B\big)\big) = \phi_A\big(\pi_A\big(N^B\big)\big)$$

Let us now define a *single* function from noumenal to phenomenal states of *all* systems. In order to deal with the possibility that the same noumenal or phenomenal state could belong to different systems,[8] this function takes two arguments: a system and a noumenal state of this system. For such a function to be useful for our purposes, it has to satisfy two conditions, which are encapsulated in the following definition.

Definition 3.4.6 (Universal noumenal-phenomenal epi/homomorphism)**.** Let ϕ be a function of two arguments. The first argument can be an arbitrary system A and the second an arbitrary noumenal state of system A. For each system A, this function ϕ gives rise to a function $\phi_A : \textsf{Noumenal-Space}^A \to \textsf{Phenomenal-Space}^A$ defined by $\phi_A(N^A) = \phi(A, N^A)$. We say that ϕ is a *universal noumenal-phenomenal homomorphism* if two conditions hold:

1. function ϕ_A is a noumenal-phenomenal homomorphism for all systems A; and

[7] The difference between *family* $(\phi_A)_{A\in\mathscr{S}}$ and the more familiar notation for what could be *set* $\{\phi_A\}_{A\in\mathscr{S}}$ is that each family element ϕ_A retains its association with the corresponding index, in this case system A. Formally, the notation for this family is shorthand for $\{(A,\phi_A) : A \in \mathscr{S}\}$. Note that the more familiar notation is ambiguous as it is used interchangeably to mean either a set or a family, which is why we prefer to write $\{\phi_A : A \in \mathscr{S}\}$ for the former. For reasons of consistency, we shall sometimes write the family index as a superscript rather than a subscript.

[8] Without loss of generality, we could have imposed the condition that the set of states of any system A has to be disjoint from the set of states of any other system B, but that would not have been natural when it comes to the phenomenal states of quantum theory, for instance in the way that they will be defined in Section 3.5.1.

2. the family $(\phi_A)_{A\in\mathscr{S}}$ of noumenal-phenomenal homomorphisms is consistent, according to Definition 3.4.5.

The same concept defines a universal noumenal-phenomenal *epi*morphism if we replace "homomorphism" by "epimorphism" throughout. Note that it is not sufficient for a universal noumenal-phenomenal homomorphism φ to be surjective in order to be called a universal noumenal-phenomenal epimorphism: it must be that φ_A is surjective for each system A.

If we are given a consistent family $(\phi_A)_{A\in\mathscr{S}}$ of noumenal-phenomenal homomorphisms, there is a natural way to build a single universal noumenal-phenomenal homomorphism ϕ defined as

$$\phi(A,N^A) \stackrel{\text{def}}{=} \phi_A(N^A)\,.$$

Since there will be no ambiguity on the system under consideration, we shall simply write $\phi\left(N^A\right)$ instead of $\phi\left(A,N^A\right)$, or equivalently instead of $\phi_A\left(N^A\right)$.

Similarly, if we are given a consistent family $(\varphi_A)_{A\in\mathscr{S}}$ of noumenal-phenomenal *epi*morphisms, we can build a universal noumenal-phenomenal epimorphism φ. Note that this is consistent with the notation φ introduced in Section 3.3.

In conclusion, our requirement that there be a noumenal-phenomenal epimorphism φ_A associated to each system A, and that the family $(\varphi_A)_{A\in\mathscr{S}}$ of all these epimorphisms be consistent, is equivalent to the requirement of the existence of a single universal noumenal-phenomenal epimorphism φ.

3.4.3.5 Abstract trace

Quantum theory often mentions tracing out other systems. More generally, we can define an abstract trace from any projector. For all disjoint systems A and B, for all noumenal states N^{AB} and all phenomenal states ρ^{AB}, we define

$$\operatorname{tr}_B\left(N^{AB}\right) \stackrel{\text{def}}{=} \pi_A\left(N^{AB}\right) \text{ and } \operatorname{tr}_B\left(\rho^{AB}\right) \stackrel{\text{def}}{=} \pi_A\left(\rho^{AB}\right).$$

Again, while both traces are different functions, we denote them with the same symbols since no ambiguity can arise. Our choice of working with projectors rather than traces stems from the fact that the notion of trace belongs to linear algebra only, whereas projectors are universal mathematical objects.

3.4.3.6 Compatibility

Recall that S denotes the system that represents the entire universe under consideration. Therefore, any noumenal state N^A belonging to system A can be represented as the projection of some noumenal state N^S of the universe

$$N^A = \pi_A\left(N^S\right)$$

since $A \sqsubseteq S$ by definition of S and by surjectivity of π^S_A, remembering that π_A is shortcut for π^S_A. The following definitions formalize the notion that states are *compatible* if they can exist simultaneously in the same universe.

Definition 3.4.7 (Compatible noumenal states)**.** Consider two systems A and B. Noumenal states N^A and N^B are *compatible* if there exists a noumenal state N^S of the global system such that $N^A = \pi_A\left(N^S\right)$ and $N^B = \pi_B\left(N^S\right)$.

Definition 3.4.8 (Compatible family of states)**.** Let $\mathscr{I}$ and $\mathscr{J}$ be possibly empty sets of systems. Let N^A be a noumenal state of system A for each $A \in \mathscr{I}$ and let ρ^A be a phenomenal state of system A for each $A \in \mathscr{J}$. We say that the ordered pair of families $\left(N^A\right)_{A\in\mathscr{I}}$ and $\left(\rho^A\right)_{A\in\mathscr{J}}$, denoted

$$\mathscr{F} = \left(\left(N^A\right)_{A\in\mathscr{I}}, \left(\rho^A\right)_{A\in\mathscr{J}}\right),$$

is a *compatible family of states* if there exists a noumenal state N^S of the global system such that $N^A = \pi_A\left(N^S\right)$ for each $A \in \mathscr{I}$ and $\rho^A = \pi_A\left(\varphi\left(N^S\right)\right)$ for each $A \in \mathscr{J}$. Any such N^S is called an *underlying global state* for family $\mathscr{F}$. Note that N^S must belong to $\left(N^A\right)_{A\in\mathscr{I}}$ whenever $S \in \mathscr{I}$.

As an abuse of notation, we say that $\left(N^A\right)_{A\in\mathscr{I}}$ or $\left(\rho^A\right)_{A\in\mathscr{J}}$ are compatible families of states should they be so, according to the above definition, if coupled with the empty family on the appropriate side.

Definition 3.4.9 (Compatible states)**.** Two states are *compatible* if they form together a compatible family of states. These two states could be either two noumenal states, two phenomenal states or one noumenal state and one phenomenal state. This definition subsumes Definition 3.4.7 in the case of two noumenal states.

The following theorems illustrate consequences of the notion of compatibility.

Theorem 3.4.2. Let A be a subsystem of B. Noumenal states N^A and N^B are compatible if and only if $N^A = \pi_A\left(N^B\right)$.

Proof. We first prove ⇒: Suppose N^A and N^B are compatible states. Let N^S be such that $N^A = \pi_A\left(N^S\right)$ and $N^B = \pi_B\left(N^S\right)$.

$$\begin{aligned}
\pi_A\left(N^B\right) &= \pi_A\left(\pi_B\left(N^S\right)\right) \\
&= \left(\pi_A \circ \pi_B\right)\left(N^S\right) \\
&= \pi_A\left(N^S\right) \\
&= N^A
\end{aligned}$$

Now we prove ⇐: Suppose $N^A = \pi_A\left(N^B\right)$. By surjectivity of the noumenal projector π_B there exists N^S such that $N^B = \pi_B\left(N^S\right)$.

$$
\begin{aligned}
N^A &= \pi_A\left(N^B\right) \\
&= \pi_A\left(\pi_B\left(N^S\right)\right) \\
&= (\pi_A \circ \pi_B)\left(N^S\right) \\
&= \pi_A\left(N^S\right)
\end{aligned}
$$

□

Theorem 3.4.3. Let A be a subsystem of B. Phenomenal states ρ^A and ρ^B are compatible if and only if $\rho^A = \pi_A\left(\rho^B\right)$.

Proof. We first prove $\Rightarrow$: Suppose ρ^A and ρ^B are compatible states. Let N^S be such that $\rho^A = \pi_A\left(\varphi\left(N^S\right)\right)$ and $\rho^B = \pi_B\left(\varphi\left(N^S\right)\right)$.

$$
\begin{aligned}
\pi_A\left(\rho^B\right) &= \pi_A\left(\pi_B\left(\varphi\left(N^S\right)\right)\right) \\
&= (\pi_A \circ \pi_B)\left(\varphi\left(N^S\right)\right) \\
&= \pi_A\left(\varphi\left(N^S\right)\right) \\
&= \rho^A
\end{aligned}
$$

Now we prove $\Leftarrow$: Suppose $\rho^A = \pi_A\left(\rho^B\right)$. By surjectivity of the phenomenal projector π_B and of the noumenal-phenomenal epimorphism φ_S, there exists N^S such that $\rho^B = \pi_B\left(\varphi\left(N^S\right)\right)$.

$$
\begin{aligned}
\rho^A &= \pi_A\left(\rho^B\right) \\
&= \pi_A\left(\pi_B\left(\varphi\left(N^S\right)\right)\right) \\
&= (\pi_A \circ \pi_B)\left(\varphi\left(N^S\right)\right) \\
&= \pi_A\left(\varphi\left(N^S\right)\right)
\end{aligned}
$$

□

Theorem 3.4.4. Let A be a subsystem of B. Phenomenal state ρ^A and noumenal state N^B are compatible if and only if $\rho^A = \pi_A\left(\varphi\left(N^B\right)\right)$.

Proof. This proof is similar to the two previous ones and is left to the reader. □

Corollary 3.4.1. Phenomenal state ρ^A and noumenal state N^A are compatible if and only if $\rho^A = \varphi\left(N^A\right)$.

Proof. Immediate since $A \sqsubseteq A$ and $\rho^A = \pi_A\left(\varphi\left(N^A\right)\right) = \varphi\left(N^A\right)$. □

3.4.3.7 The join product

For all disjoint systems A and B, we require the existence of an operation, the *join product*, denoted "$\odot$",[9] such that for all noumenal states N^{AB}, the following relation holds:

$$N^A \odot N^B = N^{AB},$$

where $N^A = \pi_A\left(N^{AB}\right)$ and $N^B = \pi_B\left(N^{AB}\right)$. It follows that an arbitrary composite state N^{AB} can be represented by its decomposition $N^A \odot N^B$, which is unique according to Theorem 3.4.6 below.

Note that the join product is only defined on compatible states. Therefore, if noumenal states N^A and N^B are not both projections of some noumenal state N^{AB}, their join product is not defined. This implies that $N^A \odot N^B = N^{AB}$ if and only if N^A, N^B and N^{AB} are compatible.

As we can see, compatible states are states on which the operations of join product, noumenal and phenomenal projectors, and the noumenal-phenomenal epimorphisms are well-behaved.

Convention on compatibility. Hereinafter, we shall adopt the following convention: states without indices, for example N^A, ρ^B (rather than N^A_α, ρ^B_i), are always assumed to be compatible and to have an underlying global state N^S. Thus, we shall always assume

[9] Technically, we should write $\odot_{(A,B)}$ to denote the fact that the join product depends on systems A and B, but since there will be no confusion, as an abuse of notation, we shall not do so.

that $N^A = \pi_A(N^S)$ and $\rho^A = \pi_A(\varphi(N^S))$ for all systems A. In particular, the following propositions are implicitly assumed in the theorems below.

For all systems A,

$$\rho^A = \varphi\left(N^A\right).$$

For all systems A and B such that $A \sqsubseteq B$,

$$\rho^A = \pi_A\left(\rho^B\right)$$

$$N^A = \pi_A\left(N^B\right).$$

For all disjoint systems A and B,

$$N^A \odot N^B = N^{AB}$$

$$\pi_A\left(N^{AB}\right) = N^A$$

$$\pi_A\left(\rho^{AB}\right) = \rho^A.$$

Theorem 3.4.5. $\pi_A\left(N^A \odot N^B\right) = N^A$ and $\pi_B\left(N^A \odot N^B\right) = N^B$.

Proof. We prove only $\pi_A\left(N^A \odot N^B\right) = N^A$; the other statement is similar.

$$\pi_A\left(N^A \odot N^B\right) = \pi_A\left(N^{AB}\right) = N^A$$

□

Theorem 3.4.6 (Unique Decomposition)**.** Let A and B be disjoint systems.

$$N_1^A \odot N_1^B = N_2^A \odot N_2^B \implies N_1^A = N_2^A \text{ and } N_1^B = N_2^B$$

Proof. We prove $N_1^A = N_1^B$; the other statement is similar

$$
\begin{aligned}
&N_1^A \\
&= \pi_A \left(N_1^A \odot N_1^B \right) \\
&= \pi_A \left(N_2^A \odot N_2^B \right) \\
&= N_2^A
\end{aligned}
$$

□

Theorem 3.4.7 (Commutativity of the join product)**.** For any disjoint systems A and B,

$$N^A \odot N^B = N^B \odot N^A \,.$$

To be technically more precise, $N^A \odot_{(A,B)} N^B = N^B \odot_{(B,A)} N^A$, according to footnote 9.

Proof.

$$
\begin{aligned}
&\left(N^A \odot N^B \right) \\
&= N^{AB} \\
&= N^{BA} \\
&= \left(N^B \odot N^A \right)
\end{aligned}
$$

□

Theorem 3.4.8 (Associativity of the join product)**.** For any disjoint systems A, B and C,

$$\left(N^A \odot N^B \right) \odot N^C = N^A \odot \left(N^B \odot N^C \right) \,.$$

It follows that we can omit the parentheses and simply write $N^A \odot N^B \odot N^C$.

Proof.

$$
\begin{aligned}
&\left(N^A \odot N^B\right) \odot N^C \\
&= N^{AB} \odot N^C \\
&= N^{ABC} \\
&= N^A \odot N^{BC} \\
&= N^A \odot \left(N^B \odot N^C\right)
\end{aligned}
$$

In the above, recall that $N^{ABC} = N^{(AB)C} = N^{A(BC)}$. □

Now, we generalize the join product to an arbitrary set $\mathscr{J}$ of mutually disjoint systems provided $X = \bigsqcup_{A \in \mathscr{J}} A$ is a system. If X is in noumenal state N^X and $N^A = \pi_A\left(N^X\right)$ for each $A \in \mathscr{J}$, then we require that the generalized join product $\bigodot$, which is defined only on compatible families of states, satisfies

$$\bigodot_{A \in \mathscr{J}} N^A = N^X .$$

It follows that an arbitrary state N^X of system X can be represented by its unique decomposition $\bigodot_{A \in \mathscr{J}} N^A$.

Theorem 3.4.9. Consider a set $\mathscr{J}$ of disjoint systems such that $X = \bigsqcup_{A \in \mathscr{J}} A$ exists. Let B be a subsystem of X that is not necessarily in the set.

$$\pi_B\left(\bigodot_{A \in \mathscr{J}} N^A\right) = N^B$$

Proof.

$$\pi_B\left(\bigodot_{A \in \mathscr{J}} N^A\right) = \pi_B\left(N^X\right) = N^B$$

□

Theorem 3.4.10 (Generalized unique decomposition)**.** Let $\mathscr{J}$ be a set of mutually disjoint systems.

$$\left(\bigodot_{A\in\mathscr{J}} N_1^A = \bigodot_{A\in\mathscr{J}} N_2^A\right) \implies \left(\forall A \in \mathscr{J} \quad N_1^A = N_2^A\right)$$

Proof. Let $A \in \mathscr{J}$

$$\begin{aligned} & N_1^A \\ & = \pi_A\left(\bigodot_{A\in\mathscr{J}} N_1^A\right) \\ & = \pi_A\left(\bigodot_{A\in\mathscr{J}} N_2^A\right) \\ & = N_2^A \end{aligned}$$

□

3.4.4 Separate evolution and product of operations

Suppose we have two disjoint systems, A and B, respectively in compatible states N^A and N^B. If we apply some operation U on system A and V on system B, the new state of systems A and B will be $U\left(N^A\right)$ and $V\left(N^B\right)$, respectively. Intuitively, we have performed some operation W on joint system AB, which maps state $N^A \odot N^B$ to $U\left(N^A\right) \odot V\left(N^B\right)$. However, for this to make sense, it must be not only that $U\left(N^A\right)$ and $V\left(N^B\right)$ are compatible, but also that W itself belongs to the set of operations on system AB. We now proceed to formalize this notion.

For all disjoint systems A and B, $U \in \mathsf{Operations}^A$ and $V \in \mathsf{Operations}^B$, we require the existence of $W \in \mathsf{Operations}^{AB}$ such that for all compatible states $N^A \in \mathsf{Noumenal\text{-}Space}^A$ and $N^B \in \mathsf{Noumenal\text{-}Space}^B$,

$$W\left(N^A \odot N^B\right) = U\left(N^A\right) \odot V\left(N^B\right) .$$

This requirement justifies the introduction of a *direct product of operations*, which we denote "$\times$",[10] such that for any operation U on system A, any operation V on system B, and for any noumenal state $N^{AB} = N^A \odot N^B$, we define $U \times V$ as *the* operation on system AB that satisfies

$$(U \times V)\left(N^A \odot N^B\right) = U\left(N^A\right) \odot V\left(N^B\right). \tag{3.1}$$

Note that this equation defines $U \times V$ uniquely because we had required the noumenal action to be faithful; see Definition 3.3.2. It follows that

$$\pi_A\Big((U \times V)\left(N^{AB}\right)\Big) = U\left(N^A\right) \quad \text{and} \quad \pi_B\Big((U \times V)\left(N^{AB}\right)\Big) = V\left(N^B\right).$$

Thus, the new state of system A is simply $U\left(N^A\right)$, as it should. Crucially, we see that the operation U performed on (possibly far-away) system A has had absolutely no effect on the noumenal state of system B.

This concept is illustrated by the following commuting diagram.

$$\begin{array}{ccc} N^{AB} & \xrightarrow{\;U \times V\;} & (U \times V)\left(N^{AB}\right) \\ \pi_A \big\downarrow & & \big\downarrow \pi_A \\ N^A & \xrightarrow[\;U\;]{} & U\left(N^A\right) \end{array}$$

More generally, consider a set $\mathscr{J}$ of disjoint systems such that $X = \bigsqcup_{A \in \mathscr{J}} A$ exists and is in noumenal state $N^X = \bigodot_{A \in \mathscr{J}} N^A$. Consider also an operation U^A on each system

[10] Technically, we should denote this product of operations as $\times_{(A,B)}$ but we shall consider the dependence in A and B to be implicit. Note also that in quantum theory we would use "$\otimes$", called tensor product, but the usual quantum-mechanical construct is not a *direct product* because $U \otimes V = U' \otimes V'$ is possible when $U \neq U'$ and $V \neq V'$, unless we define equality of unitary transformations up to phase, as we shall do in Section 3.5.1. Here, it is an abstract algebraic construction that is *defined* by Eq (3.1), and indeed it is a direct product by construction.

$A \in \mathscr{J}$. Eq (3.1) generalizes to the following requirement.

$$\left(\prod_{A\in\mathscr{J}} U^A\right)\left(\bigodot_{A\in\mathscr{J}} N^A\right) = \bigodot_{A\in\mathscr{J}} U^A\left(N^A\right)$$

For any $B \in \mathscr{J}$, this implies that

$$\pi_B\left(\left(\prod_{A\in\mathscr{J}} U^A\right)\left(\bigodot_{A\in\mathscr{J}} N^A\right)\right) = U^B\left(N^B\right).$$

Again, the operations performed on all the systems from $\mathscr{J}$ other than B have no effect on the noumenal state of system B.

3.4.4.1 Immediate consequences of the definition of product of operations

We now state and prove several properties of the product of operations. These proofs hinge upon the fact that if two operations act identically on all noumenal states, then they are the same operation, again by faithfulness of the noumenal action. Recall also that any state N^{AB} can be represented as a product state $N^{AB} = N^A \odot N^B$ and that the state N^{ABC} can be represented as a product $N^{ABC} = N^A \odot \left(N^B \odot N^C\right) = \left(N^A \odot N^B\right) \odot N^C$.

Theorem 3.4.11.

$$\left(U_2 \times V_2\right)\left(U_1 \times V_1\right) = \left(U_2 U_1\right) \times \left(V_2 V_1\right)$$

Proof. Consider arbitrary compatible noumenal states N^A and N^B for systems A and B.

$$\begin{aligned}
&\left(U_2 \times V_2\right)\left(U_1 \times V_1\right)\left(N^A \odot N^B\right)\\
=&\left(U_2 \times V_2\right)\left(U_1\left(N^A\right) \odot V_1\left(N^B\right)\right)\\
=&U_2\left(U_1\left(N^A\right)\right) \odot \left(V_2\left(V_1\left(N^B\right)\right)\right)\\
=&\left(U_2 U_1\right)\left(N^A\right) \odot \left(V_2 V_1\right)\left(N^B\right)\\
=&\left(\left(U_2 U_1\right) \times \left(V_2 V_1\right)\right)\left(N^A \odot N^B\right)
\end{aligned}$$

□

Theorem 3.4.12.

$$I^A \times I^B = I^{AB}$$

Proof. Consider arbitrary compatible noumenal states N^A and N^B for systems A and B.

$$\begin{aligned}
&\left(I^A \times I^B\right)\left(N^A \odot N^B\right) \\
=&I^A\left(N^A\right) \odot I^A\left(N^B\right) \\
=&N^A \odot N^B \\
=&I^{AB}\left(N^A \odot N^B\right)
\end{aligned}$$

□

Theorem 3.4.13.

$$U \times V = V \times U$$

To be technically more precise, $U \times_{(A,B)} V = V \times_{(B,A)} U$, according to footnote 10.

Proof. Consider an arbitrary noumenal state $N^{AB} = N^{BA}$ for system $AB = BA$.

$$\begin{aligned}
&(U \times V)\left(N^{AB}\right) \\
=&U\left(N^A\right) \odot V\left(N^B\right) \\
=&V\left(N^B\right) \odot U\left(N^A\right) \\
=&(V \times U)\left(N^{BA}\right) \\
=&(V \times U)\left(N^{AB}\right)
\end{aligned}$$

□

Theorem 3.4.14.

$$U \times (V \times W) = (U \times V) \times W$$

Proof. Consider an arbitrary noumenal state N^{ABC} for system ABC.

$$
\begin{aligned}
&(U \times (V \times W))\left(N^{ABC}\right) \\
=&U\left(N^{A}\right) \odot (V \times W)\left(N^{BC}\right) \\
=&U\left(N^{A}\right) \odot \left(V\left(N^{B}\right) \odot W\left(N^{C}\right)\right) \\
=&\left(U\left(N^{A}\right) \odot V\left(N^{B}\right)\right) \odot W\left(N^{C}\right) \\
=&(U \times V)\left(N^{AB}\right) \odot W\left(N^{C}\right) \\
=&((U \times V) \times W)\left(N^{ABC}\right)
\end{aligned}
$$

□

Since both $\odot$ and $\times$ are associative (Theorems 3.4.8 and 3.4.14), we can omit the parentheses. For example,

$$(U \times V \times W)\left(N^{A} \odot N^{B} \odot N^{C}\right) = U\left(N^{A}\right) \odot V\left(N^{B}\right) \odot W\left(N^{B}\right).$$

Theorem 3.4.15.

$$\left(\prod_{A \in \mathscr{J}} U^{A}\right)\left(\prod_{A \in \mathscr{J}} V^{A}\right) = \prod_{A \in \mathscr{J}} \left(U^{A} V^{A}\right)$$

Proof. Consider arbitrary compatible noumenal states N^A for each system $A \in \mathscr{J}$.

$$
\begin{aligned}
&\left(\left(\prod_{A\in\mathscr{J}} U^A\right)\left(\prod_{A\in\mathscr{J}} V^A\right)\right)\left(\bigodot_{A\in\mathscr{J}} N^A\right) \\
=&\left(\prod_{A\in\mathscr{J}} U^A\right)\left(\left(\prod_{A\in\mathscr{J}} V^A\right)\left(\bigodot_{A\in\mathscr{J}} N^A\right)\right) \\
=&\left(\prod_{A\in\mathscr{J}} U^A\right)\left(\bigodot_{A\in\mathscr{J}} V^A\left(N^A\right)\right) \\
=&\bigodot_{A\in\mathscr{J}} U^A\left(V^A\left(N^A\right)\right) \\
=&\bigodot_{A\in\mathscr{J}}\left(U^A V^A\right)\left(N^A\right) \\
=&\prod_{A\in\mathscr{J}}\left(U^A V^A\right)\left(\bigodot_{A\in\mathscr{J}} N^A\right)
\end{aligned}
$$

□

Theorem 3.4.16. Consider a set $\mathscr{J}$ of disjoint systems such that $X = \bigsqcup_{A\in\mathscr{J}} A$ exists. We have

$$\prod_{A\in\mathscr{J}} I^A = I^X .$$

Proof. Consider arbitrary compatible noumenal states N^A for each system $A \in \mathscr{J}$.

$$
\begin{aligned}
&\prod_{A\in\mathscr{J}} I^A\left(\bigodot_{A} N^A\right) \\
=&\bigodot_{A\in\mathscr{J}} I^A\left(N^A\right) \\
=&\bigodot_{A\in\mathscr{J}} N^A \\
=&I^X\left(\bigodot_{A\in\mathscr{J}} N^A\right)
\end{aligned}
$$

□

The following two theorems hold *provided the set of operations on any given system* forms a group.

Theorem 3.4.17. Let A and B be disjoint systems. For any operation U on system A and V on system B,

$$(U \times V)^{-1} = U^{-1} \times V^{-1}.$$

Proof.

$$\begin{aligned}
&(U \times V)\left(U^{-1} \times V^{-1}\right)\\
&\left(UU^{-1}\right) \times \left(VV^{-1}\right)\\
&\left(I^A \times I^B\right)\\
&= I^{AB}
\end{aligned}$$

□

Theorem 3.4.18. Consider a set $\mathscr{J}$ of disjoint systems such that $X = \bigsqcup_{A \in \mathscr{J}} A$ exists. We have

$$\left(\prod_{A \in \mathscr{J}} U^A\right)^{-1} = \prod_{A \in \mathscr{J}} \left(U^A\right)^{-1}.$$

Proof.

$$\begin{aligned}
&\prod_{A \in \mathscr{J}} U^A \prod_{A \in \mathscr{J}} \left(U^A\right)^{-1}\\
=&\prod_{A \in \mathscr{J}} \left(U^A \left(U^A\right)^{-1}\right)\\
=&\prod_{A \in \mathscr{J}} I^A\\
=&I^X
\end{aligned}$$

□

Note for the expert: Our definition gives a *direct product* in the usual algebraic sense; see footnote 10 again. Had we not required the action to be faithful, we could have had various pathologies. For instance, it could have happened that even though both I^{AB}

and $I^A \times I^B$ do nothing on any noumenal states, $I^A \times I^B$ is *not* the neutral element of the monoid, only an element of the kernel of the action, contradicting Theorem 3.4.12.

3.4.5 No-signalling principle

One important, albeit obvious, consequence of a theory being local-realistic is that it is not possible to send a signal from one system to another if there is no interaction between the two.

Intuitively, no operation performed on some system A can have an instantaneous effect of any kind on a remote system B. It follows that no operation performed on system A can have an instantaneous *observable* effect on system B. More precisely, when we perform an operation U on system A and an operation V on system B, operation V has only affected the noumenal state of system B, without any influence on the noumenal state of system A. It follows that the phenomenal state of system A, which is a function of its noumenal state, is also unchanged. This is formalized in the following theorem.

Theorem 3.4.19 (No-Signalling Principle)**.** Let ρ^{AB} be a phenomenal state of system AB. For all operations U on system A and V on system B,

$$\pi_A\Big((U \times V)\left(\rho^{AB}\right)\Big) = U\left(\rho^A\right).$$

We call the equation above the *no-signalling principle* because it means that no operation V applied on system B can have a phenomenal (i.e. observable) effect on a remote system A.

Proof. Let N^{AB} be any noumenal state such that $\rho^{AB} = \varphi\left(N^{AB}\right)$. Its existence is guaranteed from the fact that φ is surjective.

$$
\begin{aligned}
& \pi_A\left(\left(U \times V\right)\left(\rho^{AB}\right)\right) \\
&= \pi_A\left(\left(U \times V\right)\left(\varphi\left(N^{AB}\right)\right)\right) \\
&= \pi_A\left(\varphi\left(\left(U \times V\right)\left(N^{AB}\right)\right)\right) \\
&= \varphi\left(\pi_A\left(\left(U \times V\right)\left(N^{AB}\right)\right)\right) \\
&= \varphi\left(U\left(N^{A}\right)\right) \\
&= U\left(\varphi\left(N^{A}\right)\right) \\
&= U\left(\rho^{A}\right)
\end{aligned}
$$

□

In a local-realistic structure, no-signalling is a theorem, not a postulate! Any theory that is local-realistic is automatically no-signalling. We shall later explore the converse question, whether given a no-signalling theory, it is possible to construct a local-realistic theory that gives rise to the same phenomenal observations.

Thus, a theory is no-signalling if the following diagram commutes.

$$
\begin{array}{ccc}
\rho^{AB} & \xrightarrow{\;U \times V\;} & \left(U \times V\right)\left(\rho^{AB}\right) \\
\pi_A \big\downarrow & & \big\downarrow \pi_A \\
\rho^{A} & \xrightarrow[\;U\;]{} & U\left(\rho^{A}\right)
\end{array}
$$

The no-signalling principle can be extended to arbitrary products of operations.

Theorem 3.4.20. Consider a set $\mathscr{J}$ of disjoint systems such that $X = \bigsqcup_{A \in \mathscr{J}} A$ exists, and an operation U^A on each system $A \in \mathscr{J}$. For any $B \in \mathscr{J}$, we have

$$
\pi_B\left(\left(\prod_{A \in \mathscr{J}} U^A\right)\left(\rho^X\right)\right) = U^B\left(\pi_B\left(\rho^X\right)\right) .
$$

Proof. Let N^X be any noumenal state such that $\rho^X = \varphi(N^X)$. It suffices to apply the commuting relations.

$$
\begin{aligned}
&\pi_B\left(\left(\prod_{A\in\mathscr{J}} U^A\right)(\rho^X)\right) \\
&= \pi_B\left(\left(\prod_{A\in\mathscr{J}} U^A\right)(\varphi(N^X))\right) \\
&= \pi_B\left(\varphi\left(\left(\prod_{A\in\mathscr{J}} U^A\right)(N^X)\right)\right) \\
&= \varphi\left(\pi_B\left(\left(\prod_{A\in\mathscr{J}} U^A\right)(N^X)\right)\right) \\
&= \varphi(U^B(N^B)) \\
&= U^B(\varphi(N^B)) \\
&= U^B(\rho^B)
\end{aligned}
$$

□

Our statement of the no-signalling principle in Theorem 3.4.19 is a generalization of the usual notion, which is typically formulated in terms of the probability distribution of *observation* outcomes (which would be called *measurements* in quantum theory) made in two of more remote locations. In the simplest bipartite instance, consider two observers Alice and Bob, who share some system AB. They dispose of sets of operations $\{U_i\colon i \in I\}$ that Alice can apply on A and $\{V_j\colon j \in J\}$ that Bob can apply on B. These operations may include observations that can produce outcomes x and y, respectively. Denote by $\text{Prob}[U_i \to x]$ the probability that operation U_i applied by Alice on system A produces outcome x. Similarly, $\text{Prob}[U_i \to x, V_j \to y]$ is the joint probability that Alice observes x and Bob observes y if they perform operations U_i and V_j on systems A and B, respectively.

Assume now that Alice and Bob are sufficiently far apart that their systems can be considered disjoint and non-interacting in the sense of Section 3.4.1 (possibly because they are spacelike separated). The usual no-signalling principle [30] says that, for any

$i \in I$ and any possible outcome x when operation U_i is performed by Alice on system A, $\text{Prob}[U_i \to x]$ can be well-defined as

$$\text{Prob}[U_i \to x] \;=\; \sum_y \text{Prob}[U_i \to x, V_j \to y]\,,$$

regardless of the choice of j that Bob may make. In other words, the observable outcome at Alice's of performing some operation U_i on system A must not depend on which operation V_j is performed by Bob on remote system B, including no operation at all.[11] It follows that Bob cannot signal information to Alice by a clever choice of which operation to apply (or not) to his system.

3.4.6 Dropping the surjectivity requirement

The surjectivity requirement of the noumenal-phenomenal epimorphisms will be a hindrance later, when we shall build a local-realistic model from any no-signalling theory. For this reason, it is sometimes convenient to relax this requirement. Here, we show that this can be done under the conditions established below, according to which a family of not-necessarily-surjective universal homomorphisms can be collected into a single universal epimorphism that respects all the conditions set above.

Consider again a lattice of systems $(\mathscr{S}, \sqcup, \sqcap, \bar{\cdot}, S, \emptyset)$. Associated to each system A, we still have a noumenal state space $\mathsf{Noumenal\text{-}Space}^A$, a phenomenal state space $\mathsf{Phenomenal\text{-}Space}^A$, a set of operations $\mathsf{Operations}^A$, as well as noumenal and phenomenal projectors, both denoted π_A; and associated to each disjoint pair of systems A and B, we still have a join product $\odot_{(A,B)}$ and a product of operations $\times_{(A,B)}$, the latter two simply denoted $\odot$ and $\times$ for convenience. However, instead of having a single universal noumenal-phenomenal epimorphism, we have a family of universal noumenal-phenomenal homomorphisms $(\phi_i)_{i \in I}$ for some index set I. Suppose also that for each

[11] Formally, we need the identity operation to be among Bob's choices for "including no operation at all" to hold.

phenomenal state ρ^S of the global system, there exists at least one noumenal-phenomenal homomorphism ϕ_i and a noumenal state N^S of the global system such that $\rho^S = \phi_i\left(N^S\right)$.

Now, we proceed to build a local-realistic model composed of the same lattice of systems, the same phenomenal spaces, the same operations and phenomenal actions of the operations on the phenomenal spaces and the same phenomenal projectors. However, in order to obtain the desired universal noumenal-phenomenal epimorphism, we shall need to define new noumenal state spaces, and therefore new noumenal states, noumenal projectors denoted π', join product denoted $\odot'$, and actions on noumenal states. Consider any two systems A and B.

$$\begin{cases} \textsf{New-Noumenal-Space}^A \stackrel{\text{def}}{=} \left\{\left(N^A, i\right) : N^A \in \textsf{Noumenal-Space}^A, i \in I\right\} \\ \pi'_A\left(N^B, i\right) \stackrel{\text{def}}{=} \left(\pi_A\left(N^B\right), i\right) & \text{provided } A \sqsubseteq B \\ \left(N^A, i\right) \odot' \left(N^B, i\right) \stackrel{\text{def}}{=} \left(N^A \odot N^B, i\right) & \text{provided } N^A \odot N^B \text{ is defined} \\ U\left(N^A, i\right) \stackrel{\text{def}}{=} \left(U\left(N^A\right), i\right) \end{cases}$$

As before, the new join product $\odot'$ is only defined on compatible states: $\left(N^A, i\right) \odot' \left(N^B, j\right)$ is defined under conditions that $i = j$, N^A and N^B are compatible states in the original noumenal spaces, and A and B are disjoint systems. The new universal noumenal-phenomenal epimorphism φ' is defined as follows:

$$\varphi'\left(N^A, i\right) \stackrel{\text{def}}{=} \phi_i\left(N^A\right).$$

It is easy to verify that the new noumenal spaces, join products, actions and universal noumenal-phenomenal epimorphism give rise to a local-realistic model. Let us prove for example that the new join product behaves properly, according to how it was defined at the beginning of Section 3.4.3.7. The other requirements for a local-realistic structure are proved similarly.

Theorem 3.4.21.

$$\pi'_A\left(N^{AB},i\right)\odot'\pi'_B\left(N^{AB},i\right)=\left(N^{AB},i\right)$$

Proof.

$$\begin{aligned}\pi'_A\left(N^{AB},i\right)\odot'\pi'_B\left(N^{AB},i\right)&=\left(\pi_A\left(N^{AB}\right),i\right)\odot'\left(\pi_B\left(N^{AB}\right),i\right)\\&=\left(\pi_A\left(N^{AB}\right)\odot\pi_B\left(N^{AB}\right),i\right)\\&=\left(N^{AB},i\right)\end{aligned}$$

□

Furthermore, φ' is indeed a universal noumenal-phenomenal epimorphism, which was the purpose of the entire exercise. To prove this, it suffices to show that φ' is a universal noumenal-phenomenal *homo*morphism (which is obvious) and that φ'_A, the restriction of φ' to system A, is surjective for each system A, which is the purpose of the next theorem.

Theorem 3.4.22. Consider an arbitrary system A and some phenomenal state ρ^A in $\textsf{Phenomenal-Space}^A$. There exists (N^A,i) in $\textsf{New-Noumenal-Space}^A$ such that $\varphi'_A\left(N^A,i\right)=\rho^A$.

Proof. Let ρ^S be so that $\pi_A\left(\rho^S\right)=\rho^A$. Let N^S and i be so that $\rho^S=\phi_i\left(N^S\right)$.

$$\begin{aligned}\varphi'_A\left(\pi_A\left(N^S\right),i\right)&=\phi_i\left(\pi_A\left(N^S\right)\right)\\&=\pi_A\left(\phi_i\left(N^S\right)\right)\\&=\pi_A\left(\rho^S\right)\\&=\rho^A\end{aligned}$$

□

It follows that the requirement that the universal noumenal-phenomenal homomorphism be surjective on each system (hence an epimorphism) can be dropped, provided we have a family of universal homomorphisms that has the property defined above.

3.5 No-signalling operational theories

Until now, we have defined a framework for local-realistic theories. Let us now consider theories for which there is a phenomenal world, but no explicit noumenal world, or perhaps even no noumenal world at all. Such theories, which deal with the observable, are called *operational*. More specifically, we are interested in *no-signalling* operational theories, in which no operation performed on a system *A* has any *observable* effect on a disjoint system *B*. In case there is a noumenal world, however, an operation performed on system *A* is allowed to have have an instantaneous effect on the noumenal state of remote system *B*, provided it has no effect on its phenomenal state, hence it does not lead to any observable consequences.

The key difference between a no-signalling operational theory and a local-realistic theory is the required existence of a join product in the latter, which allows us to describe the state of a composite system as a function purely of its subsystems, whereas there is no such requirement in the former. Furthermore, there is no requirement of an underlying reality in an operational theory: it does not have to be the shadow of some unspecified noumenal world.

The central purpose of this paper is to establish a link between no-signalling operational theories and local-realistic theories. We have already seen that all local-realistic theories are no-signalling (Theorem 3.4.19), but could this statement be reversed? Could it be that all no-signalling operational theories are local-realistic? The answer depends of what we mean exactly by this statement. Obviously, any operational theory, whether or not it is no-signalling, can be given a non-local interpretation (more on this in Section 3.7). The interesting question is whether, given a no-signalling operational theory, we can *construct* a corresponding local-realistic theory that makes the same operational predictions. We shall prove the affirmative for a wide class of no-signalling theories, including unitary quantum theory. But first, we introduce the explicit requirements that define a structure of no-signalling operational theory.

Definition 3.5.1 (No-signalling operational theory)**.** A no-signalling operational theory is composed of a lattice of systems $(\mathscr{S}, \sqcup, \sqcap, \bar{\cdot}, S, \emptyset)$ such that, associated to each system within the lattice, there is

1. a phenomenal state space;
2. a set of operations;
3. a faithful operation action of the operations on the phenomenal state space;
4. and a phenomenal projector onto the system.

In addition, to each disjoint pair of systems, there is

5. a product of operations.

The first four of these mathematical objects are defined exactly as in the case of local realistic theories (Section 3.4). The *product of operations*, which is very different, is formally defined below.

The faithfulness of the phenomenal action is not fundamental because any phenomenal action can be made faithful by replacing operations by equivalence classes of operations, in effect equating any two operations that act identically on all possible phenomenal states, as we shall do for instance in the specific case of quantum theory in Section 3.5.1. However, it is algebraically very useful and natural to impose phenomenal faithfulness from the outset. The reason why we had not required fidelity of the phenomenal action after Definition 3.3.2 is that such faithfulness could be incompatible with the underlying noumenal world, even if the latter exhibits a faithful action. But here, only the phenomenal world is given and we are free to build our own noumenal world to explain it. This gives us latitude to make the phenomenal action faithful if needed, before we proceed to building the noumenal world, whose action will then be automatically faithful by virtue of Theorem 3.3.1.

A no-signalling operational theory differs from a local-realistic theory in the fact that it does not come with a noumenal state space. Therefore, there are no universal noumenal-phenomenal epimorphism, noumenal projectors, nor join product of noumenal states. The latter is the essential missing ingredient in a local-realistic operational theory: there is no phenomenal counterpart for the noumenal join product, which was as the heart of local-realistic theories. Phenomenal states ρ^A and ρ^B of disjoint systems A and B could admit several distinct phenomenal states ρ^{AB} such that $\rho^A = \pi_A\big(\rho^{AB}\big)$ and $\rho^B = \pi_B\big(\rho^{AB}\big)$. In operational quantum theory, as we shall see in Section 3.5.1, the usual density matrices play the role of phenomenal states, and indeed it is generally not possible to recover ρ^{AB} from ρ^A and ρ^B when systems A and B are entangled. This is why it is usually, *albeit wrongly*, asserted that quantum theory is a nonlocal theory since, *at the phenomenal level*, the state of the whole is not determined by the state of its parts.

Product of operations. In local-realistic theories, the product of operations was completely determined at the noumenal level by Eq (3.1) in Section 3.4.4, which depended crucially on the existence of the join product, a notion that does not exist at the phenomenal level. Nevertheless, this induced a phenomenal meaning to the product of operations through the noumenal-phenomenal epimorphism. In sharp contrast, the product of operations is a primitive notion in no-signalling operational theories, which we now proceed to characterize. For all disjoint systems A and B, we require the existence of a function denoted "$\times$",[12] the *product of operations*. Given operations U and V on disjoint systems A and B, we denote by $U \times V$ an operation on system AB that satisfies the following conditions.

1. *No-signalling principle*. Given any operations U and V on disjoint systems A and B, respectively, and given any phenomenal state ρ^{AB} of joint system AB, we require that
$$\pi_A\Big((U \times V)\Big(\rho^{AB}\Big)\Big) = U\Big(\pi_A\Big(\rho^{AB}\Big)\Big)\,.$$

[12] Technically, we should denote this product of operations as $\times_{(A,B)}$ but once again we shall consider the dependence in A and B to be implicit.

2. *Associativity*. Given any operations U, V and W on mutually disjoint systems, we require that

$$U \times (V \times W) = (U \times V) \times W \,.$$

Since there is no ambiguity, we shall omit the parentheses and simply write $U \times V \times W$.

3. Given any operations U_1, U_2 on system A and V_1, V_2 on disjoint system B,

$$(U_2 \times V_2)\,(U_1 \times V_1) = U_2 U_1 \times V_2 V_1 \,.$$

This means that if we first do jointly operation U_1 on A and V_1 on B, and then we do jointly operation U_2 on A and V_2 on B, then this is equivalent to having done jointly the operation that consists of doing U_1 followed by U_2 on A and the operation that consists of doing V_1 followed by V_2 on B.

4. Given any two disjoint systems A and B,

$$I^A \times I^B = I^{AB} \,.$$

This means that if we do nothing on system A and nothing on system B, then we have done nothing on joint system AB.

5. Given operations U and V on disjoint systems A and B, respectively,

$$U^A \times V^B = V^B \times U^A \,.$$

To be technically more precise, $U^A \times_{(A,B)} V^B = V^B \times_{(B,A)} U^A$, according to footnote 12.

6. The last requirement is more technical, but nevertheless necessary. Consider three mutually disjoint systems A, B and C, and operations U^{BC} and V^{AC} on joint systems BC and AC, respectively, such that $I^A \times U^{BC} = I^B \times V^{AC}$, then there exists an

operation W^C acting on system C alone such that

$$I^A \times U^{BC} = I^B \times V^{AC} = I^{AB} \times W^C \,.$$

Intuitively, this says that if nothing is done to system A and nothing to system B, then nothing is done to system AB.

The first five of the above requirements did not have to be imposed on the product of operations when we considered local-realistic structures in Sections 3.4.4 and 3.4.5 because they were consequences of the join product and of Eq (3.1). Specifically, these five requirements correspond to Theorems 3.4.19, 3.4.14, 3.4.11, 3.4.12 and 3.4.13, respectively.

3.5.1 Unitary quantum theory is a no-signalling operational theory

Finite dimensional unitary quantum theory is a model of a no-signalling operational theory. To see this, we must define the various components of a no-signalling operational theory, as specified in Definition 3.5.1, in quantum-mechanical terms. The obvious approach outlined below does not *quite* work but it helps in order to gain intuition.

1. The phenomenal state of a quantum system is its density matrix.
2. The operations acting on those states are unitary transformation of the appropriate dimension.
3. Operation U acts on phenomenal state ρ by producing $U\rho U^\dagger$.
4. The phenomenal projector π_A on system A is the usual tracing out of the rest of the universe (see Section 3.4.3.5).
5. The product of operations is the usual tensor product of unitary transformations.

It is elementary to verify that the first five requirements for the product of operations are satisfied. The last is slightly technical and is left as an exercise for the reader.

The only problem is that this operation action is not faithful. Indeed, consider any unitary operation U and complex number η of unit norm, then if we define $V = \eta U$, it is well-known that $U\rho U^\dagger = V\rho V^\dagger$ for any density matrix ρ of matching dimension, even though *algebraically speaking*, $U \neq V$ whenever $\eta \neq 1$. In order to make the operation action faithful, we need to equate those two operations, and more generally to equate any two operations that differ only by a multiplicative complex constant factor of unit norm, known in usual quantum theory as an *irrelevant phase factor*. The clean mathematical way to do this is to define equivalence relation $\sim$ by $U \sim V$ if and only if $V = \eta U$ for some complex number η of unit norm. Then, the operations in the operational theory are no longer unitary transformations but *equivalence classes* of unitary transformations, where class $[U]$ is defined as $\{\eta U : \eta \in \mathbb{C} \text{ and } |\eta| = 1\}$ and operation $[U]$ acts on noumenal state ρ by producing $U\rho U^\dagger$. This is well defined because if $[U] = [V]$ then $U\rho U^\dagger = V\rho V^\dagger$ by definition of the equivalence classes. We also leave as an exercise for the reader to show that whenever $[U] \neq [V]$, there exists a density matrix ρ such that $U\rho U^\dagger \neq V\rho V^\dagger$, and therefore our new phenomenal action is indeed faithful. Finally, instead of taking the tensor product on unitary operations, we must take it on classes of operations. Consequently, we define the product of operations $\times$ by: $[U] \times [V] \stackrel{\text{def}}{=} [U \otimes V]$. This product is well defined and has the required properties.

3.6 From no-signalling to local realism

As stated at the beginning of Section 3.5, our main objective is to give a local-realistic interpretation to the broadest possible class of no-signalling operational theories, including unitary quantum theory. For this purpose, we need to start from the description of a no-signalling operational theory and construct a local-realistic model that gives rise to the same phenomenal behaviour. Specifically, we are given a lattice of systems. For each system in the lattice, we are given a phenomenal state space, a set of operations, and a faithful action of the operations on the phenomenal state space. We are also given a phenomenal projector π_A^B for each pair of systems $A \sqsubseteq B$. Furthermore, for each pair of disjoint systems A and B, we are given a product of operations $\times_{(A,B)}$ that respects the six requirements for the product of operations in a no-signalling operational theory. Our aim is to construct a noumenal state space and a faithful action of the operations on the noumenal state space for each system, a noumenal projector π_A^B for each pair of systems $A \sqsubseteq B$, a universal noumenal-phenomenal epimorphism φ from our noumenal state spaces onto the imposed phenomenal state spaces, and a join product. Note that we must keep the original sets of operations, from which we must define our actions on the noumenal state spaces.

We need our construction to recover the original operational theory in the sense that each product of operations induced by our join product by virtue of Eq (3.1) in Section 3.4.4 must be precisely the corresponding product of operations that was prescribed in the given no-signalling operational theory. This is Theorem 3.6.9 below. We must also make sure that φ be indeed an epimorphism: it must be surjective and for all operations U and all noumenal states N on which U can act, we must have

$$\varphi(U(N)) = U(\varphi(N)) .$$

Furthermore, for all systems A and B such that A is a subsystem of B and all noumenal states N^B, we must have

$$\pi_A\big(\varphi\big(N^B\big)\big) = \varphi\big(\pi_A\big(N^B\big)\big) .$$

Our main result is that we can achieve this goal, but with a caveat. We need to require that all operations be reversible: the set of operations on any given system must be a group. It might be possible to achieve the same goal without a group structure, which is the subject of current research, but this would most likely come at the cost of significant loss in mathematical elegance. We now proceed with our construction of a local-realistic model for any reversible no-signalling operational theory.

Let us be given an operational no-signalling theory, which consists of a lattice of systems $(\mathscr{S}, \sqcup, \sqcap, \bar{\cdot}, S, \emptyset)$ so that, associated to each system, the operations are reversible. Recall that for any system A, the set of operations that acts on A is denoted $\mathsf{Operations}^A$. Recall also that the global system is denoted S, so that $A \sqsubseteq S$ for any system $A \in \mathscr{S}$.

Definition 3.6.1 (Fundamental equivalence relation)**.** For each system A, we define equivalence relation "$\sim_A$" on $\mathsf{Operations}^S$ as follows.

$$W \sim_A W' \overset{\text{def}}{\iff} \exists V \in \mathsf{Operations}^{\overline{A}} \text{ such that } W = \left(I^A \times V\right) W'$$

Intuitively, $W \sim_A W'$ for operations W and W' that act on the global system when their action on system A is phenomenally indistinguishable.

Theorem 3.6.1. $\sim_A$ is an equivalence relation on $\mathsf{Operations}^S$.

The proof of this theorem, albeit easy, is crucial as it illustrates precisely where each requirement of the product of operations is used. In particular, the proof that this relation is symmetric is one of only two places in which the validity of our construction hinges upon the assumption that the given no-signalling operational theory is reversible, the other being to prove that the join product is well-defined in Theorem 3.6.8 below.

Proof. We need to show that $\sim_A$ is reflexive, symmetric and transitive.

$\sim_A$ **is reflexive:** For all W in $\mathsf{Operations}^S$,

$$\begin{aligned} W &= I^S W \\ &= \left(I^A \times I^{\overline{A}}\right) W \,. \end{aligned}$$

Thus $W \sim_A W$.

$\sim_A$ **is symmetric:** Suppose $W \sim_A W'$. By definition, there exists $V \in \mathsf{Operations}^{\overline{A}}$ such that: $W = \left(I^A \times V\right) W'$. Therefore,

$$\begin{aligned} W' &= I^S W' \\ &= \left(I^A \times I^{\overline{A}}\right) W' \\ &= \left(I^A I^A \times V^{-1} V\right) W' \\ &= \left(I^A \times V^{-1}\right) \left(I^A \times V\right) W' \\ &= \left(I^A \times V^{-1}\right) W . \end{aligned}$$

Since $W' = \left(I^A \times V^{-1}\right) W$ and $V^{-1} \in \mathsf{Operations}^{\overline{A}}$, we have $W' \sim_A W$.

$\sim_A$ **is transitive:** Suppose that $W \sim_A W'$ and $W' \sim_A W''$. By definition, there exist $V, V' \in \mathsf{Operations}^{\overline{A}}$ such that $W = \left(I^A \times V\right) W'$ and $W' = \left(I^A \times V'\right) W''$. Therefore,

$$\begin{aligned} W &= \left(I^A \times V\right) W' \\ &= \left(I^A \times V\right) \left(I^A \times V'\right) W'' \\ &= \left(I^A \times V V'\right) W'' . \end{aligned}$$

Since $W = \left(I^A \times V V'\right) W''$ and $V V' \in \mathsf{Operations}^{\overline{A}}$, we have $W \sim_A W''$. □

Any equivalence relation gives rise to equivalence classes. For any $W \in \mathsf{Operations}^S$, we define the *class of* W with respect to A to be

$$[W]^A \stackrel{\text{def}}{=} \left\{ W' \in \mathsf{Operations}^S \colon W' \sim_A W \right\} .$$

Noumenal states. Let A be a system. The noumenal space for system A is defined as

$$\mathsf{Noumenal\text{-}Space}^A \stackrel{\text{def}}{=} \left\{ [W]^A : W \in \mathsf{Operations}^S \right\} .$$

Noumenal projectors. Let A be a subsystem of B. The noumenal projector of a noumenal state $[W]^B$ onto system A is defined by

$$\pi_A\left([W]^B\right) \stackrel{\text{def}}{=} [W]^A .$$

For such a definition to make sense, we need to verify that it does not depend on the choice of representative for the equivalence class. The following theorem establishes that our noumenal projectors are well defined.

Theorem 3.6.2. Let A be a subsystem of B. For any $W, W' \in \mathsf{Operations}^S$, we have $W' \sim_B W \implies W' \sim_A W$.

Proof. By definition of $\sim_B$, there exists a $V \in \mathsf{Operations}^{\overline{B}}$ such that

$$\begin{aligned} W' &= \left(I^B \times V\right) W \\ &= \left(I^A \times \left(I^{B \sqcap \overline{A}} \times V\right)\right) W . \end{aligned}$$

□

Furthermore, our noumenal projectors verify the other requirements.

Theorem 3.6.3. If A is a subsystem of B, then $[W]^A$ is the projection of a noumenal state of the system B, namely $[W]^B$.

Proof. This follows directly from the definition of π_A. □

Theorem 3.6.4. If A is a subsystem of B and B is subsystem of C, we have

$$\pi_A\left([W]^C\right) = (\pi_A \circ \pi_B)\left([W]^C\right) .$$

Proof.

$$\begin{aligned}\pi_A\left([W]^C\right) &= [W]^A\\ &= \pi_A\left([W]^B\right)\\ &= \pi_A\left(\pi_B[W]^C\right)\\ &= (\pi_A \circ \pi_B)\left([W]^C\right)\end{aligned}$$

□

Noumenal action. Let A be a system and let U be an operation that acts on system A. We define the noumenal action of operation U on system A by

$$U\left([W]^A\right) \stackrel{\text{def}}{=} \left[\left(U \times I^{\overline{A}}\right) W\right]^A .$$

Again, for such a definition to make sense, we need to verify that it does not depend on the choice of representative for the equivalence class. The following theorem proves that our noumenal actions are well defined.

Theorem 3.6.5. For any system A, operation U acting on A, and for any $W, W' \in \mathsf{Operations}^S$, we have $W' \sim_A W \implies \left(U \times I^{\overline{A}}\right) W' \sim_A \left(U \times I^{\overline{A}}\right) W$.

Proof. By definition of $\sim_A$, there exists a $V \in \mathsf{Operations}^{\overline{A}}$ such that $W' = \left(I^A \times V\right) W$. Therefore,

$$\begin{aligned}&\left(U \times I^{\overline{A}}\right) W'\\ =\ &\left(U \times I^{\overline{A}}\right)\left(I^A \times V\right) W\\ =\ &(U \times V)\, W\\ =\ &\left(I^A \times V\right)\left(U \times I^{\overline{A}}\right) W .\end{aligned}$$

□

It remains to prove that this defines a proper operation action at the noumenal level. This is the purpose of the following two theorems.

Theorem 3.6.6. For all operations U and V on system A

$$(VU)\left([W]^A\right) = V\left(U\left([W]^A\right)\right)$$

Proof.

$$\begin{aligned}
&(VU)\left([W]^A\right)\\
=\ &\left[\left((VU)\times I^{\overline{A}}\right)W\right]^A\\
=\ &\left[\left(V\times I^{\overline{A}}\right)\left(U\times I^{\overline{A}}\right)W\right]^A\\
=\ &V\left(\left[\left(U\times I^{\overline{A}}\right)W\right]^A\right)\\
=\ &V\left(U\left([W]^A\right)\right)
\end{aligned}$$

□

Theorem 3.6.7.

$$I^A\left([W]^A\right) = [W]^A$$

Proof.

$$\begin{aligned}
&I^A\left([W]^A\right)\\
=\ &\left[\left(I^A\times I^{\overline{A}}\right)W\right]^A\\
=\ &\left[I^S W\right]^A\\
=\ &[W]^A
\end{aligned}$$

□

Noumenal join product. We are now ready to define the join product in our local-realistic model. First note that two noumenal states of disjoint systems A and B are compatible if and only if there exists some $W \in \mathsf{Operations}^S$ such that these states are $[W]^A$ and $[W]^B$.

Definition 3.6.2 (Join product)**.** Let $[W]^A$ and $[W]^B$ be noumenal states for disjoint systems A and B. Their join product is defined as follows.

$$[W]^A \odot [W]^B \stackrel{\text{def}}{=} [W]^{AB}$$

Once again, for such a definition to make sense, we need to verify that it does not depend on the choice of representatives for the equivalence classes. The following theorem establishes that this is the case.

Theorem 3.6.8. For any operations $W, W' \in \mathsf{Operations}^S$, if $W \sim_A W'$ and $W \sim_B W'$, then $W \sim_{AB} W'$.

Proof. Let W' be such that $W \sim_A W'$ and $W \sim_B W'$, and let $C = \overline{AB}$. This means that there exist V^{BC} and V^{AC} such that $W = \left(I^A \times V^{BC}\right) W'$ and $W = \left(I^B \times V^{AC}\right) W'$. Multiplying by the inverse of W', it follows that $I^A \times V^{BC} = I^B \times V^{AC}$. The last requirement that defines the phenomenal product of operations imposes the existence of an operation V^C such that $W = \left(I^A \times I^B \times V^C\right) W'$. Hence we have $W = \left(I^{AB} \times V^C\right) W'$, and therefore $W \sim_{AB} W'$. □

Finally, we prove that the product of operations prescribed in the given no-signalling operational theory satisfies the requirement given by Eq (3.1) to be the product of operations in the constructed local-realistic structure. This is established in Theorem 3.6.9 below, but first we need to prove that noumenal states evolve locally, in the sense that the evolution of a noumenal state does not depend on how the rest of the universe evolves.

Lemma 3.6.1. For any system A, operation U acting on A, operation V acting on $\overline{A}$, and operation W acting on S,

$$U\left([W]^A\right) = [(U \times V)\, W]^A\,.$$

Proof.

$$
\begin{aligned}
U\left([W]^A\right) &= U\left(\left[\left(I^A \times V\right)W\right]^A\right) \\
&= \left[\left(U \times I^{\overline{A}}\right)\left(I^A \times V\right)W\right]^A \\
&= [(U \times V)\,W]^A
\end{aligned}
$$

□

Theorem 3.6.9. Let $[W]^A$ and $[W]^B$ be noumenal states for disjoint systems A and B, and let U and V be operations that can act on these systems, respectively.

$$
(U \times V)\left([W]^A \odot [W]^B\right) = U\left([W]^A\right) \odot V\left([W]^B\right)
$$

Proof.

$$
\begin{aligned}
&(U \times V)\left([W]^A \odot [W]^B\right) \\
=&(U \times V)\left([W]^{AB}\right) \\
=&\left[\left(U \times V \times I^{\overline{AB}}\right)W\right]^{AB} \\
=&\left[\left(U \times V \times I^{\overline{AB}}\right)W\right]^{A} \odot \left[\left(U \times V \times I^{\overline{AB}}\right)W\right]^{B} \\
=&\,U\left([W]^A\right) \odot V\left([W]^B\right)
\end{aligned}
$$

□

The noumenal-phenomenal epimorphism.

Instead of a universal noumenal-phenomenal epimorphism, we shall construct a family of universal noumenal-phenomenal homomorphisms, which is sufficient according to the technique developed in Section 3.4.6. For each phenomenal state $\rho \in \textsf{Phenomenal-Space}^S$ of the global system, we define a universal homomorphism ϕ_ρ as follows:

$$
\phi_\rho\left([W]^A\right) \stackrel{\text{def}}{=} \pi_A(W(\rho))
$$

for any system A and noumenal state $[W]^A$.

The following theorem establishes that this definition does not depend on the choice of representative for equivalence class $[W]^A$.

Theorem 3.6.10. $W' \sim_A W \implies \pi_A(W'(\rho)) = \pi_A(W(\rho))$.

Proof. By definition of $\sim_A$, let $V \in \mathsf{Operations}^{\overline{A}}$ be such that $W' = \left(I^A \times V\right) W$.

$$\begin{aligned}
&\pi_A\left(W'(\rho)\right) \\
&= \pi_A\left(\left(\left(I^A \times V\right)(W)\right)(\rho)\right) \\
&= I^A(\pi_A(W(\rho))) \quad \text{This is where we make use of the no-signalling principle} \\
&= \pi_A(W(\rho))
\end{aligned}$$

□

Theorem 3.6.11. For each $\rho \in \mathsf{Phenomenal\text{-}Space}^S$, function ϕ_ρ is a homomorphism:

$$\begin{aligned}
U\left(\phi_\rho\left([W]^A\right)\right) &= \phi_\rho\left(U\left([W]^A\right)\right) \\
\text{and } \pi_B\left(\phi_\rho\left([W]^A\right)\right) &= \phi_\rho\left(\pi_B\left([W]^A\right)\right)
\end{aligned}$$

for any systems A and B such that $B \sqsubseteq A$, noumenal state $[W]^A$ and operation $U \in \mathsf{Operations}^A$.

Proof.

$$\begin{aligned}
&U\left(\phi_\rho\left([W]^A\right)\right) \\
&= U(\pi_A(W(\rho))) \\
&= \pi_A\left(\left(U \times I^{\overline{A}}\right)(W(\rho))\right) \\
&= \pi_A\left(\left(\left(U \times I^{\overline{A}}\right) W\right)(\rho)\right) \\
&= \phi_\rho\left(\left[\left(U \times I^{\overline{A}}\right) W\right]^A\right) \\
&= \phi_\rho\left(U\left([W]^A\right)\right)
\end{aligned}$$

and

$$
\begin{aligned}
&\pi_B\left(\phi_\rho\left([W]^A\right)\right) \\
&= \pi_B(\pi_A\left(W(\rho)\right)) \\
&= \left(\pi_B \circ \pi_A\right)\left(W(\rho)\right) \\
&= \pi_B(W(\rho)) \\
&= \phi_\rho\left([W]^B\right) \\
&= \phi_\rho\left(\pi_B\left([W]^A\right)\right)
\end{aligned}
$$

□

To apply the technique of Section 3.4.6, it remains to verify that for each phenomenal state ρ of the global system, there exists at least one universal noumenal-phenomenal homomorphism ϕ_i and a noumenal state N of the global system such that $\rho = \phi_i(N)$. This is achieved by the following theorem, with the appropriate choice of N and i.

Theorem 3.6.12. Consider any phenomenal state ρ of the global system.

$$\rho = \phi_\rho\left([I]^S\right)$$

Proof.

$$\phi_\rho\left([I]^S\right) = \pi_S(I(\rho)) = \pi_S(\rho) = \rho$$

□

All the conditions required in Section 3.4.6 being satisfied, the existence of the required universal noumenal-phenomenal epimorphism is established, which concludes our construction of a local-realistic noumenal model that corresponds to any reversible no-signalling operational theory.

3.7 Conclusions and open problems

The question of whether or not quantum theory has a local-realistic interpretation should not be answered merely by providing a local-realistic formalism for it. Indeed, such an answer, while mathematically valid, does not answer the deeper question: "But why does quantum theory have a local-realistic interpretation?". A metaphysical question deserves an answer based on metaphysical principles rather than by the power of mathematics alone. So, why does quantum theory have a local-realistic interpretation? Our answer is that deep down, it stems from the fact that it is a theory that follows the no-signalling principle.

It is obvious that any local-realistic theory can be given a *non*-local interpretation, simply by adding extraneous noumenal invisible entities that "talk" to each other instantaneously across space just for the fun of it, without having any phenomenal effect whatsoever. As pointed out to us by David Deutsch [15], it is not meaningful to claim that a theory is nonlocal simply because it has some nonlocal interpretation. Otherwise, all theories would be nonlocal! It follows that we may reasonably claim that any physical theory that *can* be given a local-realistic interpretation *is* in fact a local theory. To be truly considered nonlocal, a theory must have no possible local interpretation. We illustrate this concept with the noncontroversial fact that any graph that *can* be drawn without crossing edges *is* a planar graph regardless of how it's presented at first, and that in order to be declared nonplanar, a graph must be so that it cannot be drawn without crossing edges in a plane. Seen this way, our main theorem is that all no-signalling operational theories with a reversible dynamics, including finite-dimensional unitary quantum theory, *are* local-realistic, not merely that they can be given a local-realistic interpretation.

It is not possible to prove formally that we have captured correctly the intuitive notions of local realism and no-signalling theories. This would be like attempting to prove the Church-Turing thesis! One cardinal reason for this impossibility is that any attempt to bridge the intuitive with the formal necessarily carries aspects that are intuitive and informal. The informal parts were written in English, and therefore are prone to the ambiguity of all natural languages. The parts that appeal to the intuition contain considerations that perhaps cannot even be put into words. To establish our thesis, we sought a collection of requirements, which would be called *axioms* in the context of mathematical theories. From our perspective, each of them appeared necessary to describe the notions of local realism and no-signalling theories. Each requirement was chosen for its intuitiveness, whose clear meaning is easily apprehended from its description using simple words in a natural language. We also strived to have requirements that are mathematically natural. For example, we chose to impose the faithfulness of noumenal actions to simplify subsequent mathematics, as explained in Section 3.3. The success of our enterprise is conditioned not only on whether or not we have captured the intuitive notions correctly, but also on whether or not the mathematical structure we have developed reveals how and why *our* world is local realistic after all.

There are other topics, which we consider important, yet have not been addressed in this article because they belong to companion papers. Specifically, why are local-hidden variable theories not general enough to cover all local-realistic theories? How can a theory violate a Bell inequality and still be local-realistic? How should we address the Einstein-Podolsky-Rosen paradox? We answer these questions through the construction of a local-realistic Popescu-Rohrlich box in Ref. [6]. Why is the existence of a larger and richer noumenal world inevitable in a local quantum theory? How can we reconcile locality with Bell measurements? Or a teleportation experiment? We answer these questions in Ref. [7], which also provides a more concrete local-realistic model for quantum theory, built on linear algebra notions.

3.7.1 Further work

Our main result depended on one hypothesis that is conceptually essential neither to describe a local-realistic nor a no-signalling theory, namely the assumption of a reversible dynamics. Can this requirement be lifted? In other words, do all no-signalling theories still have an underlying local-realistic interpretation without this extraneous assumption? And even more importantly, can such a construction be achieved without the sacrifice of elegance? Will such a proof give rise to the local-realistic model derived in this paper whenever the dynamics is reversible? It is tempting to conjecture that it does, since one might think that perhaps every system could carry all information of all interactions it had with all systems in the past. However, this argument is fallacious because systems are not allowed in our framework to do such a thing. For example, if we do U followed by U^{-1} on a system, it has the same effect as simply doing I, on both the noumenal and phenomenal states of the system. It is simply impossible to distinguish between these two histories that a system might have experienced. Should it be the case that the hypothesis of reversible dynamics cannot be removed, there would be two tasks to undertake. The first would be to provide an example of a no-signalling theory without reversible dynamics that cannot have a local-realistic interpretation. The second task would be to find a minimal condition on no-signalling theories, such that this condition is essential for the theory to have an underlying local-realistic interpretation. We know that reversibility is not necessary in all cases since the PR-box has irreversible dynamics and yet it has a local-realistic explanation nevertheless [6].

We have used the assumption that systems form a boolean lattice as this allowed us to introduce various notions, such as subsystems and complementary systems. However, this framework is perhaps too restrictive. Can we weaken this assumption? For example, is it absolutely necessary that systems be closed under union or intersection?

We have shown that finite-dimensional unitary quantum theory is a model of a reversible no-signalling operational theory, which directly implies a local-realistic interpretation. What about countably infinite unitary quantum theory? Or continuous unitary quantum theory? Are these no-signalling operational theories as well? In the case of countably infinite unitary quantum theory, we are quite confident that they are. Yet, to give a hint of the potential difficulties, in any finite-dimensional vector space V, for all linear operators $A, B : V \to V$, we have that

$$AB = I \iff BA = I.$$

However, such a statement is false in infinite-dimensional vector spaces. This illustrates the necessity to verify that every step of the proofs we have used apply for infinite-dimensional spaces, and if not, how to modify the results (or the proofs) accordingly.

On a similar topic, we have developed an explicit local-realistic framework for quantum theory in a companion paper [7], which is built around matrices and Hilbert spaces. It remains to determine if this approach works for discrete infinite-dimensional quantum theory. A more difficult research direction concerns the development of a local-realistic framework for continuous degrees of freedom, such as the position and momentum of particles. This framework could be used to derive the time evolution of noumenal states based on the representations of space-time symmetry groups. The non-relativistic and relativistic symmetry groups should be independently studied. We expect that the image of the noumenal time evolution by the noumenal-phenomenal epimorphism will lead to Schrödinger's equation in the non-relativistic case and to Dirac's equation for spin-$\frac{1}{2}$ relativistic particles. Charles Alexandre Bédard is currently working on these issues [1].

Even though we have argued that the conventional no-signalling principle formulated in terms of probabilities is subsumed by the more general principle we have developed for operational no-signalling theories in Section 3.5, we have not carried out the full mathematical details. For instance, we should give a formal proof that no-signalling theories with probabilities satisfy our axioms of no-signalling operational theories. Furthermore, in the event that we cannot find a general proof that all irreversible non-signalling operational theories have a local-realistic interpretation, or if we can actually prove that some don't, we conjecture that all theories satisfying the *probabilistic* no-signalling principle have a local-realistic interpretation. A proof of this conjecture is currently within grasp, but details remain to be fleshed out.

Additionally, we intend to study limits of sequences of operations, continuity of operations and topological properties of states. The framework we have developed in this paper is inspired by universal algebra and structure theory. However, algebra does not capture topological properties like continuity. In mathematics, there exist axiomatic systems having both algebraic and topological properties, such as topological groups and topological fields, on which there are algebraic requirements and topological requirements that a topological structure must satisfy, in such a way that the algebraic and topological requirements are consistent with each other. For example, the axioms of a topological group imply that the group operation is a continuous operation. An important question is to determine the topological requirements that should be expected in a local-realistic structure if we want the operations and states to have both topological and algebraic properties. Such a framework might be necessary for continuous quantum theory or for the investigation of topological local-realistic structures.

In recent years, category theory has been used to describe frameworks that include theories more general than the quantum. For example, category theory has been used successfully to derive quantum theory from axioms built on ideas of operations, information and probabilities [11]. We think that many of the ideas we have expressed here would benefit from being written categorically.

Given that our framework of local realism is extremely general, we wonder if there are interesting examples of local-realistic models beyond quantum theory or the Popescu-Rohrlich nonlocal box, or more generally probabilistic no-signalling theories? An interesting candidate would be "Almost Quantum Theory" [34]. Furthermore, David Deutsch has suggested that qubit field theory [14], which is based on non-commutative observables, would be a purely local theory based on observables that nevertheless need not commute at equal times [15].

Any time a system of axioms is developed, many questions arise. The foundational questions include whether the axioms are independent from each others, or whether there are alternative axioms giving rise to the same structure. Another set of important questions is to determine the interesting theorems of local-realistic and no-signalling theories. Each of these theorems would give us a consequence of living in a local-realistic world or in a no-signalling world.

John Archibald Wheeler famously said “it from bit” [37]. By coining this phrase, he suggested that every physical quantity is postulated to explain our observations. As realists, we postulate an external world to explain our observations. This is also why we postulated the noumenal world: the goal is to understand the phenomenal world. Wheeler argued that “the supreme goal” is to “*deduce the quantum* from an understanding of *existence*” [37]. The question of deriving the quantum from existence has been largely abandoned in recent years in favour of models that are not realistic, and do not attempt to derive quantum theory from an independent objective world. How can we derive the quantum from existence? A path towards the answer might be to find additional metaphysical principles that go on top of a local-realistic structure and end up deriving the quantum. One such principle is the reversibility of operations. But what else? At this point, we do not know, but we would like to launch the question, and the quest for finding metaphysical principles that lead from local-realistic reversible-dynamics theories to quantum theory.

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CHAPITRE 4

A LOCAL-REALISTIC MODEL FOR QUANTUM THEORY

By Gilles Brassard and Paul Raymond-Robichaud

ABSTRACT

We show that the standard quantum mechanical description of physical reality cannot be considered complete. We complete quantum mechanics by presenting a local-realistic model for finite-dimensional unitary quantum theory.

4.1 Introduction

This article continues right where we left in the previous article. We first show that the universal wavefunction cannot be a complete description of local reality. We then provide a local-realistic formalism for finite dimensional unitary quantum theory.

4.2 The wavefunction cannot be a complete description of local reality

Could the state of the universe be described by the Everettian universal wavefunction, which would be a unit vector in the system space of the universe, evolving according to unitary evolution?

Certainly, the universal wavefunction describes everything that is observable in the universe. As such, it is at the very least a description of the phenomenal state of the universe. Should the universal wavefunction correspond also to the noumenal state of the universe, it would be in bijective correspondence with its phenomenal state, and as such the operations would act in the same way.

We shall do a simple proof by contradiction that the universal wavefunction cannot be a description of local reality. Let us consider a very simple universe, one composed of only two qubits. The proof would apply to any universe with at least two qubits.

For that purpose, suppose are given a local-realistic model of this universe. Among other things, such a model contains noumenal traces, which might differ from their phenomenal counterparts, and the model would also contain a noumenal join product.

According to our requirement, all noumenal states are separable. In particular the Bell State $|\Psi^+\rangle = \frac{1}{\sqrt{2}}\big(|01\rangle + |10\rangle\big)$ can be written as a product state:

$$|\Psi^+\rangle = [\Psi^+]^A \odot [\Psi^+]^B .$$

where $[\Psi^+]^A = \mathrm{tr}_A\left(|\Psi^+\rangle\right)$ and $[\Psi^+]^B = \mathrm{tr}_B\left(|\Psi^+\rangle\right)$. Here tr_A and tr_B are noumenal traces.

If we apply the negation gate, $N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, on both parts of the state, we get

$$(N \otimes N)\,|\Psi^+\rangle = N[\Psi^+]^A \odot N[\Psi^+]^B .$$

But since

$$\left|\Psi^+\right\rangle = (N \otimes N)\,|\Psi^+\rangle ,$$

by tracing out B, we conclude

$$[\Psi^+]^A = N[\Psi^+]^A .$$

However,

$$\left|\Phi^+\right\rangle = (N \otimes I)\left|\Psi^+\right\rangle = N[\Psi^+]^A \odot [\Psi^+]^B = [\Psi^+]^A \odot [\Psi^+]^B = \left|\Psi^+\right\rangle ,$$

where $|\Phi^+\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big) \neq |\Psi^+\rangle$. Thus, we have reached a contradiction! The conclusion is that we cannot describe a simple Bell state as a product state, even less the entire universe, by taking the usual formulation of quantum physics.

It follows that the universal wavefunction cannot be the complete description of a local universe. It merely describes what can be observed, which is the phenomenal state of the universe. Seen this way, if we believe in a local universe, the answer to the question posed in the title of the EPR 1935 paper is: "NO, [the standard] quantum-mechanical description of physical reality CANNOT be considered complete". In other words, the universal wavefunction is but a *shadow* of the real world, which can only be fully described at its noumenal level!

4.3 A local-realistic model for quantum theory

Now that we have established the mathematical requirements for quantum theory to have a local-realistic interpretation, we shall provide a mathematical model meeting all the requirements.

We shall be interested in working within a global system S, which can be composed of finitely many elementary subsystems. If A is a system, $\overline{A}$ denotes its complement, which is the disjoint system such that $A\overline{A} = S$.

4.3.1 Systems

Associated to any system A is a Hilbert Space of some dimension n.

4.3.2 Noumenal states

For a system A associated with a Hilbert Space of dimension n, its noumenal state N^A is formally defined by an *evolution matrix* $[W]^A$, which is an $n \times n$ matrix whose entry

i, j, in the basis $\left\{|i\rangle^A\right\}_{i=1}^n$ is the matrix

$$[W]_{ij}^A \stackrel{\text{def}}{=} W^\dagger \left(|j\rangle\langle i| \otimes I^{\overline{A}}\right) W$$

for some unitary W on the global state. Here $I^{\overline{A}}$ refer to the identity unitary operation applied to the rest of the global system.

Theorem 4.3.1.

$$\left[W\right]^A = \left[\left(I^A \otimes V\right) W\right]^A$$

for any unitary V acting on $\overline{A}$

Proof.

$$\begin{aligned}
\left[W\right]_{ij}^A &= W^\dagger \left(|j\rangle\langle i| \otimes I^{\overline{A}}\right) W \\
&= W^\dagger \left(|j\rangle\langle i| \otimes \left(V^\dagger I^{\overline{A}} V\right) \right) W \\
&= W^\dagger \left(I^A \otimes V^\dagger\right) \left(|j\rangle\langle i| \otimes I^{\overline{A}}\right) \left(I^A \otimes V\right) W \\
&= \left(\left(I^A \otimes V\right) W\right)^\dagger \left(|j\rangle\langle i| \otimes I^{\overline{A}}\right) \left(\left(I^A \otimes V\right) W\right) \\
&= \left[\left(I^A \otimes V\right) W\right]_{ij}^A
\end{aligned} \tag{4.1}$$

□

Definition 4.3.1 (Evolution)**.** If we have a system A in state $\left[W\right]^A$ on which we apply a unitary operation U, the system evolves to $U\left(\left[W\right]^A\right)$, which for simplicity we denote $U\left[W\right]^A$, defined as

$$\left(U\left[W\right]^A\right)_{ij} \stackrel{\text{def}}{=} \sum_{m,n} U_{im} \left[W\right]_{mn}^A U_{nj}^\dagger$$

Theorem 4.3.2.

$$U\left[W\right]^A = \left[\left(U \otimes V\right) W\right]^A$$

for any unitary V acting on $\overline{A}$.

Proof.

$$\begin{aligned}
\left(U\left[W\right]^{A}\right)_{ij} &= \sum_{m,n} U_{im}\left[W\right]^{A}_{mn} U^{\dagger}_{nj} \\
&= \sum_{m,n} \langle i|U|m\rangle \left(W^{\dagger}\left(|n\rangle\langle m| \otimes I^{\overline{A}}\right) W\right) \langle n|U^{\dagger}|j\rangle \\
&= \sum_{m,n} W^{\dagger}\left(\left(|n\rangle\langle n|\ U^{\dagger}|j\rangle\langle i|U\ |m\rangle\langle m|\right) \otimes I^{\overline{A}}\right) W \\
&= W^{\dagger}\left(U^{\dagger}|j\rangle\langle i|U \ \otimes\ I^{\overline{A}}\right) W \\
&= \left((U \otimes I)\, W\right)^{\dagger}\left(|j\rangle\langle i| \otimes I^{\overline{A}}\right)\left(U \otimes I\right) W \\
&= \left[\left(U \otimes I\right) W\right]^{A}_{ij} \\
&= \left[\left(U \otimes V\right) W\right]^{A}_{ij}
\end{aligned}$$

□

The following two theorems prove that the law of evolution is an action.

Theorem 4.3.3.

$$(VU)\left[W\right]^{A} = V\left(U\left[W\right]^{A}\right)$$

Proof. The proof is left to the reader. □

Theorem 4.3.4.

$$I^{A}\left[W\right]^{A} = \left[W\right]^{A}$$

Proof. The proof is left to the reader. □

Definition 4.3.2 (Trace)**.** The evolution matrix of a system A can be obtained from the evolution matrix of a system AB by a trace operation defined as

$$\left(\mathrm{tr}_{B}\left[W\right]^{AB}\right)_{ij} \stackrel{\text{def}}{=} \sum_{k} \left[W\right]^{AB}_{(i,k)(j,k)}.$$

Theorem 4.3.5.

$$\mathrm{tr}_{B}\left[W\right]^{AB} = \left[W\right]^{A}$$

Proof.

$$\begin{aligned}\left(\mathrm{tr}_B\left[W\right]^{AB}\right)_{ij} &= \sum_k \left[W\right]^{AB}_{(i,k)(j,k)} \\ &= \sum_k W^\dagger\left(|j\rangle\langle i|^A \otimes |k\rangle\langle k|^B \otimes I^{\overline{AB}}\right)W \\ &= W^\dagger\left(|j\rangle\langle i|^A \otimes I^B \otimes I^{\overline{AB}}\right)W \\ &= \left[W\right]^A_{ij}\end{aligned}$$

□

Theorem 4.3.6. Let *ABC* be a composite system

$$(\mathrm{tr}_B \circ \mathrm{tr}_C)\left[W\right]^{ABC} = \mathrm{tr}_{BC}\left[W\right]^{ABC}$$

Proof. The proof is left to the reader. □

Definition 4.3.3 (Join Product)**.** The state of a composite system *AB* can be obtained from the evolution matrices of systems *A* and *B* by the join product $\odot$, which is defined as

$$\left(\left[W\right]^A \odot \left[W\right]^B\right)_{(i,k)(j,\ell)} \stackrel{\text{def}}{=} \left[W\right]^A_{ij}\left[W\right]^B_{k\ell}.$$

Theorem 4.3.7.

$$\left[W\right]^A \odot \left[W\right]^B = \left[W\right]^{AB}$$

Proof.

$$\begin{aligned}&\left(\left[W\right]^A \odot \left[W\right]^B\right)_{(i,k)(j,\ell)} \\ =&\left[W\right]^A_{ij}\left[W\right]^B_{k\ell} \\ =&\left(W^\dagger\left(|j\rangle\langle i|^A \otimes I^B \otimes I^{\overline{AB}}\right)W\right)\left(W^\dagger\left(I^A \otimes |\ell\rangle\langle k|^B \otimes I^{\overline{AB}}\right)W\right) \\ =&W^\dagger\left(|j\rangle\langle i|^A \otimes I^B \otimes I^{\overline{AB}}\right)\left(I^A \otimes |\ell\rangle\langle k|^B \otimes I^{\overline{AB}}\right)W \\ =&W^\dagger\left(|j\rangle\langle i|^A \otimes |\ell\rangle\langle k|^B \otimes I^{\overline{AB}}\right)W \\ =&\left[W\right]^{AB}_{(i,k)(j,\ell)}\end{aligned}$$

□

Theorem 4.3.8.

$$U\left[W\right]^{A}\odot V\left[W\right]^{B}=\left(U\otimes V\right)\left[W\right]^{AB}$$

Proof.

$$\begin{aligned}
&U\left[W\right]^{A}\odot V\left[W\right]^{B}\\
&=\left[\left(U\otimes V\otimes I^{\overline{AB}}\right)W\right]^{A}\odot\left[\left(U\otimes V\otimes I^{\overline{AB}}\right)W\right]^{B} \qquad (4.2)\\
&=\left[\left(U\otimes V\otimes I^{\overline{AB}}\right)W\right]^{AB}\\
&=\left(U\otimes V\right)\left[W\right]^{AB}
\end{aligned}$$

□

4.3.3 Recovering the density matrices

4.3.3.1 A noumenal-phenomenal homorphism

Let ρ be a density matrix belonging to the global state S.

Let

$$\left(\varphi_{\rho}\left([W]^{A}\right)\right)_{ij}\stackrel{\text{def}}{=}\operatorname{tr}\left([W]^{A}_{i,j}\,\rho\right).$$

For simplicity we shall write $\varphi_{\rho}\left[W\right]^{A}$ instead of $\varphi_{\rho}\left(\left[W\right]^{A}\right)$.

Theorem 4.3.9.

$$\varphi_{\rho}\left[W\right]^{A}=\operatorname{tr}_{\overline{A}}\left(W\left(\rho\right)\right)$$

Proof.

$$
\begin{aligned}
\left(\varphi_\rho\left[W\right]^A\right)_{ij} &= \mathrm{tr}\left(\left[W\right]^A_{ij}\rho\right) \\
&= \mathrm{tr}\left(W^\dagger\left(|j\rangle\langle i|\otimes I^{\overline{A}}\right)W\rho\right) \\
&= \mathrm{tr}\left(W^\dagger\left(|j\rangle\langle i|\otimes\sum_k|k\rangle\langle k|\right)W\rho\right) \\
&= \mathrm{tr}\left(\sum_k\left(|j\rangle\langle i|\otimes|k\rangle\langle k|\right)W\rho W^\dagger\right) \\
&= \sum_k\left(\langle i|\otimes\langle k|\right)W\rho W^\dagger\left(|j\rangle\otimes|k\rangle\right) \\
&= \left(\mathrm{tr}_{\overline{A}}\left(W\rho W^\dagger\right)\right)_{ij} \\
&= \left(\mathrm{tr}_{\overline{A}}\left(W\left(\rho\right)\right)\right)_{ij}
\end{aligned}
$$

□

Theorem 4.3.10.

$$U\left(\varphi_\rho\left[W\right]^A\right) = \varphi_\rho\left(U\left[W\right]^A\right)$$

Proof.

$$
\begin{aligned}
& U\left(\varphi_\rho\left[W\right]^A\right) \\
&= U\left(\mathrm{tr}_{\overline{A}}\left(W\left(\rho\right)\right)\right) \\
&= \mathrm{tr}_{\overline{A}}\left(\left(\left(U\otimes I^{\overline{A}}\right)W\right)\left(\rho\right)\right) \\
&= \varphi_\rho\left(\left(U\otimes I^{\overline{A}}\right)W\right) \\
&= \varphi_\rho\left(U\left[W\right]^A\right)
\end{aligned}
$$

□

Theorem 4.3.11.

$$\mathrm{tr}_B\left(\varphi_\rho\left[W\right]^{AB}\right) = \varphi_\rho\left(\mathrm{tr}_B\left[W\right]^{AB}\right)$$

Proof. Let $C = \overline{AB}$, meaning that $S = ABC$.

$$\begin{aligned}
&\operatorname{tr}_B\left(\varphi_\rho\left[W\right]^{AB}\right)\\
&= \operatorname{tr}_B\left(\operatorname{tr}_{\overline{AB}}\left(W\left(\rho\right)\right)\right)\\
&= \operatorname{tr}_B\left(\operatorname{tr}_C\left(W\left(\rho\right)\right)\right)\\
&= \left(\operatorname{tr}_B \circ \operatorname{tr}_C\right)\left(W\left(\rho\right)\right)\\
&= \operatorname{tr}_{BC}\left(W\left(\rho\right)\right)\\
&= \operatorname{tr}_{\overline{A}}\left(W\left(\rho\right)\right)\\
&= \varphi_\rho\left[W\right]^{A}\\
&= \varphi_\rho\left(\operatorname{tr}_B\left[W\right]^{AB}\right)
\end{aligned}$$

□

4.3.4 Change of basis

The operations can be defined independently of their basis. For every evolution matrix $\left[W\right]^A$ in basis $\{|i\rangle\}_{i=1}^n$, we can define an evolution matrix $\left[W\right]^A$ in basis $\{|\phi_i\rangle\}_{i=1}^n$.

Theorem 4.3.12. Let $\{|i\rangle\}_{i=1}^n$ and $\{|\phi_i\rangle\}_{i=1}^n$ be two orthogonal bases of A. For all k and ℓ,

$$\left[W\right]_{\phi_k\phi_\ell} = \sum_{i,j}\langle\phi_k|i\rangle\left[W\right]_{ij}^{A}\langle j|\phi_\ell\rangle\,.$$

Proof.

$$
\begin{aligned}
&[W]^{A}_{\phi_k\phi_\ell} \\
&= W^\dagger \left(|\phi_\ell\rangle\langle\phi_k| \otimes I^{\overline{A}}\right) W \\
&= \sum_{i,j} W^\dagger \left(|j\rangle\langle j|\phi_\ell\rangle\langle\phi_k|i\rangle\langle i| \otimes I^{\overline{A}}\right) W \\
&= \sum_{i,j} \langle\phi_k|i\rangle W^\dagger \left(|j\rangle\langle i| \otimes I^{\overline{A}}\right) W \langle j|\phi_\ell\rangle \\
&= \sum_{i,j} \langle\phi_k|i\rangle \, [W]^{A}_{ij} \, \langle j|\phi_\ell\rangle
\end{aligned}
$$

□

4.4 Conclusion

We have shown that the universal wavefunction cannot be a complete description of local reality. Furthermore, we have provided a local-realistic model for quantum theory.

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APPENDIX A

AFFICHE SUR LES VIES PARALLÈLES

"What is proved by impossibility proofs is lack of imagination" - John Bell
"Imagination is more important than knowledge" - Albert Einstein

Abstract:

We show how local realism can be consistent with bipartite correlations that are usually considered to be nonlocal. For this purpose, we conduct a thought experiment in an imaginary world.

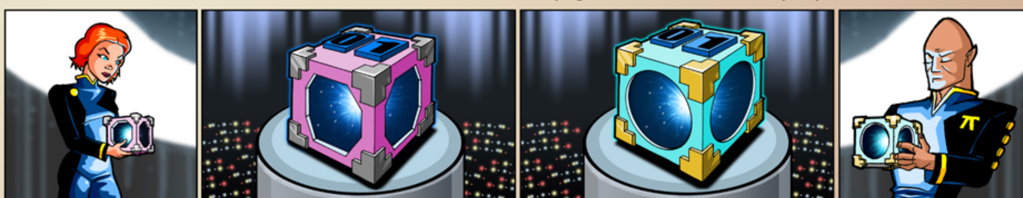
Imaginary World:

Our imaginary world follows the principles of Locality and Realism.

Principle of Locality: No action taken at a point A can have any effect at a point B at a speed faster than light.

Principle of Realism: There is a real world and observations are determined by the state of the real world.

This world has two inhabitants, Alice and Bob, which are each carrying a PR box, introduced by Popescu and Rorhlich.



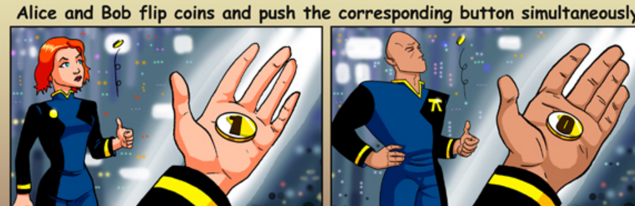
A PR box has a "0" and a "1" button. Whenever a button is pushed, it instantaneously flashes a red or green light with equal probability. If Alice and Bob both push a button, they will discover when they meet that they have seen different colours precisely when they both have pushed the "1" button.

(Note that the PR box does not enable instantaneous communication between Alice and Bob)

Alice and Bob will test the boxes with this protocol:

They travel far apart in their spaceships.

Alice and Bob flip coins and push the corresponding button simultaneously.



Once a button is pushed, the box flashes either a green or red light.

The experiment is performed with sufficient simultaneity that Alice's box cannot know the result of Bob's coin flip (hence the input to Bob's box) before it has to flash its own light, and vice versa.

After many experiments, they meet and realize that the boxes work perfectly.

The Einstein-Podolsky-Rosen Argument:

- Alice's pushing of a button cannot have any instantaneous effect on Bob's system by the principle of Locality.
- After Alice pushes her button, she can know with certainty what colour Bob will see depending on which button he pushes. *(For example, if Alice pushes "1" and sees green, she knows that if Bob pushes "0" he will see green)*
- Since it is possible for Alice to predict with certainty what light Bob will see when he pushes a button, without influencing his system, it means that his observations were predetermined.
- The observations of Bob should be described by local hidden variables B0 and B1.

B0 = 0 if Bob will observe green after pushing "0"
B0 = 1 if Bob will observe red after pushing "0"

B1 = 0 if Bob will observe green after pushing "1"
B1 = 1 if Bob will observe red after pushing "1"

- Likewise, Alice's system should be described by local hidden variables A0 and A1.

A0 = 0 if Alice will observe green after pushing "0"
A0 = 1 if Alice will observe red after pushing "0"

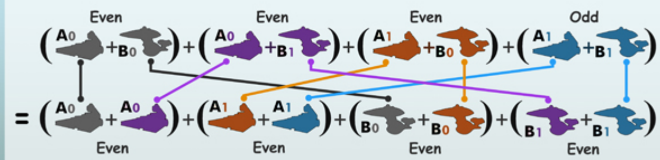
A1 = 0 if Alice will observe green after pushing "1"
A1 = 1 if Alice will observe red after pushing "1"

- A local hidden variable theory would give a local realistic explanation of this experiment.

Bell's Theorem: Local hidden variable theories can only produce PR boxes that work at most 75% of the time.

Proof: A hidden variable theory of these boxes must satisfy the following 4 equations:

Summing these equations on both sides and rearranging the terms:



This implies: Even = Odd!

It is not possible for the four equations to be all correct. At least one of the four possible choices of buttons pushed will give incorrect results.

Many people have concluded that any world that could produce PR boxes that work more than 75% of the time cannot be Local and Realistic. Remarkably, quantum mechanics enables PR boxes that work 85% of the time. Must we conclude that quantum mechanics cannot be Local and Realistic?

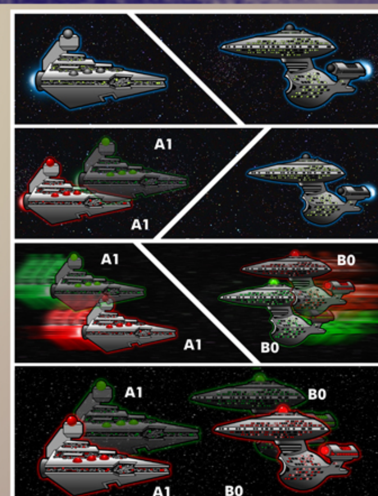
Here is how the seemingly impossible is accomplished:

Each spaceship lives inside a bubble.

When Alice pushes a button on her box (here "1"), her bubble splits into two bubbles. Each bubble contains a copy of its spaceship and its inhabitant. Inside one bubble, Alice has seen the red light flash; inside the other, she has seen the green light flash. From now on, the two bubbles are living **parallel lives**. They cannot interact in any way and will never meet again. Notice that this phenomenon is strictly local.

The same phenomenon takes place when Bob pushes his button (here "0") on the box. Let's see what happens when they travel toward each other.

Each of the two bubbles that contain Alice is allowed to interact with and see only a single bubble that contains Bob, namely the one that satisfies the equations described above.



Note that such a perfect matching is always possible. Furthermore, each bubble can "know" with which other bubble to interact provided it keeps a local memory of which button was pushed and which light flashed. Alice and Bob will be under the illusion of correlations that seem to emerge from outside of space and time.

In our imaginary world, the Einstein-Podolsky-Rosen argument does not hold because whenever Alice pushes a button and can predict something about Bob, she is really predicting, not what is happening simultaneously at Bob's place, but how their various lives will meet in the future.

Conclusion:

The virtue of our imaginary world is to demonstrate in an exceedingly simple way that local reality can produce correlations that are impossible in any classical theory based on local hidden variables.

In quantum mechanics, a theory analogous to this one can be traced back to at least Deutsch and Hayden.

Perhaps we live parallel lives?

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