Externalities and the Nucleolus*

Mikel Álvarez-Mozos[1] and Lars Ehlers[2]

[1] *Departament de Matemàtica Econòmica, Financera i Actuarial and BEAT, Universitat de Barcelona, Spain.*

[2] *Département de Sciences Économiques and CIREQ, Université de Montréal, Canada*

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Abstract

In most economic applications, externalities prevail: the worth of a coalition depends on how the other players are organized. We show that there is a unique natural way of extending the nucleolus from (coalitional) games without externalities to games with externalities. This is in contrast to the Shapley value and the core for which many different extensions have been proposed.

Keywords: Externalities; Partition function; Nucleolus; Reduced game

1 Introduction

There is an abundance of economic situations where the worth of a coalition depends on how the other players are organized. In such situations a game with externalities associates with each coalition and each possible partition of the other players a worth of that (embedded) coalition. The literature on coalitional games with externalities is still relatively limited compared to the solid foundations of the theory of games without externalities.

For classic coalitional games, the most important set-valued solution concept is the core and the two most important single-valued solution concepts are the Shapley value and the nucleolus.

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For both the core and the Shapley value many different extensions were proposed to games with externalities. For instance, for the core the recursive approach by Kóczy (2007) and the expectation formation approach by Bloch and Van den Nouweland (2014) and for the Shapley value the average approach by Macho-Stadler et al. (2007), the marginality approach by de Clippel and Serrano (2008), and the utilization of reduction and consistency by Dutta et al. (2010). All these contributions provide *families* of extensions. To date, an extension of the nucleolus is missing in the literature.

We provide a natural extension of the nucleolus from coalitional games without externalities to games with externalities: for each embedded coalition consisting of the coalition and partition of the other players, we measure the excess of this embedded coalition as the difference between the worth of the embedded coalition minus what the coalition gets in the allocation (which equals the sum of the allotments of the players in the coalition). For each allocation, then we rearrange the excesses of all embedded coalitions in non-increasing order. The nucleolus is then simply the set of allocations which lexicographically minimize the rearranged excesses of all embedded coalitions. We show that (i) the nucleolus is unique and (ii) the nucleolus of a game with externalities coincides with nucleolus of the following associated game without externalities: for each coalition we take the maximal worth among all possible organizations of the other players. Indeed, Fact (ii) is our key contribution. In the spirit of de Clippel and Serrano (2008), we have a unique "externality free" extension of the nucleolus.

We also present an axiomatic characterization of the new solution concept. Indeed, we can adapt the properties used by Sobolev (1975) in the well known characterization the prenucleolus, namely anonymity, covariance and the reduced game property, to games with externalities quite naturally. The reduced game property shapes a consistency principle and is of paramount importance in our result. Such a principle states that in the event that some agents leave the game with the proposed payoffs and the remaining agents renegotiate the sharing in a reduced game, the payoffs do not change. We consider two natural ways to extend the Davis and Maschler reduced game to our framework. The first and more naive is a classic coalitional game without externalities that enables us to characterize the nucleolus. The second inherits the externalities of the underlying game and yields a weaker property. Recently, Abe (2017) proposed other alternatives to extend consistency properties to games with externalities, among them we can find two extensions of the Davis and Maschler reduced game property. The key difference of our approach is that the reduced game we consider only depends on the payoff and the coalition of agents that leave and does not assume the formation of any coalition structure.

We proceed as follows. In Section 2 we introduce games with externalities, we extend the nucleolus from classic games to games with externalities, and we show that it is a well defined point-valued solution concept. In Section 3 we present our main result, an axiomatic characterization of the nucleolus. We also provide some intermediate results like the characterization by means of balanced collections and the coincidence with the nucleolus of an associated game without externalities. Section 4 concludes presenting other interesting properties of the nucleolus.

2 Preliminaries

Let $\mathcal{N}$ stand for the infinite set of potential players and let $N \subset \mathcal{N}$ be a finite set of players. The set of partitions of N is denoted by $\mathcal{P}(N)$.[1] An *embedded coalition* of N is a pair (S, P) where $S \subseteq N$ and $P \in \mathcal{P}(N \setminus S)$. We denote by $\mathcal{EC}^N$ the set of all embedded coalitions of N. A *coalitional game with externalities (or for short, game)* is a pair (N, v) consisting of a finite set of players $N \subset \mathcal{N}$ and a partition function $v\colon \mathcal{EC}^N \to \mathbb{R}$, satisfying $v(\emptyset, P) = 0$, for every $P \in \mathcal{P}(N)$. The set of all games is denoted by $\mathcal{G}$. Given $(N, v) \in \mathcal{G}$, we say that (N, v) is a *coalitional game without externalities* if for all $(S, P), (S, Q) \in \mathcal{EC}^N$, $v(S, P) = v(S, Q)$. In this case we may simply write $v(S)$. The set of all coalitional games without externalities is denoted by $\mathcal{CG}$.

Our purpose is to propose a point-valued solution concept for coalitional games with externalities. Given a game $(N, v) \in \mathcal{G}$, (i) an allocation for (N, v) is a vector $x = (x_i)_{i \in N} \in \mathbb{R}^N$ and (ii) $X(N, v)$ denotes the set of all efficient allocations for (N, v), i.e., $X(N, v) = \left\{x \in \mathbb{R}^N : x(N) = \sum_{i \in N} x_i = v(N, \emptyset)\right\}$.[2] Given a game $(N, v) \in \mathcal{G}$, an embedded coalition $(S, P) \in \mathcal{EC}^N$, and an efficient allocation $x \in X(N, v)$, the *excess* of (S, P) at x is defined by

$$e(S, P, x, v) = v(S, P) - x(S).$$

The excess of an embedded coalition, (S, P), at x measures the dissatisfaction of coalition S if the worth of the grand coalition is divided according to x and players are arranged according to $P \cup \{S\}$. Define $c(n) = \left|\mathcal{EC}^N\right|$, with $n = |N|$. Let $e(x, v) \in \mathbb{R}^{c(n)}$ denote the vector of excesses at x in v, i.e., $e(x, v) = (e(S, P, x, v))_{(S,P) \in \mathcal{EC}^N}$. For a given $(N, v) \in \mathcal{G}$ and $x \in X(N, v)$, we are

[1] By convenience, let $\emptyset$ be the only partition in $\mathcal{P}(\emptyset)$.

[2] We denote by $\mathbb{R}^N$, the $|N|$-dimensional Euclidean space, with coordinates indexed by the elements of N. For every $x \in \mathbb{R}^N$ and $S \subseteq N$, $x(S) = \sum_{i \in S} x_i$.

going to build a vector with all the excesses $e(x, v)$ arranged in non-increasing order. Then, the vector of ordered excesses is defined as follows:[3]

$$\theta(x,v) \in \mathbb{R}^{c(n)} \quad \text{, where} \quad \cup_{i=1}^{c(n)} \{\theta_i(x,v)\} = \cup_{i=1}^{c(n)} \{e_i(x,v)\} \text{ and}$$
$$\theta_1(x,v) \geq \theta_2(x,v) \geq \cdots \geq \theta_{c(n)}(x,v).$$

Let $\mathbb{R}^m_{\geq}$ denote the set of all vectors $x \in \mathbb{R}^m$ such that $x_1 \geq x_2 \geq \cdots \geq x_m$, i.e. the coordinates of x are arranged in non-increasing order. Let $\precsim$ denote the lexicographical ordering on $\mathbb{R}^m_{\geq}$: for all $x, y \in \mathbb{R}^m_{\geq}$, $x \precsim y$, with means that either $x = y$ or there is $1 \leq t \leq m$, such that $x_i = y_i$ for every $1 \leq i < t$ and $x_t < y_t$. We write $x \prec y$ if $x \precsim y$ and $y \not\precsim x$.

Definition 1. *The* nucleolus *of a game with externalities is the set of efficient allocations which lexicographically minimize the ordered vector of excesses:*

$$\eta(N,v) = \{x \in X(N,v) : \theta(x,v) \precsim \theta(y,v) \text{ for all } y \in X(N,v)\}.$$

The first step is to show that the nucleolus is a well defined solution. Let $\mathcal{H} \subseteq \mathcal{G}$. Formally, a *(single-valued) solution on* $\mathcal{H}$ is a mapping f that assigning allocation $f(N,v) \in \mathbb{R}^N$ for every game $(N,v) \in \mathcal{H}$.

The second step will relate the nucleolus of games with externalities to the classical pre-nucleolus of a particular game without externalities. The latter is a well known solution on $\mathcal{CG}$ that we denote by η^* and can be defined for every $(N,v) \in \mathcal{CG}$ by $\eta^*(N,v) = \eta(N,v)$ (where we use the first step, or simply η^* is the restriction of η to $\mathcal{CG}$). This result is one of our key insights: the nucleolus of a partition function form game is uniquely defined and given by the nucleolus of its associated "externality-free" max-game where for any coalition S its worth is equal to the maximum of the worths $v(S,P)$ where P is any possible organization of the other players. Formally, for any $(N,v) \in \mathcal{G}$, let $(N, v^{\max}) \in \mathcal{CG}$ be defined by for all $S \subseteq N$,

$$v^{\max}(S) = \max\{v(S,P) : P \in \mathcal{P}(N \setminus S)\}.$$

The following is our first main result.

Theorem 1. *(i) The nucleolus is a (single-valued) solution on $\mathcal{G}$.*

(ii) For all $(N,v) \in \mathcal{G}$, we have $\eta(N,v) = \eta^*(N, v^{\max})$.

Proof. (i): This proof is an adaptation of the original proof by Schmeidler (1969).

[3] Here identical numbers appear multiple times, i.e. we could have $\{2, 2, 2, 1, 1, 0, \ldots\}$.

Let $(N, v) \in \mathcal{G}$. First, given $y \in X(N,v)$ let $\alpha(y) = \max\left\{e(S,P,y,v) : (S,P) \in \mathcal{EC}^N\right\}$. Let $\alpha = \min\{\alpha(y) : y \in X(N,v)\}$. Define

$$Y_0 = \left\{x \in X(N,v) : e(S,P,x,v) \leq \alpha : \text{for every } (S,P) \in \mathcal{EC}^N\right\}.$$

Clearly, $\eta(N,v) = \{x \in Y_0 : \theta(x,v) \precsim \theta(y,v), \text{ for every } y \in Y_0\}$. It is also easy to see that Y_0 is a non-empty, compact, and convex polytope.

Second, let $(S,P) \in \mathcal{EC}^N$. Then $e(S,P,x,v)$ is a continuous function of x. Let $m = |\mathcal{EC}^N|$, note that $\theta_i(x,v)$ can be defined for every $i \in \{1,\ldots,m\}$ by

$$\theta_i(x,v) = \left\{\max\left\{\min\left\{e(S,P,x,v) : (S,P) \in \mathcal{D}\right\}\right\} : \mathcal{D} \subseteq \mathcal{EC}^N \text{ such that } |\mathcal{D}| = i\right\}.$$

Therefore, for every $i \in \{1,\ldots,m\}$, $\theta_i(x,v)$ is a continuous function on x.

Third, for every $i \in \{1,\ldots,m\}$, we define

$$Y_i = \left\{x \in Y_{i-1} : \theta_i(x,v) \leq \theta_i(y,v) \text{ for all } y \in Y_{i-1}\right\}.$$

Since Y_0 is a non-empty, compact, and convex polytope and $\theta_i(x,v)$ is continuous on x for every $i \in \{1,\ldots,m\}$, every Y_i is a non-empty, compact, and convex subpolytope of Y_{i-1}. Since, $Y_m = \eta(N,v)$, we have shown $\eta(N,v) \neq \emptyset$.

Forth, let $\delta\colon \mathbb{R}^m \to \mathbb{R}^m$ be the function whose output has the same coordinates as the input vector but in non-increasing order. For instance, $\delta\left((e(S,P,x,v))_{(S,P)\in\mathcal{EC}^N}\right) = \theta(x,v)$, for every $x \in X(N,v)$. It is easy to observe that for every $z^1, z^2 \in \mathbb{R}^m$, $\delta(z^1+z^2) \precsim \delta(z^1)+\delta(z^2)$, and that $\delta(z^1+z^2) = \delta(z^1)+\delta(z^2)$ if and only if the i^{th} highest coordinates of both z^1 and z^2 are in the same positions for every $i \in \{1,\ldots,m\}$.

Fifth, let $x, y \in X(N,v)$ be such that $x, y \in \eta(N,v)$. In particular, $\theta(x,v) = \theta(y,v)$ and $\theta(x,v) + \theta(y,v) = 2\theta(x,v)$. On the other hand,

$$2\theta\left(\frac{1}{2}(x+y), v\right) = \theta(x+y,v) = \delta\left((e(S,P,x+y,v))_{(S,P)\in\mathcal{EC}^N}\right).$$

Then, using the forth point above, $2\theta\left(\frac{1}{2}(x+y),v\right) \precsim 2\theta(x,v)$. But since $x \in \eta(N,v)$, it must be that $\theta\left(\frac{1}{2}(x+y),v\right) = \theta(x,v)$. Using again the forth point, it must be that vectors $\left((e(S,P,x,v))_{(S,P)\in\mathcal{EC}^N}\right)$ and $\left((e(S,P,y,v))_{(S,P)\in\mathcal{EC}^N}\right)$ have the i^{th} highest coordinate in the same position for every $i \in \{1,\ldots,m\}$. Finally, using again that $\theta(x,v) = \theta(y,v)$ we get $x = y$ and the proof of (i) concludes.

(ii): In order to (ii), we extend Property I of Kohlberg (1971) to our setting.

Definition 2. *Let* $(N,v) \in \mathcal{G}$*. For every* $x \in X(N,v)$ *and* $\alpha \in \mathbb{R}$*, define*

$$\mathcal{A}(\alpha,x,v) = \{S \subseteq N : e(S,P,x,v) \geq \alpha, \textit{ for some } P \in \mathcal{P}(N \setminus S)\}.$$

An efficient allocation $x \in X(N,v)$ *is said to have* Property I *with respect to* (N,v) *if the following condition is satisfied for every* $\alpha \in \mathbb{R}$ *where* $\mathcal{A}(\alpha,x,v) \neq \emptyset$*: If* $y \in \mathbb{R}^N$ *is such that* $y(N) = 0$ *and* $y(S) \geq 0$ *for every* $S \in \mathcal{A}(\alpha,x,v)$*, then* $y(S) = 0$ *for every* $S \in \mathcal{A}(\alpha,x,v)$*.*

Lemma 1. *Let* $(N,v) \in \mathcal{G}$ *and* $x \in X(N,v)$*. Then* $x = \eta(N,v)$ *if and only if* x *has Property I with respect to* (N,v)*.*

Proof. This proof is an adaptation of the original proof by Kohlberg (1971).

Necessity: Let $(N,v) \in \mathcal{G}$ and $x = \eta(N,v)$. Let $\alpha \in \mathbb{R}$ be such that $\mathcal{A}(\alpha,x,v) \neq \emptyset$ and $y \in \mathbb{R}^N$ such that $y(N) = 0$ and $y(S) \geq 0$ for every $S \in \mathcal{A}(\alpha,x,v)$. We denote by $\mathcal{B}(\alpha,x,v)$ the set of embedded coalitions whose excesses at x are no less than α, i.e., $\mathcal{B}(\alpha,x,v) = \{(S,P) \in \mathcal{EC}^N : e(S,P,x,v) \geq \alpha\}$. Note that $\mathcal{B}(\alpha,x,v)$ contains the embedded coalitions whose excesses at x are the first coordinates of $\theta(x,v)$. Define $z_\epsilon = x + \epsilon y$, where $\epsilon > 0$. Note that $z_\epsilon \in X(N,v)$. We choose $\epsilon^* > 0$ such that for every $(S,P) \in \mathcal{B}(\alpha,x,v)$ and every $(T,Q) \notin \mathcal{B}(\alpha,x,v)$,

$$e(S,P,z_{\epsilon^*},v) > e(T,Q,z_{\epsilon^*},v). \tag{1}$$

In other words, we choose $\epsilon^* > 0$ in such a way that the excesses of the embedded coalitions in $\mathcal{B}(\alpha,x,v)$ are in the first positions of $\theta(z_{\epsilon^*},v)$. Next, for every $(S,P) \in \mathcal{B}(\alpha,x,v)$,

$$e(S,P,z_{\epsilon^*},v) \leq e(S,P,x,v), \tag{2}$$

because $y(S) \geq 0$ for every $S \in \mathcal{A}(\alpha,x,v)$ and $(S,P) \in \mathcal{B}(\alpha,x,v)$ implies $S \in \mathcal{A}(\alpha,x,v)$.

Finally, suppose that x does not have Property I, i.e., there is $S \in \mathcal{A}(\alpha,x,v)$ such that $y(S) > 0$. Then, by (1) and (2), $\theta(z_{\epsilon^*},v) \prec \theta(x,v)$ which contradicts our assumption.

Sufficiency: Let $(N,v) \in \mathcal{G}$ and $x \in X(N,v)$ be a vector that has Property I w.r.t (N,v). Let $\alpha_1 > \cdots > \alpha_p$ be such that $\{e(S,P,x,v) : (S,P) \in \mathcal{EC}^N\} = \{\alpha_1,\ldots,\alpha_p\}$. Next, let $z = \eta(N,v)$ and $y = z - x$. We have to show that y is the null vector. We proceed by induction on $i = 1,\ldots,m$ and show that $y(S) = 0$ for every $S \in \mathcal{A}(\alpha_i,x,v)$. Clearly, $y \in \mathbb{R}^N$ with $y(N) = 0$. Let $S \in \mathcal{A}(\alpha_1,x,v)$, i.e., there is $P \in \mathcal{P}(N \setminus S)$ such that $e(S,P,x,v) = \alpha_1$. Then,

$$y(S) = e(S,P,x,v) - e(S,P,z,v) \leq 0,$$

where the inequality holds because $\theta(z,v) \precsim \theta(x,v)$. Since, x has Property I, $y(S) = 0$. Suppose now that $y(S) = 0$ for every $S \in \mathcal{A}(\alpha_i,x,v)$ for some $i \in \{1,\ldots,p-1\}$. Let $S \in \mathcal{A}(\alpha_{i+1},x,v) \setminus$

$\mathcal{A}(\alpha_i, x, v)$, i.e., there is $P \in \mathcal{P}(N \setminus S)$ such that $e(S, P, x, v) = \alpha_{i+1}$. Then,

$$y(S) = e(S, P, x, v) - e(S, P, z, v) \leq 0,$$

where the inequality holds by the induction hypothesis and the fact that $\theta(z, v) \precsim \theta(x, v)$. Since, x has Property I, $y(S) = 0$ and the proof concludes. □

For (ii), given $(N, v) \in \mathcal{G}$, we have to show $\eta(N, v) = \eta^*(N, v^{\max})$. Let $x \in X(N, v)$ and $\alpha \in \mathbb{R}$. By definition, we can write $\mathcal{A}(\alpha, x, v) = \{S \subseteq N : v^{\max}(S) - x(S) \geq \alpha\}$. Then, the result follows directly from Lemma 1 and the characterization of the pre-nucleolus of a characteristic function game by Kohlberg (1971). □

3 Characterization

The purpose of this section is to present our second main result, namely an axiomatic characterization of the solution introduced above.

The first property we would like to impose on a solution is the classic anonymity, stated in the general form taking into account that the games we consider have a player set belonging to an infinite universe of potential players.

Anonymity: A solution f is *anonymous* if for every $(N, v) \in \mathcal{G}$ and every injection $\pi\colon N \to \mathcal{N}$,

$$f(\pi(N), \pi v) = \pi\left(f(N, v)\right),$$

where $(\pi(N), \pi v) \in \mathcal{G}$ is defined for every $(S, P) \in \mathcal{EC}^{\pi(N)}$, by $v(S, P) = \pi v(\pi(S), \pi(P))$ with $\pi(P) = \{\pi(T) : T \in P\}$.

In words, *anonymity* states that relabeling of players should not affect the solution.

The next property is a natural generalization from cooperative game theory to our setting of games with externalities.

Covariance: A solution f is *covariant* if for every $(N, v) \in \mathcal{G}$, $\alpha > 0$, and $\beta \in \mathbb{R}^N$,

$$f(N, \alpha v + \beta) = \alpha f(N, v) + \beta,$$

where $(N, \alpha v + \beta) \in \mathcal{G}$ is defined for every $(S, P) \in \mathcal{EC}^N$, by $(\alpha v + \beta)(S, P) = \alpha v(S, P) + \beta(S)$.

Note that *covariance* entails linearity of an arbitrary game with an inessential game.

Next, we present the most important property of our paper which states that a solution should not be affected if a coalition renegotiates the sharing in a particular subgame. Given $(N, v) \in \mathcal{G}$, $\emptyset \neq S \subseteq N$, and $x \in \mathbb{R}^N$. The *reduced game* with respect to S and x is denoted by $(S, v_{S,x}) \in \mathcal{CG}$ and is defined for every $T \subseteq S$ by

$$v_{S,x}(T) = \begin{cases} 0 & \text{if } T = \emptyset, \\ v(N) - x(N \setminus S) & \text{if } T = S, \\ \max\left\{v(R, Q) - x(R \setminus T) : (R, Q) \in \mathcal{EC}^N \text{ and } R \cap S = T\right\} & \text{otherwise.} \end{cases}$$

The idea behind the above reduced game is that if agents in $N \setminus S$ leave the game with the payoff proposed by x, the remaining agents interact in a new coalitional game without externalities. In the latter game, the worth of the grand coalition, S, is determined by the remainder $v(N) - x(N \setminus S)$ and every other coalition $T \neq \emptyset$ assesses its worth by taking the maximum over all possible embedded coalitions obtained when some agents in $N \setminus S$ may join coalition T. Note that this coincides with the Davis and Maschler (1965) reduced game with the exception that instead the worth a bare coalition, say R, we consider the worth of every embedded coalition of the type (R, Q).

Reduced Game Property: A solution f satisfies the *reduced game property* if for all $(N, v) \in \mathcal{G}$, all $\emptyset \neq S \subseteq N$, and all $i \in S$ (where $x = f(N, v)$), we have

$$x_i = f_i\left(S, v_{S,x}\right).$$

If a solution fulfills the *reduced game property*, then the payoffs remain unaffected when agents in a coalition renegotiate in the reduced game.

Theorem 2. *The nucleolus, η, is the only solution on $\mathcal{G}$ satisfying anonymity, covariance, and the reduced game property.*

Before we continue, it is helpful to recall the characterization of η^* by Sobolev (1975). In order to present it, we can consider variants of the three properties we have introduced above that only apply to games without externalities. That is, let *anonymity**, *covariance**, and the *reduced game property** be the restrictions of *anonymity*, *covariance*, and the *reduced game property* to solutions on $\mathcal{CG}$, respectively.

Theorem 3. (Sobolev, 1975) *The pre-nucleolus, η^*, is the only solution on $\mathcal{CG}$ satisfying anonymity*, covariance*, and the reduced game property*.*

We are now in the position to show our characterization result.

Proof of Theorem 2.

Existence: Using Theorem 1 (ii) and Theorem 3 we get the existence from the following observations. Let $(N, v) \in \mathcal{G}$. First, for every injection $\pi\colon N \to \mathcal{N}$, $\pi(v^{\max}) = (\pi v)^{\max}$. Second, for every $\alpha > 0$ and $\beta \in \mathbb{R}^N$, $(\alpha v + \beta)^{\max} = \alpha v^{\max} + \beta$. Third, if $x = \eta(N, v)$ and $\emptyset \neq S \subseteq N$, then $(v_{S,x})^{\max} = (v^{\max})_{S,x}$.

Uniqueness: Let $(N, v) \in \mathcal{G}$ and f be a solution on $\mathcal{G}$ satisfying the three properties. If $x = f(N, v)$, then from the definition of the reduced game it follows that for every $(S, P) \in \mathcal{EC}^N$, $v_{N,x}(S, P) = v^{\max}(S)$. Then, by the *reduced game property*

$$f(N, v) = f(N, v_{N,v}) = f(N, v^{\max}).$$

Finally, using Theorem 3 and the fact that $(N, v^{\max}) \in \mathcal{CG}$,

$$f(N, v^{\max}) = \eta^*(N, v^{\max}) = \eta(N, v),$$

where the last equality follows from Theorem 1 (ii). □

Remark 1. *If the grand coalition is efficient, i.e., for every $P \in \mathcal{P}(N)$, $v(N, \emptyset) \geq \sum_{S \in P} v(S, P \setminus S)$, then the nucleolus allocates the worth of the grand coalition to all players. If the grand coalition is not efficient, then we could pick one of the efficient partitions, define a game within each of its coalitions and use the solution to obtain a sharing of the worth of each of these embedded coalitions.*

4 Discussion

In this section we discuss another interesting property of the nucleolus and its relation to different notions of the core introduced in the literature.

It could be reasonable to define a reduced game which inherits externalities from the original game. Formally, given $(N, v) \in \mathcal{G}$, $\emptyset \neq S \subseteq N$, and $x \in \mathbb{R}^N$. The *reduced game with externalities* with respect to S and x is denoted by $(S, w_{S,x}) \in \mathcal{G}$ and is defined for every $(T, P) \in \mathcal{EC}^S$ by[4]

$$w_{S,x}(T,P) = \begin{cases} 0 & \text{if } T = \emptyset, \\ v(N) - x(N \setminus S) & \text{if } T = S, \\ \max\left\{v(R,Q) - x(R \setminus T) : (R,Q) \in \mathcal{EC}^N, R \cap S = T, Q \cap S = P\right\} & \text{otherwise.} \end{cases}$$

The idea behind the above reduced game is that when $N \setminus S$ leave the game with the payoff proposed by x, the remaining agents interact in a new coalitional game with externalities. In the latter game, the worth of the grand coaltion, $(S, \emptyset)$, is determined by the remainder $v(N) - x(N \setminus S)$. Otherwise, in the event that coalition structure $P \cup \{T\}$ emerges, coalition T assesses its worth by taking the maximum over all possible ways in which some agents in $N \setminus S$ may join T and some others may form new coalitions or join any of the coalitions in the structure, assuming that agents that join coalition T are paid according to x. The above reduced game yields another version of the well known reduced game property.

Weak Reduced Game Property: A solution f satisfies the *weak reduced game property* if for all $(N, v) \in \mathcal{G}$, all $\emptyset \neq S \subseteq N$, and all $i \in S$ (where $x = f(N, v)$), we have

$$x_i = f_i\left(S, w_{S,x}\right).$$

It is easy to see how the above property also generalizes the reduced game property, introduced for point-valued solutions by Sobolev (1975). Indeed, the two versions of the reduced game property proposed here coincide for coalitional games without externalities. The difference between the two properties is the fact that the former is not affected by the externalities of the original game because it takes the maximum over all possible partitions while the latter takes the maximum only among those partitions that are consistent with the coalitional organization of the players in the reduced game.

Proposition 1. *The nucleolus satisfies the weak reduced game property.*

Proof. Let $(N, v) \in \mathcal{G}$, $x = \eta(N, v)$, and $\emptyset \neq S \subseteq N$. Using the definition of both reduced games, we can write for every $T \subseteq S$,

$$\left(w_{S,x}\right)^{\max}(T) = v_{S,x}(T). \tag{3}$$

Then, by Theorem 1 (ii), for every $i \in S$

$$\eta_i\left(S, w_{S,x}\right) = \eta_i\left(S, \left(w_{S,x}\right)^{\max}\right) = \eta_i\left(S, v_{S,x}\right) = x_i,$$

[4]Given $(R, Q) \in \mathcal{EC}^N$, let $Q \cap S = \{U \cap S : U \in P\}$.

where the second equality holds by (3) and the third is because the nucleolus satisfies the reduced game property (Theorem 2). □

Whether we can weaken the reduced game property of our characterization result in this direction is still an open question.

It is also straightforward to see how we can recover one of the auxiliary results of Sobolev (1975) to the framework of games with externalities, namely that covariance and the reduced game with externalities property imply efficiency.

Proposition 2. *Let f be a solution satisfying covariance and the reduced game with externalities property. Then, for every $(N, v) \in \mathcal{G}$, $f(N, v) \in X(N, v)$.*

Proof. This proof is an adaptation of the original proof by Sobolev (1975).

First of all, we show the result for one-person games. Let $(\{i\}, v) \in \mathcal{G}$. If $v = 0$, i.e., $v(\{i\}) = 0$, by covariance $f(\{i\}, v) = f(\{i\}, 2v) = 2f(\{i\}, v)$. Which implies $f(\{i\}, v) = 0$. Otherwise, let $v \neq 0$. Again by covariance, $f(\{i\}, v) = f(\{i\}, 0 + v(\{i\})) = f(\{i\}, 0) + v(\{i\}) = v(\{i\})$. Next, let $(N, v) \in \mathcal{G}$ with $|N| \geq 2$ and $x = f(N, v)$. Take $i \in N$, as we have just shown $f(\{i\}, v_{\{i\},x}) \in X(\{i\}, v_{\{i\},x})$, that is

$$f_i(\{i\}, v_{\{i\},x}) = v_{\{i\},x}(\{i\}) = v(N, \emptyset) - x(N \setminus \{i\}),$$

where the second equality follows from the definition of the reduced game. Finally, since f satisfies the reduced game with externalities property,

$$x_i = f_i(\{i\}, w_{\{i\},x}) = v(N, \emptyset) - x(N \setminus \{i\}),$$

which means that $x(N) = v(N)$. □

Another well-known property of the prenucleolus of coalitional games without externalities is that whenever non-empty it always lies within the core. It is interesting to analyze the behavior of the nucleolus as introduced here with respect to different notions of the core proposed in the literature. A way to pin down a particular core in the presence of externalities is to anticipate the coalitional reaction of the deviating players. This is precisely the approach of Bloch and Van den Nouweland (2014) where a large class of core notions are studied in a common framework. Formally, an *expectation formation rule* is a mapping, f, that associates to every $S \subseteq N$ a partition of $N \setminus S$, i.e., for every $S \subseteq N$, $f(S, v) \in \mathcal{P}(N \setminus S)$.[5] Then, the *core* of $(N, v) \in \mathcal{G}$ with

[5] Recall that for simplicity, we assume that the grand coalition is the most efficient arrangement of any set of players.

respect to the expectation formation rule f is defined by

$$C_f(N,v) = \{x \in X(N,v) : x(S) \geq v\left(S, f(S,v)\right) \quad \forall S \subseteq N\}.$$

The optimistic rule, f_o, originally proposed by Shenoy (1979) selects for every coalition, the most favorable partition, i.e., for every $(N,v) \in \mathcal{G}$ and $S \subseteq N$, $f_o(S,v) = \underset{P \in \mathcal{P}(N \setminus S)}{\arg\max}\; v(S,P)$. The core with respect to the optimistic rule is called the *optimistic core*.

Proposition 3. *Regarded that the optimistic core is non-empty, the nucleolus is in the core of the game with respect to any expectation formation rule.*

Proof. Let $(N,v) \in \mathcal{G}$. Note that, for every $S \subseteq N$, $v(S, f(S)) = v^{\max}(S)$. That is, the optimistic core is the core of the coalitional game without externalities $(N, v^{\max})$. Then, by Theorem 1 (ii) and the well-known fact that the prenuclolus of a game without externalities lies in the core whenever non-empty, we have that $\eta(N,v) \in C_{f_o}(N,v)$. Finally, since the optimistic core is contained in every other core (Bloch and Van den Nouweland, 2014) we get the desired result. □

A natural follow up question is whether the nucleolus is in the core of any expectation formation rule whenever non-empty. We show by a counter-example that the answer is negative.

Example 1. *Let* $N = \{1,2,3\}$ *and* $(N,v) \in \mathcal{G}$ *be defined by*[6]

$$\begin{array}{llll} v(1;2,3) = 0 & v(1;23) = 1 & v(12;3) = 2 & \\ v(2;1,3) = 0 & v(2;13) = 1 & v(13;2) = 1 & v(N;\emptyset) = 2 \\ v(3;1,2) = 2 & v(3;12) = 0 & v(23;1) = 1 & \end{array}$$

and let also the expectation formation rule be such that, $f(1) = 23$*,* $f(2) = 13$*, and* $f(3) = 12$*. That is, according to* f *each coalition expect the rest of agents to form a one coalition partition. Then it is easy to see that*

$$C_f(N,v) = \{(1,1,0)\}.$$

However, using Theorem 1 (ii) we can easily compute the nucleolus

$$\eta(N,v) = \left(\frac{3}{4}, \frac{3}{4}, \frac{1}{2}\right).$$

Still, one could wonder whether there is a necessary and sufficient condition on the expectation formation rule that guarantees the nucleolus to be a core allocation (with respect to the expectation formation rule) whenever non-empty. This is another open question for future research.

[6] For the sake of clarity we omit brackets and only use commas between coalitions.

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