

(II)legal Assignments in School Choice*

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Abstract

In public school choice, students with strict preferences are assigned to schools. Schools are endowed with priorities over students. Incorporating different constraints from applications, priorities are often modeled as choice functions over sets of students. It has been argued that the most desirable criterion *for an assignment* is fairness; there should not be a student having justified envy in the following way: he prefers some school to his assigned school and has higher priority than some student who got into that school. Justified envy could cause court cases. We propose the following fairness notion *for a set of assignments*: a set of assignments is legal if and only if any assignment outside the set has justified envy with some assignment in the set and no two assignments inside the set block each other via justified envy. We show that under very basic conditions on priorities, there always exists a unique legal set of assignments, and that this set has a structure common to the set of fair assignments: (i) it is a lattice and (ii) it satisfies the rural-hospitals theorem. This is the first contribution providing a “set-wise” solution for many-to-one matching problems where priorities are not necessarily responsive and schools are not active agents. *JEL C78, D61, D78, I20.*

1 Introduction

Centralized admissions procedures are now being used in a wide range of applications ranging from national college admissions, assigning students to public schools, to implementing auxiliary programs such as magnet schools.¹ There has been a great deal of research focused

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¹Examples of countries that use a centralized college admissions process are Turkey (Balinski and Sönmez, 1999); China (Chen and Kesten, 2016); and India (Aygün and Turhan, 2016). There is now a long literature devoted to public school assignment beginning with the seminal work of Abdulkadiroğlu and Sönmez (2003). See Pathak (2011) and Abdulkadiroğlu and Sönmez (2013) for surveys of the literature. See Dur, Hammond, and Morrill (2017) for a discussion of centralized magnet school assignment.

on the tradeoffs between efficiency, fairness, and strategic properties of candidate mechanisms. These mechanisms have received a great deal of attention in the economics literature precisely because for parents and students school choice is an important issue. Assigning objects which are valuable and yet scarce leads to contention, and contention leads to lawsuits. For example, parent groups in Seattle and Louisville filed lawsuits contesting the use of racial status in the tiebreaker of their school district’s assignment procedure. These law suites eventually led to the Supreme Court ruling (in *Parents Involved in Community Schools v. Seattle School District No. 1*, 551 U.S. 701, 2007) that race cannot be used explicitly in a school assignment procedure.

This is the basic question of our paper: which school assignments are legal? We are concerned with a parent or group of parents who file a lawsuit with the intent of changing the school assignment that is to be made.² Consider what seems to be the most straightforward application: college admissions. Typically, all students take a common exam, and a student’s score determines her priority when choosing a university. In this environment, legality may appear simple; if a student is denied admissions to a university, each student accepted to that university must have a higher score than her. However, there are two reasons why (at least in the United States) this does not correctly determine which assignments are legal.

Legal standing, or *locus standi*, is the capacity to bring suit in court. As interpreted by the United States Supreme Court:

Under modern standing law, a private plaintiff seeking to bring suit in federal court must demonstrate that he has suffered “injury in fact,” that the injury is “fairly traceable” to the actions of the defendant, and that injury will “*likely be redressed by a favorable decision.*”³

Therefore, it is not illegal to reject a student from a university (regardless of which students are accepted) unless there exists a legal way of assigning her to the university. This suggests that legality is a set-wise property of assignments. A set of assignments is legal or not as we must be able to determine which assignments are possible in order to know which assignments are legal.

The second reason why a simple comparison of students’ scores is not sufficient to determine the legality of an assignment is that typically a school’s decision on which students to admit is at least partially based on the composition of the student body. Public schools often reserve seats for minority students or students who live within a “walk-zone”.⁴ Admission to a magnet school may consider a student’s income level (Dur, Hammond and Morrill, 2017). The centralized admissions process in India incorporates which caste the

²In particular, we do not address the separate question of a parent filing a lawsuit with the purpose of receiving monetary damages. Note that a government agency typically has sovereign immunity and would not be liable for damages.

³This quote is from Hessick (2007) regarding Supreme Court case *Lujan vs. Defenders of Wildlife*, 504 U.S. 555, 560-61 (1992).

⁴For examples, see Abdulkadiroğlu and Sönmez (2003), Abdulkadiroğlu et al. (2005), Kominers and Sönmez (2016), and Dur et al. (2017).

student is a member of in their admissions process (Aygün and Turhan, 2016). In each case, admission decisions are based on a more complicated choice function than a simple rank-order list of students. Is it still possible to determine which assignments are legal in a coherent way?

A generalized choice function is just a more complicated set of rules for determining which students are admitted. We interpret these rules as conveying rights to each student. A student’s rights have been violated if the rules dictate that she should have been chosen. However, whether or not this violation is illegal is more subtle; although the student has been harmed, this violation is not illegal unless the harm is redressable.⁵ We propose a definition of legality that incorporates these two constraints. This is analogous to a “stability” notion of a set of assignments (where stability depends on the whole set). More specifically, blocking is only allowed via assignments in the set (which we deem legal). Any assignment outside the set is illegal because it is blocked by some assignment in the set. The important feature is that here blocking is defined in terms of assignments: student i blocks an assignment if i blocks it with some school *and* there exists some assignment in the set where i is assigned to the blocking school. It should be clear that in this assignment the school is not necessarily better off. It has the interpretation that there is some “legal way” of assigning i to the blocking school. More precisely, we call a set of assignments legal iff (i) any assignment not in the set is blocked by some student with an assignment in the set and (ii) no two assignments block each other.

Of course, this is related to stable sets à la von Neumann Morgenstern (vNM). A cursory reading makes one think that the two concepts are identical. They are in the sense of the formulation of (i) and (ii), but most importantly, a school might be worse off under the assignment in the set when compared to the original one. But this is not a problem as we are here in the context of public school choice where (as it has been argued) students are “active agents” and schools are “objects to be consumed”. Any legal set is a vNM-stable set where schools are “objects to be consumed”.

Our main results show that there always exists a unique legal set of assignments and that this set shares the following properties with stable assignments: (i) it is a lattice and (ii) the rural hospitals theorem is satisfied. Therefore, there always exists a student-optimal legal assignment and a school-optimal assignment. Moreover, we demonstrate that the student-optimal legal assignment is Pareto efficient. Therefore, unlike fairness, there is no tension between making a legal assignment and an efficient assignment. Finally, we relate the student-optimal legal assignment to Kesten’s efficiency adjusted deferred-acceptance (DA)-mechanism (Kesten, 2010). The efficiency adjusted DA-mechanism has not been previously defined when schools have general choice functions. We show that when schools have acceptant⁶ choice functions that the mechanism is straightforward to generalize and that the efficiency adjusted DA-mechanism chooses the student-optimal legal assignment.

⁵It is common, especially among economists, to view all harm as redressable via side payments. However, states and by extension local governments have sovereign immunity from lawsuits for damages unless the state consents to be sued.

⁶A school a ’s choice function is *acceptant* if there exists a capacity q such that a accepts all students if fewer than q apply and q students whenever q or more students apply.

As a byproduct, we offer a foundation for the generalization of Kesten’s efficiency adjusted DA-mechanism to school choice environments where priorities are given by substitutable and LAD choice functions.

Our paper is most closely related to Morrill (2017) which reinterprets which school assignments are fair. Typically, an assignment is deemed unfair if a student has justified envy.⁷ Morrill (2017) defines a student to have legitimate envy if she has justified envy at school a and it is possible to assign her to school a . Otherwise, her envy is defined to be petty. There is not a direct relationship between which schools are possible and which schools are legal since a legal assignment is allowed to be wasteful whereas in Morrill (2017) non-wasteful assignments are excluded by assumption; however, there is a close relationship between which assignments are possible and which assignments are legal. The analysis in Morrill (2017) relies critically on two assumptions: each school has responsive preferences and the school assignments considered are non-wasteful. However, in many practical applications (such as when incorporating affirmative action) these assumptions are unreasonable. Our paper demonstrates that the legal set of assignments has similar properties even when these restrictions do not hold.

Our paper also relates to several recent papers that consider alternative fairness notions to eliminating justified envy. Dur, Gitmez, and Yilmaz (2015) introduce the concept of partial fairness. Intuitively, they define an assignment to be partially fair if the only priorities that are violated are “acceptable violations”. Kloosterman and Troyan (2016) also introduce a new fairness concept called essentially stable. Intuitively, an assignment is essentially stable if resolving i ’s justified envy of school a initiates a vacancy chain that ultimately leads to i being rejected from a . Both Dur, Gitmez and Yilmaz (2016) and Kloosterman and Troyan (2016) provide characterizations of EADA using their respective fairness notion. Partial fairness and essentially stable are similar in spirit but do not directly relate to legality. Each is a pointwise concept while legality is a setwise concept. Moreover, the analysis in both Dur, Gitmez and Yilmaz (2015) and Kloosterman and Troyan (2016) relies heavily on the assumption that schools have responsive preferences. It is not clear whether or not their results would hold in the general environment considered in the current paper.⁸

In school choice with responsive priorities, Wu and Roth (2016) study the structure of assignments which are fair and individually rational (i.e. non-wastefulness may be violated). They show that this set has a lattice structure and that the student-optimal assignment coincides with the student-optimal stable assignment.

In contexts where both sides are agents, in one-to-one matching problems Ehlers (2007) studies vNM-stable sets, and Wako (2010) shows the existence and uniqueness of such sets. Klijn and Masso (2003) study bargaining sets in those problem. Note that all these papers consider one-to-one settings whereas our paper considers the most general many-to-one

⁷Student i has *justified envy* of student j if i prefers j ’s assignment to her own and i has a higher priority at that school than does j .

⁸Kesten (2004), Alcalde and Romero (2015), and Cantala and Papai (2014) also introduce alternative notions of fairness for the school assignment problem. The concepts they introduce do not directly relate to legality.

setting and provides an alternative solution concept to the set of stable assignments.

We proceed as follows. Section 2 introduces school choice and all basic notions for choice functions and assignments. Section 3 defines legal assignments. Section 3.1 generalizes the Pointing Lemma, the Decomposition Lemma and the Rural Hospital Theorem to any two individually rational assignments which do not block each other, and Section 3.2 establishes a Lattice Theorem. We then use these results to show the existence and uniqueness of a legal set in Section 3.3. Section 4 discusses our results. In Subsection 4.1 we relate legal assignments to efficiency and non-wastefulness. Section 4.2 shows that there is a unique strategy-proof and legal mechanism, namely, the student-proposing DA-mechanism. Section 4.3 shows how our results carry over to the setting of assignment with contracts. The Appendix contains all proofs omitted from the main text.

2 Model

We consider the following many-to-one matching problem. There is a finite set of students, $A = \{i, j, k, \dots\}$, to be assigned to a finite set of schools, $O = \{a, b, c, \dots\}$. Each student i has a strict preference P_i over the schools and being unassigned $O \cup \{i\}$ (where i stands for being unassigned). Then iP_ia indicates that student i prefers being unassigned to being assigned to school a and R_i denotes the weak preference relation associated with P_i .

We allow schools having general choice functions for priorities in order to incorporate various assignment constraints. Let 2^A denote the set of all subsets of A . Each school a has a choice function $C_a : 2^A \rightarrow 2^A$ such that for all $X \in 2^A$, $C_a(X) \subseteq X$. Then $C_a(X)$ denotes the set of students that school a chooses from X . Throughout we assume that C_a satisfies the following standard properties of substitutability and the law of aggregate demand (LAD): (a) substitutability rules out complementarities in the sense that students chosen from larger sets should remain chosen from smaller sets and (b) LAD requires the number of chosen students to be weakly monotonic for bigger sets of students.

Definition 1. Let $a \in A$ and $C_a : 2^A \rightarrow 2^A$ be a choice function.

- (a) The choice function C_a is **substitutable** if for all $X \subseteq Y \subseteq A$ we have $C_a(Y) \cap X \subseteq C_a(X)$.⁹
- (b) The choice function C_a satisfies the **law of aggregate demand (LAD)** if for all $X \subseteq Y \subseteq A$ we have $|C_a(X)| \leq |C_a(Y)|$.¹⁰

Throughout we fix the assignment problem $(A, O, (P_i)_{i \in A}, (C_a)_{a \in A})$.

⁹Note that this is equivalent to $i \in C_a(Y)$ and $j \in Y \setminus \{i\}$ implies $i \in C_a(Y \setminus \{j\})$ (or the same condition formulated in terms of rejected students $Y \setminus C_a(Y)$).

¹⁰Here $|X|$ denotes the cardinality of a set. LAD was introduced by Hatfield and Milgrom (2005) in a more general model of matching with contracts. Our definition of LAD is equivalent to size monotonicity introduced by Alkan and Gale (2003) and Fleiner (2003). We use the LAD terminology to be consistent with the standard matching literature.

An assignment is a function $\mu : A \rightarrow O \cup A$ from students to schools and students. Given assignment μ and $i \in A$, let $\mu_i = a$ indicate student i being assigned to school a (and $\mu_i = i$ indicate student i being unassigned). We use the convention that for each school a the set $\mu_a = \{i \in A : \mu_i = a\}$ denotes the students assigned to school a . Let \mathcal{A} denote the set of all assignments. An assignment μ is **individually rational** if for every student i , $\mu_i R_i i$ and, for every school a , $C_a(\mu_a) = \mu_a$. *Throughout we will consider individually rational assignments only.*¹¹ Let \mathcal{IR} denote the set of individually rational assignments.

Blocking is defined as follows for general choice functions. Given an assignment μ , student i and school a **block** μ if $a P_i \mu_i$ and $i \in C_a(\mu_a \cup \{i\})$. This means that student i prefers school a to his assignment and school a chooses i from its assigned students and i . There are two types of blocking: school a has an empty seat available for i or school a would like to admit i and reject a previously admitted student. These two types are distinguished below in the usual sense. An assignment μ is **non-wasteful** if (it is individually rational and) there do not exist a student i and a school a such that $a P_i \mu_i$ and $C_a(\mu_a \cup \{i\}) = \mu_a \cup \{i\}$. Given an assignment μ , student i has **justified envy** if there is a school a such that $a P_i \mu_i$, $i \in C_a(\mu_a \cup \{i\})$, and $C_a(\mu_a \cup \{i\}) \neq \mu_a \cup \{i\}$. This means that student i prefers a to his assignment and has higher “choice” priority because he is chosen from the set of students assigned to school a and including him (and some other student is rejected). An assignment is **fair** if (it is individually rational and) there is no justified envy. An assignment is **stable** if it is individually rational, non-wasteful and fair.

Stable assignments were introduced by Gale and Shapley (1962) in two-sided matching and adopted to school choice by Balinski and Sönmez (1999) and Abdulkadiroğlu and Sönmez (2002). The main difference is that in two-sided matching both sides are “agents” whereas in school choice students are “agents” and schools are “objects to be consumed”.

Nevertheless, the set of stable assignments coincide in both interpretations: the set of stable assignments is non-empty, it is a lattice and it satisfies the strong rural hospitals theorem. Furthermore, note that stability is a “point-wise” property specific to one assignment alone.

3 Legal Assignments

We will be interested in “set-wise” blocking which will depend on the whole set of assignments under consideration.

Definition 2. Let $\mu, \nu \in \mathcal{A}$ and $i \in A$.

- (a) Student i **blocks** assignment μ with assignment ν if for some school $a \in A$: (1) $a P_i \mu_i$, (2) $i \in C_a(\mu_a \cup \{i\})$ and (3) $\nu_i = a$.
- (b) Assignment μ **blocks** ν if there is a student i who blocks μ with ν .

¹¹Individual rationality can be alternatively interpreted as “feasibility” of assignments.

Note that in the usual blocking notion, both the blocking student and the school are unambiguously (myopically) better off (with respect to the original assignment) whereas here the student is only unambiguously better off (because the school's priority ranking is not clear between μ and ν). Our main solution concept only allows blocking via assignments which are in the set under consideration: (i) any assignment outside the set is blocked via some assignment inside the set and (ii) any two assignments inside the set do not block each other.

Definition 3. *Let $L \subseteq \mathcal{IR}$. Then L is **legal** if and only if*

- (i) *for all $\nu \in \mathcal{IR} \setminus L$ there exists $\mu \in L$ such that μ blocks ν , and*
- (ii) *for all $\mu, \nu \in L$, μ does not block ν .*

On first sight this is similar to stable sets à la von Neumann-Morgenstern (hereafter vNM-stability). However, under vNM-stability, both sides (often called workers and firms instead of students and schools) are considered to be agents, and all agents must be made better off in order to block. However, in the school assignment problem only the students are agents. The important fact in our definition of blocking is that only the student is made better off and the school may be made worse off.¹² One could interpret the legality of a set of assignments as the natural generalization of stable sets to school choice. Of course, this could be done to other contexts in cooperative game theory where sharing problems contain “neutral” agents with priorities.

Throughout we will use the convention that for a given legal set L , any assignment belonging to L is called **legal** and any assignment not belonging to L is called **illegal**.

Since school choice problems have a non-empty set of stable assignments (the core), the following heuristic way of finding a set of legal assignments (as already suggested by von Neumann-Morgenstern) is plausible.

Recall that \mathcal{IR} denotes the set of all individually rational assignments. We call a function $f : 2^{\mathcal{IR}} \rightarrow 2^{\mathcal{IR}}$ an **operator**. We define an operator f to be increasing if $X \subseteq Y \subseteq \mathcal{IR}$ implies $f(X) \subseteq f(Y)$, and analogously, f is decreasing if $X \subseteq Y$ implies $f(X) \supseteq f(Y)$.

The following operator will be central for finding legal assignments. Given any set of assignments $X \subseteq \mathcal{IR}$, $\pi(X)$ is the set of individually rational assignments which are not blocked by any assignment in X :

$$\pi(X) = \{\mu \in \mathcal{IR} \mid \nexists \nu \in X \text{ such that } \nu \text{ blocks } \mu\}. \quad (1)$$

The following three properties are straightforward to verify but will be useful.

Lemma 1. *The operator π defined in (1) satisfies:*

- (i) *π is decreasing.*
- (ii) *π^2 is increasing.*

¹²Note that legality and vNM-stability are equivalent when a school can be assigned at most one student.

(iii) If J is the set of stable assignments, then $J \subseteq \pi(M)$ for any set $M \subseteq \mathcal{IR}$.

Proof. If a student is able to block with more assignments, then fewer assignments will remain unblocked. Therefore, π is a decreasing operator. Consider two sets of assignments X and Y such that $X \subseteq Y \subseteq \mathcal{IR}$. Since π is decreasing, $\pi(Y) \subseteq \pi(X)$. Again, since π is decreasing, $\pi(\pi(X)) \subseteq \pi(\pi(Y))$. Therefore, π^2 is increasing. Finally, stable assignments are not blocked by any assignment. Therefore, they are not blocked by any assignment in \mathcal{IR} . \square

As it turns out, any legal set of assignments is a fixed point of the operator π (and vice versa).

Lemma 2. *Let $L \subseteq \mathcal{IR}$. Then L is a legal set if and only if $\pi(L) = L$.*

Proof. Suppose L is legal. If $\mu \in L$, then μ is not blocked by any $\nu \in L$. Therefore, $L \subseteq \pi(L)$. Similarly, if $\mu \in \pi(L)$, then by construction there does not exist $\nu \in L$ such that ν blocks μ . Therefore, $\pi(L) \subseteq L$. For the other direction, suppose that $\pi(L) = L$. Then $\mu \notin L$ if and only if $\mu \notin \pi(L)$ (since $L = \pi(L)$) if and only if there exists a $\nu \in L$ such that ν blocks μ (by the definition of π). Therefore, L is legal. \square

It is not obvious that a legal set of assignments must exist (we will show this later). Suppose that a legal set of assignments does exist. We define $S^0 = \emptyset$, and we set $B^0 = \pi(S^0)$. Note that $B^0 = \mathcal{IR}$, the set of all individually rational assignments. Continuing, we let $S^1 = \pi(B^0)$. Note that S^1 is the set of stable assignments. In general, we define:

$$\begin{aligned} S^0 &= \emptyset \\ B^k &= \pi(S^k) \\ S^{k+1} &= \pi(B^k) = \pi^2(S^k) \end{aligned}$$

Let L be a legal set of assignments. It is trivially true that $S^0 \subseteq L \subseteq B^0$. If μ is a stable assignment, then μ is not blocked by any assignment. Therefore, the set of stable assignments, S^1 , must be contained in L . Moreover, a legal set of assignments must be internally consistent. Since S^1 is contained in any legal set, no assignment blocked by an assignment in S^1 can be part of any legal set. Therefore, $L \subseteq B^1$. Similarly, if L is a legal set of assignments, and μ is not blocked by any assignment in B^1 , then μ is not blocked by any assignment in L . Therefore, by external stability, μ must be legal. Therefore, it must be that $S^2 \subseteq L$, and so on.

In general, for any k , if L is a legal set of assignments then:

$$S^0 \subseteq S^1 \subseteq \dots \subseteq S^k \subseteq L \subseteq B^k \subseteq \dots \subseteq B^1 \subseteq B^0$$

We seek a fixed point of the operator π ; however, it is not obvious that such a fixed point exists. However, since π^2 is an increasing function, a fixed point of π^2 must exist. In

particular, since there are only a finite number of possible assignments, there must be a n such that $S^n = S^{n+1}$.¹³

Furthermore, for this fixed point we have $S^n \subseteq \pi(S^n)$: Trivially, $S^0 \subseteq \pi(S^0) = B^0$. Now suppose by induction that we have $S^{k-1} \subseteq \pi(S^{k-1})$. Because π^2 is increasing, we have $\pi^2(S^{k-1}) \subseteq \pi^3(S^{k-1})$. Thus,

$$S^k = \pi^2(S^{k-1}) \subseteq \pi^3(S^{k-1}) = \pi(\pi^2(S^{k-1})) = \pi(S^k),$$

which yields the desired conclusion $S^k \subseteq \pi(S^k)$.

Thus, if S^n is a fixed point of π^2 , then the two key properties of S^n are:¹⁴

$$(1) S^n \subseteq \pi(S^n) \text{ and } (2) S^n = \pi^2(S^n).$$

Our main challenge will be to show that in fact $S^n = B^n$. This will establish the existence and uniqueness of a legal set of assignments. However, we first establish properties of S^n that will be used in our proof. We will show that any set with properties (1) and (2) is a lattice and satisfies the Rural Hospitals theorem.

For this, it will be instrumental to show for any two individually rational assignments μ and ν , which do not block each other, a Pointing Lemma, a Decomposition Lemma and the Rural Hospitals Theorem. Then we go on to show the lattice structure for these assignments. Any reader, who wants to go directly to the main results, may skip Subsections 3.1 and 3.2.

3.1 Pointing, Decomposition and Rural Hospital Theorem

Two of the classic results in matching theory are the Pointing Lemma and the Decomposition Lemma. The Pointing Lemma (attributed to Conway in Knuth, 1976) is the basis for the proof that the set of stable marriages is a lattice.¹⁵ The Pointing Lemma compares any two stable assignments μ and ν . We ask each man to point to his favorite wife under the two marriages (he is possibly unmarried or married to the same woman), and we ask each woman to point to her favorite husband. The Pointing Lemma says that no man and woman point to each other; no two men point to the same woman; and no two women point to the same man.

¹³This follows from Tarski's fixed point theorem because $2^{\mathcal{I}\mathcal{R}}$ is a partially ordered set with respect to set inclusion and π^2 is increasing. Moreover, Tarski's theorem says that the set of fixed points of π^2 is a lattice with respect to unions and intersections of sets. However, his result does not tell us anything about the structure of the assignments belonging to a fixed point of π^2 .

¹⁴This is very closely related to the concept of a subsolution defined in Roth (1976). What is now called a vNM-stable set was originally referred to by von Neumann and Morgenstern as a **solution**. Roth (1976) introduced a generalization of vNM-stability called a subsolution: A subsolution is any set S such that (1) $S \subseteq \pi(S)$ and (2) $S = \pi^2(S)$ (and we used above Roth's argument to show the existence of a subsolution). The reason we do not call our set S^n a subsolution is that the definition of blocking is different in our framework than under the traditional vNM-stability. We thank Federico Echenique for pointing out this connection.

¹⁵Following the exposition in Roth and Sotomayor (1992), we refer to it as the Pointing Lemma.

Lemma 3 (Classical Pointing Lemma). *Consider a marriage problem where the men and women have strict preferences and let μ and μ' be stable matchings. Then:*

- (i) *no man and woman point at each other unless they are matched under both μ and μ' ;*
- (ii) *no two women point at the same man; and*
- (iii) *no two men point at the same woman.*

The key implication of the Pointing Lemma is that the assignments $\mu \vee \nu$ (defined by each man is assigned to the woman he is pointing at) and $\mu \wedge \nu$ (defined as each woman is assigned to the man she is pointing at) are well defined. This is the basis of the Lattice Theorem as all that remains is to show that $\mu \vee \nu$ and $\mu \wedge \nu$ are also stable.

The Pointing Lemma is closely related to the Decomposition Lemma which is due to Gale and Sotomayor (1985).

Lemma 4 (Classical Decomposition Lemma). *Consider a marriage problem where the men and women have strict preferences and let μ and μ' be stable matchings. Let $M(\mu')$ be the set of men who prefer μ' to μ and let $W(\mu)$ be the set of women who prefer μ to μ' . Then μ' and μ map $M(\mu')$ onto $W(\mu)$.*

The Pointing Lemma generalizes to many-to-one problems in a straightforward way when schools have responsive preferences with quotas: instead of a choice function, each school a has a strict preference over sets of students, say \succ_a , and a quota q_a (of available seats at a). Then \succ_a is responsive iff for any students i, j and any set $H \subseteq A \setminus \{i, j\}$ such that $|H| \leq q_a - 1$, we have (i) $H \cup \{i\} \succ_a H \cup \{j\}$ iff $i \succ_a j$, and (ii) $H \cup \{i\} \succ_a H$ iff $i \succ_a \emptyset$; and (iii) $\emptyset \succ_a H$ for any $H \subseteq A$ with $|H| > q_a$. Now we know that the set of stable assignments of the many-to-one market corresponds to the set of stable assignments of the one-to-one market where any school a is split into q_a copies. A similar construction can be done for two assignments which do not block each other,¹⁶ and hence the pointing lemma carries over in a straightforward manner from one-to-one to many-to-one.

We will show that when schools have general choice functions that only the first two conditions of the Pointing Lemma generalize. However, the Decomposition Lemma continues to hold. To the best of our knowledge, we are the first to generalize the Pointing and Decomposition Lemmas when schools have choice functions instead of responsive preferences.

Since pointing indicates that the student is willing to form a blocking pair, the most natural way to adapt pointing to non-responsive preferences is, given two assignments μ and ν and given a student $i \in \mu_a \setminus \nu_a$, a points to i if $i \in C_a(\nu_a \cup \{i\})$.

For later purposes, instead, we define pointing using a seemingly stronger condition. We will later show (in Corollary 2) that this condition is equivalent to the weaker version of pointing.

¹⁶Simply consider $\mu \setminus \nu$ (where any school a receives students $\mu_a \setminus \nu_a$) and $\nu \setminus \mu$ with appropriately reduced capacities (where for any school a we reduce q_a by $|\mu_a \cap \nu_a|$ and the set of students is shrunk to $A \setminus (\cup_{a \in O} (\mu_a \cap \nu_a))$).

Definition 4. Given two assignments μ and ν , student i points to μ_i (ν_i) if $\mu_i R_i \nu_i$ ($\nu_i R_i \mu_i$), and school a points to student i if $i \in C_a(\mu_a \cup \nu_a)$.

It is clear that under this definition of pointing that when a school points to a student, then she is willing to form a blocking pair with that student. However, it is less clear that each student will be pointed at. We first establish a weak version of the Pointing Lemma.

Lemma 5 (Weak Pointing Lemma). *Let μ and ν be two individually rational assignments which do not block each other. Then:*

- (i) *no student and school point at each other unless they are assigned under both μ and ν , and*
- (ii) *no two schools point to the same student.*

Proof. Consider any student i such that $\mu_i \neq \nu_i$. Without loss of generality, assume $\mu_i P_i \nu_i$. By individual rationality of μ and ν , we have $\mu_i \in O$, say $\mu_i = a$. Then i points to a . By substitutability of C_a and $i \in \mu_a$, if $i \in C_a(\mu_a \cup \nu_a)$, then $i \in C_a(\nu_a \cup \{i\})$. Therefore, if a pointed to i (meaning $i \in C_a(\mu_a \cup \nu_a)$), then i would block ν with μ (because $\mu_i = a$), a contradiction. For any student i such that $\mu_i \neq \nu_i$, by $\mu_i R_i i$ and $\nu_i R_i i$, i must point to a school. Therefore, if two schools point to the same student, there must be a student and a school pointing at each other which would be a contradiction to the above. \square

Notice that we are missing the third conclusion of the Classical Pointing Lemma. The generalization to the school assignment problem would be as follows: no two students point at the same school unless they are classmates (i.e. they are both assigned to that school under either μ or ν). The following example is taken from Ehlers and Klaus (2014) and demonstrates that this result does not hold when a school does not have responsive preferences.

Example 1. Let $O = \{a, b\}$ and $A = \{s_1, s_2, j_1, j_2\}$. University a and university b are both hoping to hire two economists. They are considering two senior candidates, s_1 and s_2 , and two junior candidates, j_1 and j_2 . Candidates s_x and j_x are in the same field. University a would prefer to hire seniors to juniors, but if it must hire a mixture of the two, it would prefer to hire candidates in the same field. Specifically:

$$\{s_1, s_2\} \succ_a \{s_1, j_1\} \succ_a \{s_2, j_2\} \succ_a \{j_1, j_2\} \succ_a \{s_1, j_2\} \succ_a \{j_1, s_2\}.$$

If a is only able to hire one economist, then its preferences are: $s_1 \succ_a s_2 \succ_a j_1 \succ_a j_2$. Note that the choice function C_a induced by \succ_a satisfies substitutability and LAD, but \succ_a is not responsive because $\{s_2, j_2\} \succ_a \{s_2, j_1\}$ and $j_1 \succ_a j_2$.

University b has the opposite preferences:

$$\{j_1, j_2\} \succ_b \{s_1, j_1\} \succ_b \{s_2, j_2\} \succ_b \{s_1, s_2\} \succ_b \{s_1, j_2\} \succ_b \{j_1, s_2\},$$

and $j_1 \succ_b j_2 \succ_b s_1 \succ_b s_2$. Again the choice function C_b induced by \succ_b satisfies substitutability and LAD, but \succ_b is not responsive.

Both junior candidates prefer a to b whereas both senior candidates prefer b to a . Consider the assignments

$$\mu = \begin{pmatrix} a & b \\ \{s_1, j_1\} & \{s_2, j_2\} \end{pmatrix} \text{ and } \nu = \begin{pmatrix} a & b \\ \{s_2, j_2\} & \{s_1, j_1\} \end{pmatrix},$$

where under assignment μ , a receives $\{s_1, j_1\}$ and b receives $\{s_2, j_2\}$ (and similar for ν). It is straightforward to verify that μ and ν are both stable (and therefore, do not block each other). Note that both junior candidates point to a . Similarly, both senior candidates point to b , whereas university a points to the two senior candidates and university b points to the two junior candidates.

Our objective is to show that the pointing procedure still leads to two well-defined assignments: assigning each student to the school she points to, and assigning each student to the school pointing to her. Eventually, we will show that if the original assignments are legal, then the induced reassignments are legal. But it is interesting to note that this construction applies to any two individually rational assignments which do not block each other.

Definition 5. Given assignments μ and ν , define $\mu \wedge \nu$ by $\mu \wedge \nu_a = C_a(\mu_a \cup \nu_a)$ for all $a \in O$.

Our main focus is on any two individually rational assignments μ and ν which do not block each other. Then $\mu \wedge \nu$ is the reassignment resulting from assigning a student to the school that is pointing to her. The following lemma demonstrates that this yields a well-defined assignment.

Lemma 6. Let μ and ν be two individually rational assignments which do not block each other. Then:

- (i) $\mu \wedge \nu$ is an individually rational assignment;
- (ii) if i is assigned a school under μ , then i is assigned a school under $\mu \wedge \nu$; and
- (iii) every school receives the same number of students under μ and $\mu \wedge \nu$.

Lemma 6 is essential for our construction of alternative assignments. We will prove it formally using a counting argument; however, we first demonstrate the intuition for why the result holds. For expositional purposes, we will call a student chosen if they are pointed at by some school. First, we observe that if school a points to a student i , then i must be assigned under both μ and ν . Consider Figure 1. In this case, a student i is assigned under μ but not ν . Let $\mu_i = a$. Moreover, since i was pointed at, it must be that $i \in C_a(\mu_a \cup \nu_a)$. Therefore, so long as i prefers a to being unassigned (μ is individually rational), i is able to block ν with μ .

Note that this leaves us with only a “small” number of students that school a may point at: students that are assigned to some school under both μ and ν . But we have already

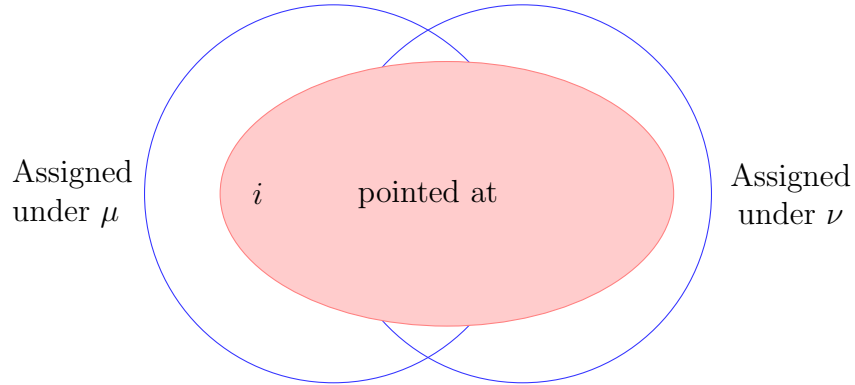


Figure 1: So long as μ is individually rational, i blocks ν with μ .

observed in the Weak Pointing Lemma that no student is pointed at by two different schools (otherwise she would block with one). Moreover, each school's choice function satisfies LAD. Therefore, each school points at weakly more students than they are assigned under either μ or ν . Therefore, the number of students that are chosen must be weakly more than the number of students assigned under either μ or ν . For example, Figure 2 illustrates a contradiction. If, as in Figure 2, there is a student that is assigned under μ but not chosen, then the number of students chosen would be strictly less than the number of students assigned under μ or ν . However, there is only one way for the set of students to both lie in the intersection of μ and ν and be weakly more than both μ and ν : the set of students chosen must equal the set of students assigned under μ which must equal the set of students assigned under ν .

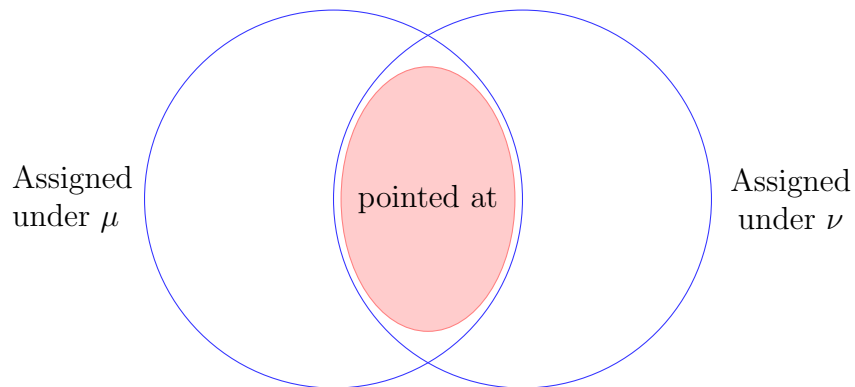


Figure 2: The number of student chosen must be less than or equal to the number of students she is assigned under μ or ν .

We now prove Lemma 6 formally.

Proof. (i): Suppose for contradiction that there is a student i and $a \neq b$ such that both

$i \in \mu \wedge \nu_a$ and $i \in \mu \wedge \nu_b$. Then $i \in C_a(\mu_a \cup \nu_a)$ and $i \in C_b(\mu_b \cup \nu_b)$. Then both a and b point to i . Because $a \neq b$, ($i \in \mu_a$ and $i \in \nu_b$) or ($i \in \mu_b$ and $i \in \nu_a$), and i must point to either a or b . Therefore, there is a student and a school pointing at each other which contradicts the Pointing Lemma. In showing that $\mu \wedge \nu$ is individually rational, we have by definition $C_a(\mu \wedge \nu_a) = \mu \wedge \nu_a$.¹⁷ Furthermore, $\mu_i R_i i$ and $\nu_i R_i i$ imply $\mu \wedge \nu_i R_i i$. Hence, $\mu \wedge \nu \in \mathcal{IR}$.

(ii) and (iii): For counting purposes, in this proof we use the convention $|\mu_i| = 1$ if $\mu_i \neq i$ and $|\mu_i| = 0$ if $\mu_i = i$. First note that if $\mu \wedge \nu_i = a$ but $\mu_i = i$, then i blocks μ with ν : by individual rationality, $\nu_i = a P_i i$; and $i \in C_a(\mu_a \cup \nu_a)$ and substitutability of C_a imply $i \in C_a(\mu_a \cup \{i\})$. Therefore, $|\mu \wedge \nu_i| = 1$ implies that $|\mu_i| = 1$ and $|\nu_i| = 1$. Hence,

$$\sum_{i \in A} |\mu \wedge \nu_i| \leq \sum_{i \in A} |\mu_i|. \quad (2)$$

By the Law of Aggregate Demand and $\mu_a \cup \nu_a \supseteq \mu_a$, $|C_a(\mu_a \cup \nu_a)| \geq |C_a(\mu_a)|$. Therefore,

$$\sum_{a \in O} |\mu \wedge \nu_a| \geq \sum_{a \in O} |\mu_a| \quad (3)$$

Note that for any assignment λ we have

$$\sum_{i \in A} |\lambda_i| = \sum_{a \in O} |\lambda_a|. \quad (4)$$

Combining the three equations yields that $\sum_{i \in A} |\mu \wedge \nu_i| = \sum_{i \in A} |\mu_i|$. Since $|\mu \wedge \nu_i| = 1$ implies that $|\mu_i| = 1$, it must also be that $|\mu_i| = 1$ implies that $|\mu \wedge \nu_i| = 1$. Similarly, since $|\mu \wedge \nu_a| \geq |\mu_a|$ for every school a and $\sum_{a \in O} |\mu \wedge \nu_a| = \sum_{a \in O} |\mu_a|$, it must be that for every school a , $|\mu_a| = |\mu \wedge \nu_a|$. \square

An immediate corollary of Lemma 6 is our version of the Rural Hospital Theorem (where hospitals correspond to schools in our context).¹⁸ The Rural Hospital Theorem is an important result for the residency matching program (Roth and Sotomayor, 1992). It says that under any stable assignment, each hospital receives the same number of doctors. It turns out that this result holds far more generally than when it is just applied to stable assignments. In any two individually rational assignments which do not block each other, each school is assigned the same number of students.

Corollary 1 (Rural Hospital Theorem). *Let μ and ν be two individually rational assignments which do not block each other. Then*

(i) *for any school a , $|\mu_a| = |\nu_a|$; and*

¹⁷Note that substitutability and LAD of C_a imply IRC: for all $X \subseteq Y$, if $C_a(Y) \subseteq X$, then $C_a(X) = C_a(Y)$.

¹⁸One could also refer to this as the ‘‘Rural Schools Theorem’’ in our context with the appropriate interpretation.

(ii) for any student i , $\mu_i = i$ if and only if $\nu_i = i$.

Proof. By Lemma 6, $|\mu_a| = |\mu \wedge \nu_a| = |\nu_a|$ (which implies (i)), and if $\mu_i \neq i$, then $\mu \wedge \nu_i \neq i$. If $\nu_i = i$, then by individual rationality of μ and ν , we have $\mu_i P_i i$, and by Lemma 6, $\mu \wedge \nu_i = \mu_i$. Thus, letting $\mu_i = a$, we have $i \in C_a(\mu_a \cup \nu_a)$ and by substitutability of C_a , $i \in C_a(\nu_a \cup \{i\})$, which implies that i blocks ν with μ , a contradiction. \square

Lemma 6 allows us to strengthen the Pointing Lemma.

Corollary 2 (Strong Pointing Lemma). *Let μ and ν be two individually rational assignments which do not block each other.*

(i) *If a student is assigned a school under either μ or ν , then she points to one school and is pointed to by one school.*

(ii) *For any school a , a points to $|\mu_a| = |\nu_a|$ students and $|\mu_a| = |\nu_a|$ students point to a .*

(iii) *Let $i \in A$ be such that $\mu_i = b$ and $\nu_i = a$. Then $i \in C_a(\mu_a \cup \{i\})$ if and only if $i \in C_a(\mu_a \cup \nu_a)$.*

Proof. (i): Consider a student i who is assigned a school under either μ or ν . By $\mu, \nu \in \mathcal{IR}$, i points to one school by strict preferences. By (ii) of Lemma 6, $\mu \wedge \nu_i \neq i$. Without loss of generality, $\mu \wedge \nu_i = \mu_i = a$. Since $i \in C_a(\mu_a \cup \nu_a)$, a points to i . Two schools cannot point to i , or else we would violate the Pointing Lemma.

(ii): This follows from the same counting exercise as in the proof of Lemma 6. If some school a had fewer than $|\mu_a|$ students pointing to it, then some school b would have to have more than $|\mu_b|$ students pointing to it. Then b would have to point to one of these students which would contradict the Pointing Lemma.

(iii): By substitutability of C_a , if $i \in C_a(\mu_a \cup \nu_a)$, then $i \in C_a(\mu_a \cup \{i\})$. In showing the other direction, suppose that $i \in C_a(\mu_a \cup \{i\})$ but $i \notin C_a(\mu_a \cup \nu_a)$. Because μ and ν do not block each other, we must have $b = \mu_i P_i \nu_i = a$ and i does not point to a . Thus, i points to $\mu_i = b$. Because $i \notin C_a(\mu_a \cup \nu_a)$, school a does not point to i . But then by (i), school b must point to i meaning $i \in C_b(\mu_b \cup \nu_b)$. Now by substitutability of C_b , we have $i \in C_b(\nu_b \cup \{i\})$. But then i blocks ν with μ , a contradiction. \square

We have already established that if we reassign each student to the school that is pointing to her that this results in a well-defined assignment. It is immediate from Corollary 2 that reassigning students to the school they are pointing to is an individually rational assignment. We refer to this assignment as $\mu \vee \nu$.

Definition 6. *Let μ and ν be two individually rational assignments which do not block each other. Define the assignment $\mu \vee \nu$ as follows: for all $i \in A$,*

$$\mu \vee \nu_i = \max_{P_i} \{\mu_i, \nu_i\}.$$

Lemma 7. *Let μ and ν be two individually rational assignments which do not block each other. Then $\mu \vee \nu$ is an individually rational assignment.*

Proof. First we show that for every school a , $C_a(\mu \vee \nu_a \cup \mu_a) = \mu_a$ (and symmetrically that $C_a(\mu \vee \nu_a \cup \nu_a) = \nu_a$). Suppose for contradiction that $C_a(\mu \vee \nu_a \cup \mu_a) \neq \mu_a$. Since μ is individually rational, we have $C_a(\mu_a) = \mu_a$. By the Law of Aggregate Demand, $|C_a(\mu \vee \nu_a \cup \mu_a)| \geq |\mu_a|$, so if $C_a(\mu \vee \nu_a \cup \mu_a) \neq \mu_a$, there must exist $i \in C_a(\mu \vee \nu_a \cup \mu_a)$ such that $i \notin \mu_a$. Therefore, $\mu \vee \nu_i = a$ and $\nu_i = a$. In words, since $\mu \vee \nu_i = a$, i prefers $\nu_i = a$ to μ_i . Since $i \in C_a(\mu \vee \nu_a \cup \mu_a)$, by substitutability of C_a , $i \in C_a(\mu_a \cup \{i\})$. Therefore, i blocks μ with ν which is a contradiction.

Second we prove the lemma. By construction, each student is assigned to only one school, and by individual rationality of μ and ν we have $\mu \vee \nu_i R_i i$. We must show that for every school a , $C_a(\mu \vee \nu_a) = \mu \vee \nu_a$. By definition, $C_a(\mu \vee \nu_a) \subseteq \mu \vee \nu_a$. Suppose $\mu \vee \nu_i = a$ and assume without loss of generality that $\mu_i = a$. We have already shown that $C_a(\mu \vee \nu_a \cup \mu_a) = \mu_a$. Since $i \in \mu_a$, $i \in C_a(\mu \vee \nu_a \cup \mu_a)$. Therefore, by substitutability of C_a and $i \in \mu \vee \nu_a$, $i \in C_a(\mu \vee \nu_a)$. \square

We conclude by showing that the Classical Decomposition Lemma generalizes to our environment. In the classical formulation, the Decomposition Lemma asks the men and women “Do you prefer μ or ν ?”. We do not know the preferences of the schools but instead know their choice functions. The analogous question (for the students) in choice language is “Do you choose your assignment under μ or ν ?”. Note that by construction, student i ’s answer is $\mu \vee \nu_i$. We cannot ask a school “Do you choose μ or ν ?” since we do not know the schools preferences. However, we can ask them the following question: “Which students do you choose among all the students you were assigned?” Note that by construction, school a ’s answer is $\mu \wedge \nu_a$. Our generalization of the Classical Decomposition Lemma is to show that there is a one-to-one mapping between the two answers.

Lemma 8 (Generalized Decomposition Lemma). *Let μ and ν be two individually rational assignments which do not block each other, and let i be a student such that $\mu_i \neq \nu_i$. Student i chooses school a if and only if school a rejects i . Formally, $\mu \vee \nu_i = a$ if and only if $i \notin \mu \wedge \nu_a = C_a(\mu_a \cup \nu_a)$.*

Proof. Suppose that $\mu_i \neq \nu_i$ and without loss of generality assume that i points to $\mu_i = a$. If i is not rejected by a ($i \in \mu \wedge \nu_a$), then a points at i . This contradicts the Weak Pointing Lemma which says that a student and a school cannot point at each other. Similarly, suppose that $\mu_i = a$ but that school a rejects i ($i \notin \mu \wedge \nu_a$). By the Strong Pointing Lemma, some school points at i . It must be that ν_i points at i . Since a school and a student cannot point at each other and ν_i points at i , it follows that i points at $\mu_i = a$. \square

3.2 Lattice Theorem

Our goal is to eventually show that the set of legal assignments is a lattice. We first show that a (weakly) larger set of assignments is a lattice.¹⁹ As a reminder, we defined $S^0 = \emptyset$ (and thus, $\pi(\emptyset) = \mathcal{IR}$), and in general let $S^k = \pi^2(S^{k-1})$ and $B^k = \pi(S^k)$. Since π^2

¹⁹We will in fact show that it is not a larger set but instead is the same set.

is increasing, eventually $S^n = S^{n+1}$ for some n . The two key properties of S^n are (1) $S^n \subseteq \pi(S^n)$ (for any two assignments $\mu, \nu \in S^n$, μ and ν do not block each other); and (2) $S^n = \pi^2(S^n)$ (if $\mu \notin S^n$, then μ is blocked by an assignment in $\pi(S^n)$).

So far we have only compared individually rational assignments which do not block each other. Next we strengthen our results by considering the additional structure inherent in S^n . We will show that S^n is a lattice under the following partial order which was inspired by Blair (1988) and Martinez et al. (2001).²⁰ Strikingly, our results are analogous to the properties of the stable set of assignments (Roth and Sotomayor, 1990) and the set of individually rational assignments that eliminate justified envy (Wu and Roth, 2016).

$$\mu \geq \nu \text{ if for every school } a \in O, C_a(\mu_a \cup \nu_a) = \nu_a \quad (5)$$

The following result from Blair (1988) will be useful.²¹

Lemma 9 (Blair 1988, Proposition 2.3). *For all $X, Y \in 2^A$ and all $a \in O$, $C_a(X \cup Y) = C_a(C_a(X) \cup Y)$.*

Proof. Let $x \in C_a(X \cup Y)$. If $x \in C_a(X) \cup Y$, then by substitutability of C_a we have $x \in C_a(C_a(X) \cup Y)$. If $x \notin C_a(X) \cup Y$, then $x \in X \setminus C_a(X)$. But this contradicts substitutability of C_a as $x \in C_a(X \cup Y)$ and $x \in X$ imply $x \in C_a(X)$. Thus, $C_a(X \cup Y) \subseteq C_a(C_a(X) \cup Y)$.

Because $C_a(X) \subseteq X$, LAD implies $|C_a(X \cup Y)| \geq |C_a(C_a(X) \cup Y)|$. Hence, $C_a(X \cup Y) = C_a(C_a(X) \cup Y)$. \square

Lemma 10. *Let μ and ν be two individually rational assignments which do not block each other. Then*

$$\mu \vee \nu \geq \mu \geq \mu \wedge \nu.$$

Proof. Let $a \in O$. By definition, $\mu \wedge \nu_a = C_a(\mu_a \cup \nu_a)$. Therefore:

$$\begin{aligned} C_a(\mu_a \cup (\mu \wedge \nu_a)) &= C_a(\mu_a \cup C_a(\mu_a \cup \nu_a)) \\ &= C_a(\mu_a \cup \mu_a \cup \nu_a) \\ &= C_a(\mu_a \cup \nu_a) \\ &= \mu \wedge \nu_a \end{aligned}$$

where the second equality follows from Lemma 9. Therefore, $\mu \geq \mu \wedge \nu$ (and of course, by symmetry, $\nu \geq \mu \wedge \nu$).

In showing $\mu \vee \nu \geq \mu$, suppose by contradiction that for some $a \in O$, $C_a(\mu_a \cup (\mu \vee \nu_a)) \neq \mu_a$. Because μ is individually rational, $C_a(\mu_a) = \mu_a$. Since $\mu_a \subseteq \mu_a \cup (\mu \vee \nu_a)$, LAD implies $|C_a(\mu_a \cup (\mu \vee \nu_a))| \geq |\mu_a|$. Thus, by $C_a(\mu_a \cup (\mu \vee \nu_a)) \neq \mu_a$, there exists $i \in C_a(\mu_a \cup (\mu \vee \nu_a)) \setminus \mu_a$. But then $\nu_i = a$ and $aP_i\mu_i$. By substitutability of C_a and $\mu_a \cup \{i\} \subseteq \mu_a \cup (\mu \vee \nu_a)$, we have $i \in C_a(\mu_a \cup \{i\})$ and i blocks μ with ν , a contradiction. Hence, $C_a(\mu_a \cup (\mu \vee \nu_a)) = \mu_a$ for all $a \in A$, and $\mu \vee \nu \geq \mu$. \square

²⁰Note that it is an immediate corollary of Tarski's Fixed Point Theorem that S^n is a lattice. However, we will be able to prove the stronger properties of S^n by using first principles.

²¹For completeness, we include its proof.

Lemma 11. *Let $S \subseteq \mathcal{IR}$ be such that (1) $S \subseteq \pi(S)$ and (2) $\pi^2(S) = S$. For any $\mu, \nu \in S$, $\mu \vee \nu \in S$ and $\mu \wedge \nu \in S$. In particular, S with partial order \geq is a lattice.*

Proof. Let $B = \pi(S)$. By assumption, $S \subseteq B$ and $S = \pi(B)$. Therefore, μ and ν are not blocked by any assignment in B , and in particular, μ and ν do not block each other. We have already shown that $\mu \vee \nu$ and $\mu \wedge \nu$ are well-defined assignments. Furthermore, by individual rationality of μ and ν and (ii) of Lemma 6, $\mu \wedge \nu_i R_i i$ for all $i \in A$, and by definition, $C_a(\mu \wedge \nu_a) = C_a(\mu_a \cup \nu_a) = \mu \wedge \nu_a$. Thus, $\mu \wedge \nu \in \mathcal{IR}$. By Lemma 7, $\mu \vee \nu \in \mathcal{IR}$. All that remains is to show that $\mu \vee \nu$ and $\mu \wedge \nu$ are not blocked by any assignment in B .

Suppose for contradiction that i blocks $\mu \wedge \nu$ with $\lambda \in B$. By individual rationality of $\mu \wedge \nu$, we have $\lambda_i \neq i$, say $\lambda_i = b$. If $\mu \wedge \nu_i = i$, then by (ii) of Lemma 6 we have $\mu_i = i$ and $\nu_i = i$. But then by substitutability of C_b and $i \in C_b(\mu \wedge \nu_b \cup \{i\}) = C_b(\mu_b \cup \nu_b \cup \{i\})$, we have $i \in C_b(\mu_b \cup \{i\})$. Because $i \notin \mu_b$, now i blocks μ with λ , a contradiction to $\mu \in S$. Thus, $\mu \wedge \nu_i \neq i$, say $\mu \wedge \nu_i = a$ and without loss of generality, assume $\mu_i = a$. Since i blocks $\mu \wedge \nu$ with b ,

$$i \in C_b(\mu \wedge \nu_b \cup \{i\}). \quad (6)$$

Note that $C_b(X \cup Y) = C_b(C_b(X) \cup Y)$ (Lemma 9). Therefore,

$$C_b(C_b(\mu_b \cup \nu_b) \cup \{i\}) = C_b(\mu_b \cup \nu_b \cup \{i\}). \quad (7)$$

By definition, $\mu \wedge \nu_b = C_b(\mu_b \cup \nu_b)$. Thus, by (6), $i \in C_b(C_b(\mu_b \cup \nu_b) \cup \{i\})$. By (7), $i \in C_b(\mu_b \cup \nu_b \cup \{i\})$. By substitutability of C_b , $i \in C_b(\mu_b \cup \{i\})$. Therefore, $bP_i \mu_i$, $i \in C_b(\mu_b \cup \{i\})$, and $\lambda_i = b$ where $\lambda \in B = \pi(S)$. Therefore, i blocks μ with λ implying that $\mu \notin \pi(B)$. This is a contradiction as $\mu \in S = \pi(B)$.

The proof for $\mu \vee \nu$ is similar. Suppose for contradiction that $\mu \vee \nu$ is blocked by student i , school a , and assignment $\lambda \in B$ where $\lambda_i = a$. We first show that there exists a student $j \in \mu \vee \nu_a$ who is rejected when a chooses from $\mu \vee \nu_a \cup \{i\}$, i.e. $j \notin C_a(\mu \vee \nu_a \cup \{i\})$. Note that $\mu \wedge \nu$ is not blocked by i and λ as otherwise $\mu \vee \nu_i P_i \mu \wedge \nu_i$ (Lemma 7) so i and a would block $\mu \wedge \nu$ with λ if $i \in C_a(\mu \wedge \nu_a \cup \{i\})$. Therefore, $i \notin C_a(\mu \wedge \nu_a \cup \{i\})$. Because $\mu \wedge \nu \in \mathcal{IR}$, we have $C_a(\mu \wedge \nu_a) = \mu \wedge \nu_a$. Thus, by LAD and substitutability of C_a , we have $C_a(\mu \wedge \nu_a \cup \{i\}) = C_a(\mu_a \cup \nu_a \cup \{i\}) = \mu \wedge \nu_a$. As a reminder, $|\mu_a| = |\mu \wedge \nu_a| = |\nu_a| = |\mu \vee \nu_a|$. By the Law of Aggregate Demand and $\mu \vee \nu \in \mathcal{IR}$,

$$|\mu \vee \nu_a| = |C_a(\mu \vee \nu_a)| \leq |C_a(\mu \vee \nu_a \cup \{i\})| \leq |C_a(\mu_a \cup \nu_a \cup \{i\})| = |\mu \wedge \nu_a| = |\mu \vee \nu_a|.$$

Now all these inequalities become equalities. Because $i \in C_a(\mu \vee \nu_a \cup \{i\})$ and $i \notin C_a(\mu_a \cup \nu_a \cup \{i\})$, there must exist $j \in \mu \vee \nu_a \setminus C_a(\mu \vee \nu_a \cup \{i\})$. Without loss of generality, $\mu_j = a$. Then $i \notin C_a(\mu_a \cup \{i\})$ or else i would block μ with λ . Because μ is individually rational, $C_a(\mu_a) = \mu_a$. Therefore, by LAD and substitutability of C_a ,

$$C_a(\mu_a \cup \{i\}) = \mu_a. \quad (8)$$

Note that

$$\begin{aligned} C_a(\mu_a \cup (\mu \vee \nu_a) \cup \{i\}) &= C_a(C_a(\mu_a \cup \mu \vee \nu_a) \cup \{i\}) \\ &= C_a(\mu_a \cup \{i\}) \\ &= \mu_a \end{aligned}$$

where the first equality follows from Lemma 9, the second equality follows from Lemma 5 ($\mu \vee \nu \geq \mu$ and therefore, $C_a(\mu_a \cup \mu \vee \nu_a) = \mu_a$), and the third inequality follows from (8). However, $j \in \mu_a$ and therefore $j \in C_a(\mu_a \cup (\mu \vee \nu_a) \cup \{i\})$. This contradicts substitutability of C_a as $j \notin C_a(\mu \vee \nu_a \cup \{i\})$ but $\mu \vee \nu_a \cup \{i\} \subseteq \mu_a \cup (\mu \vee \nu_a) \cup \{i\}$. \square

3.3 Existence and Uniqueness

We are now ready to prove the main theorem. As a reminder, we set $S^0 = \emptyset$, $S^1 = \pi^2(\emptyset)$, $S^k = \pi^2(S^{k-1})$ and $B^k = \pi(S^k)$. We defined S as the first fixed point of our construction, i.e. $S = \pi^2(S)$. Let $B = \pi(S)$. By Lemma 11, S is a lattice, and we may let μ^I denote the student-optimal assignment in S and μ^O denote the school-optimal assignment in S .

In the Appendix we establish that any such fixed point must be a legal set of assignments.

Theorem 1. *There exists a legal set of assignments.*

We can now prove that there exists a unique legal set of assignments.

Theorem 2. *There exists a unique legal set of assignments.*

Proof. By Lemma 2, L is a legal set of assignments if and only if $\pi(L) = L$.

Let $S \subseteq \mathcal{IR}$ be such that (1) $S \subseteq \pi(S)$ and (2) $S = \pi^2(S)$. By the proof of Theorem 1, we have $S = \pi(S)$. Thus, S is legal.

To show uniqueness, let L be any legal set of assignments. By (iii) of Lemma 1, $S^1 \subseteq \pi(L) = L$. By (i) of Lemma 1, π is decreasing. Therefore, $\pi(L) = L \subseteq \pi(S^1) = B^1$. Repeating this argument, we find that

$$S^0 \subseteq S^1 \subseteq \dots S^n \subseteq L \subseteq B^n \subseteq \dots B^1 \subseteq B^2.$$

Since there exists n such that $S^n = B^n$, we conclude that $L = S^n$. \square

4 Discussion

4.1 Efficiency and Non-Wastefulness

First, we discuss various properties of the student-optimal legal assignment. Because any individually rational assignment outside L is illegal, it must be that μ^I is not Pareto dominated by any individually rational assignment.

Proposition 1. *The student-optimal legal assignment μ^I is efficient among all individually rational assignments.*

Proof. Suppose that there exists $\nu \in \mathcal{IR}$ such that for all $i \in I$, $\nu_i R_i \mu_i^I$ and for some $j \in I$, $\nu_j P_j \mu_j^I$. By Lemma 13 (in the Appendix) and $L = S$, $\nu \notin L$. Since ν is illegal, there exists $\mu \in L$ which blocks ν . Thus, for some $i \in A$ we have $\mu_i P_i \nu_i R_i \mu_i^I$. But again by Lemma 13 (in the Appendix), $\mu_i^I R_i \mu_i$, which is a contradiction to transitivity of P_i . \square

The deferred-acceptance (DA) assignment is the student-optimal stable assignment and it is found by the (student-proposing) deferred-acceptance (DA) algorithm.²² To the best of our knowledge, Kesten's efficiency adjusted DA (EADA) has only been defined for responsive choice functions. Kesten's original EADA mechanism and the simplified EADA mechanism (hereafter sEADA) introduced by Tang and Yu (2014) produce the same assignment when schools have responsive choice functions. The sEADA is based on the concept of an underdemanded school. For a given assignment μ , a school a is *underdemanded* if for every student i , $\mu_i R_i a$. Tang and Yu (2014) note two facts that are critical for their mechanism. First, under the DA assignment, there is always an underdemanded school. For example, the last school that any student applies to is an underdemanded school. Second, a student assigned by DA to an underdemanded school cannot be part of a Pareto improvement. However, as Example 2 demonstrates, when choice functions are not responsive, there does not have to exist an underdemanded school. In this case, Tang and Yu's algorithm no longer produces a Pareto efficient assignment.

Example 2. Let $O = \{a, b, c, d\}$ and $A = \{1, 2, 3, 4, 5\}$, and suppose $q_a = 2$ while all other schools have a capacity of 1. Suppose the preferences of the students and the priorities of the schools (other than a) are defined as below:

R_1	R_2	R_3	R_4	R_5	\succ_b	\succ_c	\succ_d
b	a	a	c	d	3	2	4
a	c	b	d	a	1	4	5

School a has more complicated preferences. Intuitively, a chooses at most one student from students 1, 2, and 3 (where the students are ranked \succ_a : 1, 2, 3) and at most one student from 4 and 5 (where \succ'_a : 4, 5). More formally, given a set of students X ,

$$C_a(X) = (\max_{\succ_a} X \cap \{1, 2, 3\}) \cup (\max_{\succ'_a} X \cap \{4, 5\})$$

Note that C_a is substitutable and satisfies LAD. However, the DA assignment is:

$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ a & c & b & d & a \end{pmatrix}$$

However, there is no underdemanded school as 2 would prefer a , 1 would prefer b , 4 would prefer c , and 5 would prefer d . Further, the DA assignment is Pareto dominated by the following assignment:

$$\nu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ b & c & a & d & a \end{pmatrix}$$

It is straightforward to verify that ν is legal.²³

²²The proof of Lemma 13 contains a formal description of the DA-algorithm.

²³To see that it is not blocked by any legal assignment, note that the only student with justified envy is 2. However, if 2 is assigned to a , then 1 must be assigned to b or else 1 will block with the DA assignment. But if 1 is assigned to b , then 3 must be assigned to a or else she will block with the DA assignment. However, it is not individually rational to assign both 2 and 3 to a .

The following natural condition on choice functions guarantees that an underdemanded school will exist. School a 's choice rule is q -*acceptant* if $|C_a(X)| = \min\{q, |X|\}$ for all $X \in 2^A$. A choice rule is *acceptant* if it is q -acceptant for some q . The null assignment is underdemanded, so if a student is left unassigned by DA then there is an underdemanded assignment. Otherwise, DA ends when each student that applies to a new school is accepted. In particular, in this final round, each school accepts a student without rejecting another. If such a school has an acceptant choice function, then it has never rejected a student. If a school never rejects a student under DA, then it is underdemanded. Therefore, sEADA generalizes in a natural way when choice functions that are substitutable and acceptant.²⁴

The (simplified) Efficiency Adjusted Deferred Acceptance Mechanism (sEADA) when choice functions are acceptant

Round 0: Run DA on the full population. For each underdemanded school²⁵ a and each student i assigned to a , permanently assign i to a and then remove both i and a .

Round k : Run DA on the remaining population. For each underdemanded school a and each student i assigned to a , permanently assign i to a and then remove both i and a .²⁶

Lemma 12. *Let $\mu = DA(P)$ and suppose μ_i is underdemanded. Then for any individually rational and non-wasteful assignment ν such that $\nu_i P_i \mu_i$, μ blocks ν .*

Proof. Let ν be an individually rational and non-wasteful assignment such that $\nu_i P_i \mu_i$. Suppose for contradiction that μ does not block ν . Since μ has no justified envy, ν does not block μ . Therefore, the Pointing Lemma holds. Let $a = \mu_i$. Since a is underdemanded, for every $j \notin \mu_a$, $\mu_j P_j a$. In particular, if $j \in \nu_a \setminus \mu_a$, then j does not point to a . Furthermore, i does not point to a since by assumption $\nu_i P_i \mu_i = a$. Therefore, the set of students pointing to a is contained in $\mu_a \setminus \{i\}$. In particular, the number of students that point to a is strictly less than $|\mu_a|$. However, this contradicts the Pointing Lemma which says that if μ and ν do not block each other, then $|\mu_a| = |\nu_a|$ many students must point to a . Therefore, μ blocks ν . \square

²⁴To be precise, this is the definition of sEADA when all students consent to allowing their priority to be violated. It is straightforward to modify the algorithm if some students do not consent or even if students consent to some schools but not others.

²⁵Note that a student may also be unassigned. For expositional convenience, we interpret being unassigned as being assigned to the null school which has unlimited capacity. Since the DA assignment is individually rational, every student weakly prefers her assignment to being unassigned. Therefore, the null school is underdemanded.

²⁶We refer the reader to Kesten (2010) for the precise formulation of the mechanism, but for the reader who is already familiar with the mechanism, we briefly discuss the difficulty in generalizing EADA directly. EADA removes interrupters individually. Specifically, it runs DA and identifies the last interrupter pair (i, s) , *i.e.* student i is an interrupter at school s . The mechanism then removes s from i 's preference list and then reruns DA on the modified preferences. However, when preferences are not responsive, removing i from s 's choice set can have (at least theoretically) a significant impact on the students s chooses. In contrast, sEADA removes the underdemanded school and all students assigned to it. Therefore, whether that school has a general choice function is irrelevant to the definition.

Theorem 3. *Suppose each school has an acceptant choice function. The student-optimal legal assignment and sEADA coincide.*

Proof. Because the student-optimal legal assignment is the unique efficient legal assignment, it suffices to show that sEADA is legal.

We define an algorithm that is equivalent to sEADA but where instead of removing students, we change their preferences. Set $P^1 = P$, and let $\mu^1 = DA(P^1)$. We call a student i underdemanded if i is assigned to an underdemanded school. Let U^1 denote the set of underdemanded students under μ^1 . Define preference profile P^2 as follows: if $i \in U^1$, then move μ_i^1 to the top of i 's preference list P_i^2 ; and if $i \notin U^1$, then leave i 's preferences unchanged (i.e. $P_i^2 = P_i^1$). In general, given P^k , let $\mu^k = DA(P^k)$. Let U^k denote the set of underdemanded students under μ^k , and we modify P^k to create P^{k+1} as follows: if $i \in U^k$, then move μ_i^k to the top of i 's preference P_i^{k+1} ; and if $i \notin U^k$, then leave i 's preferences unchanged (i.e. $P_i^{k+1} = P_i^k$). Note that $U^k \subseteq U^{k+1}$ since if a student does not desire school a under her true preferences, then the student does not desire school a when we move her assignment to the top of her preference list. The process stops once all students are underdemanded. It is straightforward to verify that this is equivalent to the sEADA procedure.

Observe that by construction, we have $\mu_i^k R_i \mu_i^{k-1}$ for all $i \in A$ and all $k \in \mathbb{N}$.

We prove by induction that for each integer k , (a) μ^k is legal and (b) if $i \in U^k \setminus U^{k-1}$ and ν is an assignment such that $\nu_i P_i \mu_i^k$, then ν is illegal.²⁷ By definition, μ^1 is the DA assignment which is stable. Therefore $\mu^1 \in S^1$. Lemma 12 establishes part (b) of the base step.

Let $k > 1$. Note that if $U^k = U^{k-1}$ then $\mu^k = \mu^{k-1}$ and the result holds trivially. Now suppose $U^k \setminus U^{k-1} \neq \emptyset$. First, we show (a), i.e. μ^k is legal. If i blocks μ^k with some other assignment under P , then $P_i^k \neq P_i$ (since μ^k is stable under P^k). Therefore, by construction $i \in U^{k-1}$. Now there exists $l \in \{1, \dots, k-1\}$ such that $i \in U^l \setminus U^{l-1}$. However, if $i \in U^l$ and ν is an assignment such that $\nu_i P_i \mu_i^k$, then by $\mu_i^k R_i \mu_i^l$ we have $\nu_i P_i \mu_i^l$. But then by the inductive hypothesis for l , ν is illegal. Since μ^k can only be blocked by illegal assignments, μ^k is legal.

Next, we show part (b) of the inductive hypothesis. Our argument is analogous to the proof of Lemma 12. Let $i \in U^k \setminus U^{k-1}$, and let ν be any assignment such that $\nu_i P_i \mu_i^k$. Let $a = \mu_i^k$. By definition of U^k , a is an underdemanded school. Specifically, for every student j such that $\mu_j^k \neq a$, $\mu_j^k P_j^k a$. Suppose for contradiction that ν is legal. Since we have already shown that μ^k is legal, ν and μ^k do not block each other. We will show that there are not enough students who point to $a = \mu_i^k$. By assumption, $\nu_i P_i \mu_i^k$; therefore, i does not point to $a = \mu_i^k$. Consider any student $j \in \nu_a \setminus \mu_a^k$. Since a is underdemanded under P^k , $\mu_j^k P_j^k a$. If $P_j^k = P_j$, then $\mu_j^k P_j a$ and j does not point to a . If $P_j^k \neq P_j$, then $j \in U^{k-1}$ (j was an underdemanded student in an earlier round). By the inductive hypothesis, any assignment she strictly prefers to μ^k (relative to her true preferences) is illegal. Since ν is legal, it must be that $\mu_j^k P_j \nu_j = a$. Therefore, j does not point to a . Therefore, no student $j \in \nu_a \setminus \mu_a^k$ points to a . Since $i \in \mu_a^k$ does not point to a , the number of students who point to a is

²⁷We set $U^0 = \emptyset$.

smaller than $|\mu_a^k|$. However, this contradicts the Pointing Lemma. Therefore, ν must be illegal. \square

Remark 1. *First, by Theorem 3, for responsive choice functions, the student-optimal legal assignment and EADA coincide. Thus, the student-optimal legal assignment and the student-optimal “possible” assignment by Morrill (2017) coincide with the assignment made by EADA. Second, we generalize the sEADA by Tang and Yu (2014) from responsive choice functions to substitutable and acceptant choice functions. Third, the student-optimal legal assignment offers a foundation for the extension of Kesten’s EADA from responsive choice functions to choice functions satisfying substitutability and LAD.²⁸*

It is well-known that the student-optimal stable assignment is weakly efficient among all individually rational assignments. Hence, (i) of Proposition 1 describes the important advantage of the student-optimal legal assignment over the student-optimal stable assignment.

As the example below shows, efficiency of the student-optimal legal assignment is not guaranteed when Pareto domination is allowed via non-individually rational assignments (and as it is known, the student-optimal stable assignment is not necessarily weakly efficient). Furthermore, the example establishes that non-individually rationally assignments are not necessarily blocked by legal assignments, and the Pointing Lemma may be violated.

Example 3. Let $A = \{1, 2\}$, $O = \{a, b\}$, $P_1 : a1b$ (where this stands for aP_11P_1b), $P_2 : b2a$, $\succ_a : 2a1$ and $\succ_b : 1b2$ (where this stands for $C_b(\{1\}) = C_b(\{1, 2\}) = \{1\}$ and $C_b(\{2\}) = \emptyset$). Let μ^0 be such that $\mu_1^0 = 1$ and $\mu_2^0 = 2$. Then $\mathcal{IR} = \{\mu^0\}$ and $L = \{\mu^0\}$, and μ^0 is the unique stable assignment. Considering μ such that $\mu_1 = a$ and $\mu_2 = b$ we see that μ^0 is not (weakly) efficient. In addition, μ and μ^0 do not block each other but the pointing lemma is violated for these two assignments: 1 and 2 would point to a school but no school would point to a student.

One would expect legal assignments to be non-wasteful. The following example shows that non-wasteful assignments may be legal. Of course, by Proposition 1, the student-optimal legal assignment is non-wasteful (as otherwise it would not be efficient among individually rational assignments).

Example 4. Let $A = \{1, 2\}$, $O = \{a, b, c\}$, $P_1 : bca1$, $P_2 : acb1$, $\succ_a : 12a$, $\succ_b : 21b$, and $\succ_c : 12c$. Letting $\mu'_1 = b$ and $\mu'_2 = a$, it is easy to see that μ' is the only stable assignment. Letting $\mu''_1 = a$ and $\mu''_2 = b$, it is obvious that μ' and μ'' do not block each other. Thus, $\mu'' \in L$ and $L = \{\mu', \mu''\}$. Letting $\nu_1 = c$ and $\nu_2 = a$, we can see that 1 blocks ν with μ' (and $\nu \notin L$). But then $\mu'' \in L$ is wasteful because $bP_1\mu''_1$ and $C_b(\mu''_b \cup \{1\}) = C_b(\{1\}) = \{1\}$.

Non-wastefulness allows for blocking of students and “empty” slots (in the sense that adding a student to a school would result in the choice of this student and all previously assigned students). However, as we show below, legal assignments satisfy a weaker property

²⁸Note that it is even not clear what the right formulation of Kesten’s EADAM is for these environments.

of non-wastefulness (where blocking is only allowed with unassigned students and “empty” slots): μ is **weakly non-wasteful** if there exist no student i and school a such that $\mu_i = i$, $aP_i i$ and $C_a(\mu_a \cup \{i\}) = \mu_a \cup \{i\}$.

Proposition 2. *If μ is legal ($\mu \in L$), then μ is weakly non-wasteful.*

Proof. Let $\mu \in L$. Suppose that μ is weakly wasteful. Then there exists a student i and a school a such that $\mu_i = i$, $aP_i i$ and $C_a(\mu_a \cup \{i\}) = \mu_a \cup \{i\}$. Let μ' be such that $\mu'_i = a$ and $\mu'_j = \mu_j$ for all $j \in A \setminus \{i\}$. Then by the previous facts and $\mu \in \mathcal{IR}$, it follows that $\mu' \in \mathcal{IR}$. Since $|\mu'_a| = |\mu_a| + 1$ and the Rural Hospitals Theorem holds for all assignments in L , we have $\mu' \notin L$. Thus, there exist $j \in A$ and $\nu \in L$ such that j blocks μ' with ν . Thus, $\nu_j P_j \mu'_j$ and (letting $\nu_j = b$) $j \in C_b(\mu'_b \cup \{j\})$. Since $\mu_b \subseteq \mu'_b$, substitutability of C_b implies $j \in C_b(\mu_b \cup \{j\})$. If $j \neq i$, then $\mu'_j = \mu_j$ and j blocks μ with ν , a contradiction to $\mu, \nu \in L$. If $j = i$, then $bP_i aP_i i$ and by $\mu_i = i$, i blocks μ with ν , again a contradiction to $\mu, \nu \in L$. \square

4.2 Strategy-Proofness

Below we consider centralized mechanism design where students have to report their preferences to the clearinghouse. We keep everything fixed except for students’ preferences. Let \mathcal{P}^i denote the set of all i ’s strict preferences over $O \cup \{i\}$, and $\mathcal{P}^A = \times_{i \in A} \mathcal{P}^i$. Let \mathcal{M} denote the set of all assignments.

A mechanism is a function $\varphi : \mathcal{P}^A \rightarrow \mathcal{M}$ choosing for profile P assignment $\varphi(P)$. Then φ is strategy-proof if for all $i \in A$, all $P \in \mathcal{P}^A$ and all $P'_i \in \mathcal{P}^i$ we have $\varphi_i(P) R_i \varphi_i(P'_i, P_{-i})$. This means that reporting the truth is a weakly dominant strategy. A mechanism is legal if for all profiles P , $\varphi(P)$ is a legal assignment.

Let DA denote the student-proposing deferred-acceptance mechanism.

Theorem 4. *DA is the unique strategy-proof and legal mechanism.*

Proof. Because DA is stable, we have that DA is legal. Strategy-proofness of DA has been established by Roth (1982) and Dubins and Freedman (1982).

In showing the converse, let φ be strategy-proof and legal. We show that for all $P \in \mathcal{P}^A$ and all $i \in A$, $\varphi_i(P) R_i DA_i(P)$. Suppose not. Then there exists $i \in A$ such that $DA_i(P) P_i \varphi_i(P)$. Thus, by individual rationality, $DA_i(P) \neq i$, say $DA_i(P) = a$. Let $P'_i \in \mathcal{P}^i$ be such that for all $b \in O$, (i) if $bR_i a$, then $bR'_i a$ and (ii) if $aP_i b$, then $aP'_i i P'_i b$. By construction, stability of $DA(P)$ under P implies stability of $DA(P)$ under (P'_i, P_{-i}) . Thus, $DA(P)$ is legal under (P'_i, P_{-i}) . Then by the rural hospitals theorem of legal assignments, we have $\varphi_i(P'_i, P_{-i}) \neq i$. Thus, by construction of P'_i ,

$$\varphi_i(P'_i, P_{-i}) R_i a P_i \varphi_i(P),$$

which implies that φ is not strategy-proof, a contradiction.

Hence, we have shown for all $P \in \mathcal{P}^A$ and all $i \in A$, $\varphi_i(P) R_i DA_i(P)$. If $\varphi \neq DA$, then φ must Pareto-dominate DA . This is a contradiction to Abdulkadirođlu, Pathak and Roth (2009) who show that no strategy-proof mechanism can Pareto-dominate DA . \square

Thus, by Theorem 4, any strategy-proof mechanism different than DA must be illegal. In particular, the top-trading cycles algorithm is illegal (and it is easy to see that all variants of the Boston mechanism are illegal).²⁹

4.3 Assignment with Contracts

Recall that A denotes the set of students and O denotes the set of schools. Let \mathcal{X} denote the set of all contracts. Each contract $x \in \mathcal{X}$ is associated with one student $x_A \in A$ and one school $x_O \in O$. Given $Y \subseteq \mathcal{X}$, let Y_i denote the set of contracts associated with student i and Y_a denote the set of contracts associated with school a .

Each student i has a strict preference P_i over $\mathcal{X}_i \cup \{i\}$. Let C_i denote the choice function induced by P_i : for any $Y \subseteq \mathcal{X}$, let $C_i(Y) = \max_{P_i} Y_i \cup \{i\}$.

Any school a has a choice function $C_a : 2^{\mathcal{X}} \rightarrow 2^{\mathcal{X}}$ such that for any $Y \subseteq \mathcal{X}$ we have $C_a(Y) \subseteq Y_a$. Substitutability and LAD are straightforward to adapt to the setup with contracts.

A **(feasible) assignment** is a set of contracts, $\mu \subseteq \mathcal{X}$, such that each student signs only one contract: for each student i , $|\mu_i| \leq 1$. An assignment is **individually rational** if for all $i \in A$, $\mu_i = C_i(\mu)$ and for all $a \in O$, $C_a(\mu) = \mu_a$. Let \mathcal{IR} denote the set of all individually rational assignments. Given assignment μ , student i and school a **block** μ via contract x if $x P_i \mu_i$ and $x \in C_a(\mu \cup \{x\})$ (where this implies $x_A = i$ and $x_O = a$). An assignment μ is **non-wasteful** if there do not exist i and a and a contract x such that $x P_i \mu_i$ and $C_a(\mu_a \cup \{x\}) = \mu_a \cup \{x\}$. An assignment is **fair** if there do not exist i and a and a contract x such that $x P_i \mu_i$ and $x \in C_a(\mu_a \cup \{x\}) \neq \mu_a \cup \{x\}$. Now blocking among assignments carries over in a straightforward fashion: i blocks μ with ν if for some $x \in \mathcal{X}_i$, (1) $x P_i \mu_i$, (2) $x \in C_a(\mu_a \cup \{x\})$ and (3) $\nu_i = x$. Then μ blocks ν if there exists a student i who blocks μ with ν .

Now $L \subseteq \mathcal{IR}$ is a legal set of assignments if and only if (i) for all $\nu \in \mathcal{IR} \setminus L$ there exists $\mu \in L$ such that μ blocks ν and (ii) for all $\mu, \nu \in L$, μ does not block ν . The operator π is defined in the same way as in the main text, and its properties carry over without change, namely Lemma 1, Lemma 2, and that there exists n such that (1) $S^n \subseteq \pi(S^n)$ and (2) $S^n = \pi^2(S^n)$.

Regarding pointing, we let students and schools point to contracts instead of pointing to schools and students. Given two assignments μ and ν , student i points to μ_i (ν_i) if $\mu_i R_i \nu_i$ ($\nu_i R_i \mu_i$) and school a points to $x \in \mathcal{X}$ if $x \in C_a(\mu_a \cup \nu_a)$. Then Lemma 5 (Weak Pointing Lemma) carries over in the following way: let μ and ν be two individually rational assignments which do not block each other. Then (i) no student and school point to the same contract unless the contract belongs to μ and ν and (ii) no two schools point to contracts which are associated with the same student.

Now with these modifications, it is easy to see that all our results and proofs continue to hold in the assignment with contracts framework where schools' choice functions satisfy substitutability and LAD. This means that (i) there exists a unique legal set of assignments,

²⁹One may also use Alva and Manjunath (2016) to show Theorem 4.

(ii) this set is a lattice and (iii) there exists a student-optimal legal assignment which is efficient and which provides a foundation of the generalization of Kesten’s EADAM to assignment with contracts. We show these results in Appendix B.

5 Conclusion

When a school board chooses an assignment mechanism, it typically balances strategy-proofness, efficiency, and fairness. However, a critical pragmatic consideration for any board is which of the possible assignments are legal. We show that there is a unique set of legal assignments, and that there is always a unique Pareto efficient assignment that is legal. Prior to our work, it was thought that there was no “silver bullet” solution to the school assignment problem as it is impossible for a mechanism to be both efficient and eliminate justified envy (Abdulkadiroğlu and Sönmez, 2003). However, we show that the only envy of a legal assignment is either unjustified or else is petty. Therefore, the set of legal assignments satisfy a natural interpretation of fairness. Combined, our results offer a foundation of the generalization of the assignment made by Kesten’s EADA from responsive choice functions to our general framework. One may see this as the ideal school assignment. It is the unique assignment that is legal and Pareto efficient. It is fair in a meaningful way, and it Pareto dominates any other fair or legal assignment.

APPENDIX.

A Proof of Theorem 1

Let $S \subseteq \mathcal{IR}$ be such that (1) $S \subseteq \pi(S)$ and (2) $\pi^2(S) = S$. By Lemma 11, S is a lattice. Let μ^I be the student-optimal assignment in S and let μ^O be the school optimal assignment in S . The key step for the proof of Theorem 1 is to show that any individually rational assignment which is not blocked by S , must lie in between μ^I and μ^O with respect to students’ preferences.

Lemma 13. *For every $\lambda \in \pi(S)$ and every student i , $\mu_i^I R_i \lambda_i R_i \mu_i^O$.*

Proof. Let S be a set that satisfies (1) and (2) and let $B = \pi(S)$. We say that school a is **possible** for i if there exists a $\lambda \in B$ such that $\lambda_i = a$. Let

$$B(i) = \{a \in O \mid \text{there exists } \lambda \in B \text{ such that } \lambda_i = a\}$$

denote the set of possible schools for student i . Let \hat{P}_i be defined as follows: (i) for all $a \in B(i)$ and $b \in O \setminus B(i)$, $a\hat{P}_i i \hat{P}_i b$, (ii) for all $a, b \in B(i)$, $a\hat{P}_i b \Leftrightarrow aP_i b$ and (iii) for all $a, b \in O \setminus B(i)$, $a\hat{P}_i b \Leftrightarrow aP_i b$. Now we introduce a natural modification of DA which we call rDA (restricted DA): when we run DA we only allow a student to apply to a possible school and we use the profile $(\hat{P}_i)_{i \in N}$ for students to apply to schools. Formally, the rDA is defined as follows:

Step 1: Each student i proposes to his most \hat{P}_i -preferred acceptable school. Let X_a^1 denote the proposals received by school a . Then school a tentatively accepts $C_a(X_a^1)$ and rejects $X_a^1 \setminus C_a(X_a^1)$.

Step t : Any student i rejected in Step $t - 1$ proposes to his most \hat{P}_i -preferred acceptable school among the ones which did not reject i (if there is no acceptable school left for i , then i does not make any proposal). Let X_a^t denote the set of proposals received by school a and the ones tentatively accepted by a in the previous step. Then school a tentatively accepts $C_a(X_a^t)$ and rejects $X_a^t \setminus C_a(X_a^t)$.

Stop: There are no rejected students or all rejected students have applied to all acceptable schools. Then the tentative acceptances become final assignments, which we denote by μ^I .

Note that μ^I is stable under \hat{P} , which implies $\mu^I \in S$.

We establish the result by showing that no student is rejected under rDA. This implies that for each student i , μ_i^I is i 's favorite possible school (or equivalently, μ_i^I is i 's most \hat{P}_i -preferred school).

If a student was rejected, then there would have to be a last student rejected. Call this student i and the school that rejected her a . Note that a must be possible for i , so there exists a $\nu \in B$ such that $\nu_i = a$. Because $\nu \in B$ and $\mu^I \in S$, ν and μ^I do not block each other. Thus, by the Rural Hospital Theorem, $\mu_i^I \neq i$. Let $\mu_i^I = b$. Let $X = \{j \in A \mid b \hat{R}_j \mu_j^I\}$ (in words, X is the set of students j such that b is possible for j and b is weakly preferred to her assignment under rDA). By construction and stability of μ^I under \hat{P} , $\mu_b^I = C_b(X)$. When i proposes to b , no student is rejected (since i is the last student rejected). Therefore, by substitutability of C_b ,

$$C_b(X \setminus \{i\}) = \mu_b^I \setminus \{i\}. \quad (9)$$

Since μ^I and ν do not block each other, by Lemma 6, $\nu' = \mu^I \vee \nu$ is an individually rational assignment. By the Strong Pointing Lemma, $|\nu'_b| = |\mu_b^I|$ (ν'_b is the set of students pointing at b). However, this leads us to our contradiction. By the definition of pointing, $\nu'_b \subseteq X$. Since $\nu_i P_i \mu_i^I$, $i \notin \nu'_b$. Therefore, $\nu'_b \subset X \setminus \{i\}$; consequently, by the LAD and (9), $|C_b(\nu'_b)| < |\mu_b^I|$. But ν' is an individually rational assignment meaning $C_b(\nu'_b) = \nu'_b$. Since $|\nu'_b| = |\mu_b^I|$, $|C_b(\nu'_b)| = |\mu_b^I|$ which is a contradiction.

Therefore, we conclude that no student is rejected under rDA. Since for all $\lambda \in B$ and all $i \in A$, $\mu_i^I \hat{R}_i \lambda_i$ and $\lambda_i \hat{R}_i i$. It now follows that $\mu^I \in S$ and $\mu_i^I R_i \lambda_i$ for all $i \in A$.

Similarly, when under school proposing rDA, a school a can only propose to student i if a is possible for i , which we denote by $B(a) = \{i \in A \mid \mu_i = a \text{ for some } \mu \in B\}$. Then the school proposing rDA is defined as follows:

Step 1: Each school a proposes to all students belonging to $C_a(B(a))$. Let X_i^1 denote the proposals received by student i . Then student i tentatively accepts the \hat{P}_i -preferred acceptable school from X_i^1 and rejects the rest (and i rejects all schools if all proposals are from unacceptable schools).

Step t : Let R_a^{t-1} denote the students who have rejected school a in a step before Step t . Then school a proposes to all students belonging to $C_a(B(a) \setminus R_a^{t-1})$. Let X_i^t denote

the proposals received by student i . Then student i tentatively accepts the \hat{P}_i -preferred acceptable school from X_i^t and rejects the rest (and i rejects all schools if all proposals are from unacceptable schools).

Stop: There is no rejected school. Then the tentative acceptances become final assignments, which we denote by μ^O .

Again, note that μ^O is stable under \hat{P} , which implies $\mu^O \in S$.

By an analogous argument, we show that no school is rejected under the school-proposing rDA. Let μ^O be the outcome of school proposing rDA. Suppose for contradiction that some school is rejected: suppose student i rejects school a and that this is the last time that a student rejects a school. Let student i reject school a at Step t . Then $i \in C_a(B(a) \setminus R_a^{t-1})$. Since $\mu_a^O \subseteq B(a) \setminus R_a^{t-1}$, substitutability of C_a implies $i \in C_a(\mu_a^O \cup \{i\})$. We first show that a proposes to another student j after being rejected by i . Since a was allowed to propose to i , there exists a $\nu \in B$ such that $\nu_i = a$. Since ν and μ^O do not block each other (because $\mu^O \in S$), by the Rural Hospital Theorem $|\mu^O \wedge \nu_a| = |\mu_a^O|$. Since $i \in \nu_a \setminus \mu_a^O$ and $i \in C_a(\mu_a^O \cup \{i\})$, there must exist $j \in \mu_a^O \setminus C_a(\mu_a^O \cup \{i\})$. By substitutability of C_a and $\mu_a^O \subseteq B(a) \setminus R_a^{t-1}$, we have $j \notin C_a(B(a) \setminus R_a^{t-1})$. In words, a does not propose to j until after i has rejected her. Note that if j was holding onto a proposal then i 's rejection of a would not be the last rejection. Therefore, no other school proposed to j , and in particular, $j \notin C_b(\mu_b^O \cup \{j\})$ for any school $b \in O \setminus \{a\}$. Therefore, when we apply the pointing to μ^O and ν , school ν_j does not point to j . However, we have already concluded that school a does not point to j (if $j \in C_a(\nu_a \cup \mu_a^O)$, then by $i \in \nu_a$ and substitutability of C_a , we have $j \in C_a(\mu_a^O \cup \{i\})$, a contradiction), so neither μ_j^O nor ν_j point to j . This contradicts Corollary 2 which says that each student is pointed to by one school.

Because for all $\lambda \in B$ and all $i \in A$, $\lambda_i \hat{R}_i \mu_i^O$ and $\lambda_i \hat{R}_i i$, now it follows that $\mu^O \in S$ and $\lambda_i R_i \mu_i^O$ for all $i \in A$. \square

We are now ready to prove the Theorem 1. The following two facts will be useful for the proof of the uniqueness of a legal set (which is the main result).

Lemma 14. (i) For an assignment μ and a school a , let

$$V(\mu, a) = \{i \in A \mid a R_i \mu_i \text{ and } \exists \nu \in B \text{ such that } \nu_i = a\}.$$

If $\mu \in S$, then $C_a(V(\mu, a)) = \mu_a$.

(ii) If $\mu \in S$, $\mu_j P_j a$ and a is possible for j , then $j \in C_a(\mu_a \cup \{j\})$.

Proof. In showing (i), note that $S \subseteq \mathcal{IR}$ and $C_a(\mu_a) = \mu_a$. By $\mu_a \subseteq V(\mu, a)$ and LAD, $|C_a(V(\mu, a))| \geq |\mu_a|$. If $C_a(V(\mu, a)) \neq \mu_a$, then there exists $i \in C_a(V(\mu, a)) \setminus \mu_a$. For student i we have the following: $a P_i \mu_i$ and for some $\nu \in B$, $\nu_i = a$; if $i \in C_a(\mu_a \cup \{i\})$, then i blocks μ with ν , a contradiction; thus by LAD, $C_a(\mu_a \cup \{i\}) = \mu_a$. But $\mu_a \subseteq V(\mu, a)$ and $i \in C_a(V(\mu, a))$ would contradict substitutability of C_a .

In showing (ii), since a is possible, there exists $\lambda \in B$ such that $\lambda_j = a$. By construction, μ and λ do not block each other. Therefore, $\mu \wedge \lambda$ is well defined. Moreover, $\mu \wedge \lambda_j = a$ since $\mu_j P_j \lambda_j$. Therefore, $j \in C_a(\mu_a \cup \lambda_a)$. Thus, by substitutability of C_a , $j \in C_a(\mu_a \cup \{j\})$. \square

Proof of Theorem 1: Let $S \subseteq \mathcal{IR}$ be such that (1) $S \subseteq \pi(S)$ and (2) $S = \pi^2(S)$.³⁰ We show that $S = \pi(S) = B$. Then by Lemma 2, S is a legal set of assignments.

Suppose by contradiction that there exists an assignment $\nu \in B \setminus S$. Since $\nu \notin S$, ν is blocked by some student i with assignment $\mu \in B$. Let $a = \mu_i$. Note that there does not exist $\phi \in S$ such that $\phi_i = a$ as otherwise, i would block ν with ϕ in which case $\nu \notin B$.

Thus, by Lemma 13, $\mu_i^I P_i a P_i \mu_i^O$. For student i , define the ‘‘legal’’ schools for i as

$$S(i) = \{d \in O \mid \exists \phi \in S \text{ such that } \phi_i = d\}.$$

Among i 's legal schools that she prefers to a , let b be her least favorite, i.e. $b \in S(i)$, $bP_i a$, and there does not exist $d \in S(i)$ such that $bP_i dP_i a$. Similarly, let c be i 's favorite school among her legal schools that she likes less than a , i.e. $c \in S(i)$, $aP_i c$, and there does not exist $d \in S(i)$ such that $aP_i dP_i c$. By Lemma 13, b and c are well-defined. Let $\overline{X}^b = \{\phi \in S \mid \phi_i = b\}$. Note that if $\phi, \phi' \in \overline{X}^b$, then $\phi \wedge \phi'_i = b$ and therefore $\phi \wedge \phi' \in \overline{X}^b$. Thus, \overline{X}^b has a well-defined minimum element (with respect to students' preferences). Let

$$\underline{\mu} := \min_{>} \overline{X}^b \quad (10)$$

Now we define the students' favorite assignment that is worse than $\underline{\mu}$. Let

$$\underline{X} = \{\phi \in S \mid \underline{\mu} \neq \phi \text{ and } \underline{\mu}_i R_i \phi_i \text{ for all } i \in A\}.$$

Note that by our choice of b and c we have for all $\phi \in \underline{X}$, $b = \phi_i$ or $cR_i \phi_i$. If $b = \phi_i$, then $\phi \in \overline{X}^b$, a contradiction to $\phi \neq \underline{\mu}$ and $\underline{\mu}_j R_j \phi_j$ for all $j \in A$. Thus, for all $\phi \in \underline{X}$, $cR_i \phi_i$. Now note that if $\phi, \phi' \in \underline{X}$, then $\phi \vee \phi' \in \underline{X}$ because $\underline{\mu}_i = bP_i \phi \vee \phi'_i$. Therefore, \underline{X} has a well-defined maximum assignment. Let

$$\underline{\mu} := \max_{>} \underline{X} \quad (11)$$

As shown already above, we have $cR_i \underline{\mu}_i$. If $c \neq \underline{\mu}_i$, then by $c \in S(i)$, there exists $\phi \in S$ such that $\phi_i = c$. Because S is a lattice and $bP_i c$, we have $\phi \wedge \underline{\mu} \in S$ and $\phi \wedge \underline{\mu}_i = c$. Since $\underline{\mu}_j R_j \phi \wedge \underline{\mu}_j$ for all $j \in A$ and $\underline{\mu} \neq \phi \wedge \underline{\mu}$, we have $\phi \wedge \underline{\mu} \in \underline{X}$. Hence, we must have $\underline{\mu}_i = c$.

Claim 1: $\underline{\mu}_j R_j \underline{\mu}_j$ for all $j \in A$ and consequently for every school d , $V(\underline{\mu}, d) \subseteq V(\underline{\mu}, d)$.

Claim 1 follows from our construction of $\underline{\mu}$ and $\underline{\mu}$: we have $\underline{\mu}_j R_j \underline{\mu}_j$ for all $j \in A$. Thus, $V(\underline{\mu}, d) \subseteq V(\underline{\mu}, d)$ for all $d \in O$. In particular, $\underline{\mu} \in \underline{X}$ and for every $\phi \in \underline{X}$, $\underline{\mu}_j R_j \phi_j$ for all $j \in A$.

Since $\underline{\mu}_i P_i a = \mu_i$, we have $\underline{\mu} \wedge \mu_i = a$. In particular, $i \in C_a(\underline{\mu}_a \cup \mu_a)$ and by substitutability of C_a , $i \in C_a(\underline{\mu}_a \cup \{i\})$. By the Rural Hospitals Theorem, $|\underline{\mu}_a| = |C_a(\underline{\mu}_a \cup \mu_a)|$. Thus, by LAD, $|C_a(\underline{\mu}_a \cup \{i\})| = |\underline{\mu}_a|$, and there exists a unique student $r_1 \in \underline{\mu}_a \setminus C_a(\underline{\mu}_a \cup \{i\})$.

³⁰Recall that the existence of S is assured because π^2 is an increasing function and for some n we have $S^n = \pi^2(S^n)$. As we have shown, $S^n \subseteq \pi(S^n)$ holds as well.

We show $\bar{\mu}_{r_1} \neq \underline{\mu}_{r_1}$: otherwise by definition, $\bar{\mu}_{r_1} = \underline{\mu}_{r_1} = a$. But then $\bar{\mu} \wedge \underline{\mu}_{r_1} = a$ and $r_1 \in C_a(\bar{\mu}_a \cup \underline{\mu}_a)$. If $i \in C_a(\bar{\mu} \wedge \underline{\mu}_a \cup \{i\})$, then by $\bar{\mu} \wedge \underline{\mu} = \underline{\mu}$, we have that i blocks $\underline{\mu}$ with μ , a contradiction to $\underline{\mu} \in S$. Thus,

$$i \notin C_a(\bar{\mu} \wedge \underline{\mu}_a \cup \{i\}) = C_a(C_a(\bar{\mu}_a \cup \underline{\mu}_a) \cup \{i\}) = C_a(\bar{\mu}_a \cup \underline{\mu}_a \cup \{i\}),$$

where the first equality follows from the definition of $\bar{\mu} \wedge \underline{\mu}$ and second one from Lemma 9. Thus, $i \notin C_a(\bar{\mu}_a \cup \underline{\mu}_a \cup \{i\})$ and $r_1 \in C_a(\bar{\mu}_a \cup \underline{\mu}_a)$. Now by substitutability of C_a and the LAD, we must have $r_1 \in C_a(\bar{\mu}_a \cup \underline{\mu}_a \cup \{i\})$. This is a contradiction to $r_1 \notin C_a(\bar{\mu}_a \cup \{i\})$ and substitutability of C_a . Thus, we must have $\bar{\mu}_{r_1} \neq \underline{\mu}_{r_1}$ and $\bar{\mu}_{r_1} P_{r_1} \underline{\mu}_{r_1}$.

Then $r_1 \in \bar{\mu}_a \setminus C_a(\bar{\mu}_a \cup \{i\})$ and in words, r_1 is the student a would reject if i applied to it.

We define an iterative procedure that is a variation of the vacancy chains that is inherit in the Deferred Acceptance algorithm (when students apply sequentially à la McVitie and Wilson). For each student l , define all schools that l strictly prefers to $\bar{\mu}_l$ to have rejected l . Formally, letting for student l ,

$$\bar{O}(l) = \{d \in O \mid d \in B(l) \text{ and } \bar{\mu}_l R_l d\}.$$

Then student l uses the preference \bar{P}_l defined by (i) for all $d, e \in \bar{O}(l)$, $d\bar{P}_l e \Leftrightarrow dP_l e$ and (ii) for all $d \in \bar{O}(l)$ and all $e \in O \setminus \bar{O}(l)$, $d\bar{P}_l l \bar{P}_l e$. Reject r_1 from a . This starts a vacancy chain. We only allow student l to apply to school b if b is possible for l . Whenever a student is rejected, she applies to her favorite possible school that has not yet rejected her. In other words, we use the profile $(\bar{P}_l)_{l \in A}$ for the vacancy chain (starting first with rejecting r_1 from a). Each time a school receives a new application, it chooses among all the students that have ever applied to it.

Claim 2: In the vacancy chain, no student j applies to a school worse than $\underline{\mu}_j$.

If not, then let l be the first student in the vacancy chain rejected by her school $\underline{\mu}_l$. Let $d = \underline{\mu}_l$ and let Y be all students who have applied to d . For every $j \in Y$, $dR_j \underline{\mu}_j$ since l is the first student rejected by her assignment under $\underline{\mu}$. Thus,

$$Y \subseteq V(\underline{\mu}, d) = \left\{ j \in A \mid dR_j \underline{\mu}_j \text{ and } \exists \nu \in B \text{ such that } \nu_j = d \right\}.$$

By (i) of Lemma 14, $C_d(V(\underline{\mu}, d)) = \underline{\mu}_d$. Therefore, by $\underline{\mu}_l = d$ and substitutability of C_d , l cannot be rejected by d , a contradiction.

Note that Claim 2 also holds for student r_1 because $\bar{\mu}_{r_1} P_{r_1} \underline{\mu}_{r_1}$.

In the above definition of the vacancy chain, if a student j ever applies to school a , then we pause to make sure that a is better off despite the fact that a did not voluntarily reject r_1 . For now, assume that if student j applies to a in the vacancy chain. By Claim 2, $aR_j \underline{\mu}_j$. By (i) of Lemma 14 and the LAD, a chooses exactly $|\underline{\mu}_a| = |\bar{\mu}_a|$ students (because every student applying to d belongs to $V(\underline{\mu}, d)$). Prior to j 's application, a is holding onto $|\bar{\mu}_a| - 1$ proposals (because we rejected $r_1 \in \bar{\mu}_a$). If we allowed a to choose amongst r_1, j , and the $|\bar{\mu}_a| - 1$ proposals she is holding, then she would wish to hold onto $|\bar{\mu}_a|$ proposals

and reject one student. Call this student r_2 . If $r_2 = r_1$ (the student we already rejected), then we stop (because we rejected the “right” student in first place). Otherwise, school a rejects r_2 and we continue. Note that in this case, a ’s new application did not come from i or else a would have wanted to reject r_1 by construction. Set $j = j_1$. Continue the vacancy chain with r_2 as the rejected student. In general, whenever a student j_m proposes to a , we check to see if $r_1 \in C_a(\bar{\mu}_a \cup \{j_1, \dots, j_m\})$. If $r_1 \notin C_a(\bar{\mu}_a \cup \{j_1, \dots, j_m\})$, then we stop (because we rejected the “right” student r_1 in first place). If $r_1 \in C_a(\bar{\mu}_a \cup \{j_1, \dots, j_m\})$, then $j_m \neq i$ (as otherwise substitutability would be violated by $r_1 \notin C_a(\bar{\mu}_a \cup \{i\})$), and a would prefer to reject one of her current proposals and keep r_1 . We allow a to reject this student and continue.

The process ends when $r_1 \notin C_a(\bar{\mu}_a \cup \{j_1, \dots, j_m\})$ for some m or when a student has been rejected by every school that is possible for her or when a school accepts the application without rejecting one of its current students. Let ϕ be the assignment that results from this process. By Claim 2, we have for all $j \in A$, $\phi_j R_j \underline{\mu}_j$.

Claim 3: The vacancy chain ends with an application to a .

There are only three ways for the vacancy chain to end: (1) a student applies to a , (2) a student applies to a school $b \neq a$ and b accepts the student without rejecting any student, and (3) a student is rejected by all of her possible schools.

We show that (3) does not occur. If student l is part of the vacancy chain, then $\bar{\mu}_l \neq l$. Therefore, $\underline{\mu}_l \neq l$ by the Rural Hospital Theorem. Since by Claim 2, $\phi_l R_l \underline{\mu}_l$, we have $\phi_l \neq l$. Therefore, the vacancy chain does not end with a student having been rejected by all possible schools. Similarly, (2) does not occur: for every school $b \neq a$, $|\bar{\mu}_b| = |\underline{\mu}_b|$. Since $\phi_j R_j \underline{\mu}_j$ for all $j \in A$, we have $V(\phi, b) \subseteq V(\underline{\mu}, b)$ (meaning that b has more students to choose from under $\underline{\mu}$ as the students are less happy with their assignment). By (i) of Lemma 14, we have $C_b(V(\underline{\mu}, b)) = \underline{\mu}_b$. But then $|\phi_b| > |\underline{\mu}_b|$ would violate the Law of Aggregate Demand for b to accept a student without rejecting another (because $\phi_b \subseteq V(\underline{\mu}, b)$). Therefore, (1) must occur and the vacancy chain can only conclude when a student \bar{l} applies to a .

Claim 4: $\phi \in S$.

For any school $b \neq a$, school b receives a better set of students under ϕ than under $\bar{\mu}$ as it has weakly more students to choose from. Mathematically, $C_b(\phi_b \cup \bar{\mu}_b) = \phi_b$. School a is the only school which did not voluntarily reject all of its students as a did not voluntarily reject r_1 . However, the key point is that the vacancy chain must stop with an application to a , and we only stop after an application to a if a now wants to reject r_1 . Therefore, a is made strictly better off by the vacancy chain. Consider a student j and a school b such that b is possible for j and $b P_j \phi_j$. If $b P_j \bar{\mu}_j$, then $j \notin C_b(\bar{\mu}_b \cup \{j\})$ or else $\bar{\mu}$ would be blocked. Since b did not choose j before, b does not choose j now that she has weakly more students to choose from. If $\bar{\mu}_j R_j b$ then j was rejected by b during the vacancy chain and j is not able to block ϕ with b .

Claim 5: $a P_i \phi_i$

Since $\phi \in S$, we have $\phi_i \neq a$. Suppose by contradiction that $\phi_i P_i a$. By our choice of b and c and $\phi \in S$, this can only happen if i was never rejected from b . Therefore, $b = \phi_i = \bar{\mu}_i$. Because the vacancy chain stops with an application to a where r_1 is rejected,

we must have $a = \bar{\mu}_{r_1} P_{r_1} \phi_{r_1}$. By Claim 4, $\phi \in S$ and thus, $\phi \in \bar{X}_b$. But now this is a contradiction as $\bar{\mu} = \min_{>} \bar{X}^b$.

Now Claim 5 yields the contradiction: student i applied in the vacancy chain to a before applying to ϕ_i (because $a \in \bar{O}(i)$). But when i applied to a , the vacancy chain must stop as $r_1 \in \bar{\mu}_a \setminus C_a(\bar{\mu}_a \cup \{i\})$, $i \in C_a(\bar{\mu}_a \cup \{i\})$, and thus when $j_m = i$, we must have $r_1 \notin C_a(\bar{\mu}_a \cup \{j_1, \dots, j_m\})$ as otherwise substitutability of C_a is violated. But then we must have $a = \phi_i$ which contradicts Claim 5. \square

B Assignment with Contracts

Recall that A denotes the set of students and O denotes the set of schools. Let \mathcal{X} denote the set of all contracts. Each contract $x \in \mathcal{X}$ is associated with one student $x_A \in A$ and one school $x_O \in O$. Given $Y \subseteq \mathcal{X}$, let Y_i denote the set of contracts associated with student i and Y_a denote the set of contracts associated with school a .

Each student i has a strict preference P_i over $\mathcal{X}_i \cup \{i\}$. Let C_i denote the choice function induced by P_i : for any $Y \subseteq \mathcal{X}$, let $C_i(Y) = \max_{P_i} Y_i \cup \{i\}$.

Any school a has a choice function $C_a : 2^{\mathcal{X}} \rightarrow 2^{\mathcal{X}}$ such that for any $Y \subseteq \mathcal{X}$ we have $C_a(Y) \subseteq Y_a$. Substitutability and LAD are straightforward to adapt to the setup with contracts.

Any $\mu \subseteq \mathcal{X}$ is an assignment. An assignment μ is **individually rational** if for all $i \in A$, $\mu_i = C_i(\mu)$ and for all $a \in O$, $C_a(\mu) = \mu_a$. Let \mathcal{IR} denote the set of all individually rational assignments. Again, throughout we consider only individually rational assignments. Given assignment μ , student i and school a **block** μ via contract x if $x P_i \mu_i$ and $x \in C_a(\mu \cup \{x\})$ (where this implies $x_A = i$ and $x_O = a$). An assignment μ is **non-wasteful** if there do not exist i and a and a contract x such that $x P_i \mu_i$ and $C_a(\mu_a \cup \{x\}) = \mu_a \cup \{x\}$. An assignment is **fair** if there do not exist i and a and a contract x such that $x P_i \mu_i$ and $x \in C_a(\mu_a \cup \{x\}) \neq \mu_a \cup \{x\}$. Now blocking among assignments carries over in a straightforward fashion: i blocks μ with ν if for some $x \in \mathcal{X}_i$, (1) $x P_i \mu_i$, (2) $x \in C_a(\mu_a \cup \{x\})$ and (3) $\nu_i = x$. Then μ blocks ν if there exists a student i who blocks μ with ν .

Now $L \subseteq \mathcal{IR}$ is a legal set of assignments if and only if (i) for all $\nu \in \mathcal{IR} \setminus L$ there exists $\mu \in L$ such that μ blocks ν and (ii) for all $\mu, \nu \in L$, μ does not block ν .

The operator $\pi : 2^{\mathcal{IR}} \rightarrow 2^{\mathcal{IR}}$ is defined in the same way as in the main text, and its properties carry over without change, namely Lemma 1, Lemma 2, and that there exists n such that (1) $S^n \subseteq \pi(S^n)$ and (2) $S^n = \pi^2(S^n)$.

Regarding pointing, we let students and schools point to contracts instead of pointing to schools and students. Given two assignments μ and ν , student i points to μ_i (ν_i) if $\mu_i R_i \nu_i$ ($\nu_i R_i \mu_i$) and school a points to $x \in \mathcal{X}$ if $x \in C_a(\mu_a \cup \nu_a)$. Then Lemma 5 (Weak Pointing Lemma) carries over in the following way: let μ and ν be two individually rational assignments which do not block each other. Then (i) no student and school point to the same contract unless the contract belongs to μ and ν and (ii) no two schools point to two contracts which are associated with the same student. For the remainder of the Appendix, we use Lemma A' to denote Lemma A from the main text translated to the setting of

assignment with contracts.

Lemma 5' (*Weak Pointing Lemma*) *Let μ and ν be two individually rational assignments which do not block each other. Then:*

(i) *no student and school point to the same contract unless the contract belongs to both μ and ν , and*

(ii) *no two schools point to two contracts which are associated with the same student.*

Proof. Consider any student i such that $\mu_i \neq \nu_i$. Without loss of generality, assume $\mu_i P_i \nu_i$. By individual rationality of μ and ν , we have $\mu_i \neq i$. Let $(\mu_i)_O = a$. Then i points to μ_i . By substitutability of C_a and $\mu_i \in \mu_a$, if $\mu_i \in C_a(\mu_a \cup \nu_a)$, then $\mu_i \in C_a(\nu_a \cup \{\mu_i\})$. Therefore, if a pointed to μ_i (meaning $\mu_i \in C_a(\mu_a \cup \nu_a)$), then i would block ν with μ (because $\mu_i \in \mu_a$), a contradiction. For any student i such that $\mu_i \neq \nu_i$, by $\mu_i R_i i$ and $\nu_i R_i i$, i must point to a contract. Therefore, if two schools point to two contracts associated with the same student, there must be a student and a school pointing to the same contract which would be a contradiction to the above. \square

Definition 5' Given assignments μ and ν , define $\mu \wedge \nu$ by $\mu \wedge \nu_a = C_a(\mu_a \cup \nu_a)$ for all $a \in O$.

Our main focus is on any two individually rational assignments μ and ν which do not block each other. Then $\mu \wedge \nu$ is the reassignment resulting from assigning a student to the school that is pointing to her. The following lemma demonstrates that this yields a well-defined assignment.

Lemma 6' *Let μ and ν be two individually rational assignments which do not block each other. Then:*

(i) *$\mu \wedge \nu$ is an individually rational assignment;*

(ii) *if $\mu_i \neq i$, then $\mu \wedge \nu_i \neq i$; and*

(iii) *every school receives the same number of contracts under μ and $\mu \wedge \nu$, i.e. $|\mu_a| = |\mu \wedge \nu_a|$.*

Proof. (i): Suppose for contradiction that there are two contracts $x \neq y$ with $x_A = y_A = i$ and $a, b \in O$ such that both $x \in \mu \wedge \nu_a$ and $y \in \mu \wedge \nu_b$. Then $x \in C_a(\mu_a \cup \nu_a)$ and $y \in C_b(\mu_b \cup \nu_b)$. Then a points to x and b points to y . Then $(x \in \mu_a$ and $y \in \nu_b)$ or $(x \in \mu_b$ and $y \in \nu_a)$, and i must point to either x or y . Therefore, there is a student and a school pointing to the same contract which contradicts the Pointing Lemma. In showing that $\mu \wedge \nu$ is individually rational, we have by definition $C_a(\mu \wedge \nu_a) = \mu \wedge \nu_a$.³¹ Furthermore, $\mu_i R_i i$ and $\nu_i R_i i$ imply $\mu \wedge \nu_i R_i i$. Hence, $\mu \wedge \nu \in \mathcal{IR}$.

³¹Note that substitutability and LAD of C_a imply IRC: for all $X \subseteq Y$, if $C_a(Y) \subseteq X$, then $C_a(X) = C_a(Y)$.

(ii) and (iii): For counting purposes, in this proof we use the convention $|\mu_i| = 1$ if $\mu_i \neq i$ and $|\mu_i| = 0$ if $\mu_i = i$. First note that if $\mu \wedge \nu_i = x$ but $\mu_i = i$, then i blocks μ with ν : by individual rationality, $\nu_i = xP_i i$; and $x \in C_a(\mu_a \cup \nu_a)$ and substitutability of C_a imply $x \in C_a(\mu_a \cup \{x\})$. Therefore, $|\mu \wedge \nu_i| = 1$ implies that $|\mu_i| = 1$ and $|\nu_i| = 1$. Hence,

$$\sum_{i \in A} |\mu \wedge \nu_i| \leq \sum_{i \in A} |\mu_i|. \quad (12)$$

By the Law of Aggregate Demand and $\mu_a \cup \nu_a \supseteq \mu_a$, $|C_a(\mu_a \cup \nu_a)| \geq |C_a(\mu_a)|$. Therefore,

$$\sum_{a \in O} |\mu \wedge \nu_a| \geq \sum_{a \in O} |\mu_a| \quad (13)$$

Note that for any assignment λ we have

$$\sum_{i \in A} |\lambda_i| = \sum_{a \in O} |\lambda_a|. \quad (14)$$

Combining the three equations yields that $\sum_{i \in A} |\mu \wedge \nu_i| = \sum_{i \in A} |\mu_i|$. Since $|\mu \wedge \nu_i| = 1$ implies that $|\mu_i| = 1$, it must also be that $|\mu_i| = 1$ implies that $|\mu \wedge \nu_i| = 1$. Similarly, since $|\mu \wedge \nu_a| \geq |\mu_a|$ for every school a and $\sum_{a \in O} |\mu \wedge \nu_a| = \sum_{a \in O} |\mu_a|$, it must be that for every school a , $|\mu_a| = |\mu \wedge \nu_a|$. \square

An immediate corollary of Lemma 6' is our version of the Rural Hospital Theorem for the assignment with contracts setting.³²

Corollary 1' (*Rural Hospital Theorem*) *Let μ and ν be two individually rational assignments which do not block each other. Then*

- (i) *for any school a , $|\mu_a| = |\nu_a|$; and*
- (ii) *for any student i , $\mu_i = i$ if and only if $\nu_i = i$.*

Proof. By Lemma 6', $|\mu_a| = |\mu \wedge \nu_a| = |\nu_a|$ (which implies (i)), and if $\mu_i \neq i$, then $\mu \wedge \nu_i \neq i$. Let $(\mu_i)_O = a$. If $\nu_i = i$, then by individual rationality of μ and ν , we have $\mu_i P_i i$, and by Lemma 6', $\mu \wedge \nu_i = \mu_i$. Thus, $\mu_i \in C_a(\mu_a \cup \nu_a)$ and by substitutability of C_a , $\mu_i \in C_a(\nu_a \cup \{\mu_i\})$, which implies that i blocks ν with μ , a contradiction. \square

Lemma 6' allows us to strengthen the Pointing Lemma.

Corollary 2' (*Strong Pointing Lemma*) *Let μ and ν be two individually rational assignments which do not block each other.*

- (i) *If a student is assigned a contract under either μ or ν , then she points to a contract and one school points to a contract which is associated with her.*

³²One could also refer to this as the "Rural Schools Theorem" in our context with the appropriate interpretation.

(ii) For any school a , a points to $|\mu_a| = |\nu_a|$ contracts and $|\mu_a| = |\nu_a|$ students point contracts associated with a .

Proof. (i): Consider a student i who is assigned a contract under either μ or ν . By $\mu, \nu \in \mathcal{IR}$, i points to one contract by strict preferences. By (ii) of Lemma 6', $\mu \wedge \nu_i \neq i$. Without loss of generality, $\mu \wedge \nu_i = \mu_i$ and $(\mu_i)_O = a$. Since $\mu_i \in C_a(\mu_a \cup \nu_a)$, a points to μ_i . Two schools cannot point to two contracts associated with i , or else we would violate the Pointing Lemma.

(ii): This follows from the same counting exercise as in the proof of Lemma 6'. If some school a had fewer than $|\mu_a|$ students pointing to contracts associated with a , then some school b would have to have more than $|\mu_b|$ students pointing to contracts associated with b . Then b would have to point to one of these contracts which would contradict the Pointing Lemma. \square

We have already established that if we reassign each student to the school that is pointing to her that this results in a well-defined assignment. We now show that reassigning each student to the school she is pointing to is also a well-defined assignment. We refer to this assignment as $\mu \vee \nu$.

Definition 6' Let μ and ν be two individually rational assignments which do not block each other. Define the assignment $\mu \vee \nu$ as follows: for all $i \in A$, $\mu \vee \nu_i = \max_{P_i} \{\mu_i, \nu_i\}$.

Lemma 7' Let μ and ν be two individually rational assignments which do not block each other. Then $\mu \vee \nu$ is an individually rational assignment.

Proof. First we show that for every school a , $C_a(\mu \vee \nu_a \cup \mu_a) = \mu_a$ (and symmetrically that $C_a(\mu \vee \nu_a \cup \nu_a) = \nu_a$). Suppose for contradiction that $C_a(\mu \vee \nu_a \cup \mu_a) \neq \mu_a$. Since μ is individually rational, we have $C_a(\mu_a) = \mu_a$. By the Law of Aggregate Demand, $|C_a(\mu \vee \nu_a \cup \mu_a)| \geq |\mu_a|$, so if $C_a(\mu \vee \nu_a \cup \mu_a) \neq \mu_a$, there must exist $x \in C_a(\mu \vee \nu_a \cup \mu_a)$ such that $x \notin \mu_a$. Let $x_A = i$. Therefore, $\mu \vee \nu_i = x$ and $\nu_i = x$. In words, since $\mu \vee \nu_i = x$, i prefers $\nu_i = x$ to μ_i . Since $x \in C_a(\mu \vee \nu_a \cup \mu_a)$, by substitutability of C_a , $x \in C_a(\mu_a \cup \{x\})$. Therefore, i blocks μ with ν which is a contradiction.

Second we prove the lemma. By construction, each student is assigned only one contract, and by individual rationality of μ and ν we have $\mu \vee \nu_i R_i i$. We must show that for every school a , $C_a(\mu \vee \nu_a) = \mu \vee \nu_a$. By definition, $C_a(\mu \vee \nu_a) \subseteq \mu \vee \nu_a$. Suppose $\mu \vee \nu_i = x$ and $x_O = a$. Assume without loss of generality that $\mu_i = x$. We have already shown that $C_a(\mu \vee \nu_a \cup \mu_a) = \mu_a$. Since $x \in \mu_a$, $x \in C_a(\mu \vee \nu_a \cup \mu_a)$. Therefore, by substitutability of C_a and $x \in \mu \vee \nu_a$, $x \in C_a(\mu \vee \nu_a)$. \square

Lemma 8' (Generalized Decomposition Lemma) Let μ and ν be two individually rational assignments which do not block each other, and let i be a student such that $\mu_i \neq \nu_i$. Student i chooses contract x if and only if school $a = x_O$ rejects x . Formally, $\mu \vee \nu_i = x$ if and only if $x \notin \mu \wedge \nu_a = C_a(\mu_a \cup \nu_a)$.

Proof. Suppose that $\mu_i \neq \nu_i$ and without loss of generality assume that i points to $\mu_i = x$, and $x_O = a$. If x is not rejected by a ($x \in \mu \wedge \nu_a$), then a points to x . This contradicts the Weak Pointing Lemma which says that a student and a school cannot point to the same contract. Similarly, suppose that $\mu_i = x$ but that school a rejects x ($x \notin \mu \wedge \nu_a$). Then school a does not point to x . By the Strong Pointing Lemma and since a school and a student cannot point to the same contract, it follows that i points to $\mu_i = x$. \square

As a reminder, we defined $S^0 = \emptyset$ (and thus, $\pi(\emptyset) = \mathcal{IR}$), and in general let $S^k = \pi^2(S^{k-1})$ and $B^k = \pi(S^k)$. Since π^2 is increasing, eventually $S^n = S^{n+1}$ for some n . The two key properties of S^n are (1) $S^n \subseteq \pi(S^n)$ (for any two assignments $\mu, \nu \in S^n$, μ and ν do not block each other); and (2) $S^n = \pi^2(S^n)$ (if $\mu \notin S^n$, then μ is blocked by an assignment in $\pi(S^n)$).

We define the following partial ordering over assignments:

$$\mu \geq \nu \text{ if for every school } a \in O, C_a(\mu_a \cup \nu_a) = \nu_a \quad (15)$$

Lemma 10' *Let μ and ν be two individually rational assignments which do not block each other. Then $\mu \vee \nu \geq \mu \geq \mu \wedge \nu$.*

Proof. Let $a \in O$. By definition, $\mu \wedge \nu_a = C_a(\mu_a \cup \nu_a)$. Therefore:

$$\begin{aligned} C_a(\mu_a \cup (\mu \wedge \nu_a)) &= C_a(\mu_a \cup C_a(\mu_a \cup \nu_a)) \\ &= C_a(\mu_a \cup \mu_a \cup \nu_a) \\ &= C_a(\mu_a \cup \nu_a) \\ &= \mu \wedge \nu_a \end{aligned}$$

where the second equality follows from Lemma 9. Therefore, $\mu \geq \mu \wedge \nu$ (and of course, by symmetry, $\nu \geq \mu \wedge \nu$).

In the proof of Lemma 7' we demonstrated that for every school a , $C_a(\mu \vee \nu_a \cup \mu_a)$. Therefore, by definition, $\mu \vee \nu \geq \mu$. \square

Lemma 11' *Let $S \subseteq \mathcal{IR}$ be such that (1) $S \subseteq \pi(S)$ and (2) $\pi^2(S) = S$. For any $\mu, \nu \in S$, $\mu \vee \nu \in S$ and $\mu \wedge \nu \in S$. In particular, S with partial order \geq is a lattice.*

Proof. Let $B = \pi(S)$. By assumption, $S \subseteq B$ and $S = \pi(B)$. Therefore, μ and ν are not blocked by any assignment in B , and in particular, μ and ν do not block each other. We have already shown that $\mu \vee \nu$ and $\mu \wedge \nu$ are well-defined assignments. Furthermore, by individual rationality of μ and ν and (ii) of Lemma 6', $\mu \wedge \nu_i R_i i$ for all $i \in A$, and by definition, $C_a(\mu \wedge \nu_a) = C_a(\mu_a \cup \nu_a) = \mu \wedge \nu_a$. Thus, $\mu \wedge \nu \in \mathcal{IR}$. By Lemma 7', $\mu \vee \nu \in \mathcal{IR}$. All that remains is to show that $\mu \vee \nu$ and $\mu \wedge \nu$ are not blocked by any assignment in B .

Suppose for contradiction that i blocks $\mu \wedge \nu$ with $\lambda \in B$. By individual rationality of $\mu \wedge \nu$, we have $\lambda_i \neq i$, say $\lambda_i = x$ and $x_O = b$. If $\mu \wedge \nu_i = i$, then by (ii) of Lemma 6' we have $\mu_i = i$ and $\nu_i = i$. But then by substitutability of C_b and $x \in C_b(\mu \wedge \nu_b \cup \{x\}) = C_b(\mu_b \cup \nu_b \cup \{x\})$, we have $x \in C_b(\mu_b \cup \{x\})$. Because $x \notin \mu_b$, now i blocks μ with λ , a

contradiction to $\mu \in S$. Thus, $\mu \wedge \nu_i \neq i$, say $\mu \wedge \nu_i = y$ and without loss of generality, assume $\mu_i = y$ and $y_O = a$. Since i blocks $\mu \wedge \nu$ with x ,

$$x \in C_b(\mu \wedge \nu_b \cup \{x\}). \quad (16)$$

Note that for any sets of contracts X and Y , $C_b(X \cup Y) = C_b(C_b(X) \cup Y)$ (Lemma 9). Therefore,

$$C_b(C_b(\mu_b \cup \nu_b) \cup \{x\}) = C_b(\mu_b \cup \nu_b \cup \{x\}). \quad (17)$$

By definition, $\mu \wedge \nu_b = C_b(\mu_b \cup \nu_b)$. Thus, by (16), $x \in C_b(C_b(\mu_b \cup \nu_b) \cup \{x\})$. By (17), $x \in C_b(\mu_b \cup \nu_b \cup \{x\})$. By substitutability of C_b , $x \in C_b(\mu_b \cup \{x\})$. Therefore, $x P_i \mu_i$, $x \in C_b(\mu_b \cup \{x\})$, and $\lambda_i = x$ where $\lambda \in B = \pi(S)$. Therefore, i blocks μ with λ implying that $\mu \notin \pi(B)$. This is a contradiction as $\mu \in S = \pi(B)$.

The proof for $\mu \vee \nu$ is similar. Suppose for contradiction that $\mu \vee \nu$ is blocked by student i with assignment $\lambda \in B$ where $\lambda_i = x$ and $x_O = a$. We first show that there exists a contract $y \in \mu \vee \nu_a$ which is rejected when a chooses from $\mu \vee \nu_a \cup \{x\}$, i.e. $y \notin C_a(\mu \vee \nu_a \cup \{x\})$. We have already shown that $\mu \wedge \nu$ is not blocked by i and λ (or by any other student); therefore, $x \notin C_a(\mu \wedge \nu_a \cup \{x\})$. Otherwise, i would block $\mu \wedge \nu$ since $\lambda_i P_i \mu \vee \nu_i$ implies $\lambda_i P_i \mu \wedge \nu_i$.

Because $\mu \wedge \nu \in \mathcal{IR}$, we have $C_a(\mu \wedge \nu_a) = \mu \wedge \nu_a$. Thus, by LAD and substitutability of C_a , we have $C_a(\mu \wedge \nu_a \cup \{x\}) = C_a(\mu_a \cup \nu_a \cup \{x\}) = \mu \wedge \nu_a$. As a reminder, $|\mu_a| = |\mu \wedge \nu_a| = |\nu_a| = |\mu \vee \nu_a|$. By the Law of Aggregate Demand and $\mu \vee \nu \in \mathcal{IR}$,

$$|\mu \vee \nu_a| = |C_a(\mu \vee \nu_a)| \leq |C_a(\mu \vee \nu_a \cup \{x\})| \leq |C_a(\mu_a \cup \nu_a \cup \{x\})| = |\mu \wedge \nu_a| = |\mu \vee \nu_a|.$$

Now all these inequalities become equalities. Because $x \in C_a(\mu \vee \nu_a \cup \{x\})$ and $x \notin C_a(\mu_a \cup \nu_a \cup \{x\})$, there must exist $y \in \mu \vee \nu_a \setminus C_a(\mu \vee \nu_a \cup \{x\})$. Without loss of generality, $y \in \mu_a$ and $y_A = j$. Then $x \notin C_a(\mu_a \cup \{x\})$ or else i would block μ with λ . Because μ is individually rational, $C_a(\mu_a) = \mu_a$. Therefore, by LAD and substitutability of C_a ,

$$C_a(\mu_a \cup \{x\}) = \mu_a. \quad (18)$$

Note that

$$\begin{aligned} C_a(\mu_a \cup (\mu \vee \nu_a) \cup \{x\}) &= C_a(C_a(\mu_a \cup \mu \vee \nu_a) \cup \{x\}) \\ &= C_a(\mu_a \cup \{x\}) \\ &= \mu_a \end{aligned}$$

where the first equality follows from Lemma 9, the second equality follows from Lemma 5' ($\mu \vee \nu \geq \mu$ and therefore, $C_a(\mu_a \cup \mu \vee \nu_a) = \mu_a$), and the third inequality follows from (18). However, $y \in \mu_a$ and therefore $y \in C_a(\mu_a \cup (\mu \vee \nu_a) \cup \{x\})$. This contradicts substitutability of C_a as $y \notin C_a(\mu \vee \nu_a \cup \{x\})$ but $\mu \vee \nu_a \cup \{x\} \subseteq \mu_a \cup (\mu \vee \nu_a) \cup \{x\}$. \square

Lemma 13' *Let $S \subseteq \mathcal{IR}$ be such that (1) $S \subseteq \pi(S)$ and (2) $\pi^2(S) = S$. Let μ^I be the student-optimal assignment in S and let μ^O be the school optimal assignment in S . For every $\lambda \in \pi(S)$ and every student i , $\mu_i^I R_i \lambda_i R_i \mu_i^O$.*

Proof. Let S be a set that satisfies (1) and (2) and let $B = \pi(S)$. We say that contract x is **possible** for i if there exists $\lambda \in B$ such that $\lambda_i = x$. Let

$$B(i) = \{x \in \mathcal{X}_i \mid \text{there exists } \lambda \in B \text{ such that } \lambda_i = x\}$$

denote the set of possible contracts for student i . Let \hat{P}_i be defined as follows: (i) for all $x \in B(i)$ and $y \in \mathcal{X}_i \setminus B(i)$, $x \hat{P}_i y$, (ii) for all $x, y \in B(i)$, $x \hat{P}_i y \Leftrightarrow x P_i y$ and (iii) for all $x, y \in \mathcal{X}_i \setminus B(i)$, $x \hat{P}_i y \Leftrightarrow x P_i y$. Now we introduce a natural modification of DA which we call rDA (restricted DA): when we run DA we only allow a student to propose possible contracts and we use the profile $(\hat{P}_i)_{i \in N}$ for students to propose contracts.³³ Formally, the rDA is defined as follows:

Step 1: Each student i proposes his most \hat{P}_i -preferred acceptable contract. Let X_a^1 denote the proposed contracts received by school a . Then school a tentatively accepts $C_a(X_a^1)$ and rejects $X_a^1 \setminus C_a(X_a^1)$.

Step t : Any student i rejected in Step $t - 1$ proposes his most \hat{P}_i -preferred acceptable contract among the ones which were not yet rejected (if there is no acceptable contract left for i , then i does not make any proposal). Let X_a^t denote the set of proposed contracts received by school a and the ones tentatively accepted by a in the previous step. Then school a tentatively accepts $C_a(X_a^t)$ and rejects $X_a^t \setminus C_a(X_a^t)$.

Stop: There are no rejected contracts or all rejected students have applied to all acceptable contracts. Then the tentative acceptances become final assignments, which we denote by μ^I .

Note that μ^I is stable under \hat{P} , which implies $\mu^I \in S$.

We establish the result by showing that no contract is rejected under rDA. This implies that for each student i , μ_i^I is i 's favorite possible contract (or equivalently, μ_i^I is i 's most \hat{P}_i -preferred contract). If a contract was rejected, then there would have to be a last contract rejected. Call this contract x . Let $x_A = i$ and $x_O = a$, i.e. school a rejected x . Then x must be possible for i , so there exists a $\nu \in B$ such that $\nu_i = x$. Because $\nu \in B$ and $\mu^I \in S$, ν and μ^I do not block each other. Thus, by the Rural Hospital Theorem, $\mu_i^I \neq i$. Let $\mu_i^I = y$ and $y_O = b$.

Let $Y = \{z \in \mathcal{X}_b \mid \text{for } j = z_A, z \hat{R}_j \mu_j^I\}$ (in words, Y is the set of contracts with b which are possible for some student j and weakly preferred by j to her assignment under rDA). By construction and stability of μ^I under \hat{P} , $\mu_b^I = C_b(Y)$. When i proposes contract y to b , no contract is rejected (since x is the last contract rejected). Therefore, by substitutability of C_b ,

$$C_b(Y \setminus \{y\}) = \mu_b^I \setminus \{y\}. \quad (19)$$

Since μ^I and ν do not block each other, by Lemma 6', $\nu' = \mu^I \vee \nu$ is an individually rational assignment. By the Strong Pointing Lemma', $|\nu'_b| = |\mu_b^I|$ (ν'_b is the set of students pointing to contracts associated with b). However, this leads us to our contradiction.

³³Since choice functions satisfy substitutability and LAD, the cumulative offer process and DA coincide.

By the definition of pointing, $\nu'_b \subseteq Y$. Since $\nu_i P_i \mu_i^I$, i points to x , not to any contract associated with b , i.e. $\mu_i^I \notin \nu'_b$. Therefore, $\nu'_b \subset Y \setminus \{\mu_i^I\}$; consequently, by the LAD and (19), $|C_b(\nu'_b)| < |\mu_b^I|$. But ν' is an individually rational assignment meaning $C_b(\nu'_b) = \nu'_b$. Since $|\nu'_b| = |\mu_b^I|$, $|C_b(\nu'_b)| = |\mu_b^I|$ which is a contradiction.

Therefore, we conclude that no contract is rejected under rDA. Since for all $\lambda \in B$ and all $i \in A$, $\mu_i^I \hat{R}_i \lambda_i$ and $\lambda_i \hat{R}_i i$. It now follows that $\mu^I \in S$ and $\mu_i^I R_i \lambda_i$ for all $i \in A$.

Similarly, when under school proposing rDA, a school a can only propose contract x if x is possible for a , which we denote by $B(a) = \{x \in \mathcal{X}_a \mid x \in \mu_a \text{ for some } \mu \in B\}$. Then the school proposing rDA is defined as follows:

Step 1: Each school a proposes all contracts belonging to $C_a(B(a))$. Let X_i^1 denote the proposals received by student i . Then student i tentatively accepts the \hat{P}_i -preferred acceptable contract from X_i^1 and rejects the rest (and i rejects all contracts if all proposed contracts are unacceptable).

Step t : Let R_a^{t-1} denote the contracts associated with school a which were rejected in a step before Step t . Then school a proposes all contracts belonging to $C_a(B(a) \setminus R_a^{t-1})$. Let X_i^t denote the proposals received by student i . Then student i tentatively accepts the \hat{P}_i -preferred acceptable contract from X_i^t and rejects the rest (and i rejects all contracts if all proposed contracts are unacceptable).

Stop: There is no rejected contract. Then the tentative acceptances become final assignments, which we denote by μ^O .

Again, note that μ^O is stable under \hat{P} , which implies $\mu^O \in S$.

By an analogous argument, we show that no contract is rejected under the school-proposing rDA. Let μ^O be the outcome of school proposing rDA. Suppose for contradiction that some contract is rejected: let student i be rejecting contract x , $x_O = a$, and be this the last time that a student rejects a contract. Let student i reject x at Step t . Then $x \in C_a(B(a) \setminus R_a^{t-1})$. Since $\mu_a^O \subseteq B(a) \setminus R_a^{t-1}$, substitutability of C_a implies $x \in C_a(\mu_a^O \cup \{x\})$. We first show that a proposes another contract after i rejects x . Since a was allowed to propose x , there exists a $\nu \in B$ such that $\nu_i = x$. Since ν and μ^O do not block each other (because $\mu^O \in S$), by the Rural Hospital Theorem $|\mu^O \wedge \nu_a| = |\mu_a^O|$. Since $x \in \nu_a \setminus \mu_a^O$ and $x \in C_a(\mu_a^O \cup \{x\})$, there must exist $y \in \mu_a^O \setminus C_a(\mu_a^O \cup \{x\})$. By substitutability of C_a and $\mu_a^O \subseteq B(a) \setminus R_a^{t-1}$, we have $y \notin C_a(B(a) \setminus R_a^{t-1})$. In words, a does not propose y until after i has rejected x . Note that if $y_A = j$ was holding onto a proposal then i 's rejection of x would not be the last rejection. Therefore, no other school proposed a contract associated with j , and in particular, $z \notin C_b(\mu_b^O \cup \{z\})$ for any school $b \in O \setminus \{a\}$ and $z \in \mathcal{X}_j \cap \mathcal{X}_b$. Therefore, when we apply the pointing to μ^O and ν , school $(\nu_j)_O$ does not point to ν_j . However, we have already concluded that school a does not point to y (if $y \in C_a(\nu_a \cup \mu_a^O)$, then by $x \in \nu_a$ and substitutability of C_a , we have $y \in C_a(\mu_a^O \cup \{x\})$, a contradiction), so neither a nor $(\nu_j)_O$ point to a contract associated with j . This contradicts Corollary 2' which says that one school points to a contract associated with j .

Because for all $\lambda \in B$ and all $i \in A$, $\lambda_i \hat{R}_i \mu_i^O$ and $\lambda_i \hat{R}_i i$, now it follows that $\mu^O \in S$ and $\lambda_i R_i \mu_i^O$ for all $i \in A$. \square

We are now ready to prove the main theorem. As a reminder, we set $S^0 = \emptyset$, $S^1 = \pi^2(\emptyset)$, $S^k = \pi^2(S^{k-1})$ and $B^k = \pi(S^k)$. We defined S as the first fixed point of our construction, i.e. $S = \pi^2(S)$. Let $B = \pi(S)$. The following two facts will be useful for the proof of the uniqueness of a legal set (which is the main result).

We are now ready to prove the main theorem. As a reminder, we set $S^0 = \emptyset$, $S^1 = \pi^2(\emptyset)$, $S^k = \pi^2(S^{k-1})$ and $B^k = \pi(S^k)$. We defined S as the first fixed point of our construction, i.e. $S = \pi^2(S)$. Let $B = \pi(S)$. The following two facts will be useful for the proof of the uniqueness of a legal set (which is the main result).

Lemma 14'

(i) For an assignment μ and a school a , let

$$V(\mu, a) = \{x \in \mathcal{X}_a \mid \text{for some } i \in A, xR_i\mu_i \text{ and } \exists \nu \in B \text{ such that } \nu_i = x\}.$$

If $\mu \in S$, then $C_a(V(\mu, a)) = \mu_a$.

(ii) If $\mu \in S$, $\mu_j P_j x$ and x is possible for j (where $x_O = a$), then $x \in C_a(\mu_a \cup \{x\})$.

Proof. In showing (i), note that $S \subseteq \mathcal{IR}$ and $C_a(\mu_a) = \mu_a$. By $\mu_a \subseteq V(\mu, a)$ and LAD, $|C_a(V(\mu, a))| \geq |\mu_a|$. If $C_a(V(\mu, a)) \neq \mu_a$, then there exists $y \in C_a(V(\mu, a)) \setminus \mu_a$. For student $y_A = i$ we have the following: $y P_i \mu_i$ and for some $\nu \in B$, $\nu_i = y$; if $y \in C_a(\mu_a \cup \{y\})$, then i blocks μ with ν , a contradiction; thus by LAD, $C_a(\mu_a \cup \{y\}) = \mu_a$. But $\mu_a \subseteq V(\mu, a)$ and $y \in C_a(V(\mu, a))$ would contradict substitutability of C_a .

In showing (ii), since x is possible, there exists $\lambda \in B$ such that $\lambda_j = x$. By construction, μ and λ do not block each other. Therefore, $\mu \wedge \lambda$ is well defined. Moreover, $\mu \wedge \lambda_j = x$ since $\mu_j P_j \lambda_j$. Therefore, $x \in C_a(\mu_a \cup \lambda_a)$. Thus, by substitutability of C_a , $x \in C_a(\mu_a \cup \{x\})$. \square

Theorem 1' *There exists a legal set of assignments.*

Proof. Let $S \subseteq \mathcal{IR}$ be such that (1) $S \subseteq \pi(S)$ and (2) $S = \pi^2(S)$.³⁴ We show that $S = \pi(S) = B$. Then by Lemma 2, S is a legal set of assignments.

Suppose by contradiction that there exists an assignment $\nu \in B \setminus S$. Since $\nu \notin S$, ν is blocked by some student i with assignment $\mu \in B$. Let $x = \mu_i$. Note that there does not exist $\phi \in S$ such that $\phi_i = x$ as otherwise, i would block ν with ϕ in which case $\nu \notin B$.

Thus, by Lemma 13', $\mu_i^I P_i x P_i \mu_i^O$. For student i , define the "legal" contracts for i as

$$S(i) = \{z \in \mathcal{X}_i \mid \exists \phi \in S \text{ such that } \phi_i = z\}.$$

Among i 's legal contracts that she prefers to x , let y be her least favorite, i.e. $y \in S(i)$, $y P_i x$, and there does not exist $z \in S(i)$ such that $y P_i z P_i x$. Similarly, let u be i 's favorite school among her legal contracts that she likes less than x , i.e. $u \in S(i)$, $x P_i u$, and there does not exist $z \in S(i)$ such that $x P_i z P_i u$. By Lemma 13', y and u are well-defined. Let

³⁴Recall that the existence of S is assured because π^2 is an increasing function and for some n we have $S^n = \pi^2(S^n)$. As we have shown, $S^n \subseteq \pi(S^n)$ holds as well.

$\overline{X}^y = \{\phi \in S \mid \phi_i = y\}$. Note that if $\phi, \phi' \in X^y$, then $\phi \wedge \phi'_i = y$ and therefore $\phi \wedge \phi' \in X^y$. Thus, X^y has a well-defined minimum element (with respect to students' preferences). Let

$$\overline{\mu} := \min_{>} \overline{X}^y \quad (20)$$

Now we define the students' favorite assignment that is worse than $\overline{\mu}$. Let

$$\underline{X} = \{\phi \in S \mid \overline{\mu} \neq \phi \text{ and } \overline{\mu}_i R_i \phi_i \text{ for all } i \in A\}.$$

Note that by our choice of y and u we have for all $\phi \in \underline{X}$, $y = \phi_i$ or $u R_i \phi_i$. If $y = \phi_i$, then $\phi \in \overline{X}^y$, a contradiction to $\phi \neq \overline{\mu}$ and $\overline{\mu}_j R_j \phi_j$ for all $j \in A$. Thus, for all $\phi \in \underline{X}$, $u R_i \phi_i$. Now note that if $\phi, \phi' \in \underline{X}$, then $\phi \vee \phi' \in \underline{X}$ because $\overline{\mu}_i = y P_i \phi \vee \phi'_i$. Therefore, \underline{X} has a well-defined maximum assignment. Let

$$\underline{\mu} := \max_{>} \underline{X} \quad (21)$$

As shown already above, we have $u R_i \underline{\mu}_i$. If $u \neq \underline{\mu}_i$, then by $u \in S(i)$, there exists $\phi \in S$ such that $\phi_i = u$. Because S is a lattice and $y P_i u$, we have $\phi \wedge \overline{\mu} \in S$ and $\phi \wedge \overline{\mu}_i = u$. Since $\overline{\mu}_j R_j \phi \wedge \overline{\mu}_j$ for all $j \in A$ and $\overline{\mu} \neq \phi \wedge \overline{\mu}$, we have $\phi \wedge \overline{\mu} \in \underline{X}$. Hence, we must have $\underline{\mu}_i = u$.

Let $x_O = a$, $y_O = b$ and $u_O = c$.

Claim 1: $\overline{\mu}_j R_j \underline{\mu}_j$ for all $j \in A$ and consequently for every school d , $V(\overline{\mu}, d) \subseteq V(\underline{\mu}, d)$.

Claim 1 follows from our construction of $\overline{\mu}$ and $\underline{\mu}$: we have $\overline{\mu}_j R_j \underline{\mu}_j$ for all $j \in A$. Thus, $V(\overline{\mu}, d) \subseteq V(\underline{\mu}, d)$ for all $d \in O$. In particular, $\underline{\mu} \in \underline{X}$ and for every $\phi \in \underline{X}$, $\overline{\mu}_j R_j \phi_j$ for all $j \in A$.

Since $\overline{\mu}_i P_i x = \mu_i$, we have $\overline{\mu} \wedge \mu_i = x$. In particular, $x \in C_a(\overline{\mu}_a \cup \mu_a)$ and by substitutability of C_a , $x \in C_a(\overline{\mu}_a \cup \{x\})$. By the Rural Hospitals Theorem, $|\overline{\mu}_a| = |C_a(\overline{\mu}_a \cup \mu_a)|$. Thus, by LAD, $|C_a(\overline{\mu}_a \cup \{x\})| = |\overline{\mu}_a|$, and there exists a unique contract $t_1 \in \overline{\mu}_a \setminus C_a(\overline{\mu}_a \cup \{x\})$. Let $(t_1)_A = r_1$.

We show $\overline{\mu}_{r_1} \neq \underline{\mu}_{r_1}$: otherwise by definition, $\overline{\mu}_{r_1} = \underline{\mu}_{r_1} = t_1$. But then $\overline{\mu} \wedge \underline{\mu}_{r_1} = t_1$ and $t_1 \in C_a(\overline{\mu}_a \cup \underline{\mu}_a)$. If $x \in C_a(\overline{\mu} \wedge \underline{\mu}_a \cup \{x\})$, then by $\overline{\mu} \wedge \underline{\mu} = \underline{\mu}$, we have that i blocks $\underline{\mu}$ with μ , a contradiction to $\underline{\mu} \in S$. Thus,

$$x \notin C_a(\overline{\mu} \wedge \underline{\mu}_a \cup \{x\}) = C_a(C_a(\overline{\mu}_a \cup \underline{\mu}_a) \cup \{x\}) = C_a(\overline{\mu}_a \cup \underline{\mu}_a \cup \{x\}),$$

where the first equality follows from the definition of $\overline{\mu} \wedge \underline{\mu}$ and the second one from Lemma 9. Thus, $x \notin C_a(\overline{\mu}_a \cup \underline{\mu}_a \cup \{x\})$ and $t_1 \in C_a(\overline{\mu}_a \cup \underline{\mu}_a)$. Now by substitutability of C_a and the LAD, we must have $t_1 \in C_a(\overline{\mu}_a \cup \underline{\mu}_a \cup \{x\})$. This is a contradiction to $t_1 \notin C_a(\overline{\mu}_a \cup \{x\})$ and substitutability of C_a . Thus, we must have $\overline{\mu}_{r_1} \neq \underline{\mu}_{r_1}$ and $\overline{\mu}_{r_1} P_{r_1} \underline{\mu}_{r_1}$.

Then $t_1 \in \overline{\mu}_a \setminus C_a(\overline{\mu}_a \cup \{x\})$ and in words, t_1 is a contract a would reject if $\overline{\mu}_a \cup \{x\}$ is proposed.

We define an iterative procedure that is a variation of the vacancy chains that is inherit in the Deferred Acceptance algorithm (when students propose sequentially à la McVitie

and Wilson). For each student l , define all contracts that l strictly prefers to $\bar{\mu}_l$ to have been rejected. Formally, letting for student l ,

$$\bar{O}(l) = \{z \in \mathcal{X}_l \mid z \in B(l) \text{ and } \bar{\mu}_l R_l z\}.$$

Then student l uses the preference \bar{P}_l defined by (i) for all $v, w \in \bar{O}(l)$, $v\bar{P}_l w \Leftrightarrow vP_l w$ and (ii) for all $v \in \bar{O}(l)$ and all $w \in O \setminus \bar{O}(l)$, $v\bar{P}_l \bar{P}_l w$. Let school a reject contract t_1 . This starts a vacancy chain. We only allow student l to propose contracts which are possible for l . Whenever a student is rejected, she proposes her favorite contract that has not been rejected. In other words, we use the profile $(\bar{P}_l)_{l \in A}$ for the vacancy chain (starting first with rejecting t_1 by a). Each time a school receives a new application, it chooses among all the contracts that have ever applied to it.

Claim 2: In the vacancy chain, no student j proposes a contract worse than $\underline{\mu}_j$.

If not, then let l be the first student in the vacancy chain such that $\underline{\mu}_l$ is rejected. Let $d = (\underline{\mu}_l)_O$ and let Y be all contracts who have been proposed to d . For every $z \in Y$ with $z_A = j$, $z R_j \underline{\mu}_j$ since l is the first student rejected by her assignment under $\underline{\mu}$. Thus,

$$Y \subseteq V(\underline{\mu}, d).$$

By (i) of Lemma 14, $C_d(V(\underline{\mu}, d)) = \underline{\mu}_d$. Therefore, by $\underline{\mu}_l \in \underline{\mu}_d$ and substitutability of C_d , $\underline{\mu}_l$ cannot be rejected by d , a contradiction.

Note that Claim 2 also holds for student r_1 because $\bar{\mu}_{r_1} P_{r_1} \underline{\mu}_{r_1}$.

In the above definition of the vacancy chain, if a student j ever proposes a contract associated with school a , then we pause to make sure that school a is better off despite the fact that a did not voluntarily reject t_1 . For now, assume that student j proposes z_j such that $(z_j)_O = a$ in the vacancy chain. By Claim 2, $z_j R_j \underline{\mu}_j$. By (i) of Lemma 14' and LAD, a chooses exactly $|\underline{\mu}_a| = |\bar{\mu}_a|$ contracts (because every student proposing a contract associated with a , the contract then belongs to $V(\underline{\mu}, a)$). Prior to j 's proposal of z_j , a is holding onto $|\bar{\mu}_a| - 1$ proposals (because we rejected $t_1 \in \bar{\mu}_a$). If we allowed a to choose amongst t_1 , z_j , and the $|\bar{\mu}_a| - 1$ proposals she is holding, then she would wish to hold onto $|\bar{\mu}_a|$ proposals and reject one contract. Call this contract t_2 and $(t_2)_A = r_2$. If $t_2 = t_1$ (the contract we already rejected), then we stop (because we rejected the "right" contract in first place). Otherwise, school a rejects t_2 and we continue. Note that in this case, the new proposed contract z_j did not come from r_1 , or else (for $j = r_1$) we have $\bar{\mu}_j = t_1 P_j z_j$ and by (ii) of Lemma 14', $z_j \in C_a(\bar{\mu}_a \cup \{z_j\})$, and a would have wanted to reject t_1 by construction. Set $j = j_1$ and j_1 proposed z_{j_1} . Continue the vacancy chain with t_2 as the rejected contract. In general, whenever a student j_m proposes a contract z_{j_m} associated with a , we check to see if $t_1 \in C_a(\bar{\mu}_a \cup \{z_{j_1}, \dots, z_{j_m}\})$. If $t_1 \notin C_a(\bar{\mu}_a \cup \{z_{j_1}, \dots, z_{j_m}\})$, then we stop (because we rejected the "right" contract t_1 in first place). If $t_1 \in C_a(\bar{\mu}_a \cup \{z_{j_1}, \dots, z_{j_m}\})$, then $j_m \neq r_1$ (as otherwise (ii) of Lemma 14' and substitutability would be violated by $t_1 \notin C_a(\bar{\mu}_a \cup \{x\})$), and a would prefer to reject one of her current proposals and keep t_1 . We allow a to reject this contract and continue.

The process ends when $t_1 \notin C_a(\bar{\mu}_a \cup \{z_{j_1}, \dots, z_{j_m}\})$ for some m or when a student's possible contracts all have been rejected or when a school accepts the application without

rejecting one of its current contracts. Let ϕ be the assignment that results from this process. By Claim 2, we have for all $j \in A$, $\phi_j R_j \underline{\mu}_j$.

Claim 3: The vacancy chain ends with a proposed contract associated with a .

There are only three ways for the vacancy chain to end: (1) a student proposes a contract associated with a , (2) a student proposes a contract associated with school $b \neq a$ and b accepts the contract without rejecting any contract, and (3) a student's possible contracts are all rejected.

We show that (3) does not occur. If student l is part of the vacancy chain, then $\bar{\mu}_l \neq l$. Therefore, $\underline{\mu}_l \neq l$ by the Rural Hospital Theorem. Since by Claim 2, $\phi_l R_l \underline{\mu}_l$, we have $\phi_l \neq l$. Therefore, the vacancy chain does not end with a student's possible contracts all having been rejected. Similarly, (2) does not occur: for every school $b \neq a$, $|\bar{\mu}_b| = |\underline{\mu}_b|$. Since $\phi_j R_j \underline{\mu}_j$ for all $j \in A$, we have $V(\phi, b) \subseteq V(\underline{\mu}, b)$ (meaning that b has more contracts to choose from under $\underline{\mu}$ as the students are less happy with their assignment). By (i) of Lemma 14', we have $\bar{C}_b(V(\underline{\mu}, b)) = \underline{\mu}_b$. But then $|\phi_b| > |\underline{\mu}_b|$ would violate the Law of Aggregate Demand for b to accept a contract without rejecting another (because $\phi_b \subseteq V(\underline{\mu}, b)$). Therefore, (1) must occur and the vacancy chain can only conclude when a student l proposes a contract associated with a .

Claim 4: $\phi \in S$.

For any school $b \neq a$, school b receives a better set of contracts under ϕ than under $\bar{\mu}$ as it has weakly more contracts to choose from. Mathematically, $C_b(\phi_b \cup \bar{\mu}_b) = \phi_b$. School a is the only school which did not voluntarily reject all of its contracts as a did not voluntarily reject t_1 . However, the key point is that the vacancy chain must stop with an application to a , and we only stop after an application to a if a now wants to reject t_1 . Therefore, a is made strictly better off by the vacancy chain. Consider a student j , contract z_j and school $b = (z_j)_O$ such that z_j is possible for j and $z_j P_j \phi_j$. If $z_j P_j \bar{\mu}_j$, then $z_j \notin C_b(\bar{\mu}_b \cup \{z_j\})$ or else $\bar{\mu}$ would be blocked. Since b did not choose z_j before, b does not choose z_j now that she has weakly more contracts to choose from. If $\bar{\mu}_j R_j z_j$ then z_j was rejected by b during the vacancy chain and j is not able to block ϕ with b and contract z_j .

Claim 5: $x P_i \phi_i$

Since $\phi \in S$, we have $\phi_i \neq x$. Suppose by contradiction that $\phi_i P_i x$. By our choice of y and u and $\phi \in S$, this can only happen if y was never rejected by school b . Therefore, $y = \phi_i = \bar{\mu}_i$. Because the vacancy chain stops with an application to a where t_1 is rejected, we must have $t_1 = \bar{\mu}_{r_1} P_{r_1} \phi_{r_1}$. By Claim 4, $\phi \in S$ and thus, $\phi \in \bar{X}_b$. But now this is a contradiction as $\bar{\mu} = \min_{>} \bar{X}^b$.

Now Claim 5 yields the contradiction: student i proposed in the vacancy chain to x before proposing to ϕ_i (because $x \in \bar{O}(i)$). But when i proposed x , the vacancy chain must stop as $t_1 \in \bar{\mu}_a \setminus C_a(\bar{\mu}_a \cup \{x\})$, $x \in C_a(\bar{\mu}_a \cup \{x\})$, and thus when $j_m = i$, we must have $t_1 \notin C_a(\bar{\mu}_a \cup \{z_{j_1}, \dots, z_{j_m}\})$ as otherwise substitutability of C_a is violated. But then we must have $x = \phi_i$ which contradicts Claim 5. \square

The proof of Theorem 2 carries over unchanged to the assignment with contracts framework, i.e. there exists a unique legal set of assignments. Since this set is a lattice, there

exists a student-optimal legal assignment. Using Lemma 13' and the same logic as in Proposition 1, again it follows that this assignment is efficient among all individually rational assignments.

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