Strategyproof Choice of Acts: Beyond Dictatorship*

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Abstract

We model social choices as acts mapping states of nature to (public) outcomes. A social choice function (or SCF) assigns an act to every profile of subjective expected utility preferences over acts. A SCF is strategyproof if no agent ever has an incentive to misrepresent her beliefs about the states of nature or her valuation of the outcomes; it is ex-post efficient if the act selected at any given preference profile picks a Pareto-efficient outcome in every state of nature. We offer a complete characterization of all strategyproof and ex-post efficient SCFs. The chosen act must pick the most preferred outcome of some (possibly different) agent in every state of nature. The set of states in which an agent's top outcome is selected may vary with the reported belief profile; it is the union of all the states assigned to her by a collection of bilaterally dictatorial and bilaterally consensual assignment rules.

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1 Introduction

We address the problem of designing incentive-compatible mechanisms for making social choices under uncertainty. Following Savage (1954), we model such choices as acts mapping states of nature to outcomes, and we assume that agents compare acts according to the subjective expected utility they yield. Society chooses acts on the basis of the preferences of its members: a social choice function asks agents to report their preferences over acts, and assigns an act to every preference profile. Outcomes are public in nature: they are of interest to all agents. Applications range from a democratic government choosing social policies in an uncertain environment to a manager making investment decisions on behalf of the shareholders of the firm. Since individual preferences are (a priori) private information, it is important that a social choice function be incentive-compatible. This paper focuses on the property of strategyproofness, which requires that reporting one's true preferences be a dominant strategy: no agent should ever have an incentive to misrepresent her beliefs or her valuation of the outcomes. Because subjective expected utility preferences form a restricted domain,[1] the Gibbard-Satterthwaite theorem (Gibbard (1973), Satterthwaite (1975)) does not apply. This raises the problem of describing the set of strategyproof social choice functions.

To the best of our knowledge, this problem has not been studied. The related literature may be divided into three strands. The first strand belongs to the field of statistics. It is concerned with the problem of eliciting an agent's assessment of the likelihood of uncertain events. The best known incentive-compatible elicitation procedures are Savage's (1971) proper scoring rules; see Gneiting and Raftery (2007) for a survey of the literature on the topic. Other procedures include de Finetti's (1974) promissory notes method and Karni's (2009) direct revelation mechanism. These methods do not elicit the agent's valuation of the outcomes and do not address the social choice problem of selecting an act based on the preferences of several individuals.

The second and most closely related strand studies strategyproofness in the context of risk, that is, when society chooses lotteries rather than acts. The seminal contribution is due to Gibbard (1977), who analyzes social choice rules asking agents to report their preferences over sure outcomes only. Hylland (1980), Dutta, Peters and Sen (2007, 2008), and Nandeibam (2013) allow agents to report full-fledged von Neumann-Morgenstern preferences over lotteries. A central finding in this literature is that every strategyproof and ex-post efficient social choice function is a random dictatorship. Ex-post efficiency requires that the chosen lottery attaches zero probability to every Pareto-dominated sure outcome. A random dictatorship selects each agent's most preferred outcome with a probability that does not depend on the reported preference profile.

[1] With two states of nature and three outcomes, there are 362 880 linear preference orderings over the 9 possible acts, of which only 96 are of the expected utility type.

Finally, let us mention that the issue of preference aggregation under uncertainty has received a good deal of attention: see Hylland and Zeckhauser (1979), Mongin (1995), Gilboa, Samet and Schmeidler (2004), and Gilboa, Samuelson and Schmeidler (2014), among others. This literature, which is normative in nature, is not concerned with the incentive-compatibility issue and is therefore only tangentially related to our work. It shows that utilitarian aggregation of preferences is problematic; it also questions the desirability of Pareto efficiency when individual beliefs differ, and proposes weakened versions of it.

In line with the literature on strategyproofness under risk, we restrict attention to social choice functions that are ex-post efficient. Under uncertainty, ex-post efficiency means that the act selected at a given preference profile should recommend a Pareto efficient outcome in every state of nature. This does not imply that the chosen act is Pareto efficient.

We offer a complete characterization of all strategyproof and ex-post efficient social choice functions.

We begin by proving that every such function must be a *top selection:* at every preference profile, the chosen act must pick in each state of nature the most preferred outcome of some agent (possibly picking different agents in different states). A top selection is fully characterized by its associated *assignment rule* determining in which states of nature each agent dictates the outcome.

We then describe which assignment rules do generate strategyproof social choice functions. Constant assignment rules are one obvious possibility; the social choice functions they generate are analogous to the random dictatorships identified in the literature on strategyproof choice of lotteries. But, in contrast to the findings in that literature, there exist here more flexible rules. It turns out that if the agents' valuations cannot be used in assigning states, their *beliefs* can. To some extent, the mechanism designer can exploit the differences in subjective probabilities to make sure that each agent selects the outcome in states that she finds relatively more likely.

This can be done in two primitive ways, which turn out to constitute the building blocks of all ex-post efficient strategyproof social choice functions. A *bilaterally dictatorial assignment rule* lets one agent, say 1, choose from an exogenous menu of events (i.e., subsets of states) the one she considers most likely – leaving the complement event to some other predetermined agent, say 2. The corresponding social choice function then picks 1's top outcome in the event she declared most likely, and 2's top outcome otherwise. Under a *bilaterally consensual assignment rule,* the state space is exogenously partitioned into two events. The first is tentatively assigned to, say, agent 1, and its complement is assigned to, say, 2. However, if agent 1 reports that the second event is more likely than the first *and* agent 2 reports the opposite belief, they exchange events. The social choice function picks an agent's reported top outcome in every state that the bilaterally consensual assignment rule has assigned to her.

Under the basic rules described above, only two agents have a say in the final decision. But such rules can be combined as follows (if the mechanism designer wants all agents to affect the decision). Fix an exogenous partition of the state space into events. For each event belonging to that partition, choose a (possibly different) pair of agents. Use a bilaterally dictatorial or a bilaterally consensual "sub-rule" to assign the states belonging to that event on the basis of these two agents' conditional beliefs over these states. Compute the overall event assigned to an agent by taking the union of the events assigned to her by all these assignment sub-rules. Our theorem asserts that every strategyproof and ex-post efficient social choice function is a top selection based on a such a union of bilaterally dictatorial or bilaterally consensual assignment sub-rules.

Two remarks are in order. The first is a point of (re)interpretation. Assignment rules, which are mappings from profiles of beliefs into partitions of the state space, are mathematically equivalent to rules for allocating (valuable) indivisible objects to agents having additively separable preferences over bundles of objects.[2] It is easy to see that the assignment rule associated with a strategyproof and ex-post efficient social choice function must itself be strategyproof: by misrepresenting her *beliefs,* an agent cannot obtain an event she considers more likely than the one she gets by reporting truthfully. When there are only two agents, a social choice function is strategyproof and ex-post efficient if and only if it is generated by a strategyproof assignment rule. In that particular case, as a by-product, our theorem solves the problem of characterizing all strategyproof assignment rules for allocating indivisible objects between two agents with additively separable preferences: these rules are precisely the constant, dictatorial, or consensual unions defined in Section 4. This two-agent result was proved independently by Amanatadis et al. (2017), who do not study at all the problem of choosing social outcomes under uncertainty, nor consider n-agent assignment rules.

The second remark is technical. The set of acts is a Cartesian product, and subjective expected utility preferences over acts are additively separable. It is known that when individual preferences over a product set of social alternatives are separable *and* form a suitably rich domain, strategyproof social choice rules are products of strategyproof "sub-rules" defined on the marginal profiles of preferences over the components of the social alternatives. Le Breton and Sen (1999) offer general theorems of this type; earlier papers proving variants of the result include Border and Jordan (1983), Barberà, Sonnenschein and Zhou (1991), and Barberà, Gul and Stacchetti (1993). This decomposition property does not hold in our setting. The reason is that subjective expected utility preferences do not form a rich domain. Le Breton and Sen's (1999) richness condition requires that for any collection of admissible preferences over the components of the social alternatives there exists a preference over the social alternatives which induces marginal preferences over components coinciding

[2] Reinterpret states of nature as objects and observe that beliefs define additively separable "preferences" over subsets of states. Pápai (2007) studies various subclasses of n-agent strategyproof allocation rules for arbitrary monotonic preferences.

with the ones in that collection. Since in our setting all state-contingent preferences over outcomes induced by a subjective expected utility preference over acts are identical, Le Breton and Sen's condition is violated. It is this lack of richness that makes it possible to define non-decomposable rules where beliefs affect the states where an agent's top outcome is selected.

2 The problem

There is a finite set of agents $N = \{1, ..., n\}$ with $n \geq 2$, a finite set of states of nature Ω with $|\Omega| \geq 2$, and a finite set of outcomes X with $|X| \geq 3$. Outcomes should be interpreted as public alternatives (such as policies or allocations) that are of interest to all agents. Subsets of Ω are called events. An *act* is a function $f \in X^\Omega$. Agent i's preference ordering $\succcurlyeq_i$ over acts is assumed to be of the subjective expected utility type: there exist a valuation function $v_i : X \to \mathbb{R}$ and a subjective probability measure p_i on the set of events such that for all $f, g \in X^\Omega$,

$$f \succcurlyeq_i g \Leftrightarrow \sum_{\omega \in \Omega} p_i(\omega) v_i(f(\omega)) \geq \sum_{\omega \in \Omega} p_i(\omega) v_i(g(\omega)).$$

Note that we write ω instead of $\{\omega\}$ and i instead of $\{i\}$; we will often omit curly brackets to alleviate notation. Of course, since the set of acts is finite, neither the valuation function v_i nor the subjective probability measure p_i representing the preference ordering $\succcurlyeq_i$ are determined uniquely.

Throughout the paper, we assume that $\succcurlyeq_i$ is a *linear* ordering. This is a reasonable assumption given that the set of acts is finite. It implies that for any (v_i, p_i) representing $\succcurlyeq_i$, (i) v_i is injective and (ii) p_i is injective: for all $E, E' \subseteq \Omega$, $p_i(E) = p_i(E') \Rightarrow E = E'$. Because $p_i(\emptyset) = 0$, it follows from (ii) that $p_i(\omega) > 0$ for all $\omega \in \Omega$. We further assume, without loss of generality, that v_i is normalized: $\min_X v_i = 0 < \max_X v_i = 1$. We denote by $\mathcal{V}$ the set of normalized injective valuation functions v_i and by $\mathcal{P}$ the set of (necessarily positive) injective measures p_i, which we call *beliefs*. Formally, the domain of preferences is the set of pairs (v_i, p_i) that generate a linear ordering of the set of acts, that is to say,

$$\mathcal{D} = \left\{ (v_i, p_i) \in \mathcal{V} \times \mathcal{P} : \sum_{\omega \in \Omega} p_i(\omega) v_i(f(\omega)) \neq \sum_{\omega \in \Omega} p_i(\omega) v_i(g(\omega)), \forall f, g \in X^\Omega \text{ s.t. } f \neq g \right\}.$$

A *social choice function* (or *SCF*) is a function $\varphi : \mathcal{D}^N \to X^\Omega$. We denote the ordered list $((v_1, p_1), ..., (v_n, p_n)) \in \mathcal{D}^N$ by (v, p). In principle, our formulation allows a SCF φ to choose different acts for profiles (v, p) and (v', p') that represent the same profile of preferences $(\succcurlyeq_1, ..., \succcurlyeq_n)$. Of course, the requirement of strategyproofness defined below will rule this out. With a slight abuse of terminology, we therefore call any $(v, p) \in \mathcal{D}^N$ a *preference profile*. We call $v = (v_1, ..., v_n) \in \mathcal{V}^N$ a *valuation profile* and $p = (p_1, ..., p_n) \in \mathcal{P}^N$ a *belief profile*. For every preference profile $(v, p) \in \mathcal{D}^N$ and every $\omega \in \Omega$, we denote by $\varphi(v, p; \omega)$ the outcome chosen by the act $\varphi(v, p)$ in state ω.

We emphasize that the chosen act is allowed to change when an agent's valuation function is replaced with another that generates the same ranking of the outcomes but a different ordering of the acts: no information about individual preferences is a priori discarded.

As usual, $v_{-i} \in \mathcal{V}^{N \setminus i}$ and $p_{-i} \in \mathcal{P}^{N \setminus i}$ denote the valuation and belief sub-profiles obtained by deleting v_i from v and p_i from p, respectively. A SCF φ is *strategyproof* if, for all $i \in N$, all $(v, p) \in \mathcal{D}^N$, and all $(v_i', p_i') \in \mathcal{D}$,

$$\sum_{\omega \in \Omega} p_i(\omega) v_i(\varphi(v, p; \omega)) \geq \sum_{\omega \in \Omega} p_i(\omega) v_i(\varphi((v_i', v_{-i}), (p_i', p_{-i}); \omega)).$$

This means that distorting one's preferences –by misrepresenting one's valuation function or one's beliefs– is never profitable.

A SCF φ is *ex-post efficient* if for all $(v, p) \in \mathcal{D}^N$ and all $\omega \in \Omega$, there is no $x \in X$ such that $v_i(x) > v_i(\varphi(v, p; \omega))$ for all $i \in N$. In words, ex-post efficiency says that a social outcome that all agents value less than some other outcome x should never be picked. This requirement does not imply that the acts chosen by φ are (ex-ante Pareto) efficient at all preference profiles.

The purpose of this paper is to describe the class of all strategyproof and ex-post efficient SCFs.

3 A preliminary result: the Top Selection lemma

An *assignment* is an ordered list $\mathbf{A} = (A_1, ..., A_n)$ of subsets of Ω such that $A_i \cap A_j = \emptyset$ whenever $i \neq j$, and $\cup_{i \in N} A_i = \Omega$. We refer to the condition that $A_1, ..., A_n$ partition Ω as *feasibility.* Let $\mathcal{S}$ denote the set of assignments. An *assignment rule* is a function $s : \mathcal{P}^N \to \mathcal{S}$ assigning to each belief profile p an assignment $s(p) = (s_1(p), ..., s_n(p))$. We refer to $s_i(p)$, the event assigned to agent i at p, as i's *share.* Note that an agent's share may be empty.

For all $v_i \in \mathcal{V}$, let $\tau(v_i)$ denote the unique maximizer (or *top*) of v_i in X.

Top Selection Lemma. *If a SCF φ is strategyproof and ex-post efficient, then there exists a unique assignment rule $s : \mathcal{P}^N \to \mathcal{S}$ such that, for all $(v, p) \in \mathcal{D}^N$, $\omega \in \Omega$, and $i \in N$, we have*

$$\omega \in s_i(p) \Rightarrow \varphi(v, p; \omega) = \tau(v_i). \tag{1}$$

We say that the assignment rule s in (1) is *associated* with (or *generates*) φ; and we call φ a *top selection.*

The Top Selection lemma really contains two statements. The first is that every strategyproof and ex-post efficient SCF can only choose acts that pick in every state of nature some agent's top outcome. This forbids choosing acts that select "compromise outcomes". As an illustration, suppose that $N = \{1, 2\}$, $X = \{a, b, c\}$ and consider a preference profile (v, p) such that $v_1(a) = v_2(c) = 1$, $v_1(b) = v_2(b) = .99$, and $v_1(c) = v_2(a) = 0$. The Top Selection lemma tells us that the natural compromise outcome b cannot be picked in any state of nature at this profile. The only admissible form of compromise (at a fixed belief profile p) consists in allowing different agents to choose the final outcome in different states of nature. An obvious corollary is that no strategyproof SCF is (ex-ante Pareto) efficient.

The second statement contained in the Top Selection lemma is that the set of states in which an agent's top outcome is selected depends only upon the profile of beliefs: the valuation profile v is not an argument of the function s. An immediate corollary is that a strategyproof and ex-post efficient SCF is *tops-only*:[3] if $(v, p), (v', p) \in \mathcal{D}^N$ and $\tau(v_i) = \tau(v'_i)$ for all $i \in N$, then $\varphi(v, p) = \varphi(v', p)$. The chosen act depends only upon the belief profile and the tops of the valuation functions.

The proof of the Top Selection lemma is in Appendix A but it may be worth sketching the main lines of the argument here. We proceed by induction. We first show that every two-agent strategyproof and ex-post efficient SCF must be a tops-only top selection (Lemma 4). We then focus on the case $n \geq 3$ and, making use of the induction hypothesis, we show in Lemma 5 that a strategyproof and ex-post efficient SCF must select top outcomes whenever two agents (or more) report the same top. Finally, we show in Lemma 6 that under our two axioms (a) the chosen act must select only top outcomes (for all preference profiles and in every state); (b) the tops-only property holds: the decision must remain the same if some agents change their valuations of any non-top outcomes. It thus follows from these results that, at each preference profile (v, p), every strategyproof and ex-post efficient SCF must partition the state space Ω by assigning some event $s_i(v, p)$ to each agent i. We conclude the proof by arguing that, in fact, each agent's share of the state space does not vary with the valuation profile v. Three consequences of strategyproofness (stated in Lemmas 1-3) are pervasive in the proof sketched above.

4 Statement of the theorem

The Top Selection lemma is not a characterization result yet. The SCF generated by an assignment rule is ex-post efficient but need not be strategyproof. Our task is now to determine which assignment rules do indeed generate a strategyproof SCF.

In order to state our main theorem, we need to extend some of our notation to subsets of events. Fix $\emptyset \neq \Omega' \subseteq \Omega$. A *belief on* Ω' is an injective probability measure p_i defined on $2^{\Omega'}$ and the set of beliefs on Ω' is denoted $\mathcal{P}(\Omega')$; note that $\mathcal{P}(\Omega) = \mathcal{P}$. An *assignment of* Ω' is an ordered list of non-intersecting subsets of Ω' that cover Ω' and $\mathcal{S}(\Omega')$ denotes the set of assignments of Ω'; note that $\mathcal{S}(\Omega) = \mathcal{S}$. An Ω'-*assignment*

[3] The cumbersome term "valuations tops-only" would be more precise: the SCFs identified in the Top Selection lemma may certainly use more information than just the tops of the preference orderings $\succcurlyeq_1, ..., \succcurlyeq_n$ in the set of acts.

rule is a function $s : \mathcal{P}(\Omega')^N \to \mathcal{S}(\Omega')$.

If $p_i \in \mathcal{P}$, we denote by $p_i \mid \Omega'$ the conditional belief generated by p_i on $\mathcal{P}(\Omega')$, namely, $(p_i \mid \Omega')(A) = p_i(A)/p_i(\Omega')$ for all $A \subseteq \Omega'$. If $p \in \mathcal{P}^N$, we write $p \mid \Omega' = (p_1 \mid \Omega', ..., p_n \mid \Omega')$.

Three types of assignment rules will be central in our analysis: the constant, bilaterally dictatorial, and bilaterally consensual rules. An Ω'-assignment rule s is *constant* if there exists an assignment $\mathbf{A}$ of Ω' such that $s(p) = \mathbf{A}$ for all $p \in \mathcal{P}(\Omega')^N$.

A *proper covering* of Ω' is a family $\mathcal{A}$ of subsets of Ω' such that $A \setminus B$ and $B \setminus A$ are nonempty for all distinct $A, B \in \mathcal{A}$, $\cup_{A \in \mathcal{A}} A = \Omega'$, and $\cap_{A \in \mathcal{A}} A = \emptyset$. For any belief p_i on Ω', we denote by $\underset{\mathcal{A}}{\operatorname{argmax}}\, p_i$ the event maximizing p_i in the family $\mathcal{A}$. If $i, j \in N$ are two distinct agents, a rule s is called (i, j)-*dictatorial* if there exists a proper covering $\mathcal{A}$ of Ω' such that $s_i(p) = \underset{\mathcal{A}}{\operatorname{argmax}}\, p_i$ and $s_j(p) = \Omega' \setminus \underset{\mathcal{A}}{\operatorname{argmax}}\, p_i$ for all $p \in \mathcal{P}(\Omega')^N$. Note that, by feasibility, $s_k(p) = \emptyset$ for all $k \neq i, j$ and all p. Note also that, because $\mathcal{A}$ is a proper covering of Ω', an (i, j)-dictatorial rule s is not constant; moreover, there is no ordered pair $(i', j') \neq (i, j)$ for which s is (i', j')-dictatorial. We call s *bilaterally dictatorial* if it is (i, j)-dictatorial for some (unique) ordered pair of agents (i, j).

Finally, we say that s is $\{i, j\}$-*consensual* if there exists a nonempty set $A \subset \Omega'$ (where $\subset$ denotes strict inclusion) such that

$$(s_i(p), s_j(p)) = \begin{cases} (\Omega' \setminus A, A) & \text{if } p_i(\Omega' \setminus A) > p_i(A) \text{ and } p_j(A) > p_j(\Omega' \setminus A), \\ (A, \Omega' \setminus A) & \text{otherwise.} \end{cases}$$

Again, feasibility implies $s_k(p) = \emptyset$ for all $k \neq i, j$ and all p. We call s *bilaterally consensual* if it is $\{i, j\}$-consensual for some (unique) unordered pair of agents $\{i, j\}$.

Bilaterally dictatorial and bilaterally consensual rules exploit beliefs in different ways. The former allow the mechanism designer to extract detailed information about the beliefs of a single agent; their range may be large. The latter have a binary range but allow the designer to exploit differences in beliefs between agents.

An assignment rule s is a *union of constant, bilaterally dictatorial, or bilaterally consensual rules* (or a *C-BD-BC union*), if there is a partition $\{\Omega^t\}_{t=1}^T$ of Ω, and, for each $t = 1, ..., T$, a constant, bilaterally dictatorial, or bilaterally consensual Ω^t-assignment rule s^t such that

$$s_i(p) = \cup_{t=1}^T s_i^t(p \mid \Omega^t)$$

for all $p \in \mathcal{P}^N$ and all $i \in N$. Merging cells of the partition if necessary, we may assume without loss of generality that there is at most one t for which s^t is constant and, for each ordered pair of agents (i, j), at most one t for which s^t is (i, j)-dictatorial. This is the canonical representation of a C-BD-BC union.

Theorem. *A SCF φ is strategyproof and ex-post efficient if and only if it is a top selection whose associated assignment rule s is a C-BD-BC union.*

Note that our assumption $|X| \geq 3$ is needed for this result. When there are two outcomes and an odd number of agents, majority voting between the two constant acts defines a strategyproof and ex-post efficient SCF.

Remark also that a dictatorial assignment rule —which allows a single agent to select the assignment that maximizes the subjective probability of her own share (over the range of the rule)– does not generate a strategyproof SCF if it is not the union of *bilaterally* dictatorial sub-rules. As an example, suppose that $N = \{1, 2, 3\}$, $\Omega = \{\omega_1, \omega_2, \omega_3\}$, X is arbitrary, and define the assignment rule $s : \mathcal{P}^N \to \mathcal{S}$ by

$$s(p_1, p_2, p_3) = \begin{cases} (\{\omega_1\}, \{\omega_2\}, \{\omega_3\}) \text{ if } \operatorname{argmax}_\Omega p_1 = \omega_1, \\ (\{\omega_2\}, \{\omega_3\}, \{\omega_1\}) \text{ if } \operatorname{argmax}_\Omega p_1 = \omega_2, \\ (\{\omega_3\}, \{\omega_1\}, \{\omega_2\}) \text{ if } \operatorname{argmax}_\Omega p_1 = \omega_3. \end{cases}$$

It is easy to see that s is not a union of bilaterally dictatorial sub-rules. To see why the top selection SCF φ generated by s is not strategyproof, consider a preference profile (v, p) such that $p_1(\omega_1) = .52$, $p_1(\omega_2) = .12$, $p_1(\omega_3) = .36$, $v_1(\tau(v_2)) = 1$, and $v_1(\tau(v_3)) = 0$. If all agents report their preferences truthfully, the selected act $\varphi(v, p) = (\varphi(v, p; \omega_1), \varphi(v, p; \omega_2), \varphi(v, p; \omega_3)) = (\tau(v_1), \tau(v_2), \tau(v_3))$ yields to agent 1 an expected utility of .64. If agent 1 reports (v_1, p_1') with $\operatorname{argmax}_\Omega p_1' = \omega_3$, the selected act $\varphi(v, (p_1', p_2, p_3)) = (\tau(v_2), \tau(v_3), \tau(v_1))$ yields an expected utility of .88, which is higher. A similar example can be constructed to see the importance of bilaterality for consensual assignment rules.

5 Proof of the theorem: local bilaterality

It is easy to check that every SCF generated by a C-BD-BC union is strategyproof and ex-post efficient. In order to prove the converse statement, given the Top Selection lemma, it suffices to prove that the assignment rule s associated with a strategyproof and ex-post efficient SCF φ is a C-BC-CD union. In the current section, we show that s satisfies a strong incentive-compatibility property –dubbed *super-strategyproofness*– and we use this property to characterize the local behavior of s. It turns out that this behavior is bilateral: an elementary change in an agent's belief may only affect her own share and that of *one* other agent.

Call an assignment rule $s : \mathcal{P}^N \to \mathcal{S}$ *strategyproof* if $p_i(s_i(p)) \geq p_i(s_i(p_i', p_{-i}))$ for all $i \in N$, $p \in \mathcal{P}^N$, and $p_i' \in \mathcal{P}$: no agent can increase the likelihood of the event assigned to her by misrepresenting her belief.

For any assignment $\mathbf{A} = (A_1, ..., A_n) \in \mathcal{S}$ and any $M \subseteq N$, write $A_M = \cup_{i \in M} A_i$. Call s *super-strategyproof* if $p_i(s_M(p)) \geq p_i(s_M(p_i', p_{-i}))$ for all i, M such that $i \in M \subset N$, all $p \in \mathcal{P}^N$, and all $p_i' \in \mathcal{P}$: by misrepresenting her belief, an agent can never increase the likelihood of the event assigned to any group to which she belongs.

For any $\omega \in \Omega$ and $p \in \mathcal{P}^N$, it will be convenient to let $a_\omega(p)$ denote the agent to whom s assigns ω at the belief profile p, that is,

$$a_\omega(p) = i \Leftrightarrow \omega \in s_i(p). \tag{2}$$

Super-strategyproofness Lemma. *The assignment rule s associated with a strategyproof and ex-post efficient SCF φ is super-strategyproof.*

Proof. Let φ be a strategyproof and ex-post efficient SCF and let s be the assignment rule associated with it through (1). Suppose, by way of contradiction, that there exist i, M such that $i \in M \subset N$, $p \in \mathcal{P}^N$, and $p'_i \in \mathcal{P}$ such that $p_i(s_M(p'_i, p_{-i})) > p_i(s_M(p))$. Choose $v \in \mathcal{V}^N$ such that $(v, p), (v, (p'_i, p_{-i})) \in \mathcal{D}^N$ and $v_i(\tau(v_j)) = 1$ for all $j \in M$ and $v_i(\tau(v_j)) = 0$ for all $j \in N \setminus M$. Then,

$$\begin{aligned}
\sum_{\omega \in \Omega} p_i(\omega) v_i(\varphi(v, (p'_i, p_{-i}); \omega)) &= \sum_{\omega \in \Omega} p_i(\omega) v_i(\tau_{a_\omega(p'_i, p_{-i})}) \\
&= \sum_{\omega \in \Omega : a_\omega(p'_i, p_{-i}) \in M} p_i(\omega) \\
&= p_i(s_M(p'_i, p_{-i})) \\
&> p_i(s_M(p)) \\
&= \sum_{\omega \in \Omega : a_\omega(p) \in M} p_i(\omega) \\
&= \sum_{\omega \in \Omega} p_i(\omega) v_i(\tau_{a_\omega(p)}) \\
&= \sum_{\omega \in \Omega} p_i(\omega) v_i(\varphi(v, p; \omega)),
\end{aligned}$$

contradicting the assumption that φ is strategyproof. □

Call an assignment rule s *non-bossy* if, for all $i \in N$, $p \in \mathcal{P}^N$ and $p'_i \in \mathcal{P}$, $s_i(p) = s_i(p'_i, p_{-i}) \Rightarrow s(p) = s(p'_i, p_{-i})$. Non-bossiness says that no agent can affect another agent's share without affecting her own.

Non-Bossiness Corollary. *The assignment rule s associated with a strategyproof and ex-post efficient SCF φ is non-bossy.*

Proof. Given the super-strategyproofness lemma, it suffices to show that every super-strategyproof assignment rule s is non-bossy. Let s be super-strategyproof and suppose, by way of contradiction, that there exist $i, j \in N$, $p \in \mathcal{P}^N$ and $p'_i \in \mathcal{P}$ such that $s_i(p) = s_i(p'_i, p_{-i})$ and $s_j(p) \neq s_j(p'_i, p_{-i})$. By super-strategyproofness applied to $M = \{i, j\}$ and because p_i is injective, $p_i(s_{ij}(p)) > p_i(s_{ij}(p'_i, p_{-i}))$, hence $p_i(s_j(p)) > p_i(s_j(p'_i, p_{-i}))$. Since such a strict inequality holds for every j such that $s_j(p) \neq s_j(p'_i, p_{-i})$, we have $1 = \sum_{j \in N} p_i(s_j(p)) > \sum_{j \in N} p_i(s_j(p'_i, p_{-i})) = 1$, a contradiction. □

We are now ready to characterize the *local* behavior of a super-strategyproof assignment rule. Define $\mathcal{H} = \{\{A, B\} : \emptyset \neq A, B \subset \Omega \text{ and } A \cap B = \emptyset\}$, the set of pairs of disjoint nonempty events. Two beliefs $p_i, q_i \in \mathcal{P}$ will be called $\{A, B\}$-*adjacent* if

$$\begin{aligned}
(p_i(A) - p_i(B))(q_i(A) - q_i(B)) &< 0 \text{ and} \\
(p_i(C) - p_i(D))(q_i(C) - q_i(D)) &> 0 \text{ for } \{C, D\} \in \mathcal{H} \setminus \{\{A, B\}\}.
\end{aligned}$$

We say that p_i, q_i are *adjacent* if they are $\{A, B\}$-adjacent for some $\{A, B\} \in \mathcal{H}$.

Adjacency is an ordinal property. Every belief $p_i \in \mathcal{P}$ generates a likelihood ordering $R(p_i)$ over events defined by $AR(p_i)B \Leftrightarrow p_i(A) \geq p_i(B)$. Call two beliefs p_i, q_i *ordinally equivalent* if $R(p_i) = R(q_i)$. If p_i, q_i are adjacent and p'_i is ordinally equivalent to p_i, then p'_i, q_i are adjacent. Two beliefs are adjacent if the likelihood orderings they generate differ on a single pair of disjoint nonempty events.

Example 1. *Let $\Omega = \{1, 2, 3\}$ and consider the simplex Δ depicted in Figure 1. Every point in Δ implicitly defines a measure $p_i \in \overline{\mathcal{P}}$, where $\overline{\mathcal{P}}$ denotes the closure of $\mathcal{P}$ in $[0, 1]^{2^\Omega}$. Every line segment corresponds to (the intersection with Δ of) the hyperplane $p_i(A) = p_i(B)$ generated by some pair of disjoint events $\{A, B\} \in \mathcal{H}$. Each connected component of the complement of (the union of) these line segments defines a region of ordinally equivalent beliefs: the shaded area is an example. Two beliefs are adjacent if they lie on the same side of all but one hyperplane. For instance, the beliefs p_i^1, p_i^2, which lie on the same side of all hyperplanes except $p_i(\{2\}) = p_i(\{3\})$, are $\{\{2\}, \{3\}\}$-adjacent. These beliefs generate the likelihood relations*

$$\begin{aligned} R(p_i^1) &= \{1,2,3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1\}, \{2\}, \{3\}, \\ R(p_i^2) &= \{1,2,3\}, \{1,3\}, \{1,2\}, \{2,3\}, \{1\}, \{3\}, \{2\}, \end{aligned}$$

where events are listed in decreasing order of likelihood. Note that $R(p_i^1)$ and $R(p_i^1)$ disagree not only on $\{2\}, \{3\}$ but, as a consequence, also on $\{1, 2\}, \{1, 3\}$: this does not contradict the definition of adjacency because $\{1, 2\}, \{1, 3\}$ intersect.

Local Bilaterality Lemma. *Let s be a super-strategyproof assignment rule. Let $\{A, B\} \in \mathcal{H}$ and let $i \in N$, $p \in \mathcal{P}^N$, $p'_i \in \mathcal{P}$ be such that p_i, p'_i are $\{A, B\}$-adjacent and $p_i(A) > p_i(B)$. Then, either (i) $s(p) = s(p'_i, p_{-i})$ or (ii) there exists $j \in N \setminus i$ such that*

$$\begin{aligned} s_i(p) \setminus s_i(p'_i, p_{-i}) &= A = s_j(p'_i, p_{-i}) \setminus s_j(p), \\ s_i(p'_i, p_{-i}) \setminus s_i(p) &= B = s_j(p) \setminus s_j(p'_i, p_{-i}), \\ s_k(p) &= s_k(p'_i, p_{-i}) \text{ for all } k \in N \setminus \{i, j\}. \end{aligned}$$

This is a complete description of the local behavior of s. By reporting a belief adjacent to her own, an agent i can only change the event that is assigned to her and *one* other agent j. Moreover, if the assignment is indeed modified, i and j must precisely exchange the disjoint events that have been switched in i's likelihood ordering.

Proof. Fix a super-strategyproof (hence also non-bossy) assignment rule s. Let $\{A, B\} \in \mathcal{H}$ and let $i \in N$, $p \in \mathcal{P}^N$, $p'_i \in \mathcal{P}$ be such that p_i, p'_i are $\{A, B\}$-adjacent and $p_i(A) > p_i(B)$.

Step 1. We show that for all $M \subseteq N$ such that $i \in M$, either (i) $s_M(p) = s_M(p'_i, p_{-i})$ or (ii) $s_M(p) \setminus s_M(p'_i, p_{-i}) = A$ and $s_M(p'_i, p_{-i}) \setminus s_M(p) = B$.

To see this, suppose (i) fails. Define $A_M = s_M(p) \setminus s_M(p'_i, p_{-i})$ and $B_M = s_M(p'_i, p_{-i}) \setminus s_M(p)$. These sets are disjoint and super-strategyproofness of s implies that both are nonempty; hence, they belong to $\mathcal{H}$. Suppose, by way of contradiction, that $A_M \neq A$ or $B_M \neq B$. Since p_i, p'_i are $\{A, B\}$-adjacent, their associated likelihood orderings must agree on the ranking of A_M, B_M : either (a) $p_i(A_M) > p_i(B_M)$ and $p'_i(A_M) > p'_i(B_M)$, or (b) $p_i(A_M) < p_i(B_M)$ and $p'_i(A_M) < p'_i(B_M)$. If (a) holds, then $p'_i(s_M(p)) > p'_i(s_M(p'_i, p_{-i}))$ whereas if (b) holds, then $p_i(s_M(p'_i, p_{-i})) > p_i(s_M(p))$. Each of these two inequalities contradicts super-strategyproofness.

Step 2. Applying Step 1 with $M = \{i\}$, either (i) $s_i(p) = s_i(p'_i, p_{-i})$ or (ii) $s_i(p) \setminus s_i(p'_i, p_{-i}) = A$ and $s_i(p'_i, p_{-i}) \setminus s_i(p) = B$.

If (i) holds, non-bossiness of s implies $s(p) = s(p'_i, p_{-i})$, and we are done.

If (ii) holds, let $j \in N \setminus i$. Applying Step 1 with $M = \{i, j\} = ij$, we have either (a) $s_{ij}(p) = s_{ij}(p'_i, p_{-i})$ or (b) $s_{ij}(p) \setminus s_{ij}(p'_i, p_{-i}) = A$ and $s_{ij}(p'_i, p_{-i}) \setminus s_{ij}(p) = B$. If (a) holds, then (ii) implies

$$s_j(p'_i, p_{-i}) \setminus s_j(p) = A \text{ and } s_j(p) \setminus s_j(p'_i, p_{-i}) = B \tag{3}$$

whereas if (b) holds, (ii) implies

$$s_j(p) = s_j(p'_i, p_{-i}). \tag{4}$$

By feasibility, (3) can hold for at most one agent $j \in N \setminus i$. Because of (ii), it must hold for exactly one such agent. Since (4) holds for every other agent $j \in N \setminus i$, the proof is complete. □

6 Proof of the theorem: the bilateral consensus and bilateral dictatorship lemmas

The rest of the proof of the theorem consists in exploiting the Local Bilaterality lemma to show that a super-strategyproof assignment rule is a C-BD-BC union. Fix a super-strategyproof assignment rule $s : \mathcal{P}^N \to \mathcal{S}$. Let $\Omega_0, \Omega_1, \Omega_2$ denote the sets of states whose assignment is either constant, varies with the belief of a single agent, or with the beliefs of at least two agents. That is, using the definition of a_ω in (2),

(i) $\omega \in \Omega_0 \Leftrightarrow a_\omega$ is constant on $\mathcal{P}^N$,

(ii) $\omega \in \Omega_1 \Leftrightarrow \big[$there exist $i \in N,\ p \in \mathcal{P}^N,$ and $p'_i \in \mathcal{P}$ such that $a_\omega(p) \neq a_\omega(p'_i, p_{-i})\big]$ and $\big[a_\omega(., p_{-j})$ is constant on $\mathcal{P}$ for all $j \neq i$ and $p_{-j} \in \mathcal{P}^{N \setminus j}\big]$,

(iii) $\omega \in \Omega_2 \Leftrightarrow$ there exist distinct agents $i, j \in N$, $p, q \in \mathcal{P}^N$, and $p'_i, q'_j \in \mathcal{P}$ such that $a_\omega(p) \neq a_\omega(p'_i, p_{-i})$ and $a_\omega(q) \neq a_\omega(q'_j, q_{-j})$.

By definition, $\{\Omega_0, \Omega_1, \Omega_2\}$ is a partition of Ω. In particular, the definition in (iii) allows the assignment of states in Ω_2 to vary with the beliefs of more than two agents. Moreover, the set of agents to whom a state in Ω_2 may potentially be assigned is a priori unrestricted.

We proceed by considering the states in Ω_2 first. We show that, in fact, these states can only be assigned to two distinct agents, and the assignment must be based on the beliefs of these two agents only. More specifically, states in Ω_2 must be assigned through bilateral consensus:

Bilateral Consensus Lemma. *For every $\omega \in \Omega_2$ there exists a unique event $E^\omega \subseteq \Omega_2$ containing ω and there exists a bilaterally consensual E^ω-assignment rule s^ω such that*

$$s_i(p) \cap E^\omega = s_i^\omega(p \mid E^\omega)$$

for all $p \in \mathcal{P}^N$ and $i \in N$.

The long proof of this lemma is relegated to Appendices B and C, but here is a quick overview. The proof is "by contagion".

Appendix B derives a "semi-global" characterization. For any given state $\omega \in \Omega_2$, we fix a profile π of beliefs over $\Omega \setminus \omega$, and we consider the sub-domain $\mathcal{P}^N(\pi)$ of all belief profiles on Ω generating the same profile of likelihood orderings as π on the subsets of $\Omega \setminus \omega$. Using the Local Bilaterality lemma, we show that there exist two disjoint events A, B, whose union contains ω, such that the restriction of s to $A \cup B$ coincides with a bilaterally consensual $(A \cup B)$-assignment rule *on the sub-domain* $\mathcal{P}^N(\pi)$.

In Appendix C, we consider every belief profile (π_i', π_{-i}) over $\Omega \setminus \omega$ such that π_i' is adjacent to π_i for some agent i and, in a series of "contagion lemmas", we describe how the behavior of the restriction of s to $A \cup B$ on the sub-domain $\mathcal{P}^N(\pi_i', \pi_{-i})$ is linked to the behavior of its restriction to $A \cup B$ on $\mathcal{P}^N(\pi)$. Using the connectedness of the set of all beliefs on $\Omega \setminus \omega$, we conclude that the restriction of s to $A \cup B$ must be bilaterally consensual on the whole domain $\mathcal{P}^N$. The claim follows by setting $E^\omega = A \cup B$.

The Bilateral Consensus lemma fully determines the behavior of s on Ω_2. For any two states $\omega, \omega' \in \Omega_2$, since there exist a bilaterally consensual E^ω-rule s^ω and a bilaterally consensual $E^{\omega'}$-rule $s^{\omega'}$ such that $s_i(p) \cap E^\omega = s_i^\omega(p \mid E^\omega)$ and $s_i(p) \cap E^{\omega'} = s_i^{\omega'}(p \mid E^{\omega'})$ for all $i \in N$, we must have either (i) $E^\omega = E^{\omega'}$ and $s^\omega = s^{\omega'}$, or (ii) $E^\omega \cap E^{\omega'} = \emptyset$. This delivers at once the following corollary:

Bilateral Consensus Corollary. *There exists a partition $\{\Omega^t\}_{t=1}^{T_2}$ of Ω_2 and, for each $t = 1, ..., T_2$, a bilaterally consensual Ω^t-assignment rule s^t such that*

$$s_i(p) \cap \Omega_2 = \cup_{t=1}^{T_2} s_i^t(p \mid \Omega^t)$$

for all $p \in \mathcal{P}^N$ and $i \in N$.

Next, we turn next to the assignment of the states in Ω_1. Let Ω_{11} be the subset of those states in Ω_1 whose assignment varies with the beliefs of agent 1. We show that these states are assigned by bilateral dictatorship of agent 1.

Bilateral Dictatorship Lemma. *There exist a set $N_1 \subseteq N \setminus 1$, a partition $\left\{\Omega_{11}^j\right\}_{j \in N_1}$ of Ω_{11}, and for each $j \in N_1$ a $(1, j)$-dictatorial Ω_{11}^j-assignment rule s^j*

such that

$$s_i(p) \cap \Omega_{11} = \cup_{j \in N_1} s_i^j(p \mid \Omega_{11}^j) \tag{5}$$

for all $p \in \mathcal{P}^N$ *and* $i \in N$.

The proof is in Appendix D, but let us outline it here. Consider the family of all subsets of Ω_{11} that are assigned to agent 1 at some belief profile. We begin by showing that $s_1(p) \cap \Omega_{11}$ maximizes p_1 over that family whenever p_1 is a so-called Ω_{11}-dominant belief –one in which only the probability differences between events in Ω_{11} are large. We then use the Local Bilaterality lemma to extend this observation to all belief profiles p. The next and crucial step consists in proving that every state in Ω_{11} can only be allocated to a single agent other than 1. The set Ω_{11} can therefore be partitioned into a collection of subsets $\left\{\Omega_{11}^j\right\}$ such that every state in Ω_{11}^j is allocated to either 1 or j, and super-strategyproofness can be used to show that $s_1(p) \cap \Omega_{11}^j$ maximizes p_1 over the family of all subsets of Ω_{11}^j that are assigned to agent 1 at some belief profile. The argument is completed by appealing to non-bossiness.

We have stated the Bilateral Dictatorship lemma for agent 1, but a corresponding lemma obviously holds for every agent. It now follows from these Bilateral Dictatorship lemmas, the Bilateral Consensus corollary, and the definition of Ω_0, that s is a C-BD-BC union. Together with the Top Selection lemma, this completes the proof of the Theorem.

7 Concluding comments

We have shown that strategyproof and ex-post efficient social choice functions are top selections generated by assignment rules that are unions of constant, bilaterally dictatorial, or bilaterally consensual sub-rules. Thus, under uncertainty, strategyproofness and ex-post efficiency are compatible with a form of consensuality that cannot be achieved under risk. This generates efficiency gains: any random dictatorship (that is, any SCF generated by a constant assignment rule) is Pareto-dominated by some SCF generated by a consensual rule.

We conclude by mentioning some open problems.

(1) How should we *choose* between the social choice functions identified in our theorem? Assuming a given (for instance uniform) distribution over the set of all preference profiles, one could search for social choice functions that maximize some measure of expected welfare –the expected sum of normalized utilities for instance. Alternatively, one could proceed axiomatically and impose properties that complement strategyproofness and ex-post efficiency. It is a corollary of our theorem, however, that no strategyproof SCF is (ex-ante Pareto) efficient. Anonymity and neutrality are also impossible. On the other hand, it also follows from our theorem that all strategyproof and ex-post efficient SCFs are group-strategyproof: the members of a group cannot all benefit from jointly misrepresenting their preferences. It would be interesting to explore what lower bounds can be guaranteed on each agent's welfare.

(2) Strategyproof SCFs that are not ex-post efficient deserve to be studied. If there is an odd number of agents, majority voting between two pre-specified acts is clearly strategyproof. But more flexible strategyproof SCFs are possible. Partition the state space into a collection of events. For each event specify two "sub-acts", that is, two mappings from that event into the set of outcomes, and apply majority voting to choose between these two sub-acts. Let the chosen act be the concatenation of all the chosen sub-acts. The additive separability of subjective expected utility preferences guarantees that this SCF is strategyproof; it is also anonymous. Non-anonymous variants of such SCFs can be defined by using a committee rule (rather than majority voting) to decide between the two pre-specified sub-acts on each event.

(3) We conducted our analysis under the assumption that all acts are feasible. While this unconstrained social choice framework is a natural benchmark, applications will typically require imposing *constraints* on the set of feasible acts. The class of strategyproof and ex-post efficient social choice functions will generally depend in a subtle way upon these feasibility constraints, but our results certainly provide a good starting point for the study of any such problem. A similar generalization to constrained sets of alternatives was successfully achieved in the literature on strategyproofness on rich domains of additively separable preferences originally defined over product sets: see in particular Barberà, Massó and Neme (2005) and Reffgen and Svensson (2012).

(4) In many contexts, it will also be natural to impose *restrictions on preferences.* An interesting case is that of shareholders of a firm choosing acts with monetary outcomes –the profits to be shared. Here all agents have the same monotonic preference ordering over outcomes but not necessarily the same valuation functions or the same beliefs. While the unconstrained problem is uninteresting –the constant act choosing the highest profit level in all states is dominant–, the problem of choosing acts under constraints is entirely nontrivial.

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8 Appendix A: proof of the Top Selection lemma

We first define two consequences of strategyproofness. Given a pair (v_i, p_i), let $E_{v_i}^{p_i}(f) := \sum_{\omega\in\Omega} p_i(\omega)v_i(f(\omega))$ denote agent i's expected utility associated with the act f. We say that a SCF $\tilde{\varphi} : \mathcal{D}^N \to X^\Omega$ is *misvaluation-proof* if no agent ever benefits from distorting her valuation function (while reporting her actual belief), that is, for all $i \in N$, $(v,p) \in \mathcal{D}^N$ and $v_i' \in \mathcal{V}^i$,

$$E_{v_i}^{p_i}\left(\tilde{\varphi}(v,p)\right) \geq E_{v_i}^{p_i}\left(\tilde{\varphi}((v_i', v_{-i}), p)\right).$$

Likewise $\tilde{\varphi} : \mathcal{D}^N \to X^\Omega$ will be called *misbelief-proof* if: for all $i \in N$, $(v,p) \in \mathcal{D}^N$ and $p_i' \in \mathcal{P}^i$,

$$E_{v_i}^{p_i}\left(\tilde{\varphi}(v,p)\right) \geq E_{v_i}^{p_i}\left(\tilde{\varphi}(v,(p_i', p_{-i}))\right)$$

This says that no agent should ever benefit from distorting her belief p_i (while truthfully reporting v_i). Obviously, just as misvaluation-proofness, misbelief-proofness is implied by strategyproofness.

Let $\varphi : \mathcal{D}^N \to X^\Omega$ be a strategyproof and ex-post efficient SCF. Unless explicitly specified otherwise, we assume in what follows that $p \in \mathcal{P}^N$ is fixed (but arbitrary), and we write $\varphi(v)$ and E_{v_i} instead of the respective $\varphi(v,p)$ and $E_{v_i}^{p_i}$. Given the fixed p, the set of i's admissible valuation functions is $\mathcal{V}_{p_i} := \{v \in \mathcal{V} \mid (v_i, p_i) \in \mathcal{D}\}$; and (with a slight abuse of notation) we write $\mathcal{V}_p := \mathcal{V}_{p_1} \times \ldots \times \mathcal{V}_{p_n}$. For any $x \in X$ and $f \in X^\Omega$, we let $f^x := \{\omega \in \Omega \mid f(\omega) = x\}$. Likewise, we will often write $\varphi^x(v,p)$.

The preliminary result below says that, if the chosen acts at two given profiles (v, v') disagree only in states where either a_1 or a_2 is selected, then they must coincide as long as every agent's ordering of a_1 and a_2 does not change from v to v'.

Lemma 1. *Invariance for binary-differentiated acts*
If $a_1, a_2 \in X$ and $v, v' \in \mathcal{V}_p$ are such that $(v_i(a_1) - v_i(a_2))(v_i'(a_1) - v_i'(a_2)) > 0$ for all $i \in N$, then $[\varphi^x(v) = \varphi^x(v), \forall x \neq a_1, a_2] \Rightarrow [\varphi(v) = \varphi(v')]$.

Proof. The result follows from the fact that an agent's preferences over binary-differentiated acts f, f' (that is, acts that may only differ in states where a_1 or a_2 is chosen) remain unchanged as long as her ordering of these two outcomes is the same. □

The Monotonicity lemma below states that, if the chosen act changes as agent i's reported valuation of the outcome a increases (all else equal), then the probability assigned to this outcome a in the chosen act must increase as well.

Lemma 2. *Monotonicity*
If $a \in X$, $i \in N$, $v, w \in \mathcal{V}_p$ are such that $v_{-i} = w_{-i}$, $v_i(x) = w_i(x)$ for all $x \neq a$, and $1 > v_i(a) > w_i(a) \geq 0$, then $[\varphi(v) \neq \varphi(w)] \Rightarrow [p_i(\varphi^a(v)) > p_i(\varphi^a(w))]$.

Proof. Fix i, a, v, w as in the statement of Lemma 2 and suppose that $\varphi(v) = f \neq g = \varphi(w)$. Next, for any $x \in X$ and $z \in [0,1)$, let $v_i^z(x) = \begin{cases} v_i(x), & \text{if } x \neq a \\ z, & \text{if } x = a \end{cases}$ and define the following function of z:

$$\Delta_{fg}(z) := \sum_{\omega \in \Omega} p_i(\omega) \left[v_i^z(f(\omega)) - v_i^z(g(\omega)) \right].$$

Factoring out z and reshuffling, one can rewrite $\Delta_{fg}(z)$ as

$$\Delta_{fg}(z) = \underbrace{[p_i(f^a) - p_i(g^a)]}_{\alpha} \cdot z + \underbrace{\sum_{\omega \notin f^a} p_i(\omega) v_i(f(\omega)) - \sum_{\omega \notin g^a} p_i(\omega) v_i(g(\omega))}_{\beta}.$$

Thus, $\Delta_{fg}(z) = \alpha \cdot z + \beta$ is a linear function of $z \in [0,1)$.

Moreover, observe that $v_i^z = \begin{cases} w_i, & \text{if } z = w_i(a) \\ v_i, & \text{if } z = v_i(a) \end{cases}$. Therefore, misvaluation-proofness implies: (i) $\Delta_{fg}(w_i(a)) < 0$ and (ii) $\Delta_{fg}(v_i(a)) > 0$. Given that Δ_{fg} is linear and $w_i(a) < v_i(a)$, we necessarily have that the slope is positive, that is to say, $\alpha = p_i(\varphi^a(v)) - p_i(\varphi^a(w)) > 0$. □

The next lemma asserts the following: ceteris paribus, as an agent i's reported valuation of her second-best outcome a_2 gets infinitely close to 1 (the valuation of her top a_1), there necessarily comes a point where (i) the chosen act becomes constant and (ii) for the two possible orders of i's two top outcomes a_1 and a_2, the respective outcomes chosen must be the same in each state where a_1 and a_2 are not selected.

Lemma 3. *Invariance at the bottom (with close tops)*
Consider $a_1, a_2 \in X$, with $a_1 \neq a_2$, and fix $i \in N$, $v \in \mathcal{V}_p$ such that $v_i(a_1) = 1$. Let $\hat{v}_i^m, \bar{v}_i^m \in \mathcal{V}_{p_i}$ be such that $\begin{cases} \hat{v}_i^m(a_1) = 1 > \hat{v}_i^m(a_2) = 1 - 1/m > \hat{v}_i^m(x) = v_i(x), \\ \bar{v}_i^m(a_2) = 1 > \bar{v}_i^m(a_1) = 1 - 1/m > \bar{v}_i^m(x) = v_i(x), \end{cases}$ for any $m \geq m_0^{v_i} > 1$ and any $x \notin \{a_1, a_2\}$. Then the following statements hold:

(i) $\exists \hat{f}, \bar{f} \in X^\Omega$ *and* $\exists \tilde{m} \in \mathbb{N}$ *such that:* $m > \tilde{m} \Rightarrow \left[\varphi(\hat{v}_i^m, v_{-i}) = \hat{f} \text{ and } \varphi(\bar{v}_i^m, v_{-i}) = \bar{f} \right]$.

(ii) *For all $x \in X \setminus \{a_1, a_2\}$, we have $\hat{f}^x = \bar{f}^x$.*

Proof. Let $a, b \in X$, $i \in N$, $v \in \mathcal{V}_p$, $\hat{v}_i^m, \bar{v}_i^m \in \mathcal{V}_{p_i}$ satisfy the conditions of the statement of Lemma 3.

(i) Suppose by contradiction that *(i)* is false. Then one of the two sequences $\hat{f}_m := \varphi(\hat{v}_i^m, v_{-i}), \bar{f}_m := \varphi(\hat{v}_i^m, v_{-i})$ is not *stationary*.[4] Assuming without loss of generality that $\hat{f}^m$ is not stationary, there exists a subsequence of $\hat{f}_m$ (say, $\hat{f}_{m_k}$) such that $\hat{f}_{m_k} \neq f_{m_{k+1}}$, for all $k \in \mathbb{N}$ (where m_k increases with k). Given that

[4]We say that a sequence $(q_m)_{m \in \mathbb{N}}$ is *stationary* if there exists $\bar{m} \in \mathbb{N}$ s.t.: $m > \bar{m} \Rightarrow q_m = q_{m+1}$. In words, a stationary sequence is one that becomes constant after a finite number of steps.

$\hat{v}_i^{m_{k+1}}(a_2) = 1 - 1/m_{k+1} > 1 - 1/m_k = \hat{v}_i^{m_k}(a_2)$ and $\hat{v}_i^{m_k}(x) = \hat{v}_i^{m_{k+1}}(x)\ \forall x \neq b$, it comes from Lemma 2 that $p_i(\hat{f}_{m_{k+1}}) > p_i(\hat{f}_{m_k})$. Hence, for any $\hat{k} \geq 1$, we may write:

$$p_i\left(\hat{f}^{a_2}_{m_0}\right) < p_i\left(\hat{f}^{a_2}_{m_1}\right) < \ldots < p_i\left(\hat{f}^{a_2}_{m_{\hat{k}-1}}\right) < p_i\left(\hat{f}^{a_2}_{m_{\hat{k}}}\right) < p_i\left(\hat{f}^{a_2}_{m_{\hat{k}+1}}\right) < \ldots$$

Since p_i is injective, we have thus found an infinite sequence $(\hat{f}_{m_k})_{k\geq 0}$ of pairwise distinct acts. But this is impossible because the set of acts, X^Ω, is finite.

(ii) By way of contradiction, let us assume that $\hat{f}^x \neq \bar{f}^x$ for some $x \in X \setminus \{a_1, a_2\}$. We define two new acts $\hat{g}$ and $\bar{g}$ as follows:

$$\begin{cases} \hat{g}(\omega) = \hat{f}(\omega), \forall \omega \notin \hat{f}^{a_1} \cup \hat{f}^{a_2}; \\ \hat{g}(\omega) = a_1, \forall \omega \in \hat{f}^{a_1} \cup \hat{f}^{a_2}; \end{cases} \quad \text{and} \quad \begin{cases} \bar{g}(\omega) = \bar{f}(\omega), \forall \omega \notin \bar{f}^{a_1} \cup \bar{f}^{a_2}; \\ \bar{g}(\omega) = a_1, \forall \omega \in \bar{f}^{a_1} \cup \bar{f}^{a_2}. \end{cases} \tag{6}$$

Note that $\hat{g} \neq \bar{g}$ because there exists $x \neq a_1, a_2$ such that $\hat{f}^x \neq \bar{f}^x$. Thus, since the pair (v_i, p_i) defines a linear ordering over the set of acts, we must have $E_{v_i}(\hat{g}) \neq E_{v_i}(\bar{g})$:

$$\sum_{x \neq a,b} p_i(\hat{g}^x) v_i(x) + p_i(\hat{f}^{a_1} \cup \hat{f}^{a_2}) v_i(a_1) \neq \sum_{x \neq a,b} p_i(\bar{f}^x) v_i(x) + p_i(\bar{f}^{a_1} \cup \bar{f}^{a_2}) v_i(a_1).$$

Without loss of generality, suppose that $E_{v_i}(\hat{g}) - E_{v_i}(\bar{g}) > 0$. It then comes from the above equation, and the fact that $v_i(a_1) = 1$, that

$$\sum_{x \neq a,b} \left(p_i(\hat{g}^x) - p_i(\bar{g}^x)\right) v_i(x) + \left(p_i(\hat{f}^{a_1} \cup \hat{f}^{a_2}) - p_i(\bar{f}^{a_1} \cup \bar{f}^{a_2})\right) > 0.$$

Hence, since $\hat{g}^x = \hat{f}^x, \forall x \neq a_1, a_2$ ($\bar{g}^x = \bar{f}^x, \forall x \neq a_1, a_2$) from Equation (6), we have

$$\varepsilon := \sum_{x \neq a,b} \left(p_i(\hat{f}^x) - p_i(\bar{f}^x)\right) v_i(x) + p_i(\hat{f}^{a_1} \cup \hat{f}^{a_2}) - p_i(\bar{f}^{a_1} \cup \bar{f}^{a_2}) > 0. \tag{7}$$

Consider now $m \geq \tilde{m}$, where $\tilde{m}$ is defined in *(i)*. Then we have $\hat{f}_m = \hat{f}$ and $\bar{f}_m = \bar{f}$ and, given that $\bar{v}_i(a_2) = 1$, $\bar{v}_i^m(a_1) = 1 - 1/m$, it follows that

$$\begin{aligned}
E_{\bar{v}_i^m}(\hat{f}) - E_{\bar{v}_i^m}(\bar{f}) &= \sum_{x \in X} \left(p_i(\hat{f}^x) - p_i(\bar{f}^x)\right) \bar{v}_i(x) \\
&= \sum_{x \neq a_1, a_2} (p_i(\hat{f}^x) - p_i(\bar{f}^x)) \bar{v}_i(x) + (p_i(\hat{f}^{a_1}) - p_i(\bar{f}^{a_1})) \bar{v}_i^m(a_1) \\
&\quad + (p_i(\hat{f}^{a_1}) - p_i(\bar{f}^{a_2})) \bar{v}_i^m(a_2) \\
&= \sum_{x \neq a_1, a_2} (p_i(\hat{f}^x) - p_i(\bar{f}^x)) v_i(x) + p_i(\hat{f}^{a_1} \cup \hat{f}^{a_2}) - p_i(\bar{f}^{a_1} \cup \bar{f}^{a_2}) \\
&\quad - \frac{1}{m}(p_i(\hat{f}^{a_1}) - p_i(\bar{f}^{a_1})) \\
&= \varepsilon - \frac{1}{m}(p_i(\hat{f}^{a_1}) - p_i(\bar{f}^{a_1})).
\end{aligned}$$

Since $\varepsilon > 0$ from (7), and $\lim_{m\to\infty} \frac{1}{m}(p_i(\hat{f}^{a_1}) - p_i(\bar{f}^{a_1})) = 0$, there exists $m^* \geq \tilde{m}$ such that $\frac{1}{m}(p_i(\hat{f}^{a_1}) - p_i(\bar{f}^{a_1})) < \varepsilon$, for $m \geq m^*$. Hence, for all $m \geq m^* \geq \tilde{m}$, we finally get $E_{\bar{v}_i^m}(\hat{f}) - E_{\bar{v}_i^m}(\bar{f}) > 0$. But this is a contradiction since, together, misvaluation-proofness, $\bar{f} = \bar{f}^m = \varphi(\bar{v}_i^m, v_{-i})$ and $\hat{f} = \hat{f}_m = \varphi(\hat{v}_i^m, v_{-i})$ (for all $m \geq m^*$) imply that we must rather have $E_{\bar{v}_i^m}(\bar{f}) - E_{\bar{v}_i^m}(\hat{f}) > 0$. □

Lemma 4. *Top selection in the case of two agents.*
Let $N = \{1, 2\}$, $a_1, a_2 \in X$, and $v, v' \in \mathcal{V}_p$. If $\tau(v_i) = \tau(v'_i) = a_i$ for all $i \in N$, then $\varphi(v) = \varphi(v') \in \{a_1, a_2\}^\Omega$.

Proof. Fix $N = \{1, 2\}$, $a_1, a_2 \in X$, and $v, v' \in \mathcal{V}_p$ such that $\tau(v_i) = \tau(v'_i) = a_i$ for $i = 1, 2$. If $a_1 = a_2$, ex-post efficiency alone delivers the desired result. In what follows, assume $a_1 \neq a_2$. We prove the claims below.

Claim 1. For any $x, y \in X$, let $\mathcal{T}_{xy} := \{w \in \mathcal{V}_p : w_1(x) = w_2(y) = 1\}$ and call $\mathcal{DOM}_{xy}$ the subset of $\mathcal{V}$ containing all profiles $w \in \mathcal{T}_{xy}$ such that any $z \in X \setminus \{x, y\}$ is ex-post dominated by x or y. Then there exists $f_{xy} \in \{x, y\}^\Omega$ such that

$$\varphi(w) = \varphi(w') = f_{xy}, \ \forall x, y \in X, \forall w, w' \in \mathcal{DOM}_{xy}.$$

To prove Claim 1, fix $x, y \in X$ and $\tilde{w} \in \mathcal{DOM}_{xy}$ and observe that, by ex-post efficiency, we have $f_{xy} := \varphi(\tilde{w}) \in \{x, y\}^\Omega$. In addition, remark that for any agent $i \in \{1, 2\}$, we have $(w_i(x) - w_i(y))(w'_i(x) - w'_i(y)) > 0$ for all $w, w' \in \mathcal{DOM}_{xy}$. Thus, Lemma 1 yields the desired result: $\varphi(w) = \varphi(w') = f_{xy}$, for all $w, w' \in \mathcal{DOM}_{xy}$.

Note that, in particular, Claim 1 implies that $\varphi(w) = \varphi(w') = f_{a_1a_2} \in \{a_1, a_2\}^\Omega$ for all $w, w' \in \mathcal{DOM}_{a_1a_2}$. To prove Lemma 4, it thus suffices to show that $\varphi(w) = f_{a_1a_2}$ for any $w \in \mathcal{T}_{a_1a_2} \setminus \mathcal{DOM}_{a_1a_2}$. Let us then consider a fixed $v \in \mathcal{T}_{a_1a_2} \setminus \mathcal{DOM}_{a_1a_2}$.

By way of contradiction, suppose that $\varphi(v) \neq f_{a_1a_2}$. Then there exists $b \in X \setminus \{a_1, a_2\}$ such that $\varphi^b(v) \neq \emptyset$ —otherwise, Lemma 1 would yield $\varphi(v) = f_{a_1a_2}$. Moreover, by monotonicity (Lemma 2), remark that it is not restrictive to assume that b is the second-best outcome for both players, that is,

$$\begin{aligned} v_1(a_1) = 1 > v_1(b) > v_1(x), \forall x \neq a_1, b; \\ v_2(a_2) = 1 > v_2(b) > v_2(x), \forall x \neq a_2, b. \end{aligned} \tag{8}$$

Next, we define $\hat{v}_2^m, \bar{v}_2^m \in \mathcal{V}_{p_2}$ by:[5]

$$\hat{v}_2^m(a_2) \ = \ 1 > \hat{v}_2^m(b) = 1 - 1/m > \hat{v}_2^m(x) = v_2(x) \tag{9}$$
$$\bar{v}_2^m(b) \ = \ 1 > \bar{v}_2^m(a_2) = 1 - 1/m > \bar{v}_2^m(x) = v_i(x), \tag{10}$$

[5]Since the set of acts X^Ω is finite, note that the starting point $m_0^{v_2}$ of the sequence $\{\hat{v}_2\}_{m \geq m_0^{v_2}}$, can be conveniently chosen so as to have $\hat{v}_2^m \in \mathcal{V}_{p_2}$ for all $m \geq m_0^{v_2}$ (and likewise for $\bar{v}_2^m$).

for any $m \geq m_0^{v_2}$ and any $x \notin \{a_2, b\}$. We then prove the additional claims below.

Claim 2. For all $m \geq m_0^{v_2}$ and $u_1 \in \mathcal{V}_{p_1}$ such that $(u_1, \hat{v}_2^m) \in \mathcal{T}_{a_1a_2}$, we have:

$$\varphi(u_1, \hat{v}_2^m) \neq f_{a_1a_2} \Rightarrow \left[p_1(\varphi^{a_2}(u_1, \hat{v}_2^m)) < p_1(f_{a_1a_2}^{a_2}) \text{ and } p_2(\varphi^{a_1}(u_1, \hat{v}_2^m)) < p_2(f_{a_1a_2}^{a_1})\right].$$

Let us prove Claim 2. Consider $m \geq m_0^{v_2}$ and $u_1 \in \mathcal{V}_{p_1}$ such that $(u_1, \hat{v}_2^m) \in \mathcal{T}_{a_1a_2}$; and suppose that $\varphi(u_1, \hat{v}_2^m) \neq f_{a_1a_2}$. Then we know from Claim 1 that $(u_1, \hat{v}_2^m) \notin \mathcal{DOM}_{a_1a_2}$; but defining w by

$$\begin{cases} w_2(a_1) = 1 - 1/(2m) > w_2(b) = 1 - 1/m \\ w_2(x) = \hat{v}_2^m(x) \text{ for all } x \neq a_1, \end{cases}$$

we get $(u_1, w_2) \in \mathcal{DOM}_{a_1a_2}$ and hence, by Claim 1, $\varphi(u_1, w_2) = f_{a_1a_2}$. Observe that w_2 obtains from $\hat{v}_2^m$ by merely raising the value of a_1: thus, it follows from monotonicity (Lemma 2) that $p_2(\varphi^{a_1}(u_1, \hat{v}_2^m)) < p_2(f_{a_1a_2}^{a_1}) = p_2(\varphi(u_1, w_2))$. The proof of $p_1(\varphi^{a_2}(u_1, \hat{v}_2^m)) < p_1(f_{a_1a_2}^{a_2})$ is similar and will be omitted.

Claim 3. There exists $\varepsilon^* \in (0, 1)$ such that, for any $u_1 \in \mathcal{V}_{p_1}$, we have:

$$[u_1(a_1) = 1 > \varepsilon^* \geq u_1(x),\ \forall x \neq a_1] \Rightarrow [\varphi(u_1, \hat{v}_2^m) = f_{a_1a_2},\ \forall m \geq m_0^{v_2}].$$

To prove Claim 3, define $\alpha := \min_{\substack{E, E' \subseteq \Omega \\ E \neq E'}} |p_1(E) - p_1(E')|$ and let $\varepsilon^* = \frac{\alpha}{|X|} > 0$. Let us fix $m \geq m_0^{v_2}$ and $u_1 \in \mathcal{V}^1$ such that $u_1(a_1) = 1 > \varepsilon^* > u_1(x)$ for all $x \neq a_1$; and by contradiction suppose that $\varphi(u_1, \hat{v}_2^m) \neq f_{a_1a_2}$. Then it follows from Claim 2 that $p_2(\varphi^{a_1}(u_1, \hat{v}_2^m)) < p_2(f_{a_1a_2}^{a_1})$ and, therefore, $\varphi^{a_1}(u_1, \hat{v}_2^m) \neq f_{a_1a_2}^{a_1}$. Since p_1 is injective, this means that either $p_1(\varphi^{a_1}(u_1, \hat{v}_2^m)) < p_1(f_{a_1a_2}^{a_1})$ or $p_1(\varphi^{a_1}(u_1, \hat{v}_2^m)) > p_1(f_{a_1a_2}^{a_1})$. We show below that either case leads to a contradiction.

Suppose first that $p_1(\varphi^{a_1}(u_1, \hat{v}_2^m)) < p_1(f_{a_1a_2}^{a_1})$. Then, recalling the definition of α, we have $p_1(f_{a_1a_2}^{a_1}) - p_1(\varphi^{a_1}(u_1, \hat{v}_2^m)) \geq \alpha$; and one can hence write

$$\begin{aligned}
& E_{u_1}(f_{a_1a_2}) - E_{u_1}(\varphi(u_1, \hat{v}_2^m)) \\
= \; & p_1(f_{a_1a_2}^{a_1}) - p_1(\varphi^{a_1}(u_1, \hat{v}_2^m)) + \sum_{x \neq a_1} [p_1(f_{a_1a_2}^x) - p_1(\varphi^x(u_1, \hat{v}_2^m))] \overbrace{u_1(x)}^{\leq \varepsilon^*} \\
\geq \; & \underbrace{p_1(f_{a_1a_2}^{a_1}) - p_1(\varphi^{a_1}(u_1, \hat{v}_2^m))}_{\geq \alpha = \varepsilon^* |X|} - \varepsilon^* \sum_{x \neq a_1} \underbrace{\left|p_1(f_{a_1a_2}^x) - p_1(\varphi^x(u_1, \hat{v}_2^m))\right|}_{\leq 1} \\
\geq \; & \varepsilon^* |X| - \varepsilon^* (|X| - 1) = \varepsilon^* > 0.
\end{aligned}$$

But this contradicts misvaluation-proofness: agent 1 will deviate from u_1 to u_1' such that $(u_1', \hat{v}_2^m) \in \mathcal{DOM}_{a_1a_2}$ and obtain the preferred act $f_{a_1a_2}$.

Suppose now that $p_1(\varphi^{a_1}(u_1, \hat{v}_2^m)) > p_1(f_{a_1a_2}^{a_1})$ and consider $w_1 \in \mathcal{V}_{p_1}$ such that $w_1(a_1) = 1 > \varepsilon^* > w_2(a_2) > w_2(x)$ for all $x \neq a_2$. Note that $(w_1, \hat{v}_2^m) \in \mathcal{DOM}_{a_1a_2}$, hence, $\varphi(w_1, \hat{v}_2^m) = f_{a_1a_2}$. But then agent 1 prefers reporting u_1 to telling the truth when receiving w_1:

$$\begin{aligned}
& E_{w_1}(\varphi(u_1, \hat{v}_2^m)) - E_{w_1}(f_{a_1a_2}) \\
= \; & p_1(\varphi^{a_1}(u_1, \hat{v}_2^m)) - p_1(f_{a_1a_2}^{a_1}) + \sum_{x \neq a_1} [p_1(\varphi^x(u_1, \hat{v}_2^m)) - p_1(f_{a_1a_2}^x)] \overbrace{w_1(x)}^{\leq \varepsilon^*} \\
\geq \; & \underbrace{p_1(\varphi^{a_1}(u_1, \hat{v}_2^m)) - p_1(f_{a_1a_2}^{a_1})}_{\geq \alpha = \varepsilon^*|X|} - \varepsilon^* \sum_{x \neq a_1} \underbrace{\left| p_1(f_{a_1a_2}^x) - p_1(\varphi^x(u_1, \hat{v}_2^m)) \right|}_{\leq 1} \\
\geq \; & \varepsilon^*|X| - \varepsilon^*(|X| - 1) = \varepsilon^* > 0.
\end{aligned}$$

This also contradicts misvaluation-proofness of φ; and Claim 3 is shown.

To conclude the proof of Lemma 4, let us now fix $u_1 \in \mathcal{V}_{p_1}$ such that

$$u_1(a_1) = 1 > \varepsilon^* = \frac{\alpha}{|X|} \geq u_1(x), \; \forall x \neq a_1; \tag{11}$$

and consider the sequences defined by $\varphi(v_1, \hat{v}_2^m)$ and $\varphi(u_1, \hat{v}_2^m)$, for $m \geq m_0^{v_2}$. By Lemma 3-(i), there exist $\hat{f}_{v_1} \in X^\Omega$ and $\tilde{m}_{v_1} \geq m_0^{v_2}$ such that: $\varphi(v_1, \hat{v}_2^m) = \hat{f}_{v_1}$, for all $m \geq \tilde{m}_{v_1}$. And Lemma 3-(ii) then gives

$$p_2(\varphi^{a_1}(v_1, \hat{v}_2^m)) = p_2(\hat{f}_{v_1}^{a_1}) = p_2(\varphi^{a_1}(v_1, \bar{v}_2^m)), \; \forall m \geq \tilde{m}_{v_1}. \tag{12}$$

On the other hand, it follows from Claim 3 that the sequence $\varphi(u_1, \hat{v}_2^m)$ is constant. Precisely, $\varphi(u_1, \hat{v}_2^m) = f_{a_1a_2}$ for any m; and hence, applying Lemma 3-(ii), we get

$$p_2(\hat{f}_{u_1}^{a_1}) = p_2(f_{a_1a_2}^{a_1}) = p_2(\varphi^{a_1}(u_1, \bar{v}_2^m)), \forall m \geq \tilde{m}_{u_1}. \tag{13}$$

Next, since $\varphi^b(v) \neq \emptyset$, note from monotonicity (Lemma 2) that $p_2(\varphi^b(v_1, \hat{v}_2^m) > 0$ and therefore $\varphi(v_1, \hat{v}_2^m) \neq f_{a_1a_2}$ for any $m \geq \tilde{m}_{v_1}$; and it then follows from Claim 2 that $p_2(\varphi^{a_1}(v_1, \hat{v}_2^m)) = p_2(\hat{f}_{v_1}^{a_1}) < p_2(f_{a_1a_2}^{a_1})$. Plugging this inequality in (12)-(13) finally gives

$$p_2(\hat{f}_{v_1}^{a_1}) = p_2(\varphi^{a_1}(v_1, \bar{v}_2^m)) < p_2(\varphi^{a_1}(u_1, \bar{v}_2^m)) = p_2(f_{a_1a_2}^{a_1}), \forall m \geq max\{\tilde{m}_{v_1}, \tilde{m}_{u_1}\} \tag{14}$$

But note from (14) that the inequality $p_2(\varphi^{a_1}(v_1, \bar{v}_2^m)) < p_2(\varphi^{a_1}(u_1, \bar{v}_2^m))$ contradicts Claim 1. Indeed, remark from (8), (9) and (11) that $(v_1, \bar{v}_2^m), (u_1, \bar{v}_2^m) \in \mathcal{DOM}_{a_1b}$; and by Claim 1 we should rather have $p_2(\varphi^{a_1}(v_1, \bar{v}_2^m)) = p_2(\varphi^{a_1}(u_1, \bar{v}_2^m))$ for any m. $\square$

We emphasize that our proof of Lemma 4 only makes use of misvaluation-proofness and ex-post efficiency; it does not require the full force of strategyproofness. Indeed, up to now, we have kept the belief profile fixed.

Next, using an induction argument, we prove in Lemma 5 below that φ selects only tops at each profile where two players report the same top. The statement and proof of this result require variations of the belief profile p; we hence return to our original notation, writing $\varphi(v, p)$ and $E_{v_i}^{p_i}$ rather than just $\varphi(v)$ and E_{v_i}.

Lemma 5. *Induction lemma*

Suppose that $|N| = n \geq 3$. Assume by induction that for all S such that $|S| \leq n-1$ every misvaluation-proof, misbelief-proof and ex-post efficient SCF $\tilde{\varphi} : \mathcal{D}^S \to X^\Omega$ is a top and tops-only selection. Then, for any distinct $k, l \in N$ and any $(v,p) \in \mathcal{D}^N$, we have: $[\tau(v_k) = \tau(v_l)] \Rightarrow \varphi(v,p) \in \{\tau(v_i) : i \in N\}^\Omega$.

Proof. We will prove Lemma 5 in two steps. Suppose that the conditions in the statement are satisfied; and let us fix $k, l \in N$ such that $k \neq l$.

Step 1. For any $(v,p) \in \mathcal{D}^N$ such that $v_k = v_l$ and $p_k = p_l$, we have $\varphi(v,p) \in \{\tau(v_i) : i \in N\}^\Omega$. Moreover, we have $\varphi(v',p) = \varphi(v,p)$ for any $v' \in \mathcal{V}_p$ such that $v'_k = v'_l$ and $(\tau(v_i))_{i \in N} = (\tau(v'_i))_{i \in N}$.

Proof. Let $N_{-l} := N \setminus l$ and consider $\tilde{\varphi} : \mathcal{D}^{N_{-l}} \to X^\Omega$, defined by:

$$\forall (w,q) \in \mathcal{D}^{N_{-l}},\ \tilde{\varphi}(w,q) = \varphi(\overbrace{(w, \underbrace{w_k}_{l})}^{\in \mathcal{V}^N}, \overbrace{(q, \underbrace{q_k}_{l})}^{\in \mathcal{P}^N}). \tag{15}$$

That is to say, $\tilde{\varphi}(w,q)$ obtains as the decision under φ at the profile of $\mathcal{D}^N$ constructed from (w,q) by assigning to agent l the same valuation function and beliefs as agent k. It is straightforward to see from its definition that $\tilde{\varphi}$ is ex-post efficient (since φ is). We show next that $\tilde{\varphi}$ is also misvaluation-proof and misbelief-proof.

It is easy to see from (15) that misreporting v_i or p_i will never benefit any agent $i \in N_{-l} \setminus k$ (it would contradict strategyproofness of φ). To show that agent $k \in N_{-l}$ cannot profitably misreport either, pick an arbitrary pair $(w,q) \in \mathcal{D}^{N_{-l}}$ and let $(w'_k, q'_k) \in \mathcal{D}$. Since φ is misvaluation-proof, agent k cannot profitably misreport w'_k when receiving (w_k, q_k):

$$E^{q_k}_{w_k}(\tilde{\varphi}((w_{-k}, w'_k), q) = E^{q_k}_{w_k}(\varphi((w_{-k}, \underbrace{w'_k}_{k}, \underbrace{w'_k}_{l}), (q, q_k))) \leq E^{q_k}_{w_k}(\varphi((w_{-k}, w_k, w'_k), (q, q_k))) \tag{16}$$

Likewise, agent l cannot profitably misreport w'_k when receiving (w_k, q_k), that is,

$$E^{q_k}_{w_k}(\varphi((w_{-k}, w_k, w'_k), (q, q_k))) \leq E^{q_k}_{w_k}(\varphi((w_{-k}, w_k, w_k), (q, q_k)) = E^{q_k}_{w_k}(\tilde{\varphi}(w,q)) \tag{17}$$

Combining (16) and (17) thus gives $E^{q_k}_{w_k}(\tilde{\varphi}((w_{-k}, w'_k), q) \leq E^{q_k}_{w_k}(\tilde{\varphi}(w,q))$, which shows that $\tilde{\varphi}$ is misvaluation proof.

Using the same procedure, we also get $E^{q_k}_{w_k}(\tilde{\varphi}((w, (q_{-k}, q'_k))) \leq E^{q_k}_{w_k}(\tilde{\varphi}(w,q))$; and hence $\tilde{\varphi}$ is misbelief-proof. It thus follows from the induction hypothesis in the statement of Lemma 5 that $\tilde{\varphi}$ is a top (and tops-only) selection. That is to say, for any $(v,p) \in \mathcal{D}^N$ such that $v_k = v_l$ and $p_k = p_l$, the following results hold: (i) $\tilde{\varphi}(v_{-l}, p_{-l}) = \varphi(v,p) \in \{\tau(v_i), i \in N\}^\Omega$; (ii) $\tilde{\varphi}(v'_{-l}, p_{-l}) = \varphi(v',p) = \varphi(v,p)$ for any $v' \in \mathcal{V}_p$ such that $v'_k = v'_l$ and $(\tau(v_i))_{i \in N} = (\tau(v'_i))_{i \in N}$

Step 2. For all $(v,p) \in \mathcal{D}^N$ such that $v_k = v_l$, we have $\varphi(v,p) \in \{\tau(v_i) : i \in N\}^\Omega$.

Proof. In order to complete this step, let us first state some preliminary results.

Preliminary 1. Let $(u, q) \in \mathcal{D}^N$ and suppose that $x^* \in X$ satisfies $u_i(x^*) \in [0, 1)$, for any $i \in N$. Then there exists $\varepsilon_u > 0$ such that: $(u', q) \in \mathcal{D}^N$ and $\varphi(u', q) = \varphi(u, q)$, whenever $u' \in \mathcal{V}^N$ satisfies $\begin{cases} u_i'(x) = u_i(x), \text{ if } x \neq x^* \\ |u_i'(x^*) - u_i(x^*)| < \varepsilon_u \end{cases}$ for each $i \in N$.

The proof of Preliminary 1 is left to the reader: it follows from the facts that (i) the expected utility operator $E_{u_i}^{q_i}(\cdot)$ is a continuous function of u_i; and (ii) all players have identical preferences under (u, q) and (u', q) if u and u' are sufficiently close (the decision must hence be the same by misvaluation-proofness). It is important to note that ε_u may vary with u, but not with p.

Preliminary 2. Suppose that $(u_i, q_i), (u_i, q_i') \in \mathcal{D}$. Then there exists a finite sequence of beliefs $\{q_i^t : t = 0, \dots, T\}$ such that: (i) $q_i^0 = q_i$ and $q_i^T = q_i'$; (ii) $(u_i, q_i^t) \in \mathcal{D}$, for every $t = 0, \dots, T$; for all $t = 0, \dots, T-1$, we have *at most* one $\bar{q}_i \in [q_i^t, q_i^{t+1}]$ such that $(u_i, \bar{q}_i) \notin \mathcal{D}$.

The proof of Preliminary 2 is also omitted: it obtains as well from the continuity of the expected utility function $E_{u_i}^{q_i}$, and the fact that the set of acts X^Ω is finite. In words, Preliminary 2 means that, given a fixed u_i, any deviation from a belief q_i to another belief q_i' can be decomposed as a sequence of deviations $q_i^t \to q_i^{t+1}$ that are elementary in the sense that the segment $[q_i^t, q_i^{t+1}]$ contains at most one $\bar{q}_i$ such that $(u_i, \bar{q}_i) \notin \mathcal{D}$.

Let us now proceed with the proof of Step 2. Fix $(v, p) \in \mathcal{D}^N$ such that $v_k = v_l$ *and* $p_k \neq p_l$; and suppose by contradiction that $f := \varphi(v, p) \notin \{\tau(v_i) : i \in N\}^\Omega$. Using Preliminary 2,[6] we will without loss of generality assume that there exists a unique $\bar{p}_i \in [p_k, p_l]$ such that $(v_k, \bar{p}_k) \notin \mathcal{D}$.

First, let $g := \varphi(v, (p_{-kl}, p_k, p_k))$, $h := \varphi(v, (p_{-kl}, p_l, p_l))$; and remark that, for any $p_k^* \in (p_k, \bar{p}_k)$, misbelief-proofness or φ gives

$$\varphi(v, (p_{-kl}, p_k^*, p_l)) = \varphi(v, (p_{-kl}, p_k, p_l)) = f. \tag{18}$$

Indeed, note that (v_k, p_k^*) and (v_k, p_l) necessarily yield the same ranking of all acts because there is no $\bar{p}_k' \in [p_k, p_l] \setminus \bar{p}_k \supset (p_k, \bar{p}_k) \ni p_k^*$ such that $(v_k, \bar{p}_k') \notin \mathcal{D}$. Likewise, using misbelief-proofness of $\tilde{\varphi}$ (established in Step 1) yields

$$\underbrace{\varphi(v, (p_{-kl}, p_k^*, p_k^*))}_{=\tilde{\varphi}(v_{-l}, (p_{-kl}, p_k^*))} = \underbrace{\varphi(v, (p_{-kl}, p_k, p_k))}_{=\tilde{\varphi}(v_{-l}, (p_{-kl}, p_k))} = g, \quad \forall p^* \in (p_k, \bar{p}_k) \tag{19}$$

Second, we get from Step 1 that $g = \varphi(v, (p_{-kl}, p_k, p_k)) = \tilde{\varphi}(v_{-l}, p_{-l}) \in \{\tau(v_i), i \in N\}^\Omega$ and also $h = \varphi(v, (p_{-kl}, p_l, p_l)) = \tilde{\varphi}(v_{-l}, (p_{-kl}, p_l)) \in \{\tau(v_i), i \in N\}^\Omega$. Note

[6] Note from Preliminary 2 that we are ignoring here the case where there exists no $\bar{p}_k \in [p_k, p_l]$ such that $(v_k, \bar{p}_k) \notin \mathcal{D}$. In this case, (v_k, p_k) and (v_k, p_l) generate exactly the same ranking over the set of acts X^Ω; and hence misbelief-proofness of φ trivially gives the desired contradiction. We thus focus on the interesting case, where there exists exactly one $\bar{p}_k \in [p_k, p_l]$ such that $(v_k, \bar{p}_k) \notin \mathcal{D}$.

that $f \notin \{g, h\}$ since both g and h are top selections (whereas f is not). Since $(v, p) = (v, (p_{-kl}, p_k, p_l))$ obtains from $(v, (p_{-kl}, p_k, p_k))$ when agent l changes her reported belief from p_k to p_l, misbelief-proofness of φ implies **(a.1)** $E^{p_k}_{v_k}(g) > E^{p_k}_{v_k}(f)$ and **(a.2)** $E^{p_l}_{v_k}(f) > E^{p_l}_{v_k}(g)$. Doing the same for agent k, from $(v, (p_{-kl}, p_k, p_l))$ to $(v, (p_{-kl}, p_l, p_l))$, we may write **(b.1)** $E^{p_k}_{v_k}(f) > E^{p_k}_{v_k}(h)$ and **(b.2)** $E^{p_l}_{v_k}(h) > E^{p_l}_{v_k}(f)$. Finally, using misbelief-proofness of $\tilde{\varphi}$, which has been established in Step 1, we obtain **(c.1)** $E^{p_k}_{v_k}(g) \geq E^{p_k}_{v_k}(h)$ and **(c.2)** $E^{p_k}_{v_k}(g) \geq E^{p_k}_{v_k}(h)$ —where the equalities hold only if $g = h$. We distinguish two cases below.

Suppose first that $g = h$. Then observe that (a.1) and (b.1) above respectively become $E^{p_k}_{v_k}(g) > E^{p_k}_{v_k}(f)$ and $E^{p_k}_{v_k}(f) > E^{p_k}_{v_k}(g)$, and we obviously have a contradiction.

Suppose now that $g \neq h$. Combining the intermediate value theorem with the fact that $\bar{p}_k$ is the only belief in $[p_k, p_l]$ such that$(v_k, \bar{p}_k) \notin \mathcal{D}$, we get from (a.1) and (a.2) that $E^{\bar{p}_k}_{v_k}(g) = E^{\bar{p}_k}_{v_k}(f)$. By the same token, using (c.1) and (c.2), note that we must as well have $E^{\bar{p}_k}_{v_k}(g) = E^{\bar{p}_k}_{v_k}(h)$. That is to say,

$$E^{\bar{p}_k}_{v_k}(g) \;:= \textstyle\sum_{w\in\Omega} \bar{p}_k(w) v_k(g(\omega)) = \sum_{w\in\Omega} \bar{p}_k(w) v_k(f(\omega)) =: E^{\bar{p}_k}_{v_k}(f); \tag{20}$$

$$E^{\bar{p}_k}_{v_k}(g) \;:= \textstyle\sum_{w\in\Omega} \bar{p}_k(w) v_k(g(\omega)) = \sum_{w\in\Omega} \bar{p}_k(w) v_k(h(\omega)) =: E^{\bar{p}_k}_{v_k}(h). \tag{21}$$

Given that $f := \varphi(v, p) \notin \{\tau(v_i) : i \in N\}^\Omega$, there necessarily exists $(x^*, \omega^*) \in X \times \Omega$ such that $v_k(x^*) \in [0, 1)$ and $f(\omega^*) = x^*$. Next, recall the definition of ε_v (in Preliminary 1) and define $v' \in \mathcal{V}^N$ by $\begin{cases} v'_i(x) = v_i(x), \text{ if } x \neq x^* \\ v'_i(x^*) = v_i(x^*) + \varepsilon_v/2 \end{cases}$ for all $i \in N$. Note from this definition of v' that $v'_i(x) = v_i(x)$ for all $i \in N$ and all $x \in \{\tau(v_j) : j \in N\}$. Combining that observation with the fact that $g, h \in \{\tau(v_j) : j \in N\}^\Omega$, we may use (20) to write

$$E^{\bar{p}_k}_{v'_k}(g) = E^{\bar{p}_k}_{v_k}(g) := \sum_{w\in\Omega} \bar{p}_k(w) v_k(g(\omega)) = \sum_{w\in\Omega} \bar{p}_k(w) v_k(h(\omega)) =: E^{\bar{p}_k}_{v_k}(h) = E^{\bar{p}_k}_{v'_k}(h). \tag{22}$$

Also, since $v'_i(x) \geq v_i(x)$ for any $x \in X$ [with the strict inequality for $x^* \in f(\Omega)$], it comes from (22) that

$$E^{\bar{p}_k}_{v'_k}(g) = \sum_{w\in\Omega} \bar{p}_k(w) \overbrace{v_k(g(\omega))}^{=v'_k(g(\omega))} = \sum_{w\in\Omega} \bar{p}_k(w) v_k(f(\omega)) < \sum_{w\in\Omega} \bar{p}_k(w) v'_k(f(\omega)) =: E^{\bar{p}_k}_{v'_k}(f). \tag{23}$$

Using the continuity of $E^{p'_k}_{v_k}$ with respect to the belief p'_k, Equation (23) implies that there exists $p^*_k \in (p_k, \bar{p}_k)$ such that $E^{p^*_k}_{v'_k}(g) < E^{p^*_k}_{v'_k}(f)$. In addition, note that $(v_k, p^*_k) \in \mathcal{D}$ —because of our assumption that there exists no $\bar{p}'_k \in [p_k, p_l] \backslash \bar{p}_k$ such that $(v_k, \bar{p}'_k) \notin \mathcal{D}$. Hence, applying Preliminary 1 also gives $(v'_k, p^*_k) \in \mathcal{D}$ and

$$\varphi(v', (p_{-kl}, p^*_k, p_l)) = \varphi(v, (p_{-kl}, p^*_k, p_l)) = f, \tag{24}$$

where the last equality comes from (18).

Finally, recalling from Step 1 that $\tilde{\varphi}$ is a tops-only selection —and noting that $\tau(v_i) = \tau(v'_i)$ for all $i \in N$, we get

$$\varphi\left(v', (p_{-kl}, p_k, p_k)\right) = \tilde{\varphi}(v'_{-l}, p_{-l}) = \tilde{\varphi}(v_{-l}, p_{-l}) = \varphi(v, (p_{-kl}, p_k, p_k)). \qquad (25)$$

Combining (25) and (19) then gives $\varphi\left(v', (p_{-kl}, p^*_k, p^*_k)\right) = g = \varphi(v, (p_{-kl}, p_k, p_k))$. But remark that this is a contradiction to misbelief-proofness of φ. Indeed, agent l will profitably deviate from $(v', (p_{-kl}, p^*_k, p^*_k))$ to $(v', (p_{-kl}, p^*_k, p_l))$ since we have $v'_l = v'_k$ and

$$E^{p^*_k}_{v'_l}(g) = E^{p^*_k}_{v'_k}(g) < E^{p^*_k}_{v'_k}(f) = E^{p^*_k}_{v'_l}(f);$$
$$\varphi\left(v', (p_{-kl}, p^*_k, p^*_k)\right) = g;$$
$$\varphi\left(v', (p_{-kl}, p^*_k, p_l)\right) = f \text{ [from (24)]}.$$

This concludes the proof of Step 2. Combining Step 1 and Step 2, we thus have $\varphi(v,p) \in \{\tau(v_i) : i \in N\}^{\Omega}$ for all $(v,p) \in \mathcal{D}^N$ such that $v_k = v_l$. □

For the case where $n = 2$, the top (and tops-only) property has been shown in Lemma 4. The following lemma states this property for $n \geq 3$.

Lemma 6. *Top Selection (and tops-only) for* $n \geq 3$
Let $|N| = n \geq 3$, $A_n = (a_1, \ldots, a_n) \in X^N$ *and fix* $p \in \mathcal{P}^N$. *If* $v, v' \in \mathcal{V}_p$ *are such that* $\tau(v_i) = \tau(v'_i) = a_i$ *for all* $i \in N = \{1, \ldots, n\}$, *then* $\varphi(v,p) = \varphi(v',p) \in \{a_1, \ldots, a_n\}^{\Omega}$.

Proof. Suppose that $n \geq 3$; and fix $A_n = (a_1, \ldots, a_n) \in X^N$ and $p \in \mathcal{P}^N$. Next, define $\mathcal{T}_{A_n} := \{v \in \mathcal{V}_p : v_i(a_i) = 1, \forall i \in N\}$ and, for any $v \in \mathcal{T}_{A_n}$, let $\hat{f}_v := \lim_{\infty} \varphi((\hat{v}^m_1, v_{-1}), p)$. That is, $\hat{f}_v$ is the value taken by the stationary sequence $\varphi((\hat{v}^m_1, v_{-1}), p)$ for m large enough [recall Lemma 3-(i)]. We will prove the result by showing two claims: (1) $\exists v \in \mathcal{T}_{A_n}$ such that $\varphi(v,p) \in \{a_1, \ldots, a_n\}^{\Omega}$; (2) $\varphi(v,p) = \varphi(v',p)$ for all $v, v' \in \mathcal{T}_{A_N}$.

Claim 1. There exists $v'' \in \mathcal{T}_{A_n}$ such that $\varphi(v'', p) \in \{a_1, \ldots, a_n\}^{\Omega}$.

To prove Claim 1, we distinguish two cases. Fix any $v \in \mathcal{T}_{A_n}$.
Case 1. Suppose that $a_k = a_l$ for some distinct $k, l \in N$. Then the result of Claim 1 holds by Lemma 5.
Case 2. Suppose now that $a_i \neq a_j$, for any distinct $i, j \in N$, that is, A_n consists of n distinct tops. Take $i = 1$ in Lemma 3 and recall that $\bar{v}^m$ is such that $\bar{v}^m_1(a_2) = 1 > 1 - 1/m = \bar{v}^m_1(a_1) > \bar{v}^m_1(x)$ for $x \neq a_1, a_2$. Since $\tau(\bar{v}^m_1) = a_2 = \tau(v_2)$, we do not have n distinct tops at $(\bar{v}^m_1, v_{-1})$; and it thus comes from Lemma 4 and Lemma 3-(i) that, for m large enough, $\varphi(\bar{v}^m_1, v_{-1}) = \bar{f}_v \in \{a_2, \ldots, a_n\}^{\Omega}$. That is to say, $\bar{f}^x_v = \emptyset$, for all $x \notin \{a_2, \ldots, a_n\}$. Moreover, Lemma 3-(ii) tells us that $\hat{f}^x_v = \bar{f}^x_v$ for all $x \neq a_1, a_2$. Therefore, we have $\hat{f}^x_v = \bar{f}^x_v = \emptyset$, for all $x \notin \{a_1, \ldots, a_n\}$. That is to say, there exists (a sufficiently large) $m'' \in \mathbb{N}$ such that $\varphi((\hat{v}^{m''}_1, v_{-1}), p) = \hat{f}_v \in \{a_1, \ldots, a_n\}^{\Omega}$. It thus suffices to take $v'' = (\hat{v}^{m''}_1, v_{-1})$ to see that Claim 1 is satisfied.

Claim 2. For any $v, v' \in \mathcal{T}_{A_N}$, we have $\varphi(v,p) = \varphi(v',p)$.

To prove Claim 2, let us state two additional preliminaries.

Preliminary 3. Let $i \in N$ and suppose that $(w_i, q_i), (w_i', q_i) \in \mathcal{D}$, with $\tau(w_i) = \tau(w_i')$. Then there exist two finite sequences $w_i^1, w_i^2, \ldots, w_i^T \in \mathcal{V}_q$ and $x^1, x^2, \ldots, x^T \in X \setminus \tau(w_i)$ such that:[7] (a) $w^1 = w$ and $w^T = w'$; (b) for all $t = 2, \ldots, T$, and $w_i^t(x) = w_i^{t-1}(x)$ for every $x \neq x^t$ and $(w_i, q_i) \in \mathcal{D}$.

The proof of Preliminary 3 is easy (and left to the reader). This preliminary means that we can always find a path $w_i^1, \ldots, w_i^T$ of valuation functions (starting at w_i and leading to w_i') such that, for each $t = 2, \ldots, T$, w_i^t and w_l^{t-1} disagree on at most one $x^t \in X$ that is not i's top.

Let us introduce some notation before the next preliminary. Consider distinct $\omega_1, \omega_2 \in \Omega$. For any $\alpha \geq 0$, $i \in N$ and $q_i, q_i' \in \mathcal{P}^i$, we write $q_i' = q_i \oplus \alpha\omega_1 \ominus \alpha\omega_2$ if, for all $\omega \in \Omega$, we have

$$q_i'(\omega) = \begin{cases} q_i(\omega), & \text{if } \omega \neq \omega_1, \omega_2, \\ q_i(\omega) + \alpha, & \text{if } \omega = \omega_1, \\ q_i(\omega) - \alpha, & \text{if } \omega = \omega_2. \end{cases}$$

Preliminary 4. Suppose that $(u, q) \in \mathcal{D}$. Then there exists $\alpha_q > 0$ such that: $(u, q') \in \mathcal{D}$ and $\varphi(u, q') = \varphi(u, q)$ whenever $q' \in \mathcal{P}^N$ satisfies $q'_{-i} = q_{-i}$ and $q_i' = q_i \oplus \alpha\omega_1 \ominus \alpha\omega_2$ for some $i \in N$, $\alpha \in (0, \alpha_q)$ and distinct $\omega_1, \omega_2 \in \Omega$.

Preliminary 4 obtains as the analog of Preliminary 1 when one slightly varies the belief profile at (u, q). Its proof follows from the fact that the expected utility operator $E_{u_i}^{q_i'}$ is a continuous function of the belief q_i'.

We are now ready to prove Claim 2. Using Preliminary 3, it suffices to show that, starting from a profile $v \in \mathcal{T}_{A_N}$, the decision does not change if any single agent changes her valuation of one non-top outcome. By contradiction, suppose that $h := \varphi(v,p) \neq h' := \varphi(v',p)$ for two profiles $(v,p), (v',p) \in \mathcal{D}^N$ such that $v_k = v_l$ and, for some $i \in N$ and $a \in X \setminus \tau(v_i)$, satisfying: **(i)** $v'_{-i} = v_{-i}$; **(ii)** $v_i'(x) = v_i(x)$ if $x \neq a$; **(iii)** $v_i'(a) > v_i(a)$. In addition,[8] since the set of acts is finite, it is not restrictive to assume that there exists *at most* one $\bar{z} \in (v_i(a), v_i'(a))$ such that $(v_i^{\bar{z}}, p_i) \notin \mathcal{D}$. Note that, if $\bar{z}$ indeed exists, this assumption (along with ordinality) implies

$$\varphi\left((v_{-i}, v_i^{z^*}), p\right) = \varphi\left((v_{-i}, v_i), p\right) = h, \quad \forall z^* \in (v_i(a), \bar{z}). \tag{26}$$

Next, remark that misvaluation-proofness of φ requires: (1) $E_{v_i^z}^{p_i}(h') - E_{v_i^z}^{p_i}(h) < 0$ when $z = v_i(x)$; and (2) $E_{v_i^z}^{p_i}(h) - E_{v_i^z}^{p_i}(h') > 0$ when $z = v_i'(x)$. Since $E_{v_i^z}^{p_i}$ is a continuous function of z, we may use the intermediate value theorem to claim that

[7] Remark that (b) implies $\tau(w_i^t) = \tau(w_i) = \tau(w_i')$.

[8] Recall from the proof of Lemma 2 that $v_i^z \in \mathcal{V}^i$ is defined by $v_i^z(x) = \begin{cases} v_i(x), & \text{if } x \neq a \\ z, & \text{if } x = a. \end{cases}$

there exists $\bar{z} \in (v_i(z), v_i'(x))$ such that $E^{p_i}_{v_i^{\bar{z}}}(h') - E^{p_i}_{v_i^{\bar{z}}}(h) = 0$, that is to say,

$$\sum_{\omega \in \Omega'_+} p_i(\omega)[v_i^{\bar{z}}(h'(\omega)) - v_i^{\bar{z}}(h(\omega))] = \sum_{\omega \in \Omega_-} p_i(\omega)[v_i^{\bar{z}}(h(\omega)) - v_i^{\bar{z}}(h'(\omega))], \tag{27}$$

where $\Omega'_+ := \{\omega \in \Omega : v_i^{\bar{z}}(h'(\omega)) > v_i^{\bar{z}}(h(\omega))\}$ and $\Omega'_- := \{\omega \in \Omega : v_i^{\bar{z}}(h'(\omega)) < v_i^{\bar{z}}(h(\omega))\}$. Note that $\Omega'_+ \neq \emptyset$ and $\Omega'_- \neq \emptyset$. Indeed, we have $p_i(h'^a) > p_i(h^a)$ from Lemma 2 (monotonicity); and hence $\emptyset \neq h'^a \setminus h^a \subseteq \Omega'_+ \cup \Omega'_-$. Then, assuming that $\Omega'_+ \neq \emptyset$ (or $\Omega'_- \neq \emptyset$), we may use (27) [and $p_i(\omega) > 0$ for all $w \in \Omega$)] to see that $\Omega'_- \neq \emptyset$ (or $\Omega'_+ \neq \emptyset$) must also hold.

Next, pick any $\omega_1 \in \Omega'_+$ and $\omega_2 \in \Omega'_-$; and define $p_i' = p_i \oplus \frac{\alpha_p}{2}\omega_1 \ominus \frac{\alpha_p}{2}\omega_2$, where α_p comes from Preliminary 4. Note from (27) that $\sum_{\omega \in \Omega'_+} p_i'(\omega)[v_i^{\bar{z}}(h'(\omega)) - v_i^{\bar{z}}(h(\omega))] > \sum_{\omega \in \Omega_-} p_i'(\omega)[v_i^{\bar{z}}(h(\omega)) - v_i^{\bar{z}}(h'(\omega))]$, that is to say, $E^{p_i'}_{v_i^{\bar{z}}}(h') - E^{p_i'}_{v_i^{\bar{z}}}(h) > 0$. Hence, since $E^{p_i'}_{v_i^{z}}(\cdot)$ is a continuous function of z, we can claim that there exists $z^* \in (v_i(a), \bar{z})$ such that $E^{p_i'}_{v_i^{z^*}}(h') - E^{p_i'}_{v_i^{z^*}}(h) > 0$. In other words, i prefers h' to h at $(v_i^{z^*}, p_i')$. But this is a contradiction to misvaluation-proofness. Indeed, observe that $(v_i^{z^*}, p_i) \in \mathcal{D}$ —since there exists no $\bar{z}' \in (v_i(a), v_i'(a)) \setminus \{\bar{z}\}$ such that $(v_i^{\bar{z}'}, p_i) \notin \mathcal{D}$. Therefore, we have $\left((v_i^{z^*}, v_{-i}), p\right) \in \mathcal{D}^N$. It then follows from Preliminary 4 that: $\left((v_i^{z^*}, v_{-i}), (p_i', p_{-i})\right) \in \mathcal{D}^N$ and $\varphi\left((v_i^{z^*}, v_{-i}), (p_i', p_{-i})\right) = \varphi\left((v_i^{z^*}, v_{-i}), p\right)$. Since $\varphi\left((v_i^{z^*}, v_{-i}), p\right) = h$ from (26), it holds that $\varphi\left((v_i^{z^*}, v_{-i}), (p_i', p_{-i})\right) = h$. Moreover, given that $((v_i', v_{-i}), p) = (v', p) \in \mathcal{D}^N$, Preliminary 4 once again gives: $(v', (p_i', p_{-i})) \in \mathcal{D}^N$ and $\varphi(v', (p_i', p_{-i})) = \varphi(v', p) = h'$. Thus, agent i can profitably manipulate φ at $((v_i^{z^*}, v_{-i}), (p_i', p_{-i}))$ by misreporting (v_i', p_i') in order to get h' (which she prefers to h).

Therefore, we must have $\varphi(v, p) = \varphi(v', p) = \varphi(v'', p) \in \{a_1, \ldots, a_n\}^\Omega$ for all $v, v' \in \mathcal{T}_{A_n}$; and Lemma 6 is proved. □

Lemmas 4 and 6 imply that, if a SCF is strategyproof and ex-post efficient, then at each preference profile the state space must be partitioned into a collection of events $\{E_i \in 2^\Omega : i \in N\}$ such that agent i dictates the outcome in all states $\omega \in E_i$. Note that (i) some E_i may be empty (ii) E_i may vary if we change the beliefs p or the valuations v —or more precisely, if we change the tops $(\tau(v_1), \ldots, \tau(v_n))$). That is to say, there exist functions $\sigma : \mathcal{D}^N \to (2^\Omega)^N$ such that, for all $(v, p) \in \mathcal{D}^N$,

$$\cup_{i \in N} \sigma_i(v, p) = \Omega \text{ and } \varphi(v, p; \omega) = \tau(v_i) \text{ if } \omega \in \sigma_i(v, p). \tag{28}$$

Remark that there exist many functions σ satisfying (28); but for any two of them (say, σ' and σ''), we must have:[9] $\sigma'(v, p) = \sigma''(v, p)$ at any $(v, p) \in \mathcal{D}^N$ such that $\tau(v_i) \neq \tau(v_j)$ if $i \neq j$. To conclude the proof of the Top Selection lemma, it just remains to notice that there is a unique function $s(p) = \sigma^*(v, p)$ that satisfies (28) **and** does not

[9]That is to say, $\sigma'(v, p) \neq \sigma''(v, p)$ may occur only at profiles (v, p) where some distinct agents have the same top.

vary with v. Since we have shown the tops-only property in Lemmas 4 and 6, we slightly abuse notation and conveniently write $\varphi((a_1, \ldots, a_n), p)$ to refer to the chosen act at each $(v, p) \in \mathcal{D}^N$ such that $(\tau(v_i))_{i \in N} = (a_1, \ldots, a_n)$. We then define $s(p) = \sigma^*(v, p)$ as follows. For any distinct $a_1, a_2 \in X$, let $s_1^{a_1 a_2}(p) = \varphi^{a_1}((a_1, a_2, \ldots, a_2), p)$. Define $s_i^{a_1 a_2}(p)$ in a similar way for all $i \neq 1$; and write $s(p) := s^{a_1 a_2}(p) = (s_i^{a_1 a_2}(p))_{i \in N}$. We leave it to the reader to check [by using Lemma 3-(ii) and the now established top-and-tops-only property] that we have: **(i)** $s^{a_1 a_2}(p) = s^{a_3 a_4}(p) = s(p)$, for all $a_1, a_2, a_3, a_4 \in X$ ($a_1 \neq a_2$ and $a_3 \neq a_4$) and all $p \in \mathcal{P}$; **(ii)** $\varphi((\bar{a}_1, \ldots, \bar{a}_n), p; \omega) = \bar{a}_i$ if $\omega \in s_i(p)$, for all $p \in \mathcal{P}^N$ and $(\bar{a}_1, \ldots, \bar{a}_n) \in X^N$. Observing that s meets the feasibility constraint $\cup_{i \in N} s_i(p) = \Omega$ (for all $p \in \mathcal{P}^N$) then allows to conclude. □

9 Appendix B: semi-global results

Let $\widetilde{\omega} \in \Omega$. This state is fixed throughout this appendix. It will be convenient to further simplify notation as follows: we write $\widetilde{\Omega}$ instead of $\Omega \setminus \widetilde{\omega}$, $\widetilde{\mathcal{P}}$ instead of $\mathcal{P}(\widetilde{\Omega})$, and $\widetilde{a}$ instead of $a_{\widetilde{\omega}}$. For any $\pi_i \in \widetilde{\mathcal{P}}$, define

$$\mathcal{P}(\pi_i) = \left\{ p_i \in \mathcal{P} : p_i \mid \widetilde{\Omega} \ \approx \pi_i \right\}.$$

This is the set of beliefs on Ω generating on $\widetilde{\Omega}$ a likelihood ordering that coincides with that generated by π_i.

For any two beliefs $p_i, q_i \in \mathcal{P}$, we write $p_i \approx q_i$ if p_i, q_i are ordinally equivalent, that is, if $R(p_i) = R(q_i)$. We abuse this notation and, for any profiles $p, q \in \mathcal{P}^N$, we write $p \approx q$ if $p_i \approx q_i$ for all $i \in N$. We write $p_i J q_i$ if p_i, q_i are adjacent according to the definition in Section 5. The adjacency relation J is obviously a symmetric binary relation. If $p_i, q_i \in \mathcal{P}' \subseteq \mathcal{P}$, *a J-path between p_i and q_i in $\mathcal{P}'$* is a finite sequence $\mathbf{p}_i = (\mathbf{p}_i^t)_{t=1}^T$ such that $\mathbf{p}_i^1 = p_i$, $\mathbf{p}_i^T = q_i$, $\mathbf{p}_i^t J \mathbf{p}_i^{t+1}$ for $t = 1, ..., T-1$, and $\mathbf{p}_i^t \in \mathcal{P}'$ for $t = 1, ..., T$. We call $\mathcal{P}'$ *connected* if such a J-path exists between any two beliefs in $\mathcal{P}'$.

Finally, define the relation $\widetilde{J}$ on $\mathcal{P}(\pi_i)$ by

$$p_i \widetilde{J} q_i \Leftrightarrow p_i, q_i \text{ are } \{A, B\}\text{-adjacent for some } \{A, B\} \in \mathcal{H}, \widetilde{\omega} \in A, \text{ and } p_i(A) > p_i(B).$$

This is a sub-relation of the adjacency relation J. Contrary to J, the relation $\widetilde{J}$ is not symmetric. For an illustration, see Figure 2, where an arrow stands for $\widetilde{J}$. Observe that if two beliefs $p_i, q_i \in \mathcal{P}(\pi_i)$ are $\{A, B\}$-adjacent, then $\widetilde{\omega} \in A \cup B$: this is because the likelihood relations generated by p_i, q_i coincide on $\widetilde{\Omega}$. Just like J, the relation $\widetilde{J}$ is ordinal: if $p_i \widetilde{J} q_i$, $p_i' \approx p_i$ and $q_i' \approx q_i$, then $p_i' \widetilde{J} q_i'$. All its maximal elements in $\mathcal{P}(\pi_i)$ are ordinally equivalent; any such maximal element p_i^+ is characterized by the property that

$$p_i^+(\widetilde{\omega}) > p_i^+(\widetilde{\Omega}). \tag{29}$$

Likewise, all the minimal elements of $\widetilde{J}$ are ordinally equivalent and any such minimal element p_i^- is characterized by the property that

$$p_i^-(C \cup \widetilde{\omega}) < p_i^-(D) \text{ whenever } \pi_i(C) < \pi_i(D).$$

Example 2. *If* $\Omega = \{1,2,3\}$, $\widetilde{\omega} = 1$, *and* π_i *is a belief on* $\{2,3\}$ *generating the ordering* $\{2,3\},\{2\},\{3\}$, *then any belief on* $\{1,2,3\}$ *generating the ordering*

$$\{\mathbf{1},2,3\},\{\mathbf{1},2\},\{\mathbf{1},3\},\{\mathbf{1}\},\{2,3\},\{2\},\{3\}$$

is a maximal element p_i^+ *of* $\widetilde{J}$ *on* $\mathcal{P}(\pi_i)$, *and any belief on* $\{1,2,3\}$ *generating the ordering*

$$\{\mathbf{1},2,3\},\{2,3\},\{\mathbf{1},2\},\{2\},\{\mathbf{1},3\},\{3\},\{\mathbf{1}\}.$$

is a minimal element p_i^- *of* $\widetilde{J}$ *on* $\mathcal{P}(\pi_i)$. *See again Figure 2 for an illustration.*

A *complete* $\widetilde{J}$*-path* in $\mathcal{P}(\pi_i)$, or simply a complete path, is a finite sequence $\mathbf{p}_i = (\mathbf{p}_i^t)_{t=1}^T$ such that $\mathbf{p}_i^1$ is a maximal element of $\widetilde{J}$ (in $\mathcal{P}(\pi_i)$), $\mathbf{p}_i^T$ is a minimal element, $\mathbf{p}_i^t \widetilde{J} \mathbf{p}_i^{t+1}$ for $t = 1,...,T-1$, and $\mathbf{p}_i^t \in \mathcal{P}(\pi_i)$ for $t = 1,...,T$.

Observation 1. *For each complete* $\widetilde{J}$*-path* $\mathbf{p}_i = (\mathbf{p}_i^t)_{t=1}^T$ *in* $\mathcal{P}(\pi_i)$, $T = \mid \{\{A,B\} \in \mathcal{H} : \widetilde{\omega} \in A \cup B\} \mid$.

This is because any maximal and minimal elements p_i^+, p_i^- lie (i) on opposite sides of every hyperplane $p_i(A) = p_i(B)$ such that $\widetilde{\omega} \in A \cup B$, and (ii) on the same side of every hyperplane $p_i(A) = p_i(B)$ such that $\widetilde{\omega} \notin A \cup B$.

Observation 2. *For each complete* $\widetilde{J}$*-path* $\mathbf{p}_i$ *in* $\mathcal{P}(\pi_i)$ *and each* $t \in \{1,...,T-1\}$, *there is a unique* $\{A^t, B^t\} \in \mathcal{H}$ *such that* $\mathbf{p}_i^t, \mathbf{p}_i^{t+1}$ *are* $\{A^t, B^t\}$*-adjacent. Moreover,* $\{A^t, B^t\} \neq \left\{A^{t'}, B^{t'}\right\}$ *if* $t \neq t'$.

Observation 3. *Each belief* $p_i \in \mathcal{P}(\pi_i)$ *lies on some complete* $\widetilde{J}$*-path in* $\mathcal{P}(\pi_i)$: *there exist* $\mathbf{p}_i$ *and* $t \in \{1,...,T\}$ *such that* $p_i = \mathbf{p}_i^t$.

The proofs of observations 2 and 3 are straightforward and left to the reader.

Lemma 7. *For all* $i \in N$, $\pi_i \in \widetilde{\mathcal{P}}$, *and* $p_{-i} \in \mathcal{P}^{N\backslash i}$, *either (a)* $s_i(., p_{-i})$ *is constant on* $\mathcal{P}(\pi_i)$, *or (b) there exist disjoint sets* $A_i(\pi_i, p_{-i}), B_i(\pi_i, p_{-i}), C_i(\pi_i, p_{-i}) \subseteq \Omega$ *such that* $\widetilde{\omega} \in A_i(\pi_i, p_{-i})$, $\pi_i(A_i(\pi_i, p_{-i}) \setminus \widetilde{\omega}) < \pi_i(B_i(\pi_i, p_{-i}))$, *and for all* $p_i \in \mathcal{P}(\pi_i)$,

$$s_i(p_i, p_{-i}) = \begin{cases} A_i(\pi_i, p_{-i}) \cup C_i(\pi_i, p_{-i}) & \text{if } p_i(A_i(\pi_i, p_{-i})) > p_i(B_i(\pi_i, p_{-i})), \\ B_i(\pi_i, p_{-i}) \cup C_i(\pi_i, p_{-i}) & \text{otherwise.} \end{cases}$$

The inequality $\pi_i(A_i(\pi_i, p_{-i}) \setminus \widetilde{\omega}) < \pi_i(B_i(\pi_i, p_{-i}))$ implies that the function $s_i(., p_{-i})$ in statement *(b)* is not constant: the assignment actually varies with agent i's beliefs.

Proof. Let $i \in N$, $\pi_i \in \widetilde{\mathcal{P}}$, $p_{-i} \in \mathcal{P}^{N\backslash i}$. Since π_i, p_{-i} are fixed throughout the proof, we omit them from our notation. It is important to keep in mind, however, that the sets whose existence is asserted in Lemma 7 may depend on our choice of π_i, p_{-i}. Let $T = |\{\{A,B\} \in \mathcal{H} : \widetilde{\omega} \in A \cup B\}|$.

Step 1. We claim that for any complete $\widetilde{J}$-path $\mathbf{p}_i = (\mathbf{p}_i^t)_{t=1}^T$ in $\mathcal{P}(\pi_i)$, one of the following statements hold:

(α) $s_i(\mathbf{p}_i^1) = s_i(\mathbf{p}_i^2) = ... = s_i(\mathbf{p}_i^T)$,
(β) there exist disjoint sets $A_i(\mathbf{p}_i), B_i(\mathbf{p}_i), C_i(\mathbf{p}_i) \subseteq \Omega$ such that $\widetilde{\omega} \in A_i(\mathbf{p}_i)$, $\pi_i(A_i(\mathbf{p}_i) \setminus \widetilde{\omega}) < \pi_i(B_i(\mathbf{p}_i))$, and there exists $t^*(\mathbf{p}_i) \in \{1, ..., T-1\}$ such that

$$s_i(\mathbf{p}_i^t) = \begin{cases} A_i(\mathbf{p}_i) \cup C_i(\mathbf{p}_i) & \text{if } t \leq t^*(\mathbf{p}_i), \\ B_i(\mathbf{p}_i) \cup C_i(\mathbf{p}_i) & \text{if } t > t^*(\mathbf{p}_i). \end{cases} \tag{30}$$

To prove this claim, fix a complete $\widetilde{J}$-path $\mathbf{p}_i$ in $\mathcal{P}(\pi_i)$. For each $t = 1, ..., T-1$, let $\{A^t, B^t\}$ be the unique pair in $\mathcal{H}$ such that $\mathbf{p}_i^t, \mathbf{p}_i^{t+1}$ are $\{A^t, B^t\}$-adjacent. By definition of $\widetilde{J}$, $\widetilde{\omega} \in A^t$ and $\mathbf{p}_i^t(A^t) > \mathbf{p}_i^t(B^t)$. By the Local Bilaterality lemma, one of the following statements holds:

(i) $s_i(\mathbf{p}_i^t) = s_i(\mathbf{p}_i^{t+1})$,
(ii) $s_i(\mathbf{p}_i^t) \setminus s_i(\mathbf{p}_i^{t+1}) = A^t$ and $s_i(\mathbf{p}_i^{t+1}) \setminus s_i(\mathbf{p}_i^t) = B^t$.

If (i) holds for $t = 1, ..., T-1$, then statement (α) is true. Otherwise, let t^* be the smallest $t \in \{1, ..., T-1\}$ such that $s_i(\mathbf{p}_i^t) \neq s_i(\mathbf{p}_i^{t+1})$. By (ii), $s_i(\mathbf{p}_i^{t^*}) \setminus s_i(\mathbf{p}_i^{t^*+1}) = A^{t^*}$. Since $\widetilde{\omega} \in A^{t^*}$, we have $\widetilde{\omega} \notin s_i(\mathbf{p}_i^{t^*+1})$. This means that statement (ii) cannot hold for any $t = t^* + 1, ..., T$. Hence, $s_i(\mathbf{p}_i^t) = s_i(\mathbf{p}_i^{t^*+1})$ for $t = t^* + 1, ..., T$. Defining $A_i(\mathbf{p}_i) = A^{t^*}$, $B_i(\mathbf{p}_i) = B^{t^*}$, $C_i(\mathbf{p}_i) = s_i(p_i^1) \setminus A^{t^*}$, we obtain (30).

Step 2. Let p_i^+ and p_i^- be maximal and minimal elements of $\widetilde{J}$ in $\mathcal{P}(\pi_i)$.

If $s_i(p_i^+) = s_i(p_i^-)$, define $C_i = s_i(p_i^+) = s_i(p_i^-)$. For any $p_i \in \mathcal{P}(\pi_i)$ there exists some path $\mathbf{p}_i$ and some $t \in \{1, ..., T\}$ such that $p_i = \mathbf{p}_i^t$ (Observation 3). By Step 1, $s_i(p_i) = s_i(\mathbf{p}_i^t) = C_i$, that is, statement *(a)* in Lemma 7 holds.

If $s_i(p_i^+) \neq s_i(p_i^-)$, we know from Step 2 that statement (β) holds for every complete $\widetilde{J}$-path $\mathbf{p}_i = (\mathbf{p}_i^t)_{t=1}^T$ in $\mathcal{P}(\pi_i)$. We claim that the sets $A_i(\mathbf{p}_i), B_i(\mathbf{p}_i), C_i(\mathbf{p}_i)$ do not change with $\mathbf{p}_i$. To see why, let $\mathbf{p}_i, \mathbf{q}_i$ be two paths. If $A_i(\mathbf{p}_i) \neq A_i(\mathbf{q}_i)$ or $C_i(\mathbf{p}_i) \neq C_i(\mathbf{q}_i)$, then $s_i(p_i^+) = s_i(\mathbf{p}_i^1) = A_i(\mathbf{p}_i) \cup C_i(\mathbf{p}_i) \neq A_i(\mathbf{q}_i) \cup C_i(\mathbf{q}_i) = s_i(\mathbf{q}_i^1) = s_i(p_i^+)$, a contradiction. Thus $A_i(\mathbf{p}_i) = A_i(\mathbf{q}_i)$ and $C_i(\mathbf{p}_i) = C_i(\mathbf{q}_i)$. Next, if $B_i(\mathbf{p}_i) \neq B_i(\mathbf{q}_i)$, then $s_i(p_i^-) = s_i(\mathbf{p}_i^T) = B_i(\mathbf{p}_i) \cup C_i(\mathbf{p}_i) = B_i(\mathbf{p}_i) \cup C_i(\mathbf{q}_i) \neq B_i(\mathbf{q}_i) \cup C_i(\mathbf{q}_i) = s_i(\mathbf{q}_i^T) = s_i(p_i^-)$, again a contradiction.

Let A_i, B_i, C_i be the sets such that $A_i(\mathbf{p}_i) = A_i$, $B_i(\mathbf{p}_i) = B_i$, and $C_i(\mathbf{p}_i) = C_i$ for all complete $\widetilde{J}$-paths $\mathbf{p}_i$ in $\mathcal{P}(\pi_i)$. For any $p_i \in \mathcal{P}(\pi_i)$ there exist some path $\mathbf{p}_i$ and some $t \in \{1, ..., T\}$ such that $p_i = \mathbf{p}_i^t$, and, by Step 1, an integer $t^*(\mathbf{p}_i) \in \{1, ..., T-1\}$ such that

$$s_i(\mathbf{p}_i^t) = \begin{cases} A_i \cup C_i & \text{if } t \leq t^*(\mathbf{p}_i), \\ B_i \cup C_i & \text{if } t > t^*(\mathbf{p}_i). \end{cases} \tag{31}$$

This integer may –and typically does– change with the path $\mathbf{p}_i$, as Figure 2 illustrates.

If $p_i(A_i) = \mathbf{p}_i^t(A_i) > \mathbf{p}_i^t(B_i) = p_i(B_i)$, then $t \leq t^*(\mathbf{p}_i)$: otherwise (31) would imply $s_i(p_i) = B_i \cup C_i$, hence $p_i(s_i(\mathbf{p}_i^1)) = p_i(A_i \cup C_i) > p_i(B_i \cup C_i) = p_i(s_i(p_i))$, contradicting strategyproofness. Since $t \leq t^*(\mathbf{p}_i)$, (31) implies $s_i(p_i) = A_i \cup C_i$.

Likewise, if $p_i(A_i) < p_i(B_i)$, then $t > t^*(\mathbf{p}_i)$ and (31) imply $s_i(p_i) = B_i \cup C_i$. We conclude that statement *(b)* in Lemma 7 holds with $A_i(\pi_i, p_{-i}) = A_i$, $B_i(\pi_i, p_{-i}) = B_i$, and $C_i(\pi_i, p_{-i}) = C_i$. $\square$

We record below two immediate consequences of Lemma 7 that will be used later.

Corollary 1. *For all $i \in N$, $\pi_i \in \widetilde{\mathcal{P}}$, $p_i, p_i' \in \mathcal{P}(\pi_i)$, and $p_{-i} \in \mathcal{P}^{N\setminus i}$,*
(a) $\widetilde{\omega} \in s_i(p_i, p_{-i}) \cap s_i(p_i', p_{-i}) \Rightarrow s_i(p_i, p_{-i}) = s_i(p_i', p_{-i})$,
(b) $\widetilde{\omega} \notin s_i(p_i, p_{-i}) \cup s_i(p_i', p_{-i}) \Rightarrow s_i(p_i, p_{-i}) = s_i(p_i', p_{-i})$.

Given the other agents' beliefs, i's assignment is fully determined by whether it contains $\widetilde{\omega}$ or not.

Corollary 2. *For all $i \in N$, $\pi_i \in \widetilde{\mathcal{P}}$, $p_{-i} \in \mathcal{P}^{N\setminus i}$, and any maximal and minimal elements p_i^+, p_i^- of $\widetilde{J}$ in $\mathcal{P}(\pi_i)$, if $s(., p_{-i})$ is not constant on $\mathcal{P}(\pi_i)$, then $\widetilde{\omega} \in s_i(p_i^+, p_{-i}) \setminus s_i(p_i^-, p_{-i})$.*

We now show that the sets $A_i(\pi_i, p_{-i})$, $B_i(\pi_i, p_{-i})$, $C_i(\pi_i, p_{-i})$ in Lemma 7 do not vary with p_{-i} as long as the ordering generated on $\widetilde{\mathcal{P}}$ by each p_j, $j \in N \setminus i$, remains unchanged. If $\pi \in \widetilde{\mathcal{P}}^N$ and $i \in N$, we write $\mathcal{P}^N(\pi) = \Pi_{k \in N}\mathcal{P}(\pi_k)$ and $\mathcal{P}^{N\setminus i}(\pi_{-i}) = \Pi_{k \neq i}\mathcal{P}(\pi_k)$.

Lemma 8. *For all $i \in N$ and $\pi \in \widetilde{\mathcal{P}}^N$, there exist disjoint sets $A_i(\pi)$, $B_i(\pi)$, $C_i(\pi) \subseteq \Omega$ such that $\widetilde{\omega} \in A_i(\pi)$, $\pi_i(A_i(\pi) \setminus \widetilde{\omega}) < \pi_i(B_i(\pi))$, and, for all $p_{-i} \in \mathcal{P}^{N\setminus i}(\pi_{-i})$, either (a) $s_i(., p_{-i})$ is constant on $\mathcal{P}(\pi_i)$, or (b) for all $p_i \in \mathcal{P}(\pi_i)$,*

$$s_i(p_i, p_{-i}) = \begin{cases} A_i(\pi) \cup C_i(\pi) & \text{if } p_i(A_i(\pi)) > p_i(B_i(\pi)), \\ B_i(\pi) \cup C_i(\pi) & \text{otherwise.} \end{cases}$$

We emphasize that Lemma 8 does *not* assert that $s_i(p_i, .)$ is constant over $\mathcal{P}^{N\setminus i}(\pi_{-i})$.

Proof. Let $i \in N$ and let $\pi \in \widetilde{\mathcal{P}}^N$. Define the set

$$\mathcal{P}_*^{N\setminus i}(\pi_{-i}) = \left\{p_{-i} \in \mathcal{P}^{N\setminus i}(\pi_{-i}) : s_i(., p_{-i}) \text{ is not constant on } \mathcal{P}(\pi_i)\right\}. \tag{32}$$

Let $p_{-i}, q_{-i} \in \mathcal{P}_*^{N\setminus i}(\pi_{-i})$. By Lemma 7 –and dropping π_i from the notation– there exist disjoint sets $A_i(p_{-i})$, $B_i(p_{-i})$, $C_i(p_{-i}) \subseteq \Omega$ such that $\widetilde{\omega} \in A_i(p_{-i})$, $\pi_i(A_i(p_{-i}) \setminus \widetilde{\omega}) < \pi_i(B_i(p_{-i}))$, and

$$\text{for all } p_i \in \mathcal{P}(\pi_i),\ s_i(p_i, p_{-i}) = \begin{cases} A_i(p_{-i}) \cup C_i(p_{-i}) & \text{if } p_i(A_i(p_{-i})) > p_i(B_i(p_{-i})), \\ B_i(p_{-i}) \cup C_i(p_{-i}) & \text{otherwise,} \end{cases} \tag{33}$$

and there exist disjoint sets $A_i(q_{-i})$, $B_i(q_{-i})$, $C_i(q_{-i}) \subseteq \Omega$ such that $\widetilde{\omega} \in A_i(q_{-i})$, $\pi_i(A_i(q_{-i}) \setminus \widetilde{\omega}) < \pi_i(B_i(q_{-i}))$, and

$$\text{for all } p_i \in \mathcal{P}(\pi_i),\ s_i(p_i, q_{-i}) = \begin{cases} A_i(q_{-i}) \cup C_i(q_{-i}) & \text{if } p_i(A_i(q_{-i})) > p_i(B_i(q_{-i})), \\ B_i(q_{-i}) \cup C_i(q_{-i}) & \text{otherwise.} \end{cases} \tag{34}$$

We must prove that $A_i(p_{-i}) = A_i(q_{-i})$, $B_i(p_{-i}) = B_i(q_{-i})$, and $C_i(p_{-i}) = C_i(q_{-i})$.

There is obviously no loss of generality in assuming that there exists some $j \neq i$ such that $p_k = q_k$ for all $k \in N \setminus \{i, j\}$. We therefore drop the beliefs of the agents other than i, j from our notation. Moreover, since $\mathcal{P}(\pi_j)$ is connected, there is no loss in assuming that p_j, q_j are adjacent.

Let p_i^+, p_i^- be maximal and minimal elements of $\widetilde{J}$ in $\mathcal{P}(\pi_i)$. By Corollary 2,

$$\begin{aligned} \widetilde{\omega} &\in s_i(p_i^+, p_j) \setminus s_i(p_i^-, p_j), \\ \widetilde{\omega} &\in s_i(p_i^+, q_j) \setminus s_i(p_i^-, q_j). \end{aligned}$$

Since $\widetilde{\omega} \notin s_j(p_i^+, p_j) \cup s_j(p_i^+, q_j)$, Corollary 1 implies $s_j(p_i^+, p_j) = s_j(p_i^+, q_j)$. By non-bossiness, $s_i(p_i^+, p_j) = s_i(p_i^+, q_j)$. Since $\widetilde{\omega} \in s_i(p_i^+, p_j) \cap s_i(p_i^+, q_j)$, it follows from (34) that

$$A_i(p_j) \cup C_i(p_j) = A_i(q_j) \cup C_i(q_j).$$

Next, we claim that either $\widetilde{\omega} \in s_j(p_i^-, p_j) \cap s_j(p_i^-, q_j)$ or $\widetilde{\omega} \notin s_j(p_i^-, p_j) \cup s_j(p_i^-, q_j)$. Suppose, on the contrary, that, say, $\widetilde{\omega} \in s_j(p_i^-, p_j) \setminus s_j(p_i^-, q_j)$. Since $\widetilde{\omega} \notin s_i(p_i^-, q_j)$, there exists $k \in N \setminus \{i, j\}$ such that $\widetilde{\omega} \in s_k(p_i^-, q_j) \setminus s_k(p_i^-, p_j)$. By the Local Bilaterality lemma, $s_i(p_i^-, q_j) = s_i(p_i^-, p_j)$, that is,

$$B_i(p_j) \cup C_i(p_j) = B_i(q_j) \cup C_i(q_j).$$

Since $A_i(p_j)$, $B_i(p_j)$, $C_i(p_j)$ are disjoint and $A_i(q_j)$, $B_i(q_j)$, $C_i(q_j)$ are disjoint, these equalities imply $A_i(p_j) = A_i(q_j)$, $B_i(p_j) = B_i(q_j)$, and $C_i(p_j) = C_i(q_j)$. $\square$

We are now ready to describe the structure of s on any sub-domain $\mathcal{P}^N(\pi)$.

Terminology. Given $\pi \in \widetilde{\mathcal{P}}^N$, we say that *$s$ varies only with agent i's beliefs (on $\mathcal{P}^N(\pi)$)* if there exists $p_{-i} \in \mathcal{P}^{N \setminus i}(\pi_{-i})$ such that $s(., p_{-i})$ is not constant on $\mathcal{P}(\pi_i)$ but $s(., p_{-j})$ is constant on $\mathcal{P}(\pi_j)$ for every $j \neq i$ and every $p_{-j} \in \mathcal{P}^{N \setminus j}(\pi_{-j})$. We say that *$s$ varies with the beliefs of agents i and j (on $\mathcal{P}^N(\pi)$)* if there exist $p_{-i} \in \mathcal{P}^{N \setminus i}(\pi_{-i})$ such that $s(., p_{-i})$ is not constant on $\mathcal{P}(\pi_i)$ and there exists $p_{-j} \in \mathcal{P}^{N \setminus j}(\pi_{-j})$ such that $s(., p_{-j})$ is not constant on $\mathcal{P}(\pi_j)$. We emphasize that this definition allows s to potentially vary with the beliefs of agents other than i, j as well.

We say that $\{A, B\} \in \mathcal{H}$ *cuts* $\mathcal{P}(\pi_i)$ if there exist $p_i, q_i \in \mathcal{P}(\pi_i)$ such that $(p_i(A) - p_i(B))(q_i(A) - q_i(B)) < 0$. Observe that if $\widetilde{\omega} \in A$, then $\{A, B\}$ cuts $\mathcal{P}(\pi_i)$ if and only if $\pi_i(A \setminus \widetilde{\omega}) < \pi_i(B)$.

Lemma 9. *For every $\pi \in \widetilde{\mathcal{P}}^N$ there exists a partition $\{A(\pi), B(\pi), C_1(\pi), ..., C_n(\pi)\}$ of Ω such that $\widetilde{\omega} \in A(\pi) \cup B(\pi)$ and*
(a) if s varies only with agent 1's beliefs on $\mathcal{P}^N(\pi)$, then $\{A, B\}$ cuts $\mathcal{P}(\pi_1)$ and there exists an agent $i \in N \setminus 1$, say agent 2, such that for all $p \in \mathcal{P}^N(\pi)$,

$$s(p) = \begin{cases} (A(\pi) \cup C_1(\pi),\ B(\pi) \cup C_2(\pi),\ C_3(\pi), ...,\ C_n(\pi)) & \text{if } p_1(A(\pi)) > p_1(B(\pi)), \\ (B(\pi) \cup C_1(\pi),\ A(\pi) \cup C_2(\pi),\ C_3(\pi), ...,\ C_n(\pi))) & \text{otherwise,} \end{cases}$$

(b) if s varies with the beliefs of agents 1 and 2 on $\mathcal{P}^N(\pi)$, then $\{A, B\}$ cuts $\mathcal{P}(\pi_1), \mathcal{P}(\pi_2)$ and for all $p \in \mathcal{P}^N(\pi)$,

$$s(p) = \begin{cases} (A(\pi) \cup C_1(\pi),\ B(\pi) \cup C_2(\pi),\ C_3(\pi), ...,\ C_n(\pi)) & \text{if } p_1(A(\pi)) > p_1(B(\pi)) \\ & \text{and } p_2(A(\pi)) < p_2(B(\pi)), \\ (B(\pi) \cup C_1(\pi),\ A(\pi) \cup C_2(\pi),\ C_3(\pi), ...,\ C_n(\pi)) & \text{otherwise.} \end{cases}$$

Remark 1. *(a) We stated Lemma 9 with reference to agents 1 and 2 for notational convenience but of course the result holds, up to a relabeling, for any pair of agents.*

(b) Statement (b) does not assume that the assignment is independent of the beliefs of agents $3, ..., n$. Rather, it is a corollary to Lemma 9 that, on $\mathcal{P}^N(\pi)$, (i) the assignment may vary with the beliefs of at most two agents and (ii) only the events assigned to two agents may change.

Proof. Let $\pi \in \widetilde{\mathcal{P}}^N$. This profile is fixed throughout the proof and dropped from the notation whenever this causes no confusion.

Step 1. Suppose first that s varies only with agent 1's beliefs.

Recall the definition of $\mathcal{P}_*^{N\backslash 1}(\pi_{-1})$ in (32). By Lemma 8, there exist disjoint sets A_1, B_1, C_1 such that for all $p_1 \in \mathcal{P}(\pi_1)$ and all $p_{-1} \in \mathcal{P}_*^{N\backslash 1}(\pi_{-1})$,

$$s_1(p_1, p_{-1}) = \begin{cases} A_1 \cup C_1 & \text{if } p_1(A_1) > p_1(B_1), \\ B_1 \cup C_1 & \text{otherwise.} \end{cases}$$

Moreover, $\widetilde{\omega} \in A_1$ and $\pi_1(A_1 \setminus \widetilde{\omega}) < \pi_1(B_1(\pi))$, implying that $\{A_1, B_1\}$ cuts $\mathcal{P}(\pi_1)$.

Since s does not vary with the beliefs of agents $2, ..., n$, the above expression must, in fact, hold for all $(p_1, p_{-1}) \in \mathcal{P}^N(\pi)$. Statement *(a)* now follows from the Local Bilaterality lemma and non-bossiness.

Step 2. Suppose next that s varies with the beliefs of agents 1 and 2 on $\mathcal{P}^N(\pi)$.

Since $\mathcal{P}(\pi_1)$, $\mathcal{P}(\pi_2)$ are connected, there are adjacent beliefs $p_1, p_1' \in \mathcal{P}(\pi_1)$, adjacent beliefs $p_2, p_2' \in \mathcal{P}(\pi_2)$, and sub-profiles $p_{-1} \in \mathcal{P}^{N\backslash 1}(\pi_{-1})$, $q_{-2} \in \mathcal{P}^{N\backslash 2}(\pi_{-2})$ such that

$$s(p_1, p_{-1}) = \alpha \neq \alpha' = s(p_1', p_{-1}), \tag{35}$$

$$s(q_2, q_{-2}) = \beta \neq \beta' = s(q_2', q_{-2}). \tag{36}$$

Sub-step 2.1. We show that the assignment varies locally with two agents' beliefs: there exist two agents $i, j \in N$, two adjacent beliefs $p_i, p_i' \in \mathcal{P}(\pi_i)$, two adjacent beliefs $p_j, p_j' \in \mathcal{P}(\pi_j)$, and a sub-profile $p_{-ij} \in \mathcal{P}^{N\backslash ij}(\pi_{-ij})$ such that $s(p_i', p_j, p_{-ij}) \neq s(p_i, p_j, p_{-ij}) \neq s(p_i, p_j', p_{-ij})$.

Suppose not. Then (35) implies

$$s(p_1, p'_j, p_{-1j}) = \alpha \neq \alpha' = s(p'_1, p'_j, p_{-1j})$$

for all $j \neq 1$ and all p'_j adjacent to p_j. Since $\mathcal{P}(\pi_j)$ is connected, it follows that

$$s(p_1, p'_{-1}) = \alpha \neq \alpha' = s(p'_1, p'_{-1}) \tag{37}$$

for all $p'_{-1} \in \mathcal{P}^{N\backslash 1}(\pi_{-1})$.

By the same token, (36) implies

$$s(q_2, q'_{-2}) = \alpha \neq \alpha' = s(q'_2, q'_{-2}) \tag{38}$$

for all $q'_{-2} \in \mathcal{P}^{N\backslash 2}(\pi_{-2})$.

Statement (37) implies $s(p_1, q_2, p_{-12}) = s(p_1, q'_2, p_{-12})$ and statement (38) implies $s(p_1, q_2, p_{-12}) \neq s(p_1, q'_2, p_{-12})$, a contradiction.

Sub-step 2.2. We show that there exist disjoint sets $A, B, C_1, ..., C_n$ such that $A, B \neq \emptyset$, $\widetilde{\omega} \in A \cup B$, and, for all $k \neq i, j$,

$$\begin{aligned} (s_i, s_j, s_k)(p_i, p_j, p_{-ij}) &= (A \cup C_i,\ B \cup C_j,\ C_k), \\ (s_i, s_j, s_k)(p'_i, p_j, p_{-ij}) = (s_i, s_j, s_k)(p'_i, p_j, p_{-ij}) &= (B \cup C_i,\ A \cup C_j,\ C_k). \end{aligned} \tag{39}$$

Since p_{-ij} is fixed, let us drop it from the notation. By Sub-step 2.1 and Lemma 8, there exist disjoint sets A_i, B_i, C_i and disjoint sets A_j, B_j, C_j such that $\widetilde{\omega} \in A_i \cap A_j$, $B_i, B_j \neq \emptyset$, and

$$[s_i(p_i, p_j) = A_i \cup C_i,\ s_i(p'_i, p_j) = B_i \cup C_i] \text{ or } [s_i(p_i, p_j) = B_i \cup C_i,\ s_i(p'_i, p_j) = A_i \cup C_i]$$

and

$$\left[s_j(p_i, p_j) = A_j \cup C_j,\ s_j(p_i, p'_j) = B_j \cup C_j\right] \text{ or } \left[s_j(p_i, p_j) = B_j \cup C_j,\ s_j(p_i, p'_j) = A_j \cup C_j\right].$$

Since $\widetilde{\omega} \in A_i \cap A_j$ and $s_i(p_i, p_j) \cap s_j(p_i, p_j) = \emptyset$, we need only consider three cases.

Case 1. (i) $s_i(p_i, p_j) = A_i \cup C_i$, (ii) $s_i(p'_i, p_j) = B_i \cup C_i$, (iii) $s_j(p_i, p_j) = B_j \cup C_j$, (iv) $s_j(p_i, p'_j) = A_j \cup C_j$.

Define $A = A_i$, $B = B_j$, $C_k = s_k(p_i, p_j)$ for $k \neq i, j$. By the Local Bilaterality lemma, (i), (iii), and (iv) imply $A_j = A$, $B_i = B$, $s_i(p_i, p'_j) = B \cup C_i$, and $s_k(p_i, p'_j) = C_k$ for $k \neq i, j$.

Next, since $s_i(p_i, p_j) = A \cup C_i$, $s_i(p'_i, p_j) = B \cup C_i$, and $s_j(p_i, p_j) = B \cup C_j$, the Local Bilaterality lemma implies $s_j(p'_i, p_j) = A \cup C_j$ and $s_k(p'_i, p_j) = C_k$ for $k \neq i, j$, establishing (39).

Case 2. (i) $s_i(p_i, p_j) = B_i \cup C_i$, (ii) $s_i(p'_i, p_j) = A_i \cup C_i$, (iii) $s_j(p_i, p_j) = A_j \cup C_j$, (iv) $s_j(p_i, p'_j) = B_j \cup C_j$.

Define $A = B_i$, $B = A_j$, $C_k = s_k(p_i, p_j)$ for $k \neq i, j$. Statement (39) follows by the same argument as in Case 1, mutatis mutandis.

Case 3. (i) $s_i(p_i, p_j) = B_i \cup C_i$, (ii) $s_i(p'_i, p_j) = A_i \cup C_i$, (iii) $s_j(p_i, p_j) = B_j \cup C_j$, (iv) $s_j(p_i, p'_j) = A_j \cup C_j$.

This case is impossible. To see why, note first that (i), (ii), (iii), and the Local Bilaterality lemma imply $s_j(p'_i, p_j) = B_j \cup C_j$ whereas (i), (iii), (iv) and the Local Bilaterality lemma imply $s_i(p_i, p'_j) = B_i \cup C_i$.

Since $(s_i, s_j)(p'_i, p_j) = (A_i \cup C_i, B_j \cup C_j)$ and $(s_i, s_j)(p_i, p'_j) = (B_i \cup C_i, A_j \cup C_j)$, Lemma 3 implies that one of the following statements holds:

$$\begin{aligned}(s_i, s_j)(p'_i, p'_j) &= (A_i \cup C_i,\ B_j \cup C_j),\\ (s_i, s_j)(p'_i, p'_j) &= (B_i \cup C_i,\ A_j \cup C_j).\end{aligned}$$

In either case, the Local Bilaterality lemma requires $A_i = A_j$ and $B_i = B_j$. The latter equality implies that $s_i(p_i, p_j) \cap s_j(p_i, p_j) \neq \emptyset$, violating feasibility.

Sub-step 2.3. Assume from now on that $\widetilde{\omega}$ belongs to the set A in (39). The case where $\widetilde{\omega}$ belongs to B is identical up to a permutation of agents i and j. We show that for all $(q_i, q_j) \in \mathcal{P}(\pi_i) \times \mathcal{P}(\pi_j)$ and all $k \neq i, j$,

$$(s_i, s_j, s_k)(q_i, q_j, p_{-ij}) = \begin{cases} (A \cup C_i,\ B \cup C_j,\ C_k) & \text{if } q_i(A) > q_i(B) \text{ and } q_j(A) < q_j(B),\\ (B \cup C_i,\ A \cup C_j,\ C_k) & \text{otherwise.}\end{cases} \tag{40}$$

Since p_{-ij} is fixed, let us drop it again from the notation. By Sub-step 2.2 and Lemma 8, $p_i(A) > p_i(B)$ and $p_j(A) < p_j(B)$, and it follows that (40) holds for the case where $q_i = p_i$ or $q_j = p_j$.

Next, for any q_i such that $q_i(A) < q_i(B)$, the fact that $s_j(q_i, p_j) = A \cup C_j$ implies that $s_j(q_i, .)$ is constant, hence, by non-bossiness, $(s_i, s_j, s_k)(q_i, q_j) = (B \cup C_i,\ A \cup C_j,\ C_k)$.

Similarly, for any q_j such that $q_j(A) > q_j(B)$, the fact that $s_i(p_i, q_j) = B \cup C_i$ implies that $s_i(., q_j)$ is constant, hence, by non-bossiness, $(s_i, s_j, s_k)(q_i, q_j) = (B \cup C_i,\ A \cup C_j,\ C_k)$.

Finally, for any (q_i, q_j) such that $q_i(A) > q_i(B)$ and $q_j(A) < q_j(B)$, the fact that $s_i(., q_j)$ and $s_j(., q_i)$ are not constant, together with non-bossiness, implies (s_i, s_j, s_k) $(q_i, q_j) = (A \cup C_i,\ B \cup C_j,\ C_k)$, completing the proof of (40).

Sub-step 2.4. We show that for all $q \in \mathcal{P}^N(\pi)$ and all $k \neq i, j$,

$$(s_i, s_j, s_k)(q) = \begin{cases} (A \cup C_i, B \cup C_j, C_k) & \text{if } q_i(A) > q_i(B) \text{ and } q_j(A) < q_j(B),\\ (B \cup C_i,\ A \cup C_j,\ C_k) & \text{otherwise.}\end{cases} \tag{41}$$

Let $q \in \mathcal{P}^N(\pi)$. Given Sub-step 2.3 and because each $\mathcal{P}(\pi_k)$ is connected, we may assume without loss of generality that there exists some $k \neq i, j$ such that q_k is adjacent to p_k and $q_{k'} = p_{k'}$ for all $k' \neq i, j, k$. In what follows, we drop $q_{-ijk} = p_{-ijk}$ from our notation. Suppose, by way of contradiction, that $s(q_i, q_j, q_k) \neq s(q_i, q_j, p_k)$.

If $(s_i, s_j, s_k)(q_i, q_j, p_k) = (A \cup C_i,\, B \cup C_j,\, C_k)$, non-bossiness implies $s_k(q_i, q_j, q_k) \neq s_k(q_i, q_j, p_k)$. Since $p_k, q_k \in \mathcal{P}(\pi_k)$, the pair of events $\{E, E'\}$ for which p_k, q_k are $\{E, E'\}$-adjacent is such that $\widetilde{\omega} \in E \cup E'$. Since $\widetilde{\omega} \in A \cup C_i = s_i(q_i, q_j, p_k)$, we must therefore have $s_i(q_i, q_j, q_k) \neq s_i(q_i, q_j, p_k)$ and Lemma 8 implies $s_i(q_i, q_j, q_k) = B \cup C_i$. By the Local Bilaterality lemma, $s_j(q_i, q_j, q_k) = s_i(q_i, q_j, p_k) = B \cup C_j$. This means that $s_i(q_i, q_j, q_k) \cap s_j(q_i, q_j, q_k) \neq \emptyset$, contradicting feasibility.

If $(s_i, s_j, s_k)(q_i, q_j, p_k) = (B \cup C_i,\, A \cup C_j,\, C_k)$, exchanging the roles of i and j in the above argument yields the same contradiction.

Sub-step 2.5. Since s varies with the beliefs of agents 1 and 2 on $\mathcal{P}^N(\pi)$, (41) must hold with $\{i, j\} = \{1, 2\}$, completing the proof of statement *(b)*. $\square$

Terminology. Given $\pi \in \widetilde{\mathcal{P}}^N$, a rule s of the type identified in part *(a)* of Lemma 9 is called $(1,2)$-*dictatorial (with respect to* $\{A(\pi), B(\pi)\}$*) on* $\mathcal{P}^N(\pi)$: the assignment varies only with agent 1's beliefs and only the events allocated to agents 1 and 2 change. For such a rule, there is no loss of generality in assuming that $\widetilde{\omega} \in A(\pi)$: we maintain that convention throughout.

A rule of the type identified in part *(b)* is called $\{1,2\}$-*consensual (with respect to* $\{A(\pi), B(\pi)\}$*) on* $\mathcal{P}^N(\pi)$. We call it $(1,2)$-consensual if $\widetilde{\omega} \in B(\pi)$ and $(2,1)$-consensual if $\widetilde{\omega} \in A(\pi)$: under an (i,j)-consensual rule, the "default option" assigns state $\widetilde{\omega}$ to agent i.

We call the sets $C_1(\pi), ..., C_n(\pi)$ *residuals.*

10 Appendix C: proof of the Bilateral Consensus lemma

10.1 Contagion results

As in Appendix B, $\widetilde{\omega} \in \Omega$ remains fixed throughout this sub-section, and we keep the notation $\widetilde{\Omega} = \Omega \setminus \widetilde{\omega}$ and $\widetilde{\mathcal{P}} = \mathcal{P}(\widetilde{\Omega})$. For any fixed belief profile $\pi \in \widetilde{\mathcal{P}}$, Lemma 9 describes the structure of s on the sub-domain $\mathcal{P}^N(\pi)$. We will now describe how this structure varies with π. We begin with two "contagion lemmas" and an "independence lemma", which link the behavior of s across "adjacent" sub-domains. These lemmas require extending the notion of adjacency to beliefs defined over an arbitrary *subset* of Ω. For any $\Omega' \subseteq \Omega$ (e.g., $\Omega' = \widetilde{\Omega}$), let $\mathcal{H}(\Omega') = \{\{A, B\} : \emptyset \neq A, B \subset \Omega' \text{ and } A \cap B = \emptyset\}$ and say that $\pi_i, \sigma_i \in \mathcal{P}(\Omega')$ are $\{A, B\}$-*adjacent* if $(\pi_i(A) - \pi_i(B))(\sigma_i(A) - \sigma_i(B)) < 0$ and $(\pi_i(C) - \pi_i(D))\ (\sigma_i(C) - \sigma_i(D)) > 0$ for all $\{C, D\} \in \mathcal{H}(\Omega') \setminus \{\{A, B\}\}$. With a slight abuse of notation, we use J to denote the adjacency relation between beliefs on any Ω'. Connectedness of a subset of $\mathcal{P}(\Omega')$ is defined in the obvious way.

First, an intermediate result.

Lemma 10. *Let* $\pi \in \widetilde{\mathcal{P}}^N$*, let* $\sigma_1, \sigma_2 \in \widetilde{\mathcal{P}}$ *be adjacent to* π_1, π_2*, respectively, and let* s *be* $(2,1)$*-consensual with respect to* $\{A, B\}$ *on* $\mathcal{P}^N(\pi)$ *with residuals* $C_1, ..., C_n$*.*
(a) If s *is* $(2,1)$*-consensual with respect to some* $\{A', B'\}$ *on* $\mathcal{P}^N(\sigma_2, \pi_{-2})$*, then* $\{A', B'\}$ *cuts* $\mathcal{P}(\pi_2)$ *and* $\{A, B\}$ *cuts* $\mathcal{P}(\sigma_2)$*.*
(b) If s *is* $(2,1)$*-consensual with respect to some* $\{A', B'\}$ *on* $\mathcal{P}^N(\sigma_1, \pi_{-1})$*, then* $\{A', B'\}$ *cuts* $\mathcal{P}(\pi_1)$ *and* $\{A, B\}$ *cuts* $\mathcal{P}(\sigma_1)$*.*

Remark 2. *We stated Lemma 10 for the ordered pair* $(2,1)$ *for notational simplicity only: up to a relabeling, the result applies to any ordered pair* (i,j) *of agents. This comment applies also to the results below.*

Proof. We only prove statement *(a)*. Although statement *(b)* is *not* a mere permutation of statement *(a)* (because s is $(2,1)$-consensual in both cases), its proof is almost identical and therefore omitted. Fix $\pi \in \widetilde{\mathcal{P}}^N$ and $\sigma_2 \in \widetilde{\mathcal{P}}$ adjacent to π_2. Suppose s is $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi)$ with residuals $C_1, ..., C_n$, and $(2,1)$-consensual with respect to $\{A', B'\}$ on $\mathcal{P}^N(\sigma_2, \pi_{-2})$ with residuals $C'_1, ..., C'_n$. Fix an arbitrary sub-profile $p_{-12} \in \mathcal{P}^{N \setminus 12}(\pi_{-12})$ and drop it from the notation. Then, for all $p = (p_1, p_2) \in \mathcal{P}(\pi_1) \times \mathcal{P}(\pi_2)$,

$$(s_1, s_2)(p_1, p_2) = \begin{cases} (A \cup C_1, B \cup C_2) & \text{if } p_1(A) > p_1(B) \text{ and } p_2(A) < p_2(B), \\ (B \cup C_1, A \cup C_2) & \text{otherwise,} \end{cases} \tag{42}$$

and for all $(p_1, q_2) \in \mathcal{P}(\pi_1) \times \mathcal{P}(\sigma_2)$,

$$(s_1, s_2)(p_1, q_2) = \begin{cases} (A' \cup C'_1, B' \cup C'_2) & \text{if } p_1(A') > p_1(B') \text{ and } q_2(A') < q_2(B'), \\ (B' \cup C'_1, A' \cup C'_2) & \text{otherwise,} \end{cases} \tag{43}$$

where $\widetilde{\omega} \in A \cap A'$, $\{A, B\}$ cuts $\mathcal{P}(\pi_1), \mathcal{P}(\pi_2)$, and $\{A', B'\}$ cuts $\mathcal{P}(\pi_1), \mathcal{P}(\sigma_2)$. In particular, writing $\widetilde{A} := A \setminus \widetilde{\omega}$, $\widetilde{A}' := A' \setminus \widetilde{\omega}$, we have

$$\pi_2(\widetilde{A}) < \pi_2(B). \tag{44}$$

$$\sigma_2(\widetilde{A}') < \sigma_2(B'). \tag{45}$$

Let p_1^+, p_2^+, q_2^+ and p_1^-, p_2^-, q_2^- be, respectively, maximal and minimal elements of $\widetilde{J}$ in, respectively, $\mathcal{P}(\pi_1), \mathcal{P}(\pi_2)$, and $\mathcal{P}(\sigma_2)$. Let $\{E, E'\} \in \mathcal{H}(\widetilde{\Omega})$ be the unique pair of disjoint subsets of $\widetilde{\Omega}$ such that π_2 and σ_2 are $\{E, E'\}$-adjacent with, say, $\pi_2(E) > \pi_2(E')$. Recall that π_2, σ_2 are beliefs on $\widetilde{\Omega} = \Omega \setminus \widetilde{\omega}$; this implies that $\widetilde{\omega} \notin E \cup E'$. Observe now that p_2^+, q_2^+ are $\{E, E'\}$-adjacent beliefs on Ω: this follows directly from the characteristic inequality (29). In contrast, p_2^-, q_2^- need not be adjacent, as Figure 2 illustrates.

We will only prove that $\{A', B'\}$ cuts $\mathcal{P}(\pi_2)$; the proof that $\{A, B\}$ cuts $\mathcal{P}(\sigma_2)$ is the same, mutatis mutandis. Suppose, by way of contradiction, that

$$\pi_2(\widetilde{A}') > \pi_2(B'). \tag{46}$$

We first claim that for every $\widehat{\omega} \in E \cup E'$,

$$p_2^- \mid \widehat{\Omega} \approx q_2^- \mid \widehat{\Omega}, \tag{47}$$

where $\widehat{\Omega} := \Omega \setminus \widehat{\omega}$. To see why, fix disjoint events $C, D \subseteq \widehat{\Omega}$ and observe that

$$\begin{aligned} p_2^-(C) < p_2^-(D) \quad &\Leftrightarrow \quad \pi_2(C \setminus \widetilde{\omega}) < \pi_2(D \setminus \widetilde{\omega}) \\ &\Leftrightarrow \quad \sigma_2(C \setminus \widetilde{\omega}) < \sigma_2(D \setminus \widetilde{\omega}) \\ &\Leftrightarrow \quad q_2^-(C) < q_2^-(D). \end{aligned}$$

The first equivalence holds by definition of p_2^-. The second holds because $\widehat{\omega} \in E \cup E'$ and $\widehat{\omega} \notin C \cup D$ imply that $\{C \setminus \widetilde{\omega}, D \setminus \widetilde{\omega}\}$ differs from $\{E, E'\}$, the unique pair of disjoints subsets of $\widetilde{\Omega}$ on which the likelihood orderings generated by π_2, σ_2 disagree. The third equivalence holds by definition of q_2^-.

Next, let $\widehat{\pi}_2$ be a belief on $\widehat{\Omega}$ such that $p_2^- \mid \widehat{\Omega} \approx q_2^- \mid \widehat{\Omega} \approx \widehat{\pi}_2$. We emphasize that the belief $\widehat{\pi}_2$ is not defined on the same event as π_2, σ_2, which are beliefs on $\widetilde{\Omega}$. Define $\mathcal{P}(\widehat{\pi}_2) = \left\{p_2 \in \mathcal{P} : p_2 \mid \widehat{\Omega} \approx \widehat{\pi}_2\right\}$. For every $\alpha \in [0,1]$, define

$${}^{\alpha}q_2 = \alpha p_2^- + (1-\alpha) q_2^-.$$

Observe that ${}^{\alpha}q_2 \in \overline{\mathcal{P}(\widehat{\pi}_2)} \cap (\overline{\mathcal{P}(\sigma_2)} \cup \overline{\mathcal{P}(\pi_2)})$ for every $\alpha \in [0,1]$, where the upperbar denotes the closure operator. Furthermore, because we assumed that $\{A', B'\}$ does not cut $\mathcal{P}(\pi_2)$ (i.e., (46) holds), there exists some $\alpha \in [0,1]$ such that

$${}^{\alpha}q_2 \in \mathcal{P}(\sigma_2) \text{ and } {}^{\alpha}q_2(A') > {}^{\alpha}q_2(B'). \tag{48}$$

We omit the easy proof for brevity.

Pick $p_1 \in \mathcal{P}(\pi_1)$ such $p_1(A) > p_1(B)$ and $p_1(A') > p_1(B')$. By definition of q_2^- and thanks to (45), $q_2^-(A') < q_2^-(B')$, hence from (43),

$$s_2(p_1, {}^{0}q_2) = s_2(p_1, q_2^-) = B' \cup C_2'. \tag{49}$$

Choosing α such that (48) holds, (43) again implies

$$s_2(p_1, {}^{\alpha}q_2) = A' \cup C_2'. \tag{50}$$

But since ${}^{\beta}q_2 \in \overline{\mathcal{P}(\widehat{\pi}_2)}$ for all $\beta \in [0,1]$, (49), (50), and Lemma 8, applied with $\widehat{\Omega}$ instead of $\widetilde{\Omega}$, imply

$$s_2(p_1, {}^{1}q_2) = s_2(p_1, p_2^-) = A' \cup C_2'.$$

However, by definition of p_2^- and thanks to (44), $p_2^-(A) < p_2^-(B)$, hence from (42),

$$s_2(p_1, p_2^-) = B \cup C_2,$$

contradicting the previous equality since $\widetilde{\omega} \in (A' \cup C_2') \setminus (B \cup C_2)$. □

First Contagion Lemma. *Let $\pi \in \widetilde{\mathcal{P}}^N$, let $\sigma_2 \in \widetilde{\mathcal{P}}$ be adjacent to π_2, and let s be $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi)$ with residuals $C_1, ..., C_n$.*
(a) If $\{A, B\}$ cuts $\mathcal{P}(\sigma_2)$, then s is $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\sigma_2, \pi_{-2})$.
(b) If $\{A, B\}$ does not cut $\mathcal{P}(\sigma_2)$, then $s(p) = (B \cup C_1, A \cup C_2, C_3, ..., C_n)$ for all $p \in \mathcal{P}^N(\sigma_2, \pi_{-2})$.

Remark 3. *Statement (a) does not assert that the residuals $C_1', ..., C_n'$ associated with the $(2,1)$-consensual rule s on $\mathcal{P}^N(\sigma_2, \pi_{-2})$ coincide with the residuals $C_1, ..., C_n$ on $\mathcal{P}^N(\pi)$: in fact, they generally do not.*

Statement (b), on the other hand, asserts that s is constant on $\mathcal{P}^N(\sigma_2, \pi_{-2})$ and the residuals are the same as on $\mathcal{P}^N(\pi)$: the assignment outside $A \cup B$ remains constant when 2's beliefs switch from $\mathcal{P}(\pi_2)$ to $\mathcal{P}(\sigma_2)$. It may be worth explaining why a C-BD-BC union indeed possesses this property. The reason is the following. Since we have assumed that s is $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi)$, we know that $\{A, B\}$ cuts $\mathcal{P}(\pi_2)$, that is, $\pi_2(\widetilde{A}) < \pi_2(B)$. On the other hand, since $\{A, B\}$ does not cut $\mathcal{P}(\sigma_2)$, we have $\sigma_2(\widetilde{A}) > \sigma_2(B)$. It follows that the adjacent beliefs π_2, σ_2 must, in fact, be $\{\widetilde{A}, B\}$-adjacent. This means that any two beliefs $p_2 \in \mathcal{P}(\pi_2), q_2 \in \mathcal{P}(\sigma_2)$ agree on the ranking of all events $C, D \subseteq \Omega \setminus (A \cup B)$. As a result, the assignment outside $A \cup B$ remains unchanged under a C-BD-BC union.

Proof. Fix $\pi \in \widetilde{\mathcal{P}}^N$, $\sigma_2 \in \widetilde{\mathcal{P}}$ such that π_2, σ_2 are adjacent. Suppose s is $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi)$ with residuals $C_1, ..., C_n$: (42) holds for all $p \in \mathcal{P}^N(\pi)$, $\widetilde{\omega} \in A$, and $\{A, B\}$ cuts $\mathcal{P}(\pi_2)$, i.e., (44) holds. For any $k \in N$, let p_k^+, p_k^- denote maximal and minimal elements of $\widetilde{J}$ in $\mathcal{P}(\pi_k)$, q_2^+, q_2^- be maximal and minimal elements of $\widetilde{J}$ in $\mathcal{P}(\sigma_2)$, and let E, E' be the disjoint subsets of $\widetilde{\Omega}$ such that π_2 and σ_2 are $\{E, E'\}$-adjacent with $\pi_2(E) > \pi_2(E')$. Recall that $\widetilde{\omega} \notin E \cup E'$.

Step 1. We show that for every agent $k \neq 2$ and every $k' \neq k$, s is neither (k, k')-dictatorial nor (k, k')-consensual on $\mathcal{P}^N(\sigma_2, \pi_{-2})$.

Fix $k \neq 2$, $k' \neq k$. Fix a sub-profile $p_{-2k} \in \mathcal{P}^{N \setminus 2k}(\pi_{-2k})$ and drop it from the notation. Since s is $(2,1)$-consensual on $\mathcal{P}^N(\pi)$, we have $\widetilde{\omega} \in s_2(p_2^+, p_k^+)$. If s is (k, k')-dictatorial or (k, k')-consensual on $\mathcal{P}^N(\sigma_2, \pi_{-2})$, then $\widetilde{\omega} \in s_k(q_2^+, p_k^+)$. These two statements contradict the Local Bilaterality lemma because p_2^+, q_2^+ are $\{E, E'\}$-adjacent and $\widetilde{\omega} \notin E \cup E'$.

Step 2. We prove statement *(a)*.

Suppose $\{A, B\}$ cuts $\mathcal{P}(\sigma_2)$, that is,

$$\sigma_2(\widetilde{A}) < \sigma_2(B). \tag{51}$$

Sub-step 2.1. We show that s varies with the beliefs of agents 1 and 2 on $\mathcal{P}^N(\sigma_2, \pi_{-2})$.

Fix a sub-profile $p_{-12} \in \mathcal{P}^{N\setminus 12}(\pi_{-12})$ and drop it from the notation. Because $\{A, B\}$ cuts $\mathcal{P}(\sigma_2)$, there exist adjacent beliefs $\overline{p}_2 \in \mathcal{P}(\pi_2)$ and $\overline{q}_2 \in \mathcal{P}(\sigma_2)$ such that $\overline{p}_2(A) < \overline{p}_2(B)$. These beliefs are, in fact, $\{E, E'\}$-adjacent.

Choose $p_1 \in \mathcal{P}(\pi_1)$ such that $p_1(A) > p_1(B)$. From (42), $s_2(p_1, p_2^+) = A \cup C_2$ and $s_2(p_1, \overline{p}_2) = B \cup C_2$. By the Local Bilaterality lemma,

$$\begin{aligned} s_2(p_1, q_2^+) &= A \cup C_2 \text{ or } (A \cup C_2 \cup E') \setminus E, \\ s_2(p_1, \overline{q}_2) &= B \cup C_2 \text{ or } (B \cup C_2 \cup E') \setminus E. \end{aligned}$$

It follows that $\widetilde{\omega} \in s_2(p_1, q_2^+) \setminus s_2(p_1, \overline{q}_2)$: s varies with agent 2's beliefs.

Next, choose $q_1 \in \mathcal{P}(\pi_1)$ such that $q_1(A) < q_1(B)$. From (42), $s_2(q_1, \overline{p}_2) = A \cup C_2$. By the Local Bilaterality lemma,

$$s_2(q_1, \overline{q}_2) = A \cup C_2 \text{ or } (A \cup C_2 \cup E') \setminus E.$$

Thus $\widetilde{\omega} \in s_2(q_1, \overline{q}_2) \setminus s_2(p_1, \overline{q}_2)$: s varies with agent 1's beliefs.

Sub-step 2.2. Since s varies with the beliefs of agents 1 and 2 on $\mathcal{P}^N(\sigma_2, \pi_{-2})$, Lemma 9 and Step 1 imply that s is $(2, 1)$-consensual with respect to some $\{A', B'\}$ on $\mathcal{P}^N(\sigma_2, \pi_{-2})$ with, say, residuals $C_1', ..., C_2'$. Thus, (43) holds for all $(p_1, q_2) \in \mathcal{P}(\pi_1) \times \mathcal{P}(\sigma_2)$, $\widetilde{\omega} \in A'$, and $\{A', B'\}$ cuts $\mathcal{P}(\pi_1), \mathcal{P}(\sigma_2)$. In particular, (45) holds. To complete the proof of statement *(a),* it remains to prove that $\{A, B\} = \{A', B'\}$.

Suppose, contrary to our claim, that $\{A, B\} \neq \{A', B'\}$. Define the positive numbers

$$\begin{aligned} \delta &= \pi_1(B) - \pi_1(\widetilde{A}), \\ \delta' &= \pi_1(B') - \pi_1(\widetilde{A}'). \end{aligned}$$

Assume $\delta \neq \delta'$. This is without loss of generality: if $\delta = \delta'$, simply replace π_1 with an ordinally equivalent belief for which the two corresponding numbers differ. Either $\delta < \delta'$ or $\delta' < \delta$. We will only treat the former case; the latter is identical, mutatis mutandis.

For each $\alpha \in [0, 1]$, define $p_1^\alpha \in \overline{\mathcal{P}(\pi_1)}$ by

$$p_1^\alpha(\widetilde{\omega}) = \alpha \text{ and } p_1^\alpha(\omega) = (1 - \alpha)\pi_1(\omega) \text{ for all } \omega \in \widetilde{\Omega}.$$

Elementary algebra shows that $p_1^\alpha(A) < p_1^\alpha(B) \Leftrightarrow \alpha < \frac{\delta}{1+\delta}$ and $p_1^\alpha(A') < p_1^\alpha(B') \Leftrightarrow \alpha < \frac{\delta'}{1+\delta'}$. Since $\delta < \delta'$, we have $\frac{\delta}{1+\delta} < \frac{\delta'}{1+\delta'}$. Choosing $\frac{\delta}{1+\delta} < \alpha < \frac{\delta'}{1+\delta'}$, we have

$$p_1^\alpha(A) > p_1^\alpha(B) \text{ and } p_1^\alpha(A') < p_1^\alpha(B'). \tag{52}$$

Because of (44) and (51), there exist adjacent beliefs $p_2 \in \mathcal{P}(\pi_2)$ and $q_2 \in \mathcal{P}(\sigma_2)$ such that $p_2(A) < p_2(B)$. This is illustrated in Figure 3 with $A = \{1\}, B = \{2\}$; we omit the easy proof for brevity. From this inequality, (42), and the first inequality in (52), we obtain

$$s_2(p_1^\alpha, p_2) = B \cup C_2.$$

From (43) and the second inequality in (52),

$$s_2(p_1^\alpha, q_2) = A' \cup C_2'.$$

It follows that $\widetilde{\omega} \in s_2(p_1^\alpha, q_2) \setminus s_2(p_1^\alpha, p_2)$, contradicting the Local Bilaterality lemma because p_2, q_2 are $\{E, E'\}$-adjacent and $\widetilde{\omega} \notin E \cup E'$.

Step 3. We prove statement *(b).*

Suppose $\{A, B\}$ does not cut $\mathcal{P}(\sigma_2)$, that is,

$$\sigma_2(\widetilde{A}) > \sigma_2(B). \tag{53}$$

Sub-step 3.1. We prove that s is neither $(2,k)$-dictatorial nor $(2,k)$-consensual on $\mathcal{P}^N(\sigma_2, \pi_{-2})$ for any $k \neq 2$.

Suppose it is.

Case 1: $\{A', B'\}$ cuts $\mathcal{P}(\pi_2)$, that is, $\pi_2(\widetilde{A}') < \pi_2(B')$.

Fix a sub-profile $p_{-2k} \in \mathcal{P}^{N \setminus 2k}(\pi_{-2k})$ and drop it from the notation. Because of (53), there exist adjacent $p_2 \in \mathcal{P}(\pi_2)$ and $q_2 \in \mathcal{P}(\sigma_2)$ such that $p_2(A) > p_2(B)$ and $q_2(A') < q_2(B')$.

Choose $p_k \in \mathcal{P}(\pi_k)$ such that $p_k(A') > p_k(B')$. From (42), $\widetilde{\omega} \in s_2(p_2, p_k)$. But since s is $(2,k)$-dictatorial or $(2,k)$-consensual on $\mathcal{P}^N(\sigma_2, \pi_{-2})$, $\widetilde{\omega} \in s_k(q_2, p_k)$, contradicting the Local Bilaterality lemma.

Case 2: $\{A', B'\}$ does not cut $\mathcal{P}(\pi_2)$, that is, $\pi_2(\widetilde{A}') > \pi_2(B')$.

Fix a sub-profile $p_{-2} \in \mathcal{P}^{N \setminus 2}(\pi_{-2})$ such that $p_1(A) > p_1(B)$ and $p_k(A') > p_k(B')$ (where 1 and k may coincide). Drop this sub-profile from the notation.

We derive a contradiction using a variant of the argument in Lemma 10. Fix $\widehat{\omega} \in E \cup E'$. As we proved in Lemma 10, there exists a belief $\widehat{\pi}_2$ on $\Omega \setminus \widehat{\omega}$ such that $p_2^- \mid \widehat{\Omega} \approx q_2^- \mid \widehat{\Omega} \approx \widehat{\pi}_2$ and there exists $\alpha \in [0,1]$ such that ${}^\alpha q_2 := \alpha p_2^- + (1-\alpha) q_2^- \in \mathcal{P}(\sigma_2)$ and ${}^\alpha q_2(A') > {}^\alpha q_2(B')$.

Since $q_2^-(A') < q_2^-(B')$ and s is $(2,k)$-dictatorial or $(2,k)$-consensual on $\mathcal{P}^N(\sigma_2, \pi_{-2})$,

$$\begin{aligned} s_2({}^0 q_2) &= s_2(q_2^-) = B' \cup C_2', \\ s_2({}^\alpha q_2) &= A' \cup C_2'. \end{aligned}$$

Since ${}^\beta q_2 \in \overline{\mathcal{P}(\widehat{\pi}_2)}$ for all $\beta \in [0,1]$, these equalities and Lemma 8 imply

$$s_2({}^1 q_2) = s_2(p_2^-) = A' \cup C_2'.$$

But (42) implies $s_2(p_2^-) = B \cup C_2$, a contradiction.

Sub-step 3.2. Step 1, Sub-step 3.1, and Lemma 9 together imply that s is constant on $\mathcal{P}^N(\sigma_2, \pi_{-2})$. To complete the proof of statement *(b)*, we need to show that the constant assignment prescribed by s is $(B \cup C_1, A \cup C_2, C_3, ..., C_n)$.

Fix again $\widehat{\omega} \in E \cup E'$ and $\widehat{\pi}_2 \approx p_2^- \mid \widehat{\Omega} \approx q_2^- \mid \widehat{\Omega}$. Because $\{A, B\}$ does not cut $\mathcal{P}(\sigma_2)$, there exists $\alpha \in [0,1]$ such that ${}^{\alpha}q_2 := \alpha p_2^- + (1-\alpha) q_2^- \in \mathcal{P}(\pi_2)$ and ${}^{\alpha}q_2(A) > {}^{\alpha}q_2(B)$. Pick $\overline{p}_1 \in \mathcal{P}(\pi_1)$ such $\overline{p}_1(A) > \overline{p}_1(B)$. Fix p_{-12} and drop it from the notation. From (42),

$$\begin{aligned} s_2(\overline{p}_1, {}^1 q_2) &= s_2(\overline{p}_1, p_2^-) = B \cup C_2, \\ s_2(\overline{p}_1, {}^{\alpha} q_2) &= A \cup C_2. \end{aligned}$$

Since ${}^{\beta}q_2 \in \overline{\mathcal{P}(\widehat{\pi}_2)}$ for all $\beta \in [0,1]$, Lemma 8 implies

$$s_2(\overline{p}_1, {}^0 q_2) = s_2(\overline{p}_1, q_2^-) = A \cup C_2,$$

hence, since s is constant on $\mathcal{P}^N(\sigma_2, \pi_{-2})$, $s_2(p_1, q_2) = A \cup C_2$ for all $(p_1, q_2) \in \mathcal{P}(\pi_1) \times \mathcal{P}(\sigma_2)$. The claim now follows from non-bossiness. $\square$

Second Contagion Lemma. *Let $\pi \in \widetilde{\mathcal{P}}^N$ and let $\sigma_1 \in \widetilde{\mathcal{P}}$ be adjacent to π_1.*
(a) If s is $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi)$ and $\{A, B\}$ cuts $\mathcal{P}(\sigma_1)$, then s is $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\sigma_1, \pi_{-1})$.
(b) If s is $(2,1)$-consensual or $(2,1)$-dictatorial with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi)$ and $\{A, B\}$ does not cut $\mathcal{P}(\sigma_1)$, then s is $(2,1)$-dictatorial with respect to $\{A, B\}$ on $\mathcal{P}^N(\sigma_1, \pi_{-1})$.

Remark 4. *Statement (a) is not the permutation of statement (a) in the First Contagion lemma because the rule is assumed to be $(2,1)$-consensual in both cases.*

Proof. Fix $\pi \in \widetilde{\mathcal{P}}^N$ and $\sigma_1 \in \widetilde{\mathcal{P}}$ adjacent to π_1. For any $k \in N$, let p_k^+, p_k^- denote maximal and minimal elements of $\widetilde{J}$ in $\mathcal{P}(\pi_k)$, let q_1^+, q_1^- be maximal and minimal elements of $\widetilde{J}$ in $\mathcal{P}(\sigma_1)$, and let now E, E' denote the disjoint subsets of $\widetilde{\Omega}$ such that π_1 and σ_1 are $\{E, E'\}$-adjacent with $\pi_1(E) > \pi_1(E')$. Again, $\widetilde{\omega} \notin E \cup E'$.

Step 1. We show that if s is $(2,1)$-consensual or $(2,1)$-dictatorial on $\mathcal{P}^N(\pi)$, then for every $k \neq 2$ and $k' \neq k$, s is neither (k, k')-dictatorial nor (k, k')-consensual on $\mathcal{P}^N(\sigma_1, \pi_{-1})$.

Fix $k \neq 2$, $k' \neq k$. Fix a profile $p \in \mathcal{P}^N(\pi)$ such that $p_1 = p_1^+$, $p_2 = p_2^+$, and $p_k = p_k^+$ (where k may coincide with 1). Since s is $(2,1)$-consensual or $(2,1)$-dictatorial on $\mathcal{P}^N(\pi)$, we have $\widetilde{\omega} \in s_2(p)$. If s is (k, k')-dictatorial or (k, k')-consensual on $\mathcal{P}^N(\sigma_1, \pi_{-1})$, then $\widetilde{\omega} \in s_k(q_1^+, p_{-1})$. These two statements contradict the Local Bilaterality lemma because p_1^+, q_1^+ are $\{E, E'\}$-adjacent and $\widetilde{\omega} \notin E \cup E'$.

Step 2. We show that if s is $(2,1)$-consensual or $(2,1)$-dictatorial on $\mathcal{P}^N(\pi)$, then s it is not constant on $\mathcal{P}^N(\sigma_1, \pi_{-1})$.

Fix a sub-profile $p_{-12} \in \mathcal{P}^{N\setminus 12}(\pi_{-12})$ and drop it from the notation. If s is $(2,1)$-consensual or $(2,1)$-dictatorial on $\mathcal{P}^N(\pi)$, there exist disjoint sets A, B, C_2 such that $\widetilde{\omega} \in A$ and

$$\begin{aligned} s_2(p_1^+, p_2^+) &= A \cup C_2, \\ s_2(p_1^+, p_2^-) &= B \cup C_2 \end{aligned}$$

and the Local Bilaterality lemma implies

$$\begin{aligned} s_2(q_1^+, p_2^+) &= A \cup C_2 \text{ or } (A \cup C_2 \cup E) \setminus E', \\ s_2(q_1^+, p_2^-) &= B \cup C_2 \text{ or } (B \cup C_2 \cup E) \setminus E'. \end{aligned}$$

Hence, $\widetilde{\omega} \in s_2(q_1^+, p_2^+) \setminus s_2(q_1^+, p_2^-)$, proving that s is not constant on $\mathcal{P}^N(\sigma_1, \pi_{-1})$.

Step 3. We prove statement *(a).*

Suppose s is $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi)$ with, say, residuals $C_1, ..., C_n$, and $\{A, B\}$ cuts $\mathcal{P}(\sigma_1)$. Fix $p_{-12} \in \mathcal{P}^{N\setminus 12}(\pi_{-12})$ and drop it from the notation. By assumption, (42) holds for all $(p_1, p_2) \in \mathcal{P}(\pi_1) \times \mathcal{P}(\pi_2)$ and $\sigma_1(\widetilde{A}) < \sigma_1(B)$.

Sub-step 3.1. We show that s varies with agent 1's beliefs on $\mathcal{P}^N(\sigma_1, \pi_{-1})$.

Because $\{A, B\}$ cuts $\mathcal{P}(\sigma_1)$, there exist adjacent beliefs $\overline{p}_1 \in \mathcal{P}(\pi_1)$ and $\overline{q}_1 \in \mathcal{P}(\sigma_1)$ such that $\overline{p}_1(A) < \overline{p}_1(B)$. These beliefs are, in fact, $\{E, E'\}$-adjacent.

Choose $p_2 \in \mathcal{P}(\pi_2)$ such that $p_2(A) < p_2(B)$. From (42), $s_2(p_1^+, p_2) = B \cup C_2$ and $s_2(\overline{p}_1, p_2) = A \cup C_2$. By the Local Bilaterality lemma,

$$\begin{aligned} s_2(q_1^+, p_2) &= B \cup C_2 \text{ or } (B \cup C_2 \cup E') \setminus E, \\ s_2(\overline{q}_1, p_2) &= A \cup C_2 \text{ or } (A \cup C_2 \cup E') \setminus E. \end{aligned}$$

It follows that $\widetilde{\omega} \in s_2(\overline{q}_1, p_2) \setminus s_2(q_1^+, p_2)$: s varies with agent 1's beliefs.

Sub-step 3.2. By Step 1, Sub-step 3.1, and Lemma 9, s is $(2,1)$-consensual on $\mathcal{P}^N(\sigma_1, \pi_{-1})$ with respect to some $\{A', B'\}$ and residuals $C_1', ..., C_n'$. For all $(q_1, p_{-1}) \in \mathcal{P}^N(\sigma_1, \pi_{-1})$,

$$s(q_1, p_{-1}) = \begin{cases} (A' \cup C_1', B' \cup C_2', C_3', ..., C_n') & \text{if } q_1(A') > q_1(B') \text{ and } p_2(A') < p_2(B'), \\ (B' \cup C_1', A' \cup C_2', C_3', ..., C_n') & \text{otherwise,} \end{cases} \tag{54}$$

where $\widetilde{\omega} \in A'$ and $\{A', B'\}$ cuts $\mathcal{P}(\sigma_1), \mathcal{P}(\pi_2)$. It remains to prove that $\{A', B'\} = \{A, B\}$.

Fix $p_{-12} \in \mathcal{P}^{N\setminus 12}(\pi_{-12})$ and drop it from the notation. If $\{A', B'\} \neq \{A, B\}$, define the positive numbers

$$\begin{aligned} \delta &= \pi_2(B) - \pi_2(\widetilde{A}), \\ \delta' &= \pi_2(B') - \pi_2(\widetilde{A}') \end{aligned}$$

and assume without loss of generality $\delta \neq \delta'$.

If $\delta < \delta'$, there exists $p_2 \in \mathcal{P}(\pi_2)$ such that $p_2(A) > p_2(B)$ and $p_2(A') < p_2(B')$. From (42), $s_2(p_1^+, p_2) = A \cup C_2$ and from (54), $s_2(q_1^+, p_2) = B' \cup C_2'$, contradicting the Local Bilaterality lemma.

If $\delta' < \delta$, there exists $p_2 \in \mathcal{P}(\pi_2)$ such that $p_2(A) < p_2(B)$ and $p_2(A') > p_2(B')$. From (42), $s_2(p_1^+, p_2) = B \cup C_2$ and from (54), $s_2(q_1^+, p_2) = A' \cup C_2'$, contradicting the Local Bilaterality lemma again.

Step 4. We prove statement *(b).*

Sub-step 4.1. Suppose first that s is $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi)$ and $\{A, B\}$ does not cut $\mathcal{P}(\sigma_1)$.

By Steps 1, 2, and Lemmas 9 and 10, s is $(2,1)$-dictatorial on $\mathcal{P}^N(\sigma_1, \pi_{-1})$ with respect to some $\{A', B'\}$ and residuals $C_1', ..., C_n'$. For all $(q_1, p_{-1}) \in \mathcal{P}^N(\sigma_1, \pi_{-1})$,

$$s(q_1, p_{-1}) = \begin{cases} (A' \cup C_1', B' \cup C_2', C_3', ..., C_n') & \text{if } p_2(B') > p_2(A'), \\ (B' \cup C_1', A' \cup C_2', C_3', ..., C_n') & \text{otherwise,} \end{cases} \tag{55}$$

where $\widetilde{\omega} \in A'$ and $\{A', B'\}$ cuts $\mathcal{P}(\sigma_1)$. It remains to prove that $\{A', B'\} = \{A, B\}$.

If $\{A', B'\} \neq \{A, B\}$, consider again the numbers δ, δ' defined in Sub-step 3.2 and assume without loss of generality $\delta \neq \delta'$. Note that δ' may now be negative as $\{A', B'\}$ need no longer cut $\mathcal{P}(\pi_2)$. This, however, does not affect the rest of the argument: combining (42) with (55) rather than (54) delivers the same contradiction to the Local Bilaterality lemma.

Sub-step 4.2. Suppose next that s is $(2,1)$-dictatorial with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi)$ and $\{A, B\}$ does not cut $\mathcal{P}(\sigma_1)$.

By Steps 1, 2, and Lemma 9, s is either $(2,1)$-consensual or $(2,1)$-dictatorial on $\mathcal{P}^N(\sigma_1, \pi_{-1})$.

If s is $(2,1)$-consensual on $\mathcal{P}^N(\sigma_1, \pi_{-1})$, it must be with respect to some $\{A', B'\} \neq \{A, B\}$ since $\{A, B\}$ does not cut $\mathcal{P}(\sigma_1)$.

Suppose first that $\{A', B'\}$ does not cut $\mathcal{P}(\pi_1)$: exchanging the roles of $\{A, B\}$, $\{A', B'\}$ and π_1, σ_1 in the argument in Sub-step 4.1 leads to the conclusion that s is $(2,1)$-dictatorial with respect to $\{A', B'\}$ on $\mathcal{P}^N(\pi)$, contradicting the assumption of the current sub-step.

Suppose next that $\{A', B'\}$ cuts $\mathcal{P}(\pi_1)$: exchanging the roles of $\{A, B\}$, $\{A', B'\}$ and π_1, σ_1 in statement *(a)* leads to the conclusion that s is $(2,1)$-consensual with respect to $\{A', B'\}$ on $\mathcal{P}^N(\pi)$, again a contradiction.

We conclude that s is $(2,1)$-dictatorial on $\mathcal{P}^N(\sigma_1, \pi_{-1})$. The proof that it must in fact be $(2,1)$-dictatorial with respect to $\{A, B\}$ proceeds in the same way as in Sub-step 4.1. □

Independence Lemma. *Let $\pi \in \widetilde{\mathcal{P}}^N$, $k \in N \setminus \{1, 2\}$, and let $\sigma_k \in \widetilde{\mathcal{P}}$ be adjacent to π_k. If s is $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi)$, then s is $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\sigma_k, \pi_{-k})$.*

Proof. Fix $\pi \in \widetilde{\mathcal{P}}^N$ and suppose s is $(2,1)$-consensual with respect to $\{A,B\}$ on $\mathcal{P}^N(\pi)$: there exists a partition $\{A,B,C_1,...,C_n\}$ of Ω such that $\widetilde{\omega} \in A$, $\{A,B\}$ cuts $\mathcal{P}(\pi_1), \mathcal{P}(\pi_2)$, and, for all $p \in \mathcal{P}^N(\pi)$,

$$s(p) = \begin{cases} (A \cup C_1, B \cup C_2, C_3, ..., C_n) & \text{if } p_1(A) > p_1(B) \text{ and } p_2(A) < p_2(B), \\ (B \cup C_1, A \cup C_2, C_3, ..., C_n) & \text{otherwise.} \end{cases} \tag{56}$$

Fix $k \in N \setminus \{1,2\}$, say, $k = 3$, and let $\sigma_3 \in \widetilde{\mathcal{P}}$ be adjacent to π_3.

By calibrating the probability assigned to $\widetilde{\omega}$, we can find $\{A,B\}$-adjacent beliefs $p_1, p_1' \in \mathcal{P}(\pi_1)$ and $\{A,B\}$-adjacent beliefs $p_2, p_2' \in \mathcal{P}(\pi_2)$ with, say, $p_1(A) > p_1(B)$ and $p_2(A) < p_2(B)$. Let $p_{-123} \in \mathcal{P}^{N\setminus 123}(\pi_{-123})$. This sub-profile is fixed throughout the argument and therefore omitted from the notation. Let p_3^+, q_3^+ be maximal elements of $\widetilde{J}$ in $\mathcal{P}(\pi_3), \mathcal{P}(\sigma_3)$.

By (56),

$$\begin{aligned} s(p_1,p_2,p_3^+) &= (A \cup C_1, B \cup C_2, C_3, ..., C_n), \\ s(p_1',p_2,p_3^+) &= (B \cup C_1, A \cup C_2, C_3, ..., C_n), \\ s(p_1,p_2',p_3^+) &= (B \cup C_1, A \cup C_2, C_3, ..., C_n). \end{aligned} \tag{57}$$

Step 1. We show that there exists a partition $\{C_1',...,C_n'\}$ of $\Omega \setminus (A \cup B)$ such that

$$s(p_1,p_2,q_3^+) = (A \cup C_1', B \cup C_2', C_3', ..., C_n'). \tag{58}$$

By definition, p_3^+, q_3^+ are adjacent. By the Local Bilaterality lemma and the first equality in (57), there are only three cases.

Case 1. There exists some $j \neq 1,2,3$ such that $s_j(p_1,p_2,q_3^+) \cap s_3(p_1,p_2,p_3^+) \neq \emptyset$, $s_3(p_1,p_2,q_3^+) \cap s_j(p_1,p_2,p_3^+) \neq \emptyset$, and $s_i(p_1,p_2,q_3^+) = s_i(p_1,p_2,p_3^+)$ for all $i \neq j,3$.

In this case (58) holds with $C_i' = C_i$ for all $i \neq j,3$.

Case 2. $s_1(p_1,p_2,q_3^+) \cap s_3(p_1,p_2,p_3^+) \neq \emptyset$, $s_3(p_1,p_2,q_3^+) \cap s_1(p_1,p_2,p_3^+) \neq \emptyset$, and $s_i(p_1,p_2,q_3^+) = s_i(p_1,p_2,p_3^+)$ for all $i \neq 1,3$.

If $A \nsubseteq s_1(p_1,p_2,q_3^+)$, then since p_1, p_1' are $\{A,B\}$-adjacent with $p_1(A) > p_1(B)$, the Local Bilaterality lemma implies $s(p_1',p_2,q_3^+) = s(p_1,p_2,q_3^+)$. Comparing with (57),

$$\begin{aligned} s_1(p_1',p_2,q_3^+) \cap B &= \emptyset \text{ and } s_1(p_1',p_2,p_3^+) \cap B \neq \emptyset, \\ s_2(p_1',p_2,q_3^+) \cap B &\neq \emptyset \text{ and } s_2(p_1',p_2,p_3^+) \cap B = \emptyset, \\ s_3(p_1',p_2,q_3^+) \cap A &\neq \emptyset \text{ and } s_3(p_1',p_2,p_3^+) \cap A = \emptyset, \end{aligned}$$

implying $s_i(p_1',p_2,q_3^+) \neq s_i(p_1',p_2,p_3^+)$ for $i = 1,2,3$, contradicting the Local Bilaterality lemma.

This shows that $A \subseteq s_1(p_1,p_2,q_3^+)$. Then (58) holds with $C_i' = C_i$ for all $i \neq 1,3$.

Case 3. $s_2(p_1,p_2,q_3^+) \cap s_3(p_1,p_2,p_3^+) \neq \emptyset$, $s_3(p_1,p_2,q_3^+) \cap s_2(p_1,p_2,p_3^+) \neq \emptyset$, and $s_i(p_1,p_2,q_3^+) = s_i(p_1,p_2,p_3^+)$ for all $i \neq 2,3$.

If $B \nsubseteq s_2(p_1, p_2, q_3^+)$, then since p_2, p_2' are $\{A, B\}$-adjacent with $p_2(A) < p_2(B)$, the Local Bilaterality lemma implies $s(p_1, p_2', q_3^+) = s(p_1, p_2, q_3^+)$. Comparing with (57),

$$\begin{aligned}
s_1(p_1, p_2', q_3^+) \cap A &\neq \emptyset \text{ and } s_1(p_1, p_2', p_3^+) \cap A = \emptyset, \\
s_2(p_1, p_2', q_3^+) \cap A &= \emptyset \text{ and } s_2(p_1, p_2', p_3^+) \cap A \neq \emptyset, \\
s_3(p_1, p_2', q_3^+) \cap B &\neq \emptyset \text{ and } s_3(p_1, p_2', p_3^+) \cap B = \emptyset,
\end{aligned}$$

implying $s_i(p_1, p_2', q_3^+) \neq s_i(p_1, p_2', p_3^+)$ for $i = 1, 2, 3$, contradicting the Local Bilaterality lemma again.

This shows that $B \subseteq s_2(p_1, p_2, q_3^+)$, Then (58) holds with $C_i' = C_i$ for all $i \neq 2, 3$.

Step 2. We show that

$$s(p_1', p_2, q_3^+) = s(p_1, p_2', q_3^+) = (B \cup C_1', A \cup C_2', C_3', ..., C_n'). \tag{59}$$

Since p_1, p_1' are $\{A, B\}$-adjacent, Step 1 and the Local Bilaterality lemma imply that either (i) $s(p_1', p_2, q_3^+) = (A \cup C_1',\ B \cup C_2',\ C_3', ..., C_n')$ or (ii) $s(p_1', p_2, q_3^+) = (B \cup C_1', A \cup C_2', C_3', ..., C_n')$. Statement (i) and the second statement in (57) together contradict the Local Bilaterality lemma, hence (ii) must hold. Likewise, the third statement in (57) and the Local Bilaterality lemma imply that $s(p_1, p_2', q_3^+) = (B \cup C_1', A \cup C_2', C_3', ..., C_n')$.

Step 3. Combining statements (58), (59), and statement *(b)* in Lemma 9, we obtain that for all $(q_1, q_2, q_3) \in \mathcal{P}(\pi_1) \times \mathcal{P}(\pi_2) \times \mathcal{P}(\sigma_3)$,

$$s(q_1, q_2, q_3) = \begin{cases} (A \cup C_1', B \cup C_2', C_3', ...,\ C_n') & \text{if } q_1(A) > q_1(B) \text{ and } q_2(A) < q_2(B), \\ (B \cup C_1', A \cup C_2', C_3', ...,\ C_n') & \text{otherwise.} \end{cases}$$

Since p_{-123} was chosen arbitrarily in $\mathcal{P}^{N \setminus 123}(\pi_{-123})$, this proves that s is $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\sigma_3, \pi_{-3})$. $\square$

Next, we derive two corollaries of the above results which link the behavior of s across sub-domains that need not be adjacent.

First Contagion Corollary. *Let $\pi \in \widetilde{\mathcal{P}}^N$, let $\sigma_2 \in \widetilde{\mathcal{P}}$, and let s be $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi)$ with residuals $C_1, ..., C_n$.*
(a) If $\{A, B\}$ cuts $\mathcal{P}(\sigma_2)$, then s is $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\sigma_2, \pi_{-2})$.
(b) If $\{A, B\}$ does not cut $\mathcal{P}(\sigma_2)$, then there exists a partition $\{C_1', ..., C_n'\}$ of $\Omega \setminus (A \cup B)$ such that $s(p) = (B \cup C_1', A \cup C_2', C_3', ..., C_n')$ for all $p \in \mathcal{P}^N(\sigma_2, \pi_{-2})$.

Proof. Let $\pi \in \widetilde{\mathcal{P}}^N$, $\sigma_2 \in \widetilde{\mathcal{P}}$, and suppose s is $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi)$ with residuals $C_1, ..., C_n$. Define

$$\begin{aligned}
\widetilde{\mathcal{P}}_+ &= \{\sigma_2 \in \widetilde{\mathcal{P}} : \sigma_2(\widetilde{A}) < \sigma_2(B)\}, \\
\widetilde{\mathcal{P}}_- &= \{\sigma_2 \in \widetilde{\mathcal{P}} : \sigma_2(\widetilde{A}) > \sigma_2(B)\}.
\end{aligned}$$

These sets partition $\widetilde{\mathcal{P}}$: $\sigma_2 \in \widetilde{\mathcal{P}}_+$ if and only if $\{A, B\}$ cuts $\mathcal{P}(\sigma_2)$. Clearly, $\widetilde{\mathcal{P}}_+$ and $\widetilde{\mathcal{P}}_-$ are connected: any two beliefs in one set are linked by a J-path of adjacent beliefs in that set. Since s is $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi)$, we have $\pi_2 \in \widetilde{\mathcal{P}}_+$.

Step 1. We prove statement *(a)*.

Let $\sigma_2 \in \widetilde{\mathcal{P}}_+$. Let $(\sigma_2^t)_{t=1}^T$ be a J-path in $\widetilde{\mathcal{P}}_+$ with $\sigma_2^1 = \pi_2$ and $\sigma_2^T = \sigma_2$. Since s is $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\sigma_2^1, \pi_{-2})$, repeated application of statement *(a)* in the First Contagion lemma implies that s is $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\sigma_2^T, \pi_{-2}) = \mathcal{P}^N(\sigma_2, \pi_{-2})$.

Step 2. We prove statement *(b)*.

Call two distinct events $C, D \subseteq \widetilde{\Omega}$ *adjacent in* $\sigma_2 \in \widetilde{\mathcal{P}}$ if $(\sigma_2(C) - \sigma_2(E))(\sigma_2(D) - \sigma_2(E)) > 0$ for all $E \subseteq \widetilde{\Omega}$ different from C, D. Define

$$\begin{aligned}
\widetilde{\mathcal{P}}^* &= \{\sigma_2 \in \widetilde{\mathcal{P}} : \widetilde{A}, B \text{ are adjacent in } \sigma_2\}, \\
\widetilde{\mathcal{P}}^*_+ &= \widetilde{\mathcal{P}}_+ \cap \widetilde{\mathcal{P}}^*, \\
\widetilde{\mathcal{P}}^*_- &= \widetilde{\mathcal{P}}_- \cap \widetilde{\mathcal{P}}^*.
\end{aligned}$$

We will first prove that statement *(b)* holds if $\sigma_2 \in \widetilde{\mathcal{P}}^*_-$, then show that it holds for all $\sigma_2 \in \widetilde{\mathcal{P}}_-$. The argument is illustrated in Figure 4.

Sub-step 2.1. If $\sigma_2 \in \widetilde{\mathcal{P}}^*_-$, then σ_2 is $\{\widetilde{A}, B\}$-adjacent to some belief $\sigma_2' \in \widetilde{\mathcal{P}}^*_+$. By statement *(a)*, s is $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\sigma_2', \pi_{-2})$. Statement *(b)* now follows from statement *(b)* in the First Contagion lemma.

Sub-step 2.2. If $\sigma_2 \in \widetilde{\mathcal{P}}_- \setminus \widetilde{\mathcal{P}}^*_-$, recall first that, since $\{A, B\}$ does not cut $\mathcal{P}(\sigma_2)$, we have $\sigma_2(\widetilde{A}) > \sigma_2(B)$. Fix $p = (p_2, p_{-2}) \in \mathcal{P}^N(\sigma_2, \pi_{-2})$. Consider, for each $\alpha \in (0,1)$, the probability measure σ_2^α defined over the subsets of $\widetilde{\Omega}$ by

$$\sigma_2^\alpha(E) = \alpha \frac{\sigma_2\left(E \cap \widetilde{A}\right)}{\sigma_2\left(\widetilde{A}\right)} + (1-\alpha) \frac{\sigma_2\left(E \cap \overline{\widetilde{A}}\right)}{\sigma_2\left(\overline{\widetilde{A}}\right)} \text{ for all } E \subseteq \widetilde{\Omega}, \tag{60}$$

where $\overline{\widetilde{A}} := \widetilde{\Omega} \setminus \widetilde{A}$. Each σ_2^α is a variant of the belief σ_2 where the probability of the states in $\widetilde{A}$ relative to those outside $\widetilde{A}$ is modified, but the conditional beliefs on the subsets of $\widetilde{A}$, as well as on the subsets of $\overline{\widetilde{A}}$, are kept unchanged. If $\alpha = \sigma_2(\widetilde{A})$, then σ_2^α coincides with σ_2. If $\alpha = \frac{\sigma_2(B)}{1+\sigma_2(B)-\sigma_2(\widetilde{A})}$, then $\sigma_2^\alpha(\widetilde{A}) = \sigma_2^\alpha(B)$. This means that if α is sufficiently close to $\frac{\sigma_2(B)}{1+\sigma_2(B)-\sigma_2(\widetilde{A})}$, the belief σ_2^α belongs to $\widetilde{\mathcal{P}}^*_-$. Elementary algebra shows that $\sigma_2(\widetilde{A}) > \frac{\sigma_2(B)}{1+\sigma_2(B)-\sigma_2(\widetilde{A})}$.

Write $p_2(\widetilde{\omega}) = \gamma$ and define, for each $\alpha \in (0,1)$, the measure p_2^α over the subsets of Ω by

$$p_2^\alpha(E) = \gamma \; \mathit{1}(E \cap \{\widetilde{\omega}\}) + (1-\gamma) \; \sigma_2^\alpha\left(E \cap \widetilde{\Omega}\right) \text{ for all } E \subseteq \Omega, \tag{61}$$

where $1(E \cap \{\widetilde{\omega}\}) = 1$ if $\widetilde{\omega} \in E$ and 0 otherwise.

Choose an increasing sequence of numbers $\alpha(1), ..., \alpha(T)$ in $(0,1)$ such that (i) $\sigma_2^{\alpha(t)}$ is adjacent to $\sigma_2^{\alpha(t+1)}$ for all $t = 1, ..., T-1$, (ii) $\sigma_2^{\alpha(1)} \in \widetilde{\mathcal{P}}_-^*$, and (iii) $\sigma_2^{\alpha(T)} = \sigma_2$. Define the J-path $(\sigma_2^t)_{t=1}^T$ in $\widetilde{\mathcal{P}}_-$ by $\sigma_2^t = \sigma_2^{\alpha(t)}$ for $t = 1, ..., T$. Define the associated finite sequence $(\mathbf{p}_2^t)_{t=1}^T$ in $\mathcal{P}$ by $\mathbf{p}_2^t = p_2^{\alpha(t)}$ for $t = 1, ..., T$. Observe that $\mathbf{p}_2^T = p_2$ and $\mathbf{p}_2^t \in \mathcal{P}(\sigma_2^t)$ for each t, but $\mathbf{p}_2^t, \mathbf{p}_2^{t+1}$ need not be adjacent. Finally, for each $t = 1, ..., T$, let $\mathbf{y}_2^t$ be a maximal element of $\widetilde{J}$ in $\mathcal{P}(\sigma_2^t)$. Observe that $\mathbf{y}_2^t, \mathbf{y}_2^{t+1}$ *are* adjacent and write $\mathbf{y}_2^T = y_2$.

Since $y_2^1 \in \mathcal{P}(\sigma_2^1)$ and $\sigma_2^1 \in \widetilde{\mathcal{P}}_-^*$, Sub-step 2.1 implies that there exists a partition $\{C_1', ..., C_n'\}$ of $\Omega \setminus (A \cup B)$ such that $s(\mathbf{y}_2^1, p_{-2}) = (B \cup C_1', A \cup C_2', C_3', ..., C_n')$. We will show that $s(p) = s(p_2, p_{-2}) = (B \cup C_1', A \cup C_2', C_3', ..., C_n')$. By non-bossiness, it suffices to prove $s_2(p) = A \cup C_2'$.

We have

$$s_2(\mathbf{y}_2^1, p_{-2}) = A \cup C_2'.$$

Proceeding now by induction, fix $t \in \{1, ..., T-1\}$ and suppose that

$$s_2(\mathbf{y}_2^t, p_{-2}) = A \cup C_2'.$$

Let $\{E^t, E^{t+1}\} \in \mathcal{H}(\widetilde{\Omega})$ be the pair of disjoint events such that $\sigma_2^t, \sigma_2^{t+1}$ are $\{E^t, E^{t+1}\}$-adjacent with $\sigma_2^t(E^t) > \sigma_2^t(E^{t+1})$. Because $\sigma_2^t, \sigma_2^{t+1}$ coincide on $\widetilde{A}$ as well as on $\overline{\widetilde{A}}$,

$$E^t \cap \overline{\widetilde{A}} \neq \emptyset \text{ and } E^{t+1} \cap \widetilde{A} \neq \emptyset.$$

If $s_2(\mathbf{y}_2^{t+1}, p_{-2}) \neq s_2(\mathbf{y}_2^t, p_{-2})$, the Local Bilaterality lemma implies $s_2(\mathbf{y}_2^{t+1}, p_{-2}) \setminus s_2(\mathbf{y}_2^t, p_{-2}) = E^{t+1}$. Since $A \subseteq s_2(\mathbf{y}_2^t, p_{-2})$, we conclude $E^{t+1} \cap \widetilde{A} = \emptyset$, a contradiction. Therefore $s_2(\mathbf{y}_2^{t+1}, p_{-2}) = A \cup C_2'$, and finally

$$s_2(y_2, p_{-2}) = A \cup C_2'. \tag{62}$$

Next, we claim that

$$s_2(p) = s_2(p_2, p_{-2}) = A \cup C_2'.$$

First, observe that since $\mathbf{p}_2^1 \in \mathcal{P}(\sigma_2^1)$ and $\sigma_2^1 \in \widetilde{\mathcal{P}}_-^*$, we have

$$s_2(\mathbf{p}_2^1, p_{-2}) = A \cup C_2'$$

Next, suppose, by way of contradiction, that $s_2(p_2, p_{-2}) = D \neq A \cup C_2'$. By Lemma 9, $\widetilde{\omega} \notin D$.

Case 1. $\frac{\sigma_2(C_2' \setminus D)}{\sigma_2\left(\overline{\widetilde{A}}\right)} < \frac{\sigma_2\left(\widetilde{A} \setminus D\right)}{\sigma_2\left(\widetilde{A}\right)}$.

By strategyproofness, $p_2(s_2(p_2, p_{-2})) > p_2(s_2(y_2, p_{-2}))$, hence by (62), $\mathbf{p}_2^T(D) > \mathbf{p}_2^T(A \cup C_2')$. Given (61), this means

$$\frac{\sigma_2^T\left(\widetilde{A} \cup C_2'\right) - \sigma_2^T(D)}{1 + \sigma_2^T(D) - \sigma_2^T\left(\widetilde{A} \cup C_2'\right)} < -\gamma. \tag{63}$$

From (60),

$$\sigma_2^T(\widetilde{A} \cup C_2') - \sigma_2^T(D)) = \alpha(T)\left(\frac{\sigma_2\left(\widetilde{A} \setminus D\right)}{\sigma_2\left(\widetilde{A}\right)}\right) + (1-\alpha(T))\left(\frac{\sigma_2(C_2') - \sigma_2\left(D \cap \overline{\widetilde{A}}\right)}{\sigma_2\left(\overline{\widetilde{A}}\right)}\right).$$

By assumption of Case 1, the second term of this convex combination is smaller than the first. Since $\alpha(1) < \alpha(T)$, it follows that $\sigma_2^1(\widetilde{A} \cup C_2') - \sigma_2^1(D) < \sigma_2^T(\widetilde{A} \cup C_2') - \sigma_2^T(D)$, hence from (63),

$$\frac{\sigma_2^1\left(\widetilde{A} \cup C_2'\right) - \sigma_2^1(D)}{1 + \sigma_2^1(D) - \sigma_2^1\left(\widetilde{A} \cup C_2'\right)} < -\gamma,$$

which, given (61), implies $\mathbf{p}_2^1(D) > \mathbf{p}_2^1(A \cup C_2')$, that is, $\mathbf{p}_2^1(s_2(q_2, p_{-2})) > \mathbf{p}_2^1(s_2(\mathbf{p}_2^1, p_{-2}))$, contradicting strategyproofness.

Case 2. $\frac{\sigma_2(C_2' \setminus D)}{\sigma_2\left(\overline{\widetilde{A}}\right)} \geq \frac{\sigma_2\left(\widetilde{A} \setminus D\right)}{\sigma_2\left(\widetilde{A}\right)}$.

Define $\overline{C_2'} := \widetilde{\Omega} \setminus C_2'$. Because $\sigma_2\left(C_2'\right) < \sigma_2\left(\overline{\widetilde{A}}\right)$ and $\sigma_2\left(\widetilde{A}\right) < \sigma_2\left(\overline{C_2'}\right)$,

$$\frac{\sigma_2\left(\widetilde{A} \setminus D\right)}{\sigma_2\left(\overline{C_2'}\right)} < \frac{\sigma_2(C_2' \setminus D)}{\sigma_2\left(C_2'\right)}.$$

Notice that this is the very same inequality as the one defining Case 1 –except that the roles of C_2' and $\widetilde{A}$ have been exchanged.

For each $\alpha \in (0,1)$, define the probability measure τ_2^α over the subsets of $\widetilde{\Omega}$ by

$$\tau_2^\alpha(E) = \alpha\frac{\sigma_2\left(E \cap C_2'\right)}{\sigma_2\left(C_2'\right)} + (1-\alpha)\frac{\sigma_2\left(E \cap \overline{C_2'}\right)}{\sigma_2\left(\overline{C_2'}\right)} \text{ for all } E \subseteq \widetilde{\Omega}$$

and the measure r_2^α over the subsets of Ω by

$$r_2^\alpha(E) = \gamma \; 1(E \cap \{\widetilde{\omega}\}) + (1-\gamma)\; \tau_2^\alpha\left(E \cap \widetilde{\Omega}\right) \text{ for all } E \subseteq \Omega.$$

These constructions are the same as in (60) and (61), except that C_2' plays the role of $\widetilde{A}$.

Choose an increasing sequence $\alpha(1), ..., \alpha(T)$ in $(0,1)$ such that (i) $\tau_2^{\alpha(t)}$ is adjacent to $\tau_2^{\alpha(t+1)}$ for all t, (ii) $\tau_2^{\alpha(1)} \in \widetilde{\mathcal{P}}_-^*$, and (iii) $\tau_2^{\alpha(T)} = \sigma_2$. Define the path $(\tau_2^t)_{t=1}^T$ in $\widetilde{\mathcal{P}}_-$ by

$\tau_2^t = \tau_2^{\alpha(t)}$ for all t, and define the sequence $(\mathbf{r}_2^t)_{t=1}^T$ in $\mathcal{P}$ by $\mathbf{r}_2^t = r_2^{\alpha(t)}$ for all t. Finally, for each t, let $\mathbf{z}_2^t$ be a maximal element of $\widetilde{J}$ in $\mathcal{P}(\tau_2^t)$ and let $\mathbf{z}_2^T = z_2$.

Since $\tau_2^1 \in \widetilde{\mathcal{P}}_-^*$, Sub-step 2.1 implies that there exists a partition $\{C_1'', ..., C_n''\}$ of $\Omega \setminus (A \cup B)$ such that $s(\mathbf{z}_2^1, p_{-2}) = (B \cup C_1'', A \cup C_2'', C_3'', ..., C_n'')$. In particular,

$$s_2(\mathbf{z}_2^1, p_{-2}) = A \cup C_2''.$$

By the same inductive argument as in Case 1, we obtain

$$s_2(z_2, p_{-2}) = A \cup C_2''.$$

But since both z_2 and y_2 are maximal elements of $\widetilde{J}$ in $\mathcal{P}(\sigma_2)$, we have $s_2(z_2, p_{-2}) = s_2(y_2, p_{-2})$, hence (62) implies

$$s_2(z_2, p_{-2}) = A \cup C_2'.$$

The proof that $s_2(p_2, p_{-2}) = A \cup C_2'$ now follows by the same argument as in Case 1, provided that we exchange the roles of $\widetilde{A}$ and C_2'. □

Second Contagion Corollary. *Let $\pi \in \widetilde{\mathcal{P}}^N$, let $\sigma_1 \in \widetilde{\mathcal{P}}$, and let s be $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi)$ with residuals $C_1, ..., C_n$.*
(a) If $\{A, B\}$ cuts $\mathcal{P}(\sigma_1)$, then s is $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\sigma_1, \pi_{-1})$.
(b) If $\{A, B\}$ does not cut $\mathcal{P}(\sigma_1)$, then s is $(2,1)$-dictatorial with respect to $\{A, B\}$ on $\mathcal{P}^N(\sigma_1, \pi_{-1})$.

Proof. Let $\pi \in \widetilde{\mathcal{P}}^N$, $\sigma_1 \in \widetilde{\mathcal{P}}$, and let s be $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi)$ with residuals $C_1, ..., C_n$. Define $\widetilde{\mathcal{P}}_+, \widetilde{\mathcal{P}}_-, \widetilde{\mathcal{P}}_+^*, \widetilde{\mathcal{P}}_-^*$ as in the proof of the previous corollary. By assumption, $\pi_1 \in \widetilde{\mathcal{P}}_+$. The argument below is illustrated in Figure 5.

Step 1. To prove statement *(a)*, let $\sigma_1 \in \widetilde{\mathcal{P}}_+$ and let $(\sigma_1^t)_{t=1}^T$ be a J-path in $\widetilde{\mathcal{P}}_+$ with $\sigma_1^1 = \pi_1$ and $\sigma_1^T = \sigma_1$. Since s is $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\sigma_1^1, \pi_{-1})$, repeated application of statement *(a)* in the Second Contagion lemma implies that s is $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\sigma_1^T, \pi_{-1}) = \mathcal{P}^N(\sigma_1, \pi_{-1})$.

Step 2. To prove statement *(b)*, we proceed again in two stages.

If $\sigma_1 \in \widetilde{\mathcal{P}}_-^*$, there exists a belief $\sigma_1' \in \widetilde{\mathcal{P}}_+^*$ to which σ_1 is $\{\widetilde{A}, B\}$-adjacent. By Step 1, s is $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\sigma_1', \pi_{-1})$. By statement *(b)* in the Second Contagion lemma, it follows that s is $(2,1)$-dictatorial with respect to $\{A, B\}$ on $\mathcal{P}^N(\sigma_1, \pi_{-1})$.

If $\sigma_1 \in \widetilde{\mathcal{P}}_- \setminus \widetilde{\mathcal{P}}_-^*$, let $(\sigma_1^t)_{t=1}^T$ be a J-path in $\widetilde{\mathcal{P}}_-$ with $\sigma_1^1 \in \widetilde{\mathcal{P}}_-^*$ and $\sigma_1^T = \sigma_1$. Since s is $(2,1)$-dictatorial with respect to $\{A, B\}$ on $\mathcal{P}^N(\sigma_1^1, \pi_{-1})$, repeated application of statement *(b)* in the Second Contagion lemma implies that s is $(2,1)$-dictatorial with respect to $\{A, B\}$ on $\mathcal{P}^N(\sigma_1^T, \pi_{-1}) = \mathcal{P}^N(\sigma^1, \pi_{-1})$. □

10.2 Proof of the Bilateral Consensus lemma

We are finally ready to prove the Bilateral Consensus lemma. Let $\widetilde{\omega} \in \Omega_2$. This state is again fixed throughout the sub-section, but observe that we now assume that its assignment varies with the beliefs of at least two agents.

We must show that there exist an event $E^{\widetilde{\omega}} \subseteq \Omega_2$ such that $\widetilde{\omega} \in E^{\widetilde{\omega}}$, and a bilaterally consensual $E^{\widetilde{\omega}}$-assignment rule $s^{\widetilde{\omega}}$ such that

$$s_i(p) \cap E^{\widetilde{\omega}} = s_i^{\widetilde{\omega}}(p \mid E^{\widetilde{\omega}}) \text{ for all } i \in N \tag{64}$$

and all $p \in \mathcal{P}^N$.

Recall the definition of $a_{\widetilde{\omega}}$ in (2) and the notation $\widetilde{a} = a_{\widetilde{\omega}}$.

Step 1. There exist $\pi^0 \in \widetilde{\mathcal{P}}^N$, two distinct agents $i, j \in N$, $p, q \in \mathcal{P}^N(\pi^0)$, and $p_i' \in \mathcal{P}(\pi_i^0), q_j' \in \mathcal{P}(\pi_j^0)$ such that $\widetilde{a}(p) \neq \widetilde{a}(p_i', p_{-i})$ and $\widetilde{a}(q) \neq \widetilde{a}(q_j', q_{-j})$.

By definition of Ω_2, there exist two agents, say $1, 2$, profiles $p, q \in \mathcal{P}^N$, and beliefs $p_1', q_2' \in \mathcal{P}$ such that

$$\widetilde{a}(p) \neq \widetilde{a}(p_1', p_{-1}) \text{ and } \widetilde{a}(q) \neq \widetilde{a}(q_2', q_{-2}). \tag{65}$$

Because $\mathcal{P}$ is connected, we assume without loss of generality that p_1, p_1' are adjacent and p_2, p_2' are adjacent. Let $\{E, E'\}$ be the pair of events such that p_1, p_1' are $\{E, E'\}$-adjacent. By the Local Bilaterality lemma and the first inequality in (65), $\widetilde{\omega} \in E \cup E'$, hence, $(p_1(C) - p_1(D))(p_1'(C) - p_1'(D)) > 0$ for all distinct $C, D \subseteq \widetilde{\Omega}$. This means that there exists $\pi_1^0 \in \widetilde{\mathcal{P}}$ such that $p_1 \mid \widetilde{\Omega} \approx p_1' \mid \widetilde{\Omega} \approx \pi_1^0$, that is, $p_1, p_1' \in \mathcal{P}(\pi_1^0)$. By the same token, there exists $\pi_2^0 \in \widetilde{\mathcal{P}}$ such that $p_2, p_2' \in \mathcal{P}(\pi_2^0)$.

To keep notation simple, suppose $n = 3$; the argument is easily extended to any number of agents. Suppose first that $p_3 = q_3$. Dropping that belief from the notation, (65) reads

$$\widetilde{a}(p_1, p_2) \neq \widetilde{a}(p_1', p_2) \text{ and } \widetilde{a}(q_1, q_2) \neq \widetilde{a}(q_1, q_2').$$

Case 1: $\widetilde{a}(p_1', q_2) \neq \widetilde{a}(p_1, q_2) \neq \widetilde{a}(p_1, q_2')$. In this case the claim is trivially true.

Case 2: (i) $\widetilde{a}(p_1, q_2) = \widetilde{a}(p_1', q_2)$ or (ii) $\widetilde{a}(p_1, q_2) = \widetilde{a}(p_1, q_2')$.

Assume (i); the argument is the same, up to a relabeling, if (ii) holds. Let $(\mathbf{p}_2^t)_{t=1}^T$ be a J-path between $\mathbf{p}_2^1 = p_2$ and $\mathbf{p}_2^T = q_2$. From (65) and (i), there exists an integer t such that

$$\widetilde{a}(p_1, \mathbf{p}_2^t) \neq \widetilde{a}(p_1', \mathbf{p}_2^t) \text{ and } \widetilde{a}(p_1, \mathbf{p}_2^{t+1}) = \widetilde{a}(p_1', \mathbf{p}_2^{t+1}) \tag{66}$$

Using the Local Bilaterality lemma, the same argument as in Sub-step 1.1 shows that there exists π_2^0 such that $\mathbf{p}_2^t \mid \widetilde{\Omega} \approx \mathbf{p}_2^{t+1} \mid \widetilde{\Omega} \approx \pi_2^0$, that is, $\mathbf{p}_2^t, \mathbf{p}_2^{t+1} \in \mathcal{P}(\pi_2^0)$. Moreover, statement (66) implies

$$\widetilde{a}(p_1', \mathbf{p}_2^t) \neq \widetilde{a}(p_1, \mathbf{p}_2^t) \neq \widetilde{a}(p_1, \mathbf{p}_2^{t+1})$$

or

$$\widetilde{a}(p_1, \mathbf{p}_2^{t+1}) \neq \widetilde{a}(p_1', \mathbf{p}_2^{t+1}) \neq \widetilde{a}(p_1', \mathbf{p}_2^{t}).$$

In either case the claim is true.

Finally, let us drop the assumption that $p_3 = q_3$. Suppose that there exist $p_3 \neq q_3$ such that

$$\widetilde{a}(p_1, p_2, p_3) \neq \widetilde{a}(p_1', p_2, p_3) \text{ and } \widetilde{a}(q_1, q_2, q_3) \neq \widetilde{a}(q_1, q_2', q_3).$$

and

$$\widetilde{a}(p_1, p_2, q_3) = \widetilde{a}(p_1', p_2, q_3) \text{ and } \widetilde{a}(q_1, q_2, p_3) = \widetilde{a}(q_1, q_2', p_3).$$

Let $(\mathbf{p}_3^t)_{t=1}^T$ be a J-path between $\mathbf{p}_3^1 = p_3$ and $\mathbf{p}_3^T = q_3$. There exists an integer t such that

$$\widetilde{a}(p_1, p_2, \mathbf{p}_3^t) \neq \widetilde{a}(p_1', p_2, \mathbf{p}_3^t) \text{ and } \widetilde{a}(p_1, p_2, \mathbf{p}_3^{t+1}) = \widetilde{a}(p_1', p_2, \mathbf{p}_3^{t+1}). \tag{67}$$

By the Local Bilaterality lemma again, there exists π_3^0 such that $\mathbf{p}_3^t \mid \widetilde{\Omega} \approx \mathbf{p}_3^{t+1} \mid \widetilde{\Omega}$ $\approx \pi_3^0$, that is, $\mathbf{p}_3^t, \mathbf{p}_3^{t+1} \in \mathcal{P}(\pi_3^0)$. Moreover, statement (67) implies

$$\widetilde{a}(p_1, p_2, \mathbf{p}_3^t) \neq \widetilde{a}(p_1', p_2, \mathbf{p}_3^t) \neq \widetilde{a}(p_1', p_2, \mathbf{p}_3^{t+1})$$

or

$$\widetilde{a}(p_1, p_2, \mathbf{p}_3^{t+1}) \neq \widetilde{a}(p_1, p_2, \mathbf{p}_3^t) \neq \widetilde{a}(p_1', p_2, \mathbf{p}_3^t).$$

In either case the claim is again true.

Step 2. Step 1 has established that there is some $\pi^0 \in \widetilde{\mathcal{P}}^N$ such that s varies with the beliefs of two distinct agents, say 1 and 2, on $\mathcal{P}^N(\pi^0)$. By statement *(b)* in Lemma 9, s is bilaterally consensual on $\mathcal{P}^N(\pi^0)$ and we may assume without loss of generality (in light of Remark 2) that s is $(2,1)$-consensual on that domain: there exists a partition $\{A, B, C_1, ..., C_n\}$ of Ω such that $\widetilde{\omega} \in A$, $\{A, B\}$ cuts $\mathcal{P}(\pi_1^0)$, $\mathcal{P}(\pi_2^0)$, and for all $p \in \mathcal{P}^N(\pi^0)$,

$$s(p) = \begin{cases} (A \cup C_1, B \cup C_2, C_3, ..., C_n) & \text{if } p_1(A) > p_1(B) \text{ and } p_2(A) < p_2(B), \\ (B \cup C_1, A \cup C_2, C_3, ..., C_n) & \text{otherwise.} \end{cases} \tag{68}$$

Define $E^{\widetilde{\omega}} := A \cup B$ and define the bilaterally consensual $E^{\widetilde{\omega}}$-assignment rule $s^{\widetilde{\omega}}$ as follows: for all $\widetilde{p} \in \mathcal{P}(E^{\widetilde{\omega}})^N$,

$$s^{\widetilde{\omega}}(\widetilde{p}) = \begin{cases} (A, B, \emptyset, ..., \emptyset) & \text{if } \widetilde{p}_1(A) > \widetilde{p}_1(B) \text{ and } \widetilde{p}_2(A) < \widetilde{p}_2(B), \\ (B, A, \emptyset, ..., \emptyset) & \text{otherwise.} \end{cases}$$

We claim that (64) holds for all $p \in \mathcal{P}^N$.

By definition, statement (64) is true for all $p \in \mathcal{P}^N(\pi^0)$. Next, fix an arbitrary sub-profile $\pi_{-12} \in \widetilde{\mathcal{P}}^{N\setminus 12}$.

Sub-step 2.1. By repeated application of the Independence lemma, s is $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi_1^0, \pi_2^0, \pi_{-12})$, hence, (64) is true for all $p \in \mathcal{P}^N(\pi_1^0, \pi_2^0, \pi_{-12})$.

Sub-step 2.2. For any profile $(\pi_1, \pi_2) \in \widetilde{\mathcal{P}}_+ \times \widetilde{\mathcal{P}}_+$, combining Sub-step 2.1 with part *(a)* of the First Contagion Corollary and part *(a)* of the Second Contagion Corollary shows that s is $(2,1)$-consensual with respect to $\{A, B\}$ on $\mathcal{P}^N(\pi_1, \pi_2, \pi_{-12})$, hence, (64) is true for all $p \in \mathcal{P}^N(\pi_1, \pi_2, \pi_{-12})$.

Sub-step 2.3. For any profile $(\pi_1, \sigma_2) \in \widetilde{\mathcal{P}}_+ \times \widetilde{\mathcal{P}}_-$, Sub-step 2.2 and part *(b)* of the First Contagion Corollary imply that there is a partition $\{C_1', ..., C_n'\}$ of $\Omega \setminus (A \cup B)$ such that $s(p) = (B \cup C_1', A \cup C_2', C_3', ..., C_n')$ for all $p \in \mathcal{P}^N(\pi_1, \sigma_2, \pi_{-12})$. Since $\{A, B\}$ does not cut $\mathcal{P}(\sigma_2)$, we have $p_2(A) > p_2(B)$ for all $p_2 \in \mathcal{P}(\sigma_2)$, hence (64) is true for all $p \in \mathcal{P}^N(\pi_1, \sigma_2, \pi_{-12})$.

Sub-step 2.4. For any profile $(\sigma_1, \pi_2) \in \widetilde{\mathcal{P}}_- \times \widetilde{\mathcal{P}}_+$, Sub-step 2.2 and part *(b)* of the Second Contagion Corollary imply that s is $(2,1)$-dictatorial on $\mathcal{P}^N(\sigma_1, \pi_2, \pi_{-12})$. Since $\{A, B\}$ does not cut $\mathcal{P}(\sigma_1)$, we have $p_1(A) > p_1(B)$ for all $p_1 \in \mathcal{P}(\sigma_1)$, hence (64) is true for all $p \in \mathcal{P}^N(\sigma_1, \pi_2, \pi_{-12})$.

Sub-step 2.5. Consider finally a profile $(\sigma_1, \sigma_2) \in \widetilde{\mathcal{P}}_- \times \widetilde{\mathcal{P}}_-$. By definition, $\sigma_2(\widetilde{A}) > \sigma_2(B)$. For each $\alpha \in (0,1)$, consider again the measure ${}_\alpha\sigma_2$ defined on $\widetilde{\Omega}$ by (60). Recall that ${}_\alpha\sigma_2$ coincides with σ_2 for $\alpha = \sigma_2(\widetilde{A})$ and observe that ${}_\alpha\sigma_2 \in \widetilde{\mathcal{P}}_+$ for any generic $\alpha < \frac{\sigma_2(B)}{1+\sigma_2(B)-\sigma_2(\widetilde{A})}$.

Choose an increasing sequence of numbers $\alpha(1), ..., \alpha(T)$ such that (i) ${}_{\alpha(t)}\sigma_2$ is adjacent to ${}_{\alpha(t+1)}\sigma_2$ for all $t = 1, ..., T-1$, (ii) ${}_{\alpha(1)}\sigma_2 \in \widetilde{\mathcal{P}}_+$, and (iii) ${}_{\alpha(T)}\sigma_2 = \sigma_2$. Consider the J-path $(\sigma_2^t)_{t=1}^T$ in $\widetilde{\mathcal{P}}_-$ defined by $\sigma_2^t =_{\alpha(t)} \sigma_2$ for $t = 1, ..., T$.

Since $\sigma_2^1 \in \widetilde{\mathcal{P}}_+$, Sub-step 2.3 implies that there exists a partition $\{C_1', ..., C_n'\}$ of $\Omega \setminus (A \cup B)$ such that $s(p) = (B \cup C_1', A \cup C_2', C_3', ..., C_n')$ for all $p \in \mathcal{P}^N(\sigma_1, \sigma_2^1, \pi_{-12})$. The same argument as in Sub-step 2.2 of the proof of the First Contagion Corollary then establishes that $s(p) = (B \cup C_1', A \cup C_2', C_3', ..., C_n')$ for all $p \in \mathcal{P}^N(\sigma_1, \sigma_2^T, \pi_{-12}) = \mathcal{P}^N(\sigma_1, \sigma_2, \pi_{-12})$.

Since $\{A, B\}$ does not cut $\mathcal{P}(\sigma_2)$, we have $p_2(A) > p_2(B)$ for all $p_2 \in \mathcal{P}(\sigma_2)$, hence (64) is true for all $p \in \mathcal{P}^N(\sigma_1, \sigma_2, \pi_{-12})$.

Since $\mathcal{P} = \cup_{\pi_i \in \widetilde{\mathcal{P}}} \mathcal{P}(\pi_i)$, the proof of the Bilateral Consensus lemma is complete. □

11 Appendix D: proof of the Bilateral Dictatorship lemma

Let Ω_{11} be the set of states whose assignment varies only with the beliefs of agent 1, namely,

$$\omega \in \Omega_{11} \Leftrightarrow \left[\text{there exist } p \in \mathcal{P}^N \text{ and } p_1' \in \mathcal{P} \text{ such that } a_\omega(p) \neq a_\omega(p_1', p_{-1})\right] \text{ and } \left[a_\omega(., p_{-j}) \text{ is constant on } \mathcal{P} \text{ for all } j \neq 1 \text{ and } p_{-j} \in \mathcal{P}^{N\setminus j}\right].$$

To avoid triviality, assume $\Omega_{11} \neq \emptyset$. Let $\widetilde{\omega} \in \Omega_{11}$. We must show that there exist a set $N_1 \subseteq N \setminus 1$, a partition $\left\{\Omega_{11}^j\right\}_{j\in N_1}$ of Ω_{11}, and for each $j \in N_1$ a $(1, j)$-dictatorial Ω_{11}^j-assignment rule s^j such that

$$s_i(p) \cap \Omega_{11} = \cup_{j\in N_1} s_i^j(p \mid \Omega_{11}^j) \tag{69}$$

for all $p \in \mathcal{P}^N$ and $i \in N$.

Define the family

$$\begin{aligned} \mathcal{A}_{11} &= \left\{A \subseteq \Omega_{11} : \exists p \in \mathcal{P}^N \text{ such that } s_1(p) \cap \Omega_{11} = A\right\} \\ &= \left\{A \subseteq \Omega_{11} : \exists p_1 \in \mathcal{P} \text{ such that } s_1(p_1, p_{-1}) \cap \Omega_{11} = A \text{ for all } p_{-1} \in \mathcal{P}^{N\setminus 1}\right\}, \end{aligned}$$

where the first equality constitutes the definition and the second follows from the definition of Ω_{11}.

Let $\overline{\Omega}_{11} = \Omega \setminus \Omega_{11}$. Call a belief $p_1 \in \mathcal{P}$ Ω_{11}-*dominant* if $|p_1(A) - p_1(B)| > |p_1(A') - p_1(B')|$ for all distinct $A, B \subset \Omega_{11}$ and all distinct $A', B' \subset \overline{\Omega}_{11}$ (or, equivalently, $|p_1(\omega) - p_1(\omega')| > p_1(\overline{\Omega}_{11})$ for all distinct $\omega, \omega' \in \Omega_{11}$). In such a belief, the probability *differences* within Ω_{11} overwhelm the differences outside Ω_{11}. To see that such beliefs exist, write $\Omega_{11} = \{1, ..., m\}$ and observe that any belief p_1 such that $p_1(1) > p_1(\Omega \setminus 1)$, $p_1(2) > p_1(\Omega \setminus 12)$, ..., and $p_1(m) > p_1(\Omega \setminus 1...m-1)$, is Ω_{11}-dominant. Let $\mathcal{P}_{11}$ denote the set of Ω_{11}-dominant beliefs.

Step 1. We show that

$$s_1(p) \cap \Omega_{11} = \underset{\mathcal{A}_{11}}{\operatorname{argmax}}\, p_1 \tag{70}$$

for all $p = (p_1, p_{-1}) \in \mathcal{P}_{11} \times \mathcal{P}^{N\setminus 1}$.

The claim is obviously true if $\Omega_{11} = \Omega$; in what follows we assume $\Omega_{11} \neq \Omega$. For any two beliefs $p_1, q_1 \in \mathcal{P}$ and for any $p_{-1} \in \mathcal{P}^{N\setminus 1}$, we claim that

$$\left[p_1 \mid \overline{\Omega}_{11} = q_1 \mid \overline{\Omega}_{11}\right] \Rightarrow \left[s_1(p_1, p_{-1}) \cap \overline{\Omega}_{11} = s_1(q_1, p_{-1}) \cap \overline{\Omega}_{11}\right]. \tag{71}$$

To see why this is true, fix $p_1, q_1 \in \mathcal{P}$, $p_{-1} \in \mathcal{P}^{N\setminus 1}$, and note that the definitions of Ω_0 and Ω_{1j} for $j \neq 1$ trivially imply

$$s_1(p_1, p_{-1}) \cap [\Omega_0 \cup \cup_{j\neq 1}\Omega_{1j}] = s_1(q_1, p_{-1}) \cap [\Omega_0 \cup \cup_{j\neq 1}\Omega_{1j}].$$

Moreover, by the Bilateral Consensus corollary, agent 1's share of Ω_2 is determined by bilateral consensus, hence does not depend on her belief outside Ω_2. Therefore,

$$\left[p_1 \mid \overline{\Omega}_{11} = q_1 \mid \overline{\Omega}_{11}\right] \Rightarrow [s_1(p_1, p_{-1}) \cap \Omega_2 = s_1(q_1, p_{-1}) \cap \Omega_2],$$

and (71) follows.

Let now $p = (p_1, p_{-1}) \in \mathcal{P}_{11} \times \mathcal{P}^{N\backslash 1}$. Since p_{-1} is fixed in the argument below, we drop it from the list of arguments of s_1. Suppose, contrary to the claim, that $s_1(p_1) \cap \Omega_{11} \neq \underset{\mathcal{A}_{11}}{\operatorname{argmax}}\, p_1$. Choosing $q_1 \in \mathcal{P}$ such that $s_1(q_1) \cap \Omega_1 = \underset{\mathcal{A}_{11}}{\operatorname{argmax}}\, p_1$, we have

$$p_1(s_1(q_1) \cap \Omega_{11}) > p_1(s_1(p_1) \cap \Omega_{11}).$$

Because p_1 is Ω_{11}-dominant,

$$\begin{aligned} & p_1(s_1(q_1) \cap \Omega_{11}) - p_1(s_1(p_1) \cap \Omega_{11}) \\ > \ & p_1(s_1(p_1) \cap \overline{\Omega}_{11}) - p_1(s_1(q_1) \cap \overline{\Omega}_{11}). \end{aligned}$$

Combining these inequalities yields $p_1(s_1(q_1)) > p_1(s_1(p_1))$, contradicting strategyproofness.

Step 2. We prove that (70) holds for all $p \in \mathcal{P}^N$.

Let $p = (p_1, p_{-1}) \in \mathcal{P}^N$ and drop again p_{-1} from the list of arguments of s_1. For each $\alpha \in (0, 1)$, define the probability measure ${}_\alpha p_1$ over the subsets of Ω by

$$_\alpha p_1(A) = \alpha \frac{p_1(A \cap \Omega_{11})}{p_1(\Omega_{11})} + (1 - \alpha) \frac{p_1(A \cap \overline{\Omega}_{11})}{p_1(\overline{\Omega}_{11})} \text{ for all } A \subseteq \Omega. \tag{72}$$

If $\alpha = p_1(\Omega_{11})$, then ${}_\alpha p_1$ coincides with p_1. If α is sufficiently close to 1, then ${}_\alpha p_1$ is Ω_{11}-dominant. For every α, ${}_\alpha p_1 \mid \Omega_{11} = p_1 \mid \Omega_{11}$ and ${}_\alpha p_1 \mid \overline{\Omega}_{11} = p_1 \mid \overline{\Omega}_{11}$.

Choose an increasing sequence of numbers $\alpha(1), ..., \alpha(T)$ such that (i) ${}_{\alpha(t)}p_1$ is adjacent to ${}_{\alpha(t+1)}p_1$ for all $t = 1, ..., T-1$, (ii) ${}_{\alpha(1)}p_1 = p_1$, and (iii) ${}_{\alpha(T)}p_1$ is Ω_{11}-dominant. Consider the J-path $(\mathbf{p}_1^t)_{t=1}^T$ in $\mathcal{P}$ defined by $\mathbf{p}_1^t =_{\alpha(t)} p_1$ for $t = 1, ..., T$.

Let $A^t = s_1(\mathbf{p}_1^t) \cap \Omega_{11}$ for $t = 1, ..., T$. Suppose, contrary to the claim, that $A^1 \neq \underset{\mathcal{A}_{11}}{\operatorname{argmax}}\, p_1$. Since $\mathbf{p}_1^T$ is Ω_{11}-dominant and $\mathbf{p}_1^T \mid \Omega_{11} = p_1 \mid \Omega_{11}$, Step 1 implies $A^T = \underset{\mathcal{A}_{11}}{\operatorname{argmax}}\, p_1$. Let t be the largest integer in $\{1, ..., T-1\}$ such that $A^t \neq \underset{\mathcal{A}_{11}}{\operatorname{argmax}}\, p_1$. Let $\{E^t, E^{t+1}\}$ be the pair of disjoint events such that $\mathbf{p}_1^t, \mathbf{p}_1^{t+1}$ are $\{E^t, E^{t+1}\}$-adjacent and $\mathbf{p}_1^t(E^t) > \mathbf{p}_1^t(E^{t+1})$. Because $\mathbf{p}_1^t \mid \Omega_{11} = \mathbf{p}_1^{t+1} \mid \Omega_{11}$ and $\mathbf{p}_1^t \mid \overline{\Omega}_{11} = \mathbf{p}_1^{t+1} \mid \overline{\Omega}_{11}$,

$$E^t \cap \overline{\Omega}_{11} \neq \emptyset \text{ and } E^{t+1} \cap \Omega_{11} \neq \emptyset.$$

By the Local Bilaterality lemma,

$$s_1(\mathbf{p}_1^t) \setminus s_1(\mathbf{p}_1^{t+1}) = E^t \text{ and } s_1(\mathbf{p}_1^{t+1}) \setminus s_1(\mathbf{p}_1^t) = E^{t+1}.$$

It follows that $(s_1(\mathbf{p}_1^t) \setminus s_1(\mathbf{p}_1^{t+1})) \cap \overline{\Omega}_{11} \neq \emptyset$, that is, $s_1(\mathbf{p}_1^t) \cap \overline{\Omega}_{11} \neq s_1(\mathbf{p}_1^{t+1}) \cap \overline{\Omega}_{11}$, contradicting (71).

Step 3. We show that for all $p, q \in \mathcal{P}_{11} \times \mathcal{P}^{N\backslash 1}$,

$$[p_1 \mid \Omega_{11} = q_1 \mid \Omega_{11}] \Rightarrow [s_i(p) \cap \Omega_{11} = s_i(q) \cap \Omega_{11} \text{ for all } i \in N].$$

Let $p, q \in \mathcal{P}_{11} \times \mathcal{P}^{N\setminus 1}$. Since we are only concerned with the restriction of s to Ω_{11}, we may assume $p_{-1} = q_{-1}$ and omit that sub-profile from the notation. Suppose $p_1 \mid \Omega_{11} = q_1 \mid \Omega_{11}$. By Step 1,

$$s_1(p_1) \cap \Omega_{11} = s_1(q_1) \cap \Omega_{11} = \operatorname*{argmax}_{\mathcal{A}_{11}} p_1. \tag{73}$$

Because $p_1, q_1 \in \mathcal{P}_{11}$, (73) and super-strategyproofness imply

$$s_i(p_1) \cap \Omega_{11} = s_i(q_1) \cap \Omega_{11} \text{ for all } i \in N.$$

Indeed, if, say, $s_2(p_1) \cap \Omega_{11} \neq s_2(q_1) \cap \Omega_{11}$, then (73) and the assumption $p_1 \mid \Omega_{11} = q_1 \mid \Omega_{11}$ imply that either (i) $p_1(s_{12}(p_1) \cap \Omega_{11}) > p_1(s_{12}(q_1) \cap \Omega_{11})$ and $q_1(s_{12}(p_1) \cap \Omega_{11}) > q_1(s_{12}(p_1) \cap \Omega_{11})$, or (ii) both of these two strict inequalities are reversed. Because $p_{1,} q_1$ are Ω_{11}-dominant, each of (i) and (ii) violates super-strategyproofness.

Step 4. We claim that for every $\omega \in \Omega_{11}$ there is a unique $j \neq 1$ such that $a_\omega(\mathcal{P}_{11} \times \mathcal{P}^{N\setminus 1}) = \{1, j\}$.

From Step 3, the assignment of all states in Ω_{11} depends only on the conditional beliefs of agent 1 over Ω_{11}. We may thus drop p_{-1} from the notation and regard s as a function from $\mathcal{P}(\Omega_{11})$ to $\mathcal{S}(\Omega_{11})$. By assumption, s is super-strategyproof (hence also non-bossy) and it is not constant on $\mathcal{P}(\Omega_{11})$.

We want to show that

$$s_j(p_1) \cap s_k(q_1) = \emptyset \text{ for any distinct } j, k \in N \setminus 1 \tag{74}$$

and any $p_1, q_1 \in \mathcal{P}(\Omega_{11})$. For any $\tilde{\Omega}_{11} \subset \Omega_{11}$, an $\tilde{\Omega}_{11}$-assignment rule $\tilde{s} : \mathcal{P}(\tilde{\Omega}_{11}) \to \mathcal{S}(\tilde{\Omega}_{11})$ will be called *1-C-BD union* if it is a *union of constant or bilaterally 1-dictatorial rules* on $\tilde{\Omega}_{11}$, namely, if there is a partition $\left\{\Omega_{11}^l\right\}_{t=1}^L$ of Ω_{11} such that, for all $p_1 \in \mathcal{P}(\tilde{\Omega}_{11})$,

$$\tilde{s}_i(p_1) = \cup_{l=1}^L s_i^l(p_1 \mid \Omega_{11}^l) \text{ for all } i \in N, \tag{75}$$

where each s^l is a constant or $(1, j^l)$-dictatorial Ω_{11}^l-assignment rule. With a slight abuse of terminology, we will call (the restriction to $\bar{\mathcal{P}}$ of) $\tilde{s}$ a *1-C-BD union over* $\bar{\mathcal{P}}$ if (75) is satisfied for all $p_1 \subset \bar{\mathcal{P}} \subset \mathcal{P}(\tilde{\Omega}_{11})$. We prove Step 4 by induction on the size of Ω_{11}.

Sub-step 4.1. Suppose that $|\Omega_{11}| = 2$ and consider a super-strategyproof assignment rule $\tilde{s} : \mathcal{P}(\Omega_{11}) \to \mathcal{S}(\Omega_{11})$. Then there exists $j \in N \setminus 1$ such that $\tilde{s}_{1j}(p) = \Omega_{11}$ for all $p_1 \in \mathcal{P}(\Omega_{11})$. It follows that $\tilde{s}$ is a 1-C-BD union.

Indeed, suppose that $\Omega_{11} = \{\omega_1, \omega_2\}$ and let $\tilde{p_1} \in \mathcal{P}(\Omega_{11})$. If we have either $\tilde{s}_1(\tilde{p_1}) = \emptyset$ or $\tilde{s}_1(\tilde{p_1}) = \Omega_{11}$, then $\tilde{s}$ is constant over $\mathcal{P}(\Omega_{11})$ and the result of Sub-step 4.1 trivially holds. Without loss of generality, suppose now that $\tilde{s}_1(\tilde{p_1}) = \{\omega_1\}$. Then there exists some agent $j \neq 1$ such that $\omega_2 \in s_j(\tilde{p_1})$ and obviously $\tilde{s}_{1j}(\tilde{p_1}) = \Omega_{11}$. By super-strategyproofness of $\tilde{s}$, we have $p_1(\tilde{s}_{1j}(p_1)) \geq p_1(\tilde{s}_{1j}(\tilde{p_1})) = p_1(\Omega_{11}) = 1$, hence, $p_1(\tilde{s}_{1j}(\tilde{p_1})) = 1$, for all $p \in \mathcal{P}(\Omega_{11})$, meaning that $\tilde{s}$ is $(1,j)$-dictatorial. Thus, in all possible cases, $\tilde{s}$ is a 1-C-BD union.

Suppose now that $|\Omega_{11}| = K \geq 3$ and assume by induction that every assignment rule $\tilde{s} : \mathcal{P}(\tilde{\Omega}_{11}) \to \mathcal{S}(\tilde{\Omega}_{11})$ such that $|\tilde{\Omega}_{11}| \leq K-1$ is a 1-C-BD union.

Recalling that the range of $s_1(\cdot)$ is $\mathcal{E} \equiv \{E \subset \Omega_{11} : s_1(p_1) = E \text{ for some } p_1 \in \mathcal{P}(\Omega_{11})\}$, strategyproofness of s obviously implies $s_1(p_1) = \underset{\mathcal{E}}{\operatorname{argmax}}\, p_1$ for all $p_1 \in \mathcal{P}(\Omega_{11})$.

Given any $\omega \in \Omega_{11}$, define the set of ω-*lexicographic* beliefs $\mathcal{L}(\omega) := \{p_1 \in \mathcal{P}(\Omega_{11}) : p_1(\omega_1) > p_1(\Omega_{11} \setminus \omega)\}$. For any $q_1 \in \mathcal{P}(\Omega_{11}) \cup \mathcal{P}(\Omega_{11} \setminus \omega)$, let $\mathcal{L}^{q_1}(\omega) := \{p_1 \in \mathcal{L}(\omega) : p_1 \mid (\Omega_{11} \setminus \omega) = p_1 \mid (\Omega_{11} \setminus \omega)\}$ and, for any $\alpha \in (\frac{1}{2}, 1)$, define $q_1^{\omega,\alpha} \in \mathcal{L}^{q_1}(\omega)$ as follows: for all $\omega' \in \Omega_{11}$,

$$q_1^{\omega,\alpha}(\omega') := \begin{cases} \alpha & \text{if } \omega' = \omega, \\ \frac{q_1(\omega')}{1-\alpha} & \text{if } \omega' \neq \omega. \end{cases}$$

Sub-step 4.2. Consider $q_1 \in \Omega_{11}$ such that $\omega \in s_1(q_1)$; and suppose that $p_1 \in \mathcal{L}^{q_1}(\omega)$. Then we have $s(p_1) = s(q_1)$.

The proof of Sub-step 4.2. is rather straightforward, and left to the reader. It follows from non-bossiness of s and the fact that $p_1(\omega) > 1/2$ for all $p_1 \in \mathcal{L}^{q_1}(\omega)$.

Sub-step 4.3. Fix $\bar{\omega} \in \Omega_{11}$ and $\alpha \in (\frac{1}{2}, 1)$. Define the mapping ${}_{\alpha}\tilde{s}^{-\bar{\omega}} : \mathcal{P}(\Omega_{11} \setminus \bar{\omega}) \to \mathcal{S}(\Omega_{11} \setminus \bar{\omega})$ as follows: **(i)** ${}_{\alpha}\tilde{s}_1^{-\bar{\omega}}(q_1) = s_1(q_1^{\bar{\omega},\alpha}) \setminus \bar{\omega}$; **(ii)** ${}_{\alpha}\tilde{s}_i^{-\bar{\omega}}(q_1) = s_i(q_1^{\bar{\omega},\alpha}), \forall i \neq 1$. Then ${}_{\alpha}\tilde{s}^{-\bar{\omega}}$ is an $(\Omega_{11} \setminus \bar{\omega})$-assignment rule and a 1-C-BD union.

To prove Sub-step 4.3, note first that $\bar{\omega} \in s_1(p_1)$ for all $p_1 \in \mathcal{L}(\bar{\omega})$. Indeed, since the range $\mathcal{E}$ of $s_1(\cdot)$ is a proper covering of Ω_{11}, there exists $\bar{p}_1 \in \mathcal{P}(\tilde{\Omega}_{11})$ such that $\bar{\omega} \in s_1(\bar{p}_1)$. Therefore, if $\bar{\omega} \notin s_1(p_1)$ for some $p_1 \in \mathcal{L}(\bar{\omega})$, we would have $p_1(s_1(\bar{p}_1)) \geq p_1(\bar{\omega}) > \frac{1}{2} > p_1(s_1(p_1))$, contradicting strategyproofness.

Building on this result, observe from (i)-(ii) above that the mapping ${}_{\alpha}\tilde{s}^{-\bar{\omega}}$ satisfies the feasibility constraint. Indeed, for any $q_1 \in \mathcal{P}(\Omega_{11} \setminus \bar{\omega})^1$, since $q_1^{\bar{\omega},\alpha} \in \mathcal{L}(\bar{\omega})$, we get from the feasibility of s that

$$\cup_{i \in N}\ {}_{\alpha}\tilde{s}_i^{-\bar{\omega}}(q_1) = \overbrace{(\underbrace{s_1(q_1^{\bar{\omega},\alpha})}_{\bar{\omega} \in} \setminus \bar{\omega})}^{=\ {}_{\alpha}\tilde{s}_1^{-\bar{\omega}}(q_1)} \cup [\cup_{i \in N \setminus i} \underbrace{s_i(q_1^{\bar{\omega},\alpha})}_{\bar{\omega} \notin}] = \Omega_{11} \setminus \bar{\omega}.$$

Thus, the mapping ${}_{\alpha}\tilde{s}^{-\bar{\omega}}$ is a well-defined $(\Omega_{11} \setminus \bar{\omega})$-assignment rule. Moreover, it is super-strategyproof (because s is), and since $|\Omega_{11} \setminus \bar{\omega}| = K - 1 < K$, our induction hypothesis implies that ${}_{\alpha}\tilde{s}^{-\bar{\omega}}$ is a 1-C-BD union.

Sub-step 4.4. Fix $\bar{\omega} \in \Omega_{11}$. The mapping $\bar{s}^{\bar{\omega}} : \mathcal{L}(\bar{\omega}) \to \mathcal{S}(\Omega_{11} \setminus \bar{\omega})$, defined as the restriction of s to $\mathcal{L}(\bar{\omega})$, is a 1-C-BD union over $\mathcal{L}(\bar{\omega})$. As a consequence, (74) must hold for all $p_1, q_1 \in \mathcal{L}(\bar{\omega})$.

This follows from the combination of Sub-step 4.2 and Sub-step 4.3. Indeed, fix any $\alpha > 1/2$; and note from Sub-step 4.2 that, for all $q_1 \in \mathcal{L}(\bar{\omega})$, we have $\bar{s}^{\bar{\omega}}(q_1) = s(q_1) = s(q_1^{\bar{\omega},\alpha})$ because $q_1^{\bar{\omega},\alpha} \in \mathcal{L}^{q_1}(\bar{\omega})$. That is to say,

$$\bar{s}_1^{\bar{\omega}}(q_1) = \bar{\omega} \cup {}_{\alpha}\tilde{s}_1^{-\bar{\omega}}(q_1 \mid (\Omega_{11} \setminus \bar{\omega})) \text{ and } \bar{s}_i^{\bar{\omega}}(q_1) = {}_{\alpha}\tilde{s}_i^{-\bar{\omega}}(q_1 \mid (\Omega_{11} \setminus \bar{\omega})), \forall i \neq 1. \quad (76)$$

Recalling from Sub-step 4.3 that ${}_{\alpha}\tilde{s}$ is a 1-C-BD union, there exists a partition $\{\Omega_{11}^1, \dots, \Omega_{11}^L\}$ of $\Omega_{11} \setminus \bar{\omega}$ and L Ω^l-assignment rules $s^1, \dots, s^L$ such that ${}_{\alpha}\tilde{s}_i^{-\bar{\omega}}(q_1 \mid (\Omega_{11} \setminus \bar{\omega})) = \cup_{l=1}^L s_i^l(q_1 \mid \Omega_{11}^l)$ and each s^l is constant or $(1, j^l)$-dictatorial for some $j^l \neq 1$. Substituting this in (76) thus gives: for all $q_1 \in \mathcal{L}(\bar{\omega})$ and $i \in N$,

$$\bar{s}_i^{\bar{\omega}}(q_1) = \begin{cases} \cup_{l=1}^L s_i^l(q_1 \mid \Omega_{11}^l) & \text{if } i \neq 1, \\ \bar{\omega} \cup \left(\cup_{l=1}^L s_i^l(q_1 \mid \Omega_{11}^l)\right) & \text{if } i = 1. \end{cases} \quad (77)$$

Observe from (77) that $\bar{s}^{\bar{\omega}}$, the restriction of s to $\mathcal{L}(\bar{\omega})$ is expressed as the union of the $L+1$ sub-rules $s^0, s^1, \dots, s^L$, where s^0 is the constant Ω^0-assignment rule which always assigns $\Omega_{11}^0 := \{\bar{\omega}\}$ to agent 1. This concludes the proof of Sub-step 4.4.

We are now ready to proceed with the proof of Step 4. Since $\mathcal{P}(\Omega_{11})$ is connected, there is a J-path $(\mathbf{p}_1^t)_{t=1}^T$ in $\mathcal{P}(\Omega_{11})$ between any two beliefs $p_1, q_1 \in \mathcal{P}(\Omega_{11})$. If the *length* $T-1$ of this path is equal to 1, then p_1, q_1 are adjacent and the Local Bilaterality lemma implies $s_j(p_1) \cap s_k(q_1) = \emptyset$ for any distinct $j, k \in N \setminus 1$. Next, proceeding by induction, we assume that (74) is true whenever p_1, q_1 are connected by some J-path of length $T' - 1 < T - 1$ (with $T \geq 3$) and we prove that (74) also holds for any p_1, q_1 that are connected by some J-path of length $T - 1$.

By contradiction, suppose that there exist $\omega^* \in \Omega_{11}$ and $p_1'', p_1''' \in \mathcal{P}(\Omega_{11})$ such that, say, $\omega^* \in s_2(p_1'') \cap s_3(p_1''')$ and p_1'', p_1''' are connected by some J-path $\mathbf{q}_1 = (\mathbf{q}_1^t)_{t=1}^T$. Combining the Local bilaterality lemma with our induction hypothesis that (74) holds for all p_1, q_1 that are connected by some J-path of length $T' \leq T - 1$, we obtain

$$w^* \in s_1(\mathbf{q}_1^{T-1}) \setminus s_1(\mathbf{q}_1^T) = s_3(\mathbf{q}_1^T) \setminus s_3(\mathbf{q}_1^{T-1}) \neq \emptyset \quad (78)$$

$$s_i(\mathbf{q}_1^{T-1}) = s_i(\mathbf{q}_1^T), \ \forall i \neq 1, 3 \quad (79)$$

$$s_3(\mathbf{q}_1^{T-1}) \cap s_i(p_1'') = \emptyset, \ \forall i \neq 1, 3. \quad (80)$$

To see why (78) holds, note that $w^* \in s_1(\mathbf{q}_1^{t-1}) \setminus s_1(\mathbf{q}_1^t)$ for some $t \leq T - 1$ would imply a violation of our induction hypothesis on the J-path $\{\mathbf{q}_1^1, \dots, \mathbf{q}_1^t\}$, which is of length $t - 1 < T - 1$. Statement (80) holds for the same reason. Finally, (79) follows from (78) and the Local Bilaterality lemma. In addition, observe that combining (79) and (80) gives

$$s_i(p_1''') \cap s_3(p_1'') = s_i(\mathbf{q}_1^T) \cap s_3(p_1'') = s_i(\mathbf{q}_1^{T-1}) \cap s_3(p_1'') = \emptyset, \ \forall i \neq 1, 3. \quad (81)$$

Sub-step 4.5. There exist $\omega_3 \in s_1(p_1''') \cap s_3(p_1'')$ and $\omega_2 \in s_1(p_1'') \cap s_3(p_1''')$.

To prove Sub-step 4.5, first note that, together, $\omega^* \in s_2(p_1'') \cap s_3(p_1''')$ and the super-strategyproofness of s imply that $p_1'''(s_{N\backslash 3}(p_1''')) > p_1'''(s_{N\backslash 3}(p_1''))$. Thus, there exists $\hat{\omega} \in \Omega_{11}$ such that

$$\hat{\omega} \in s_{N\backslash 3}(p_1''') \setminus s_{N\backslash 3}(p_1'') = s_{N\backslash 3}(p_1''') \cap s_3(p_1''). \tag{82}$$

It thus suffices now to remark that $s_{N\backslash 3}(p_1''') \cap s_3(p_1'') = s_1(p_1''') \cap s_3(p_1'')$. Indeed, given that we have $s_{N\backslash 3}(p_1''') := \cup_{i\neq 3} s_i(p_1''')$, we can write

$$s_{N\backslash 3}(p_1''') \cap s_3(p_1'') = [s_1(p_1''') \cap s_3(p_1'')] \cup [\cup_{i\neq 1,3} \underbrace{(s_i(p_1''') \cap s_3(p_1''))}_{=\emptyset \text{ by } (81)}] = s_1(p_1''') \cap s_3(p_1''').$$

Thus, $\hat{\omega} \in s_{N\backslash 3}(p_1''') \cap s_{N\backslash 3}(p_1'') = s_1(p_1''') \cap s_3(p_1''')$. A symmetric argument shows that there exists $\omega_2 \in s_1(p_1'') \cap s_3(p_1''')$; and this ends the proof of Sub-step 4.4.

Recall from what precedes that $\omega^* \in s_2(p_1'') \cap s_3(p_1''')$, $\omega_3 \in s_1(p_1''') \cap s_3(p_1'')$ and $\omega_2 \in s_1(p_1'') \cap s_3(p_1''')$. The states $\omega^*, \omega_2, \omega_3$ are thus necessarily (pairwise) distinct. We show a few additional sub-steps below.

Fix any $q_1'' \in \mathcal{L}^{p_1''}(\omega_2)$ (see Figure 6) and $q_1''' \in \mathcal{L}^{p_1'''}(\omega_3)$, and define ${}^t q_1''' \in \mathcal{L}(\omega_3)$ by ${}^t q_1'''(\omega_3) = q_1'''(\omega_2)$, ${}^t q_1'''(\omega_2) = q_1'''(\omega_3)$ and ${}^t q_1'''(\omega) = q_1'''(\omega), \forall \omega \neq \omega_2, \omega_3$. In addition, call $\pi_{\omega_3}^{\omega_2}$ the probability measure over Ω_{11} defined by:[10]

$$\pi_{\omega_3}^{\omega_2}(\omega_2) = \pi_{\omega_3}^{\omega_2}(\omega_2) = 1/2; \quad \text{and } \pi_{\omega_3}^{\omega_2}(\omega) = 0 \text{ for all } \omega \neq \omega_2, \omega_3.$$

Define the two sequences $\{q_1^m\}_{m \geq \bar{m}_q}$ and $\{\bar{q}_1^m\}_{m \geq \bar{m}_{\bar{q}}}$ as follows: for any $\omega \in \Omega_{11}$,

$$\begin{aligned} q_1^m(\omega) =& \frac{1}{m} q_1''' + (1 - \frac{1}{m}) \pi_{\omega_3}^{\omega_2}; \\ \bar{q}_1^m(\omega) =& \frac{1}{m}\ {}^t q_1''' + (1 - \frac{1}{m}) \pi_{\omega_3}^{\omega_2}. \end{aligned} \tag{83}$$

Figure 6 gives an illustration of the construction of the beliefs $q_1^m, \bar{q}^m$ starting from $p_1'' \in \mathcal{L}(\omega_2)$. It is important to remark that, by definition, we have $q_1^m \in \mathcal{L}(\omega_2)$ and $\bar{q}_1^m \in \mathcal{L}(\omega_3)$.[11]

Sub-step 4.6. There exist $\tilde{m} \in \mathbb{N}$ (with $\tilde{m} \geq \bar{m}_q, \bar{m}_{\bar{q}}$) and $\mathbf{A}, \bar{\mathbf{A}} \in \mathcal{S}(\Omega_{11})$ such that

$$[m \geq \tilde{m}] \Rightarrow [s(q_1^m) = \mathbf{A} \text{ and } s(\bar{q}_1^m) = \bar{\mathbf{A}}].$$

The proof of Sub-step 4.6 is similar to that of Lemma 3-(i), and therefore left to the reader.

[10] Obviously, $\pi_{\omega_3}^{\omega_2}$ is not an injective probability measure (i.e., $\pi_{\omega_3}^{\omega_2} \notin \mathcal{P}(\Omega_{11})$); but this does not affect the validity of our upcoming argument —which is based on the study of sequences of injective probability measures that converge to $\pi_{\omega_3}^{\omega_2}$.

[11] There may exist only a finite number of integers m such that $q_1^m, \bar{q}_1^m$ are not injective; and this issue is taken care of by conveniently starting the sequence at a rank $\bar{m}_q$ (or $\bar{m}_{\bar{q}}$) that is higher than any such integer.

Sub-step 4.7. For any $m \geq \tilde{m}$, we have $\omega^* \in s_2(q_1^m)$; and it follows that $\mathbf{A} \neq \bar{\mathbf{A}}$. We showed in Sub-step 4.4 that $\bar{s}^{\omega_2}$, the restriction of s to $\mathcal{L}(\omega_2)$, can be written as

$$\bar{s}_i^{\omega_2}(q_1) = \begin{cases} \cup_{l=1}^L s_i^l(q_1 \mid \Omega_{11}^l) & \text{if } i \neq 1, \\ \omega_2 \cup \left(\cup_{l=1}^L s_i^l(q_1 \mid \Omega_{11}^l)\right) & \text{if } i = 1, \end{cases} \tag{84}$$

where each s^l is constant or $(1, j^l)$-dictatorial for some $j^l \neq 1$. Call $\Omega_{11}^{w^*}$ the unique event in the partition $\{\underbrace{\Omega_{11}^0}_{=\{\omega_2\}}, \Omega_{11}^1, \ldots, \Omega_{11}^L\}$ of Ω_{11} such that $\omega^* \in \Omega_{11}^{w^*}$. Since $q_1'' \in \mathcal{L}^{p_1''}(\omega_2) \subset \mathcal{L}(\omega_2)$, it follows from Sub-step 4.2 that $\omega^* \in s_2(p_1'') = s_2(q_1'') = \bar{s}_2^{\omega_2}(q_1'')$; and we may then conclude from (84) that $j^{\omega^*} = 2$ and s^{ω^*} is (1,2)-dictatorial over $\Omega_{11}^{\omega^*}$. We get in the same way that $j^{\omega_3} = 3$ and s^{ω_3} is (1,3)-dictatorial over $\Omega_{11}^{\omega_3}$. It thus follows that $\omega_3, \omega_2 \notin \Omega_{11}^{\omega^*}$ –obviously, $\omega^2 \notin \Omega_{11}^{\omega^*}$ since $\Omega_{11}^0 = \{\omega_2\}$. Using (84) and the fact that s^{ω^*} is (1,2)-dictatorial, we may assert that $\omega^* \in s_2(q_1)$ for any $q_1 \in \mathcal{L}(\omega_2)$ such that $q_1 \mid \Omega_{11}^{\omega^*} = q_1'' \mid \Omega_{11}^{\omega^*}$. One can then see that $\omega^* \in s_2(q_1^m)$ by combining (83) and $\omega_2, \omega_3 \notin \Omega_{11}^{\omega^*}$ to deduce that we indeed have: $q_1^m \mid \Omega_{11}^{\omega^*} = q_1'' \mid \Omega_{11}^{\omega^*}$, for all $m \geq \bar{m}_q$.

We conclude the proof of Sub-step 4.7 by noting that we necessarily have $\mathbf{A} \neq \bar{\mathbf{A}}$. Indeed, since $\tilde{m} \geq \bar{m}_q$, we have $\omega^* \in A_2 = s_2(q_1^{\tilde{m}})$. Assuming that $\mathbf{A} = \bar{\mathbf{A}}$ would thus give $\omega^* \in \mathbf{A}_2 = \bar{\mathbf{A}}_2 = s_2(\bar{q}^{\tilde{m}})$. But this would contradict the fact that $\bar{s}^{\omega_3}$ is a 1-C-BD union over $\mathcal{L}(\omega_3)$ (established in Sub-step 4.4), which requires (74) to hold for $\bar{q}^{\tilde{m}}, q_1''' \in \mathcal{L}(\omega_3)$ —recall that $\omega^* \in s_3(q_1''')$.

Sub-step 4.8. There exist disjoint subsets $E, \bar{E} \subset \Omega \setminus \{\omega_2, \omega_3, \omega^*\}$ such that

$$\begin{aligned} \mathbf{A}_1 \setminus \bar{\mathbf{A}}_1 &= \omega_2 \cup E = \bar{\mathbf{A}}_3 \setminus \mathbf{A}_3, \\ \bar{\mathbf{A}}_1 \setminus \mathbf{A}_1 &= \omega_3 \cup \bar{E} = \mathbf{A}_3 \setminus \bar{\mathbf{A}}_3, \\ \mathbf{A}_i &= \bar{\mathbf{A}}_i \text{ for all } i \neq 1, 3. \end{aligned}$$

We start the proof of Sub-step 4.8 by noting that: $\exists \hat{m} \geq \tilde{m}$ such that, for any $\{F, \bar{F}\} \in \mathcal{H}$ and any $m \geq \hat{m}$, $\left[w_2 \notin F \text{ or } \omega_3 \notin \bar{F}\right] \Rightarrow \left[(q_1^m(F) - q_1^m(\bar{F}))(\bar{q}_1^m(F) - \bar{q}_1^m(\bar{F})) > 0\right]$. This implication holds by construction since $\lim_{m\to\infty} q_1^m = \lim_{m\to\infty} \bar{q}_1^m = \pi_{\omega_3}^{\omega_2}$ and $\pi_{\omega_3}^{\omega_2}(\omega_2) = \pi_{\omega_3}^{\omega_2}(\omega_3) = 1/2$. In words: when m is large enough, the segment $[q_1^m, \bar{q}_1^m]$ cuts only hyperplanes $\{F, \bar{F}\} \in \mathcal{H}$ such that $\omega_2 \in F$ and $\omega_3 \in \bar{F}$ (see Figure 7), and $q_1^m, \bar{q}_1^m$ are on the same side of all other hyperplanes.

Second, recall from (83) that $q_1^m \mid (\Omega_{11} \setminus \{\omega_2, \omega_3\}) = \bar{q}_1^m \mid (\Omega_{11} \setminus \{\omega_2, \omega_3\}) = q_1'' \mid (\Omega_{11} \setminus \{\omega_2, \omega_3\})$, for any $m \geq \hat{m}$. It hence follows that the set of hyperplanes of the form $\{\omega_2 \cup E, \omega_3 \cup \bar{E}\}$ is totally ordered along the segment $[q_1^{\hat{m}}, \bar{q}_1^{\hat{m}}]$. Calling T the number of such hyperplanes, we may thus write

$$\{\{F, \bar{F}\} \in \mathcal{H} \mid F = \omega_2 \cup E, \bar{F} = \omega_3 \cup \bar{E}\} = \{\{\omega_2 \cup E_1, \omega_3 \cup \bar{E}_1\}, \ldots, \{\omega_2 \cup E_T, \omega_3 \cup \bar{E}_T,\}\},$$

where E^t $[t = 1, \ldots, T]$ is the t^{th} hyperplane cut on the way from $q_1^{\hat{m}}$ to $\bar{q}_1^{\hat{m}}$. Using this notation, we may then consider a J-path $\{\mathbf{p}_1^t\}_{t=1}^{T+1}$ satisfying the properties: (i)

$\mathbf{p}_1^1 = q_1^{\hat{m}}$, $\mathbf{p}_1^{T+1} = \bar{q}_1^{\hat{m}}$; (ii) $\mathbf{p}_1^t$ and $\mathbf{p}_1^{t+1}$ are $\{\omega_2 \cup E_t, \omega_3 \cup \bar{E}_t\}$-adjacent for any $t = 1, \ldots, T$.

We conclude the proof of Sub-step 4.8 by showing that there exists a unique $t^* \in \{1, T\}$ such that: (a) $s(\mathbf{p}_1^t) = s(q_1^{\hat{m}}), \forall t \in \{1, \ldots, t^*\}$ and (b) $s(\mathbf{p}_1^t) = s(\bar{q}_1^{\hat{m}}), \forall t \in \{t^* + 1, \ldots, T + 1\}$. First, note that the assignment may change only once along the J-path $\mathbf{p}$. Indeed, if $s(\mathbf{p}_1^{t^*}) \neq s(\mathbf{p}_1^{t^*+1})$ then we get from the Local Bilaterality lemma that $s_1(\mathbf{p}_1^{t^*}) \setminus s_1(\mathbf{p}_1^{t^*+1}) = \omega_2 \cup E_{t^*}$; and (given that $\omega_2 \notin s_1(\mathbf{p}_1^{t^*+1})$), the Local Bilaterality lemma requires that $s(\mathbf{p}_1^t) = s(\bar{q}_1^{\hat{m}}), \forall t \in \{t^* + 1, \ldots, T + 1\}$.

Second, recall from Sub-step 4.7 (and $\hat{m} \geq \tilde{m}$) that $s(q_1^{\hat{m}}) = \mathbf{A} \neq \bar{\mathbf{A}} = s(\bar{q}_1^{\hat{m}})$. Hence, there must indeed exist a unique $t^* \in \{1, \ldots, T\}$ such that $s(\mathbf{p}_1^{t^*}) \neq s(\mathbf{p}_1^{t^*+1})$. The Local Bilaterality lemma, applied to the adjacent beliefs $\mathbf{p}_1^{t^*}, \mathbf{p}_1^{t^*+1}$, then gives the desired result: $\mathbf{A}_1 \setminus \bar{\mathbf{A}}_1 = \omega_2 \cup E_{t^*} = \bar{\mathbf{A}}_3 \setminus \mathbf{A}_3$; $\bar{\mathbf{A}}_1 \setminus \mathbf{A}_1 = \omega_3 \cup \bar{E}_{t^*} = \mathbf{A}_3 \setminus \bar{\mathbf{A}}_3$; $\mathbf{A}_i = \bar{\mathbf{A}}_i, \forall i \neq 1, 3$. Recalling from Sub-step 4.7 that $\omega^* \in s_2(q_1^{\hat{m}}) = \mathbf{A}_2$, we obtain that $E_{t^*}, \bar{E}_{t^*} \subset \Omega \setminus \{\omega_2, \omega_3, \omega^*\}$.

We are finally ready to clinch the proof of Step 4. We have shown in Sub-step 4.8 that $\omega^* \in s_2(q_1^{\hat{m}}) = \mathbf{A}_2 = \bar{\mathbf{A}}_2 = s_2(\overbrace{\bar{q}_1^{\hat{m}}}^{\in \mathcal{L}(\omega_3)})$. But this is a contradiction given that $\omega^* \in s_3(\overbrace{q_1'''}^{\in \mathcal{L}(\omega_3)})$. Indeed, this violation of (74) contradicts the fact that (the restriction to $\mathcal{L}(\omega_3)$ of) s is a 1-C-BD union over $\mathcal{L}(\omega_3)$ —which was established in Sub-step 4.4. Thus, it never holds that $\omega^* \in s_j(p_1'') \cap s_k(p_1''')$ for any ω^*, p_1'', p_1''' and distinct $j, k \neq 1$. Given that s is not constant on $\mathcal{P}(\Omega_{11})$, for any $\omega \in \Omega_{11}$, we thus have, $a_\omega(\mathcal{P}(\Omega_{11})) = \{1, j\}$ for some $j \neq 1$.

Step 5. We show that for every $\omega \in \Omega_{11}$ there is a unique $j \neq 1$ such that $a_\omega(\mathcal{P}^N) = \{1, j\}$.

Let $\omega \in \Omega_{11}$. By Step 4, there is a unique $j \neq 1$ such that $a_\omega(\mathcal{P}_{11} \times \mathcal{P}^{N \setminus 1}) = \{1, j\}$. We claim that $a_\omega(\mathcal{P}^N) = \{1, j\}$. Suppose, by contradiction, that there exists some $k \neq 1, j$ and some $p \in \mathcal{P}^N$ such that $\omega \in s_k(p)$. Drop p_{-1} from the notation. Consider an Ω_{11}-dominant belief $p_1^* \in \mathcal{P}_{11}$ such that $p_1^* \mid \Omega_{11} = p_1 \mid \Omega_{11}$ and $p_1^* \mid \overline{\Omega}_{11} = p_1 \mid \overline{\Omega}_{11}$. Such a belief can be constructed by taking α close to 1 in (72). Since $a_\omega(\mathcal{P}_{11} \times \mathcal{P}^{N \setminus 1}) = \{1, j\}$, we have $\omega \notin s_k(p_1^*)$. By Step 2, $s_1(p_1) \cap \Omega_{11} = s_1(p_1^*) \cap \Omega_{11}$. By (71), $s_1(p_1) = s_1(p_1^*)$. By non-bossiness, $s(p_1) = s(p_1^*)$, contradicting $\omega \in s_k(p_1) \setminus s_k(p_1^*)$ and completing Step 5.

For every $j \neq 1$, define $\Omega_{11}^j = \left\{\omega \in \Omega_{11} : a_\omega(\mathcal{P}^N) = \{1, j\}\right\}$. Let $N_1 = \{j \in N \setminus 1 : \Omega_{11}^j \neq \emptyset\}$. By definition, $\left\{\Omega_{11}^j : j \in N_1\right\}$ is a partition of Ω_{11}. For each $j \in N_1$, let

$$\mathcal{A}_{11}^j = \left\{A^j \subseteq \Omega_{11}^j : \exists p \in \mathcal{P}^N \text{ such that } s_1(p) \cap \Omega_{11}^j = A^j\right\}.$$

Step 6. We show that $\mathcal{A}_{11}$ is a product family. Namely, for any collection of events $\{A^j : j \in N_1\}$,

$$\left[A^j \in \mathcal{A}_{11}^j \text{ for all } j \in N_1\right] \Rightarrow \left[\cup_{j \in N_1} A^j \in \mathcal{A}_{11}\right].$$

Suppose $A^j \in \mathcal{A}^j_{11}$ for all $j \in N_1$ and write $N_1 = \{2, ..., n_1\}$. Call a belief p_1 *lexicographically* $(\Omega^2_{11}, ..., \Omega^{n_1}_{11})$*-dominant* if $|p_1(A) - p_1(B)| > |p_1(A') - p_1(B')|$ for all distinct $A, B \subset \Omega^j_{11}$, all $A', B' \subset \Omega \setminus (\cup^j_{k=1}\Omega^k_{11})$, and all $j = 2, ..., n-1$. Consider a lexicographically $(\Omega^2_{11}, ..., \Omega^{n_1}_{11})$-dominant belief p_1 such that

$$\underset{\mathcal{A}^j_{11}}{\operatorname{argmax}}\, p_1 = A^j$$

for all $j = 2, ..., n-1$. Fix $p_{-1} \in \mathcal{P}^{N\setminus 1}$ and drop it from the notation.

Strategyproofness implies

$$s_1(p_1) \cap \Omega^2_{11} = A^2.$$

This is because there is some q_1 such that $s_1(q_1) \cap \Omega^2_{11} = A^2$, $\underset{\mathcal{A}^2_{11}}{\operatorname{argmax}}\, p_1 = A^2$, and p_1 is Ω^2_{11}-dominant.

Next, proceed inductively. Suppose we have shown that $s_1(p_1) \cap \Omega^j_{11} = A^j$ for $j = 2, ..., k-1$. We claim that

$$s_1(p_1) \cap \Omega^k_{11} = A^k. \tag{85}$$

Since $A^k \in \mathcal{A}^k_{11}$, there is some q_1 such that $s_1(q_1) \cap \Omega^k_{11} = A^k$. If $s_1(p_1) \cap \Omega^k_{11} = B^k \neq A^k$, then

$$\begin{aligned}
p_1(s_{\{1,...,k-1\}}(p_1) \cap (\cup^k_{j=2}\Omega^j_{11})) &= p_1(\cup^{k-1}_{j=2}\Omega^j_{11} \cup B^k) \\
&< p_1(\cup^{k-1}_{j=2}\Omega^j_{11} \cup A^k) \\
&= p_1(s_{\{1,...,k-1\}}(q_1) \cap (\cup^k_{j=2}\Omega^j_{11})),
\end{aligned}$$

contradicting super-strategyproofness and proving (85).

We conclude that $s_1(p_1) \cap \Omega^j_{11} = A^j$ for all $j \in N_1$, which implies that $s_1(p_1) \cap \Omega_{11} = \cup_{j \in N_1} A^j$, hence $\cup_{j \in N_1} A^j \in \mathcal{A}_{11}$.

Step 7. Step 6 ensures that $\underset{\mathcal{A}_{11}}{\operatorname{argmax}}\, p_1 = \cup_{j \in N_1} \underset{\mathcal{A}^j_{11}}{\operatorname{argmax}}\, p_1$ for all $p_1 \in \mathcal{P}$. Combining this with Step 2,

$$s_1(p) \cap \Omega_{11} = \cup_{j \in N_1} \underset{\mathcal{A}^j_{11}}{\operatorname{argmax}}\, p_1$$

for all $p \in \mathcal{P}^N$. Defining for each $j \in N_1$ the $(1, j)$-dictatorial Ω^j_{11}-assignment rule s^j by

$$s^j_i(\widetilde{p}) = \begin{cases} \underset{\mathcal{A}^j_{11}}{\operatorname{argmax}}\, \widetilde{p}_1 & \text{if } i = 1, \\ \Omega^j_{11} \setminus \underset{\mathcal{A}^j_{11}}{\operatorname{argmax}}\, \widetilde{p}_1 & \text{if } i = j, \\ \emptyset & \text{if } i \neq 1, j \end{cases}$$

for all $\widetilde{p} \in \mathcal{P}(\Omega_{11}^j)^N$, statement (69) holds for $p \in \mathcal{P}^N$ and $i \in N$.

To complete the proof, it only remains to check that $\mathcal{A}_{11}^j$ is a proper covering of Ω_{11}^j for every $j \in N_1$.

Fix $j \in N_1$. To check that $\cup_{A^j \in \mathcal{A}_{11}^j} A^j = \Omega_{11}^j$, fix $\omega \in \Omega_{11}^j$. Since, by definition of Ω_{11}^j, $a_\omega(\mathcal{P}^N) = \{1, j\}$, there is some $p \in \mathcal{P}^N$ such that $\omega \in s_1(p)$, hence some $A^j \in \mathcal{A}_{11}^j$ such that $\omega \in A^j$.

To check that $A^j \setminus B^j \neq \emptyset$ for all distinct $A^j, B^j \in \mathcal{A}_{11}^j$, suppose on the contrary that $A^j \subset B^j$. By Step 6, this implies that there exist $A, B \in \mathcal{A}_{11}$ such that $A \subset B$. But by definition of $\mathcal{A}_{11}$ and Step 1, there is some p such that $A = \underset{\mathcal{A}_{11}}{\operatorname{argmax}}\, p_1$, contradicting the fact that $p_1(A) < p_1(B)$.

To check that $\cap_{A^j \in \mathcal{A}_{11}^j} A^j = \emptyset$, suppose on the contrary that $\omega \in \cap_{A^j \in \mathcal{A}_{11}^j} A^j$. Then $\omega \in s_1(p)$ for all $p \in \mathcal{P}^N$, contradicting the fact that $a_\omega(\mathcal{P}^N) = \{1, j\}$. □

12 Appendix E: Figures

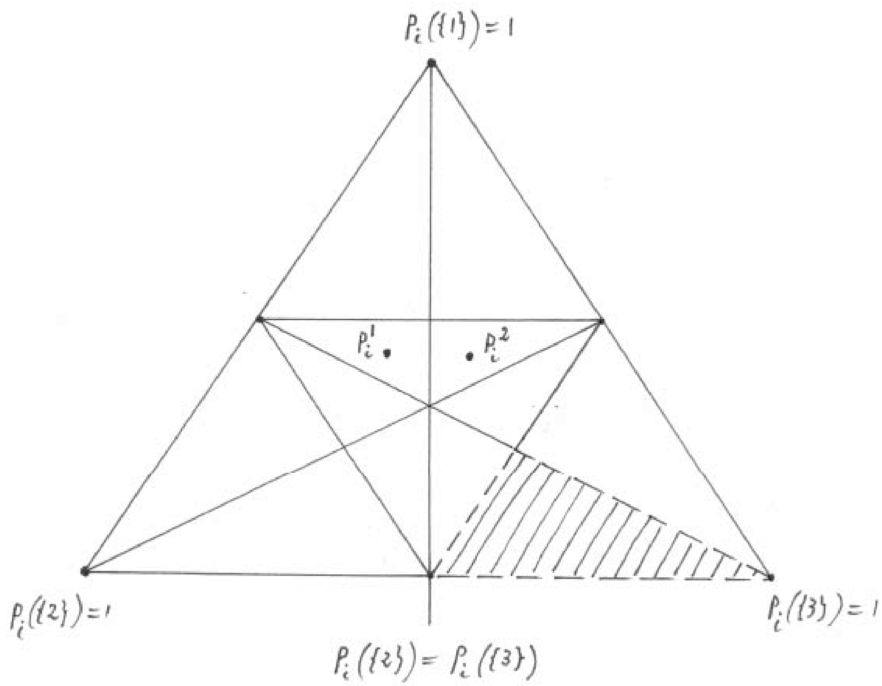


Figure 1: Beliefs, likelihood orderings, and adjacency

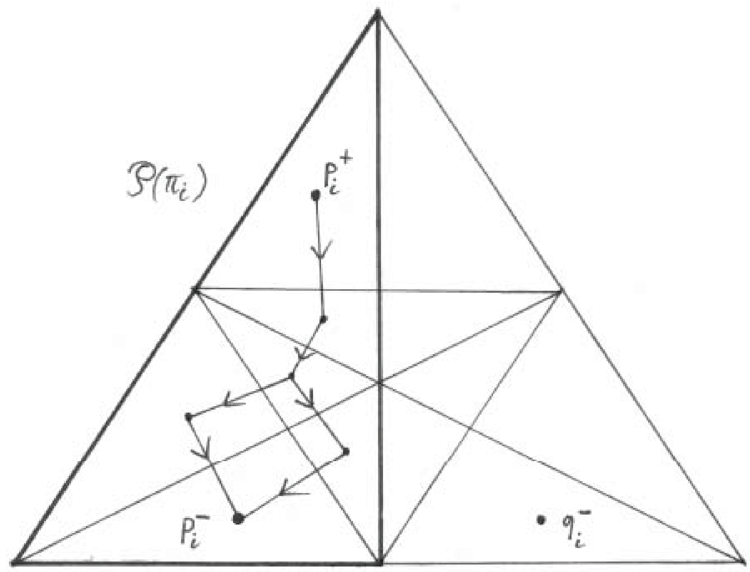


Figure 2: The binary relation $\widetilde{J}$

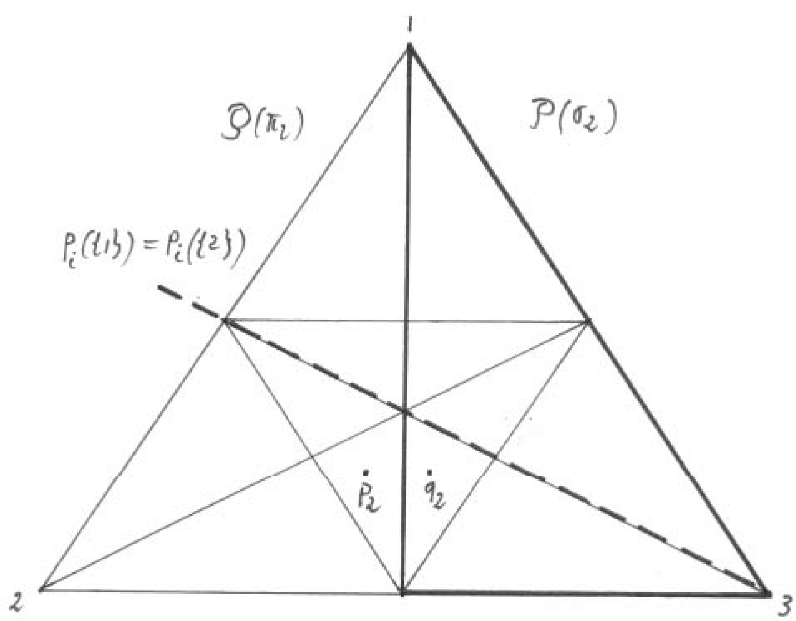


Figure 3: Illustration of the proof of the first contagion lemma

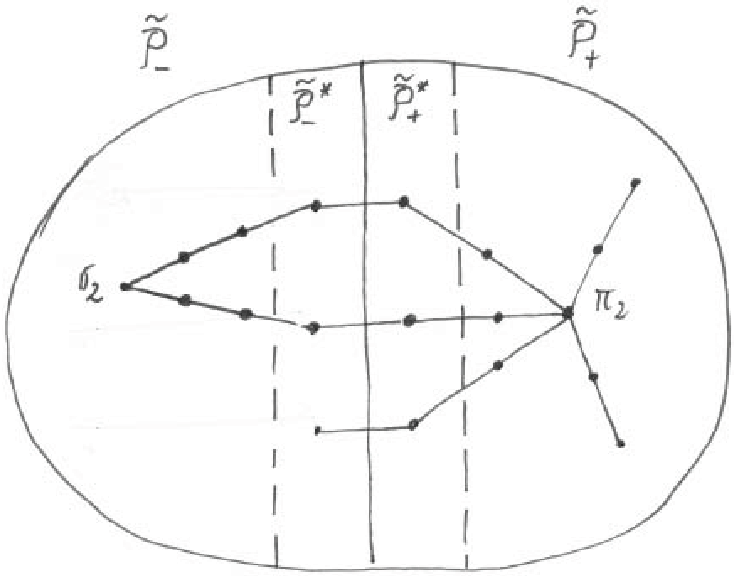


Figure 4: Illustration of the proof of the first contagion corollary

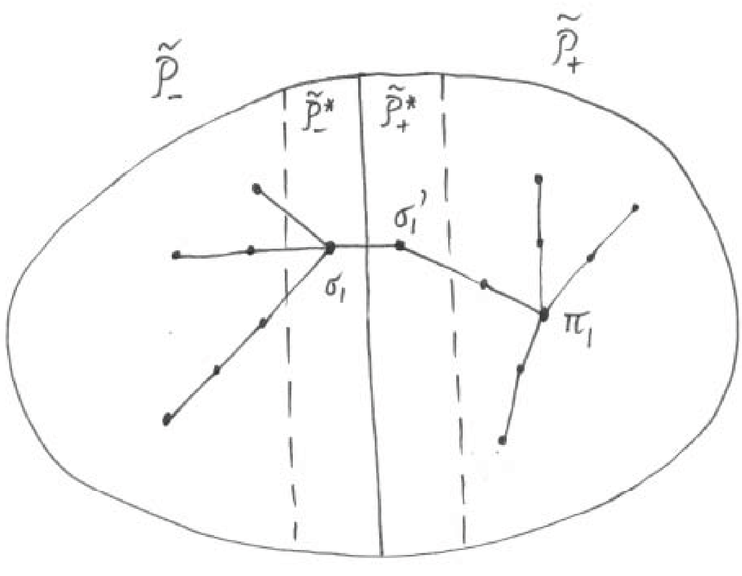


Figure 5: Illustration of the proof of the second contagion corollary

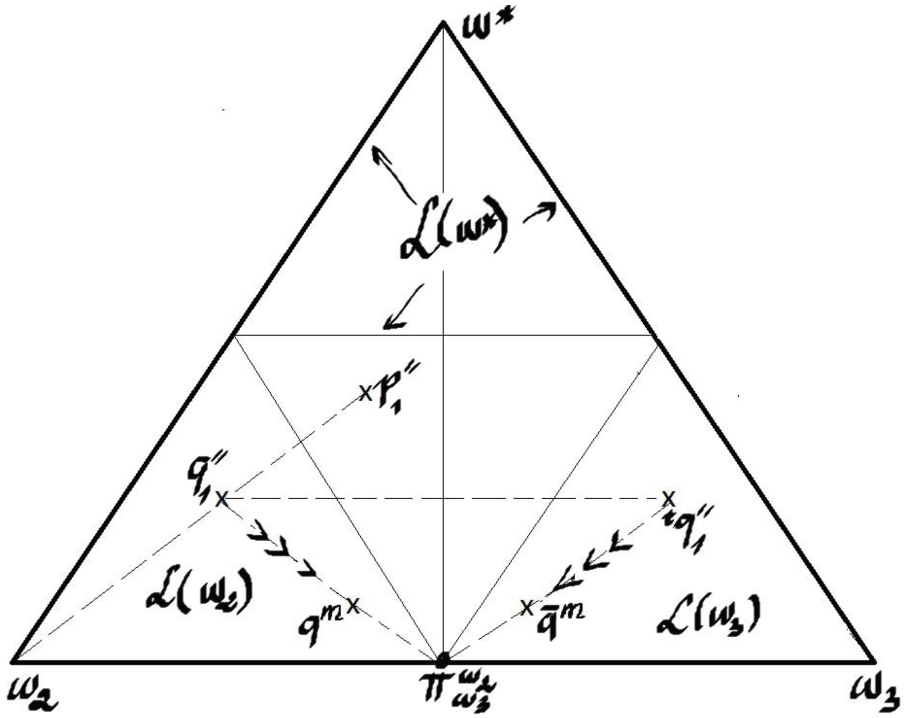
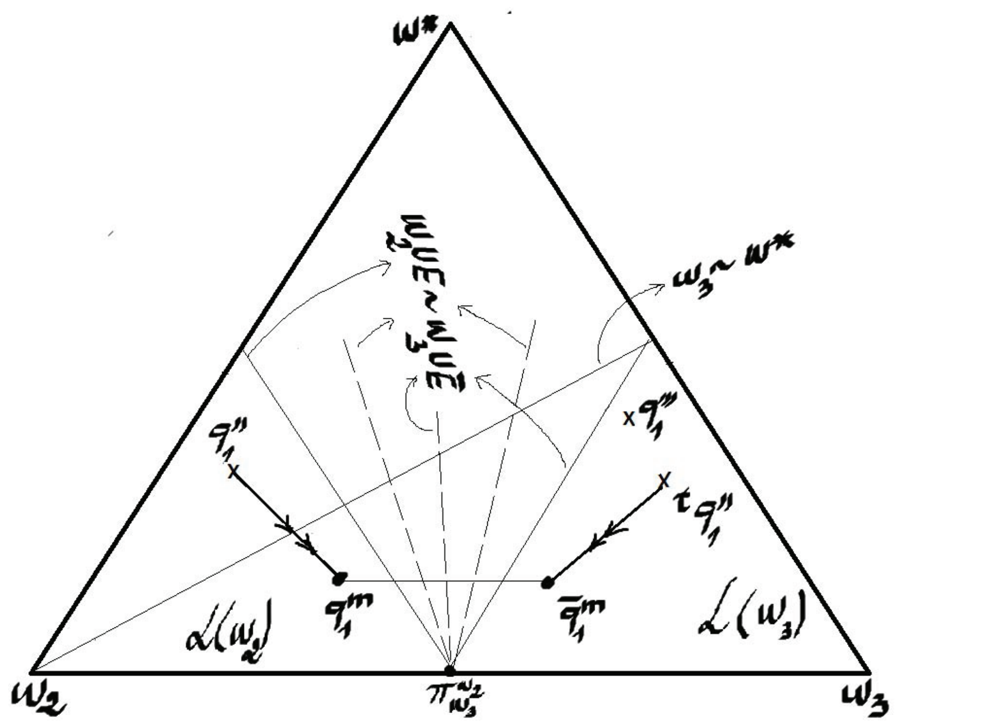


Figure 6: Construction of q_1^m and $\bar{q}_1^m$.



For m large, $[q_1^m, \bar{q}_1^m]$ cuts only hyperplanes of the form $\{\omega_2 \cup E, \omega_3 \cup \bar{E}\}$.
Note in this example that $[q_1'', q_1''']$ — but not $[q_1^m, \bar{q}_1^m]$– cuts $\{\omega_3, \omega^*\} \in \mathcal{H}$.

Figure 7: Hyperplanes cut by $[q_1^m, \bar{q}_1^m]$.