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Solvency considerations in the gamma-omega surplus model

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RÉSUMÉ

Ce mémoire de maîtrise traite de la théorie de la ruine, et plus spécialement des modèles actuariels avec surplus dans lesquels sont versés des dividendes. Nous étudions en détail un modèle appelé modèle $\gamma - \omega$, qui permet de jouer sur les moments de paiement de dividendes ainsi que sur une ruine non-standard de la compagnie. Plusieurs extensions de la littérature sont faites, motivées par des considérations liées à la solvabilité. La première consiste à adapter des résultats d'un article de 2011 à un nouveau modèle modifié grâce à l'ajout d'une contrainte de solvabilité. La seconde, plus conséquente, consiste à démontrer l'optimalité d'une stratégie de barrière pour le paiement des dividendes dans le modèle $\gamma - \omega$. La troisième concerne l'adaptation d'un théorème de 2003 sur l'optimalité des barrières en cas de contrainte de solvabilité, qui n'était pas démontré dans le cas des dividendes périodiques. Nous donnons aussi les résultats analogues à l'article de 2011 en cas de barrière sous la contrainte de solvabilité. Enfin, la dernière concerne deux différentes approches à adopter en cas de passage sous le seuil de ruine. Une liquidation forcée du surplus est mise en place dans un premier cas, en parallèle d'une liquidation à la première opportunité en cas de mauvaises prévisions de dividendes. Un processus d'injection de capital est expérimenté dans le deuxième cas. Nous étudions l'impact de ces solutions sur le montant des dividendes espérés. Des illustrations numériques sont proposées pour chaque section, lorsque cela s'avère pertinent.

Mots-clés : Dividendes périodiques, optimalité, équation de Hamilton-Jacobi-Bellman, liquidation, injections de capital, ruine oméga, lemme de vérification

ABSTRACT

This master thesis is concerned with risk theory, and more specifically with actuarial surplus models with dividends. We focus on an important model, called $\gamma - \omega$ model, which is built to enable the study of both periodic dividend distributions and a non-standard type of ruin. We make several new extensions to this model, which are motivated by solvency considerations. The first one consists in adapting results from a 2011 paper to a new model built on the assumption of a solvency constraint. The second one, more elaborate, consists in proving the optimality of a barrier strategy to pay dividends in the $\gamma - \omega$ model. The third one deals with the adaptation of a 2003 theorem on the optimality of barrier strategies in the case of solvency constraints, which was not proved right in the periodic dividend framework. We also give analogous results to the 2011 paper in case of an optimal barrier under the solvency constraint. Finally, the last one is concerned with two non-traditional ways of dealing with a ruin event. We first implement a forced liquidation of the surplus in parallel with a possibility of liquidation at first opportunity in case of bad prospects for the dividends. Secondly, we deal with injections of capital into the company reserve, and monitor their implications to the amount of expected dividends. Numerical illustrations are provided in each section, when relevant.

Key-words : Periodic dividends, optimality, Hamilton-Jacobi-Bellman equation, liquidation, capital injections, omega ruin, verification lemma

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INTRODUCTION

In this thesis, we aim at discussing several issues about solvency of insurance companies. We develop three chapters, related to three different situations. Each chapter has its own original contribution. The framework is the following.

In risk theory, we aim at studying ruin for companies that hold a stochastic capital. We use a simple model for the capital (we try to develop a model accurate enough to provide good insights about the real behaviour of an actual company) and monitor it as a function of time. Ruin plays a crucial role as it is the first parameter we have to define in this model. We use a special type of ruin to distinguish simple ruin from bankruptcy. This comes from the fact that some companies keep doing business although technically ruined (state-owned companies for example, but not only). This special type of ruin is called $\omega-$ruin.

The danger zone, which is the area between the threshold of simple ruin and the bankruptcy value, begins below a deterministic horizontal threshold that we denote by a_1, and which is called a solvency constraint. In the framework of insurance companies, it can be seen as the liquidation value and reprensents the price to pay to tranfert the portfolio to another company when the business closes. The threshold of bankruptcy is set at 0, and the area between 0 and a_1 is then the danger zone, where the risk of bankruptcy is non-negligeable.

When the capital of the company hits a_1 and enters this danger zone, we say that the company goes through an $\omega-$event because we add an ω function to this area, where the ω function is positive and increases with ruin, which represents the probability of going bankrupt. One of the main goals of this thesis is to study two possible solvency outcomes for the business in case of an $\omega-$event. The shareholders are expected to make a major move : either they are forced to liquidate the business, or they are forced to inject capital into it, because a capital under a_1 means that the company does not hold enough money to keep doing business as usual.

The priority of this work is to discuss dividends in this context. They will be assumed periodic, which means that they can only be paid at some discrete decision times. Here the decision times are random, because determined by inter-exponential distributions. A parameter $\gamma > 0$ is associated to the exponential decision times (and the mean is then $1/\gamma$). A lot of situations have already been studied in the case where dividends are paid continuously, but periodic dividends are more realistic and also more recent. An exponential distribution is a first step towards Erlang distributions, which provide even more realistic decision times because they can lead to computing deterministic intervals of decisions. The dividends can only be paid if the capital is above a determined amount of cash we call b. There are a few possibilities for where b is set. The simplest case would be to set it above a_1, so it creates three different areas in the model. This will be discussed in Chapter 2. Dividend distribution is allowed at decision times, provided that the capital is above b and also above a_1. No dividend can be paid below a_1. Besides $[0, a_1)$ and $[b, +\infty)$, $b > a_1$ creates a third area : $[a_1, b)$. This is considered as the middle part and has nothing special. Dividend payment is not allowed because it is below b and the ω coefficient is 0 because it is above a_1. It then looks like this :

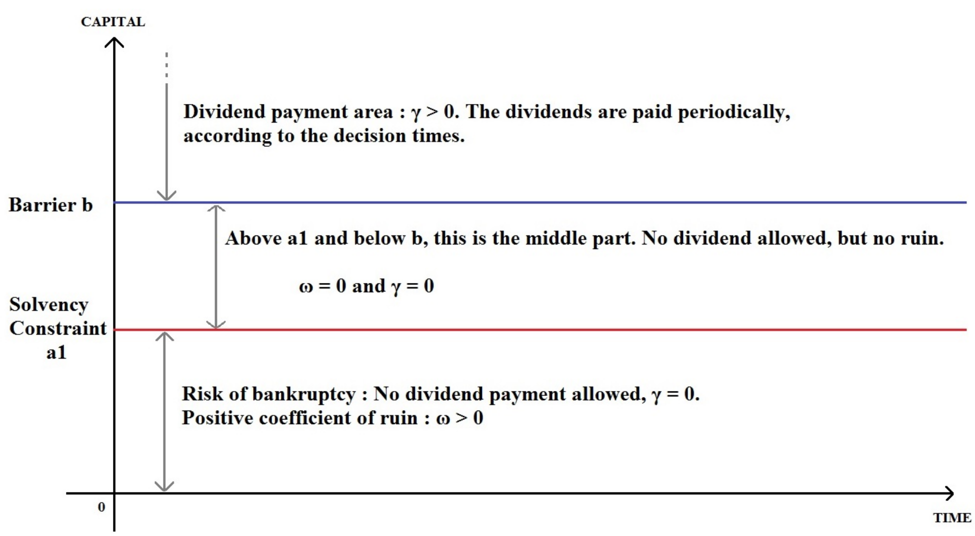


FIGURE 0.1. Provided $b > a_1$, this figure illustrates the $\gamma - \omega$ model.

We are also interested in what happens for example when $b \in [0, a_1)$. The parameter b is not chosen arbitrarily. We can find an optimal b, denoted b^* that maximizes the aggregate amount of dividends function paid until ruin and we explain in details how to obtain it in Chapter 3. We compute the function for any b, then look at the maximal one, called value function, under b^*.

This thesis is concerned with the different decisions that the shareholders can make, which happen in case of a surplus going below $a1$. In Chapter 4, We study two cases. The first one is concerned with a forced liquidation outcome : the shareholders are forced to close the company by an external regulator as soon as the capital reaches a_1. The second one is another option for the shareholders : they are forced to inject capital up to a_1 in the business to make the difference, with a penalty κ proportional to the amount of capital injected. In that case, we assume that because the shareholders are forced to inject capital, then the business can never close. This case is the last one we study and is an opening towards further models because this is not an ω model since bankruptcy cannot happen anymore.

Both considerations have a liquidation at first opportunity outcome. Indeed, when confronted to the possibility of a forced liquidation of their business, the shareholders can take the lead and liquidate the remaining surplus at the first decision time they get. We study this possibility in case of bad dividend prospects for the company. In that case, if liquidation at first opportunity is triggered, the shareholders share the difference between the surplus and a_1 as a final dividend, then the business is closed by the regulator because the surplus is brought back to a_1, which triggers bankruptcy, hence the term “liquidation”. In the second case with capital injections, in case of bad prospects, the shareholders can also make the surplus go to a_1 thanks to a final dividend or a final injection and close the business.

Main contributions

This thesis provides extensions to [Albrecher, Gerber, and Shiu, 2011]. In this paper, the ω ruin is introduced for the first time, according to the assumption of a company that keeps doing business as usual although ruined. In this paper, [Albrecher et al., 2011] start by providing the equations of the surplus, then obtain the explicit solutions to these equations, and finally provide the optimal dividend barrier in the γ periodic dividend framework.

The work of Avanzi, Tu, and Wong [2014] does not involve this alternative definition of ruin but focuses on a periodic dividend framework to provide general answers to the question of optimality of strategy in such models. Like in [Albrecher et al., 2011], the equations of the model are solved (in a case of a jump / diffusion instead of a pure diffusion). A powerful theorem is then developed to ensure a barrier strategy is optimal. Liquidation at first opportunity is also discussed.

One of our goals is to link these two papers together by implementing the ω monioring parameter of [Albrecher et al., 2011] into the work of [Avanzi et al., 2014], for the pure diffusion case.

The former paper already has an extension, published in 2012. In this one, [**?**] study the impact of ω as a parameter that determines an ultimate penalty at time of bankruptcy, as well as the distribution in the "red". The penalty component will not be discussed in this thesis and, consequently, we do not provide other references on that penalty framework.

To the best of my knowledge, monitoring a dual event liquidation / capital injections has never been done in the field of periodic dividends. Also, the idea of monitoring such model using the $\omega-$ruin is new. This type of ruin has been described in 2011 and 2012 in two papers, including one about a final penalty, but none of them involve liquidation at first opportunity, forced liquidation and /or capital injections. The verification lemma developed in the optimality section for periodic dividends has been explored in 2014 but never including the $\omega-$ruin parameter. This thesis is concerned with adapting it in the case of $\omega-$ruin. In fact, the $\omega-$ruin has not been the matter of a lot of papers despite its sufficient accuracy to describe a basic regulator, because of the complication it brings to the model. However, it brings a lot more realism to the model. Each time the barrier or the capital is below the constraint, a lot of different cases happen that need to be dealt with (and are in this thesis) and that creates unecessary dichotomy for papers that are not primarily focused on this particular issue. Last but not least, a barrier strategy in a simple $\gamma - \omega$ model had never been proven optimal before this thesis. We aim at filling important gaps, that some papers take for granted whereas it is not obvious that it is the case.

Chapitre 1

LITERATURE REVIEW

The literature on dividend-related problems in actuarial research is vast. We focus on the area that deals with our issues. In the framework of dividend distribution, a review has been done by [Avanzi, 2009] and another review more focused on optimality-related issues is [Albrecher and Thonhauser, 2009]. At the time, the research stream about periodic dividends didn't exist, so we need to review a lot of other references.

1.1. The surplus and first definitions of dividends

Our goal is to model the capital of insurance companies or any company whose capital variations fit the following description

$$U(t) = u + \mu t + \sigma W_t. \tag{1.1.1}$$

$U(t)$ is called the stochastic surplus of the company, u is the initial cash reserve held by the company at time $t = 0$, μ represents the deterministic income the company earns each unit time t and W_t is a Wiener process, or Brownian motion of mean 0 and variance σ^2, and σ is the volatility parameter. W_t plays the role of random gains and losses This model is a classic example of surplus we can find in the literature and known as pure diffusion model. Brownian motions are a standard option to model Cramér-Lundberg surplus (which contain jump processes). A good reference on how to approximate those gain/loss jump processes by Browinan motions is [Schmidli, 2008].
Note that pure diffusion processes are not new and have been studied since the second part of the twentieth century in [Gerber, 1972] for example. We consider that this surplus is adapted to a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, and we have $\mu = \mathbb{E}[U(t+1) - U(t)]$.

From that surplus, we consider a leakage function, called aggregate amount of dividend, defined as a function of time and we denote it $D(t)$. The process $\{D(t)\}$ is the amount of cash paid to the shareholders, deducted from $U(t)$. After this is done, the surplus has a new modified value, denoted :

$$X(t) = U(t) - D(t), \quad t \geq 0. \tag{1.1.2}$$

Some preliminaries remarks about the process $\{D(t)\}$: first, $D(t)$ is the cumulative value of dividends paid from the beginning ($t = 0$). It is assumed càdlàg (continu à droite, limite à gauche) with $D(0) = 0$. The process is non-decreasing and active until ultimate ruin of the company which happens at time $t = \tau$ so that :

$$\tau = \inf\{t \mid X(t) \leq 0\}. \tag{1.1.3}$$

The idea of distributing dividend is not a recent concept. It was first proposed by [de Finetti, 1957] as a criticism of the idea that it was unrealistic for a company to let grow its surplus to infinity because they wanted to minimize the probability of ruin. First because a company cannot keep all of its funds, and also (as reiterated in [Avanzi, 2009]) why should an older company hold more surplus than a younger one bearing similar risks, only because it is older ?

The potential amount of dividend distributed until bankruptcy is usually measured by the mathematical expectation known as the expected value of dividends until ruin, which is :

$$\mathbb{E}\left[\int_0^{\tau} e^{-\delta t} \mathrm{d}D(t)\right] \tag{1.1.4}$$

where δ is the force of interest parameter. But there is a major issue with that concept : allowing the company to distribute its money leads to a probability of ultimate ruin of 1.

1.2. The dividend / stability dilemma and barrier strategies

The fact that distributing dividends yields a certain bankruptcy leads to a dilemma : how can a company get some stability and distribute dividends ? A solution was proposed by [Gerber, 1974]. The idea is to pay dividends at a rate that will not induce bankruptcy in a short term (reasonable enough to get stability). The behaviour of the surplus in the long term will then be considered as not relevant. This leads us to define the notion of strategy. A dividend strategy is giving an answer to both questions "when" and "how much" dividends should be paid. The strategy proposed by [Gerber, 1974] was called a barrier and consisted in choosing a positive value, denoted b which would separate the capital in two areas : $[0, b)$ and $[b, +\infty)$. Each time the surplus should hit b and should go above this value, the corresponding difference between the surplus and b would be paid as dividends. Mathematically, the dividend D paid around the barrier is :

$$D = \begin{cases} 0 & \text{if } X(t) \in [0, b) \\ X(t) - b & \text{if } X(t) \in [b, +\infty). \end{cases}$$

This type of strategy is nowadays really common and it is the one we use in this thesis. We are interested in two things : the optimal level of b, which maximizes the the expected value of dividends, and the optimality of strategy, whose goal is to ensure that the barrier strategy is the unique strategy that maximizes the dividends. For the first time in the literature, that second type of optimality is proven for the $\gamma - \omega$ model in this thesis

Barrier is only a generic name for a lot of strategies. They can be fixed horizontal barriers, or they can be moving with time (generally increasing) but they all describe strategies that release some part of the capital after a value of the surplus is reached. In that case, if the dividends are paid continuously, the surplus never goes above b and does not grow to infinity. We are focused on this type of strategy, but it is not always the case that they are optimal. Other common strategies found in the literature and developped in [Avanzi, 2009] are threshold strategies (where the surplus over the threshold is not completely paid as dividend) or (multi)layer strategies.

1.3. Types of ruin and solvency sonstraint

Ruin is the most relevant and crucial parameter to determine in the model. We have several choices to do that. We know from section 1.1 that τ is the first time where the modified surplus becomes negative.
We know, from [Avanzi et al., 2014] (lemma 3.3 page 213) that under a barrier strategy, ultimate ruin is certain, that is, $\mathbb{P}[\tau < \infty] = 1$. We then need to define the criteria of ruin, and ruin itself.

We could define it as the lowest level of capital, denoted c_{ruin}, from which the company is in debt and cannot distribute dividends. Ultimate ruin (or bankruptcy) is the lowest level $c_{\text{bankruptcy}}$ from which the company is forced by the regulator to shut down even though the shareholders would like to continue. The first type of ruin, and the simplest, is standard ruin. In that case, $c_{\text{ruin}} = c_{\text{bankruptcy}}$ (and usually both are equal to 0).

From that, we can develop any other type of ruin. The ruin we chose to use in this thesis is a ruin where $c_{\text{ruin}} > c_{\text{bankruptcy}}$. Both values are not the same and the company can be ruined but can continue doing business until ultimate ruin. This ruin has first been described in [Albrecher et al., 2011] as the $\omega-$ruin, because between c_{ruin} and $c_{\text{bankruptcy}}$, they define a function of the initial capital $\omega(u)$, which satisfies three conditions :

(1) $\omega(u)$ is a non-increasing function.

(2) $\omega(u) \geq 0$ if $u < c_{\text{ruin}}$.

(3) $\omega(u)\mathrm{d}t$ is the probability of ultimate ruin within $\mathrm{d}t$ time units.

[Albrecher et al., 2011] use the following values in their paper :

$$c_{\text{ruin}} = 0 \quad \text{and} \quad c_{\text{bankruptcy}} = -\infty \tag{1.3.1}$$

and justify that choice by the fact that companies can hold a negative surplus and continue doing business as usual without cash reserve limit, for example state-owned companies. Building on this model, we keep the notion of $\omega-$ruin, but we define new values for c_{ruin} and $c_{\text{bankruptcy}}$ which are now

$$c_{\text{ruin}} = a_1 > 0 \quad \text{and} \quad c_{\text{bankruptcy}} = 0. \tag{1.3.2}$$

In 2013, [Albrecher and Lautscham, 2013] provide a more precise definition of ω with interpretation : a suitable locally bounded function $\omega(\cdot)$ depending on the size of the negative surplus is defined on $(-\infty, 0]$ (resp. we adapt it to $(0, a_1]$). Given some negative (resp. below a_1) surplus u and no prior bankruptcy event, the probability of bankruptcy on $[s, s+dt)$ is $\omega(u)dt$. We assume that $\omega(\cdot) > 0$ and $\omega(x) \geq \omega(y)$ for $|x| \geq |y|$ to reflect that the likelihood of bankruptcy does not decrease as the surplus becomes more negative (resp. plummets below a_1). In general, the idea is that whenever the surplus level becomes negative, there may still be a chance to survive, and it is modelled that survival is less likely the lower such a negative surplus level is. Conceptually, the replacement of the ruin concept by bankruptcy first of all removes the binary feature of the classical framework where the surplus process survives at $u = 0$, but is killed for arbitrarily small negative surplus levels $u = 0^-$ (in the non solvency constrained case). From a practical viewpoint, this is underpinned by the fact that in many jurisdictions the regulator would take control as an insurer's financial situation deteriorates, and measures would be undertaken during a rehabilitation period with the aim of curing the insurer's financial issues.

We justify that choice by the following arguments. We choose to set a ruin level at $a_1 > 0$ (which is called a solvency constraint) because we deal with insurance companies, and these companies need to hold a positive reserve of cash in case they need to close and transfer the portfolio to another company. In this case, each insurance policy will be sold for a higher price than its value, because insurance policies are risky business and another company that would agree to acquire them needs to be sure they will not be ruined by a too high amount of claims. In the end, if all the policies combined are worth a first expectation of claims, the buyer will estimate they are worth a higher expectation of claims and the difference between the two expectations is exactly a_1. In this particular case, this solvency constraint is called the liquidation value.

In this thesis, we restrict the ω function to a constant or a piecewise constant, that is $\omega(u) \equiv \omega,\ \forall u \in [0, a_1)$. This is because we choose not to focus on the ruin function itself but on its implications. This ruin function, that measures the probability of going bankrupt can be more generally related to the Parisian ruin framework, where the company is allowed to spend some time below the ruin level before declaring bankruptcy. Such framework can be found for example in [Landriault, Renaud, and Zhou, 2011]. The idea of allowing a company to spend some time in a state of pre-bankruptcy before ultimate ruin occurs is useful to model the fact that some big companies cannot know their exact value over time exactly at time t. A delay before decalring bankruptcy is then necessary to ensure consistency with what happens in reality. When the ruin function ω is a constant, their is an intersection with the Parisian literature.

What is also interesting is that Parisian delays can also model periodic dividends in some way. Indeed, the time spent above b before dividend decision happens with intensity γ is also a delay that can be part of this literature. Periodic dividends versus continuous dividends can also alternatively be explained by the fact that the company does not exactly know its exact instantaneous surplus and has to spend some time above the barrier b to declare dividend distribution.

The consideration between the going concern liability and liquidation value liability, which leads to a_1, and its optimality in this context is part of the main contribution and was not part of the literature.

However, more generally, solvency constraints have been part of the literature for some time, but not many papers choose to implement it because it brings more issues and cases to any considered model, so when the article is not focused direclty on solvency constraints, they usually choose not to include one to simplify the calculations. A good reference on optimality of barrier strategies under solvency constraints is [Paulsen, 2003]. We extend one of his main theorems in chapter 3.

It is however a great improvement for any model to consider solvency constraints because although it creates dichotomies, the realism we gain is non negligeable, and new to the literature. We are the first to shed light on a model which includes both a solvency constraint and an $\omega-$ruin function, and consider the optimality of a barrier strategy in that case. But implementing a fixed value a_1 to the model make some questions arise. The most relevant one is what happens to the areas created by the model, $[0, b)$ and $[b, +\infty)$, and how to include a_1. If $b > a_1$, it results three areas, $[0, a_1)$, $[a_1, b)$ and $[b, +\infty)$, but if $b < a_1$ the answer is not that obvious. Can a company afford to pay dividends in that case ? We provide some answers to these questions in the main development.

1.4. Periodic dividends and periodic strategies

For the moment we did not make assumptions about when dividends should be paid. We only proposed that some leakage could be distributed according to a barrier strategy, when the surplus reaches the barrier b, but this is not satisfying because in that case, we have not chosen a way to distribute dividends regardless of the barrier. A part of the dividend distribution is bound to the model itself : dividends can be released continuously or periodically. Continuous dividends have been modelled for a long time in the literature. In fact, it is the most simple case of distribution. As soon as the surplus reaches the criterion of distribution, the leakage happens and money flows from to surplus to the aggregate amount of dividends. This model is quite convenient but has major weaknesses.

First of all, in the case of a pure diffusion, unpleasant behaviour may occur around the barrier we have set for dividends. Because of the nature of the Brownian motion, the surplus can cross the barrier many times up and down and produce small dividend amounts that are not particularly interesting.

Secondly, the shareholders can never be sure when a dividend is paid : continuous leakage could mean two dividends in two days then nothing for three weeks, and so on. It is more interesting to get dividends paid at a steady rate to consider it as a reliable source of income.

Those two main issues can be resolved (or partially resolved) switching to periodic dividends. This time, dividends are not paid continuously when the surplus goes above b, but are only paid at discrete points of time, called decision times. A comprehensive paper on periodic dividends that describes this type of decision times is [Avanzi et al., 2014]. They can be seen as a sequence of times $\mathcal{T}$ such that

$$\mathcal{T} = \{T_1, T_2, T_3, \dots, T_k, \dots\} \tag{1.4.1}$$

with $T_k < \tau$ for all $k \in \mathbb{N} \setminus \{0\}$ and where each T_k is determined thanks to $\{N_\gamma\}$, a $\{\mathcal{F}_t\}-$adapted Poisson process. Decisions times occurs when the process has jumps.

The quantity $T_{k+1} - T_k$ for all $k \geq 0$ is an inter-dividend-decision time and can be assumed to be exponentially distributed with mean $1/\gamma$.

This simply means that at any time, the probability that a dividend can be paid within dt time units is γdt. We are interested in the memoryless property of the exponential distribution. This addresses the first issue, and the second one is partially resolved. Of course, the decision times are random but it gives a probability of decision thanks to the parameter γ. For the purpose of realism, and because accuracy is a criterion of choice for a model, we adopt the periodic dividends and the inter-exponential decision times. We now need a strategy that will work efficiently with this model. This is a first step towards more realism, the second one being the use of a method called "Erlangization" that considers inter-decision times are governed by multi-dimentional Erlang distributions (see for example [Avanzi, Cheung, Wong, and Woo, 2013]). This provides deterministic intervals of decisions, instead of producting random ones with a simple exponential model. This will not be considered in this thesis, but could be a crucial improvement for a further extension in this framework.

We have seen that a strategy was an answer to both questions when and how much should be paid. The above paragraphs answer the first question : the dividends are paid according to $\mathcal{T}$, but we still need to determine how much should be paid. For the moment, we provide a theoretical answer to this question : at each decision time T_k we associate a dividend ϑ_k, and similarly to the construction of $\mathcal{T}$, we can create a new sequence of dividend payouts Θ such that

$$\Theta = \{\vartheta_{T_1}, \vartheta_{T_2}, \vartheta_{T_3}, \ldots, \vartheta_{T_k}, \ldots\}. \tag{1.4.2}$$

We define Θ as a periodic strategy. Of course, Θ is not unique : any other admissible sequence is considered as an admissible periodic strategy. According to what has been done in [Avanzi et al., 2014], we denote $\mathcal{D}$ the set of admissible periodic strategies. To be admissible, a strategy Θ needs to have an associate aggregate dividend process $\{D(t)\}$ that is a non-decreasing and $\{\mathcal{F}_t\}-$adapted with càdlàg sample paths and initial value $D(0) = 0$.
The dividend payout at decision time T_k is ϑ_{T_k} for $k = 1, 2, ...$ which is measurable with respect to $\{\mathcal{F}_t\}$, and then :

$$0 \leq D(T_k) - D(t_k-) = \vartheta_{T_k} \tag{1.4.3}$$

and the process $\{D(t)\}$ can be written as

$$D(t) = \int_0^t \vartheta_s \mathrm{d}N_\gamma(s). \tag{1.4.4}$$

The modified surplus $X(t)$ can be written :

$$X(t) = u + \mu t + \sigma W_t - \sum_{k=1}^{\infty} I_{\{T_k \leq t\}} \vartheta_{T_k} \tag{1.4.5}$$

where $I_{\{A\}} = 1$ if the event A is true, and 0 otherwise.

The theoretical Expected Present Value of Dividends (EPVD) paid until ruin associated with a strategy Θ is defined as :

$$J(u, \Theta) = \mathbb{E}^u \left[\sum_{k=1}^{\infty} e^{-\delta T_k} \vartheta_{T_k} I_{\{T_k \leq \tau\}} \right], \quad u \geq 0. \tag{1.4.6}$$

This formula is the periodic analogous of equation (1.1.4), and we note that τ does not need to be in $\mathcal{T}$ because bankruptcy happens as soon as the surplus becomes null. This is called “continuous monotoring” of solvency.

1.5. Results for the Expected Present Value of Dividends in the $\gamma - \omega$ model

The Expected Present Value of Dividends, or EPVD, is the heart of the matter of this thesis. It is the dividend expectation we obtain and we want to maximize. This function needs to be continuous and at least twice differentiable (except at countably many points)to be a candidate solution function, and is a function of the initial surplus u. It also needs to be concave and increasing to be solution. Verifiction of concavity and variation is done for each section of the thesis, when we find potential solutions, to ensure that the verification theorem applies to the candidate functions. The interpretation of these conditions for the value function is that the expected dividends increase with initial capital of the company, and there cannot have jump in the function, unlike in the aggregate amount of dividends collected over time. It states that a crucial assumption on expected present value of dividends is that for any $u_1 < u_2$ initial capital level of cash held by the company, $V(u_2) - V(u_1) \to 0$ when $u_1 \to u_2$. (In the first part of the development, we aim at calculating the new EPVD resulting from the changes that have been made to [Albrecher et al., 2011]. In this paper, which is our main data source that we are trying to improve, the $\omega-$ruin function plays a role in getting the solutions of the equations for the surplus over each of the areas $[-\infty, 0)$, $[0, b)$ and $[b, +\infty)$. Recall that the level of ruin is 0 and the level of bankruptcy is $-\infty$ in their case. From the following expression, that we explain in the next section

$$G(u) + \{\gamma[l + G(u - l] - G(u)] + (\Omega - \delta)G(u)\}\, h + o(h) \tag{1.5.1}$$

where Ω is the infinitesimal operator

$$\Omega f = \frac{\sigma^2}{2} f'' + \mu f' - \omega f \tag{1.5.2}$$

and $G(u)$ is a notation to clarify that it is potentially different from $V(u)$ because $V(u)$ is the optimal function and at this point we don't know yet if $G(u) = V(u)$, we get the the three equations for each part of the surplus, which are

$$\frac{\sigma^2}{2} G''(u, b) + \mu G'(u, b) - (\omega(u) + \delta)G(u, b) = 0, \quad u \in [-\infty, 0) \tag{1.5.3}$$

$$\frac{\sigma^2}{2} G''(u, b) + \mu G'(u, b) - \delta G(u, b) = 0, \quad u \in [0, b) \tag{1.5.4}$$

$$\begin{aligned} &\frac{\sigma^2}{2} G''(u, b) + \mu G'(u, b) - \delta G(u, b) \\ &\qquad = -\gamma[u - b - G(u, b) + G(b, b)] = 0, \quad u \in [b, +\infty). \end{aligned} \tag{1.5.5}$$

We show each step to obtain these functions in the development, at the beginning of next chapter. When they obtain the functions that govern the three areas, [Albrecher et al., 2011] solve them to get the explicit EPVD, which is the piecewise function $G(u, b)$.

The middle equation is the easiest to solve for $G(u, b)$ and the solution is

$$G(u, b) = Ae^{ru} + Be^{su} \tag{1.5.6}$$

where A and B are constants to be determined and where r and s are the positive and negative roots of the characteristic equation

$$\frac{\sigma^2}{2} \xi^2 + \mu \xi - \delta = 0. \tag{1.5.7}$$

The next equation solved for $G(u,b)$ is the upper equation, and its solution according to [Albrecher et al., 2011] is

$$G(u,b) = \left(\frac{\delta}{\delta+\gamma}G(b,b) - \frac{\mu\gamma}{(\delta+\gamma)^2}\right)e^{\gamma(u-b)} + \frac{\gamma}{\delta+\gamma}[u-b+G(b,b)] + \frac{\mu\gamma}{(\delta+\gamma)^2} \quad (1.5.8)$$

where s_γ is the negative root of the associated characteristic equation.
Because a lot of calculations we have to do are similar to the ones in [Albrecher et al., 2011], we develop them in the next chapter.

This part is used to find the optimal barrier b^* which is

$$b^* = \frac{1}{r-s}\ln\left[\frac{-Bs^2(r_\gamma - r)}{Ar^2(r_\gamma - s)}\right], \quad (1.5.9)$$

where, A and B are two constants and r and r_γ the positive roots of the middle and upper associated characteristic equations. In that case, the classic result at b^* is

$$G'(b^*, b^*) = 1. \quad (1.5.10)$$

Finally, for the lower part, [Albrecher et al., 2011] find

$$G(u,b) = e^{r_\omega u} \quad (1.5.11)$$

and this part is a lot different from our work because of the new ruin condition and the solvency constraint we impose.

This thesis begins with the analogous of that work, using the new solvency conditions 0 and a_1 instead of $-\infty$ and 0. Because the EPVD is the main tool we use to compare the different outcomes developed in this thesis, we start the work by adapting their paper to obtain the new EPVD we need. This will lead to our new proof that a barrier strategy is the optimal strategy in the $\gamma - \omega$ model with solvency constraint and strengthen the work of [Albrecher et al., 2011] because a barrier may rightfully be used as the optimal strategy.

A lot of expected present value of dividends presented in this thesis have their numerical illustration (see list of figures). These graphs represent the amount of dividends paid until ruin with respect to the initial surplus of the company, u. That is why only increasing functions are to be found (the more the company holds money, the more dividends are to be paid).

1.6. Optimality of strategy, optimality of barrier and the Hamilton-Jacobi-Bellman equation

This optimality section is one of the main concerns of this thesis, because all the new results we find are based on existing results we improve. We can distinguish two kinds of optimalities, from the general-to-specific. The first one is the optimality of strategy. In that case, a strategy must be proven optimal amongst all admissible strategies. In the periodic case, it is equivalent to find which $\Theta \in \mathcal{D}$ is the best (ie the one that maximizes the expected present value of dividends). In case of periodic (gamma) dividends, [Avanzi et al., 2014] are the firsts to prove that a barrier strategy is optimal. Once a strategy is assumed to be the best one, it needs to pass a process called verification lemma to prove its uniqueness. That is why we only discuss barrier strategies in the thesis, they are the ones optimal here.
The second type of optimality is the optimality of barrier. Once a barrier strategy is proven optimal to maximize the expected present value of dividends, the second task consists in finding its optimal level, donoted b^*. Both optimalities can be done separately but usually, the second one is easier to obtain.

Let's consider the main paper [Albrecher et al., 2011] as our starting point. In this article which deals with the $\gamma - \omega$ case, they only find the optimal level for b^* but they do not know whether a barrier is optimal or not, this is only an assumption they make. To improve it, it would be useful to know whether a barrier strategy is optimal or not in the $\gamma - \omega$ model. We prove the optimality of such a strategy in Chapter 2. We add a solvency constraint to the model to make it look even more realistic and its optimality is discussed throughout chapters 2 and 3.

The second type of optimality is a consequence of the first one. In the section of the literature review dedicated to barrier strategies, we already stated that if

b is above a_1, it creates three areas : $[0, a_1)$, $[a_1, b)$ and $[b, +\infty)$. At the beginning of the article, [Albrecher et al., 2011] automatically assume it is the case : b is always above their a_1 (which is 0) and they give three equations for the expected present value of dividends, each one related to an area, and whose solutions make a continuous function of the initial surplus u. With a solvency constraint a_1, we create more possible outcomes. For example, b can be below a_1, this case is legitimate because it is proven that the optimal b^* is a logarithm of a quotient. To be exhaustive, we need to deal with this case. A convenient optimality result in that case comes from [Paulsen, 2003], whose goal is to determine and prove optimal the barrier b in case of a solvency constraint a_1, particularly when $b < a_1$. To maximize the expected present value of dividends, theorem 2.2 of [Paulsen, 2003] states that the optimal strategy is to use a barrier at $b = a_1$ if $b^* < a_1$. We cannot use this theorem in its 2003 version because it was not proven right in the case of periodic dividends. One of the main goals of chapter 3 is to prove it right in the periodic case.

Let's focus on the first type of optimality because we need to improve the existing results. Recall the expected present value of dividends $J(u, \Theta)$ from section 1.4. Our goal is to maximize this function because it is our criterion. We need to find the optimal payouts, that is, the sequence $\Theta = \{\vartheta_{T_1}, \ldots, \}$ which maximize J for all $\Theta \in \mathcal{D}$, the set of all admissible periodic strategies. This special sequence is denoted

$$\Theta^* = \{\vartheta^*_{T_1}, \vartheta^*_{T_2}, \vartheta^*_{T_3}, \ldots, \vartheta^*_{T_k}, \ldots\} \tag{1.6.1}$$

so that, mathematically

$$J(u, \Theta^*) = \sup_{\Theta \in \mathcal{D}} J(u, \Theta). \tag{1.6.2}$$

$J(u, \Theta^*)$ is denoted $V(u)$ thereafter. We will qualify a strategy with dividend payments Θ^* to be optimal if

$$V(u) = J(u, \Theta^*) = \mathbb{E}^u \left[\sum_{k=1}^{\infty} e^{-\delta T_k} \vartheta^*_{T_k} I_{\{T_k \leq \tau\}} \right], \quad u \geq 0. \tag{1.6.3}$$

The methodology to obtain optimality of strategy like in [Avanzi et al., 2014] is first to understand what happens to the surplus over a small time interval h

in terms of dividend expectation for the optimal expected present value of dividends. A dividend is usually denoted $l \geq 0$. We then need to apply the law of total probabily to determine all possible events. Because we focus on small intervals, Taylor expansions are accurate enough and the force of interest parameter $e^{-\delta h}$ can be rewritten $1 - \delta h + o(h)$.

Once we know which events happen with probability γh and which happen with probability $1 - \gamma h$, using Taylor expansions yields to a result of the form :

$$V(u) + \{\gamma[l + V(u-l] - V(u)] + (\mathcal{A} - \delta)V(u)\}\, h + o(h) \tag{1.6.4}$$

with initial condition $V(0) = 0$ for the function $V(u)$ obtained in [Avanzi et al., 2014], because bankruptcy happens at 0, and where $\mathcal{A}$ is some infinitesimal operator not crucial here. Thereafter, this leads to the construction of a mathematical tool called the Hamilton-Jacobi-Bellman equation (or HJB equation) whose role is to maximize the function V, and which comes directly from (1.6.4). The HJB in that case for V is then

$$\max_{0 \leq l \leq u} \{\gamma[l + V(u-l] - V(u)]\} + (\mathcal{A} - \delta)V(u) = 0. \tag{1.6.5}$$

This equation is crucial in the verification lemma we use to check optimality of strategy. We will need to find the HJB that comes from our model, and the same methodology will be used. Thanks to this lemma, [Avanzi et al., 2014] prove a barrier strategy to be optimal in that case, with

$$l = \begin{cases} 0 & \text{if } u \in [0, b) \\ u - b & \text{if } u \in [b, +\infty). \end{cases}$$

Formally, using the previous notation, the periodic barrier is written

$$\vartheta_{T_k} = \max\ \{X(T_k) - b, 0\} \tag{1.6.6}$$

They give another useful lemma, called lemma 3.1 in [Avanzi et al., 2014] page 212 that states : if $V(u) \in \mathbb{C}^2$ is an increasing and concave function, with a point $b > 0$ such that $V'(b) = 1$, then

$$\max_{0 \leq l \leq u} \{l + V(u-l)\} \tag{1.6.7}$$

is achieved at the above level l. Our goal is to prove a barrier strategy is also optimal in the $\gamma - \omega$ case.

1.7. Two ways of dealing with a ruin event : liquidation (forced or voluntary) and capital injections

Once the main model is set up, we would like to consider some alternative policies for the area $[0, a_1]$. In chapter 3 and 4, we focus on trouble experienced by risky businesses such as insurance companies.

For example, [Avanzi et al., 2014] has a criterion related to γ, the intensity of the dividend payout Poisson process. When prospects are not sufficient to ensure stability for the company, that is, when γ is too low (under a certain value γ_0), this triggers a "destructive" strategy called liquidation at first opportunity where the whole surplus is distributed as a final dividend, and then the company goes bankrupt and shuts down. Typically, this is because under γ_0, the probability of decision time is too low, which leads to a barrier $b^* < 0$ and the company is not sustainable for dividend payments. The shareholders can only get that final dividend at the first decision time, T_1, and not before. Hence the name "at first opportunity". To get the whole surplus, the negative optimal barrier should be set at $b^* = 0$. Considerations between dividend strategies and liquidations of type "take the money and run" have been studied for some different models, for example in [Loeffen and Renaud, 2010].

In chapter 3, we study the implications that $\gamma < \gamma_0$ yield for our model. In that case, the barrier is set at a_1 (and not at 0 because capital up to a_1 does not belong to the shareholders in our case) and it affects the EPVD. We assume that the danger zone $[0, a_1]$ is still an area when the surplus can go, but this would be a limit case : a barrier at a_1 means no buffer zone $[a_1, b)$ and a strange event occurs : two areas instead of three where the company instantly switches from being ruined to distributing dividends, which is not a really realistic situation.

In chapter 4, to address this specific issue of realism, we resort to solvency considerations based, for the first ones, on different types of liquidations. This type of ending strategies is really useful, and that is why we have chosen to implement them in our $\gamma - \omega$ model.

Regardless of barrier level, we first consider a case of forced liquidation which happens when the surplus hits a_1, that is, in case of $\omega-$event. In that case, an external regulator forces the shareholders to close the business and there is no final dividend. This is like a simple ruin but where the level of bankruptcy would be a_1. The initial surplus could not be below a_1 because it wiuld trigger an immediate liquidation (because the monitoring of solvency is continuous). In that case, $[0, a_1]$ is a no-go zone. As soon as the surplus reaches it, liquidation is forced. It is like there is no more $\omega-$ruin, or more precisely, $\omega(u) = +\infty$ so $\omega(u)\mathrm{d}t$, which is the probability of ultimate ruin when the surplus is in the omega zone is equal to 1 for $u \in [0, a_1]$. It is indeed a change of scale for a simple ruin, where bankruptcy occurs at the time of ruin.

To prevent this bad event of forced liquidation, the shareholders are allowed to liquidate at first opportunity in case of bad prospects. For example, if γ is too low and $\gamma < \gamma_0$, the barrier is set at a_1 and the shareholders decide to liquidate at first opportunity, that is, at the first decision time T_α where the surplus is above a_1 (usually T_1 provided that the surplus has not undergone an $\omega-$event in the meantime). This is an anlogous work to what is done in [Avanzi et al., 2014].

The second section of chapter 4 is dedicated to implement forced capital injections in case of a surplus below a_1 or bankruptcy. Instead of killing the business each time the surplus goes below $a1$, or that a bankruptcy should happen, we adopt the perspective of [Avanzi, Shen, and Wong, 2011] and start injecting capital into the business. To be more accurate, we consider the case where the shareholders are forced to inject capital. In this paper, [Avanzi et al., 2011] do not know how much they should inject and create an injection strategy, similar to a sequence Θ but this time this is a sequence of discrete injections, not dividend payouts. Their goal is to find the best sequence of dividend payouts and injections so that the expected present value of dividends is maximized. Injections of cash are not free, they are sanctioned by a penalty κ, traditionally worth the initial slope of the value function at time 0. Examples of this penalty are given in [Avanzi et al., 2011]. It is important to notice the penalty is proportional to the capital injected. Last but not least, if capital injections are forced in case of bankruptcy, they prevent it. In that case, because injections are immediate, there is no stopping time τ. We chose to deal with that case anyway because it is part of the periodic dividend framework. Optimality of strategies under such models where ruin does not play a role has also been studied in [Avanzi et al., 2011]

We decide to solve the issue of how much we should inject by removing the non-fixed level of injections and decide that the injections should be done up to a_1. Another aticle, [Jin and Yin, 2013] implements delays in capital injections up to a lower boundary (because usually, injecting capital takes time and is not done instantly), and verify its optimality but does not consider the periodic framework of the model. The difference is major because the lower part $[0, a_1)$ is where all changes happen.

From the optimality perspective, a capital injection is treated as a negative dividend payment : money is injected into the surplus instead of being removed. As a result of this fact, when applying the law of total probability over a small time interval h to find the equations that govern the expected present value of dividends, capital is injected at rate $c \geq 0$ with cost κ. Which leads to new terms in the equations, that we find in [Avanzi et al., 2011].

In this thesis we consider both behaviours, liquidations and capital injections as part of the solvency requirements to a_1. In the first case, the shareholders are forced to liquidate in case of $\omega-$event and in the second one they are forced to inject capital.

1.8. Summary of the different steps and contributions

We build on many papers to get our new contributions and results. This section is intended to be a summary of the following chapters, why we develop them and the papers they are built on. We try to develop the steps in an order that makes each one the logical continuation of the previous one.

(1) In chapter 2, building on [Albrecher et al., 2011], we compute the new EPVD in the $\gamma - \omega$ model with a solvency constraint $a_1 > 0$ and a bankruptcy level 0, instead of 0 and $-\infty$. We keep the assumption of the article, which is to consider the barrier only above the solvency constraint.

(2) Still in chapter 2, on the model of [Avanzi et al., 2014], we show that a barrier is optimal in the $\gamma - \omega$ model, by proving each step of the associated verification lemma.

(3) In chapter 3, we develop the calculations of [Albrecher et al., 2011], to show the structure of the optimal barrier, and observe that it is possible to have $b^* < a_1$

(4) Still in chapter 3, because of (3), we consider an extension of (1) and (2). We build on an adapted theorem from [Paulsen, 2003] that we first need to prove right in the periodic case to obtain the new EPVD of [Albrecher et al., 2011] in the case of $b^* = a_1$. We complete the adapted works of [Avanzi et al., 2014] we started in chapter 2 to prove this strategy optimal.

(5) We observe that the EPVD obtained in (4) is a limit case so we would like to consider more realistic assumption for the area $[0, a_1]$. Wich leads to two solvency considerations developed in chapter 4.

(6) In chapter 4, we first consider that $[0, a_1]$ is a no-go zone, where liquidation happens instantly in case of $\omega-$event. The shareholders are allowed to liquidate at first opportunity, on the example of [Avanzi et al., 2014] but with an additional solvency constraint. We compute the new EPVD in the case where $b > a_1$ (the business works normally until ruin) and $b < a_1$ ($b = a_1$ so the shareholders take the lead and liquidate at first opportunity if they can). We refer to this "no-go zone case" as Case 1.

(7) Still in chapter 4, we develop another solvency consideration, based on [Avanzi et al., 2011]. $[0, a_1]$ is not a no-go zone and in case surplus below $a1$ or bankruptcy, the shareholders are forced to inject capital up to a_1. Ultimate ruin can never happen unless volontarily triggered. We obtain the new EPVD for this model in both cases $b > a_1$ and $b = a_1$.

Chapitre 2

EPVD IN THE NEW $\gamma - \omega$ SURPLUS MODEL AND OPTIMALITY OF A BARRIER STRATEGY

In this chapter we study the $\gamma - \omega$ model with a periodic barrier strategy under an additional solvency constraint. By extending the works of [Albrecher et al., 2011], we first derive the value function for an arbitrary periodic barrier strategy above a_1 under the $\gamma - \omega$ model with solvency constraint. This represents our first main contribution. Subsequently, we also study the global optimality of this strategy via the establishment of an associated verification lemma.

2.1. EPVD and the three new equations in our $\gamma - \omega$ model

2.1.1. The model and definitions of the terms we use

The model we consider is the pure diffusion one, that is, we study the process

$$U(t) = u + \mu t + \sigma W_t \tag{2.1.1}$$

where these are defined at the beginning of literature review. A solvency constraint a_1 is implemented and is a deterministic horizontal threshold below which there is danger of ruin.
If the surplus goes below a_1 at a time t, this is called an $\omega-$event, because the area below a_1 is subject to an ω coefficient that measures the risk of ultimate ruin.
Dividends are paid according to a periodic barrier strategy (that we proove optimal in this chapter). The periodic barrier works as the following : each time there is a jump in the process $\{N_\gamma(t)\}$, if the surplus is below b, no dividend is paid. If the surplus is above b, the difference between the surplus and b is paid at the decision time. In this chapter we consider some events for the surplus :

We first use a model where it is allowed to go below a_1, to adapt the works of [Albrecher et al., 2011] to the new solvency constraint, because they consider a model where the surplus can be negative, and we do not want such a thing. In that case, there are three separate surplus areas because in the original paper, b is always considered positive. The following section is only an extension of the works of [Albrecher et al., 2011] and we ignore the issue of b being below a_1 on purpose for the moment. This issue has its own chapter (Chapter 3 of the thesis).

2.1.2. The application of the law of total probability to the surplus

We aim at finding the equation for the surplus, from which follow the three equations that govern the three areas $[0, a_1)$, $[a_1, b)$ and $[b, +\infty)$ introduced in [Albrecher et al., 2011]. We need to apply the law of total probability over a small time interval $[0, h)$ to analyse all the possible outcomes for the surplus within h. The argument used to build the HJB equation is then a heuristic one.

We follow the steps developed in [Avanzi et al., 2014]. Let's consider a small amount of time $[0, h)$. Over such a time interval, from the dividend decision perspective, two things can happen. Either a dividend decision is made, with probability γh, or nothing happens at all to the surplus.

If a dividend decision is made with intensity γ, and then probability γh, we denote $l \geq 0$ the amount of cash that has been released. Here l can be considered as the dividend. The variation of the model over h is thus $u + \mu t + \sigma W(h) - l$ (the stochastic surplus minus dividend). The discount factor δ plays a role under its exponential form $e^{-\delta h}$ (actualized value of money). We recall that for $h \ll 1$ we can use the Taylor expansion :

$$e^{-\delta h} = 1 - \delta h + o(h). \tag{2.1.2}$$

Then

$$\gamma h(1 - \delta h)V(u + \mu h + \sigma W_h) + o(h)$$

is the quantity that decribes the event of a dividend decision.

Taking the expectation, this can be rewritten :

$$\gamma h(1 - \delta h)\{l + \mathbb{E}[V(u + \mu h + \sigma W(h) - l)]\} + o(h). \tag{2.1.3}$$

Where $o(h)$ includes the rest of (2.1.2). On the other hand, we assume that nothing happens. In this case, it means that no dividend decision has been taken AND the company did not undergo bankruptcy.

Because the probability of decision time is γh, the probability of no decision is $1 - \gamma h$. Moreover, the probability of bankruptcy is $\omega(u)$. We then give the total probability when nothing happens :

$$1 - \gamma h - \omega(u)h. \tag{2.1.4}$$

. Keeping the same expression for the force of interest parameter and taking the expectation of the unchanged surplus $u + \mu h + \sigma W(h)$, we obtain the second part of the law of total probability :

$$(1 - \gamma h - \omega(u)h)(1 - \delta h)\mathbb{E}[V(u + \mu h + \sigma W(h))] + o(h). \tag{2.1.5}$$

When we add both we obtain the following as a result (which is everything that can happen over $[0, h)$:

$$\begin{aligned}&\gamma h(1 - \delta h)\{l + \mathbb{E}[V(u + \mu h + \sigma W(h) - l)]\} \\ &\qquad + (1 - \gamma h - \omega(u)h)(1 - \delta h)\mathbb{E}[V(u + \mu h + \sigma W(h))] + o(h).\end{aligned}$$

This is the total factorized quantity we are looking for and that we will study. Unfortunately, it is right now under its probabilistic form. Since we are working over a small time interval, we can assume that the quantities at stake are small enough to obtain good approximations using Taylor expansions.

The previous sentence motivates

$$\begin{aligned}&V(u + \mu h + \sigma W(h)) \\ &\qquad = V(u) + V'(u)(\mu h + \sigma W(h)) + V''(u)\frac{(\mu h + \sigma W(h))^2}{2} + ...\end{aligned} \tag{2.1.6}$$

and the mathematical expectation of the above is :

$$\mathbb{E}[V(u + \mu h + \sigma W(h))] = V(u) + \mu h V'(u) + \frac{\sigma^2}{2} h V''(u) + o(h), \tag{2.1.7}$$

because $\mathbb{E}[W(h)] = 0$. Furthermore, we only include terms in h in the quantity. All terms in $h^k, k \geq 2$ are included in the new $o(h)$.

The same Taylor expansions can be used to expand the case with dividends. In that case, we consider $V((u-l)+\mu h+\sigma W(h))$ and expand around $u-l$.

Introducing (2.1.7) in (2.1.6), we get :

$$\gamma h(1-\delta h)(l+V(u-l)+\mu hV'(u-l)+\frac{\sigma^2}{2}hV''(u-l)) \\ +(1-\gamma h-\omega(u)h)(1-\delta h)(V(u)+\mu hV'(u)+\frac{\sigma^2}{2}hV''(u)),$$

which is the same quantity as previously, only expanded thanks to Taylor.

We expand, and are interested in the terms in h. We neglect all terms with a $h^k, k\geq 2$ factor and include them in the new $o(h)$ instead. Then we factorize upon h to obtain the final form of the quantity we are interested in estimating :

$$V(u)+\{\gamma[l+V(u-l)-V(u)] \\ +\frac{\sigma^2}{2}V''(u)+\mu V'(u)-(\omega(u)+\delta)V(u)\}h+o(h). \quad (2.1.8)$$

This will be useful to obtain the HJB equation in the optimality section.

2.1.3. The three equations in the case of $b > a_1$

The structure of (2.1.8) motivates us to think that the optimal strategy is a periodic dividend barrier (See section 1.6 of literature review). Optimality of such a strategy is proved in section 2.2. In this section we are concerned with extending the works of [Albrecher et al., 2011] by adding a solvency constraint to the model.

We first notice that the expected present value of dividend function is a piecewise function, because each area of the model is governed by its own equation, with its own solution. However, we know that the global function needs to be continuous and at least twice differentiable (see paragraph 5 of literature review). Let's call $G(u,b)$ this function, where b is for the barrier, and u is the initial surplus.

We note that here, b is assumed to be higher than a_1, explaining why the function has three parts. The other case $b < a_1$ has its own development in chapter 3.

We write :

$$G(u,b) = \begin{cases} G_L(u,b) & \text{if } u \in [0,a_1) \\ G_M(u,b) & \text{if } u \in [a_1,b) \\ G_U(u,b) & \text{if } u \in [b,+\infty). \end{cases}$$

According to the works of [Albrecher et al., 2011], for $u \in [0,a_1)$, G_L satisfies the equation :

$$\frac{\sigma^2}{2}G''_L(u;b) + \mu G'_L(u;b) - [\delta + \omega(u)]G_L(u;b) = 0 \tag{2.1.9}$$

because it is the area where $\omega-$ruin occurs.

Between a_1 and b, in the middle area, equation (2.1.8) yields the equation :

$$\frac{\sigma^2}{2}G''_M(u;b) + \mu G'_M(u;b) - \delta G_M(u;b) = 0 \tag{2.1.10}$$

because no dividend payment occurs if the surplus is here. When the decision time happens, no surplus is distributed. This area is also above a_1, so this is essentially a buffer zone, between the ruin area and the dividend area.

Finally, above b, the equation satisfied by $G_U(u,b)$ is

$$\frac{\sigma^2}{2}G''_U(u;b) + \mu G'_U(u;b) - \delta G_U(u;b) + \gamma[u - b - G_U(u;b) + G_U(b;b)] = 0 \tag{2.1.11}$$

because dividend payments occur here.

These are the three equations that we must solve to find the expected present value of dividends. We give them right away because they are the starting point of [Albrecher et al., 2011], but they follow direclty from equation (2.1.8). In that equation, all parts are mixed together, but if we recall that our model orders $\gamma = 0$ if $u < a_1$ and $\omega(u) = 0$ if $u > a_1$, we can identify three distinct equations.

2.1.4. Initial conditions

In this subsection, we aim at explaining the initial conditions that match the previous three equations, in order to solve them. The easiest one to solve in the second one. Indeed, this equation is a simple order 2 homogeneous differential equation, and the theory to solve this type of equation is well known. It is possible to find adequate solutions thanks to the two initial conditions that are adapted from [Albrecher et al., 2011].

Firstly :

$$G_L(0, b) = 0, \tag{2.1.12}$$

which makes sense because bankruptcy occurs when $u = 0$, so the expected present value of dividends cannot have any other value than 0 at this point. Secondly we assume that $G_U(u, b)$ is linearly bounded when $u \to +\infty$.

This condition can be explained quite easily. In [Avanzi et al., 2014], it comes from the fact that $G_U(u, b)$ behaves like a linear function for an extremely large u, because in such case, the hypothesis of ruin is irrelevant (if the company holds $u \to +\infty$, then the probability of ultimate ruin is null, and the amount of dividends paid is close to u itself, plus a constant term corresponding to $G_U(b, b)$). That explains why we can think of $G_U(u, b)$ as a linear function in such case.

Finally, the last hypothesis is that $G(u, b), G'(u, b)$ and $G''(u, b)$ are continuous functions in u.

2.1.5. The Middle equation

We are interested in solving :

$$\frac{\sigma^2}{2} G''_M(u; b) + \mu G'_M(u; b) - \delta G_M(u; b) = 0. \tag{2.1.13}$$

This equation is a homogeneous differential equation of order two. Its associated characteristic equation is :

$$\frac{\sigma^2}{2} \xi^2 + \mu \xi - \delta = 0. \tag{2.1.14}$$

The roots of this equations are

$$\xi_{1,2} = \frac{-\mu \pm \sqrt{\mu^2 + 4\delta \frac{\sigma^2}{2}}}{\sigma^2}. \tag{2.1.15}$$

We notice that $\mu^2 + 4\delta\dfrac{\sigma^2}{2} > 0$, we can then call these solutions r and s where r is the positive solution and s is the negative one.

Then, the solution of this equation is

$$G_M(u,b) = Ae^{ru} + Be^{su}. \tag{2.1.16}$$

In the original paper by [Albrecher et al., 2011], they give a A and B that work in their case, that is :

$$A = \frac{G'(0,b) - sG(0,b)}{(r-s)} \qquad B = \frac{rG(0,b) - G'(0,b)}{(r-s)}. \tag{2.1.17}$$

We cannot use them because the model has changed and we have to calculate the new appropriate A and B between a_1 and b instead of 0 and b.

$$G'_M(u,b) = Are^{ru} + Bse^{su}. \tag{2.1.18}$$

At the continuity point a_1 we have

$$G_M(a_1,b) = Ae^{ra_1} + Be^{sa_1} \quad \text{and} \quad G'_M(a_1,b) = Are^{ra_1} + Bse^{sa_1}. \tag{2.1.19}$$

Now all we have to do is solve for A and B

$$\begin{cases} Ae^{ra_1} + Be^{sa_1} &= G_M(a_1,b) \\ Are^{ra_1} + Bse^{sa_1} &= G'_M(a_1,b) \end{cases}$$

$$\begin{cases} A &= G_M(a_1,b)e^{-ra_1} - Be^{(s-r)a_1} \\ G'_M(a_1,b) &= \left(G_M(a_1,b)e^{-ra_1} - Be^{(s-r)a_1}\right) re^{ra_1} + Bse^{sa_1} \end{cases}$$

We obtain the following new constants :

$$A = \frac{G'(a_1,b) - sG(a_1,b)}{e^{ra_1}(r-s)} \qquad B = \frac{rG(a_1,b) - G'(a_1,b)}{e^{sa_1}(r-s)} \tag{2.1.20}$$

because of the continuity condition that ensures $G_L(a_1^-,b) = G_M(a_1^+,b)$.

2.1.6. The Upper equation

The upper part is not different. Indeed this time we have :

$$\frac{\sigma^2}{2}G''_U(u;b)+\mu G'_U(u;b)-\delta G_U(u;b)+\gamma[u-b-G_U(u;b)+G_U(b;b)]=0 \quad (2.1.21)$$

which can can rewritten as :

$$\frac{\sigma^2}{2}G''_U(u;b)+\mu G'_U(u;b)-(\delta+\gamma)G_U(u;b)=-\gamma[u-b+G_U(b;b)]. \quad (2.1.22)$$

The solution of this equation is then the general solution of the homogeneous equation

$$\frac{\sigma^2}{2}G''_U(u;b)+\mu G'_U(u;b)-(\delta+\gamma)G_U(u;b)=0, \quad (2.1.23)$$

and a particular solution of the equation with second member. Fortunately, [Albrecher et al., 2011] already provide a particular solution, which is :

$$p(u)=\frac{\gamma}{\delta+\gamma}[u-b+G_U(b,b)]+\frac{\mu\gamma}{(\delta+\gamma)^2}. \quad (2.1.24)$$

A general solution of the homogeneous equation (2.1.23) is

$$C_3e^{s_\gamma(u-b)}+C_4e^{r_\gamma(u-b)}, \quad (2.1.25)$$

where s_γ (resp. r_γ) is the negative (resp. positive) solution of the associated characteristic equation :

$$\frac{\sigma^2}{2}\xi^2+\mu\xi-(\delta+\gamma)=0. \quad (2.1.26)$$

A global solution of the upper part is :

$$G_U(u,b)=C_3e^{s_\gamma(u-b)}+C_4e^{r_\gamma(u-b)}+p(u). \quad (2.1.27)$$

The proof is done by differentiating this function twice and replace it into (2.1.22).

However, the condition "$G_U(u,b)$ is linearly bounded when $u \to +\infty$" restricts the homogeneous part. We have $r_\gamma > 0 \Rightarrow r_\gamma(u-b) \to +\infty$ when $u \to +\infty$. The only solution we have to prevent an exponential explosion for the homogenous part is to impose $C_4 = 0$.

C_3 is still to be determined. We consider $u = b$ in (2.1.27) to obtain :

$$\begin{aligned} C_3 &= G_U(b,b) - p(b) \\ &= \frac{\delta}{\delta+\gamma} G_U(b,b) + \frac{\mu\gamma}{(\delta+\gamma)^2}. \end{aligned} \tag{2.1.28}$$

Finally, the upper part solution is :

$$\begin{aligned} G_U(u,b) = \left(\frac{\delta}{\delta+\gamma} G_U(b,b) + \frac{\mu\gamma}{(\delta+\gamma)^2} \right) e^{s_\gamma(u-b)} \\ + \frac{\gamma}{\delta+\gamma}[u - b + G_U(b,b)] + \frac{\mu\gamma}{(\delta+\gamma)^2} \end{aligned} \tag{2.1.29}$$

We now have two out of three parts of the piecewise function. In the next subsection, our goal is to determine the third and last part.

2.1.7. The Lower equation

This part is really interesting because it is where the $\omega-$ruin appears. We cannot give a general solution of the equation if we do not define ω in this context. Indeed until now, all calculations were given for an arbitrary ruin function $\omega(u)$, but we need to restrict the ruin coefficient to constant and piecewise constant functions to avoid having to solve differential equation with non-constant coefficients, which would be harder to solve.

In this subsection, we first solve the equation for $G_L(u,b)$ in the case of $\omega(u) = \omega$ constant and then we describe what happens if $\omega(u)$ becomes a piecewise constant function.

We recall that in this part, $u \in [0, a_1)$ and $G_L(0,b) = 0$.

The equation satisfied by $G_L(u,b)$ when the ruin function is a constant is

$$\frac{\sigma^2}{2} G_L''(u;b) + \mu G_L'(u;b) - [\delta+\omega] G_L(u;b) = 0. \tag{2.1.30}$$

This case is the simplest we can imagine for ω. We notice that it might not be the most efficient ruin indicator because if ω is a constant, it means that the ruin level of the company is not relevant. This model does not allow to distinguish between a small omega-event (the surplus reaches a_1 and goes a little below) from a critical near-bankruptcy event. For example, it describes an insurance company where a surplus below a_1 indicates that they will not be able to cover the next claim and go bankrupt when this happens. In that case, $\omega \mathrm{d}t$ is the probability of such claim within $\mathrm{d}t$ unit time.

The characteristic equation associated with the previous differential equation is :

$$\frac{\sigma^2}{2}\xi^2 + \mu\xi - [\delta + \omega] = 0. \tag{2.1.31}$$

Once again, let's call r_ω and s_ω the positive and negative root of the characteristic equation.

Because the equation is homogeneous, the general solution is :

$$G_L(u, b) = A_\omega e^{r_\omega u} + B_\omega e^{s_\omega u}. \tag{2.1.32}$$

Condition $G_L(0, b) = 0$ yields :

$$A_\omega e^0 + B_\omega e^0 = 0 \quad \iff \quad B_\omega = -A_\omega \tag{2.1.33}$$

and

$$G_L(u, b) = A_\omega \left(e^{r_\omega u} - e^{s_\omega u}\right). \tag{2.1.34}$$

This is where the determination of A_ω and B_ω differs from the paper. Indeed, [Albrecher et al., 2011] consider that simple ruin occurs between $-\infty$ and 0, the condition $G_L(-\infty, b) = 0$ holds $B_\omega = 0$. In this case $s_\omega < 0$ and $u < 0$ thus, $s_\omega u > 0$ and $e^{s_\omega u} \to +\infty$ when $u \to -\infty$. This is not possible given that $G_L(-\infty, b) = 0$ so the only solution is $B_\omega = 0$. And A_ω can be 1 because of this condition too. Indeed if $G_L(u, b) = A_\omega e^{r_\omega u}$, then $r_\omega > 0$ implies $e^{r_\omega u} \to 0$ when $u \to -\infty$.

However, in our case the asymptotic value is not part of the function interval and both A_ω and B_ω play a role. We may set $A_\omega = 1$ (and so $B_\omega = -1$) to obtain :

$$G_L(u,b) = e^{r_\omega u} - e^{s_\omega u} \tag{2.1.35}$$

and we can see that

$$G_L(0,b) = e^{r_\omega 0} - e^{s_\omega 0} \quad = \quad 1-1 \quad = \quad 0. \tag{2.1.36}$$

The candidate solution matches the boundary condition, which is a first step towards the actual solution.

A and B can be now explicitly determined, based on what is done in [Albrecher et al., 2011]. We must be careful comparing both solutions (ours and the one provided in the paper) because the ruin area contains most of the modifications.

[Albrecher et al., 2011] obtain $G_L(u,b) = e^{r_\omega u}$ so $G_L(0,b) = e^{r_\omega 0} = 1$ and $G'_L(0,b) = r_\omega e^{r_\omega 0} = r_\omega$. We refer to equations (2.1.17) to provide A and B when $c_{\text{ruin}} = 0$ and $c_{\text{bankruptcy}} = -\infty$, that is

$$A = \frac{r_\omega - s}{r-s} \quad \text{and} \quad B = -\frac{r_\omega - r}{r-s}. \tag{2.1.37}$$

Now this has to be adpated. Recall once again equations (2.1.20). Considering our solution $G_L(u,b) = e^{r_\omega u} - e^{s_\omega u}$, we have then

$$G'_L(u,b) = r_\omega e^{r_\omega u} - s_\omega e^{s_\omega u}. \tag{2.1.38}$$

The major difference is due to the continuity at a_1, which yields $G_L(a_1,b) = e^{r_\omega a_1} - e^{s_\omega a_1}$ and $G'_L(a_1,b) = r_\omega e^{r_\omega a_1} - s_\omega e^{s_\omega a_1}$.

Implementing this in (2.1.20) yields :

$$A = \frac{(r_\omega - s)e^{r_\omega a_1} + (s - s_\omega)e^{s_\omega a_1}}{e^{r a_1}(r-s)}, \tag{2.1.39}$$

also written as the sum

$$A = \frac{r_\omega - s}{r-s} e^{(r_\omega - r)a_1} + \frac{s - s_\omega}{r-s} e^{(s_\omega - r)a_1}. \tag{2.1.40}$$

Notice that in the above solution, if 0 is set instead of a_1, the first term of the sum is what [Albrecher et al., 2011] find as a solution. The second term is the consequence of the non-asymptotic hypothesis (the business stops at 0 instead of $-\infty$) so none of the exponentials vanish.

We determine the second constant B :

$$B = \frac{(r - r_\omega)e^{r_\omega a_1} + (s_\omega - r)e^{s_\omega a_1}}{e^{s a_1}(r - s)}, \tag{2.1.41}$$

which yields :

$$B = \frac{r - r_\omega}{r - s}e^{(r_\omega - s)a_1} + \frac{s_\omega - r}{r - s}e^{(s_\omega - s)a_1}. \tag{2.1.42}$$

In case of a constant ruin function, we have found the exhaustive expected present value of dividends because $G_L(u, b)$, $G_M(u, b)$ and $G_U(u, b)$ are now explicitly computed as functions of u.

The second type of $\omega-$ruin that follows from the first one is a piecewise constant function, that is : $\omega(u) = \omega_k, \quad k \in [1, n]$.

In this case, the interval $[0, a_1)$ is divided into n sub-intervals

$$[0 = u_0, u_1), [u_1, u_2), ..., [u_k, u_{k+1}), ..., [u_{n-1}, u_n = a_1) \tag{2.1.43}$$

and each sub-interval is bound to its ω value. For example, a surplus between 0 and u_1, is bound to $\omega(u) = \omega_1$.

In that case $\omega_k > \omega_{k+1}$ because the more the debt increases, the more the probability of ruin increases too. This model seems more efficient than the previous one to describe different situations of ruin or different $\omega-$events.

This definition of $\omega(u)$ leads to the n differential equations :

$$\frac{\sigma^2}{2}G_L''(u; b) + \mu G_L'(u; b) - [\delta + \omega_k]G_L(u; b) = 0, \quad k \in [1, n]. \tag{2.1.44}$$

Their n associated characteritic equations are thus

$$\frac{\sigma^2}{2}\xi^2 + \mu\xi - [\delta + \omega_k] = 0, \quad k \in [1,n]. \tag{2.1.45}$$

Because each equation has two distinct roots, the general solutions are :

$$G_L(u,b) = A_k e^{r_k u} + B_k e^{s_k u}, \quad k \in [1,n], \quad u \in [u_{k-1}, u_k), \tag{2.1.46}$$

where A_k and B_k are $2n$ constants to determine and r_k and s_k are the n positive and n negative solutions of the associated characteristic equations.

To determine A_k and B_k for $k \in [1,n]$, we use a recursive method. First we know that for $k \in [1, n-1]$, $r_{k+1} < r_k$ and $s_{k+1} > s_k$. And then, from (2.1.35) we have the initial condition $A = 1$ and $B = -1$.

We know the solutions of the equations must satisfy at least two continuity conditions for the function and its derivative, which yields the following results :

$$A_{k+1}e^{r_{k+1}u_k} + B_{k+1}e^{s_{k+1}u_k} = A_k e^{r_k u_k} + B_k e^{s_k u_k} \tag{2.1.47}$$

$$A_{k+1}r_{k+1}e^{r_{k+1}u_k} + B_{k+1}s_{k+1}e^{s_{k+1}u_k} = A_k r_k e^{r_k u_k} + B_k s_k e^{s_k u_k}. \tag{2.1.48}$$

When those continuity conditions are respected, we have then

$$A_{k+1}(r_{k+1} - s_{k+1})e^{r_{k+1}u_k} = A_k(r_k - s_{k+1})e^{r_k u_k} + B_k(s_k - s_{k+1})e^{s_k u_k} \tag{2.1.49}$$

$$B_{k+1}(s_{k+1} - r_{k+1})e^{s_{k+1}u_k} = A_k(r_k - r_{k+1})e^{r_k u_k} + B_k(s_k - r_{k+1})e^{s_k u_k}. \tag{2.1.50}$$

For each of these values, we need to compute the ratio $\rho = -\frac{B}{A}$ where $A = A_n$ and $B = B_n$. We aim at using a recursion for $\rho_k = -\frac{B_k}{A_k}$. From (2.1.49) and (2.1.50), we have :

$$\rho_{k+1} = e^{(r_{k+1}-s_{k+1})u_k}\frac{(r_k - r_{k+1})e^{r_k u_k} + \rho_k(r_{k+1} - s_k)e^{s_k u_k}}{(r_k - s_{k+1})e^{r_k u_k} + \rho_k(s_{k+1} - s_k)e^{s_k u_k}}, \quad k \in [1, n-1], \tag{2.1.51}$$

with starting value $\rho_1 = 1$. Indeed, in the article $B = 0$ and $A = 1$, and here $B = -1$ and $A = 1$, which explains the slight modification.

Finally, contrary to what is done in [Albrecher et al., 2011] we will not discuss other ω functions. The previous part was about discrete ruin functions but continuous functions are good candidates for $\omega(u)$ but they are not a crucial part of the main development because of two main reasons :

(1) The first one is related to the solutions of the diffential equation (2.1.35) because in the case where the coefficient $\omega(u)$ isn't a constant, the equation can't be solved using the characteristic equation method, and the solutions might be impossible to give explicitly, depending on the form of $\omega(u)$. In some cases, we could use Airy functions or Bessel functions to solve theoretically, but we are more focused on the implications of the $\omega-$ruin for the model than on the type of ω itself.

(2) The second reason is because continuous functions can be approximated by piecewise constant functions provided the interval $[0, a_1)$ is divided into enough sub-intervals. We can use this method to then find the coefficients A and B thanks to the above ratio.

For example, if we want to approximate a linear continuous $\omega(u)$, then we can divide the interval $[0, a_1)$ into n sub-interval of length $\frac{a_1}{n}$ and assign to each of the $u_k = \left[(k-1)\frac{a_1}{n},\ k\frac{a_1}{n}\right)$ the ruin intensity $\omega(u_k) = n - (k-1)$, for $k \in [1, n]$, then we could numerically look what happens for $n \longrightarrow +\infty$.

2.1.8. Numerical Illustrations for the EPVD

In this section we aim at illustrating the theorical results we have obtained. We choose to use MatLab code. The codes are available in **Annexe A**, at the end of the thesis.

We create the piecewise function epvd for expected present value of dividends. The x-axe is the initial surplus of the company, the y-axe is the EPVD obtained for each value of initial surplus u.

If we compare them to the EPVD obtained in [Albrecher et al., 2011], the changes happen in the definition interval, we begin with a capital of 0, not $-\infty$.

We choose classical parameters : $a_1 = 2$, $b = 4$, $\mu = 0.5$, $\delta = 0.05$ and try to focus on how the volatility σ, the dividend payment rate γ and the omega-ruin ω modify the graph.

Here is the expected present value of dividends for different volatilities σ. We impose $\gamma = 1000$ and $\omega = 1$.

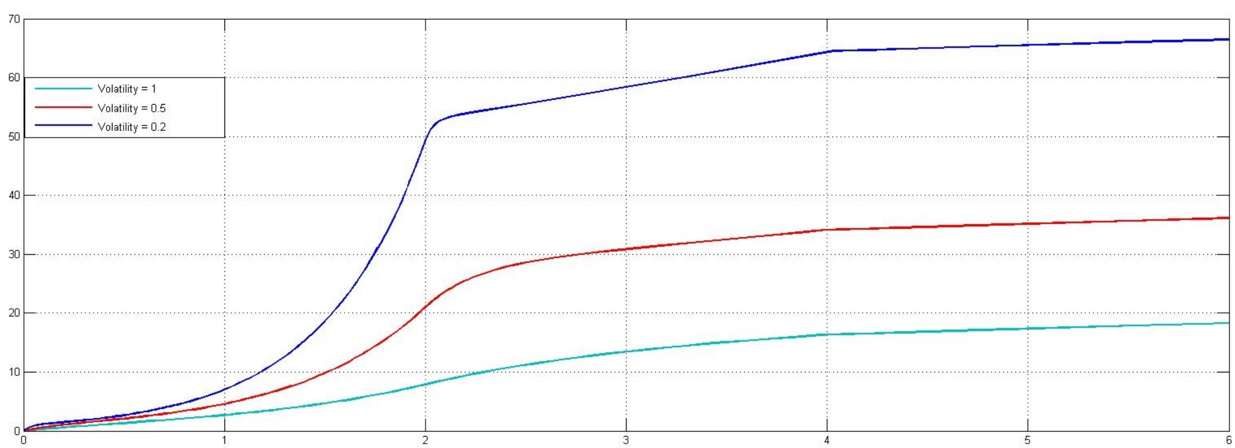


FIGURE 2.1. EPVD function between 0 and 6 for different σ

The graph shows that the EPVD is inversely proportional to σ.

We now observe the influence of ω on the graph. The parameters are $\gamma = 1000$ and $\sigma = 1$.

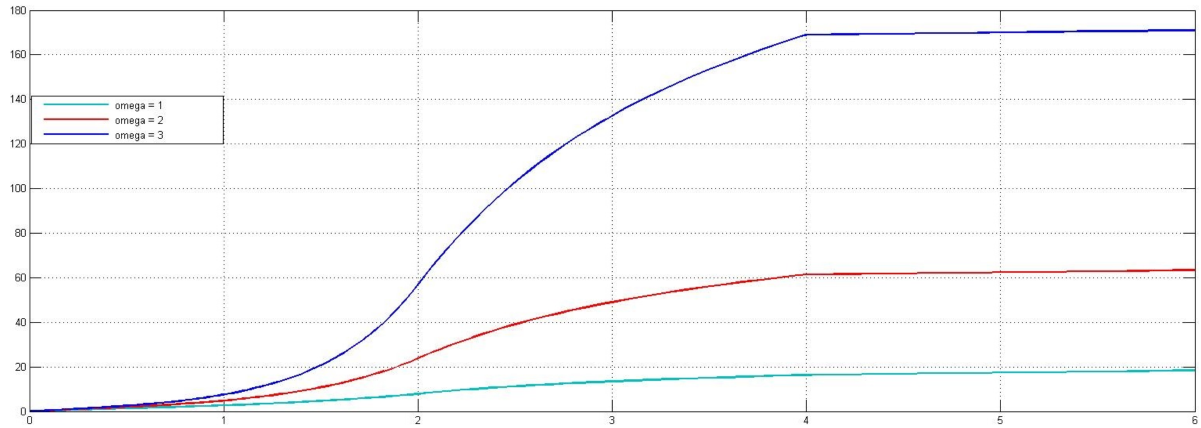


FIGURE 2.2. EPVD function between 0 and 6 for different values of ω.

We may also observe that with ω increasing, the EPVD becomes more and more concave.

Finally, the last parameter that is relevant to observe is γ. We set it to 1000 because it is a standard value but we can test with 10^5 and 1. We still set $\sigma = 1$ and $\omega = 1$.

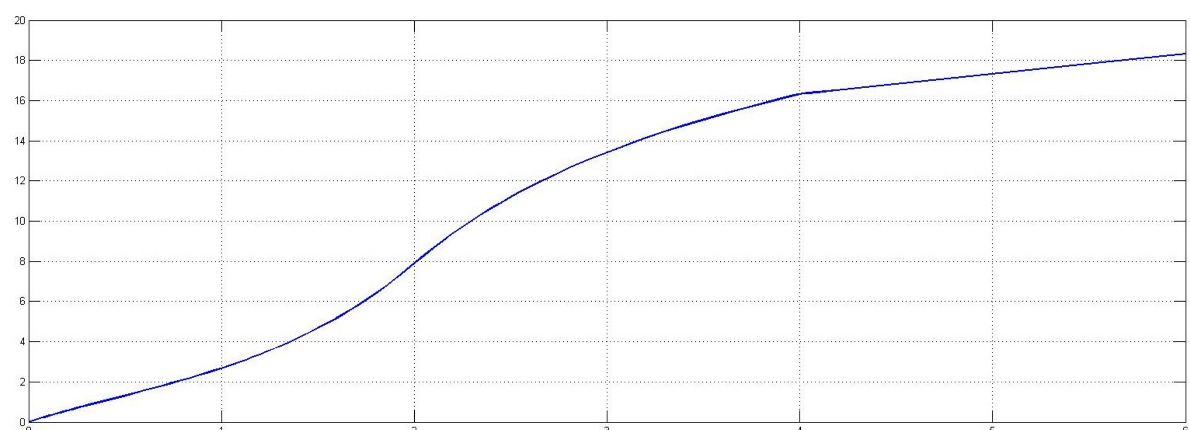
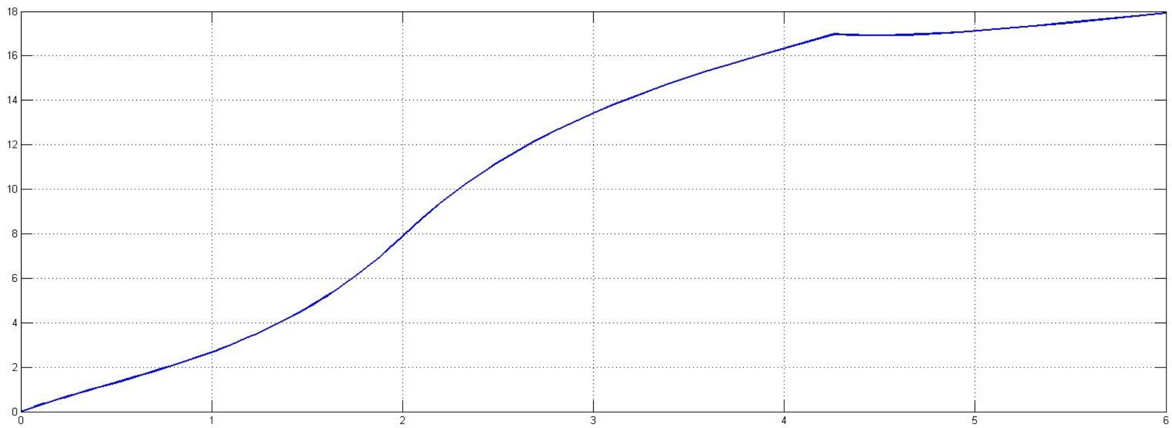


FIGURE 2.3. EPVD function between 0 and 6 for $\gamma = 10^5$.



(a) EPVD function between 0 and 6 for $\gamma = 1$. We notice that in this case, γ is too small for the model and the value function seems decreasing, which indicates that the model does not return correct results when γ tends to 0.

FIGURE 2.4. EPVD flawed because of γ too weak

Remark : Figure 2.3 does not look different from the one we obtain with the same parameters and $\gamma = 1000$. The continuous case happens for $\gamma \to \infty$ The second one however (Figure 2.4), seems a bit strange because the EPVD function is decreasing at the barrier. It seems to mean that a γ parameter too weak is not suitable for this type of strategy. This is discussed in chapter 3.

2.2. Optimal periodic strategy and Verification Lemma

2.2.1. Theoretical Expected Present Value of Dividends in the periodic framework

In this section, we recall the important steps to get the theoretical periodic expected present value of dividends. This function was first described in the periodic case as the following mathematical expectation, conditionaly to u, the initial level of capital held by the company :

$$J(u,\theta) = \mathbb{E}^u \left[\sum_{k=0}^{+\infty} e^{-\delta T_k} \vartheta_{T_k} I_{\{T_k<\tau\}} \right]. \tag{2.2.1}$$

It is quite simple to understand this formula. Its main part is the sum

$$\sum_{k=1}^{+\infty} e^{-\delta T_k} \vartheta_{T_k} I_{\{T_k<\tau\}} \tag{2.2.2}$$

$$= e^{-\delta T_1} \vartheta_{T_1} I_{\{T_1<\tau\}} \ + \ e^{-\delta T_2} \vartheta_{T_2} I_{\{T_2<\tau\}} \ + ... \tag{2.2.3}$$

where $e^{-\delta T_k}$ is the parameter related to the force of interest δ at decision time T_k. This coefficient plays the role of a "corrective term" that simply regulates the payouts at times T_k.

The dividend payouts are denoted by ϑ_{T_k} in this sum. They are the dividends that are paid (or not, depends on the strategy) at decision times T_k.

Finally $I_{\{T_k<\tau\}}$ is the characteristic function :

$$I_{\{T_k<\tau\}} = \begin{cases} 1 & \text{if } T_k < \tau \\ 0 & \text{if } T_k > \tau. \end{cases}$$

According to section 1.6 of literature review, their sequence represents the dividend strategy Θ.

$$\Theta = \{\vartheta_{T_1}, \vartheta_{T_2}, ..., \vartheta_{T_i}, ...\}. \tag{2.2.4}$$

We are interested in the optimal sequence $\Theta \in \mathcal{D}$ (the set of all admissible periodic strategies), denoted by Θ^*.

Under Θ^*, the EPVD is maximized and written

$$J(u,\Theta^*) = \mathbb{E}^u\left[\sum_{k=0}^{+\infty} e^{-\delta T_k}\vartheta^*_{T_k} I_{\{T_k<\tau\}}\right]. \quad (2.2.5)$$

We recall that the value function is defined as

$$V(u) = \sup_{\Theta\in\mathcal{D}} J(u,\Theta) = J(u,\Theta^*). \quad (2.2.6)$$

Existence of Θ^* is not that obvious. To prove that an optimal barrier exists we must show that there is at least one set of parameters for which $b^* > 0$. Essentially, since the structure of b^* is a logarithm, we need to show that the log argument is greater than 1. This can easily be verified numerically that such a model parameters set exists. In Matlab for example we have strictly positive values of b^* for standard parameters, however, given the general form of the log argument, it is very hard to show it analytically.

We need to model the periodic component of dividend payment. To do so, we introduce a mathematical tool. It is a Poisson process $\{N_\gamma(t)\}$ with parameter γ. We also recall that the aggregate amount of dividends $D(t)$ was a càdlàg process (continu a droite, limite a gauche). In this context, we can express $D(t)$ as :

$$D(t) = \int_0^t \vartheta_s dN_\gamma(s), \quad (2.2.7)$$

where $\{N_\gamma(t)\}$ is $\{\mathcal{F}_t\}$-adapted.

Dividend decisions can only occur when the process $\{N_\gamma(t)\}$ has jumps. This set of decision times is $\mathcal{T} = \{T_1, T_2, ...\}$ and the quantities $T_{k+1} - T_k \geq 0$ are the inter-dividend-decision times. In this thesis, these are assumed to be expentially distributed with mean $1/\gamma$. This is the same as in [Albrecher et al., 2011] and [Avanzi et al., 2014].

2.2.2. The Hamilton-Jacobi-Bellman equation

In this section, we aim at showing that a barrier strategy is the optimal strategy in the $\gamma - \omega$ model, which has never been done before. The methodology we use is the standard one. We assume a barrier is optimal then we test this assumption thanks to the Verification lemma, which is the main part of the test.

The last part of this section is another verification to show the uniqueness of the optimal strategy.

The first step is to determine the Hamilton-Jacobi-Bellman equation associated with our surplus. In that case, it is very easy to find thanks to the law of total probability we applied in the previous section.

From the part

$$V(u) + \{\gamma[l + V(u-l) - V(u)] \\ + \frac{\sigma^2}{2}V''(u) + \mu V'(u) - (\omega(u) + \delta)V(u)\}h + o(h),$$

which is line (2.1.8), we build the appropriate HJB equation on the model of [Avanzi et al., 2014]. The optimal dividend strategy is the l such that the expression in curly braces is maximized, which leads to the HJB equation we are looking for. Finally, we require $V(0) = 0$ in the first case because ultimate ruin (bankruptcy) occurs when the surplus is zero. This yields

$$\max_{0 \le l \le u} \{\gamma[l + V(u-l) - V(u)]\} + (\Omega - \delta)V(u) = 0 \quad \text{with } V(0) = 0, \tag{2.2.8}$$

where the operator Ω is the infinitesimal generator :

$$\Omega f = \frac{\sigma^2}{2} f'' + \mu f' - \omega(u) f. \tag{2.2.9}$$

According to the structure of this HJB, as well as in [Avanzi et al., 2014], we use the following lemma (Lemma 3.1) to assume the optimal periodic strategy is a pariodic barrier (the structure $l + V(u-l)$ is similar in both cases and motivates the assumption).

Lemma 3.1 of [Avanzi et al., 2014] If $V(u) \in \mathbb{C}^2$ is an increasing and concave function, with a point $b > 0$ such that $V'(u) = 1$, then
$\max_{0 \le l \le u} \{[l + V(u-l)]\}$ is achieved at

$$l = \begin{cases} 0 & \text{if } u \in [0, b) \\ u - b & \text{if } u \in [b, +\infty). \end{cases}$$

This lemma applies here because V is concave and a hypothesis about it is $V'(b) = 1$.

Proof We first differentiate $l + V(u-l)$ with respect to l and obtain

$$\frac{d}{dl}(l + V(u-l)) = 1 - V'(u-l). \tag{2.2.10}$$

We know that V is concave so V' is a decreasing function. Since b is the point such that $V'(b) = 1$, this means that

$$V'(u) > 1 \quad u \in [0, b) \tag{2.2.11}$$

$$V'(u) < 1 \quad u \in [b, +\infty). \tag{2.2.12}$$

When $u \in [0, b)$, we observe that for all possible values $0 \leq l \leq u$, we have $1 - V'(u - l) < 0$ and hence $l + V(u - l)$ is a strictly decreasing function of l. Since $l + V(u - l)$ is continuous, the maximum must occur at $l = 0$.

If we consider the other case, that is $u \in [b, +\infty)$, we observe that

$$1 - V'(u) < 0 \quad l \in [0, u - b), \tag{2.2.13}$$

$$1 - V'(u) \geq 0 \quad l \in [u - b, u]. \tag{2.2.14}$$

This implies that the function l+V(u-l) has a stationnary point at $l = u - b$, which is a maximum because

$$\frac{d^2}{dl^2}(l + V(u - l)) = V''(u - l) < 0. \tag{2.2.15}$$

In view of this theorem, the structure of the HJB equation suggests that the optimal periodic strategy is likely to be a periodic barrier strategy as defined below.

Definition [Periodic barrier dividend strategy, definition 3.2 of Avanzi et al. [2014]] Under a periodic barrier b, dividends payments are :

$$\vartheta_{T_i} = \max\,(X_{T_i} - b, 0), \tag{2.2.16}$$

at all dividend decision times T_i before bankruptcy.

It means that in order to pay dividends, each time that a decision time T_i happens, if the surplus is above b, then $X_{T_i} - b$ is paid, and 0 if the surplus is below b.

2.2.3. Verification lemma

In order to show that a strategy is optimal and unique, it has to pass some verification steps. These can be found in papers where optimality is the first goal, as in [Avanzi et al., 2014] or [Jeanblanc-Picqué and Shiryaev, 1995] (an old but standard reference). There are some major differences between our case and [Avanzi et al., 2014]. First, we don't need the Poisson Process part, and then we have a solvency constraint. The solvency constraint behaviour is described in the next chapter.

This lemma is built on advanced stochastic calculus like Ito's formula and martingales. We provide all the theorems and formulas used here in **Annexe B**

Theorem (Verification lemma) : Assume that we follow a periodic dividend strategy $\varTheta = \{\vartheta_{T_1}, \vartheta_{T_2}, \vartheta_{T_3}, ...\}$, where $\mathcal{T} = \{T_1, T_2, T_3, ...\}$ is the set of dividend decision times. For a non-negative function $H(u) \in C^1(\mathbb{R}^+)$ that is twice continuously differentiable except at countably many points, satisfying

(1) $\gamma \max_{0\leq l\leq u}\{l + H(u-l) - H(u)\} + (\Omega - \delta)H(u) \leq 0$,

(2) $0 \leq H'(u) < \infty, u \geq 0$,

(3) $H''(u) \leq 0, u \geq 0$,

(4) $\lim\limits_{u\to\infty} H'(u) < 1$, bounded (above) a linear function with slope less than 1

in which the integro-differential operator is defined by (2.2.9), then we have

$$H(u) \geq V(u),\ u \geq 0, \tag{2.2.17}$$

where the function $V(u)$ is defined as the optimal function (value function).
Furthermore, if there exists a point $b^* \geq 0$ such that $H(u) \in C^1(\mathbb{R}^+) \cap C^2(\mathbb{R}^+ \backslash \{b^*\})$ satisfying

$$\gamma \max_{0\leq l\leq u}\{l + H(u-l) - H(u)\} + (\Omega - \delta)H(u) = 0, \tag{2.2.18}$$

which is equivalent to

(5) $(\Omega - \delta)H(u) = 0, H'(u) \geq 1$, for $u \in [a_1, b^*]$,

(6) $(\Omega - \delta)H(u) < 0, H'(u) < 1$, for $u \in (b^*, \infty)$,

then

$$H(u) = V(u),\ u \geq 0, \tag{2.2.19}$$

and the optimal strategy is a periodic barrier strategy of level b^*.

Remark : We follow the steps presented in [Avanzi et al., 2014]. In fact we will make our first goal here to highlight the differences between this paper and our case.

Proof of Verification lemma : Firstly, we need Ito's formula for jump diffusion processes because $D(t)$ is a pure jump-diffusion process. Indeed, it depends on $\{N_\gamma(t)\}$ which is a Poisson process.

The proof starts with a slight modification in the application of the law of total probability

$$\gamma h(1-\delta h)\{l+\mathbb{E}[V(u+\mu h+\sigma W(h)-l)]\} \\ +(1-\gamma h-\omega h)(1-\delta h)\mathbb{E}[V(u+\mu h+\sigma W(h))]+o(h).$$

We notice that in the second part, $(1-\gamma h-\omega h)(1-\delta h)$ is the Taylor expansion of $e^{-\gamma h-\omega h}e^{-\delta h}$ which is equivalent to

$$e^{-\gamma h-\omega h-\delta h} \tag{2.2.20}$$

$$e^{-\delta h-\omega h}e^{-\gamma h} \tag{2.2.21}$$

which can be expanded according to Taylor :

$$(1-\delta h-\omega h)(1-\gamma h), \tag{2.2.22}$$

which could be interpreted from the dividend payments perspective : either dividends are paid with probability γh, or not, with probability $1-\gamma h$. The previous equation becomes thus :

$$\gamma h(1-\delta h)\{l+\mathbb{E}[V(u+\mu h+\sigma W(h)-l)]\} \\ +(1-\delta h-\omega h)(1-\gamma h)\mathbb{E}[V(u+\mu h+\sigma W(h))]+o(h).$$

Then according to the hypothesis of the lemma, the function we are going to develop using Ito's Formula is, for all $t \geq 0$,

$$\begin{aligned}
& e^{-(\delta+\omega)(t\wedge\tau)} H(X(t\wedge\tau)) \\
& = H(u) - \int_0^{t\wedge\tau} (\delta+\omega) e^{-(\delta+\omega)s} H(X(s)) ds \\
& + \int_0^{t\wedge\tau} e^{-(\delta+\omega)s} H'(X(s)) dX^{(\mu)}(s) \\
& + \frac{\sigma^2}{2} \int_0^{t\wedge\tau} e^{-(\delta+\omega)s} H''(X(s)) ds \\
& + \sum_{s\in\mathcal{T}, s\leq t\wedge\tau} e^{-(\delta+\omega)s} [H(X(s-) + \Delta X(s)) - H(X(s-))], \qquad (2.2.23)
\end{aligned}$$

where $X^{(\mu)}(s)$ is the continuous component of $X(s)$. Note that we have $X(s) - X(s-) \neq 0$ in the sum of (2.2.23) and $X(s) - X(s-) = 0$ in the integrals of (2.2.23). In addition, since $D(t)$ is a pure jump process, we have

$$dX^{(\mu)}(s) = \mu ds + \sigma dW(s). \qquad (2.2.24)$$

The first significant difference here is that we have no $S(t)$ process (which would be another compound Poisson process). We don't have to separate the sum part into two components because of this reason. It simplifies the proof.

The sum can be expressed using an integral. This is because a dividend decision time is made when a jump occurs in the compound Poisson process $\{N_\gamma(t)\}$.

Note that, if there is a jump in the process $\{N_\gamma(t)\}$ (i.e., $\Delta N_\gamma(t) = 1$), then a dividend decision is made ($\Delta X(t) = -\vartheta_t$). Therefore, we can rewrite the sum introduced in the previous equation as follows,

$$\begin{aligned}
& \sum_{s\in\mathcal{T}, s\leq t\wedge\tau} e^{-(\delta+\omega)s} [H(X(s-) + \Delta X(s)) - H(X(s-))] \\
& = \sum_{s\leq t\wedge\tau} e^{-(\delta+\omega)s} [H(X(s-) - \vartheta_s) - H(X(s-))] \Delta N_\gamma(s) \\
& = \sum_{s\leq t\wedge\tau} e^{-(\delta+\omega)s} [\vartheta_s + H(X(s-) - \vartheta_s) - H(X(s-))] \Delta N_\gamma(s) \\
& \quad - \sum_{s\leq t\wedge\tau} e^{-(\delta+\omega)s} \vartheta_s \Delta N_\gamma(s). \qquad (2.2.25)
\end{aligned}$$

Equation (2.2.25) can be expressed as the difference of two integrals, i.e.,

$$\begin{aligned}
&\sum_{s\in\mathcal{T},s\le t\wedge\tau} e^{-(\delta+\omega)s}[H(X(s-)-\Delta X(s))-H(X(s-))]\\
&=\int_0^{t\wedge\tau} e^{-(\delta+\omega)s}[\vartheta_s+H(X(s-)-\vartheta_s)-H(X(s-))]dN_\gamma(s)\\
&\quad-\int_0^{t\wedge\tau} e^{-(\delta+\omega)s}\vartheta_s dN_\gamma(s). \qquad (2.2.26)
\end{aligned}$$

Now that we have turned our sum into integrals we can use some stochastic calculus theorems to prove what we want :

Firstly, acknowledging that we expand the integral with respect to $dX^{(\mu)}(s)$ we can now write the Ito's formula as the following :

$$\begin{aligned}
&e^{-(\delta+\omega)(t\wedge\tau)}H(X(t\wedge\tau))\\
&=H(u)-\int_0^{t\wedge\tau}(\delta+\omega)e^{-(\delta+\omega)s}H(X(s))ds+\int_0^{t\wedge\tau}\sigma e^{-(\delta+\omega)s}H'(X(s))dW(s)\\
&+\int_0^{t\wedge\tau}e^{-(\delta+\omega)s}\mu H'(X(s))ds+\frac{\sigma^2}{2}\int_0^{t\wedge\tau}e^{-(\delta+\omega)s}H''(X(s))ds\\
&+\int_0^{t\wedge\tau}e^{-(\delta+\omega)s}[\vartheta_s+H(X(s-)-\vartheta_s)-H(X(s-))]dN_\gamma(s)\\
&-\int_0^{t\wedge\tau}e^{-(\delta+\omega)s}\vartheta_s dN_\gamma(s). \qquad (2.2.27)
\end{aligned}$$

Now we would like to find the operator Ω previously defined amongst the components of (2.2.27) to prove the first inequality.

Using the previous development we can write the formula again :

$$\begin{aligned}
&e^{-(\delta+\omega)(t\wedge\tau)}H(X(t\wedge\tau))\\
&=H(u)+\int_0^{t\wedge\tau}(\Omega-\delta)e^{-(\delta+\omega)s}H(X(s-))ds\\
&+\int_0^{t\wedge\tau}\sigma e^{-(\delta+\omega)s}H'(X(s))dW(s)\\
&+\int_0^{t\wedge\tau}e^{-(\delta+\omega)s}[\vartheta_s+H(X(s-)-\vartheta_s)-H(X(s-))]dN_\gamma(s)\\
&-\int_0^{t\wedge\tau}e^{-(\delta+\omega)s}\vartheta_s dN_\gamma(s). \qquad (2.2.28)
\end{aligned}$$

Where in the formula, $\Omega H(X(s)) = \frac{\sigma^2}{2} H(X(s)) + \mu H(X(s)) - \omega H(X(s))$.

Now we can notice that we can simplify the dividend-related term in the formula. We have :

$$\begin{aligned}
&\int_0^{t\wedge\tau} e^{-(\delta+\omega)s}[\vartheta_s + H(X(s-) - \vartheta_s) - H(X(s-))]dN_\gamma(s) \\
&= \int_0^{t\wedge\tau} e^{-(\delta+\omega)s}[\vartheta_s + H(X(s-) - \vartheta_s) - H(X(s-))]d(N_\gamma(s) - \gamma s) \\
&+ \int_0^{t\wedge\tau} e^{-(\delta+\omega)s}\gamma[\vartheta_s + H(X(s-) - \vartheta_s) - H(X(s-))]ds. \qquad (2.2.29)
\end{aligned}$$

For convient reasons we can denote :
$M_{t\wedge\tau} = \int_0^{t\wedge\tau} \sigma e^{-(\delta+\omega)s} H'(X(s))dW(s)$
$Z_{t\wedge\tau} = \int_0^{t\wedge\tau} e^{-(\delta+\omega)s}[\vartheta_s + H(X(s-) - \vartheta_s) - H(X(s-))]d(N_\gamma(s) - \gamma s)$.

This is because we need to show that $M_{t\wedge\tau}$ and $Z_{t\wedge\tau}$ are martingales. To do so, we can use the work of [Avanzi et al., 2014].

We first show that $M_{t\wedge\tau}$ is a martingale. Note that M_t is the Itō's integral of the process $\{\sigma e^{-(\delta+\omega)s} H'(X(s))\}$. In order to show that M_t is a square integrable martingale, we need to show that (Theorem 8.27 in [Klebaner, 2005])

$$\sup_{t\geq 0} \mathbb{E}^u \langle M, M\rangle(t) = \sup_{t\geq 0} \mathbb{E}^u \left[\int_0^t [\sigma e^{-(\delta+\omega)s} H'(X(s))]^2 d\langle W, W\rangle(s)\right] < \infty. \qquad (2.2.30)$$

Since we know that $d\langle W, W\rangle(s) = ds$, hence it suffices to verify that

$$\sup_{t\geq 0} \mathbb{E}^u \left[\int_0^t [\sigma e^{-(\delta+\omega)s} H'(X(s))]^2 ds\right] < \infty. \qquad (2.2.31)$$

Now because H is a concave function (around the payement area)(from condition 3), so $H'(X(s)) \leq H'(a_1)$ for all $s \geq 0$. Therefore we have

$$\begin{aligned}
\sup_{t\geq 0} \mathbb{E}^u \left[\int_0^t [\sigma e^{-(\delta+\omega)s} H'(X(s))]^2 ds\right] &\leq \sup_{t\geq 0} \mathbb{E}^u \left[\int_0^t [\sigma e^{-(\delta+\omega)s} H'(a_1)]^2 ds\right] \\
&= \sup_{t\geq 0} \int_0^t [\sigma e^{-(\delta+\omega)s} H'(a_1)]^2 ds \\
&= \sup_{t\geq 0} \frac{\sigma^2 H'(a_1)^2 (1 - e^{-2(\delta+\omega)t})}{2(\delta+\omega)} \\
&= \frac{\sigma^2 H'(a_1)^2}{2(\delta+\omega)} < \infty, \qquad (2.2.32)
\end{aligned}$$

which is true because H' is finite from condition 2. Hence M_t is a square integrable martingale. Now we can use Itō's isometry on M_t (Theorem 8.32 in [Klebaner, 2005]),

$$\begin{aligned}\sup_{t\geq 0}\mathbb{E}^u[M_t^2] &= \sup_{t\geq 0}\mathbb{E}^u\left[\int_0^t \sigma e^{-(\delta+\omega)s}H'(X(s))dW(s)\right]^2\\ &= \sup_{t\geq 0}\mathbb{E}^u\left[\int_0^t [\sigma e^{-(\delta+\omega)s}H'(X(s))]^2 d\langle W,W\rangle(s)\right]\\ &= \sup_{t\geq 0}\mathbb{E}^u\left[\int_0^t [\sigma e^{-(\delta+\omega)s}H'(X(s))]^2 ds\right] < \infty. \end{aligned} \tag{2.2.33}$$

Hence M_t is a uniformly integrable martingale (see Corollary 7.8 in [Klebaner, 2005]). Since τ is a stopping time, $M_{t\wedge\tau}$ is also a uniformly integrable martingale (see Theorem 7.14 in [Klebaner, 2005]).

Now we have to show that $Z_{t\wedge\tau}$ is a martingale.

We will first prove that Z_t is a uniformly integrable martingale. Note that Z_t is a stochastic integral with respect to a compensated Poisson process $\tilde{N}_\gamma(s) = N_\gamma(s) - \gamma s$, i.e.

$$Z_t = \int_0^t e^{-(\delta+\omega)s}[\vartheta_s + H(X(s-) - \vartheta_s) - H(X(s-))]d\tilde{N}_\gamma(s). \tag{2.2.34}$$

In order to show that Z_t is a square integrable martingale, we need to verify the following (Theorem 8.27 in [Klebaner, 2005])

$$\begin{aligned}&\sup_{t\geq 0}\mathbb{E}^u\langle Z,Z\rangle(t)\\ &= \sup_{t\geq 0}\mathbb{E}^u\left[\int_0^t e^{-(\delta+\omega)s}[\vartheta_s + H(X(s-) - \vartheta_s) - H(X(s-))]^2 d\langle \tilde{N}_\gamma,\tilde{N}_\gamma\rangle(s)\right] < \infty.\end{aligned} \tag{2.2.35}$$

We show $d\langle \tilde{N}_\gamma,\tilde{N}_\gamma\rangle(s) = \gamma ds$ in (B.5.3). Hence, to show that Z_t is a square integrable martingale, it suffices to verify

$$\begin{aligned}&\sup_{t\geq 0}\mathbb{E}^u\langle Z,Z\rangle(t)\\ &= \sup_{t\geq 0}\mathbb{E}^u\left[\int_0^t (e^{-(\delta+\omega)s}[\vartheta_s + H(X(s-) - \vartheta_s) - H(X(s-))])^2 \gamma ds\right] < \infty.\end{aligned} \tag{2.2.36}$$

Next, we show a following useful identity which is used to complete the proof. For $l \in [0, u-a_1]$ and $u \geq a_1$, we have

$$l + H(u-l) - H(u) \leq u. \tag{2.2.37}$$

The identity (2.2.37) can be easily verified. Since H is an increasing function that is defined on $\mathbb{R}_+$, then $H(u-l)-H(u)\leq 0$ and therefore $l+H(u-l)-H(u)\leq l\leq u$.

Now using (2.2.37), (2.2.36) becomes

$$\begin{aligned}\sup_{t\geq 0}\mathbb{E}^u\langle Z,Z\rangle(t) &= \sup_{t\geq 0}\mathbb{E}^u\left[\int_0^t (e^{-(\delta+\omega)s}[\vartheta_s + H(X(s-)-\vartheta_s) - H(X(s-))])^2\gamma ds\right]\\ &\leq \sup_{t\geq 0}\mathbb{E}^u\left[\int_0^t \gamma(e^{-(\delta+\omega)s}X(s))^2 ds\right] \qquad (2.2.38)\end{aligned}$$

Note that we can interchange of expectation and integral in (2.2.38) since the process $\{e^{-(\delta+\omega)s}X(s)\}$ has regular sample paths (left continuous with right limit), then using Fubini's theorem (Theorem 2.39 from [Klebaner, 2005]), we have

$$\mathbb{E}^u\left[\int_0^t \gamma e^{-2(\delta+\omega)s}X(s)^2 ds\right] = \int_0^t \mathbb{E}^u|\gamma e^{-2(\delta+\omega)s}X(s)^2|ds = \int_0^t \gamma e^{-2(\delta+\omega)s}\mathbb{E}^u(X(s)^2)ds. \qquad (2.2.39)$$

The Moment Generating Function (MGF) of $X(s)$ is

$$\begin{aligned}M_{X(s)}(t) &= e^{(u-cs)t}M_{\sigma W(s)}(t)M_{D(s)}(t)\\ &= e^{(u-cs)t}M_{\sigma W(s)}(t)M_{N_\gamma(s)}(\log M_{\theta_s}(t))\\ &= e^{(u-cs)t}e^{\frac{1}{2}\sigma^2 st^2}e^{\gamma s(M_{\theta_s}(t)-1)}. \qquad (2.2.40)\end{aligned}$$

Then the second moment of $X(s)$ can be derived using the second derivative of $M_{X(s)}(t)$ evaluated at $t=0$, i.e.

$$\begin{aligned}\mathbb{E}[X(s)^2] = M''_{X(s)}(t)\Big|_{t=0} &= M_{X(s)}(t)(\sigma^2 s + \gamma s M''_{\theta_s}(t))\Big|_{t=0}\\ &= \sigma^2 s + \gamma s M''_{\theta_s}(0)\\ &= (\sigma^2 + \gamma\mathbb{E}[\theta_s^2])s. \qquad (2.2.41)\end{aligned}$$

Since $0\leq\theta_s\leq X(s)\leq U(s)$ when $X(s)\geq 0$ and $\theta_s = 0$ when $X(s)<0$, then $\mathbb{E}[\theta_s^2]\leq\mathbb{E}[U(s)^2]$ for all $s\geq 0$. Hence using the result derived above, we have

$$\begin{aligned}\mathbb{E}[X(s)^2] &\leq (\sigma^2 + \gamma\mathbb{E}[U(s)^2])s\\ &= \sigma^2(s+\gamma s^2). \qquad (2.2.42)\end{aligned}$$

As a result, (2.2.38) becomes

$$=\gamma\sigma^2\sup_{t\geq 0}\left[\int_0^t (s+\gamma s^2)e^{-2(\delta+\omega)s}ds\right]$$

$$=\gamma\sigma^2\sup_{t\geq 0}\left[\frac{1-e^{-2(\delta+\omega)t}-2(\delta+\omega)te^{-2(\delta+\omega)t}}{4(\delta+\omega)^2} + \gamma\frac{1-e^{-2(\delta+\omega)t}-2(\delta+\omega)^2t^2e^{-2(\delta+\omega)t}-2(\delta+\omega)te^{-2(\delta+\omega)t}}{4(\delta+\omega)^3}\right]$$

$$=\gamma\sigma^2\left[\frac{\delta+\omega+\gamma}{4(\delta+\omega)^3}\right]<\infty.$$

Hence Z_t is a square integrable martingale (Theorem 8.27 in [Klebaner, 2005]). Furthermore, since $\sup_{t\geq 0}\mathbb{E}^u[Z_t^2] = \sup_{t\geq 0}\mathbb{E}^u\langle Z, Z\rangle(t) < \infty$ (Corollary 7.8 of [Klebaner, 2005]), then Z_t is a uniformly integrable martingale. Since τ is a stopping time, $Z_{t\wedge\tau}$ is also a uniformly integrable martingale (Theorem 7.14 in [Klebaner, 2005]).

Lastly for $Z_{t\wedge\tau}$, we first note that, if there is a jump in the process $\{N_\gamma(t)\}$ (i.e., $\Delta N_\gamma(t) = 1$), then a dividend decision is made ($\Delta X(t) = \vartheta_t$), which has the following impact on the process $\{\vartheta_s + H(X(s-) - \vartheta_s) - H(X(s-))\}$,

$$\vartheta_s + H(X(s-) - \vartheta_s) - H(X(s-)) = 0 \quad \Delta N_\gamma(s) = 0; \tag{2.2.44}$$

$$\vartheta_s + H(X(s-) - \vartheta_s) - H(X(s-)) \geq 0 \quad \Delta N_\gamma(s) = 1. \tag{2.2.45}$$

Equation (2.2.44) holds because when it is not a dividend decision time, no dividend will be issued, i.e. $\vartheta_s = 0$ when $\Delta N_\gamma(s) = 0$. When a dividend decision time arrives, dividends will be distributed when $\vartheta_s + H(X(s-) - \vartheta_s) - H(X(s-)) > 0$. Otherwise no dividends are to be paid and $\vartheta_s + H(X(s-) - \vartheta_s) - H(X(s-)) = 0$.

Now since Z_t is stochastic integral with respect to a compensated Poisson process, then by the properties of stochastic integral with respect to martingales [see page 216, property 1 in Klebaner, 2005], Z_t is a local martingale. In addition, since (2.2.44) and (2.2.45) establish that Z_t is also a non-negative local martingale. Then by Theorem 7.23 in Klebaner [2005], Z_t is a supermartingale, which makes the stopped process $\{Z_{t\wedge\tau}\}$ also a supermartingale [see page 9 point 17 in Borodin and Salminen, 2002].

Now since Z_t is uniformly bounded below by a uniformly integrable martingale, therefore, Z_t is a supermartingale, which makes the stopped process $\{Z_{t\wedge\tau}\}$ also a supermartingale [see page 9 point 17 in Borodin and Salminen, 2002].

In addition, since H is an increasing function, we have

$$\vartheta_s + H(X(s-) - \vartheta_s) - H(X(s-)) \leq \vartheta_s \leq X(s). \tag{2.2.46}$$

For $Z_{t\wedge\tau_n}$, since $X(t)$ is bounded by n and we compute the following integral

$$\int_0^{t\wedge\tau_n} |e^{-(\delta+\omega)s}[\vartheta_s + H(X(s-) - \vartheta_s) - H(X(s-))]|ds$$

$$= \int_0^{t\wedge\tau_n} e^{-(\delta+\omega)s} |[\vartheta_s + H(X(s-) - \vartheta_s) - H(X(s-))]| ds$$
$$\leq \int_0^{t\wedge\tau_n} e^{-(\delta+\omega)s} |X(s)| ds < \infty. \tag{2.2.47}$$

Hence by the martingale property of a compensated Poisson integral, $Z_{t\wedge\tau}$ is a martingale.

Now that we have $\mathbb{E}^u[M_{t\wedge\tau}] = 0$ and $\mathbb{E}^u[Z_{t\wedge\tau}] = 0$, and from condition 1, we know that

$$\gamma[\vartheta_s + H(X(s-) - \vartheta_s) - H(X(s-))] + (\Omega - \delta)H(X(s)) \leq 0. \tag{2.2.48}$$

Hence, taking expectation yields

$$H(u) \geq \mathbb{E}^u[e^{-(\delta+\omega)(t\wedge\tau)} H(X(t\wedge\tau))] + \mathbb{E}^u\left[\int_0^{t\wedge\tau} e^{-(\delta+\omega)s} \vartheta_s dN_\gamma(s)\right]. \tag{2.2.49}$$

For the first term of the right hand side of (2.2.49), using Fatou's lemma (Theorem 2.17 in [Klebaner, 2005]) and taking $t \to \infty$, we have

$$\liminf_{t\to\infty} \mathbb{E}^u[e^{-(\delta+\omega)(t\wedge\tau)} H(X(t\wedge\tau))] \geq \mathbb{E}^u[\liminf_{t\to\infty} e^{-(\delta+\omega)(t\wedge\tau)} H(X(t\wedge\tau))]. \tag{2.2.50}$$

By conditioning on the value of τ, we have

$$\begin{aligned}
&\liminf_{t\to\infty} e^{-(\delta+\omega)(t\wedge\tau)} H(X(t\wedge\tau)) \\
=&\liminf_{t\to\infty} e^{-(\delta+\omega)(t\wedge\tau)} H(X(t\wedge\tau)) 1_{\{\tau<\infty\}} + \liminf_{t\to\infty} e^{-(\delta+\omega)(t\wedge\tau)} H(X(t\wedge\tau)) 1_{\{\tau=\infty\}} \\
=&e^{-(\delta+\omega)(\tau)} H(X(\tau)) 1_{\{\tau<\infty\}} + \liminf_{t\to\infty} e^{-(\delta+\omega)(t\wedge\tau)} H(X(t\wedge\tau)) 1_{\{\tau=\infty\}} \\
\geq&e^{-(\delta+\omega)(\tau)} H(X(\tau)) 1_{\{\tau<\infty\}} \\
=&e^{-(\delta+\omega)(\tau)} H(0) 1_{\{\tau<\infty\}} \geq 0.
\end{aligned} \tag{2.2.51}$$

Therefore, we have

$$\liminf_{t\to\infty} \mathbb{E}^u[e^{-(\delta+\omega)(t\wedge\tau)} H(X(t\wedge\tau))] \geq 0. \tag{2.2.52}$$

For the second term on the right hand side of (2.2.49), we first observe that it is monotonely increasing as t increases. In addition, since $dN_\gamma(s)$ and ϑ_s are both non-negative, by the monotone convergence theorem (Theorem 2.16 in [Klebaner, 2005]) and conditioning on the value of τ we have

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{E}^u\left[\int_0^{t\wedge\tau} e^{-(\delta+\omega)s} \vartheta_s dN_\gamma(s)\right] \\
=& \lim_{t\to\infty} \mathbb{E}^u\left[\int_0^{t\wedge\tau} e^{-(\delta+\omega)s} 1_{\{\tau<\infty\}} \vartheta_s dN_\gamma(s)\right]
\end{aligned}$$

$$+\lim_{t\to\infty}\mathbb{E}^u\left[\int_0^{t\wedge\tau} e^{-(\delta+\omega)s}\vartheta_s 1_{\{\tau=\infty\}}dN_\gamma(s)\right]$$
$$\geq \mathbb{E}^u\left[\int_0^{\tau} e^{-(\delta+\omega)s}\vartheta_s 1_{\{\tau<\infty\}}dN_\gamma(s)\right]. \quad (2.2.53)$$

Therefore, combining (2.2.52) and (2.2.53), we find that

$$H(u)\geq \mathbb{E}^u\left[\int_0^{\tau} e^{-(\delta+\omega)s}\vartheta_s 1_{\{\tau<\infty\}}dN_\gamma(s)\right]=J(u;\Theta). \quad (2.2.54)$$

Since Θ is arbitrary, we have $H(u)\geq J(u,\Theta^*)$.

We now complete the proof by showing that $H(u)\leq J(u,\Theta^*)$. When an optimal strategy Θ^* is applied, condition 5 first implies that the integrals with respect to ds in Ito's formula is zero, i.e.,

$$\int_0^{t\wedge\tau} e^{-(\delta+\omega)s}(\Omega-\delta)H(X(s))ds$$
$$+\int_0^{t\wedge\tau} e^{-(\delta+\omega)s}\gamma[\vartheta_s^*+H(X(s-)-\vartheta_s^*)-H(X(s-))]ds=0. \quad (2.2.55)$$

Secondly from Lemma 3.1, we know that the maximum of condition 4 is attained when the dividend strategy is of a barrier type. In addition, Lemma 3.3 (see section 1.3 of literature review) also shows that probability of ruin is one for a periodic barrier strategy. Therefore ruin is certain when Θ^* is applied.

Now observe that since H is an increasing function, we have

$$\begin{aligned}Z_{t\wedge\tau}&=\int_0^{t\wedge\tau} e^{-(\delta+\omega)s}[\vartheta_s^*+H(X(s-)-\vartheta_s^*)-H(X(s-))]d(N_\gamma(s)-\gamma s)\\&\leq\int_0^{t\wedge\tau} e^{-(\delta+\omega)s}[\vartheta_s^*+H(X(s-)-\vartheta_s^*)-H(X(s-))]dN_\gamma(s)\\&\leq\int_0^{t\wedge\tau} e^{-(\delta+\omega)s}\vartheta_s^* dN_\gamma(s)\\&\leq\int_0^{\tau} e^{-(\delta+\omega)s}\vartheta_s^* 1_{\{\tau<\infty\}}dN_\gamma(s),\end{aligned} \quad (2.2.56)$$

which has finite expectation. Since $Z_{t\wedge\tau}$ is a supermartingale that is bounded above by a random variable Y with

$$\mathbb{E}^u[Y]=\mathbb{E}^u\left[\int_0^{\tau} e^{-(\delta+\omega)s}\vartheta_s^* 1_{\{\tau<\infty\}}dN_\gamma(s)\right]<\infty. \quad (2.2.57)$$

Then $Z_{t\wedge\tau}$ is a martingale with $\mathbb{E}^u[Z_{t\wedge\tau}]=0$.

After taking expectation with optimal strategy Θ^* applied, we arrive at

$$H(u) = \mathbb{E}^u[e^{-(\delta+\omega)(t\wedge\tau)}H(X(t\wedge\tau))] + \mathbb{E}^u\left[\int_0^{t\wedge\tau} e^{-(\delta+\omega)s}\vartheta_s^* dN_\gamma(s)\right]. \qquad (2.2.58)$$

Now for the first term on the right hand side of (2.2.58), we need to show the following

$$\lim_{t\to\infty} \mathbb{E}^u[e^{-(\delta+\omega)(t\wedge\tau)}H(X(t\wedge\tau))] = 0, \qquad (2.2.59)$$

which can be proven by using the dominated convergence theorem. We need to show that there is a random variable Y such that $e^{-(\delta+\omega)(t\wedge\tau)}H(X(t\wedge\tau)) < Y$ and $E[Y] < \infty$, then by the dominated convergence theorem (Theorem 2.18 in [Klebaner, 2005]), we have shown that (2.2.59) is true.

Firstly, since ruin is certain (from lemma 3.3),then ruin time is finite, i.e. $\tau < \infty$. Then by considering all possible values of t compared to τ, we have

$$\begin{aligned} e^{-(\delta+\omega)(t\wedge\tau)}H(X(t\wedge\tau)) &= e^{-(\delta+\omega)(t\wedge\tau)}H(X(t\wedge\tau))1\{t<\tau\} \\ &+ e^{-(\delta+\omega)(t\wedge\tau)}H(X(t\wedge\tau))1\{t\geq\tau\} \\ &= e^{-(\delta+\omega)t}H(X(t))1\{t<\tau\} + e^{-(\delta+\omega)\tau}H(X(\tau))1\{t\geq\tau\} \\ &= e^{-(\delta+\omega)t}H(X(t))1\{t<\tau\}. \end{aligned} \qquad (2.2.60)$$

Since H is linearly bounded from condition 4, there exists two constants a and b such that

$$\begin{aligned} e^{-(\delta+\omega)(t\wedge\tau)}H(X(t\wedge\tau)) &= e^{-(\delta+\omega)t}H(X(t))1\{t<\tau\} \\ &\leq ae^{-(\delta+\omega)t}X(t)1\{t<\tau\} + be^{-(\delta+\omega)t}1\{t<\tau\} \\ &\leq aX(t)1\{t<\tau\} + b. \end{aligned} \qquad (2.2.61)$$

Therefore now it suffices to show that $X(t)1\{t<\tau\}$ is bounded by an integrable random variable to show (2.2.59). Firstly, since ruin is certain (lemma 3.1 of [Avanzi et al., 2014]), therefore ruin time is finite, i.e. $\tau < \infty$. Then by considering possible values of t compared to τ, we have

$$\begin{aligned} e^{-(\delta+\omega)(t\wedge\tau)}H(X(t\wedge\tau)) &= e^{-(\delta+\omega)(t\wedge\tau)}H(X(t\wedge\tau))1\{t<\tau\} \\ &+ e^{-(\delta+\omega)(t\wedge\tau)}H(X(t\wedge\tau))1\{t\geq\tau\} \\ &= e^{-(\delta+\omega)t}H(X(t))1\{t<\tau\} + e^{-(\delta+\omega)\tau}H(X(\tau))1\{t\geq\tau\} \\ &= e^{-(\delta+\omega)t}H(X(t))1\{t<\tau\}. \end{aligned} \qquad (2.2.62)$$

Since H is linearly bounded, then there exists two constants a and b such that

$$\begin{aligned} e^{-(\delta+\omega)(t\wedge\tau)}H(X(t\wedge\tau)) &= e^{-(\delta+\omega)t}H(X(t))1\{t<\tau\} \\ &\leq ae^{-(\delta+\omega)t}X(t)1\{t<\tau\} + be^{-(\delta+\omega)t}1\{t<\tau\} \\ &\leq aX(t)1\{t<\tau\} + b. \end{aligned} \tag{2.2.63}$$

Therefore it suffices to show that $X(t)1\{t<\tau\}$ is bounded by an integrable random variable to show (2.2.59).

We make the following observation first. Since τ is finite almost surely, then there are finitely many dividend decisions made before ruin. Without loss of generality, we consider an event that n dividend decisions are made prior to ruin and occur at time $\mathcal{T} = \{T_1, T_2, ..., T_n\}$, the finite ruin time implies n is also finite. Also since τ is necessarily less than the potential $n+1$th dividend decision time, therefore we have

$$X(t)1\{t<\tau\} \leq \sup_{t\in[0,\tau]} X(t) \leq \sup_{t\in[0,T_{n+1}]} X(t). \tag{2.2.64}$$

In addition, the nature of a periodic barrier strategy implies that the modified surplus process $X(t)$ is bounded by b at every dividend decision time, i.e.,

$$0 \leq X(T_k) \leq b \quad \text{for } k = 1, 2, ..., n, n+1. \tag{2.2.65}$$

Furthermore, the modified surplus process $X(t)$ is the original surplus process $U(t)$ in-between all dividend decision times before ruin, then it suffices to look at the process $U(t)$, for $t \in [0, T_1) \cup [T_1, T_2) \cup ... \cup [T_{n-1}, T_n) \cup [T_n, T_{n+1})$. Also since at each dividend decision time, $X(t)$ takes a maximum value of b, thus the original surplus process $U(t)$ resets at most to level b at each dividend decision time $\mathcal{T} = \{T_1, T_2, ..., T_n\}$. As a result, we have

$$\sup_{t\in[0,T_{n+1}]} X(t) \leq \sup_{t\in[0,T_{n+1}]} U_n(t), \tag{2.2.66}$$

where $U_n(t)$ is defined as follows,

$$\begin{aligned} U_n(t) = U(t)&1\{t\in[0,T_1)\}|U(0)=u \\ &+ \sum_{k=1}^{n} U(t)1\{t\in[T_k,T_{k+1})\}|U(T_k)=b. \end{aligned} \tag{2.2.67}$$

When taking the supremum of the first term in (2.2.67), we make the following decomposition,

$$\begin{aligned}\sup_{t\in(0,T_1)}\{U(t)|U(0)=u\} &= \sup_{t\in(0,T_1)}\{u+\mu t+\sigma W(t)\}\\ &\leq \sup_{t\in(0,T_1)}\{u+\mu t\}+\sigma\sup_{t\in(0,T_1)}\{W(t)\}\\ &= u+\mu T_1+\sigma\sup_{t\in(0,T_1)}\{W(t)\}. \end{aligned} \tag{2.2.68}$$

So when taking expectation of (2.2.68), we have $E[u+\mu T_1]=u+\mu T_1$. Furthermore, we know that from Equation 1.1.1 of [Borodin and Salminen, 2002], we have

$$E\left[\exp(t\sup_{0\leq s\leq T_1}W(s))\right]=\frac{\sqrt{2\gamma}}{\sqrt{2\gamma}-t}, \tag{2.2.69}$$

which give

$$E\left[\sup_{0\leq s\leq T_1}W(s)\right]=\frac{d}{dt}E\left[\exp(t\sup_{0\leq s\leq T_1}W(s))\right]\Big|_{t=0}=\frac{d}{dt}\frac{\sqrt{2\gamma}}{\sqrt{2\gamma}-t}\Big|_{t=0}=\frac{1}{\sqrt{2\gamma}}. \tag{2.2.70}$$

Hence substituting (2.2.70) and (2.2.68) into (2.2.68) yields

$$E\left[\sup_{t\in(0,T_1)}U(t)|U(0)=u\right]\leq u+\mu T_1+\frac{\sigma}{\sqrt{2\gamma}}. \tag{2.2.71}$$

For the second term in (2.2.67), the exponential arrival time and its Markovian structure of the surplus process, we have, for $k=1,2,3,...,n$,

$$E\left[\sup_{t\in(T_k,T_{k+1})}U(t)|U(T_k,)=b\right]=E\left[\sup_{t\in(0,T_1)}U(t)|U(0)=b\right]\leq b+\frac{\sigma}{\sqrt{2\gamma}}. \tag{2.2.72}$$

Hence taking expectation of the supremum of (2.2.67), and using (2.2.71) and (2.2.72) , we have

$$E\left[\sup_{t\in(0,T_{n+1})}U_n(t)\right]\leq(u+b+\mu T_{n+1})+(n+1)\frac{\sigma}{\sqrt{2\gamma}}. \tag{2.2.73}$$

Therefore we have

$$X(t)1\{t<\tau\}\leq\sup_{t\in[0,T_{n+1}]}U_n(t),\quad\text{with}\quad E\left[\sup_{t\in(0,T_{n+1})}U_n(t)\right]<\infty. \tag{2.2.74}$$

Hence by letting $Y = \sup_{t\in(0,T_{n+1})}\{aU_n(t)+b|U(0)=u\}$, we identify an integrable random variable such that the process $\{e^{-(\delta+\omega)(t\wedge\tau)}H(X(t\wedge\tau))\}$ is dominated by Y and $E[Y]<\infty$. The proof of (2.2.59) is complete.

For the second term on the right hand side of (2.2.58), observe that it is a monotonely increasing t increases. In addition, $dN_\gamma(s)$ and ϑ_s^* are both non-negative. Hence when taking $t\to\infty$, by the monotone convergence theorem, it suffices to only consider the case when $\tau<\infty$, i.e.,

$$\begin{aligned}\lim_{t\to\infty}\mathbb{E}^u\left[\int_0^{t\wedge\tau}e^{-(\delta+\omega)s}\vartheta_s^*dN_\gamma(s)\right] &= \lim_{t\to\infty}\mathbb{E}^u\left[\int_0^{t\wedge\tau}e^{-(\delta+\omega)s}\vartheta_s^*1_{\{\tau<\infty\}}dN_\gamma(s)\right]\\ &= \mathbb{E}^u\left[\int_0^{\tau}e^{-(\delta+\omega)s}\vartheta_s^*1_{\{\tau<\infty\}}dN_\gamma(s)\right]. \quad (2.2.75)\end{aligned}$$

Hence, combining (2.2.59) and (2.2.75), we have for (2.2.58),

$$H(u)\le J(u;\Theta^*). \quad (2.2.76)$$

2.2.4. Verification of optimality

We have shown that a barrier strategy is optimal in the case of periodic dividends with $\omega-$ruin. Now, a final verification needs to be done. The optimal barrier b^* is given in the literature review at equation (1.5.9). We want to find out whether our candidate solution $G(u,b^*)$ is the one optimal or not. To do so, we show that the candidate function $G(u,b^*)$ satisfies the HJB equation, and is increasing, concave and linear bounded (above). That is, for $u\ge 0$

(1) $\max\limits_{0\le l\le u}\{\gamma[l+G(u-l,b^*)-G(u,b^*)]\}+(\Omega-\delta)G(u,b^*)=0$

(2) $0<G'(u,b^*)<\infty$

(3) $G''(u,b^*)<0$

(4) $\lim_{u\to\infty}G''(u,b^*)=0$

The proof is the following :

Proof

(1) By construction, the value function $G(u, b^*)$ is a solution the three differential equations. Therefore $G(u, b^*)$ satisfies the HJB equation and attains maximum.

(2) $G'_L(u, b^*) = r_\omega e^{r_\omega u} - s_\omega e^{s_\omega u}$. By construction, s_ω is the negative root of the characteritic equation so $-s_\omega > 0$ and $G'_L(u, b^*) > 0$.

$G'_M(u, b^*) = rAe^{ru} + sBe^{su}$ where A and B are :

$$A = \frac{(r_\omega - s)e^{r_\omega a_1} + (s - s_\omega)e^{s_\omega a_1}}{e^{ra_1}(r - s)} \text{ and } B = \frac{(r - r_\omega)e^{r_\omega a_1} + (s_\omega - r)e^{s_\omega a_1}}{e^{sa_1}(r - s)}.$$

We have $r_\omega > r$ and $s_\omega < s$. From which we deduce that A is positive and B is negative. Then, $rA > 0$ and $sB > 0$ because $s < 0$ too. Finally, $G'_M(u, b^*) > 0$.

Lastly, $G_U(u, b^*) = \left(\frac{\delta}{\delta + \gamma} G_U(b^*, b^*) + \frac{\mu\gamma}{(\delta + \gamma)^2} \right) e^{s_\gamma (u - b^*)}$
$+ \frac{\gamma}{\delta + \gamma}[u - b^* + G_U(b^*, b^*)] + \frac{\mu\gamma}{(\delta + \gamma)^2}$.

When $u = b^*$, then $e^{s_\gamma(u-b^*)} = 1$ and $G'_U(u, b^*) > 0$. Then, when $u \longrightarrow +\infty$, $e^{s_\gamma(u-b^*)} \longrightarrow 0$ because $s_\gamma < 0$. $G'_U(u, b^*)$ solely depends on the term in $[u - b^* + G_U(u, b^*)]$ which is increasing so $G_U(u, b^*)$ is a striclty increasing function of u and $G'_U(u, b^*) > 0$

(3) The function $G_L(u, b^*)$ isn't concave but convex. It dosen't matter because the only interesting part are around the barrier, so the verification needs to be done for $G_M(u, b^*)$ and $G_U(u, b^*)$.

We first show that $G'_M(u; b^*)$ is strictly decreasing. We already know that $G'_M(u; b^*) > 0$. This function is monotonic (because it is a sum of exponentials over a positive interval). Because we know the function is monotonic, we simply have to show that for two points $u_1 < u_2$, we have $G'_M(u_2, b^*) - G'_M(u_1, b^*) < 0$. We chose the points a_1 and b^* because $G'_M(b^*, b^*) = 1$ (see classic result (1.5.10) in the literature review).

$$G'_M(b^*, b^*) - G'_M(a_1, b^*) = 1 - (r_\omega e^{r_\omega a_1} - s_\omega e^{s_\omega a_1}) \tag{2.2.77}$$

and $r_\omega e^{r_\omega a_1} - s_\omega e^{s_\omega a_1} > 1$ so $G'_M(u, b^*)$ is stricty decreasing, which is equivalent to say $G''_M(u, b^*) < 0$, and the function is concave.

Then, $\lim_{u \to +\infty} G_U(u, b^*) = \dfrac{\gamma}{\delta + \gamma} < 1$ and $G'_U(b^*, b^*) = 1$ so, because it is monotonic it suffices that

$$\lim_{u \to +\infty} G_U(u, b^*) - G'_U(b^*, b^*) < 0 \tag{2.2.78}$$

and $G''_U(u, b^*)$ is concave.

Finally, because $G(u, b^*)$ is twice continuously diffentiable, there is no issue at $u = b^*$ and the global function is concave around b^*.

(4) This part is about $G''_U(u, b^*) = s_\gamma^2 \left(\dfrac{\delta}{\delta + \gamma} G_U(b^*, b^*) + \dfrac{\mu\gamma}{(\delta + \gamma)^2} \right) e^{s_\gamma (u - b^*)}$.
Immediately, when $u \longrightarrow +\infty$, $e^{s_\gamma (u - b^*)} \longrightarrow 0$.

Chapitre 3

THE OPTIMAL PERIODIC BARRIER STRATEGY AND A NEW OPTIMALITY THEOREM WITH SOLVENCY CONSTRAINT

In this chapter, we aim at extending the works of [Albrecher et al., 2011]. They only consider the barrier above a_1 (or in their case, above 0) but we build on their works to show that it is possible, according to the structure of b^*, that $b^* < a_1$. We now show how to deal with such a situation, by adapting an important theorem from [Paulsen, 2003] to our case. This theorem has to be adapted for periodic dividends and shows that the optimal strategy in the case $b^* < a_1$ to maximize the EPVD is to use a barrier strategy at $b^* = a_1$. This represents a new contribution, because this theorem have never been proved correct in the periodic framework or $\gamma - \omega$ framework. We then aim at obtaining the new equations for the surplus described in [Albrecher et al., 2011] in the case where the barrier is now equal to the solvency contraint.

3.1. Optimal periodic barrier strategy

In this section, we aim at finding b^*, the optimal periodic dividend barrier. mathematically, the problem is similar to the study of a function and the determination of its stationary point(s). All we need to do is differentiate the EPVD function on $[b, +\infty)$ because it is the part of the function where b plays a role.

This section is similar to the one in [Albrecher et al., 2011] because $G_U(u, b)$ is the same function defined over the same interval of existence in both cases, but it is not useless to mention how it is found.

3.1.1. Finding the optimal barrier b^*

We recall the expression of the EPVD for $u \in [b, +\infty)$

$$G_U(u,b) = \left(\frac{\delta}{\delta+\gamma}G_U(b,b) + \frac{\mu\gamma}{(\delta+\gamma)^2}\right)e^{s_\gamma(u-b)} + \frac{\gamma}{\delta+\gamma}[u-b+G_U(b,b)] + \frac{\mu\gamma}{(\delta+\gamma)^2}. \quad (3.1.1)$$

We now differentiate this function :

$$\frac{dG_U(u,b)}{du} = s_\gamma\left(\frac{\delta}{\delta+\gamma}G_U(b,b) - \frac{\mu\gamma}{(\delta+\gamma)^2}\right)e^{s_\gamma(u-b)} + \frac{\gamma}{\delta+\gamma}. \quad (3.1.2)$$

The continuity condition of the EPVD function at $u = b$ yields

$$\left.\frac{dG_U(u,b)}{du}\right|_{u=b} = G'_U(b,b) = s_\gamma\left(\frac{\delta}{\delta+\gamma}G_U(b,b) - \frac{\mu\gamma}{(\delta+\gamma)^2}\right) + \frac{\gamma}{\delta+\gamma}, \quad (3.1.3)$$

then rearranging the terms yields the expected equation :

$$G'_U(b,b) - s_\gamma\frac{\delta}{\delta+\gamma}G_U(b,b) = -s_\gamma\frac{\mu\gamma}{(\delta+\gamma)^2} + \frac{\delta\gamma+\gamma^2}{(\delta+\gamma)^2}. \quad (3.1.4)$$

To obtain another solvable equation, we can differentiate each side once more, and then b^* is the value of b that satisfies :

$$G''_U(b^*,b^*) - s_\gamma\frac{\delta}{\delta+\gamma}G'_U(b^*,b^*) = 0. \quad (3.1.5)$$

We recall that we know explicit expressions for $G'_U(b^*, b^*)$ and $G''_U(b^*, b^*)$. Indeed, solving both the Middle and the Lower equation in the previous section yields explicit new values for A and B. We have :

$$G'_U(b^*, b^*) = Are^{rb^*} + Bse^{sb^*} \tag{3.1.6}$$

$$G''_U(b^*, b^*) = Ar^2e^{rb^*} + Bs^2e^{sb^*}. \tag{3.1.7}$$

These results are the consequence of the continuity condition at $u = b$ and in particular $u = b^*$ and we can then use the (known) value of G_M to find them.

Using (3.1.6) and (3.1.7) in (3.1.5) yields :

$$Ar^2e^{rb^*} + Bs^2e^{sb^*} - s_\gamma \frac{\delta}{\delta+\gamma} Are^{rb^*} - s_\gamma \frac{\delta}{\delta+\gamma} Bs^2e^{sb^*}. \tag{3.1.8}$$

That is, after rearranging the terms and simplifying with the natural logarithm :

$$rb^* + \ln\left(Ar^2 - s_\gamma \frac{\delta}{\delta+\gamma} Ar\right) = \ln\left(-Bs^2 + s_\gamma \frac{\delta}{\delta+\gamma} Bs\right) + sb^* \tag{3.1.9}$$

which finally yields

$$b^* = \frac{1}{r-s} \ln\left[\frac{B\left(-s^2 + s_\gamma \frac{\delta}{\delta+\gamma} s\right)}{A\left(r^2 - s_\gamma \frac{\delta}{\delta+\gamma} r\right)}\right]. \tag{3.1.10}$$

Following the example of [Albrecher et al., 2011], for both associated characteritic equations to the upper and middle parts, we can use the relation roots / coefficients (also known as Vieta's formulas) to obtain :

$$\frac{\delta}{\delta+\gamma} = \frac{rs}{r_\gamma s_\gamma}. \tag{3.1.11}$$

The barrier b^* can thus be written

$$b^* = \frac{1}{r-s} \ln\left[\frac{-Bs^2(r_\gamma - r)}{Ar^2(r_\gamma - s)}\right]. \tag{3.1.12}$$

Because of the logarithm properties, we can as well write this expression as a sum :

$$b^* = \frac{1}{r-s}\ln\left[\frac{s^2}{r^2}\right] + \frac{1}{r-s}\ln\left[\frac{-B}{A}\right] + \frac{1}{r-s}\ln\left[\frac{r_\gamma - r}{r_\gamma - s}\right]. \tag{3.1.13}$$

Remark : All the expressions for b^* are the same as these given in [Albrecher et al., 2011]. However, the value of these expressions is very different from the one in the article because A and B are modified in our case, explaining why the value of b^* is completely different despite an unmodified formula.

3.1.2. Results for the value function and its derivative at b^*

Now that we have determined b^*, our goal is to give some results for the value function under such barrier.

The first result is that

$$G'_M(b^*, b^*) = G'_U(b^*, b^*) = 1. \tag{3.1.14}$$

This result comes from the fact that equation :

$$\max_{0 \leq l \leq u} \{\gamma[l + G(u-l) - G(u)]\} + (\Omega - \delta)G(u) = 0 \text{ with } G(0) = 0 \tag{3.1.15}$$

is maximized when G' is equal to 1. This is proved by Lemma 3.1 in [Avanzi et al., 2014] (see section 1.6 of literature review)

Another expected result is the explicit value of $G(b^*, b^*)$. We recall equation

$$\frac{\sigma^2}{2}G''_M(u;b) + \mu G'_M(u;b) - \delta G_M(u;b) = 0, \tag{3.1.16}$$

which is also valid at b^* :

$$\frac{\sigma^2}{2}G''_M(b^*;b^*) + \mu G'_M(b^*;b^*) - \delta G_M(b^*;b^*) = 0. \tag{3.1.17}$$

In the above equation, we replace $G'_M(b^*;b^*)$ with its value 1. We also had, for b^* (equation (3.1.5))

$$G''_U(b^*, b^*) - s_\gamma \frac{\delta}{\delta + \gamma} G'_U(b^*, b^*) = 0, \tag{3.1.18}$$

which is now

$$G''_U(b^*, b^*) = s_\gamma \frac{\delta}{\delta + \gamma}. \tag{3.1.19}$$

All these simplifications lead to the expected result :

$$G(b^*, b^*) = \frac{\sigma^2}{2} s_\gamma \frac{1}{\delta + \gamma} + \frac{\mu}{\delta}. \tag{3.1.20}$$

The last step is to aknowledge that s_γ is a solution of the characteristic equation for this upper part, that is :

$$\frac{\sigma^2}{2} s_\gamma^2 = -\mu s_\gamma + (\delta + \gamma). \tag{3.1.21}$$

Upon subsitution in the previous equation, we get the expected result under its final form :

$$G(b^*, b^*) = \frac{\mu}{\delta} - \frac{\mu}{\delta + \gamma} + \frac{1}{s_\gamma}. \tag{3.1.22}$$

Remark : An important thing to observe is that the expected present value of dividends at b^* under the optimal barrier strategy does not depend on the omega-ruin function $\omega(u)$ whereas b^* depends on it.

We can interpret the result based upon its different parameters as it follows

Arccording to the different papers on the topic, the classical result in case of a continuous dividend payment rate ($\gamma \to +\infty$) is :

$$G(b^*, b^*) = \frac{\mu}{\delta}. \tag{3.1.23}$$

What is observed theoretically confirms the numerical results : if γ is too low, then we have some issues because the second term is too close to the first one and since s_γ depends on γ, G might become negative which is not allowed by the model.

3.1.3. Numerical issues for b^*

The following part is not in [Albrecher et al., 2011]. In the previous section, we have determined the optimal value for b^*, which is :

$$b^* = \frac{1}{r-s} \ln \left[\frac{-Bs^2(r_\gamma - r)}{Ar^2(r_\gamma - s)} \right], \tag{3.1.24}$$

where A and B are known constants. The logarithm function is positive on the interval $[1, +\infty)$ but negative on $(0, 1)$, so it is possible that b^* becomes negative. The issue is more precisely located in the numerator of the log function.

Recall that r_γ is the positive solution of the characteristic equation

$$\frac{\sigma^2}{2}\xi^2 + \mu\xi - (\delta + \gamma) = 0 \tag{3.1.25}$$

and r is the positive solution of

$$\frac{\sigma^2}{2}\xi^2 + \mu\xi - \delta = 0. \tag{3.1.26}$$

If $\gamma \longrightarrow 0$, then the former equation turns into the latter and the consequence is that $r_\gamma \longrightarrow r$, which leads to $r_\gamma - r = 0$ in the log numerator . In that case, and because $Ar^2(r_\gamma - s) \neq 0$, we have successively $b^* < a_1$ then $b^* < 0$. Even worse, $b^* \longrightarrow -\infty$.

It is quite obvious that a company cannot expect sustainability for its dividends outcomes when $b^* = 0$. This level is quite significant, but not too important in term of impact on the model because the threshold that matters the most is the a_1 threshold. Because the log function is continuous on $(0, +\infty)$, and stricly increasing, there exists exactly one γ, denoted γ_0 for which $b^* = a_1$.

Remark : In the case where $\gamma \longrightarrow 0$, equation (3.1.22) becomes

$$G(b^*, b^*) = \frac{1}{s_\gamma} = \frac{1}{s}. \tag{3.1.27}$$

We now understand that the case $b^* > a_1$, analog to the one discussed in [Albrecher et al., 2011] is valid for $\gamma > \gamma_0$. The next section describes the process when $\gamma < \gamma_0$.

3.2. The case $b^* < a_1$

3.2.1. The new optimal result from Paulsen in the case of periodic dividends when the barrier is a_1

Consequently to the last result, we assume that $\gamma < \gamma_0$, that is $b^* < a_1$. The $\gamma - \omega$ model areas are now $[0, b^*)$, $[b^*, a_1)$ and $[a_1, +\infty)$. The first problem that arises is that dividend distribution is not allowed below a_1, so the $[b^*, a_1)$ part is not clear. Fortunately, this issue is addressed by an optimality result from [Paulsen, 2003]

Theorem [Theorem 2.2 from Paulsen (2003)]

In the case where $b < +\infty$, if $a_1 > b$ then the optimal policy is to use a barrier strategy at a_1.

However, we cannot use this theorem in its 2003 version because :

— It does not consider the $\omega-$ruin.

— It does not consider periodic dividends, only continuous dividends.

The proof needs to be adapted. It is based upon the previous verification lemma. This is an extension of the previous verification lemma where we consider any b,

Theorem : Periodic version of [Paulsen, 2003]

(1) if the periodic barrier is so that $b > a_1$, the optimal strategy is a barrier of level b

(2) in the case $b < a_1$, the EPVD is maximized using a periodic barrier strategy at a_1

Proof : (1) is the previous verification lemma. (2) Let's develop Ito's formula for $H_{a_1}(X(t))$, the candidate for optimality under a barrier at a_1.

The new HJB equation, and the new condition 1 of verification lemma would be in such case :

$$\gamma \max_{0 \leq l \leq u} \{l + H_{a_1}(u - l) - H_{a_1}(u)\} + (\Omega - \delta) H_{a_1}(u) \leq 0, \tag{3.2.1}$$

with $l = 0$ below a_1 and $l = u - a_1$ above a_1

$$
\begin{aligned}
&e^{-(\delta+\omega)(t\wedge\tau)}H_{a_1}(X(t\wedge\tau)) = H_{a_1}(u)\\
&+\int_0^{t\wedge\tau} e^{-(\delta+\omega)s}\left[\frac{\sigma^2}{2}H''_{a_1}(X(s)) + \mu H'_{a_1}(X(s)) - (\delta+\omega)H_{a_1}(X(s))\right]ds\\
&+\int_0^{t\wedge\tau} e^{-(\delta+\omega)s}\sigma H'_{a_1}(X(s))dW_s\\
&+\int_0^{t\wedge\tau} e^{-(\delta+\omega)s}[\vartheta_s + H(X(s-)-\vartheta_s) - H(X(s-))]dN_\gamma(s) - \int_0^{t\wedge\tau} e^{-(\delta+\omega)s}\vartheta_s dN_\gamma(s).
\end{aligned} \tag{3.2.2}
$$

These terms are the same we used in the first verification lemma, so the consequence is quite straightforward here :

Because we proved that

$$\mathbb{E}^u\left[\int_0^{t\wedge\tau} e^{-(\delta+\omega)s}\sigma H'_{a_1}(X(s))dW_s\right] = 0 \tag{3.2.3}$$

and also that condition 1 of verification lemma yields

$$\gamma[\vartheta_s + H_{a_1}(X(s-)-\vartheta_s) - H_{a_1}(X(s-))] + (\Omega-\delta) \leq 0. \tag{3.2.4}$$

Taking expectation leads to

$$H_{a_1}(u) \geq \mathbb{E}^u[e^{-\delta+\omega}H_{a_1}(X(t\wedge\tau))] + \mathbb{E}^u\left[\int_0^{t\wedge\tau} e^{-(\delta+\omega)s}\vartheta_s dN_\gamma(s)\right]. \tag{3.2.5}$$

Then, using $t \longrightarrow +\infty$ and using the same Fatou's lemma we use in the previous chapter, we obtain

$$H_{a_1}(u) \geq \mathbb{E}^u\left[\int_0^{\tau} e^{-(\delta+\omega)s}\vartheta_s 1_{\{\tau<+\infty\}} dN_\gamma(s)\right], \tag{3.2.6}$$

from which the result follows since the ϑ in the strategy are arbirary.

3.2.2. The new equation for the surplus when the barrier is set at a_1

The direct consequence is that only two areas of the $\gamma - \omega$ model are generated by $\gamma < \gamma_0$, $[0, a_1)$ and $[a_1, +\infty)$ because we use $b^* = a_1$. This is a limit case from the dividend perspective. Indeed, this means that either the company is ruined, with probability of ultimate bankruptcy ωdt, or they pay dividends immediately above a_1. We recall that maximizing the expected present value of dividend using a barrier strategy yields a probability of ultimate ruin of 1 (see for reference Lemma 3.3. in [Avanzi et al., 2014]) (so this policy is not inconsistent with the dividend criterion)

Thanks to the update that now $b = a_1$, the equations (2.1.9), (2.1.10) and (2.1.11) are combined into :

$$\frac{\sigma^2}{2} G_L''(u; a_1) + \mu G_L'(u; a_1) - [\delta + \omega(u)] G_L(u; a_1) = 0, \quad u \in [0, a_1) \tag{3.2.7}$$

$$\begin{aligned}\frac{\sigma^2}{2} G_U''(u; a_1) &+ \mu G_U'(u; a_1) - \delta G_U(u; a_1) \\ &+ \gamma[u - a_1 - G_U(u; a_1) + G_U(a_1; a_1)] = 0, \quad u \in [a_1, +\infty).\end{aligned} \tag{3.2.8}$$

The continuity condition for G and G' at a_1 yields

$$G_L(a_1^-, a_1) = G_U(a_1^+, a_1) \tag{3.2.9}$$

$$G_L'(a_1^-, a_1) = G_U'(a_1^+, a_1). \tag{3.2.10}$$

The lower part solution is the same as the one we found last chapter, and the upper one is slighlty modified, because b is replaced by a_1. The solution is now :

$$\begin{aligned}G_U(u, a_1) = &\left(\frac{\delta}{\delta + \gamma} G_U(a_1, a_1) + \frac{\mu\gamma}{(\delta + \gamma)^2} \right) e^{s_\gamma (u - a_1)} \\ &+ \frac{\gamma}{\delta + \gamma}[u - a_1 + G_U(a_1, a_1)] + \frac{\mu\gamma}{(\delta + \gamma)^2}.\end{aligned} \tag{3.2.11}$$

3.2.3. Numerical illustration

The Matlab code is provided in **Annexe A**.

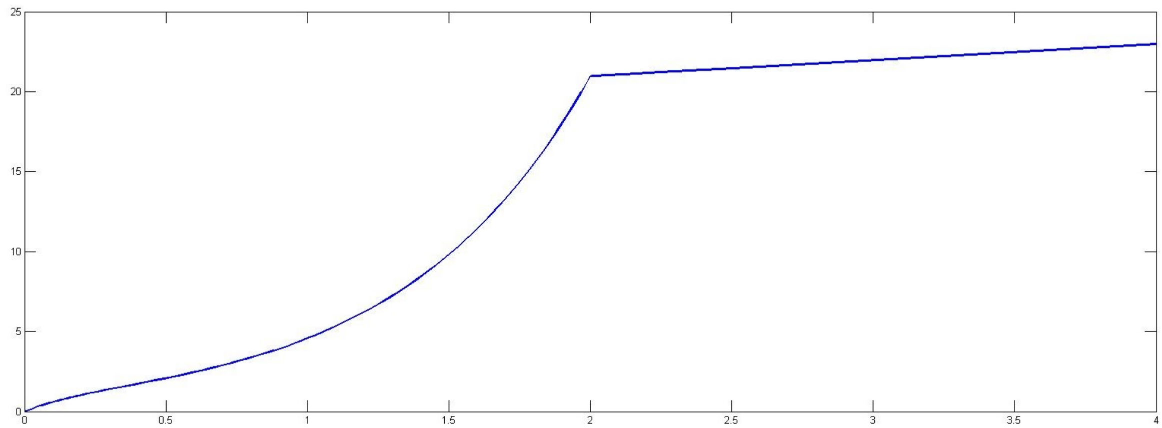


FIGURE 3.1. EPVD is the case of $b = a_1$

remark : We notice that the graph is similar to the first EPVD but without the middle part : as it is a limit case, the middle part vanishes and we find the previous lower and upper parts direclty connected to each other at a_1.

3.2.4. Verification of optimality

Now that we have found a solution that works, according to the adaptation of [Paulsen, 2003], we need to test the optimality thanks to the last step of the verification lemma. Those steps are :

(1) $\max_{0\leq l\leq u} \{\gamma[l + G(u-l, a_1) - G(u, a_1)]\} + (\Omega - \delta)G(u, a_1) = 0$
(2) $0 < G'(u, a_1) < \infty$
(3) $G''(u, a_1) < 0$
(4) $\lim_{u\to\infty} G''(u, a_1) = 0$.

We verify them all :

(1) By construction, the function $G(u, a_1)$ is solution of the two differential equations. Therefore $G(u, a_1)$ satisfies the modified HJB equation (1) and attains maximum.

(2) There are only two parts instead of three in the definition of the new $G(u; a_1)$. We still need to show that $0 < G'(u, a_1) < \infty$.

$G'_L(u, a_1) = r_\omega e^{r_\omega u} - s_\omega e^{s_\omega u}$. Because this part isn't affected by the change, the previous argument of positivity is still valid. Also, $G_U(u, a_1) = Ee^{s_\gamma} + \frac{\gamma}{\delta+\gamma}[u - a_1 + G_U(u, a_1)] + \frac{\mu\gamma}{(\delta+\gamma)^2}$ has the sames properties as the upper function in the previous section.The positivity argument is still valid.

(3) We have the following consideration : the function is quasiconcave but not concave. (Concavity around the barrier) The lower part is first concave, then convex.

(4) When appling the same argument we have used at (4) in the previous paragraph, the argument is still valid and $\lim_{u\longrightarrow\infty} G''(u, a_1) = 0$.

3.2.5. Interpretation of the case $b^* = a_1$

This transition chapter was intended to be a follow-up to the works of [Albrecher et al., 2011] regarding the surplus equations. In their model, the surplus can go freely in the ruin area $[0, a_1]$, ($(-\infty, 0]$ in their case). We observe that considering the case where the barrier is equal to the solvency constraint leads to a limit case, which is not really meant to happen to a company. Either they are ruined with probability ωdt, or they immediately pay dividends with probability γdt. This $\gamma - \omega$ model does not seem really credible, in a sense that it would be a very risk-taking model for the company.

To address the issue $b^* = a_1$, we need to impose some solvency considerations on the area $[0, a_1]$ because the problem is not well-defined otherwise.

That is why next (and last) chapter is concerned with two potential solvency considerations which give meaning to a $\gamma - \omega$ model where the barrier can be above a_1 and a_1 itself, and allow alternative strategies such as liquidations.

Chapitre 4

ON PERIODIC BARRIER STRATEGY UNDER TWO SOLVENCY CONSIDERATIONS : FORCED LIQUIDATION AND $\omega-$CAPITAL INJECTIONS

In this chapter, we study two solvency considerations. We focus on the $[0, a_1]$ area. The surplus we use is the same one as previously, but this time we consider two successive hypothesis for the danger zone below a_1.

In the first section, we define $[0, a_1]$ as a no-go zone. If the surplus hits a_1, then the shareholders are forced to liquidate the company. For example, an external regulator forces the company to close. We develop the new EPVD equations in that case. Because there is a risk of forced liquidation, the shareholders can choose to take the lead and liquidate at first opportunity, that is, at the first decision time where the surplus is above a_1, denoted T_α, if prospects are bad. The shareholders decide to share the surplus $X(T_\alpha) - a_1$ as a final dividend. We use the term "liquidation at first opportunity" strategy or "take the money and run" strategy because distributing the whole surplus above a_1 at that decision time brings the surplus back to a_1 and the regulator instantly closes the business (because monitoring of solvency is continuous).

In the second section, we explore another approach of the $\gamma - \omega$ model. The area $[0, a_1]$ is not a no-go zone. The surplus can go below a_1 without leading to bankruptcy. In this case, each time there is an $\omega-$event, or a bankrputcy, the shareholders are forced to inject capital up to a_1, with cost κ.

An $\omega-$event is the analogous of a $\gamma-$event, that is, a random decision time for capital injection, since our injection process mimics the dividend one, leading to a symmetry of injections / dividends.

This is a study case where bankruptcy cannot happen, and the first consequence is that τ, the time of bankruptcy is equal to $+\infty$. There cannot be bankruptcy in this model, and the business goes on forever. In that case, the shareholders cannot consider any kind of liquidation because it would not make sense : if they decided to liquidate and distribute either the surplus above a_1 or the whole surplus, the regulator would force them to inject capital up to a_1 and continue.

This dual monitoring of the danger area $[0, a_1]$ have never been considered in the literature before. We aim at giving key for the studying of solvency areas.

4.1. Forced liquidation and liquidation at first opportunity

4.1.1. Forced liquidation

In this section we study a first type of liquidation. It is forced, for example, by an external regulator. The model is the following :

Each time the surplus hits a_1, the company cannot afford to keep doing business and is forced to close. a_1 represents the liquidation value and needs to be paid to cover the transfer penalty to another company.

Because solvency is monitored continuously, the company is closed as soon as the surplus reaches a_1.

Mathematically, this liquidation is quite simple : despite a stricly positive surplus, because a_1 doesn't belong to the shareholders, no final dividend is paid when the surplus reaches a_1 and the company closes. In that case, there is no $\omega-$ruin, and

$$c_{\text{ruin}} = c_{\text{liquidation}} = a_1. \tag{4.1.1}$$

It is only a change of scale from a non-omega diffusion with simple ruin, because in that case we define the new time of ruin :

$$\tau_{a_1} = \inf\{t \mid X(t) \leq a_1\} \tag{4.1.2}$$

instead of τ.

Two things can then be deduced from the previous paragraph. First, because the monitoring of solvency is continuous, $u < a_1$ is impossible. Indeed, an initial surplus below closing level does not make sense, because it would trigger a liquidation at time $t = 0$. Second, in that case only two areas remain for the surplus, $[a_1, b)$ and $[b, +\infty)$. Let's denote by H the expected present value of dividends in that case, and H_M and H_U its two parts. Between a_1 and b the first equation is :

$$\frac{\sigma^2}{2} H_M''(u,b) + \mu H_M'(u,b) - \delta H_M(u,b) = 0 \tag{4.1.3}$$

whereas the upper area admits the following equation

$$\frac{\sigma^2}{2}H''_U(u,b) + \mu H'_U(u,b) - \delta H_U(u,b)$$
$$= -\gamma[u - b - H_U(u,b) + H_U(b,b)]. \quad (4.1.4)$$

A solution of the first equation is still

$$H_M(u,b) = Ae^{ru} + Be^{su}, \quad (4.1.5)$$

where r and s are the positive and negative roots of the associated characteristic equation.

Liquidation happens at a_1 so from the dividend value perspective, we can fix $H(a_1) = 0$. From that we have

$$H(a_1) = Ae^{ra_1} + Be^{sa_1} = 0 \quad (4.1.6)$$

then this yields

$$B = -Ae^{(r-s)a_1} \quad (4.1.7)$$

and if we set $A = 1$ we find both constants

$$A = 1 \quad \text{and} \quad B = -e^{(r-s)a_1}. \quad (4.1.8)$$

The upper part solution has a structure that is similar to the first upper solution. The initial condition of linear bound for $u \to +\infty$ yields

$$H_U(u,b) = C_5 e^{s_\gamma(u-b)} + \frac{\gamma}{\delta+\gamma}[u - b + H_U(b,b)] + \frac{\mu\gamma}{(\delta+\gamma)^2} \quad (4.1.9)$$

and C_5 can be determined using the continuity condition at $u = b$. In that case we find

$$\begin{aligned} C_5 &= H_U(b,b) - p(b) \\ &= \frac{\delta}{\delta+\gamma} H_U(b,b) + \frac{\mu\gamma}{(\delta+\gamma)^2}, \end{aligned} \tag{4.1.10}$$

where $p(b)$ is a particular solution of the non-homogeneous upper equation. The complete solution is then :

$$\begin{aligned} H_U(u,b) = \left(\frac{\delta}{\delta+\gamma} H_U(b,b) + \frac{\mu\gamma}{(\delta+\gamma)^2} \right) e^{s_\gamma(u-b)} \\ + \frac{\gamma}{\delta+\gamma}[u - b + H_U(b,b)] + \frac{\mu\gamma}{(\delta+\gamma)^2}. \end{aligned} \tag{4.1.11}$$

4.1.2. Numerical illustration

We provide an illustration of the EPVD in case of forced liquidation by a regulator. Standard parameters are chosen, that is : $\mu = 0.5$, $\sigma = 0.5$, $\delta = 0.05$, $\gamma = 1000$. We choose $a_1 = 2$ and $b = 4$.

The code is provided in **Annexe A**

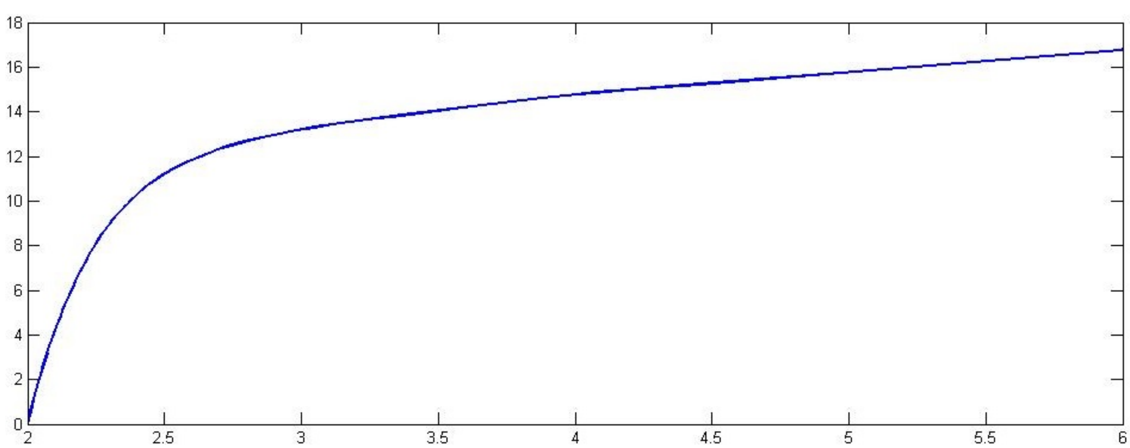


Figure 4.1. Forced liquidation with $b > a_1 = 2$ between 2 and 6

4.1.3. Liquidation at first opportunity at a_1 when the barrier is at $b = a_1$

We consider the case where prospects are not good, and $b^* < a_1$. We saw in the previous chapter that it was possible for b^* to be found below a_1 because of the structure of the equation for b^*, which was a logarithm. According to [Paulsen, 2003] and his (now) adapted theorem in the case of periodic dividends, we apply the optimal policy to set $b^* = a_1$.

In that case, when the shareholders see that the optimal barrier is equal to the liquidation value, they know that prospects are bad, and it would be better to close the buisness to get a final dividend (the difference between the surplus at the liquidation time and a_1) instead of waiting for the regulator to close it without a final dividend when the surplus reaches a_1.

Liquidation is forced when the surplus hits a_1, which creates only one admissible area for the surplus, the area $[a_1, +\infty)$ (because $a_1 = b$, no other area is created, recall that $[0, a_1]$ is a no-go zone in this case). This type of case has been studied in [Avanzi et al., 2014], but with $b = 0$, because there was no partial solvency issue. The shareholders liquidate at the first opportunity they get, which is T_α. Usually, $T_\alpha = T_1$, provided the surplus has not undergone an $\omega-$event in the meantime.

The expected present value of dividends in such case is theoretically :

$$F(u) = \mathbb{E}^u \left[e^{-\delta T_\alpha} (X(T_\alpha) - a_1) I_{\{T_k < \tau_{a_1}\}} \right]. \tag{4.1.12}$$

Distributing the whole surplus above a_1 leads the new surplus to be equal to a_1 and then killed by the regulator. We develop the law of total probability that creates the new equation.

We can expect several changes from [Avanzi et al., 2014] in the following equations. The first one is that this time, the final dividend is not $l = u - 0$ but $l = u - a_1$, because of the solvency constraint. We have :

$$\begin{aligned} F(u) = \gamma h(1 - \delta h)\{u - a_1 + \mathbb{E}[F(u + \mu h + \sigma W(h) - (u - a_1))]\} \\ + (1 - \gamma h - \omega(u)h)(1 - \delta h)\mathbb{E}[F(u + \mu h + \sigma W(h))] + o(h), \end{aligned} \tag{4.1.13}$$

which can be simplified :

$$F(u) = \gamma h(1-\delta h)\{u - a_1 + \mathbb{E}[F(a_1+\mu h+\sigma W(h))]\} \\ + (1-\gamma h-\omega(u)h)(1-\delta h)\mathbb{E}[F(u+\mu h+\sigma W(h))] + o(h). \quad (4.1.14)$$

Using once again Taylor series expansion on $\mathbb{E}[F(u+\mu h+\sigma W(h))]$ and on $\mathbb{E}[F(a_1+\mu h+\sigma W(h))]$, we have :

$$\mathbb{E}[F(u+\mu h+\sigma W(h))] = F(u) + \mu h F'(u) + \frac{\sigma^2}{2} h F''(u) + o(h). \quad (4.1.15)$$

Factorizing terms in h yields :

$$F(u) = \left\{\gamma[u - a_1 + F(a_1) - F(u)] + \frac{\sigma^2}{2}F''(u) - \mu F'(u) - (\omega(u)+\delta)F(u)\right\} h \\ + F(u) + o(h).$$

Because of the continuous monitoring of solvency, stopping time happens at τ_{a_1} and the business has to shut down. Also, the new condition is $F(a_1) = 0$ because it is where the liquidation takes place.

In that case, there is only one equation that governs the surplus between a_1 and $+\infty$, that is :

$$\frac{\sigma^2}{2}F''(u) + \mu F'(u) - (\delta+\gamma)F(u) = -\gamma[u - a_1 - F(u) + 0], \quad (4.1.16)$$

because we know that $F(a_1) = 0$. A solution is :

$$F(u) = C_6 e^{s_\gamma(u-a_1)} + \frac{\gamma}{\delta+\gamma}[u-a_1] + \frac{\mu\gamma}{(\delta+\gamma)^2}. \quad (4.1.17)$$

C_6 can be determined using the initial condition at $u = a_1$. In that case we find

$$
\begin{aligned}
C_6 &= F(a_1) - p(a_1) \\
&= 0 - p(a_1) = -\frac{\mu\gamma}{(\delta+\omega)^2},
\end{aligned}
\tag{4.1.18}
$$

where p is the usual particular solution.

The complete solution is then

$$
F(u) = \frac{\mu\gamma}{(\delta+\gamma)^2}[1 - e^{s_\gamma(u-a_1)}] + \frac{\gamma}{\delta+\gamma}[u - a_1], \tag{4.1.19}
$$

which is analogous to the result of [Avanzi et al., 2014]. Replacing a_1 with 0 (their case) leads to the same solution.

4.1.4. Numerical illustration

The code is provided in **Annexe A**. This EPVD is the one found in case of $b = a_1$ when liquidation is chosen by the shareholders at first opportunity τ_{a_1}.

We choose the parameters $\sigma = \mu = 0.5$, $\delta = 0.05$, $\gamma = 0.01$ (if $b < a_1$ it is because γ is too weak). Finally, $a_1 = 2$.

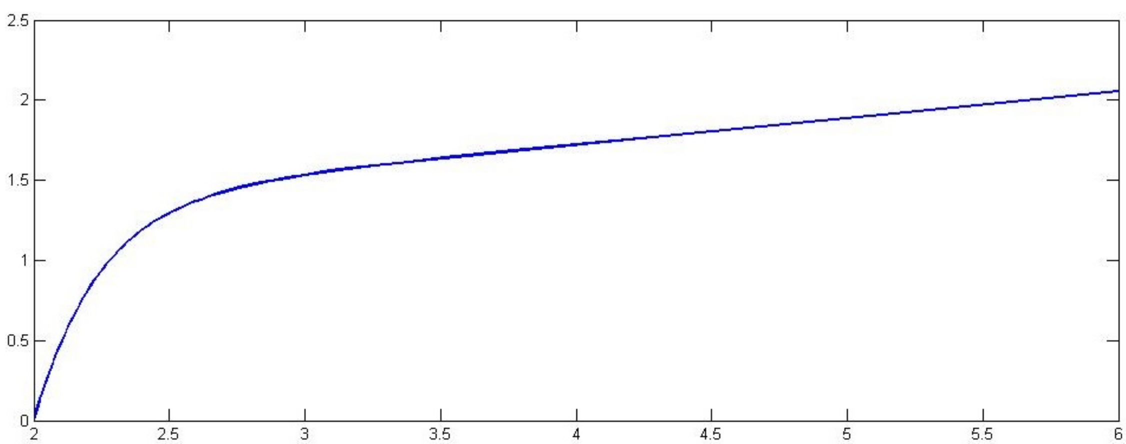


FIGURE 4.2. EPVD for liquidation at first opportunity when $b = a_1 = 2$ between 2 and 6

4.2. A model of periodic capital injections to carry out the solvency requirement

4.2.1. Construction of the new equations for the surplus

We consider a standard ruin model with a solvency constraint a_1, where capital injection below a_1 are allowed up to a_1. We first decribe the new three equations in the case of capital injected in case of $\omega-$event or bankruptcy, then we study the influence of the barrier on the EPVD, with the usual two cases : $b > a_1$ and $b = a_1$. The latter case holds a liquidation at first opportunity strategy.

In the former case, we first assume that prospects are good and $b > a_1$. In order to obtain the new three equations that govern the areas $[0, a_1)$, $[a_1, b)$ and $[b, +\infty)$, we need to apply once more the law of total probability to the surplus, to know what outcomes can happen. The heuristic argument needs to be updated with another event : capital injection in case of an $\omega-$event.

Mathematically, we consider capital injections as negative dividends. Instead of paying dividends, the company adds money. These injections are not free, and a penalty κ is paid, proportionally to the amount of capital injected. It represents the cost of injecting 1\$ into the surplus. The force of interest δ is discounted at a rate $e^{-\delta h} = 1 - \delta h + o(h)$ We then have three things to consider over the small time interval $[0, h)$ to apply the law of total probability. Once again, we use the notation V for the value function, to assure overall consistency with Chapter 2. Note that the quantity c represents for the moment any capital injection, but we only describe the case where $c = a_1 - u > 0$ (capital is injected up to a_1 in case of an ω-event) thereafter.

To mimic the negative dividend process, we require that the injections happen at a rate ω that works exactly like γ for the dividend process. $\omega \mathrm{d}t$ is the probability of capital injection within $\mathrm{d}t$ time units. Exception occurs at capital 0 where this probability equals 1 so bankruptcy never happens.

For this to work, we require that the monitoring of solvency below a_1 is not continuous, because if it was, then the shareholders would be forced to inject capital continuously when the surplus should go below a_1, and that is not the type of injections we are dealing with. The exception is at bankruptcy, where capital is injected immediately up to a_1 to prevent it.

The three possible outcomes are :

(1) A dividend is paid at rate $l \geq 0$ with probability γh. This can be quantified $\gamma h(1-\delta h)\{l + \mathbb{E}[V(u+\mu h+\sigma W(h)-l)]\}$.

(2) A capital injection happens at rate $c \geq 0$ with probability ωh, that is $\omega h(1-\delta h)\{-\kappa c + \mathbb{E}[V(u+\mu h+\sigma W(h)+c)]\}$. The term $\kappa \in [1,+\infty)$ comes from the aditional cost required to inject capital into the surplus (penalty). To inject c into the surplus, the shareholders need in fact to pay κc.

(3) Nothing happens with probability $1-\gamma h-\omega h$, leading to $(1-\gamma h-\omega h)(1-\delta h)\mathbb{E}[V(u+\mu h+\sigma W(h))] + o(h)$.

The sum of all four probabilities yields the quantity we need.
Using Taylor expansions on V, that is :

$$\begin{aligned} &V(u+\mu h+\sigma W(h)) \\ &\qquad = V(u) + V'(u)(\mu h+\sigma W(h)) + V''(u)\frac{(\mu h+\sigma W(h))^2}{2} + o(h), \end{aligned} \tag{4.2.1}$$

then,

$$\mathbb{E}[V(u+\mu h+\sigma W(h))] = V(u) + \mu h V'(u) + \frac{\sigma^2}{2} h V''(u) + o(h). \tag{4.2.2}$$

After basic algebra we find that the EPVD can be reduced to

$$\begin{aligned} &V(u) + \{\gamma[l + V(u-l) - V(u)] + \omega[-\kappa c + V(u+c) - V(u)] \\ &\qquad\qquad + \mu V'(u) + \frac{\sigma^2}{2} V''(u) - \delta V(u)\}h + o(h). \end{aligned} \tag{4.2.3}$$

This previous expression yields the new equations we need for the EPVD, but we could be interested in getting the HJB : dividing by h and considering the expression between brackets, we have the HJB equation

$$\max_{l\geq 0, c\geq 0} \{\gamma[l + V(u-l) - V(u)] + \omega[-\kappa c + V(u+c) - V(u)]\} \\ + \mathcal{A}V(u) = 0, \quad (4.2.4)$$

with $\mathcal{A}f = \frac{\sigma^2}{2}f'' + \mu f' - \delta f$ as infinitesimal operator, and with the initial condition $V(0) = 0$.

4.2.2. The three equations in the case of $b > a_1$

The structure of the HJB equations motivates us to think that the optimal strategy is a periodic dividend barrier. In this section, we adopt the same structure as Chapter 2, that is, we aim at finding the new three equations for the three areas $[0, a_1)$, $[a_1, b)$ and $[b, +\infty)$. Indeed, because $\omega = 0$ above a_1, and $\gamma = 0$ below a_1, we find again three equations.

We introduce the new piecewise EPVD. Let's call the full function $G(u, b)$, where b symbolizes the barrier.

We consider :

$$G(u,b) = \begin{cases} G_L(u,b) & \text{if } u \in [0, a_1) \\ G_M(u,b) & \text{if } u \in [a_1, b) \\ G_U(u,b) & \text{if } u \in [b, +\infty). \end{cases}$$

According to the works of [Albrecher et al., 2011] we adapt in case of capital injection, for $u \in [0, a_1)$, G_L satisfies the equation :

$$\frac{\sigma^2}{2}G''_L(u;b) + \mu G'_L(u;b) - \delta G_L(u;b) \\ = -\omega[-\kappa c + G_L(u+c) - G_L(u)], \quad (4.2.5)$$

because it is the area where $\omega-$ruin occurs.

The quantity c that we chose to inject to the surplus corresponds to the difference between the surplus (below a_1) and a_1. Because this quantity needs to be positive, we define

$$c = a_1 - u. \quad (4.2.6)$$

The first EPVD equation becomes :

$$\frac{\sigma^2}{2}G''_L(u;b) + \mu G'_L(u;b) - \delta G_L(u;b) \\ = -\omega[\kappa(u-a_1) + G_L(u+a_1-u) - G_L(u)]. \quad (4.2.7)$$

Between a_1 and b, in the middle area, the HJB gives the equation :

$$\frac{\sigma^2}{2}G''_M(u;b) + \mu G'_M(u;b) - \delta G_M(u;b) = 0, \quad (4.2.8)$$

because no dividend payment occurs if the surplus is here. When the decision time happens, no surplus is distributed.

Finally, above b, the equation satisfied by $G_U(u,b)$ is

$$\frac{\sigma^2}{2}G''_U(u;b) + \mu G'_U(u;b) - \delta G_U(u;b) \\ + \gamma[u-b-G_U(u;b) + G_U(b;b)] = 0, \quad (4.2.9)$$

because dividend payment occurs here.

4.2.3. The middle and upper parts

In this section, there are no changes to make because these two parts are not affected by capital injections. Indeed, only an omega-event or a bankruptcy can trigger a capital injection.

The previous solutions still hold. The solution to

$$\frac{\sigma^2}{2}G''_M(u;b) + \mu G'_M(u;b) - \delta G_M(u;b) = 0 \quad (4.2.10)$$

is

$$G_M(u) = Ae^{ru} + Be^{su}, \quad (4.2.11)$$

with r and s the postive and negative root of the associate characteristic equation, and

$$A = \frac{G'(a_1,b) - sG(a_1)}{e^{ra_1}(r-s)} \qquad B = \frac{rG(a_1,b) - G'(a_1,b)}{e^{sa_1}(r-s)} \tag{4.2.12}$$

have already been determined in the previous section.

About the upper part : a solution of

$$\frac{\sigma^2}{2}G_U''(u;b) + \mu G_U'(u;b) - \delta G_U(u;b) = -\gamma[u - b - G_U(u;b) + G_U(b;b)] \tag{4.2.13}$$

is

$$G_U(u,b) = \left(\frac{\delta}{\delta+\gamma}G_U(b,b) + \frac{\mu\gamma}{(\delta+\gamma)^2}\right)e^{s_\gamma(u-b)} + \frac{\gamma}{\delta+\gamma}[u - b + G_U(b,b)] + \frac{\mu\gamma}{(\delta+\gamma)^2}. \tag{4.2.14}$$

This was also in chapter 2 and is still valid. Changes happen in the next section.

4.2.4. The lower part

The lower part of $G(u,b)$ on $(0,a_1)$ is

$$\frac{\sigma^2}{2}G_L''(u;b) + \mu G_L'(u;b) - (\delta+\omega)G_L(u;b) = -\omega[\kappa(u-a_1) + G_L(a_1)], \tag{4.2.15}$$

with the requirement (immediate injection at the time of bankruptcy)

$$G(0) = -\kappa a_1 + G(a_1). \tag{4.2.16}$$

Firstly, we solve the homogeneous associate equation to find the first part of the soution. If we consider the homogeneous part, we find a lower equation not very different of the previous one :

$$Hom(u) = A_\omega e^{r_\omega u} + B_\omega e^{s_\omega u}. \tag{4.2.17}$$

Now we have to find a particular solution. We build it on the model of the upper one, that is

$$\frac{\omega}{\delta+\omega}[-\kappa(a_1-u)+G(a_1)]+\frac{\mu\kappa\omega}{(\delta+\omega)^2}. \tag{4.2.18}$$

The last step to get the complete solution is to determine the constants A_ω and B_ω. They can be determined thanks to the initial condition

$$G'(0)=\kappa=r_\omega A_\omega+s_\omega B_\omega+\frac{\omega}{\delta+\omega}[-\kappa a_1+G(a_1)]+\frac{\mu\kappa\omega}{(\delta+\omega)^2}. \tag{4.2.19}$$

From that we deduce, if we set $A_\omega = 1$:

$$B_\omega=\frac{1}{s_\omega}\left[\kappa-r_\omega-\frac{\omega\kappa}{\delta+\omega}\right]. \tag{4.2.20}$$

The solution is complete.

Now because we have the lower part solution, as in the first case in the first chapter, we can get the complete A and B for the middle part. We have then found explicit values for $G(a_1,b)$ and $G'(a_1,b)$. They are :

$$G(a_1,b)=e^{r_\omega a_1}+B_\omega e^{s_\omega a_1}+\frac{\omega}{\delta+\omega}G(a_1)+\frac{\mu\omega\kappa}{(\delta+\omega)^2}. \tag{4.2.21}$$

We also evaluate $G'(a_1,b)$ knowing the value of $G(a_1,b)$ and B_ω :

$$G'(a_1,b)=r_\omega e^{r_\omega a_1}+s_\omega B_\omega e^{s_\omega a_1}+\frac{\omega\kappa}{\delta+\omega}. \tag{4.2.22}$$

The last step in injecting those values in

$$A=\frac{G'(a_1,b)-sG(a_1,b)}{e^{ra_1}(r-s)} \qquad B=\frac{rG(a_1,b)-G'(a_1,b)}{e^{sa_1}(r-s)}. \tag{4.2.23}$$

4.2.5. Numerical illustration

We provide an illustration of this capital injection case for standard parameters. As well as in Case 1, the function might not be concave on all of its domain but is always concave around the barrier.
We provide the code in **Annexe A**

This figure is obtained with parameters $\sigma = 5$, $\pi = 1$, $\gamma = 1000$, $\mu = 0.2$, $\delta = 0.05$, $a_1 = 2$, $b = 4$ and $\kappa = 1.2$.

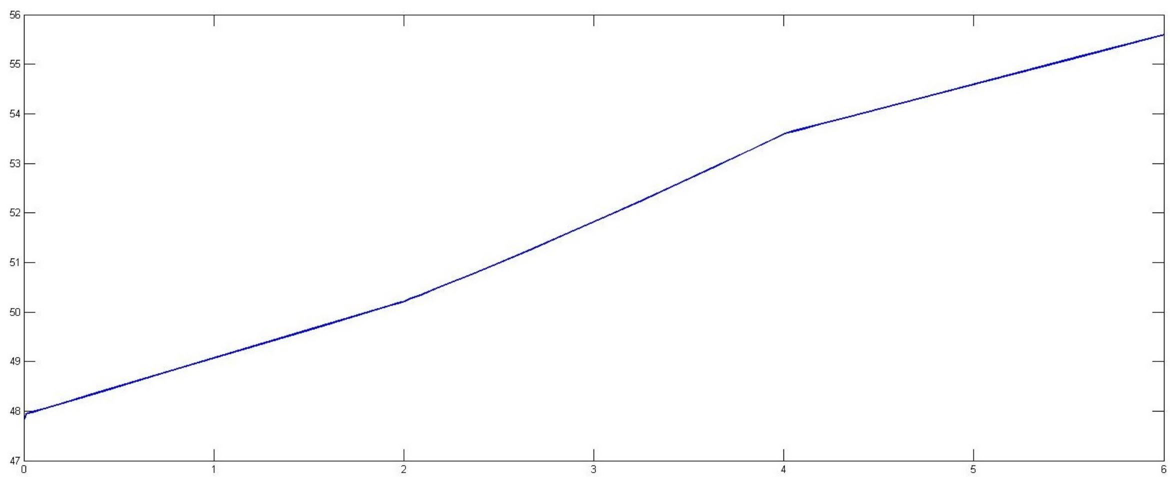


FIGURE 4.3. EPVD when capital is injected up to a_1 if the surplus goes below a_1 and an injection is triggered

4.2.6. The case $b = a_1$

We know that b^* can be found between 0 and a_1, but we cannot use [Paulsen, 2003] recommendation because capital injections are not at stake in his theory. Recall that it is forbidden to distribute dividends below a_1. The highest barrier we can have is a_1 itself, but this time under capital injections, we lose money to inject capital, so we make the assuption that in that case, a barrier strategy at a_1 should not be optimal.

Instead, we look at a liquidation at first opportunity strategy, since the money loss discourages holding the business for too long in case of bad dividend prospects.

The model is quite simple : depending on where the surplus is, either there is a final injection or a final dividend, and we stop at $a1$ with $G(a_1) = 0$. The equations coming from the law of total probability are slightly modified because this time $G(a_1) = 0$, but their structure remains unchanged.

There is no middle part and the lower equation is

$$\frac{\sigma^2}{2}G''(u) + \mu G'(u) - (\delta + \omega)G(u) = -\omega[-\kappa(a_1 - u)] \qquad (4.2.24)$$

if $u \in (0, a_1)$. This equation has solution

$$G(u) = A_\omega e^{r_\omega u} + B_\omega e^{s_\omega u} + \frac{\omega}{\delta + \omega}[-\kappa(a_1 - u)] + \frac{\kappa\omega\mu}{(\delta + \omega)^2}, \qquad (4.2.25)$$

with

$$G(0) = -\kappa a_1. \qquad (4.2.26)$$

This upper equation is the same as usual (see liquidation at first opportunity in previous section), and admits for solution

$$G(u) = +\frac{\gamma\mu}{(\delta + \gamma)^2}[1 - e^{s_\gamma(u - a_1)}] + \frac{\gamma}{\delta + \gamma}[u - a_1]. \qquad (4.2.27)$$

The last step is to find both lower constant to ensure continuity at a_1. Setting $A_\omega = 1$, we know that

$$G(a_1) = 0 = e^{r_\omega a_1} + B_\omega e^{s_\omega a_1} + \frac{\mu\kappa\omega}{(\delta + \omega)^2}, \qquad (4.2.28)$$

leading to

$$B_\omega = \frac{1}{e^{s_\omega a_1}}\left[-\frac{\mu\kappa\omega}{(\delta + \omega)^2} - e^{r_\omega a_1}\right]. \qquad (4.2.29)$$

4.2.7. Numerical illustration

We illustrate the liquidation at first opportunity with a numerical application. Recall that γ has to be chosen low because the barrier is below a_1. We chose $\gamma = 0.1$, which yields a concave function

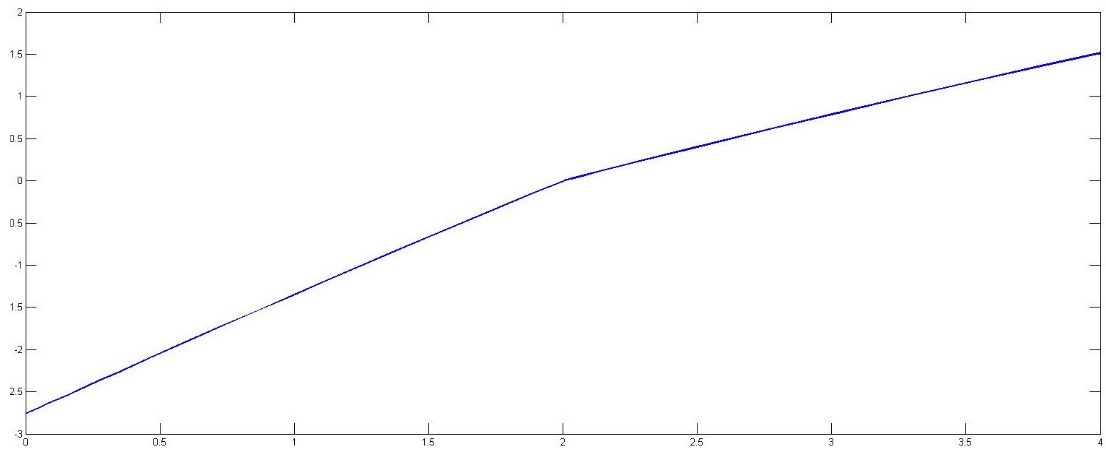


Figure 4.4. EPVD for a liquidation at first opportunity with respect to the initial surplus

.

CONCLUSION

In actuarial sciences, the field of dividends is really wide. We had to focus only on what seemed interesting to develop throughout this one year work. The $\omega-$ruin allows more realism in the modelling of surplus and we have seen that the introduction of a simple solvency constraint leads to multiple questions and issues. Some issues that could seem obvious in the case of continuous dividends might mathematically be not that easy to deal with and the field of periodic dividends in still to be extended. We believe that the forced liquidation and capital injections are important, and had to be part of the solvency considerations we studied in this model.

We tried to provide answers to these concerns by making some contributions, and tried to do it in the order that was the most meaningful, each step after the other. In Chapter 2, we started by making a small contribution, by adapting the works of [Albrecher et al., 2011] in the case of a solvency constraint. This had to be done to model the surplus of companies whose surplus cannot become negative. Then we made another deeper contribution with the proof of a barrier strategy being optimal in the $\gamma - \omega$ model. In chapter 3, we considered that the works of [Albrecher et al., 2011] were not complete because of the nature of the equation for b^* that could be found below a_1. We then made a contribution adapting the equations to the case $b^* = a_1$. This could not have been done without adapting the crucial theorem from [Paulsen, 2003] to the case of periodic dividends, that is now proven in the periodic framework thanks to this thesis. Finally, we wanted to know where the danger zone $[0, a_1]$ could lead, because that area, specific to [Albrecher et al., 2011] had not been studied in detail before this thesis. We also saw that studying the case $b^* < a_1$ in the framework of [Albrecher et al., 2011] without modifications lead to an inconclusive $\gamma - \omega$ model. To address this issue, we proposed to focus on the study of the danger zone. The first idea considered a_1 as the threshold of liquidation, where the company had to be closed in case of $\omega-$event. To escape that fate, the shareholders were allowed to liquidate at first opportunity in case of bad prospects. The second case considered the field of capital injections inside the $\gamma - \omega$ model but in the periodic dividend framework, motivated by the works of [Avanzi et al., 2011]. This dual monitoring of two completely different outcomes in the ruin area represents also a new contribution.

But this thesis might only be the first step towards more realism. Indeed, there are a lot of improvements we could add to our model.
The first one to come in mind is to add an instant loss component, that is, a compound Poisson process which would model downward jumps. This would be very intersting because in that case, there would be a lot of things at stake : the size of jumps compared to the size of the danger area for example, and it would completely change the liquidation issues.
The second crucial improvement that could be done is studying others inter-decision times ; In this thesis, only inter-exponential decision times with parameter γ were considered. But to improve realism, we could consider Erlang distributions. That would enable us to obtain deterministic decision-time intervals and would lead to more accuracy in the insights that the models provide us. (Of course, in such case we would need to find another name for the "γ"$-\omega$ model).
The third and last key improvement we could consider is the possibility of a penalty at the time of ruin. This penalty could be according the time spent in the $\omega-$ruin zone, or according to the total amount of capital that have been below the danger zone or maybe both.

The $\gamma - \omega$ model is really promising and will lead to even more surprising results in the future. This is why it was really important to study these first steps in this thesis.

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Annexe A

MATLAB CODES

In this chapter, we give the codes that are used to obtain each figure.

This is the code for **Subsection 2.1.8** :

```
function value=epvd(u,a1,b,mu,delta,sigma,gamma,omega)

r = (-mu+sqrt(mu*mu+4*delta*((sigma*sigma)/2)))/(sigma*sigma);
s = (-mu-sqrt(mu*mu+4*delta*((sigma*sigma)/2)))/(sigma*sigma);
rom = (-mu+sqrt(mu*mu+4*(delta+omega)*((sigma*sigma)/2)))/(sigma*sigma);
som = (-mu-sqrt(mu*mu+4*(delta+omega)*((sigma*sigma)/2)))/(sigma*sigma);
sgam = (-mu-sqrt(mu*mu+4*(delta+gamma)*((sigma*sigma)/2)))/(sigma*sigma);

if ((u>=0)&&(u<a1))
    value = exp(rom*u)-exp(som*u);
end
if ((u>=a1)&&(u<b))
    value = (((rom-s)/(r-s))*exp((rom-r)*a1)
    +((s-som)/(r-s))*exp((som-r)*a1))*exp(r*u)
    +(((r-rom)/(r-s))*exp((rom-s)*a1)
    +((som-r)/(r-s))*exp((som-s)*a1))*exp(s*u);
end
if (u>=b)
    value = ((delta/(delta+gamma))
    *epvd(b-0.001,a1,b,mu,delta,sigma,gamma,omega)
    +(mu*gamma/((delta+gamma)*(delta+gamma))))*exp(sgam*(u-b))
    + (gamma/(delta+gamma))
    *(u-b+epvd(b-0.001,a1,b,mu,delta,sigma,gamma,omega))
    + (mu*gamma/((delta+gamma)*(delta+gamma)));
end
end
```

This is the code for **Subsection 3.2.3** :

```
function valuebelowa1 = epvdbelowa1(u,a1,mu,delta,sigma,gamma,omega)

rom = (-mu+sqrt(mu*mu+4*(delta+omega)*((sigma*sigma)/2)))/(sigma*sigma);
som = (-mu-sqrt(mu*mu+4*(delta+omega)*((sigma*sigma)/2)))/(sigma*sigma);
rgam = (-mu+sqrt(mu*mu+4*(delta+gamma)*((sigma*sigma)/2)))/(sigma*sigma);
sgam = (-mu-sqrt(mu*mu+4*(delta+gamma)*((sigma*sigma)/2)))/(sigma*sigma);
Fa1 = exp(rom*a1)-exp(som*a1);
E = ((delta/(delta+gamma))*Ga1 - (mu*gamma)
/((delta+gamma)*(delta+gamma)))/exp(sgam*a1);

if ((u>=0)&&(u<a1))
    valuelbelowa1 = (exp(rom*u)-exp(som*u));
end

if (u>=a1)
    valuebelowa1 = E*exp(sgam*u)+((gamma)/(gamma+delta))
    *(u-a1+Fa1)+(mu*gamma)/((delta+gamma)*(delta+gamma));
end
end
```

This is the code for **Subsection 4.1.2** :

```
function valueforcedliq=forcedliq(u,a1,b,mu,delta,sigma,gamma)

r = (-mu+sqrt(mu*mu+4*delta*((sigma*sigma)/2)))/(sigma*sigma);
s = (-mu-sqrt(mu*mu+4*delta*((sigma*sigma)/2)))/(sigma*sigma);
sgam = (-mu-sqrt(mu*mu+4*(delta+gamma)*((sigma*sigma)/2)))/(sigma*sigma);

A=10; # "10" factor chosen to fix issues of scale
B=-10*exp((r-s)*a1);

if ((u>=a1)&&(u<b))
    valueforce = A*exp(r*u)+B*exp(s*u);

end

if (u>=b)
    valueforcedliq = ((delta/(delta+gamma))
    *forcedliq(b-0.000001,a1,b,mu,delta,sigma,gamma)
    +((mu*gamma)/((delta+gamma)*(delta+gamma))))*exp(sgam*(u-b))
    + (gamma/(delta+gamma))
    *(u-b+forcedliq(b-0.000001,a1,b,mu,delta,sigma,gamma))
    + (mu*gamma/((delta+gamma)*(delta+gamma)));
end
end
```

This is the code for **Subsection 4.1.4** :

```
function valuefirstopp=firstopp(u,a1,mu,delta,sigma,gamma)

sgam = (-mu-sqrt(mu*mu+4*(delta+gamma)*((sigma*sigma)/2)))/(sigma*sigma);

if (u>=a1)
    valuefirstopp = ((mu*gamma)/((gamma+delta)^2))*(1-exp(sgam*(u-a1)))
    + (gamma/(delta+gamma))*(u-a1);
end
end
```

This is the code for **Section 4.2.5** :

```
function value=epvdinject1(u,a1,b,mu,delta,sigma,gamma,omega,kappa)

r = (-mu+sqrt(mu*mu+4*delta*((sigma*sigma)/2)))/(sigma*sigma);
s = (-mu-sqrt(mu*mu+4*delta*((sigma*sigma)/2)))/(sigma*sigma);
rom = (-mu+sqrt(mu*mu+4*(delta+omega)*((sigma*sigma)/2)))/(sigma*sigma);
som = (-mu-sqrt(mu*mu+4*(delta+omega)*((sigma*sigma)/2)))/(sigma*sigma);
sgam = (-mu-sqrt(mu*mu+4*(delta+gamma)*((sigma*sigma)/2)))/(sigma*sigma);
rgam = (-mu+sqrt(mu*mu+4*(delta+gamma)*((sigma*sigma)/2)))/(sigma*sigma);

dGa1 = rom*exp(rom*a1)+(kappa-rom-(omega*kappa)/(delta+omega))*exp(som*a1)
 + (omega*kappa)/(delta+omega);
Bom = (kappa-rom-(omega*kappa)/(delta+omega))*(1/som);
Ga1 = (exp(rom*a1) + Bom*exp(som*a1)
+ (mu*kappa*omega)/((delta+omega)^2))*((delta+omega)/delta) ;

A = (dGa1 - s*Ga1)/((exp(r*a1))*(r-s));
B = (r*Ga1 - dGa1)/((exp(s*a1))*(r-s));

if (u==0)
    value = -kappa*a1 + Ga1 ;
end
if ((u>0)&&(u<a1))
    value = exp(rom*a1) + Bom*exp(som*a1)
    + (omega/(delta+omega))*(-kappa*(a1-u) + Ga1)
    + (mu*omega*kappa/((delta+omega)^2));
end
if ((u>=a1)&&(u<b))
    value = A*exp(r*u)+B*exp(s*u) ;
   end

if (u>=b)
    value = ((delta/(delta+gamma))
    *epvdinject1(b-0.001,a1,b,mu,delta,sigma,gamma,omega,kappa)
    +(mu*gamma/((delta+gamma)*(delta+gamma))))*exp(sgam*(u-b))
    + (gamma/(delta+gamma))
    *(u-b+epvdinject1(b-0.001,a1,b,mu,delta,sigma,gamma,omega,kappa))
     + (mu*gamma/((delta+gamma)*(delta+gamma)));
    end
end
```

This is the code for **Subsection 4.2.7** :

```
function value=epvdinjectlafo(u,a1,mu,delta,sigma,gamma,omega,kappa)

rom = (-mu+sqrt(mu*mu+4*(delta+omega)*((sigma*sigma)/2)))/(sigma*sigma);
som = (-mu-sqrt(mu*mu+4*(delta+omega)*((sigma*sigma)/2)))/(sigma*sigma);
sgam = (-mu-sqrt(mu*mu+4*(delta+gamma)*((sigma*sigma)/2)))/(sigma*sigma);
rgam = (-mu+sqrt(mu*mu+4*(delta+gamma)*((sigma*sigma)/2)))/(sigma*sigma);

Bom =  (-exp(rom*a1) - (mu*omega*kappa/((delta+omega)^2)))*(1/exp(som*a1)) ;

if (u==0)
    value = -kappa*a1 ;
end

if ((u>=0)&&(u<a1))
    value = exp(rom*u)+Bom*exp(som*u)
    + (omega/(delta+omega))*(-kappa*(a1-u))
    + (mu*omega*kappa/((delta+omega)^2));
end

if (u>=a1)
    value = (mu*gamma)/((delta+gamma)^2)*(1-exp(sgam*(a1-u)))
     + (gamma/(delta+gamma))*(u-a1);
end
end
```

Annexe B

STOCHASTIC CALCULUS AND MARTINGALES THEORY

B.1. Uniform/Square Integrable Martingales

From Klebaner [2005]

Corollary 7.8

If $X(t)$ is square integrable, that is, $\sup_t EX^2(t) < \infty$, then it is uniformly integrable.

Theorem 18 : Let X be a uniformly integrable continuous martingale and let τ be a stopping time. Then $X^\tau = (X_{t\wedge\tau})_{0\leq t\leq\infty}$ is also uniformly integrable right continuous martingale.

B.2. Supermartingale argument

From Klebaner [2005], Chapter 7.5,

Local Martingale **Definition** An adapted process M(t) is called a local martingale if there exits a sequence of stopping times τ_n, such that $\tau_n \to \infty$ and for each n the stopped processes $M(t \wedge \tau_n)$ is a uniformly integrable martingale in t.

Theorem 7.21

Let $M(t)$, $0 \leq t > \infty$, be a local martingale such that $|M(t)| \leq Y$, with $EY < \infty$. Then M is a uniformly integrable martingale.

Corollary 7.22

Let $M(t), 0 \leq t > \infty$, be a local martingale such that for all t, $E(\sup_{s\leq t} |M(s)|) < \infty$. Then it is a martingale, and as such it is uniformly integrable on any finite interval $[0, T]$. If in addition $E(\sup_{t\geq 0} |M(t)|) < \infty$, then $M(t), t \geq 0$, is uniformly integrable on $[0, \infty)$.

B.3. Stopped Martingale

Theorem 7.14 If $M(t)$ is a martingale and τ is a stopping time, then the stopped process $M(\tau \wedge t)$ is a martingale. Moreover,

$$EM(\tau \wedge t) = EM(0). \tag{B.3.1}$$

Also

If $M(t)$ is a sub- or supermartingale and τ is a stopping time, then the stopped process $M(\tau \wedge t)$ is also a sub- or supermartingale. In particular,

$$EM(\tau \wedge t) \leq EM(0) \quad EM(\tau \wedge t) \geq EM(0). \tag{B.3.2}$$

B.4. Fubini's Theorem

Theorem 2.39

Let $X(t)$ be a stochatic process $0 \leq t \leq T$ (for all t, $X(t)$ is a random variable), with regular sample parths (for all ω at any point t, $X(t)$ has left and right limits). Then

$$\int_0^T E|X(t)|dt = E\left(\int_0^T |X(t)|dt\right). \tag{B.4.1}$$

Furthermore if this quantity is finite, then

$$E\left(\int_0^T X(t)dt\right) = \int_0^T E(X(t))dt. \tag{B.4.2}$$

B.5. Sharp Bracket

From Theorem 8.24 in Klebaner [2005],

Théorème B.5.1. *Let M be a square integrable martingale. Then the sharp bracket process $\langle M, M\rangle(t)$ is the unique predictable increasing process for with $M^2(t) - \langle M, M\rangle(t)$ is a martingale.*

We are interested in the following quantity first

$$d\langle \tilde{N}_\gamma, \tilde{N}_\gamma\rangle(s), \tag{B.5.1}$$

where $\{\tilde{N}_\gamma(t), t \geq 0\}$ is a compensated Poisson process with $\tilde{N}_\gamma(t) = N_\gamma(t) - \gamma t$, then we can find the sharp bracket $\langle \tilde{N}_\gamma, \tilde{N}_\gamma\rangle(t)$. We make use the fact that $\tilde{N}_\gamma^2(t) - \langle \tilde{N}_\gamma, \tilde{N}_\gamma\rangle(t)$ is a martingale (from Theorem 8.24), then

$$\langle \tilde{N}_\gamma, \tilde{N}_\gamma\rangle(t) = \gamma t. \tag{B.5.2}$$

This is verified in Example 8.16 of Klebaner [2005]. Therefore

$$d\langle \tilde{N}_\gamma, \tilde{N}_\gamma\rangle(s) = \gamma ds. \tag{B.5.3}$$

B.6. Itō's formula for diffusion with jumps

From Cont and Tankov [2004], Section 8.3.2

Consider now a jump-diffusion process

$$X_t = \sigma W_t + \mu t = X^{\mu}(t), \tag{B.6.1}$$

where X^c is the continuous part of X. Define $Y_t = f(X_t)$ where $f \in C^2(\mathbb{R})$, the total change in Y_t can be written as

$$f(X_t) - f(X_0) = \int_0^t f'(X_s) dX_s^c + \int_0^t \frac{\sigma^2}{2} f''(X_s) ds + \sum_{0 \leq s \leq t}^{\Delta X_s \neq 0} [f(X_{s_-} + \Delta X_s) - f(X_{s_-})]. \tag{B.6.2}$$