Université de Montréal

Trois essais en théorie microéconomique

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Résumé en francais

Chapitre 1

Dans ce chapitre, je modélise un monopoleur multi-produit qui doit décider de monitorer ou non les achats de ses clients. Un contrat monitoré ne peut être acheté plus d'une fois tandis qu'un contrat non-monitoré peut être acheté le nombre de fois désiré. Je trouve que le monopoleur va toujours offrir aux consommateurs au moins un contrat non-monitoré.

Chapitre 2

J'étudie la composition de l'ensemble des allocations parétiennes dans le contexte d'allocation d'un nombre fini de biens indivisibles entre un même nombre d'agents. Chacun des agents reçoit un bien et aucune compensation monétaire n'est permise. Ce problème est typiquement connu comme le problème d'allocation de maisons (*house allocation problem*). Pour analyser la rationalisation d'un sous-ensemble d'allocations, j'introduis le concept de cycle. Un cycle consiste en une série d'allocations où chaque allocation est liée à la suivante par la même règle de transition. Avec le concept de cycle, je trouve certaines contraintes sur la composition d'un sous-ensemble d'allocations pour qu'il soit rationalisable.

Chapitre 3

Thomas et Worrall (1988) étudient le problème de design de contrat entre un travailleur averse au risque et une firme neutre au risque lorsque qu'ils peuvent briser le contrat à tout moment. Dans ce chapitre, j'utilise la même approche pour expliquer les fusions. J'utilise des fonctions d'utilité de type CARA, ce qui permet de dériver explicitement le contrat optimal. Ensuite, j'ajoute quelques hypothèses pour évaluer les effets d'une fusion entre deux firmes ayant des revenus aléatoires. Pour ce faire, nous effectuons des simulations numériques. De part les résultats, une fusion est souhaitable seulement lorsque les agents ont un bas facteur d'escompte.

Mots-Clés : Monitoring, Monopoleur multi-produit, Préférences multidimensionnelles, Biens indivisibles, Cycles, Rationalisabilité, Contraintes auto-excécutoires, Fusionnement, Contrats optimaux.

English summary

Chapter 1

The main purpose of the paper is to introduce the decision to monitor sales or not in the multiproduct monopoly decision problem. To do so, I introduce the concept of a monitored contract as contract that consumers can buy only once. On the other hand, a non-monitored contract could be purchased in any quantity. Obviously, to offer a monitored contract, the monopoly should be able to observe and to control consumers' choice. I find that the multiproduct monopoly will always offer at least one non-monitored contract to consumers.

Chapter 2

I study the composition of the Paretian allocation set in the context of a finite number of agents and a finite number of indivisible goods. Each agent receives at most one good and no monetary compensation is possible (typically called the house allocation problem). I introduce the concept of a cycle which is a sequence of allocations where each allocation is linked to the following allocation in the sequence by the same switch of goods between a subset of agents. I characterize the profiles of agent preferences when the Paretian set has cycles.

Chapter 3

Thomas and Worrall (1988) study the problem of designing a contract between risk-averse workers and risk-neutral firms when both of them could break the contract at any time. In this paper, I use the same approach to study mergers. I model a CARA utility function to derive explicitly the optimal contract and the value function for both agents in the case where only two states of nature are possible. I use this approach to explain the reason for a firm to merge with another one. Because the analytic solution is too difficult to derive explicitly with more than two states of nature, numerical simulations are used to illustrate these cases. I find that mergers will occur only when agents have a low discount rate.

Keywords : Monitoring, Multiproduct monopoly, Multidimensional Preferences, Indivisible goods, Cycles, Rationalisability, Self-enforcing constraints, merger, Optimal Contracts.

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Dédicace

À Lucien et Marielle...
pour la personne que je suis,

À Véronique...
pour la personne que je veux être,

À Daphnée et ses frères et soeurs...
pour la personne que je serai.

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En second lieu, je ne peux passer sous silence l'impact de Yves Richelle sur mon cheminement académique. Il fut un élément important du succès de mon doctorat. J'ai le privilège de l'avoir comme ami.

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Introduction

A distinctive feature of microeconomic theory is that it aims to model economic activity as an interaction of individual economic agents pursuing their private interests.

Mas-Colell, Whinston and Green [8].

Les incitations économiques sont à la base des décisions des individus dans la vie de tous les jours. Que ce soit pour l'achat ou la vente de biens, l'offre de travail ou d'élection, les individus choisissent l'option qui, en fonction de la situation à laquelle ils font face, maximise leur bien-être. La modélisation microéconomique des décisions individuelles des agents constitue donc un outil privilégié pour l'analyse de questions aussi intéressantes que diversifiées comme le design de contrats d'assurance, les stratégies de mise en marché ou encore les problèmes d'allocation de biens indivisibles entre individus.

Dans le premier essai de ma thèse, j'étudie le comportement stratégique d'une firme ayant un pouvoir de monopole sur plusieurs marchés. Plus précisément, l'objectif est de mieux comprendre comment la firme décide d'effectuer du monitoring ou non. Je définis le monitoring comme la capacité pour la firme de suivre et de contrôler les achats de ses biens.

Pour vendre ses produits, la firme multiproduit peut avoir recours à la vente groupée (bundling) ou à la vente séparée. Par exemple, la plupart des chaînes de restauration rapide offre la possibilité d'acheter divers biens sous forme de trios. Lorsque les demandes ne sont pas unitaires[1], la firme peut décider d'offrir des contrats qui ne peuvent être achetés plus d'une fois. Dans ce cas, nous dirons que le contrat est monitoré. Si la firme ne contrôle pas le nombre de fois que les consommateurs peuvent acheter un contrat donné, alors nous dirons que ce contrat est non-monitoré.

Jusqu'à maintenant, l'étude du monitoring se faisait au niveau des conséquences de

[1]Nous disons qu'une demande est unitaire quand les consommateurs obtiennent un gain d'utilité seulement de la première unité de bien consommée. Une laveuse et une sécheuse sont des exemples de biens pour lesquels les consommateurs ont des demandes unitaires.

monitorer ou non. La quasi-majorité des articles sur la tarification non-linéaire[2] utilisent implicitement (ou explicitement) l'hypothèse que la firme est capable de suivre et de restreindre les possibilités d'achat des consommateurs. La possibilité de ne pas monitorer fut étudiée exclusivement par Katz [5]. Cependant, aucun article ne modélise la prise de décision de monitorer ou non.

Dans cet essai, je présente un modèle qui incorpore la décision de monitorer ou non pour un monopoleur multiproduit. Une firme utilise le monitoring si elle offre au moins un contrat monitoré. Cependant, il est important de souligner que la décision de monitorer n'impose pas de monitorer tous les contrats. Il est toujours possible pour la firme d'utiliser des stratégies mixtes lors de la mise en marché des contrats. Un premier résultat est que, peu importe la fonction de coûts administratifs, la firme va toujours offrir au moins un contrat non-monitoré. Ce résultat est cohérent avec les observations. Il semble que l'ensemble des firmes offrent toujours la possibilité aux consommateurs d'acheter un type de contrat sans contrôle sur la quantité de fois qu'ils achètent ce dit-contrat.

Dans le second essai, le sujet d'étude est la rationalisation d'un ensemble de réalisations dans le cadre d'allocation de biens indivisibles (*House Allocation Problem*). J'entends par rationalisation d'un ensemble A l'existence d'un profil de préférences individuelles qui a comme ensemble des optimums de Pareto l'ensemble A. L'allocation de biens indivisibles est un problème commun dans la vie de tous les jours. On peut penser à la répartition des chambres parmi des colocataires, les charges de cours entre professeurs ou aux espaces de bureaux entre collègues de travail. Ce type de problème fut introduit par Shapley et Scarf [12] et étudié par de nombreux auteurs dont Roth et Postlewaite [11], Svensson [13] et Ehlers [3].[3]

L'objectif de cet essai est d'introduire le concept de rationalisation dans un cadre d'allocation de biens indivisibles. Pour ce faire, j'introduis le concept de cycle qui consiste en une série d'allocations où chaque allocation est liée à la suivante par la même règle de transition. Un premier résultat découlant de la présence d'un cycle dans l'ensemble des optimums de Pareto est que tous les individus doivent avoir les mêmes préférences sur les biens qui se suivent dans le cycle. Deuxièmement, si le cycle est composé d'un nombre premier d'individus, alors tous ces individus doivent avoir les mêmes préférences sur les biens qui composent ce cycle. Troisièmement, je trouve que si l'ensemble des allocations parétiennes contient un nombre minimal de cycles composés des mêmes individus et des mêmes biens, alors tous ces individus doivent avoir les mêmes préférences sur ces biens. Comme quatrième résultat important, je trouve des conditions sur le nombre d'allocations que l'ensemble des optimums de Pareto doit contenir.

[2]Voir par exemple Goldman, Leland and Sibley [4], Mirman and Sibley [10], McAfee and McMillan [9] ou Armstrong [1].

[3]Cette liste n'est pas exhaustive et ne figure qu'à titre de référence.

Le troisième essai étudie les conséquences d'une fusion dans le cadre d'un modèle d'assurance avec contraintes auto-exécutoires. Nous disons qu'un contrat est auto-exécutoire si, pour tous les états de la nature et pour toutes les périodes, les deux agents (l'assureur et l'assuré) ont un gain à respecter le contrat. Sans contraintes auto-exécutoires, le manque d'engagement devient un problème. Lorsque les coûts de faire respecter le contrat sont élevés et que les coûts de changer de contrat est bas, un agent peut avoir intérêt à briser le contrat suite à la révélation de l'état de la nature alors qu'il était optimal ex-ante. Dans le but d'éliminer ce type de problème, j'utilise la même approche que celle introduite par Thomas et Worrall [14].

Dans la première partie de l'essai, je suppose que les individus ont des préférences qui peuvent être représentées par des fonctions de type CARA (*Constant Absolute Risk Aversion*). Cette modélisation se distancie de celle de Thomas et Worrall [14] et se rapproche de celle de Kocherlakota [6] en ce sens que les deux agents sont averses au risque. Avec cette hypothèse, je suis en mesure de solutionner explicitement le contrat optimal en supposant que les deux individus ont le même coefficient d'aversion au risque. Sans cette hypothèse, je ne peux expliciter la solution. Puisque nous trouvons le contrat optimal dans toutes les situations, je peux définir et tracer les frontières des optimums de Pareto selon les valeurs des paramètres. Les graphiques illustrent clairement que les frontières sont continues mais non pas dérivables en tout point.[4]

Dans un deuxième temps, je me suis intéressé aux effets d'une fusion entre deux firmes ayant des revenus aléatoires en présence de contrats auto-exécutoires. Pour ce faire, j'ai modélisé deux firmes averses au risque qui ont un revenu aléatoire et un agent neutre au risque (le marché). Une des firmes peut décider de ne pas fusionner ou d'acheter l'autre firme au prix donné par l'équivalent certain. Dans le cas de la fusion (acquisition) ou de la situation ex-ante, les deux firmes ont la possibilité de signer des contrats d'assurance auto-exécutoires. Avec l'aide de simulations numériques, je trouve qu'une fusion peut être profitable lorsque le taux d'escompte est bas même lorsque les revenus des firmes sont corrélés.

[4]Kocherlakota [6] avançait faussement que les frontières des optimums de Pareto étaient continues en tout point. Ceci fut corrigé par Koeppl [7].

Chapitre 1

Monitoring Costs for a Multiproduct Monopoly

1.1 Introduction

Firms combine different methods to sell their products. For example, many fast-food restaurants offer discount coupons on a specific meal while they allow consumers to buy any quantities of meals at regular prices. In construction material stores, small buyers face regular prices while big buyers have special discounts on their purchases. Defining monitoring as the control of consumers' purchases, these examples suggest that firms use monitoring in combination with usual non-monitored sales methods to maximize their profit.

The first context where such control on consumer purchasing is studied in the economic literature is the case of bundling that offers consumers the possibility to buy a package in addition to the possibility of buying goods separately. The first complete model that deals with the ability of a monopoly to offer bundles was proposed by Adams & Yellen [1]. These authors study a market for two goods where consumers have unit demands for both products and they find that bundling can be efficiently used to increase firm profits even though consumers' utility for each good are unrelated. Some extensions to the Adams & Yellen's paper were made by introducing a joint distribution of consumer preferences over the two goods as in Schmalensee's [9] model. Monitoring in such a context gives the monopoly the ability to restrain the set of possibilities available to consumers. For instance, with monitoring, a monopoly that wants to sell two goods can offer these goods separately as well as in a bundle, but can force consumers who want to buy both goods to purchase the bundle. The profitability of such possibilities to restrict the opportunities available to consumers are analyzed in McAfee, McMillan and Whinston [7]. These authors present sufficient conditions over the joint distribution on consumers preferences under which bundling gives more profits than selling goods separately when consumers are monitored.

A second context where monitoring could be interesting to use is the case where firms can practice some form of nonlinear pricing (usually called second-degree price discrimination). Nonlinear pricing exists when a firm in a single market sets different unit prices for different amounts of goods purchased. Spence [10] presents a model in which a central planner must maximize the aggregate consumer surplus without having the ability to identify the consumer's type but with the ability to monitor consumers, by observing the quantities they buy. Goldman, Leland and Sibley [3] study explicitly the role of constraints on the price structure. They find that the price could be either upward or downward discontinuous in quantity with smooth and well-behaved demand and cost functions. Since the 1980's, many papers deal with the use of nonlinear pricing by a multiproduct monopoly. Mirman and Sibley [8] assume that consumers differ by only one taste parameter while McAfee and McMillan [6] examine the case where multidimensional consumer preferences can be represented by a single variable. In this last case, the analysis becomes identical to Mirman and Sibley's. The first paper considering multidimensional preferences in a nonlinear pricing context is Armstrong [2]. Armstrong examines the decision problem of a monopoly over many goods facing consumers with multidimensional preferences and finds a method to resolve the mechanism design problem for some specific classes of cases.

Except for the paper by Katz [5], which studies the case when purchases are not observable by the firm, all papers on nonlinear pricing assume that the firm is able to monitor purchases. In Katz's [5] paper, the case of a firm with monopoly power on a single market which is not able to observe consumer purchases is analyzed and a characterization of the optimal price schedule is obtained.

To my knowledge, no paper examines the ability of a multiproduct monopoly to decide to monitor consumers purchases or not. The main purpose of the present paper is to analyze a model where the decision to monitor or not to monitor is an endogenous decision. To do so, I first define a contract as a vector specifying a quantity for each good and a price that will be paid by the consumer in exchange of the specified quantities. Then a contract will be said to be monitored whenever consumers can buy such a contract only once, while a non-monitored contract is a contract that consumers can buy without restrictions. In such a framework, the decision to monitor corresponds to the decision to offer a monitored contract. However, monitoring is not an all or nothing decision. Indeed, the monopoly can actually propose monitored contracts together with non-monitored ones. The main result is that the set of contracts offered by the monopoly will always contain a non-monitored contract. This accords with the examples given above.

The paper is organized as follows. In Section 1.2, I introduce the model. The theorem of existence and some characterization of the optimal strategy of the monopoly are described in Section 1.3. Section 1.4 contains discussions on the basic assumptions and I conclude in Section 1.5.

1.2 Model

I consider a situation where a multiproduct monopoly faces N consumers. Let N be a natural number. The purpose of this section is to introduce the concept of monitored and non-monitored contracts as well as the assumptions relative to the behavior of the monopoly and of the consumers.

1.2.1 Multiproduct monopoly

A monopoly produces L goods and sells these goods through contracts. I assume that the firm is risk neutral. The problem of the monopoly is to determine the number of contracts as well as the composition of the contracts it will offer to the consumers. I define a contract as a vector that specifies a quantity for each good as well as a price that the consumer who accepts the contract will pay in exchange for the quantities specified in the contract. Precisely, a typical contract k is given by $(q_1, ..., q_l, ..., q_L, P)$ where q_l stands for the quantity of good l that will be sold if the contract is accepted and P is the price paid whenever the contract is accepted.

I also assume that the price element P is greater than or equal to ϵ with $\epsilon > 0$. As we shall see, this assumption is quite innocuous but will facilitate some of the arguments made below.

I begin by describing the kind of contracts that can be proposed by the monopoly.

Monitored and non-monitored contracts

In this paper, I say that a contract k^g is monitored if consumers cannot buy this contract more than once. In addition, I assume that consumers can buy only one monitored contract.

For non-monitored contracts, I assume that these contracts can be bought many times and in combination with other contracts. To illustrate how monitored and non-monitored contracts work, consider the following example.

Suppose the monopoly offers two monitored contracts k^a and k^b and two non-monitored contracts k^α and k^β. Following the definition of a non-monitored contract, the contracts given by δk^α, σk^β or $\delta k^\alpha + \sigma k^\beta$ for $\delta, \sigma = 1, 2, ...$ can be bought by consumers. Following the definition of monitored contracts, the contracts k^a and k^b are offered to consumers but not $k^a + k^b$ or δk^a, σk^b or $\delta k^a + \sigma k^b$ since consumers can buy at most one monitored contracts and do so only once.

In addition to these contracts, it is possible for consumers to buy a combination on non-monitored contracts and one monitored contract. So the contracts given by $k^a + \delta k^\alpha$, $k^a + \delta k^\beta$, $k^a + \delta k^\alpha + \delta k^\beta$, $k^b + \delta k^\alpha$, $k^b + \delta k^\beta$ and $k^b + \delta k^\alpha + \delta k^\beta$ are offered to consumers.

To summarize, if the monopoly offers two monitored contracts k^a and k^b and two non-monitored contracts k^α and k^β, then the following contracts are in effect offered to consumers for $\delta, \sigma = 1, 2, 3, ...$

$$\begin{aligned} k^a, \quad k^b, \quad k^\alpha, \quad k^\beta \\ \delta k^\alpha, \quad \sigma k^\beta, \quad \delta k^\alpha + \sigma k^\beta \\ k^a + \delta k^\alpha, \quad k^a + \sigma k^\beta, \quad k^a + \delta k^\alpha + \sigma k^\beta \\ k^b + \delta k^\alpha, \quad k^b + \sigma k^\beta, \quad k^b + \delta k^\alpha + \sigma k^\beta \end{aligned}$$

Let K^m and K^{nm} be respectively the set of monitored contracts and the set of non-monitored contracts. With these sets, it is possible to construct the contract set $K(K^m, K^{nm})$ which is the set of contracts which can be bought by consumers. Considering the preceding example, the set of monitored contracts K^m is given by $\{k^a, k^b\}$, the set of non-monitored contracts by $\{k^{\alpha}, k^{\beta}\}$ and the set of offered contracts by :

$$K(K^m, K^{nm}) = \begin{Bmatrix} k^a, k^b, k^{\alpha}, k^{\beta} \\ \delta k^{\alpha}, \sigma k^{\beta}, \delta k^{\alpha} + \sigma k^{\beta} \quad \delta, \sigma = 1, 2, ... \\ k^a + \delta k^{\alpha}, k^a + \sigma k^{\beta}, k^a + \delta k^{\alpha} + \sigma k^{\beta} \quad \delta, \sigma = 1, 2, ... \\ k^b + \delta k^{\alpha}, k^b + \sigma k^{\beta}, k^b + \delta k^{\alpha} + \sigma k^{\beta} \quad \delta, \sigma = 1, 2, ... \end{Bmatrix}$$

These definitions have three immediate implications that must be noted. First, whenever the set of non-monitored contracts is empty, the set of proposed contracts coincide with the set of monitored contracts, i.e., $K^m = K(K^m, \emptyset)$. This follows from the definition of a monitored contract. A second implication is that any contract $k^h \in K(K^m, K^{nm})$ is either a monitored contract and belongs to K^m or an non-monitored contract and belongs to K^{nm} or a combination of non-monitored contracts and at most one monitored contract.

Thirdly, since the all elements of a contract are a real number, the sets of monitored and non-monitored contracts are countable. This implies immediately that the contract set is also countable. Furthermore, if $K^m = K^{nm} = \emptyset$, then $K(K^m, K^{nm}) = \emptyset$. This means that inaction is possible for the monopoly.

The next step in the description of the model is to present assumptions relative to the multiproduct monopoly cost structure.

Monopoly's production and administration costs

Total costs for the monopoly consist of a production cost function which, as usual, gives the cost associated with the provision of a given quantity of goods to the consumers, and of an administration cost function which gives the cost to manage the set of proposed contracts.

The function $V : \mathbb{R}^l_+ \to \mathbb{R}_+$ gives the production cost which only depends on the total quantity of goods provided to consumers. Let $Q(N, K^m, K^{nm})$ be the vector of total quantity of each good produced. I assume further that the marginal production cost is constant.[1] So the production cost function becomes

$$V(Q(N, K^m, K^{nm})) = \sum_{l=1}^{L} (c_l * Q_l(N, K^m, K^{nm}))$$

where c_l is the marginal cost of good l and $Q_l(N, K^m, K^{nm})$ is the total quantity of good l produced. I assume also that $c_l > 0$ for all l.

Note that I assume that the production costs do not depend directly on the number as well as the kind of contracts in the set of proposed contracts. Only the total quantity

[1] This assumption can be relaxed but to keep the presentation simple, I use this assumption.

of goods matters.

The function A represents the administration cost. I shall define A^{nm} as the cost function to administrate the non-monitored contracts while the cost function to manage the monitored contracts will be denoted by A^{m}. I assume that the cost of providing non-monitored contracts and the cost of providing monitored contracts are (directly) independent. I then assume that administration costs are additive in the non-monitored contract cost function A^{nm} and the monitored contract cost function A^{m}.

Regarding the non-monitored contract cost function, I assume that the cost to provide one more non-monitored contract is constant. This means that

$$A^{nm}\left(K^{nm}\right) \quad = \quad |K^{nm}|\, a^{nm}$$

with $a^{nm} > 0$ and where $|X|$ denotes the number of elements of the set X.

For the cost to administrate monitored contracts, I assume that the number of consumers buying a monitored contracts matters. This comes from the fact that, in order to make such contracts effective, the monopoly must follow each consumer to prevent multiple purchases of these monitored contracts, which imposes a cost to the monopoly.

Let $N^{m}(N, K^{m}, K^{nm})$ be the number of consumers who choose a monitored contract proposed in $K(K^{m}, K^{nm})$ or a contract in $K(K^{m}, K^{nm})$ that is a linear combination of contracts, one of them being a monitored contract. This number is unknown by the firm since the firm does not know consumer types. Nevertheless, once the firm determines the sets of non-monitored and monitored contracts, consumers make their choice and their action generates the monitored contract cost function. I have mentioned above that the cost of providing a monitored contract does not relate directly on the set of non-monitored contracts. With the last assumption, the monitored contract cost function now depends indirectly on the set of non-monitored contracts since the latter affects the number of consumers buying a monitored contract.

Next, once again for simplicity, I assume that the monitored contract cost function is given by

$$A^{m}\left(N^{m}(N, K^{m}, K^{nm}), K^{m}\right) \quad = \quad |K^{m}|\, a^{m}\left(N^{m}\left(N, K^{m}, K^{nm}\right)\right)$$

Note that the assumption that the administrative cost is increasing in the number of consumers buying a monitored contract implies that $a^{m}(.)$ is also increasing in N. I further assume that the unit administrative cost of a non-monitored contract, a^{nm}, is strictly smaller than the unit administrative cost of a monitored contract $a^{m}(.)$, whatever the number of consumers buying a monitored contract N.

Accordingly, the larger the number of consumers choosing a monitored contract, the larger are the costs associated with the management of monitored contracts and therefore

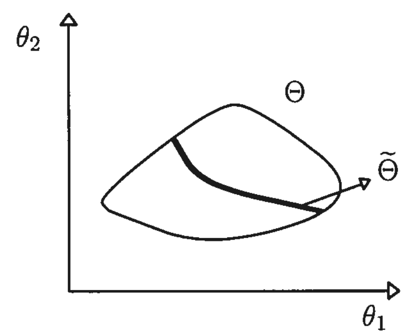

FIG. 1.1 – Non-Degeneration of the Set of Preferences

I have $A^m\left(N^1, K^m\right) > A^m\left(N^2, K^m\right)$ whenever $N^1 > N^2$.

To sum up, assumptions imply that the administrative cost function of the monopoly can be written as

$$A\left(N^m(N, K^m, K^{nm}), K^m, K^{nm}\right) = |K^{nm}|\, a^{nm} + |K^m|\, a^m\left(N^m(N, K^m, K^{nm})\right)$$

with $a^{nm} > 0$.

1.2.2 Consumers

The utility level attained by consumer i whenever he buys contract $k^h = (q_1^h, \cdots, q_L^h, P^h)$ is given by $u\left(\theta^i, q_1^h, ..., q_L^h\right) - P^h$ where $\theta^i = (\theta_1^i, \cdots, \theta_S^i)$ is a vector of preference parameters, i.e. preferences of consumer i with $S \geq L - 1$. I shall assume that u is continuous in q^h and in θ^i, increasing and strictly concave in q^h and satisfies

$$\lim_{q_l \to \infty} \frac{\partial u\left(\theta, q_1, ..., q_L\right)}{\partial q_l} = 0 \quad l = 1, ..., L \quad \forall q_j \geq 0,\ j \neq l, \quad \forall \theta \in \Theta$$

Note that individuals with the preference vector θ^i have the same utility function. The set of all preference vector, denoted by Θ, is a compact subset of R^S. Preference vectors are $i.i.d$ according to the continuous probability distribution $F(\theta)$. I also assume that $F(\theta)$ is non-degenerative, i.e., the probability that $\theta^i \in \widetilde{\Theta}$ for all $\widetilde{\Theta} \subseteq \Theta$ with $Dim[\widetilde{\Theta}] < Dim[\Theta]$ is zero. This assumption is commonly used in the economic literature. For instance, if Θ has only one dimension and the distribution is non-degenerative, the probability of getting a specific $\overline{\theta}$ is equal to zero. Figure 1.1 represents a case where $\widetilde{\Theta}$ is of dimension one while Θ is of dimension 2.

In this paper, I also assume that consumers could buy nothing if they wanted. In this case, I say that the consumer buys the null contract $k_0 = (0, 0, 0, ..., 0)$. Then, the problem of the consumer is to maximize his utility by choosing a contract in the set K_0 which is the set of contracts offered by the firm $K(K^m, K^{nm})$ plus the null contract i.e. $K_0(K^m, K^{nm}) = K(K^m, K^{nm}) \cup \{k_0\}$.

Let me denote by $K^*\left(\theta^i, K^m, K^{nm}\right)$ the set of contracts in the set K_0 which maximizes the utility of consumers whose preference vector is θ^i. I can immediately ask if there exist consumers whose $K^*\left(\theta^i, K^m, K^{nm}\right)$ contains several contracts. To avoid this possibility, I must add an assumption on utility function. The monotonicity in utility difference will guarantee that the set of optimal contracts is a singleton for almost all types of consumers.

Definition 1.1 *A utility function* $u\left(\theta^i, q^h\right)$ *is* Δ_u*-monotone if, for all* q^1, q^2 *such that* $q^1 \neq q^2$, $\exists s \in \{1, ..., S\}$ *such that* $\forall \theta^i \in \Theta$, $u\left(\theta^i, q^1\right) - u\left(\theta^i, q^2\right)$ *is strictly monotone in* θ^i_s.

The Δ_u-monotonicity says that, for any pair of contracts k^1, k^2, if there is a $\overline{\theta}$ such that k^1 and k^2 give the same utility, then an infinitesimal change in $\overline{\theta}_s$ increases differently the utility of each contract. One can say that if a function f is Δ_u-monotone, then

$$\frac{\partial^2 f(\theta, q)}{\partial q_l \partial \theta_s} > 0 \quad or \quad \frac{\partial^2 f(\theta, q)}{\partial q_l \partial \theta_s} < 0$$

In fact, Δ_u-monotonicity is more than that. Take the function $f(\theta, q) = \theta_1(q_1 + q_2)^2$. This function has positive cross-derivative if q_1 and q_2 are positive but it does not respect the Δ_u-monotonicity. Take the contracts $q^1 = (1, 2)$ and $q^2 = (2, 1)$ for example. For all values of θ, the difference in utility with those contracts will remain 0. If a function is Δ_u-monotone, then each marginal utility associated with a given good is affected differentially by a change in a specific preference parameter.

By assuming that the consumer utility functions are Δ_u-monotone, then I obtain the following result.

Lemma 1.1 *The probability of finding a profile* $\theta^i \in \Theta$ *such that* $K^*\left(\theta^i, K_0\right)$ *contains more than one contract is equal to 0.*

Proof. *Take two contracts* k^1 *and* k^2 *belonging to* K. *Let* $\overline{\Theta}$ *be the set of all preference vectors such that, for all* $\overline{\theta} \in \overline{\Theta}, k^1, k^2 \in K^*\left(\overline{\theta}, K_0\right)$ *and let* θ^1 *belong to* $\overline{\Theta}$.

Following the definition of $K^*\left(\theta^1, K_0\right)$, *if* $k^1, k^2 \in K^*\left(\theta^1, K_0\right)$, *then :*

$$u\left(\theta^1, q^1_1, ..., q^1_L\right) - P^1 = u\left(\theta^1, q^2_1, ..., q^2_L\right) - P^2$$

Following the definition of the Δ_u*-monotonicity, there is a* $s \in \{1, 2, ..., S\}$ *such that* $u\left(\theta^i, q^1\right) - u\left(\theta^i, q^2\right)$ *is monotone in* θ^i_s.

Now, suppose there is an element θ^2 *belonging to* $\overline{\Theta}$ *such that* $\theta^2_s \neq \theta^1_s$. *Because* $\theta^2 \in \overline{\Theta}$ *and by* Δ_u*-monotonicity, there is another preference parameter* $t \in \{1, 2, ..., S\}, t \neq s$ *such* *that $\theta_t^2 \neq \theta_t^1$. If not, θ^2 can not belong to $\overline{\Theta}$. This means that any change in the parameter s leads to a change in some other parameter(s) to maintain the equality*

$$u\left(\theta^2, q_1^1, ..., q_L^1\right) - P^1 = u\left(\theta^2, q_1^2, ..., q_L^2\right) - P^2$$

Then, the set $\overline{\Theta}$ has a dimensionality lower than Θ. By the non-degeneration of Θ, the probability of having an agent with a preference vector $\overline{\theta}$ belonging to $\overline{\Theta}$ is equal to 0.

Since the contract set is countable, I can conclude that the probability of having preferences such that there exist more than one optimal contract is 0. ■

Note that this assumption of Δ_u-monotonicity does not constrain too much as shown by following example

Example 1.1 *Take the case where the utility function is represented by a square root function. Now, take two contracts k^1, k^2 such that $q^1 \neq q^2$ and suppose that there is $\theta^1 \in \Theta$ such that q^1 and q^2 belong to $K^*\left(\theta^1, K_0\right)$. This means*

$$\theta_1^1\left(q_1^1\right)^{1/2} + \theta_2^1\left(q_2^1\right)^{1/2} = \theta_1^1\left(q_1^2\right)^{1/2} + \theta_2^1\left(q_2^2\right)^{1/2}$$

Because $q^1 \neq q^2$, there is at least one $l = 1, 2, ..., L$ such that $q_l^1 \neq q_l^2$. Without loss of generality, suppose that $q_2^1 > q_2^2$ Then,

$$\theta_1^1\left(q_1^1\right)^{1/2} + \theta_2^1\left(q_2^1\right)^{1/2} - \theta_1^1\left(q_1^2\right)^{1/2} + \theta_2^1\left(q_2^2\right)^{1/2}$$

increases if θ_2 increases.

Because the contract set is countable, the probability of having more than one optimal contract is 0.

Note also that the Cobb-Douglas utility function $u\left(\theta^i, q^h\right) = \left(q_1^h\right)^{\theta_1^i}\left(q_2^h\right)^{\theta_2^i}$ does not respect the property of Δ_u-monotonicity since when one quantity equals zero the utility levels are equal to zero irrespective of the value of θ^i. However the log transformation of the Cobb-Douglas utility will respect the Δ_u-monotonicity property.[2]

1.3 Results

The firm's problem can be described as choosing the number of monitored and non-monitored contracts and the composition of each of them. I shall denote by $\pi(N)$ the maximal profit the monopoly can obtain when it faces N consumers. By assuming that

[2] If I define the weak Δ_u-monotonicity with adding that q_1 and q_2 must be composed of positive elements, i.e. $q_1 \gg 0$ and $q_2 \gg 0$, then the Cobb-Douglass utility function satisfies the weak Δ_u-monotonicity.

the firm is risk-neutral, I can write the profit function in the following way :

$$\pi(N) = \max_{K^m, K^{nm}} \quad E[R(N, K^m, K^{nm}) - C(N, K^m, K^{nm})]$$

where $R(.)$ and $C(.)$ are respectively total revenues and total costs for the firm when the chosen monitored and non-monitored contract sets are K^m, K^{nm} and the number of consumers N.

I suppose that the firm is unable to observe consumer preference profiles and consumers are not allowed to resell the quantity bought from the firm. Let the term $\Pr\left(k^h, K^m, K^{nm}\right)$ be the probability that a consumer buys the contract k^h, i.e. $\Pr\left(k^h, K^m, K^{nm}\right) = \Pr\left(\theta^i \in \Theta | k^h \in K^*\left(\theta^i, K^m, K^{nm}\right)\right)$. Then, expected revenues can be written as follows

$$E[R(N, K^m, K^{nm})] = N * \sum_{k^h \in K} \left(P^h * \Pr\left(k^h, K^m, K^{nm}\right)\right).$$

In the previous subsection 1.2.1, I define the cost function like the sum of the production costs and the administration costs, i.e.,

$$C(N, K^m, K^{nm}) = V\left(Q(N, K^m, K^{nm})\right) + A\left(N^m(N, K^m, K^{nm}), K^m, K^{nm}\right)$$

By assuming that the production cost is linear in quantities, the production cost function is given by $V(Q) = \sum_{l=1}^{L} \left(c_l * Q_l(N, K^m, K^{nm})\right)$. With the assumption of risk neutrality by the firm, the expected value of the production cost, $E[V(Q)]$, is given by

$$E[V(Q)] = \sum_{l=1}^{L} \left(c_l * E[Q_l(N, K^m, K^{nm})]\right)$$

$E[Q_l(N, K^m, K^{nm})]$ denotes the expected total quantity of good l produced by the firm and equals

$$N \sum_{k^h \in K(K^m, K^{nm})} \left(q_l^h * \Pr\left(k^h, K^m, K^{nm}\right)\right)$$

In other words, the expected total quantity of good l is given by the sum (over the contracts belonging to the contract set K) of the quantity of good l specified in a contract times the expected number of consumers buying this contract. It follows that $E[V(Q)]$ can be written as

$$E[V(Q)] = N * \sum_{l=1}^{L} \sum_{k^h \in K(K^m, K^{nm})} \left(c_l * q_l^h * \Pr\left(k^h, K^m, K^{nm}\right)\right)$$

As specified in Section 1.2.1, the administrative cost function is the sum of the monitored contract cost function and the non-monitored contract cost function. The expected administrative cost function will therefore be given by

$$E[A\left(N^m(N,K^m,K^{nm}),K^m,K^{nm}\right)] = |K^{nm}|\,a^{nm} + |K^m|\,E[a^m\left(N^m(N,K^m,K^{nm})\right)]$$

Let $a_E^m\left(N,K^m,K^{nm}\right)$ be the expected value of $a^m\left(N^m(N,K^m,K^{nm})\right)$. With this notation, the expected administration cost function becomes :

$$\begin{aligned} E[C\left(N,K^m,K^{nm}\right)] = N * \sum_{l=1}^{L} \sum_{k^h \in K(K^m,K^{nm})} & \left(c_l q_l^h \Pr\left(k^h,K^m,K^{nm}\right)\right) \\ & + |K^{nm}|\,a^{nm} + |K^m|\,a_E^m\left(N,K^m,K^{nm}\right) \end{aligned}$$

To sum up, the maximal profit the monopoly, $\pi(N)$, can be written as

$$\begin{aligned} \pi\left(N\right) = \max_{K^m,K^{nm}} \quad N * & \sum_{k^h \in K(K^m,K^{nm})} \Pr\left(k^h,K^m,K^{nm}\right)\left(P^h - \sum_{l=1}^{L}\left(c_l q_l^h\right)\right) \\ & - |K^{nm}|\,a^{nm} - |K^m|\,a_E^m\left(N,K^m,K^{nm}\right) \end{aligned} \tag{1.1}$$

Expressed in this way, there could be situations where $\pi(N)$ does not exist since the maximization problem has no solution. Indeed, whenever the contract set K contains a non-monitored contract, K has an infinite number of elements so that the function to be maximized involves a sum over an infinite number of elements and this sum will not necessarily give a real number. I must therefore address this problem immediately. We have already seen that each contract can be expressed as a combination of non-monitored contracts and no more than one monitored contract. I can then rewrite (1.1) in terms of non-monitored and monitored contracts instead of the whole set of contracts. However there are still many possibilities whereby non-monitored contracts and monitored contracts can be combined to obtain the same k^h belonging to $K(K^m,K^{nm})$. The first possibility is when there are β^1,β^2 such that, for a given k^g belonging to $K^m \cup \{k_0\}$, k^h can be written as

$$k^h \quad = \quad k^g + \sum_{k^j \in K^{nm}} \beta_j^1 k^j \quad = \quad k^g + \sum_{k^j \in K^{nm}} \beta_j^2 k^j$$

The second possibility is when k^h can be expressed as two different combinations of monitored and non-monitored contracts :

$$k^h \quad = \quad k^g + \sum_{k^j \in K^{nm}} \beta_j^1 k^j \quad = \quad k^f + \sum_{k^j \in K^{nm}} \beta_j^2 k^j$$

with k^g, k^f belonging to $K^m \cup \{k_0\}$.

If no other structure is added, I could have a problem with double counting. Let me first define the lexicographic dominance of a vector. I say that a vector β^1 is lexicographically dominated by β^2 if

$$\begin{array}{lc} & \beta_1^1 < \beta_1^2 \\ or & \beta_1^1 = \beta_1^2 \quad and \quad \beta_2^1 < \beta_2^2 \\ or & \beta_1^1 = \beta_1^2, \quad \beta_2^1 = \beta_2^2 \quad and \quad \beta_3^1 < \beta_3^2 \\ & \cdots \end{array}$$

Let $K_0^m = K^m \cup \{k_0\}$ be the set of monitored contracts and the null contract. I define $W\left(k^h, K^m, K^{nm}\right)$, for any k^h belonging to $K(K^m, K^{nm})$, as the set of all pairs of k^g belonging to K_0^m and $\beta \in N_0^{|K^{nm}|}$ such that $k^h = k^g + \sum_{k^i \in K^{nm}} \beta^i k^i$. Let $\omega\left(k^h, K^m, K^{nm}\right)$ be the pair (k^g, β) belonging to $W\left(k^h, K^m, K^{nm}\right)$ such that β is lexicographically dominated by $\tilde{\beta}$ for all other pairs $\left(k^f, \tilde{\beta}\right)$ in $W\left(k^h, K^m, K^{nm}\right)$. $\omega\left(k^h, K^m, K^{nm}\right)$ is unique because the lexicographic ordering is complete and transitive and if $\beta = \tilde{\beta}$, then $k^g = k^f$.

Let $\widetilde{W}\left(K^m, K^{nm}\right) = \left\{\omega\left(k^h, K^m, K^{nm}\right) | k^h \in K(K^m, K^{nm})\right\}$ and let $\Psi(\eta) = \{\tilde{K} \subset \mathbb{R} \mid |\tilde{K}| = \eta\}$.

With all the definitions introduced, I can write the profit maximization problem like a double maximization where the first one is made on the number of non-monitored and monitored contracts and the second on the composition of those contracts. Formally,

$$\begin{aligned} \pi(N) = & \max_{\eta_m, \eta_{nm}} \max_{K^m \in \Psi(\eta_m), K^{nm} \in \Psi(\eta_{nm})} \\ & N \sum_{k^g \in K_0^m} \sum_{\beta \in N^{\eta_{nm}}} I\left[(k^g, \beta) \in \widetilde{W}\left(K^m, K^{nm}\right)\right] \Pr\left(k^h, K^m, K^{nm}\right) \\ & * \left(P^g + \sum_{j=1}^{\eta_{nm}} \beta_j P^j - \sum_{l=1}^{L} c_l * \left(q_l^g + \sum_{j=1}^{\eta_{nm}} \beta_j q_l^j\right)\right) \\ & - |K^{nm}|\, a^{nm} - |K^m|\, a_E^m\left(N, K^m, K^{nm}\right) \end{aligned} \tag{1.2}$$

where $I\left[(k^g, \beta) \in \widetilde{W}\left(K^m, K^{nm}\right)\right]$ is an indicator function which takes the value 1 if the condition is respected and 0 otherwise.

I have once again a summation over an infinite number of elements, but many of them are irrelevant for the problem. Indeed, since the marginal utility goes to zero when the quantity goes to infinity, for any θ^i belonging to Θ the utility converges to a level $\overline{u}\left(\theta^i\right)$ when quantities go to infinity. The utility function being continuous in θ^i, the function $\overline{u}\left(\theta^i\right)$ is also continuous in θ^i. Accordingly, there is a $\theta^i \in \Theta$ maximizing $\overline{u}\left(\theta^i\right)$. Let $\overline{u}_{MAX}$ be the maximum utility a consumer could obtain when quantities go to infinity.

Because the utility function is quasilinear in price, the maximum revenue the firm could earn is given by $N\overline{u}_{MAX}$. For a number η of contracts offered, the minimal administration cost[3] is given by ηa^{nm}. This implies that the maximum number of non-monitored and monitored contracts is equal to $\frac{N\overline{u}_{MAX}}{a^{nm}}$ since, otherwise, the firm's profit will be negative with probability 1. Let $\widehat{B} = \{\beta \in N_0^{\eta_{nm}} | \sum_{j=1}^{\eta_{nm}} \beta_j \leq \frac{N\overline{u}_{MAX}}{a^{nm}}\}$. It now follows that Problem (1.2) can be equivalently written as follows :

$$\begin{aligned}
\pi(N) = & \max_{\eta_m, \eta_{nm}} \quad \max_{K^m \in \Psi(\eta_m), K^{nm} \in \Psi(\eta_{nm})} \\
& N * \sum_{k^g \in K_0^m} \sum_{\beta \in \widehat{B}} I\left[(k^g, \beta) \in \widetilde{W}(K^m, K^{nm})\right] \Pr\left(k^h, K^m, K^{nm}\right) \\
& * \left(P^g + \sum_{j=1}^{\eta_{nm}} \beta_j P^j - \sum_{l=1}^{L} c_l * \left(q_l^g + \sum_{j=1}^{\eta_{nm}} \beta_j q_l^j\right)\right) \\
& - |K^{nm}| a^{nm} - |K^m| a_E^m(N, K^m, K^{nm})
\end{aligned} \tag{1.3}$$

The maximization problem is now well defined. I now show that this problem has a solution.

Proposition 1.1 *For any finite number of consumers, there is a solution to the profit maximization problem (1.3).*

Proof. *I proceed in four steps. The first step is finding the contract $k^*(\theta^i)$ which is the contract $(q^*(\theta^i), P^*(\theta^i))$ such that*

$$q^*(\theta^i) = \arg\max_{q \geq 0} \quad u(\theta^i, q) - \sum_{l=1}^{L} c_l q_l \tag{1.4}$$

$$P^*(\theta^i) = u(\theta^i, q^*(\theta^i)) \tag{1.5}$$

By the concavity of the utility function and by the assumption that the marginal utility goes to 0 when the quantity goes to infinity, there is a solution to (1.4) and (1.5). Also, with the assumption of strict concavity, $k^*(\theta^i)$ is unique[4]. I define $\pi^*(\theta^i)$ the profit given by the contract $k^*(\theta^i)$, i.e.,

$$\pi^*(\theta^i) = P^*(\theta^i) - \sum_{l=1}^{L} c_l q_l^*(\theta^i)$$

Let q_l^{MAX} be the maximum quantity of goods l a consumer of type $\theta^i \in \Theta$ obtained in

[3]Remember that the cost to offer a monitored contract exceeds the cost to offer a non-monitored contract.

[4]$k^*(\theta^i)$ is the perfect price discrimination contract.

$k^*\left(\theta^i\right)$, i.e.

$$q_l^{MAX} = \max_{\theta^i\in\Theta} q_l^*\left(\theta^i\right).$$

By the continuity of the utility function in θ^i and the compactness of Θ, q_l^{MAX} is upper-bounded. I can proceed by the same approach with P^{MAX} and π^{MAX}.

$$\begin{aligned} P^{MAX} &= \max_{\theta^i\in\Theta} P^*\left(\theta^i\right) \\ \pi^{MAX} &= \max_{\theta^i\in\Theta} \pi^*\left(\theta^i\right) \end{aligned}$$

I then define the set Δ.

$$\Delta = \left\{(q,P) \mid q_l \in \left[0, q_l^{MAX}\right] \quad \forall l = 1,2,...,L, \quad \text{and} \quad P \in \left[0, P^{MAX}\right]\right\}$$

Note that Δ is a compact set by construction.

As already discussed above, I show in the second step that the firm could only offer a finite number of monitored and non-monitored contracts. If the number of consumers is N, then the maximum profit the firm could obtain without counting the administration cost $A\left(N^m(N,K^m,K^{nm}),K^m,K^{nm}\right)$ is $N\pi^{MAX}$. I define η the maximum number of monitored and non-monitored contracts the firm could offer with the possibility to make a profit. With the assumption that $a^{nm} < a^m\left(N^m\left(N,K^m,K^{nm}\right)\right)$, the maximum number of monitored and non-monitored contracts is given by :

$$\begin{aligned} N\pi^{MAX} - \eta a^{nm} &> 0 \\ N\pi^{MAX} - (\eta+1)\,a^{nm} &\leq 0 \end{aligned}$$

Because $N\pi^{MAX}$ is upper bounded, that means the monopoly will never offer an infinite number of non-monitored or monitored contracts ($\eta < \infty$).

The third step consists of proving that

$$\begin{aligned} & N * \sum_{k^g\in K_0^m}\sum_{\beta\in\widehat{B}} I\left[\left(k^g,\beta\right)\in\widetilde{W}\left(K^m,K^{nm}\right)\right]\Pr\left(k^h,K^m,K^{nm}\right) \\ & * \left(P^g + \sum_{j=1}^{\eta_{nm}}\beta_j P^j - \sum_{l=1}^{L} c_l * \left(q_l^g + \sum_{j=1}^{\eta_{nm}}\beta_j q_l^j\right)\right) \\ & - \left|K^{nm}\right| a^{nm} - \left|K^m\right| a_E^m\left(N,K^m,K^{nm}\right) \end{aligned} \tag{1.6}$$

is continuous in q^g and P^g for all k^g belonging to K^m and continuous in q^j and P^j for all k^j belonging to K^{nm}.

The term $|K^{nm}|\, a^{nm}$ is obviously continuous since it is a constant. The expected monitored contract cost function $a^m_E(N, K^m, K^{nm})$ is function of the number of consumers buying a monitored contract. Because I take the expectation and the distribution function of θ^i is non-degenerative, then $a^m_E(N, K^m, K^{nm})$ is continuous in q^z and P^z.

I now examine the terms in the summation. To do this let me begin by defining δ_t as the vector $\left(\{\delta^j_t\}^{\eta_{nm}}_{j=1}, \{\delta^g_t\}^{\eta_{nm}+\eta_m}_{g=\eta_{nm}+1}\right)$.

Then $\overline{K^m}$ and $\overline{K^{nm}}$ be the contract sets with η_m monitored contracts and η_{nm} non-monitored contracts. I define $\overline{K^m}_t$ and $\overline{K^{nm}}_t$ the monitored and non-monitored contract sets such that $\overline{k}^g_t = \overline{k}^g + \delta^g_t$ for all $\overline{k}^g$ belonging to $\overline{K^m}$ and $\overline{k}^j_t = \overline{k}^j + \delta^j_t$ for all $\overline{k}^j$ belonging to $\overline{K^{nm}}$.

To prove the continuity of the terms in the summation in (1.6), I must discuss two cases.

The first case is when, for all $\overline{k}^g$ belonging to $\overline{K^m}$, for all β belonging to $\hat{B}$ and for any sequence $\{\delta_t\}$ that converges to the zero vector whenever t tends to infinity, the indicator function $I\left[\left(\overline{k}^g_t, \beta\right) \in \widetilde{W}\left(\overline{K^m}_t, \overline{K^{nm}}_t\right)\right] = I\left[\left(\overline{k}^g, \beta\right) \in \widetilde{W}\left(\overline{K^m}, \overline{K^{nm}}\right)\right]$ for all t. In this case, the sum of the indicator function times the probability becomes a sum of probabilities.

By definition, if the indicator function is equal to 1 for a given pair (k^g, β), then there is no other pair $\left(k^f, \beta^f\right)$ with k^f belonging to $\overline{K^m}$ and β belonging to $\hat{B}$ such that $k^g + \sum^{\eta_{nm}}_{j=1} \beta_j k^j = k^f + \sum^{\eta_{nm}}_{j=1} \beta^f_j k^j$ and such that β^f is lexicographically dominated by β.

Then, I have to analyze the effect of changes in q^g, P^g, q^j and P^j on the probability $\Pr\left(k^h, K^m, K^{nm}\right)$ evaluated at $k^g = \overline{k}^g$, $k^j = \overline{k}^j$ and $K^m = \overline{K^m}, K^{nm} = \overline{K^{nm}}$.

Let $\overline{k}^h = \overline{k}^g + \sum^{\eta_{nm}}_{j=1} \beta_j \overline{k}^j$. By definition, if $\overline{k}^h \in K^*\left(\theta^i, \overline{K^m}, \overline{K^{nm}}\right)$, then :

$$u\left(\theta, \overline{q}^h_1, ..., \overline{q}^h_L\right) - \overline{P}^h \geq \max_{\overline{k}^z \in \overline{K}_0} \left[u\left(\theta, \overline{q}^z_1, ..., \overline{q}^z_L\right) - \overline{P}^z\right]$$

By Lemma 1.1, the probability that an agent has a preference profile such that there are two contracts belonging to $K^*\left(\theta^i, K^m, K^{nm}\right)$ is 0. Then,

$$\Pr\left(k^h, K^m, K^{nm}\right) =$$
$$Pr\left(\quad u\left(\theta, \overline{q}^h_1, ..., \overline{q}^h_L\right) - \overline{P}^h > \max_{\overline{k}^z \in \overline{K}_0} \left[u\left(\theta, \overline{q}^z_1, ..., \overline{q}^z_L\right) - \overline{P}^z\right] \quad \right)$$

Because $u\left(\theta^i, \overline{q}^h\right)$ is continuous in θ^i and in $\overline{q}^h$ and because the distribution function of the θ's is non-degenerative, I can use the Slutsky Theorem[5] *to prove that the probability is continuous in $\overline{q}^g, \overline{P}^g$ for all $\overline{k}^g$ belonging to K^m and in $\overline{q}^j, \overline{P}^j$ for all $\overline{k}^j$ belonging to K^{nm}.*

The second case is when there is a pair $\left(\overline{k}^g, \beta\right)$ with $\overline{k}^g$ belonging to $\overline{K^m}$ and β belonging to $\hat{B}$ and a sequence of δ_t such that, for all $t > \tilde{t}$, $I\left[\left(k^g_t, \beta\right) \in \widetilde{W}\left(\overline{K^m}_t, \overline{K^{nm}}_t\right)\right] \neq$

[5] See, for instance, Jacod & Protter [4] page 161.

$I\left[(k^g,\beta) \in \widetilde{W}\left(\overline{K^m},\overline{K^{nm}}\right)\right]$.

To begin with, suppose that, for the pair $\left(\overline{k}^g,\beta\right)$, $I\left[\left(\overline{k}^g,\beta\right) \in \widetilde{W}\left(\overline{K^m},\overline{K^{nm}}\right)\right] = 1$ *and* $I\left[\left(\overline{k}^g_t,\beta\right) \in \widetilde{W}\left(\overline{K^m}_t,\overline{K^{nm}}_t\right)\right] = 0$ *for all* $t > \tilde{t}$. *If* $I\left[\left(\overline{k}^g,\beta\right) \in \widetilde{W}\left(\overline{K^m},\overline{K^{nm}}\right)\right] = 1$, *then this means that, for all pairs* $\left(\overline{k}^f,\beta^f\right)$ *with* $\overline{k}^f$ *belonging to* $\overline{K^m}$ *and* β^f *belonging to* $\hat{B}$ *such that* $\overline{k}^g + \sum_{j=1}^{\eta_{nm}} \beta_j \overline{k}^j = \overline{k}^f + \sum_{j=1}^{\eta_{nm}} \beta_j^f \overline{k}^j$, β *is lexicographically dominated by* β^f.

If $I\left[\left(\overline{k}^g_t,\beta\right) \in \widetilde{W}\left(\overline{K^m}_t,\overline{K^{nm}}_t\right)\right] = 0$ *for* $t > \tilde{t}$, *this means that there exists a sequence of pair* $\{(\overline{k}^z_t,\beta^z)\}_{t>\tilde{t}}$ *with* $\overline{k}^z_t$ *belonging to* $\overline{K^m}_t$ *and* β^z *belonging to* $\hat{B}$ *such that* $\overline{k}^g_t + \sum_{j=1}^{\eta_{nm}} \beta_j \overline{k}^j_t = \overline{k}^z_t + \sum_{j=1}^{\eta_{nm}} \beta_j^z \overline{k}^j_t$ *and* β^z *is lexicographically dominated by* β. *Note however that*

$$\begin{aligned}
\lim_{t\to\infty} \overline{k}^g_t + \sum_{j=1}^{\eta_{nm}} \beta_j \overline{k}^j_t &= \overline{k}^g + \sum_{j=1}^{\eta_{nm}} \beta_j \overline{k}^j \\
\lim_{t\to\infty} \overline{k}^z_t + \sum_{j=1}^{\eta_{nm}} \beta_j^z \overline{k}^j_t &= \overline{k}^z + \sum_{j=1}^{\eta_{nm}} \beta_j^z \overline{k}^j.
\end{aligned}$$

Accordingly, I find that $\overline{k}^g + \sum_{j=1}^{\eta_{nm}} \beta_j \overline{k}^j$ *is equal to* $\overline{k}^z + \sum_{j=1}^{\eta_{nm}} \beta_j^z \overline{k}^j$ *with* β^z *lexicographically dominated by* β. *This leads a contradiction which implies that there does not exist a pair* $\left(\overline{k}^g,\beta\right)$ *for which, for all* $t > \tilde{t}$, $I\left[\left(\overline{k}^g,\beta\right) \in \widetilde{W}\left(\overline{K^m},\overline{K^{nm}}\right)\right] = 1$ *and* $I\left[\left(\overline{k}^g_t,\beta\right) \in \widetilde{W}\left(\overline{K^m}_t,\overline{K^{nm}}_t\right)\right] = 0$.

Now, suppose that, for the pair $\left(\overline{k}^g,\beta\right)$ *and for all* $t > \tilde{t}$, $I\left[\left(\overline{k}^g,\beta\right) \in \widetilde{W}\left(\overline{K^m},\overline{K^{nm}}\right)\right] = 0$ *and* $I\left[\left(\overline{k}^g_t,\beta\right) \in \widetilde{W}\left(\overline{K^m}_t,\overline{K^{nm}}_t\right)\right] = 1$. *This means that there is a pair* $\left(\overline{k}^f,\beta^f\right)$ *with* $\overline{k}^f$ *belonging to* $\overline{K^m}$ *and* β^f *belonging to* $\hat{B}$ *such that* $\overline{k}^g + \sum_{j=1}^{\eta_{nm}} \beta_j \overline{k}^j = \overline{k}^f + \sum_{j=1}^{\eta_{nm}} \beta_j^f \overline{k}^j$ *and* β^f *lexicographically dominated by* β. *Without lost of generality, suppose there is only one such pair. If* $I\left[\left(\overline{k}^g_t,\beta\right) \in \widetilde{W}\left(\overline{K^m}_t,\overline{K^{nm}}_t\right)\right] = 1$ *for all* $t > \tilde{t}$, *this means* $\overline{k}^g_t + \sum_{j=1}^{\eta_{nm}} \beta_j \overline{k}^j_t$ *is not equal to* $\overline{k}^f_t + \sum_{j=1}^{\eta_{nm}} \beta_j^f \overline{k}^j_t$. *Then, the probability that a consumer has preferences* θ^i *such that either* $\left(\overline{k}^g_t + \sum_{j=1}^{\eta_{nm}} \beta_j \overline{k}^j_t\right)$ *or* $\left(\overline{k}^f_t + \sum_{j=1}^{\eta_{nm}} \beta_j^f \overline{k}^j_t\right)$ *belongs to* $K^*\left(\theta^i,\overline{K^m}_t,\overline{K^{nm}}_t\right)$ *is given by*

$$\begin{aligned}
&\Pr\left(\theta^i \in \Theta | \overline{k}^g_t + \sum_{j=1}^{\eta_{nm}} \beta_j \overline{k}^j_t \in K^*\left(\theta^i,\overline{K^m}_t,\overline{K^{nm}}_t\right)\right) \\
&+\Pr\left(\theta^i \in \Theta | \overline{k}^f_t + \sum_{j=1}^{\eta_{nm}} \beta_j^f \overline{k}^j_t \in K^*\left(\theta^i,\overline{K^m}_t,\overline{K^{nm}}_t\right)\right) \\
&-\Pr\left(\theta^i \in \Theta | \left(\overline{k}^g_t + \sum_{j=1}^{\eta_{nm}} \beta_j \overline{k}^j_t\right) \text{ and } \left(\overline{k}^f_t + \sum_{j=1}^{\eta_{nm}} \beta_j^f \overline{k}^j_t\right) \text{ belong to } K^*\left(\theta^i,\overline{K^m}_t,\overline{K^{nm}}_t\right)\right)
\end{aligned}$$

By Lemma 1.1, if $\overline{k}_t^g + \sum_{j=1}^{\eta_{nm}} \beta_j \overline{k}_t^j$ *is not equals to* $\overline{k}_t^f + \sum_{j=1}^{\eta_{nm}} \beta_j^f \overline{k}_t^j$*, then the probability that a consumer has a preference profile such that* $\overline{k}_t^g + \sum_{j=1}^{\eta_{nm}} \beta_j \overline{k}_t^j$ *and* $\overline{k}_t^f + \sum_{j=1}^{\eta_{nm}} \beta_j^f \overline{k}_t^j$ *belongs to* $K^*\left(\theta^i, \overline{K^m}_t, \overline{K^{nm}}_t\right)$ *is zero. But by taking the limit of the preceding sum of probabilities, I obtain*

$$\begin{aligned}
\lim_{t\to\infty} \quad & \Pr\left(\theta^i \in \Theta | \overline{k}_t^g + \sum_{j=1}^{\eta_{nm}} \beta_j \overline{k}_t^j \in K^*\left(\theta^i, \overline{K^m}_t, \overline{K^{nm}}_t\right)\right) \\
& + \Pr\left(\theta^i \in \Theta | \overline{k}_t^f + \sum_{j=1}^{\eta_{nm}} \beta_j^f \overline{k}_t^j \in K^*\left(\theta^i, \overline{K^m}_t, \overline{K^{nm}}_t\right)\right) \\
& - \Pr\left(\theta^i \in \Theta | \left(\overline{k}_t^g + \sum_{j=1}^{\eta_{nm}} \beta_j \overline{k}_t^j\right), \left(\overline{k}_t^f + \sum_{j=1}^{\eta_{nm}} \beta_j^f \overline{k}_t^j\right) \in K^*\left(\theta^i, \overline{K^m}_t, \overline{K^{nm}}_t\right)\right)
\end{aligned}$$

$$\begin{aligned}
= \quad & \Pr\left(\theta^i \in \Theta | \overline{k}^g + \sum_{j=1}^{\eta_{nm}} \beta_j \overline{k}^j \in K^*\left(\theta^i, \overline{K^m}, \overline{K^{nm}}\right)\right) \\
& + \Pr\left(\theta^i \in \Theta | \overline{k}^f + \sum_{j=1}^{\eta_{nm}} \beta_j^f \overline{k}^j \in K^*\left(\theta^i, \overline{K^m}, \overline{K^{nm}}\right)\right) \\
& - \Pr\left(\theta^i \in \Theta | \left(\overline{k}^g + \sum_{j=1}^{\eta_{nm}} \beta_j \overline{k}^j\right), \left(\overline{k}_t^f + \sum_{j=1}^{\eta_{nm}} \beta_j^f \overline{k}^j\right) \in K^*\left(\theta^i, \overline{K^m}, \overline{K^{nm}}\right)\right) \\
= \quad & \Pr\left(\theta^i \in \Theta | \overline{k}^f + \sum_{j=1}^{\eta_{nm}} \beta_j^f \overline{k}^j \in K^*\left(\theta^i, \overline{K^m}, \overline{K^{nm}}\right)\right)
\end{aligned}$$

because $\overline{k}^g + \sum_{j=1}^{\eta_{nm}} \beta_j \overline{k}^j = k^f + \sum_{j=1}^{\eta_{nm}} \beta_j^f \overline{k}^j$.

Then, by introducing the indicator function, I avoid double counting. I therefore have

$$\begin{aligned}
\lim_{t\to\infty} \quad & I\left[\left(\overline{k}_t^g,\beta\right)\in\widetilde{W}\left(\overline{K^m}_t,\overline{K^{nm}}_t\right)\right]\Pr\left(\theta^i\in\Theta|\overline{k}_t^g+\sum_{j=1}^{\eta_{nm}}\beta_j\overline{k}_t^j\in K^*\left(\theta^i,\overline{K^m}_t,\overline{K^{nm}}_t\right)\right)\\
& +I\left[\left(\overline{k}_t^f,\beta\right)\in\widetilde{W}\left(\overline{K}_t\right)\right]\Pr\left(\theta^i\in\Theta|\overline{k}_t^f+\sum_{j=1}^{\eta_{nm}}\beta_j^f\overline{k}_t^j\in K^*\left(\theta^i,\overline{K^m}_t,\overline{K^{nm}}_t\right)\right)\\
= \quad & I\left[\left(\overline{k}^g,\beta\right)\in\widetilde{W}\left(\overline{K}\right)\right]\Pr\left(\theta^i\in\Theta|\overline{k}^g+\sum_{j=1}^{\eta_{nm}}\beta_j\overline{k}^j\in K^*\left(\theta^i,\overline{K^m},\overline{K^{nm}}\right)\right)\\
& +I\left[\left(\overline{k}^f,\beta\right)\in\widetilde{W}\left(\overline{K}\right)\right]\Pr\left(\theta^i\in\Theta|\overline{k}^f+\sum_{j=1}^{\eta_{nm}}\beta_j^f\overline{k}^j\in K^*\left(\theta^i,\overline{K^m},\overline{K^{nm}}\right)\right)
\end{aligned}$$

where $I\left[\left(\overline{k}_t^g,\beta\right)\in\widetilde{W}\left(\overline{K^m}_t,\overline{K^{nm}}_t\right)\right]$ *and* $I\left[\left(\overline{k}_t^f,\beta\right)\in\widetilde{W}\left(\overline{K^m}_t,\overline{K^{nm}}_t\right)\right]$ *are equal to one while* $I\left[\left(\overline{k}^g,\beta\right)\in\widetilde{W}\left(\overline{K^m},\overline{K^{nm}}\right)\right]$ *is equal to zero and* $I\left[\left(\overline{k}^f,\beta\right)\in\widetilde{W}\left(\overline{K^m},\overline{K^{nm}}\right)\right]$ *is equal to one.*

Then, (1.6) is continuous in q^g *and* P^g *for all* k^g *belonging to* K^m *and in* q^j *and* P^j *for all* k^j *belonging to* K^{nm}*.*

For the fourth step, let $\tilde{\pi}\left(\eta^{nm},\eta^m\right)$ *be the maximum profit obtained when the optimal contract must be composed of* η^{nm} *non-monitored contracts and* η^m *monitored contracts.*

$$\begin{aligned}
\tilde{\pi}\left(N,\eta^{nm},\eta^m\right)= & \max_{K^m\in\Psi(\eta_m),K^{nm}\in\Psi(\eta_{nm})}\\
& N*\sum_{k^g\in K_0^m}\sum_{\beta\in\widehat{B}}I\left[\left(k^g,\beta\right)\in\widetilde{W}\left(K^m,K^{nm}\right)\right]\Pr\left(k^g+\sum_{j=1}^{\eta_{nm}}\beta_jk^j,K^m,K^{nm}\right)\\
& *\left(P^g+\sum_{j=1}^{\eta_{nm}}\beta_jP^j-\sum_{l=1}^{L}c_l*\left(q_l^g+\sum_{j=1}^{\eta_{nm}}\beta_jq_l^j\right)\right)\\
& -\left|K^{nm}\right|a^{nm}-\left|K^m\right|a_E^m\left(N,K^m,K^{nm}\right)
\end{aligned}$$

In the previous step, I prove that the function (1.6) is continuous in q^z *and* P^z *for all* k^z *belonging to* K^{nm} *or to* K^m*. Moreover, non-monitored and monitored contracts must belong to* Δ*. Then, I use the theorem of the maximum to prove that* $\tilde{\pi}\left(N,\eta^{nm},\eta^m\right)$ *exists.*

Since η *is finite, then the number of combinations of* $\left(\eta^{nm},\eta^m\right)$ *with* $\eta^{nm}+\eta^m\leq\eta$ *is also finite. Consequently, there is a maximal element in the set*

$$\left\{\tilde{\pi}\left(N,\eta^{nm},\eta^m\right)\quad|\quad\eta^{nm}+\eta^m\leq\eta\right\}$$

■

I can now examine if the maximal profit $\pi(N)$ is strictly positive for any N. This could not necessarily be the case since I have assumed above that the monitoring cost function is increasing in N. It could for instance happen that the profit is strictly positive when there are 10 consumers but equal to zero with 11 consumers. However, as shown by the following result, I can state under certain conditions that profits will stay strictly positive when the number of consumers exceeds a critical level.

Proposition 1.2 *Suppose that $\exists l \in \{1, 2, ..., L\}$ such that, for all q^h with $q_l^h = 0$,*

$$\Pr\left(\theta^i \in \Theta \quad | \quad \frac{\partial u\left(\theta^i, q^h\right)}{\partial q_l^h} > c_l\right) > 0$$

There is a $\overline{N}$ such that $\forall N \geq \overline{N}$, the profit is strictly positive.

Proof. *Suppose that $K^m = \emptyset$ and K^{nm} has only one element which maximizes :*

$$\sum_{k^h \in K(\emptyset, K^{nm})} \Pr\left(k^h, \emptyset, K^{nm}\right) * \left(P^h - \sum_{l=1}^{L}\left(c_l * q_l^h\right)\right)$$

By assumption, I know that this term is positive. Then, I can set $\overline{N}$ such that

$$\begin{aligned}
\overline{N} * \sum_{k^h \in K(\emptyset, K^{nm})} \Pr\left(k^h, \emptyset, K^{nm}\right) * \left(P^h - \sum_{l=1}^{L}\left(c_l * q_l^h\right)\right) - a^{nm} &\leq 0 \\
\left(\overline{N} + 1\right) * \sum_{k^h \in K(\emptyset, K^{nm})} \Pr\left(k^h, \emptyset, K^{nm}\right) * \left(P^h - \sum_{l=1}^{L}\left(c_l * q_l^h\right)\right) - a^{nm} &> 0
\end{aligned}$$

■

The condition for a strictly positive maximal profit is not a strong one. It says that if there is a positive probability to finding a consumer with a marginal utility over at least one good that is higher than the marginal cost to produce this good, then the firm makes profit when the number of consumers is high enough. The intuition for the proof is simple. When the number of consumers increases, the administrative cost to provide one non-monitored contract, which is constant with respect to N, becomes negligible. It will then become possible to make strictly positive profits whenever the number of consumers becomes sufficiently large.

Let me now study if the maximal profit increases when the number of consumers increases. The following result shows that this will depend on the form of the monitored contract cost function.

Proposition 1.3 *If $a_E^m(N, K^m, K^{nm})$ is concave in N, then the profit function is non-decreasing in N. If there exists an $\widetilde{N}$ for which the profit is positive, then the profit function is increasing for all $N \geq \widetilde{N}$.*

Proof. *Let $\widetilde{K^m}(N)$ and $\widetilde{K^{nm}}(N)$ be the optimal monitored and non-monitored contract set when the number of consumers is N and let $\widetilde{K} = K\left(\widetilde{K^m}(N), \widetilde{K^{nm}}(N)\right)$. By definition,*

$$\begin{aligned}\pi(N+1) \geq\ & (N+1) \sum_{k^h \in \widetilde{K}} \Pr\left(k^h, \widetilde{K^m}(N), \widetilde{K^{nm}}(N)\right)\left(P^h - \sum_{l=1}^{L}\left(c_l * q_l^h\right)\right) \\ & - \left|K^{nm}\right| a^{nm} - \left|K^m\right| a_E^m\left(N+1, \widetilde{K^m}(N), \widetilde{K^{nm}}(N)\right)\end{aligned}$$

where the right hand side is the profit when the number of consumers is $N+1$ and the contract sets are $\widetilde{K^m}(N)$ and $\widetilde{K^{nm}}(N)$.

$$\begin{aligned}\pi(N+1) \geq\ & (N+1)\left[\sum_{k^h \in \widetilde{K}} \Pr\left(k^h, \widetilde{K^m}(N), \widetilde{K^{nm}}(N)\right)\left(P^h - \sum_{l=1}^{L}\left(c_l * q_l^h\right)\right)\right] \\ & - \left|K^{nm}\right| a^{nm} - (N+1)\frac{\left|K^m\right| a_E^m\left(N+1, \widetilde{K^m}(N), \widetilde{K^{nm}}(N)\right)}{N+1}\end{aligned}$$

By the concavity of $a_E^m\left(N, \widetilde{K^m}(N), \widetilde{K^{nm}}(N)\right)$ in N, then

$$\frac{a_E^m\left(N, \widetilde{K^m}(N), \widetilde{K^{nm}}(N)\right)}{N} \geq \frac{a_E^m\left(N+1, \widetilde{K^m}(N), \widetilde{K^{nm}}(N)\right)}{N+1}$$

and therefore

$$\begin{aligned}\pi(N+1) \geq\ & (N+1)\left[\sum_{k^h \in \widetilde{K}} \Pr\left(k^h, \widetilde{K^m}(N), \widetilde{K^{nm}}(N)\right)\left(P^h - \sum_{l=1}^{L}\left(c_l * q_l^h\right)\right)\right] \\ & - \left|K^{nm}\right| a^{nm} - (N+1)\frac{a_E^m\left(N, \widetilde{K^m}(N), \widetilde{K^{nm}}(N)\right)}{N} \\ \pi(N+1) \geq\ & N\left[\sum_{k^h \in \widetilde{K}} \Pr\left(k^h, \widetilde{K^m}(N), \widetilde{K^{nm}}(N)\right)\left(P^h - \sum_{l=1}^{L}\left(c_l * q_l^h\right)\right)\right] \\ & - \left|K^{nm}\right| a^{nm} - N\frac{a_E^m\left(N, \widetilde{K^m}(N), \widetilde{K^{nm}}(N)\right)}{N}\end{aligned}$$

with a strict inequality if $\pi(N) > 0$. Note that the .right hand side of this inequality is simply $\pi(N)$ so that the results follow. ■

Without any assumptions on the concavity of the monitored contract cost function, it is impossible to obtain any conclusions about the evolution of the maximal profit $\pi(N)$ with respect to the number of consumers. It could happen that the increase in the cost of monitoring contracts is more important than the increase in the revenue minus the production cost. In this case, the profit will decrease.

The second interesting question regards the composition of the optimal contract set. The first result found is that the monopolist will always offer at least one non-monitored contract. The following proposition demonstrates this result.

Proposition 1.4 *Suppose $\widetilde{K^m}$ and $\widetilde{K^{nm}}$ are the optimal monitored and non-monitored contract set. Then, there is at least one non-monitored contract, i.e. $\widetilde{K^{nm}} \neq \emptyset$.*

Proof. *Suppose that $\widetilde{K^{nm}} = \emptyset$. By definition, $K\left(\widetilde{K^m}, \emptyset\right) = \widetilde{K^m}$. In this case the profit function is :*

$$N \sum_{k^h \in \widetilde{K^m}} \left[\Pr\left(k^h, K^m, K^{nm}\right) \left(P^h - \sum_{l=1}^{L} \left(c_l q_l^h\right)\right) - a_E^m \left(N, \widetilde{K^m}, \emptyset\right) \right]$$

First, I know that the number of monitored contracts if finite. If not, the fixed cost will be infinite. consequently, for each monitored contract k^h belonging to $\widetilde{K^m}$,

$$\Pr\left(k^h, K^m, K^{nm}\right) \left(P^h - \sum_{l=1}^{L} \left(c_l q_l^h\right)\right) - a_E^m \left(N, \widetilde{K^m}, \emptyset\right) > 0 \tag{1.7}$$

If there is a contract in which this condition is not respected, I could drop this contract out without decreasing the profit.

I define $\overline{k} \in \widetilde{K^m}$ as the monitored contract maximizing $\left(P^h - \sum_{l=1}^{L} \left(c_l * q_l^h\right)\right)$. Let $\overline{K^m} = \widetilde{K^m} \setminus \left\{\overline{k}\right\}$ and $\overline{K^{nm}} = \left\{\overline{k}\right\}$. The profit with $\overline{K^m}$ and $\overline{K^{nm}}$ is given by :

$$\begin{aligned} & N \sum_{k^g \in \overline{K_0^m}} \sum_{\beta \in \widehat{B}} I\left[\left(k^g, \beta\right) \in \widetilde{W}\left(\overline{K^m}, \overline{K^{nm}}\right)\right] \Pr\left(k^h, \overline{K^m}, \overline{K^{nm}}\right) \\ & \left(P^g + \sum_{j=1}^{\eta_{nm}} \beta_j P^j - \sum_{l=1}^{L} c_l * \left(q_l^g + \sum_{j=1}^{\eta_{nm}} \beta_j q_l^j\right)\right) \\ & - a^{nm} - \left|\overline{K^m}\right| a_E^m \left(N, \overline{K^m}, \overline{K^{nm}}\right) \end{aligned}$$

By assumption, $a_E^m(.) > a^{nm}$. Hence, the administration cost of offering $\overline{K^m}$ and $\overline{K^{nm}}$ is lower than the cost of offering $\widetilde{K^m}$ and $\emptyset$ respectively as the monitored and non-monitored contract sets. Then, if the first term with the contract sets $\overline{K^m}$ and $\overline{K^{nm}}$ is not smaller than with $\widetilde{K^m}$ and $\emptyset$, the point is proven.

Consumers who buy a contract under the contract set $K(\widetilde{K^m}, \emptyset)$ can do three things *under the contract set $K(\overline{K^m}, \overline{K^{nm}})$: they can continue to buy the same contract, they can stop buying or the can change their choice for a new contract. By construction, $K(\widetilde{K^m}, \emptyset) \subset K(\overline{K^m}, \overline{K^{nm}})$. Consumers who buy a contract under $K(\widetilde{K^m}, \emptyset)$ will never choose not to buy a contract when they face the contract set $K(\overline{K^m}, \overline{K^{nm}})$ because they decide to buy a contract belonging to $K^*\left(\theta, \widetilde{K^m}, \emptyset\right)$ and this contract is still available in $K(\overline{K^m}, \overline{K^{nm}})$.*

Then, consumers must choose between a contract belonging to $K^*\left(\theta, \widetilde{K^m}, \emptyset\right)$ or another contract. It is sure with probability equal to 1 that consumers will not switch to another contract $k^h \in \widetilde{K^m} \backslash \{\widetilde{k}\}$. If they do, this means $K^*\left(\theta, \widetilde{K^m}, \emptyset\right)$ contains two contracts but the probability that $K^*\left(\theta, \widetilde{K^m}, \emptyset\right)$ has two contracts is 0.[6]

Consumers really only have two possibilities : keep buying the contract they buy with the contract set $K(\widetilde{K^m}, \emptyset)$ or buy a contract $\widehat{k}$ which is a combination of the non-monitored contract and of at most one monitored contract.

$$\widehat{k} = k^h + \beta\overline{k} \quad k^h \in \widetilde{K}, \beta \in \{1, 2, ...\}$$

But $\widehat{P} - \sum_{l=1}^{L} (c_l * \widehat{q_l}) \geq \overline{P} - \sum_{l=1}^{L} (c_l * \overline{q_l})$ which means that the first term of (1.7) is greater with $\overline{K}$ than with $\widetilde{K}$. ∎

This result is very interesting. It says that every firm will offer at least one non-monitored contract, which is what happens in the real economy. Many stores offer some special discounts to big buyers and offer to others the possibility of buying without being monitored. The stores open an account for big buyers and offer a discount depending of the total purchases.

Which form of administration cost function is more likely to occur? The intuition says that concavity for the monitored contract cost function is a realistic assumption. The biggest cost of implementing monitoring structure is more a form of fixed cost. Some observations strengthen this intuition. Convenience stores almost always offer non-monitored contracts and supermarkets offer discounts on a specific quantity of goods. Also, if there is an important fixed cost to implement monitoring, only big surface selling stores will use monitored contracts. On the other hand, one can argue that it becomes more complicated to keep track of consumers as their number increases. Nevertheless, some technological implements can contribute to diminishing the monitored contract costs.

1.4 Remarks on assumptions

To develop the model, I use strong assumptions on the utility function, the production cost function and the monitoring cost function. I impose those assumptions to simplify

[6]See the proof of Proposition 1.

the presentation of proofs. Many of these assumptions could be relaxed to more general functions, however.

Regarding the utility function and the production cost function, I need assumptions which would guarantee the existence of a solution to the profit maximization problem. To have a solution, it must be the case that, there is a $\overline{q}$ such that, for all $q \geq \overline{q}$, for all $Q \geq q$ and for all $\theta^i \in \Theta$,

$$\frac{dv\left(Q_1,...,Q_L\right)}{dQ_l} > \frac{du\left(\theta^i,q_1,...,q_L\right)}{dq_l} \qquad \forall l = 1,2,...,L$$

In other words, the marginal utility at $\overline{q}$ must be lower than the marginal cost to provide at least $\overline{q}$. For consumers, the assumption of strict concavity in q is not necessary. The concavity is enough to have existence.

The assumptions about the monitoring cost function are more problematic to discuss. In fact, it is very difficult to specify the form of this function. It makes sense to have a monitoring cost function increasing in the number of base contracts but the linearity doesn't look too realistic. But I do not need the linearity to prove its existence. I simply need the assumption specifying an increase of the monitoring cost when the number of base contracts increases.

$$A\left(N^m(N,K^m,K^{nm}),K^m,K^{nm}\right) > A\left(N^m(N,\widetilde{K^m},\widetilde{K^{nm}}),\widetilde{K^m},\widetilde{K^{nm}}\right)$$
$$if \quad \begin{array}{l} (\eta_{nm},\eta_m) \geq (\widetilde{\eta_{nm}},\widetilde{\eta_m}) \\ (\eta_{nm},\eta_m) \neq (\widetilde{\eta_{nm}},\widetilde{\eta_m}) \end{array}$$

where η_{nm} is the number of non-monitored contracts and η_m is the number of monitored contracts. Without this assumption, I am not able to upper-bound the number of base contracts. In this case the number of contracts could be infinite and it is impossible to guarantee a solution to the profit maximization problem.

1.5 Conclusion

In this paper, I try to model the firm's decision when it involves monitoring. I find some sufficient conditions to get an existence proposition. It appears that the main assumptions to guarantee a solution are more about consumer preferences and the production function than about the monitoring cost function. I also find some results on the characterization of the optimal contract set under specific assumptions.

Another issue involves the definition of monitoring. I define monitoring as the capacity to constrain consumers to buy no more than one contract, when in fact, other definitions could also be used. For example, I could assume that a monitored contract could be bought with or without another monitored contract. But using alternative definitions complicates the presentation of the results without providing additional insights on the problem of monitoring.

In my opinion, future work on monitoring should study the uniqueness of the monopoly solution problem. Uniqueness is not guaranteed under the assumptions introduced in this paper and it seems that stronger assumptions on the utility function would have to be made in order to obtain it. Another way to extend the present study of monitoring is by developing the characterization of the optimal contract set. Also, the approach proposed in this paper could be taken in the context of a regulated monopoly context in order to examine whether the use of monitored contracts could increase welfare.

Finally, introducing monitoring in a duopoly, oligopoly or perfectly competitive framework seems to be the natural next step. But it seems to me that this step will be very difficult to take. The multidimensional preference profiles and good vectors complicate the analysis of the stability of any optimal strategies. Furthermore, firms will not compete only on quantities and on price, but also on the number of contracts and on their nature.

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Chapitre 2

Cycles and the House Allocation Problem

2.1 Introduction

The house allocation problem consists of the assignment of indivisible goods to a set of agents who can receive only one object in the final allocation. Such problems are very common : allocation of rooms between roommates, lectures between professors, offices between colleagues, etc.

This class of problems was introduced by Shapley and Scarf [8]. In their model, agents own all goods collectively. While Shapley and Scarf prove the existence of a competitive equilibrium, Roth and Postlewaite [7] show that this competitive equilibrium is unique when preferences are strict over the set of goods. Roth [6] proves that this unique solution can be implemented by a strategy-proof allocation mechanism. Furthermore, there is a unique strategy-proof, individually rational and Pareto optimal allocation mechanism leading to the unique core allocation (Ma [5]). Abdulkadiroğlu and Sönmez [1] show the equivalence between the competitive allocation from random endowments and the random serial dictatorship while Svensson [9] proves that all mechanisms that are strategy-proof, nonbossy and neutral must be serially dictatorial. Abdulkadiroğlu and Sönmez [2] model the case where there exists at the same time tenants and new comers on the same market. They introduce the top trading mechanism in this set-up and show that it is Pareto efficient, individually rational and strategy-proof. Ehlers [4] introduces the possibility of having weak preferences over the set of goods and shows some restrictions on agent preferences with which efficiency and coalitional strategy-proofness are compatible.[1]

The purpose of this paper is to look at rationalizability in the context of the house allocation problem. In other word, I am interested in answering the following question : is it possible to say if, for a given set of allocations, there is a preference profile which supports this set as a Paretian allocation set ? In existing papers on the house allocation problem, only the paper by Ben-Shoham, Serrano and Volij [3] mentions explicitly the composition of the Paretian allocation set. They show that for any two allocations in the Paretian set, there exists a sequence of allocations belonging to the Paretian set such that they are pairwise connected, i.e. there are only two agents switching their goods and all others stay with the same good. This means that a set with two allocations that are not pairwise connected cannot be rationalized.

To go further on rationalizability, I need to introduce the concept of a cycle. A cycle is a subset of allocations in which a subset of agents switch their goods according to a specific scheme. The presence of cycles in a given set of allocations which is presumingly a Paretian set gives us information on the potential preference profiles which would support this set as a Paretian allocation set. With the concept of a cycle, I derive some conditions regarding the number of allocations that have to belong to an allocation set in order for it

[1]This list of papers treating of the house allocation problem is not exhaustive.

to be a Paretian allocation set. Also, by using cycles, I am able to say if some allocations must belong to the set of Paretian allocations.

The paper is organized as follows. In Section 2.2, I present the house allocation problem and I define the concept of a cycle. Section 2.3 talks about the properties of the cycle and Section 2.4 presents the implication of the presence of cycles in the Paretian allocation set. Section 2.5 concludes.

2.2 Definitions and Notations

Let $N = \{1, 2, ..., |N|\}$ denote the set of agents with $|N| \geq 2$. The set of goods is $X = \{x_1, x_2, ..., x_{|N|}\}$ where all goods are different. I define an allocation $a = (a_1, ..., a_h, ..., a_{|N|})$ where $a_h \in X$ is the good allocated to agent h with $a_i \neq a_j$ for $i \neq j$. For any set of agents $I \subseteq N$ and for any set of goods $Y \subseteq X$ with $|I| = |Y|$, $A(I, Y)$ denotes the set of all possible allocations of goods in Y to agents in I.

Agent h's preferences are represented by a binary relation P_h which is complete, transitive and antisymmetric (strict preference). Given $x_1, x_2 \in X$, $x_1 \ P_h \ x_2$ means that agent h strictly prefers x_1 to x_2. Also, $P_h|_Y = P_g|_Y$ means agents h and g have the same preferences over the set Y. I define a profile as $P = (P_1, ..., P_{|N|})$ and the domain of all possible profiles is denoted by $\mathbb{P}(N, X)$.

Definition 2.1 *An allocation a is Pareto optimal for a given profile P if $\nexists b \in A(N, X)$ such that*

$$\begin{aligned} b_h \ P_h \ a_h & \qquad \textit{for at least one } h \in N \\ b_k \ P_k \ a_k \quad \textit{or} \quad b_k = a_k & \qquad \forall k = 1, 2, ..., n \end{aligned}$$

I denote by $PO(P)$ the set of all Paretian allocations when the profile is P. Then, $PO(P)$ must be an element of $\mathbb{A}(N, X)$ which is the set of all non-empty subsets of $A(N, X)$. It is important to note here that, for all preference profiles P, the set $PO(P)$ is never empty. This means that, for every preference profile P, there is at least one allocation which is not Pareto dominated by another allocation.

I say that a set T is rationalizable if there is a preference profile P such that the Paretian allocation set for P is T, i.e. $T = PO(P)$.

The interesting question is : under which conditions can a set T be rationalizable? If there are few goods, it could be possible to infer directly if there exists a profile supporting the set. But when the number is higher than 4, the direct inference is quite complicated.[2] Consequently, another way must be found to solve the problem.

Before doing so, I must define some concepts. The first concept I introduce is the cycle.

[2]The numbers of preference profiles with 4 goods is given by $(4!)^4 = 331776$.

Definition 2.2 *Let the set I be a subset of N and Y be a subset of X with $|Y| = |I|$. Let $i = (i_1, i_2, ..., i_{|I|})$ with $i_1, i_2, ..., i_{|I|} \in I$ and $y = (y_1, y_2, ..., y_{|I|})$ with $y_1, y_2, ..., y_{|I|} \in Y$. I say that a set $T \subseteq A(N, X)$ has a cycle $C(i, y)$ if $\exists S = \{a^1, a^2, ..., a^{|I|}\} \subseteq T$ such that*

$$\begin{aligned}
a^1_{i_1} &= y_1, \quad a^1_{i_2} = y_2, \quad \dots \quad a^1_{i_{|I|}} = y_{|I|} \\
a^2_{i_1} &= y_2, \quad a^2_{i_2} = y_3, \quad \dots \quad a^2_{i_{|I|}} = y_1 \\
&\dots \\
a^{|I|-1}_{i_1} &= y_{|I|-1}, \quad a^{|I|-1}_{i_2} = y_{|I|}, \quad \dots \quad a^{|I|-1}_{i_{|I|}} = y_{|I|-2} \\
a^{|I|}_{i_1} &= y_{|I|}, \quad a^{|I|}_{i_2} = y_1, \quad \dots \quad a^{|I|}_{i_{|I|}} = y_{|I|-1}
\end{aligned}$$

For example, if the set T has the cycle $C((1, 2, 3), (x_1, x_2, x_3))$, this means there are three allocations a^1, a^2 and a^3 in T such that

$$\begin{aligned}
a^1_1 &= x_1, \quad a^1_2 = x_2, \quad a^1_3 = x_3 \\
a^2_1 &= x_2, \quad a^2_2 = x_3, \quad a^2_3 = x_1 \\
a^3_1 &= x_3, \quad a^3_2 = x_1, \quad a^3_3 = x_2
\end{aligned}$$

It is important to underline that i and y respectively are vectors and not subsets of N and Y respectively. To illustrate the importance of this distinction, consider the two following sets :

$$\begin{aligned}
T_1 &= \{(x_1, x_2, x_3), (x_2, x_3, x_1), (x_3, x_1, x_2)\} \\
T_2 &= \{(x_2, x_1, x_3), (x_1, x_3, x_2), (x_3, x_2, x_1)\}
\end{aligned}$$

The set T_1 has the cycle $C((1, 2, 3), (x_1, x_2, x_3))$ and T_2 the cycle $C((1, 2, 3), (x_1, x_3, x_2))$. But those two cycles are different. For this reason, vectors must be used to define a cycle.

Also, it should be noted that it is possible to write the same cycle in many ways. Lemma 2.1 gives the number of ways to write the same cycle. Before presenting Lemma 2.1, I need the modulo operator. Let $\mathbb{N}$ be the natural number set. For $a, b \in \mathbb{N}$, $mod_a b$ is the remainder of the division of b by a.

Lemma 2.1 *Any cycle of $|I|$ elements can be written in $(|I| - |R_{|I|}|)|I|^2$ ways where*

$$R_{|I|} = \{r \in \{1, 2, ..., |I| - 1\} \quad | \quad \exists q \in \{1, 2, ..., |I| - 1\} \quad with \quad mod_{|I|} rq = 0\}$$

Proof. Suppose the set T has a cycle $C(i, y)$ with $i = (1, 2, ..., |I|)$ and $y = (x_1, x_2, ...x_{|I|})$.

Then, the set T contains $|I|$ allocations such that :

$$\begin{array}{r} a_1^1 = x_1, \quad a_2^1 = x_2, \ldots \quad a_{|I|}^1 = x_{|I|} \\ a_1^2 = x_2, \quad a_2^2 = x_3, \ldots \quad a_{|I|}^2 = x_1 \\ \ldots \\ a_1^{|I|} = x_{|I|}, a_2^{|I|} = x_1, \ldots \quad a_{|I|}^{|I|} = x_{|I|-1} \end{array}$$

I can write this cycle by using $y' = (x_2, x_3, ..., x_{|I|}, x_1)$. Then, all cycles $C(i, y')$ with y' which has its components switching neighbor to neighbor relative to y give the same cycle. This gives $|I|$ different ways to write the same cycle. I can do the same thing by switching elements of i and I find also $|I|$ ways to write the cycle.

Now, consider the number ρ which is a positive integer strictly lower than $|I|$. Suppose ρ does not belong to R, i.e. there are no pairs of positive integers q, s which are strictly lower such that $\rho q = s|I|$. Let $i'' = (1, \rho + 1, mod_{|I|}(2\rho) + 1, ..., mod_{|I|}((|I| - 1)\rho) + 1)$ and $y'' = \left(x_1, x_{\rho+1}, x_{mod_{|I|}(2\rho)+1}, ..., x_{mod_{|I|}((|I|-1)\rho+1)}\right)$ and consider the cycle $C(i'', y'')$. Since ρ does not belong to R, this means all components of i'' and y'' are different. So, the cycle $C(i'', y'')$ is the same as $C(i, y)$. This is true for all ρ's which are positive integers strictly lower than $|I|$ and do not belong to R.

Finally, I obtain $\left(|I| - R_{|I|}\right) |I|^2$. ■

The following example illustrates this fact.

Example 2.1 *Suppose I have the set* $T = \{(x_1, x_2, x_3), (x_2, x_3, x_1), (x_3, x_1, x_2)\}$. *Then, by Lemma 2.1, there are 18 ways to write the cycle :*

$$\begin{array}{lll} C((1,2,3),(x_1,x_2,x_3)) & C((1,2,3),(x_2,x_3,x_1)) & C((1,2,3),(x_3,x_1,x_2)) \\ C((2,3,1),(x_1,x_2,x_3)) & C((2,3,1),(x_2,x_3,x_1)) & C((2,3,1),(x_3,x_1,x_2)) \\ C((3,1,2),(x_1,x_2,x_3)) & C((3,1,2),(x_2,x_3,x_1)) & C((3,1,2),(x_3,x_1,x_2)) \\ C((3,2,1),(x_3,x_2,x_1)) & C((3,2,1),(x_2,x_1,x_3)) & C((3,2,1),(x_1,x_3,x_2)) \\ C((2,1,3),(x_3,x_2,x_1)) & C((2,1,3),(x_2,x_1,x_3)) & C((2,1,3),(x_1,x_3,x_2)) \\ C((1,3,2),(x_3,x_2,x_1)) & C((1,3,2),(x_2,x_1,x_3)) & C((1,3,2),(x_1,x_3,x_2)) \end{array}$$

It must be noted that $|I| - |R_{|I|}|$ is always higher or equal to 2 when $|I|$ is higher or equal to 3. The number 1 and $|I| - 1$ never belong to $R_{|I|}$.

To simplify the presentation, I propose using the lexicographic ordering to have a unique notation for a given cycle.

Definition 2.3 *For two vectors v and w of l components, I say that v is lexicographically* *dominated by w if*

$$\begin{aligned} w_1 > v_1 \quad & or \\ w_1 = v_1, w_2 > v_2 \quad & or \\ \ldots \\ w_1 = v_1, w_2 = v_2, \ldots, w_l > v_l \end{aligned}$$

The first step is to choose from all possible ways of writing a given cycle the ways for which the vector i is lexicographically dominated by (or equal to) the others. Secondly, from those variants, I choose the one for which the component subscripts of y are lexicographically dominated by the other vector y.

Let's apply this process to the cycle in Example 2.1. The first step tells us to select the vector i which is lexicographically dominated by the others. This vector is $(1,2,3)$. Then, from the different ways to write the cycle with $i = (1,2,3)$, which are $C((1,2,3),(x_1,x_2,x_3))$, $C((1,2,3),(x_2,x_3,x_1))$ and $C((1,2,3),(x_3,x_1,x_2))$, I must choose the one which has the vector y whose component subscripts are lexicographically dominated by the component subscripts of the other y's. I find that the unique solution is $C((1,2,3),(x_1,x_2,x_3))$.

Consider another example.

Example 2.2 *Suppose the set T is composed of the following allocations.*

$$\begin{aligned} a^1 &= (x_1, x_4, x_2, x_3) \\ a^2 &= (x_3, x_2, x_1, x_4) \\ a^3 &= (x_4, x_1, x_3, x_2) \\ a^4 &= (x_2, x_3, x_4, x_1) \end{aligned}$$

Then, the set T has the cycle $C(i,y)$ with $i = (1,3,2,4)$ and the vector $y = (x_1, x_2, x_4, x_3)$.

Now, I can answer an interesting question : how many different cycles could set T have for a given vector of agents i and a given subset of goods Y? There are $|I|!$ different vectors i and $|I|!$ different possible vectors y. There are $(|I|!)^2$ possibilities. But, I have already shown that there are $(|I| - |R_{|I|}|)|I|^2$ ways to write the same cycle. So there are $\frac{(|I|-1)!^2}{|I|-|R_{|I|}|}$ different cycles for a given subset of agents I and a subset of goods Y.

I have to emphasize that the definition of a cycle is independent of what other agents get. For example, consider the two following sets :

$$\begin{aligned} T_1 &= \{(x_1, x_2, x_3, x_4, x_5), (x_1, x_2, x_4, x_5, x_3), (x_1, x_2, x_5, x_3, x_4)\} \\ T_2 &= \{(x_2, x_1, x_3, x_4, x_5), (x_2, x_1, x_4, x_5, x_3), (x_1, x_2, x_5, x_3, x_4)\} \end{aligned}$$

These sets have the same cycle $C((3,4,5),(x_3,x_4,x_5))$ even if they do not have the same allocations.

Let $\mathbb{S}(i,y)$ be the set of all allocations a^s such that :

$$\begin{array}{lllll} a^s_{i_1} = y_1, & a^s_{i_2} = y_2, & ..., & a^s_{i_{|I|}} = y_{|I|} & or \\ a^s_{i_2} = y_2, & a^s_{i_3} = y_3, & ..., & a^s_{i_{|I|}} = y_1 & or \\ & & & & ... \\ a^s_{i_{|I|}} = y_{|I|}, & a^s_{i_1} = y_1, & ..., & a^s_{i_{|I|}} = y_{|I|-1} & \end{array}$$

Definition 2.4 *Suppose that the set $T \subseteq A(X,N)$ has a cycle $C(i,y)$. An allocation $a^c \in T$ is a cycle allocation for $C(i,y)$ if a^c belongs to $\mathbb{S}(i,y)$. The set of all cycle allocations for $C(i,y)$ is denoted $S_T(i,y)$.*

Then, two sets could have the same cycle while they do not have the same cycle allocations.

Also, it is possible that a Paretian set contains more than one cycle. In particular, it could happen that the Paretian set $PO(P)$ has two cycles : $C(i,y)$ and $C(\underline{i},\underline{y})$ with $\underline{I} \subset I$ and $\underline{Y} \subset Y$. To examine this case, I define the concept of subcycle.

Definition 2.5 *Suppose that the set $T \subseteq A(X,N)$ has a cycle $C(i,y)$ where $i = (i_1, i_2, ..., i_{|I|})$ and $y = (y_1, y_2, ..., y_{|I|})$. I say that $C(i^s, y^s)$ is a subcycle of $C(i,y)$ if $i^s = \left(i^s_1, i^s_2, ..., i^s_{|I^s|}\right)$ with $i^s_1, i^s_2, ..., i^s_{|I^s|} \in I^s \subset I$, $y^s = \left(y^s_1, y^s_2, ..., y^s_{|I^s|}\right)$ with $y^s_1, y^s_2, ..., y^s_{|I^s|} \in Y^s \subset Y$ and $C(i^s, y^s)$ is a cycle for $S_T(i,y)$.*

Consider the next example to illustrate a subcycle.

Example 2.3 *Suppose that the set T has a cycle $C((1,2,3,4),(x_1,x_2,x_3,x_4))$. Then, $S_T(i,y)$ is the set of allocations a^s belonging to T such that*

$$\begin{array}{llllll} a^s_1 &= x_1, & a^s_2 = x_2, & a^s_3 = x_3, & a^s_4 = x_4 & or \\ a^s_1 &= x_2, & a^s_2 = x_3, & a^s_3 = x_4, & a^s_4 = x_1 & or \\ a^s_1 &= x_3, & a^s_2 = x_4, & a^s_3 = x_1, & a^s_4 = x_2 & or \\ a^s_1 &= x_4, & a^s_2 = x_1, & a^s_3 = x_2, & a^s_4 = x_3 & \end{array}$$

Then, this cycle contains 4 different subcycles : $C((1,3),(x_1,x_3))$, $C((1,3),(x_2,x_4))$, $C((2,4),(x_1,x_3))$ *and* $C((2,4),(x_2,x_4))$

To know if the cycle $C(i,y)$ has subcycles, I study the set $R_{|I|}$. The next lemma tells us the condition necessary for $|I|$ to have subcycles.

Lemma 2.2 *If* $R_{|I|} \neq \{|I|\}$*, then* $C(i,y)$ *has subcycles.*

Proof. Without lost of generality (WLOG), let's take the cycle $C(i,y)$ with $i = (1,2,...,|I|)$ and $y = (x_1, x_2, ...x_{|I|})$.

Consider ρ which belongs to $R_{|I|}$ and suppose ρ is not equal to $|I|$. If ρ belongs to $R_{|I|}$, this means there is a positive integer q strictly lower than $|I|$ such that $mod_{|I|}\rho q = 0$.

Now, let's take $i' = (1, \rho+1, mod_{|I|}(2\rho)+1, ..., mod_{|I|}((q-1)\rho)+1)$ and $y' = (x_1, x_{\rho+1}, x_{mod_{|I|}(2\rho)+1}, ..., x_{mod_{|I|}((q-1)\rho)+1})$. Since ρ is not equal to $|I|$ and q is strictly lower then $|I|$, the set I' is not equal to I. I obtain the cycle $C(i',y')$ which is a subcycle of $C(i,y)$. ■

The following example illustrates the result of Lemma 2.2.

Example 2.4 *Suppose a set* T *has the cycle* $C(i,y)$ *with* $i = (1,2,3,4,5,6)$ *and* $y = (x_1, x_2, x_3, x_4, x_5, x_6)$. *This means there are 6 allocations* $a^1, ..., a^6$ *belonging to* T *such that :*

$$\begin{array}{cccccc}
a_1^1 = x_1, & a_2^1 = x_2, & a_3^1 = x_3, & a_4^1 = x_4, & a_5^1 = x_5, & a_6^1 = x_6 \\
a_1^2 = x_2, & a_2^2 = x_3, & a_3^2 = x_4, & a_4^2 = x_5, & a_5^2 = x_6, & a_6^2 = x_1 \\
 & & & & & \cdots \\
a_1^6 = x_6, & a_2^6 = x_1, & a_3^6 = x_2, & a_4^6 = x_3, & a_5^6 = x_4, & a_6^6 = x_5
\end{array}$$

Then set R_6 *is given by* $\{2,3,4,6\}$. *Take* $\rho = 3$. *Then, cycle* $C(i',y')$ *with* $i' = (1,4)$ *and* $y' = (x_1, x_4)$ *is a subcyle of* $C(i,y)$.

The last definition concerning cycles is the following :

Definition 2.6 *I say that* $T \subseteq A(X,N)$ *has a complete cycle* $C_c(I,Y)$ *with* $I \subseteq N$ *and* $Y \subseteq X$ *where* $|Y| = |I|$ *if for all* $i = (i_1, i_2, ..., i_{|I|})$ *with* $i_1, i_2, ..., i_{|I|} \in I$ *and* $y = (y_1, y_2, ..., y_{|I|})$ *with* $y_1, y_2, ..., y_{|I|} \in Y$, T *contains the cycle* $C(i,y)$.

In other words, there is a complete cycle $C_c(I,Y)$ when a set of goods are allocated in all possible combination to a set of agents, i.e., there is a complete cycle if $\exists S =$

$\{a^1, a^2, ..., a^{|I|!-1}, a^{|I|!}\} \subset T$ such that

$$\begin{array}{rcl} a_i^1 & = & y_1, \quad a_j^1 = y_2, \quad \dots \quad a_t^1 = y_{|I|-2}, \quad a_u^1 = y_{|I|-1}, \quad a_v^1 = y_{|I|} \\ a_i^2 & = & y_1, \quad a_j^2 = y_2, \quad \dots \quad a_t^2 = y_{|I|-2}, \quad a_u^2 = y_{|I|}, \quad a_v^2 = y_{|I|-1} \\ a_i^3 & = & y_1, \quad a_j^3 = y_2, \quad \dots \quad a_t^3 = y_{|I|-1}, \quad a_u^3 = y_{|I|-2}, \quad a_v^3 = y_{|I|} \\ a_i^4 & = & y_1, \quad a_j^4 = y_2, \quad \dots \quad a_t^4 = y_{|I|-1}, \quad a_u^4 = y_{|I|}, \quad a_v^4 = y_{|I|-2} \\ a_i^5 & = & y_1, \quad a_j^5 = y_2, \quad \dots \quad a_t^5 = y_{|I|}, \quad a_u^5 = y_{|I|-2}, \quad a_v^5 = y_{|I|-1} \\ a_i^6 & = & y_1, \quad a_j^6 = y_2, \quad \dots \quad a_t^6 = y_{|I|}, \quad a_u^6 = y_{|I|-1}, \quad a_v^6 = y_{|I|-2} \\ & & \dots \\ a_i^{|I|!} & = & y_{|I|}, \quad a_j^{|I|!} = y_{|I|-1}, \quad \dots \quad a_t^{|I|!} = y_3, \quad a_u^{|I|!} = y_2, \quad a_v^{|I|!} = y_1 \end{array}$$

with $i, j, t, u, v \in I$.

For the definition of a complete cycle, the arguments in the function $C_c(\cdot)$ are sets. A complete cycle contains all possible allocations of goods in Y between agents in I. In this case, it is not necessary to mention a specific order of agents or goods.

It must be noted that if a set has a cycle, this does not imply that the set has a complete cycle. This point is discussed in the next section.

2.3 Properties of cycles and complete cycles

The presence of a cycle $C(i, y)$ in a Paretian set $PO(P)$ gives information about the preferences of agents. The first insight given by a cycle is about pairs of goods which are neighbors in the vector y.

Proposition 2.1 *Let the set I be a subset of N and Y a subset of X with $|Y| = |I|$. Let $i = (i_1, i_2, ..., i_{|I|})$ with $i_1, i_2, ..., i_{|I|} \in I$ and $y = (y_1, y_2, ..., y_{|I|})$ with $y_1, y_2, ..., y_{|I|} \in Y$. If $PO(P)$ has a cycle $C(i, y)$, then $\forall k, l \in I$*

$$\begin{array}{rc} & P_k|_{y_{|I|}, y_1} = P_l|_{y_{|I|}, y_1} \\ \textit{and} & P_k|_{y_h, y_{h+1}} = P_l|_{y_h, y_{h+1}} \end{array}$$

$\forall h = 1, 2, 3, ..., |I| - 1.$

Proof. WLOG, suppose that $i = (1, 2, ..., |I|)$ and $y = (x_1, x_2, ...x_{|I|})$. Consider x_h, x_{h+1} where $h = 1, 2, ..., |I| - 1$. Suppose that agent 1 prefers x_{h+1} to x_h. Because $PO(P)$ has the cycle $C(i, y)$, there is an allocation belonging to $PO(P)$ such that x_{h+1} is allocated to agent 2 and x_h to agent 1. Since this allocation belongs to $PO(P)$, then agent 2 must also prefer x_{h+1} to x_h. Again, because $PO(P)$ has the cycle $C(i, y)$, there is an allocation belonging to $PO(P)$ such that x_{h+1} is allocated to agent 3 and x_h to agent 2. Since this allocation belongs to $PO(P)$, then agent 3 must prefer x_{h+1} to x_h. If I continue for all agents belonging to I, I find that all agents belonging to I must have similar preferences for all pairs x_h, x_{h+1} with $h = 1, 2, ..., |I| - 1$ and for the pair $x_1, x_{|I|}$.

■

With this proposition, I get information on the profile P by using the presence of a cycle in the Paretian set. However, I only have information on preferences over each pair (y_h, y_{h+1}) and the pair $(y_{|I|}, y_1)$, so I cannot make a conclusion about the preferences over all pairs of goods belonging to the set Y. The following example demonstrates the problem.

Example 2.5 *Suppose the cycle $C((1,2,3,4), (x_1, x_2, x_3, x_4))$ belongs to the Paretian set $PO(P)$. Then the following profile supports the cycle.*

P_1	P_2	P_3	P_4
x_1	x_3	x_1	x_3
x_3	x_1	x_3	x_1
x_2	x_2	x_2	x_2
x_4	x_4	x_4	x_4

Then when the good x_1 is allocated to someone who belongs to $\{1,3\}$, the good x_3 is allocated to the other agent in that set. The cycle does not contain an allocation where the good x_1 is allocated to someone in $\{1,3\}$ and the good x_3 to someone in $\{2,4\}$. This means that agents in $\{1,3\}$ could have different preferences over the set $\{x_1, x_3\}$ than agents in $\{2,4\}$. The same is true for the set of goods $\{x_2, x_4\}$.

To analyze preferences over a pair of goods which are not neighbors to each other in the vector y, I use the concept of subcycle. In Section 2.2, I showed that a subcycle is a cycle. So, if a cycle has subcycles, Proposition 2.1 can be used to infer agents' preferences.

Proposition 2.2 *Let the set I be a subset of N and Y a subset of X with $|Y| = |I|$. Let $i = (i_1, i_2, ..., i_{|I|})$ with $i_1, i_2, ..., i_{|I|} \in I$ and $y = (y_1, y_2, ..., y_{|I|})$ with $y_1, y_2, ..., y_{|I|} \in Y$. Suppose $PO(P)$ has a cycle $C(i, y)$. Let $q(r)$ be the smallest integer such that $mod_{|I|}(q(r)r) = 0$ for r belonging to $R_{|I|}$ and not equal to $|I|$. Then, for all pairs $y_\alpha, y_{\alpha+r}$ with $\alpha = 1, 2, ..., |I| - r$,*

$$
\begin{aligned}
P_{i_1}|_{\{y_\alpha, y_{\alpha+r}\}} &= P_{i_{[mod_{|I|}(\beta r)]+1}}|_{\{y_\alpha, y_{\alpha+r}\}} \quad \beta = 1, 2, ..., q(r) - 1 \\
P_{i_2}|_{\{y_\alpha, y_{\alpha+r}\}} &= P_{i_{[mod_{|I|}(\beta r)]+2}}|_{\{y_\alpha, y_{\alpha+r}\}} \quad \beta = 1, 2, ..., q(r) - 1 \\
&\dots \\
P_{i_r}|_{\{y_\alpha, y_{\alpha+r}\}} &= P_{i_{[mod_{|I|}(\beta r)]+r}}|_{\{y_\alpha, y_{\alpha+r}\}} \quad \beta = 1, 2, ..., q(r) - 1
\end{aligned}
$$

Proof. WLOG, suppose that $i = (1, 2, ..., |I|)$ and $y = (x_1, x_2, ...x_{|I|})$. Let $q(r)$ be the smallest integer such that, for $r \in R_{|I|}$, $mod_{|I|}(q(r)r) = 0$. If r belongs to $R_{|I|}$ and is not equal to $|I|$, then for every $\gamma = 1, 2, ..., r$ and every $\beta = 1, 2, ..., q(r) - 1$, the cycle $C(i^s, y^s)$ with $i^s = (\gamma, \gamma + r, \gamma + mod_{|I|}(2r), ..., \gamma + mod_{|I|}((q-1)r))$ and $y^s = \left(x_\beta, x_{\beta+r}, x_{\beta+mod_{|I|}(2r)}, ..., x_{\beta+mod_{|I|}((q-1)r)}\right)$ is a subcyle of $C(i, y)$. Because $C(i^s, y^s)$ is a subcycle of $C(i, y)$, $PO(P)$ must have the cycle $C(i^s, y^s)$. I can then apply Proposition 2.1. ■

Let's apply this proposition to the following example.

Example 2.6 *Suppose $PO(P)$ has a cycle $C(i, y)$ with $i = (1, 2, 3, 4, 5, 6)$ and $y = (x_1, x_2, x_3, x_4, x_5, x_6)$. Then, this means there are six allocations $a^1, a^2, a^3, a^4, a^5, a^6 \in PO(P)$ such that*

$$\begin{array}{llll}
a_1^1 = x_1, & a_2^1 = x_2, & \ldots & a_6^1 = x_6 \\
a_1^2 = x_2, & a_2^2 = x_3, & \ldots & a_6^2 = x_1 \\
a_1^3 = x_3, & a_2^3 = x_4, & \ldots & a_6^3 = x_2 \\
a_1^4 = x_4, & a_2^4 = x_5, & \ldots & a_6^4 = x_3 \\
a_1^5 = x_5, & a_2^5 = x_6, & \ldots & a_6^5 = x_4 \\
a_1^6 = x_6, & a_2^6 = x_1, & \ldots & a_6^6 = x_5
\end{array}$$

Then, if I take $r = 3$ and $q = 2$, I get

$$\begin{array}{ll}
a_1^1 = x_1, & a_4^1 = x_4 \\
a_1^4 = x_4, & a_4^4 = x_1
\end{array}$$

I obtain that agents 1 and 4 have same preferences over the $\{x_1, x_4\}$. I can continue this way and I find that

1. *Agents in $\{1, 2, 3, 4, 5, 6\}$ have the same preferences over sets $\{x_1, x_2\}$, $\{x_2, x_3\}$, $\{x_3, x_4\}$, $\{x_4, x_5\}$, $\{x_5, x_6\}$ and $\{x_1, x_6\}$.*
2. *Agents in $\{1, 3, 5\}$ have the same preferences over sets $\{x_1, x_3\}$, $\{x_2, x_4\}$, $\{x_3, x_5\}$, $\{x_4, x_6\}$, $\{x_1, x_5\}$ and $\{x_2, x_6\}$.*
3. *Agents in $\{2, 4, 6\}$ have the same preferences over sets $\{x_1, x_3\}$, $\{x_2, x_4\}$, $\{x_3, x_5\}$, $\{x_4, x_6\}$, $\{x_1, x_5\}$ and $\{x_2, x_6\}$.*
4. *Agents in $\{1, 4\}$ have the same preferences over sets $\{x_1, x_4\}$, $\{x_2, x_5\}$ and $\{x_3, x_6\}$.*
5. *Agents in $\{2, 5\}$ have the same preferences over sets $\{x_1, x_4\}$, $\{x_2, x_5\}$ and $\{x_3, x_6\}$.*
6. *Agents in $\{3, 6\}$ have the same preferences over sets $\{x_1, x_4\}$, $\{x_2, x_5\}$ and $\{x_3, x_6\}$.*

This result gives additional information about the profile P since it provides information on preferences over pairs of goods which are not neighbors in the cycle. Subcycles can be analysed on their own since they are themselves distinct cycles, but they could be supported by different preference profiles across agents than the larger cycle. However, by using subcycles, I can only show that agents which are neighbors in a subcycle have the same preferences over all pairs of goods which are neighbor in this subcycle and it is possible that two distinct subsets of agents in the cycle hold different preferences over the same subset of goods.

While Proposition 2.2 gives us information about preferences over pairs of goods that are neighbors in a subcycle, Proposition 2.3 deals with the other pairs.

Proposition 2.3 *Let the set I be a subset of N and Y a subset of X with $|Y| = |I|$. Let $i = (i_1, i_2, ..., i_{|I|})$ with $i_1, i_2, ..., i_{|I|} \in I$ and $y = (y_1, y_2, ..., y_{|I|})$ with $y_1, y_2, ..., y_{|I|} \in Y$. Suppose that $PO(P)$ has a cycle $C(i, y)$. For all pairs of goods $y_\alpha, y_\beta \in Y$ with $\beta > \alpha$ such that $(\beta - \alpha)$ does not belong to $R_{|I|}$,*

$$P_k|_{\{y_\alpha, y_\beta\}} = P_l|_{\{y_\alpha, y_\beta\}} \qquad \forall k, l \in I$$

Proof. WLOG, suppose that $i = (1, 2, ..., |I|)$ and $y = (x_1, x_2, ...x_{|I|})$. Now, take x_α and x_β with $\alpha = 1, 2, ..., |I| - 1$ and $\beta = \alpha + 1, ..., |I|$ and let $\delta = \beta - \alpha$. By assumption, δ does not belong to the set $R_{|I|}$

Suppose that $x_\alpha \ P_1 \ x_\beta$. Because $PO(P)$ has the cycle $C(i, y)$, there is an allocation belonging to $PO(P)$ such that x_α is allocated to agent $\delta + 1$ and x_β to agent 1. Since this allocation belongs to $PO(P)$, then agent $\delta + 1$ must prefer x_α to x_β. Again, because $PO(P)$ has the cycle $C(i, y)$, there is an allocation belonging to $PO(P)$ such that x_α is allocated to agent $mod_{|I|}(2\delta) + 1$ and x_β to agent $\delta + 1$. Since this allocation belongs to $PO(P)$, then agent $mod_{|I|}(2\delta) + 1$ must prefer x_α to x_β.

I can continue until I show that

$$x_\alpha \quad P_\gamma \quad x_\beta \qquad \gamma = 1, \delta + 1, mod_{|I|}(2\delta) + 1, ..., mod_{|I|}((|I| - 1)\delta) + 1$$

Since there is no positive integer $q < |I|$ such that $mod_{|I|}(\delta q) = 0$, then the set $\{1, \delta + 1, mod_{|I|}(2\delta) + 1, ..., mod_{|I|}((|I| - 1)\delta) + 1\}$ has $|I|$ elements. So all agents belonging to I have the same preferences over the set $\{x_\alpha, x_\beta\}$. ■

It must be noted that if $|I|$ is a prime number, all pairs of goods are treated by Proposition 2.3 since $R_{|I|} = \{|I|\}$. In this case, all agents in I have the same preferences over the set Y.

Corollary 2.1 *Let the set I be a subset of N and Y be a subset of X with $|Y| = |I|$. Let $i = (i_1, i_2, ..., i_{|I|})$ with $i_1, i_2, ..., i_{|I|} \in I$ and $y = (y_1, y_2, ..., y_{|I|})$ with $y_1, y_2, ..., y_{|I|} \in Y$. If*

$PO(P)$ has a cycle $C(i,y)$ and $|I|$ is a prime number, then $\forall k,l \in I$,

$$P_k|_Y = P_l|_Y$$

Proof. I apply Proposition 2.3 for all pairs of goods $y_h, y_{h'} \in Y$. ■

This result is very strong. With only one cycle, I can conclude that a subset of agents have the same preferences over a subset of goods. Unfortunately, as I have showed above, I can not extend this result to any number of individuals in I.

Another case can lead to the conclusion that agents in a subset of N have the same preferences over a subset of goods.

Proposition 2.4 *Let the set I be a subset of N and Y be a subset of X with $|Y| = |I|$. Let $i = (i_1, i_2, ..., i_{|I|})$ with $i_1, i_2, ..., i_{|I|} \in I$ and $y = (y_1, y_2, ..., y_{|I|})$ with $y_1, y_2, ..., y_{|I|} \in Y$. If $PO(P)$ has a complete cycle $C_c(I,Y)$, the agents in I have the same preferences over Y.*

Proof. Suppose the opposite is true, i.e. there exists $i, j \in I$ and $x', x'' \in y$ such that

$$\begin{aligned} x' \ &P_i \ x'' \\ x'' \ &P_j \ x' \end{aligned}$$

This means the good x' will never be allocated to j when the good x'' is allocated to i. That contradicts the existence of a complete cycle. ■

The presence of a complete cycle gives us more information about agent preferences. In fact, a cycle could give the same information if the number of elements in that cycle is a prime number. Unless it has this characteristic, a cycle by itself does not give information on preferences over all goods. But, if a single cycle cannot give the same information than a complete cycle, many cycles can provide it.

Proposition 2.5 *Let the set I be a subset of N and Y a subset of X with $|Y| = |I|$. Let $i = (i_1, i_2, ..., i_{|I|})$ with $i_1, i_2, ..., i_{|I|} \in I$ and $y = (y_1, y_2, ..., y_{|I|})$ with $y_1, y_2, ..., y_{|I|} \in Y$. Let α be the lowest prime number except 1 such that $mod_\alpha|I| = 0$. If $PO(P)$ has $\left[\left(\frac{|I|}{\alpha} - 1\right) * (|I| - 2)! + 1\right]$ cycles with same i and same Y, then the agents in I have the same preferences over Y.*

Proof. Suppose $|I|$ is prime. By Corollary 2.1, if $PO(P)$ has a cycle $C(i,y)$, then all agents have the same preferences over the set Y.

Now, suppose $|I|$ is not a prime number and let α be the smallest prime number such that $mod_\alpha(|I|) = 0$.

Suppose $x_1, x_2 \in Y$. Let I^1 and I^2 be two non-empty subsets of I such that all agents belonging to I^1 prefer goods x_1 to x_2 and all agents belonging to I^2 prefer goods x_2 to x_1.

Suppose $PO(P)$ has τ cycles $C(i,y^t)$ for $t=1,2,...,\tau$. By convention, $y_1^t = x_1$ for all t. For all cycles $C(i,y^t)$, I define β^t the positions of x_2 in the vector y^t so that $y_\beta^t = x_2$, and let $\delta^t = \beta^t - 1$.

By Proposition 2.3, if there is a t such that δ^t does not belong to $R_{|I|}$, then all agents must have the same preferences. Suppose δ^t belongs to $R_{|I|}$ for all t. Let the set Π be equal to $\{\alpha, 2\alpha, ..., |I|-\alpha\}$ which is a subset of $R_{|I|}$.

The maximum number such that all δ^t belong to Π is $\left(\frac{|I|}{\alpha}-1\right)(|I|-2)!$. If $\tau = \left(\frac{|I|}{\alpha}-1\right)(|I|-2)!$, then there is at least one cycle $C(i,y^t)$ with $\delta^t=\alpha$. By Proposition 2.2, then agents belonging to the same set $j, j+\alpha, 1+2\alpha, ..., j+(|I|-\alpha)$ for $j=1,2,...,\alpha$ have the same preferences. But, agents in different subsets could have different preferences.

If I add another cycle $C(i,y^\theta)$, then δ^θ does not belong to Π. If δ^θ does not belong to $R_{|I|}$, by Proposition 2.3, all agents must have the same preferences over $\{x_1,x_2\}$. If δ^θ belongs to $R_{|I|}$, by Proposition 2.1, for $h=1,2,...,\delta^\theta$, agents belonging to $h, h+\delta^\theta, 1+mod_{|I|}(2\delta^\theta), ..., h+(|I|-\delta^\theta)$ have the same preferences. Since δ^θ does not belong to Π, then h and $h+\delta^\theta$ does not belong to the same set $j, j+\alpha, 1+2\alpha, ..., j+(|I|-\alpha)$ for $j=1,2,...,\alpha$. So the two sets which contain agent h and $h+\delta^\theta$ must have the same preferences. I can continue to conclude that all agents must have the same preferences. ■

To illustrate the idea of this proof, consider the following example. Suppose $|X| = |N| = 6$ and suppose T has the following cycles :

- the 6 cycles given by $C((1,2,3,4,5,6),(x_1,\cdot,x_2,\cdot,x_3,\cdot))$
- the 6 cycles given by $C((1,2,3,4,5,6),(x_1,\cdot,x_2,\cdot,x_4,\cdot))$
- the 6 cycles given by $C((1,2,3,4,5,6),(x_1,\cdot,x_2,\cdot,x_5,\cdot))$
- the 6 cycles given by $C((1,2,3,4,5,6),(x_1,\cdot,x_2,\cdot,x_6,\cdot))$
- the 6 cycles given by $C((1,2,3,4,5,6),(x_1,\cdot,x_3,\cdot,x_2,\cdot))$
- the 6 cycles given by $C((1,2,3,4,5,6),(x_1,\cdot,x_4,\cdot,x_2,\cdot))$
- the 6 cycles given by $C((1,2,3,4,5,6),(x_1,\cdot,x_5,\cdot,x_2,\cdot))$
- the 6 cycles given by $C((1,2,3,4,5,6),(x_1,\cdot,x_6,\cdot,x_2,\cdot))$

If T has only these cycles, this means agents 1, 3 and 5 could have different preferences over x_1, x_2 than agents 2, 4 and 6. To have all agents with the same preferences, I must add at least one more cycle.

2.4 Cycles and Paretian sets

An interesting question concerning the composition of the Paretian set is what happens to the remaining agents. If the Paretian set has a cycle $C(i,y)$, it is interesting to know if there is an allocation in $A(N\setminus I, X\setminus Y)$ such that agents outside the cycle get the same goods in all allocations which can constitute the cycle. In other words, if I define $Y^c = X\setminus Y$, $I^c = N\setminus I$, the question is : "Is there a $z \in A(I^c, Y^c)$ such that the set composed by all allocations belonging to $PO(P)$ where agents in I^c get z has the cycle $C(i, y)$?" The answer is : I cannot guarantee the existence of such an element. Take the following example :

Example 2.7 *Suppose the preferences for 6 agents are given by*

P_1	P_2	P_3	P_4	P_5	P_6
x_1	x_1	x_1	x_1	x_6	x_6
x_3	x_3	x_3	x_3	x_2	x_4
x_2	x_4	x_2	x_4	x_5	x_5
x_4	x_2	x_5	x_5	x_1	x_1
x_5	x_5	x_4	x_2	x_3	x_3
x_6	x_6	x_6	x_6	x_4	y_2

Then

$$(x_1, x_2, x_3, x_4, x_5, x_6) \in PO(P)$$
$$(x_2, x_3, x_4, x_1, x_5, x_6) \notin PO(P)$$
$$(x_3, x_4, x_1, x_2, x_5, x_6) \in PO(P)$$
$$(x_4, x_1, x_2, x_3, x_5, x_6) \in PO(P)$$

and

$$(x_1, x_2, x_3, x_4, x_6, x_5) \in PO(P)$$
$$(x_2, x_3, x_4, x_1, x_6, x_5) \in PO(P)$$
$$(x_3, x_4, x_1, x_2, x_6, x_5) \notin PO(P)$$
$$(x_4, x_1, x_2, x_3, x_6, x_5) \in PO(P)$$

I obtain a cycle $C(i, y)$ with $i = (1, 2, 3, 4)$ and $y = (x_1, x_2, x_3, x_4)$. But the subset of $PO(P)$ in which allocations give x_5 to agent 5 and x_6 to agent 6 does not contain the cycle $C(i, y)$. This is also true for the subset of $PO(P)$ in which allocations give x_6 to agent 5 and x_5 to agent 6.

Example 2.7 shows that the existence of such elements is not guaranteed. Nevertheless if it exists and the agents in I have the same preferences over the set Y, the Paretian set contains a complete cycle $C_c(I, Y)$.

Proposition 2.6 *Let the set I be a subset of N and Y a subset of X with $|Y| = |I|$. Let $i = (i_1, i_2, ..., i_{|I|})$ with $i_1, i_2, ..., i_{|I|} \in I$ and $y = (y_1, y_2, ..., y_{|I|})$ with $y_1, y_2, ..., y_{|I|} \in Y$. Suppose that all agents in I have the same preferences over the set Y. If the subset of*

$PO(P)$ composed of allocations in which agents belonging to I^c get $z \in A(I^c, Y^c)$ has the cycle $C(i,y)$, then $PO(P)$ has a complete cycle $C_c(I,Y)$ in which I^c get z.

Proof. WLOG, suppose that $Y = \{x_1, x_2, ..., x_{|I|-1}, x_{|I|}\}$. Let the set Y^c be equal to $X \diagdown Y = \{x_{|I|+1}, x_{|I|+2}, ..., x_{|N|-1}, x_{|N|}\}$, $I\{1,2,...,|I|-1,|I|\}$ and $I^c = N \diagdown I$. By construction, $I^c = \{|I|+1, |I|+2, ..., |N|-1, |N|\}$.

Now suppose that $a \notin PO(P)$ where agents in I get goods in the allocation $a_i \in A(I,Y)$ and agents in I^c get goods in z. This means there exists an allocation $b \in A(X,I)$ such that

$$b_i \; P_i \; a_i \qquad \text{for at least one } i$$
$$b_j \; P_j \; a_j \quad \text{or} \quad b_j = a_j \qquad j = 1,2,...,|N|$$

Figure 2.1 illustrates the allocation a.

$$\begin{pmatrix} z_1 \\ \cdots \\ z_g \\ \cdots \\ z_h \\ \cdots \\ z_k \end{pmatrix} \qquad\qquad \begin{pmatrix} y_1 \\ \cdots \\ y_g \\ \cdots \\ y_h \\ \cdots \\ y_k \end{pmatrix}$$

FIG. 2.1 – Allocation z and the cycle $C(i,y)$

There are three possible cases for the allocation b. The first case consists of a reallocation between agents in I^c.

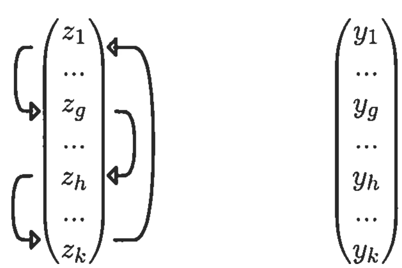

FIG. 2.2 – Allocation z : first case

But this kind of reallocation can not Pareto dominate the allocation a because there exists an allocation $\tilde{a}$ belonging to the Paretian set in which agents belonging to I^c get z. If a reallocation between agents in I^c dominates a, then the allocation $\tilde{a}$ should not belong to $PO(P)$.

The second case is a reallocation between agents in I.

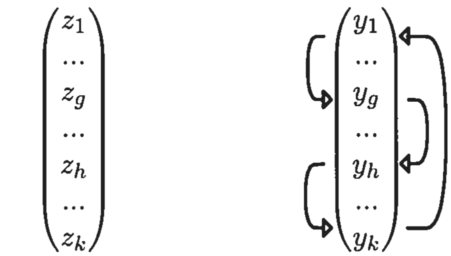

FIG. 2.3 – Allocation z : second case

Again, it is not possible for this new allocation to dominate the allocation a. I assume that all agents in I have the same preferences over goods in Y. Then no reallocation between agents in I could Pareto dominate the allocation a.

Finally, the last possibility is a reallocation between agents in both sets.

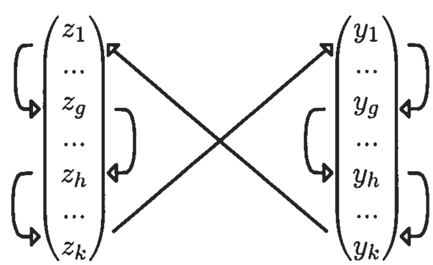

FIG. 2.4 – Allocation z : third case a

Because the agents in I have the same preferences, y_1 is preferred to y_k by all agents in I.

Suppose the agent who gets y_1 in the new allocation is agent α. Because of the cycle, there is an allocation in this cycle such that α gets good y_k. Then this allocation could not be in the Paretian set because this allocation will be dominated.

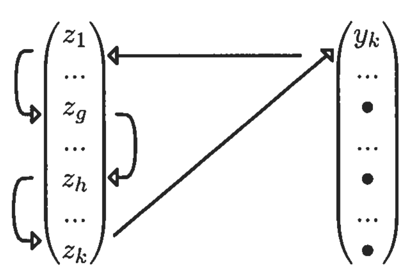

FIG. 2.5 – Allocation z : third case b

This means that the allocation where α gets y_k is Pareto dominated and contradicts the existence of a cycle $C(Y, I)$ in the set S. ■

To illustrate the proposition consider the case where the Paretian set $PO(P)$ contains the allocations $(x_1, x_2, x_3, x_4, x_5)$, $(x_2, x_3, x_1, x_4, x_5)$ and $(x_3, x_1, x_2, x_4, x_5)$. Then, $PO(P)$ has the cycle $C((1,2,3), (x_1, x_2, x_3))$. By Proposition 6, the allocations $(x_1, x_3, x_2, x_4, x_5)$, $(x_2, x_1, x_3, x_4, x_5)$ and $(x_3, x_2, x_1, x_4, x_5)$ must also belong to $PO(P)$. Then, $PO(P)$ has a complete cycle $C_c(\{1,2,3\}, \{x_1, x_2, x_3\})$.

Before presenting some constraints on the number of allocations in the Paretian set, I need the following proposition.

Proposition 2.7 *Let the set I be a subset of N and Y be a subset of X with $|Y| = |I|$. Let $i = \left(i_1, i_2, ..., i_{|I|}\right)$ with $i_1, i_2, ..., i_{|I|} \in I$ and $y = \left(y_1, y_2, ..., y_{|I|}\right)$ with $y_1, y_2, ..., y_{|I|} \in Y$. Suppose $PO(P)$ has a cycle $C(i, y)$. Let δ be an agent belonging to I and x_γ an element of Y. If all agents belonging to $I \diagdown \{\delta\}$ have the same preferences over the set $Y \diagdown \{x_\gamma\}$, then agents belonging to I have same preferences over $Y \diagdown \{x_\gamma\}$.*

Proof. For all pairs of goods belonging to $Y \diagdown \{x_\gamma\}$, I can apply Proposition 2.2 or Proposition 2.3 to find that there is at least one agent belonging to $I \diagdown \{\delta\}$ with the same preferences as δ. Because all agents belonging to $I \diagdown \{\delta\}$ have the same preferences over $Y \diagdown \{x_\gamma\}$, then all agents belonging to I have the same preferences over $Y \diagdown \{x_\gamma\}$. ■

The next proposition describes the restrictions on the number of allocations $PO(P)$ must contain.

Proposition 2.8 *If $|N| \geq 3$ and $PO(P) \neq A(N, X)$, then $|PO(P)| \leq (|N| - 1) * (|N| - 1)!$ $\forall P$. If $|PO(P)| = (|N| - 1) * (|N| - 1)!$, then there exist an agent i and a good x_l belonging to X such that there is no allocation a^h belonging to $PO(P)$ with $a_i^h = x_l$ and the preference profile is given by*

1. $P_g|_Y = P_h|_Y \qquad \forall g, h \in N \qquad Y = X \setminus \{x_l\}$
2. $P_g|_X = P_h|_X \qquad \forall g, h \in N \setminus \{i\}$
3. $P_g|_{\{x_l, x_k\}} \neq P_i|_{\{x_l, x_k\}} \qquad \forall g \in N \setminus \{i\} \qquad$ *for some* $x_k \in X \setminus \{x_l\}$

Proof. Let $\Psi = A \diagdown PO(P)$. By assumption, $|\Psi| \leq (n-1)!$.

Step 1 : Consider the good x_1. Suppose that agent 1 gets good x_1 the least often in the allocations belonging to Ψ. Then, the number of allocations in Ψ where agent 1 gets x_1 is less than

$$\frac{(|N| - 1)!}{|N|}$$

which is strictly lower than $(|N|-2)!$. This means there is at least one cycle $C(i,y)$ with $i = (2,3,...,|N|)$ and $Y = (\{x_2, x_3, ..., x_{|N|}\}$ since there are exactly $(|N|-2)!$ of such cycles.

Now take x_2. Again WLOG, suppose that x_2 is the good which is the least assigned to agent 2 in the set Ψ when good x_1 is assigned to agent 1. The number of allocations in this case is less than

$$\frac{(|N|-2)!}{(|N|-1)}$$

which is strictly lower than $(|N|-3)!$. This means there is at least one cycle $C(i,y)$ with $i = (3,...,|N|)$ and $Y = (\{x_3, ..., x_{|N|}\}$ since there are exactly $(|N|-3)!$ of such cycles.

I can continue until $|N|-t-1$ is a prime number. Let x_α belong to $\{x_t, x_{t+1}, ..., x|N|\}$ and suppose that agent α gets good x_α the most often in the allocations belonging to $PO(P)$ when x_1 is allocated to agent 1, x_2 to agent 2, ..., x_{t-1} to agent $t-1$. Then, by Corollary 2.1, all agents who belong to $\{t, t+1, ..., |N|\}\diagdown\{\alpha\}$ have the same preferences over the set $\{x_t, x_{t+1}, ..., x_{|N|}\} \diagdown \{x_\alpha\}$.

Step 2 : Now, consider the general case where agents in $\{s, s+1..., |N|\}\diagdown\beta$ have the same preferences over $\{x_s, x_{s+1}, x_{s+2}, ..., x_{|N|}\} \diagdown \{x_\beta\}$. But there is at least one cycle $C(i,y)$ with $i = (s, s+1, ..., |N|)$ and $Y = \{x_s, x_{s+1}, x_{s+2}, ..., x_{|N|}\}$. By Proposition 2.7, all agents belonging to I must have the same preferences over the set $\{x_s, x_{s+1}, x_{s+2}, ..., x_{|N|}\} \diagdown \{x_\beta\}$.

Step 3 : I can use the same approach with the two remaining x_α. Doing so, I find that all agents belonging to $\{s, s+1, ..., |N|\}$ have the same preferences over the set $\{x_s, x_{s+1}, x_{s+2}, ..., x_{|N|}\}$.

I use this approach until I find that all agents belonging to $\{2, 3, ..., |N|\}$ have the same preferences over the set $\{x_2, x_3, ..., x_{|N|}\}$.

Step 4 : If $|\Psi|$ is strictly lower than $(|N|-1)!$, this means there is at least one cycle $C(i,y)$ with $i = (1, 2, ..., |N|)$ and $Y = \{x_1, x_2, ..., x_{|N|}\}$. Then, by Proposition 2.7, all agents have the same preferences over the set $\{x_2, x_3..., x_{|N|}\}$. Now, if steps 1 to 3 are done once again with x_2 and x_3 instead of x_1, it can be seen that all agents must have the same preferences over the set $\{x_1, x_2, x_3..., x_{|N|}\}$.

Step 5 : Now suppose that $|\Psi|$ is equal to $(|N|-1)!$. Suppose that there are two allocations a^1 and a^2 belonging to Ψ such that all agents get different goods, there is no $\alpha \in \{1, 2, ..., |N|\}$ such that $a^1_\alpha = a^2_\alpha$.

Let the vector i be the cycle of goods from a^1 to a^2. In other words, the good allocated to agent i_α in the allocation a^1 goes to agent $i_{\alpha+1}$. Since there are two allocations composing the same cycle $C(i,y)$ and there are $(|N|-1)!$ allocations, this means there is at least one cycle and I obtain that all agents must have the same preferences.

The only way to avoid the possibility of having a cycle of N elements in the set $PO(P)$ is for all allocations belonging to Ψ, there is a good which is never allocated to an agent.

Suppose this good is x_γ and the agent never getting x_γ in $PO(P)$ is δ. Since all allocations belong to $PO(P)$, all agents have the same preferences over the set $X \diagdown \{x_\gamma\}$ and all agents belonging to $I \diagdown \{\delta\}$ have the same preferences over the set X.

If all agents have the same preferences, then $PO(P)$ must contain all allocations. So, this means there is at least one good belonging to $X \diagdown \{x_\gamma\}$ for which agent δ and other agents must have different preferences. ■

I can use cycles to describe the rationalizability conditions of a set further. For example, I can use the same approach to say that if $|PO(P)| < (|N|-1)(|N|-1)!$, then

1. $|PO(P)| = (|N|-2)(|N|-1)! + (|N|-2)!$ or
2. $|PO(P)| \leq (|N|-2)(|N|-1)!$

2.5 Conclusion

The rationalizability in the context of house allocation is hard to provide. Except in cases where there are only a few allocations (1, 2 or 3) or for the set of all possible allocations, it is very difficult to conclude.

The use of cycles can help to analyze the rationalizability of an allocation set. While Proposition 2.8 studies the number of elements necessary for an allocation set to be rationalizable, Proposition 2.6 presents a case where the fact that a set contains a cycle implies that it must contain some specific allocations too. Proposition 2.8 could be extended to include more conditions, but to devise a complete statement of all cases promises to be very long and complicated. From my point of view, the most interesting avenue for the use of cycles is to employ them like I do in Proposition 2.6. In short, cycles can be useful to study directly the rationalizability of an allocation set, since by using cycles it is possible to say if a given allocation set is missing some allocations to be rationalizable.

2.6 References

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Chapitre 3

Self-enforcing contracts, value functions and CARA utility functions with an application to mergers

3.1 Introduction

Why do firms merge? Two strands of the economic literature try to answer this question. Since the beginning of the 80's, Industrial Organization economists have tried to find a simple model to explain why firms merge. In their paper, Salant *et al* [17] show that under a quantity-competition framework, unless synergies are important or a majority of firms are involved (more than 80 percent of firms), merged firms (insiders) lose while other firms (outsiders) gain.[1] Deneckere and Davidson [3] state clearly the problem.

> The incentive to merge in noncooperative oligopoly models depends on the interaction of two basic forces. First, a merger allows coalition partners to absorb a negative externality. (...) Second, the merger elicits a spiral of responses from rival firms. (...) In quantity-setting games, (...) the response of other industry members tends to hurt coalition partners because in these games reaction functions are typically downward sloping.[2]

Some authors have proposed alternative approaches. Kamien and Zang [11] present a three-stage model. The first stage is the acquisition phase where firms bid to acquire other firms. In the second stage, merged firms (the parent firm) decide how many divisions (old independent firms) will produce a strictly positive quantity of goods. In the last stage, divisions of every parent firm compete in a Cournot game. This approach differs from the Salant *et al* [17] model. Implicitly, Salant *et al* [17] assume that all firms involved in a merger act post-merger as a unique entity. With their model, Kamien and Zang [11] find that 50 percent of market firms must be involved in the merger to gain from the merger. Creane and Davidson [2][3] continue in the same way and propose a model in which the parent firm can use a different strategy with their divisions. They show that the merger could be beneficial if the parent firm uses a structure in which divisions announce sequentially the quantity they will produce. This Stackelberg game, which is played by divisions in combination with a Cournot game with the other firms, leave insiders with a gain and outsiders with a loss. Moreover, they find that only a small number of firms must be involved in the merger. They argue that other kind of strategies can be used to increase the market power of the merging firm. As such, they provide an answer to the merger paradox.[4]

Finance Economists have also studied mergers. They use financial incentives to study conditions under which a merger could be beneficial to insiders.[5] While some authors look

[1]Deneckere and Davidson [3] work on a price competition model. They find that both insiders and outsiders gain but outsiders do better than insiders.

[2]Deneckere and Davidson [3], page 484.

[3]Huck, Konrad and Müller [9] present similar models with same results.

[4]Pepall, Richards and Norman [16] define the merger paradox as the difficulty to construct a simple economic model which leaves insiders with a gain even if they do not merge in a monopoly.

[5]Hubbard [8] gives a survey of the literature on financial constraints.

at the management incentives,[6] one of the most important approach relates to the optimality of using internal financing versus external financing. In a frictionless capital market framework, Modiglani and Miller [15] show that the capital structure (internal or external financing) of firms does not affect a firm market value. But some economists argue that the equivalence between internal versus external financing does not hold. Alchian [1] and Williamson [20] were the first to argue that headquarters are able to monitor production and effort more effectively than outsiders. Then, mergers could be beneficial if this problem of monitoring leads to an inefficient allocation of capital for pre-merger firms. Gertner *et al* [7] present a model in which headquarters can use the surplus of external capital from given project for financing another project. They argue that this internal capital market increases monitoring incentives, decreases entrepreneurial incentives and redeploys financial assets more efficiently. Stein [18] uses another approach. He supposes that the headquarter is able to enact a winner-picking process which consists of the allocation of the constrained capital to the division which provides a better return. Stein [18] supposes that the headquarters have a better knowledge than outsider investors to allocate more effectively. Consequently, the headquarter is able to reallocate capital as the state of nature is revealed and can reassign capital to the good project from the bad one.

Besides the question of the difference between internal and external capital, the imperfection of the financial market could explain why firms merge. The risk is transferred to the financial market and risk-averse shareholders gain from a decrease in the net revenue variance. When the financial market is not perfect, shareholders can be better off by merging their firm with another. If firms have negatively correlated revenues, the merger will decrease the firm's revenue variance by using an internal financial market. However, if firms have positively correlated revenues, it could happen that the increase in the revenue variance will decrease the effect of the financial market imperfection and leave the merged firm with a net gain.

This paper studies this question. In their paper, Inderst and Müller [10] present a model in which a firm must decide to centralize or decentralize borrowing. With the first option, investors and firms can sign a financial contract which is more efficient than contracts signed when borrowing is decentralized. Implicitly, Inderst and Müller [10] assume that the cost to enforce a contract is quite low. So, the agent must respect the contract in any period. When the cost of enforcing a contract is important and the mobility cost for an agent to quit the contract is quite low, the lack of a binding commitment becomes a problem. Indeed, one agent could have the incentive to break the ex-ante optimal contract after the state of nature is revealed. This problem of commitment in risk-sharing contracts can lead to inefficiencies. To avoid this problem, long term contracts must be self-enforcing, which means that no agent could gain by breaking the contract in all possible contingencies.

[6] For example, see McNeil, Niehaus and Powers [14].

I use this approach to study in which condition a merger could be beneficial for shareholders. Particularly, I want to study the effects of self-enforcing constraints on the efficiency of mergers. A self-enforcing contract is such that, in all possible states of nature, the firm and the borrower must have an incentive to respect the contract. This approach was first introduced by Thomas and Worrall [19]. In their model, agents agree on signing an insurance contract at time 0. Then, at the beginning of each subsequent period, the state of nature is revealed to both agents. Each agent must decide whether to respect the contract or not. If both of them decide to respect the contract, then the transfer of wealth occurs along the terms specified in the contract. If one decides to break the contract, then no wealth is transferred and it is not possible for the agent to sign another contract in the future. If a given contract, which can be viewed as a series of transfers, is such that in any state of nature and for any period, each agent gains more in respecting the contract than in breaking it, then this contract is said to be self-enforcing.

Since general results are hard to provide, I study the case where utility functions exhibit constant relative risk aversion (CARA). I begin by explicitly solving the self-enforcing contract problem when agents have CARA utility functions and there are two states of nature. From the optimal solution, I am able to draw the Pareto frontier in the context where first-best contracts are feasible and when there is no such feasible contract. Second, I look at the effects of a change in the distribution of the random revenue on the optimal contract. I show that an increase in the variance leads to an increase of the range of the discount factor for which the optimal contract is non trivial. Finally, I find that a merger may or may not be beneficial for merged firms depending on the discount rate and the correlation between firm's revenues.

The paper is divided as follows. In Section 3.2, I present the model which is then solved explicitly with CARA utility functions in Section 3.3. I analyze the effect of a change in the variance of revenues in Section 3.4. In Section 3.5, I study the benefit of a merger in the self-enforcing context. Section 3.6 provides concluding remarks.

3.2 Model

The problem is to design an insurance contract between two infinitely-lived risk-averse agents. I suppose that the state of the economy is *i.i.d.* over the finite set $S = \{1, 2, ..., |S|\}$. The revenue of agent 1 can take values $y_1, ..., y_S$ while agent 2 has a constant revenue $\overline{w}$. By convention, $y_s > y_{s-1}$. I denote by y^t the realization of agent 1's revenue in period t.

The utility functions for agents 1 and 2 are respectively $u(c_t^1)$ and $v(c_t^2)$ where c_t^i is the consumption of agent i in period t. I suppose that the utility functions are twice continuously differentiable and strictly concave. Total consumption must satisfy $c_t^1 + c_t^2 \leq y^t + \overline{w}$ for any $y^t \in \{y_1, y_2, ..., y_{|S|}\}$.

Let $h_t = (s_1, s_2, s_3, ..., s_{t-1})$ be the history of realized states of the world at period t. The insurance contract δ consists of a series of transfers which in any given period depend on the history and the current state of the world. Let $b_t(h_t, s)$ be the transfer from agent 1 to agent 2 in period t when the history is h_t and the state of nature at period t is s. The transfer could be positive or negative. Consumption in period t can then be expressed as function of the revenue and the transfer ($c_t^1 = y_s - b_t(h_t, s)$ and $c_t^2 = \overline{w} + b_t(h_t, s)$).

Now, let E_s^t be the operator expectation over s conditional on h_{t-1} and let β be the discount rate. I define $U(\delta; h_t)$ and $V(\delta; h_t)$ as the expected net gain for all periods $t, t+1, t+2, ...$ for agents 1 and 2 respectively,

$$U(\delta; h_t) = \mathrm{E}_s^t \left[\sum_{\tau=t}^{\infty} \beta^{\tau-t} \left[u\left(y_s - b_\tau(h_\tau, s)\right) - u(y_s) \right] \right]$$
$$V(\delta; h_t) = \mathrm{E}_s^t \left[\sum_{\tau=t}^{\infty} \beta^{\tau-t} \left[v\left(\overline{w} + b_\tau(h_\tau, s)\right) - v(\overline{w}) \right] \right]$$

An optimal contract is a contract δ such that agent 1 maximizes his expected utility when agent 2 obtains a given level of expected utility. This optimal contract is the solution which maximizes :

$$U(\delta, h_1) = \mathrm{E}_s^1 \left[u\left(y_s - b_1(h_1, s)\right) - u(y_s) + \beta U(\delta, h_2) \right] \tag{3.1}$$

subject to

$$V(\delta, h_1) = \mathrm{E}_s^1 \left[v\left(\overline{w} + b_1(h_1, s)\right) - v(\overline{w}) + \beta V(\delta, h_2) \right] \geq \overline{V}$$

where $\overline{V}$ is the reservation value of agent 2.

The solution to the maximization problem (3.1) is first-best. This contract is such that $u'(c_t^1)/v'(c_t^2)$ is constant for all periods t and for all states of nature s.

The first-best contract introduces a potentially large transfer from one agent to the other. In some circumstances, it is conceivable that an agent would prefer reneging on the contract rather than making a transfer to the other agent. If contract enforcement is costly, nothing can prevent an agent from doing so.

I now study this case explicitly. I suppose that each agent can leave the contract at any moment. If an agent leaves the contract, I assume the he remains in autarky forever thereafter. For the contract to hold, each agent must have incentives to respect the contract in every period and for every history. To take this into account, I must add self-enforcing constraints to the problem. The optimal self-enforcing contract is derived by solving

$$MAX \quad U(\delta, h_1) \tag{3.2}$$

subject to

$$
\begin{aligned}
V(\delta, h_1) &\geq \overline{V} \\
u(y_s - b_\tau(h_\tau, s)) - u(y_s) + \beta U(\delta, h_{\tau+1}) &\geq 0 \qquad \tau = 1, 2, \ldots \quad \forall s \in S, \quad \forall h_\tau \\
v(\overline{w} + b_\tau(h_\tau, s)) - v(\overline{w}) + \beta V(\delta, h_{\tau+1}) &\geq 0 \qquad \tau = 1, 2, \ldots \quad \forall s \in S, \quad \forall h_\tau
\end{aligned}
$$

The additional constraints state that, in any period and state, each agent must have a non-negative surplus from the relationship.

There always exists a self-enforcing contract. The contract where no transfer is made in any period is trivially self-enforcing. I call this contract the trivial self-enforcing contract (TSEC). A contract which is self-enforcing and is not the TSEC is called a non-trivial self-enforcing contract.

Let $\widetilde{b}_t(b_{t-1}, s_{t-1}, s_t)$ be the first-best transfer at period t in state s_t when the transfer at period $t-1$ was b_{t-1} and the state of nature was s_{t-1}. In other words, $\widetilde{b}_t(b_{t-1}, s_{t-1}, s_t)$ is such that

$$
\frac{u'\left(y^{t-1} - b_{t-1}\right)}{v'\left(\overline{w} + b_{t-1}\right)} = \frac{u'\left(y_s - \widetilde{b}_t(b_{t-1}, s_{t-1}, s_t)\right)}{v'\left(\overline{w} + \widetilde{b}_t(b_{t-1}, s_{t-1}, s_t)\right)}
$$

Thomas and Worrall [19] show that the optimal contract has the following characterization.

1. For any state of nature s, there exists a non-empty interval $\left[\underline{b_s}, \overline{b_s}\right]$ such that $b_t(h_t, s)$ belongs to this interval.
2. For any history h_t and state of nature s,

$$
b_t(h_t, s) = \begin{cases} \underline{b_s} & \text{if } \underline{b_s} > \widetilde{b}_t(b_{t-1}, s_{t-1}, s_t) \\ \widetilde{b}_t(b_{t-1}, s_{t-1}, s_t) & \text{if } \widetilde{b}_t(b_{t-1}, s_{t-1}, s_t) \in \left[\underline{b_s}, \overline{b_s}\right] \\ \overline{b_s} & \text{if } \overline{b_s} < \widetilde{b}_t(b_{t-1}, s_{t-1}, s_t) \end{cases} \tag{3.3}
$$

The optimal contract is as close as possible to the first-best contract subject to self-enforcing constraints which implicitly define the set of $\underline{b_s}$ and $\overline{b_s}$.

3.3 CARA utility functions

To be able to solve explicitly (3.2), I use a specific form of utility functions and add some constraints to the problem structure. In this section, I use a constant absolute risk aversion (CARA) utility function, i.e.

$$\begin{aligned} u\left(c_t^1\left(h_t, s\right)\right) &= -e^{-r\left(c_t^1\left(h_t, s\right)\right)} \\ v\left(c_t^2\left(h_t, s\right)\right) &= -e^{-q\left(c_t^2\left(h_t, s\right)\right)} \end{aligned}$$

where r and q are respectively the risk aversion parameter of agent 1 and agent 2. With this assumption about the form of the utility function, the problem becomes :

$$MAX \quad E_s^1\left[-e^{-r(y^1-b_1(h_1,s))}+e^{-ry^1}+\beta U\left(\delta, h_2\right)\right] \tag{3.4}$$

subject to

$$\begin{aligned} E_s^1\left[-e^{-q(\overline{w}+b_1(h_1,s))}+e^{-q\overline{w}}+\beta V\left(\delta, h_2\right)\right] &\geq \overline{V} \\ -e^{-r(y_s-b_\tau(h_\tau,s))}+e^{-ry_s}+\beta E_s^\tau[U\left(\delta, h_{\tau+1}\right)] &\geq 0 \qquad \tau=1,2,\ldots \quad \forall s\in S, \quad \forall h_\tau \\ -e^{-q(\overline{w}+b_\tau(h_\tau,s))}+e^{-q\overline{w}}+\beta E_s^\tau[V\left(\delta, h_{\tau+1}\right)] &\geq 0 \qquad \tau=1,2,\ldots \quad \forall s\in S, \quad \forall h_\tau \end{aligned}$$

It is possible to characterize first-best contracts using simple manipulations. To do so, I must differentiate (3.4) without the self-enforcing constraints with respect to two different states of nature at two different periods.

$$\begin{aligned} \frac{u'\left(y_s-b_t\left(h_t,s\right)\right)}{u'\left(\overline{w}+b_t\left(h_t,s\right)\right)} &= \frac{u'\left(y_z-b_\tau\left(h_\tau,z\right)\right)}{u'\left(\overline{w}+b_\tau\left(h_\tau,z\right)\right)} \\ \frac{re^{-r(y_s-b_t(h_t,s))}}{qe^{-q(\overline{w}+b_t(h_t,s))}} &= \frac{re^{-r(y_z-b_\tau(h_\tau,z))}}{qe^{-q(\overline{w}+b_\tau(h_\tau,z))}} \\ r(y_s-b_t\left(h_t,s\right))-q(\overline{w}+b_t\left(h_t,s\right)) &= r(y_z-b_\tau\left(h_\tau,z\right))-q(\overline{w}+b_\tau\left(h_\tau,z\right)) \end{aligned}$$

And I obtain :

$$b_\tau\left(h_\tau, z\right)=b_t\left(h_t, s\right)+\frac{r}{(r+q)}(y_z^\tau-y_s^t) \tag{3.5}$$

This gives the relation between each possible transfer in each possible state of nature and at every period. Equation (3.5) tells us that the optimal transfer at a specific period in a specific state of nature is linear in the revenues of both agents. Here, there are optimal contracts for special cases.

- If agent 2 has a random revenue w_s, then the first-best contract is characterized by $b_\tau\left(h_\tau, z\right)=b_t\left(h_t, s\right)+\frac{r}{(r+q)}(y_z-y_s)+\frac{q}{(r+q)}(w_s-w_z)$.
- If agent 1 and agent 2 have the same risk-aversion coefficient ($r=q$), then $b_\tau\left(h_\tau, z\right)= b_t\left(h_t, s\right)+\frac{1}{2}(y_z-y_s)$.
- If agent 2 is risk neutral ($q=0$), then $b_\tau\left(h_\tau, z\right)=b_t\left(h_t, s\right)+y_z-y_s$.

Throughout the rest of the paper, unless I explicitly suppose something else, I assume that agents have the same risk-aversion coefficient ($q = r$). This facilitates the explicit characterization of the optimal contract.[7]

Also, to be able to explicitly solve the problem, I constrain the number of states of nature to two. With more states, the problem rapidly becomes intractable.

3.3.1 Conditions for a non-trivial solution

Let's say that a contract δ' is stationary if the transfer in state 1 is b'_1 and the transfer in state 2 is b'_2, no matter what the history is. The next two lemmas are derived from Propositions 4.1 and 4.2 of Kocherlakota [12].

Lemma 3.1 *If the optimal contract δ^* is first-best, then δ^* is stationary.*

Proof. If a contract is first-best, then the transition of transfers between states of nature at any period is given by (3.3). Then, the transfer at period t is $b_1(h_t, 1) = b_1^*$ if the state of nature is 1 for any history h_t and $b_1(h_t, 2) = b_2^*$ if the state of nature is 2 for any history h_t. ■

Lemma 3.2 *If there are only two states of nature, then the optimal contract δ^* for (3.4) monotonically converges to a stationary contract δ'.*

Proof. Let the optimal contract be δ^*. By definition, the contract δ^* gives the appropriate transfer for any state of nature at period 1. Suppose that transfers at period 1 are given by $b_1^*(h_1, 1)$ and $b_1^*(h_1, 2)$.

Without loss of generality, lets assume the state of nature at period 1 is 1. By (3.3), $b_1^*(h_1, 1)$ belongs to $\left[\underline{b_1}, \overline{b_1}\right]$ and, if the state of nature is the same at period t and $t+1$, then transfers in these periods must be the same (i.e. $b_t^*(h_t, s) = b_{t+1}^*(h_{t+1}, s)$). Then, until the state of nature becomes 2, the transfer stays $b_1^*(h_1, 1)$.

Suppose that the state of nature stays 1 for period 1 to period $t-1$ and becomes 2 at period t. Then $b_t^*(h_t, 2)$ must be equal

- to $\underline{b_2}$ if $\underline{b_2} > \widetilde{b}_t\left(b_{t-1}^*(h_{t-1}, 1), 2\right)$;
- or to $\widetilde{b}_t\left(b_{t-1}^*(h_{t-1}, 1), 2\right)$ if $\widetilde{b}_t\left(b_{t-1}^*(h_{t-1}, 1), 2\right) \in \left[\underline{b_2}, \overline{b_2}\right]$;
- or to $\overline{b_2}$ if $\overline{b_2} < \widetilde{b}_t\left(b_{t-1}^*(h_{t-1}, 1), 2\right)$.

In case 2, this means that the contract is first-best and by Lemma 3.1, the contract is stable.

Suppose case 1, i.e. the transfer in state 2 is the lowest possible ($\underline{b_2}$). If I stay in state 2, then the transfer stays $\underline{b_2}$. If I return to state 1 at period $\tau > t$, then $b_\tau^*(h_\tau, 1)$ must be equal :

- to $\underline{b_1}$ if $\underline{b_1} > \widetilde{b}_\tau\left(\underline{b_2}, 1\right)$;

[7] With different risk-aversion coefficients, I obtain a system of polynomial equations of different degrees.

- or to $\widetilde{b}_\tau\left(\underline{b_2},1\right)$ if $\widetilde{b}_\tau\left(\underline{b_2},1\right) \in \left[\underline{b_1},\overline{b_1}\right]$;
- or to $\overline{b_1}$ if $\overline{b_1} < \widetilde{b}_\tau\left(\underline{b_2},1\right)$.

In case 2, this means the contract becomes stable after period τ with $\widetilde{b}_\tau\left(\underline{b_2},1\right)$ in state 1 and $\underline{b_2}$.

In case 3, this means the contract become stable after period τ with $\overline{b_1}$ in state 1 and $\underline{b_2}$

Case 1 is impossible. I have supposed that $\underline{b_2} > \widetilde{b}_t\left(b^*_{t-1}\left(h_{t-1},1\right),2\right)$. Then, $\widetilde{b}_t\left(\underline{b_2},1\right) > b^*_{t-1}\left(h_{t-1},1\right) > \underline{b_1}$.

By the structure of the process, the probability that the history h_τ contains state 1 and state 2 while τ goes to infinity is equal to one. ■

These results hold for any concave utility function. This comes from the fact that transfers in each state must belong to a closed interval. Consequently, if the first best contract transition given by $\widetilde{b}_t\left(b^*_{t-1}\left(h_{t-1},s\right),z\right)$ belongs to the interval, then there is a first-best self-enforcing contract. By definition, any first-best contract is stationary since transfers do not depend on the history but only on the actual state of nature. For any no first-best self-enforcing contract, boundaries constrain the value of transfers. In the two state case, the non-trivial self-enforcing contract (NTSEC) converges monotonically to a stationary contract where the transfer is upper bounded in state 1 or lower bounded in state 2.

In the case where the number of states of nature is higher than 2, the NTSEC does not converge to a stationary contract. The reason is transfers in intermediate states of nature (state $2,3,...,S-1$), it could be optimal to have history-dependent transfers. For example, in the 3-state case, transfer in state 2 could take different values depending of the history. But, if I define partial history-dependent stationarity, which says that transfers in any state depend only of the part of the history in which state 1 and S was realized, I can obtain a lemma similar to Lemma 3.2 using partial history-dependent stationarity for any number of states of nature.

Now, I am able to study the existence of a NTSEC. To prove the existence of such contract, I can only look for the existence of a stationary contract which satisfies the self-enforcing constraints. By Lemma 3.2, if there is a NTSEC δ^*, then this contract converges monotonically to a stationary contract δ'. The contract δ' which must be self-enforcing since a self-enforcing contract must be self-enforcing in any state of nature and at any period. Consequently, looking for the existence of a stationary self-enforcing contract is enough to prove the existence of a NTSEC.

Proposition 3.1 *Let ρ be the probability of being in the state of nature 1 and $y_2 > y_1$. If $e^{r(y_2-y_1)} \geq \left[1+\frac{1-\beta}{\beta^2\rho*(1-\rho)}\right]$, then there are some values of $\overline{V}$ for which the solution to (3.4) is not the TSEC.*

Proof. By Lemma 3.2, each optimal contract δ^* converges to a stable contract δ'. Then, if δ' is not self-enforcing, neither is δ^*.

Take δ' and assume that this contract gives at any period b_1' if the state of nature is 1 and b_2' otherwise. Let U' and V' be the gain in utility of agent 1 and 2 respectively with the contract δ'. Suppose that δ' is self-enforcing. Then,

$$\begin{aligned}
-e^{-r(y_1-b_1')} + e^{-ry_1} + \beta E_s\left[U'\right] &\geq 0 \\
-e^{-r(y_2-b_2')} + e^{-ry_2} + \beta E_s\left[U'\right] &\geq 0 \\
-e^{-r(\overline{w}+b_1')} + e^{-r\overline{w}} + \beta E_s\left[V'\right] &\geq 0 \\
-e^{-r(\overline{w}+b_2')} + e^{-r\overline{w}} + \beta E_s\left[V'\right] &\geq 0
\end{aligned}$$

I have supposed that $y_2 > y_1$. This means that agent 1 is relatively more rich in state 2 than in state 1. Then, the optimal transfer must be negative in state 1 and positive in state 2.

If I take a look at the participation constraints, I see that only two constraints are really constraining.

$$\begin{aligned}
-e^{-r(y_2-b_2')} + e^{-ry_2} + \beta E_s\left[U'\right] &\geq 0 \\
-e^{-r(\overline{w}+b_1')} + e^{-r\overline{w}} + \beta E_s\left[V'\right] &\geq 0
\end{aligned}$$

The other two are not because in those cases, the agent receives some amount. Then, they do not want to break the contract. By definition, U' and V' are stable. I can compute their value by using the Bellman equation.

$$\begin{aligned}
U' &= \rho\left(-e^{-r(y_1-b_1')} + e^{-ry_1}\right) + (1-\rho)\left(-e^{-r(y_2-b_2')} + e^{-ry_2}\right) + \beta U' \\
U' &= \frac{1}{1-\beta}\left[\rho\left(-e^{-r(y_1-b_1')} + e^{-ry_1}\right) + (1-\rho)\left(-e^{-r(y_2-b_2')} + e^{-ry_2}\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
V' &= \left[\rho\left(-e^{-rb_1'} + 1\right) + (1-\rho)\left(-e^{-rb_2'} + 1\right)\right] + \beta V' \\
V' &= \frac{1}{1-\beta}\left[\rho\left(-e^{-rb_1'} + 1\right) + (1-\rho)\left(-e^{-rb_2'} + 1\right)\right]
\end{aligned}$$

I replace U' and V' in the preceding constraints. Now, I must isolate b_2' in the first constraint.

$$\begin{aligned}
-e^{-r(y_2-b_2')} + e^{-ry_2} + \frac{\beta}{1-\beta}\left[\rho\left(-e^{-r(y_1-b_1')} + e^{-ry_1}\right) + (1-\rho)\left(-e^{-r(y_2-b_2')} + e^{-ry_2}\right)\right] &\geq 0 \\
\frac{\beta\rho}{1-\beta}\left(-e^{-r(y_1-b_1')} + e^{-ry_1}\right) + \frac{1-\beta\rho}{1-\beta}\left(-e^{-r(y_2-b_2')} + e^{-ry_2}\right) &\geq 0 \\
\beta\rho\left(-e^{-r(y_1-b_1')} + e^{-ry_1}\right) + (1-\beta\rho)\left(-e^{-r(y_2-b_2')} + e^{-ry_2}\right) &\geq 0 \\
\beta\rho\left(-e^{r(y_2-y_1)}e^{rb_1'} + e^{r(y_2-y_1)}\right) + (1-\beta\rho)\left(-e^{rb_2'} + 1\right) &\geq 0
\end{aligned}$$

And I obtain :

$$\frac{\beta\rho}{1-\beta\rho}\left(-e^{r(y_2-y_1)}e^{rb_1'} + e^{r(y_2-y_1)}\right) + 1 \geq e^{rb_2'}$$

Graphically, this condition is represented by Figure 3.1.

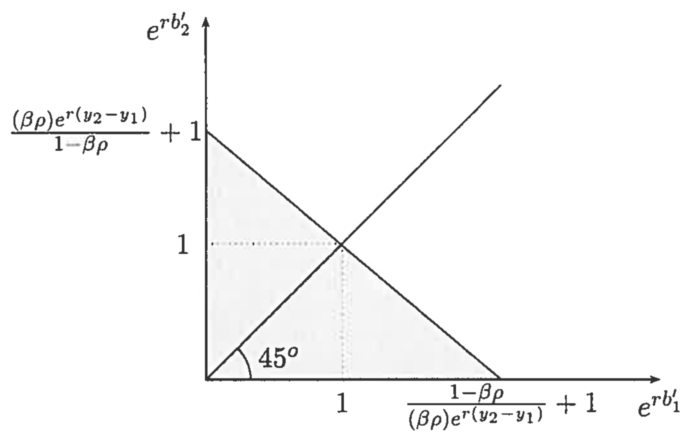

FIG. 3.1 – First Constraint

I can proceed in the same way with the second constraint.

$$\begin{aligned}
-e^{-rb_1'} + 1 + \beta\left[\frac{\rho}{1-\beta}\left(-e^{-rb_1'} + 1\right) + \frac{(1-\rho)}{1-\beta}\left(-e^{-rb_2'} + 1\right)\right] &\geq 0 \\
-e^{-rb_1'} + \beta\left[\frac{\rho}{1-\beta}\left(-e^{-rb_1'}\right) + \frac{(1-\rho)}{1-\beta}\left(-e^{-rb_2'}\right)\right] &\geq -\frac{1}{1-\beta} \\
-e^{-rb_1'} + \beta\left[\frac{\rho}{1-\beta}\left(-e^{-rb_1'}\right) + \frac{(1-\rho)}{1-\beta}\left(-e^{-rb_2'}\right)\right] &\geq -\frac{1}{1-\beta} \\
(1-\beta+\beta\rho)\left(-e^{-rb_1'}\right) + (\beta-\beta\rho)\left(-e^{-rb_2'}\right) &\geq -1
\end{aligned}$$

And I obtain :

$$\begin{aligned} \frac{1-\beta+\beta\rho}{\beta-\beta\rho}\left(-e^{-rb_1'}\right)+\frac{1}{\beta-\beta\rho} &\geq e^{-rb_2'} \\ \frac{\beta-\beta\rho}{1-(1-\beta+\beta\rho)\left(e^{-rb_1'}\right)} &\leq e^{rb_2'} \end{aligned}$$

Now, I can graph this condition (See Figure 3.2)

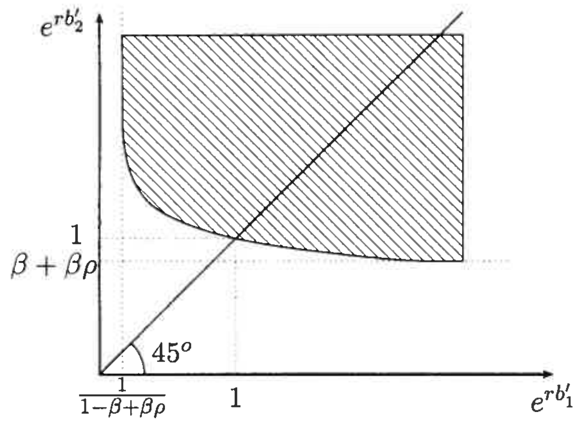

FIG. 3.2 – Second constraint

I know that the frontier must have the point $(1,1)$ since the TSEC is self-enforcing. If I combine the two constraints, I obtain Figure 3.3.

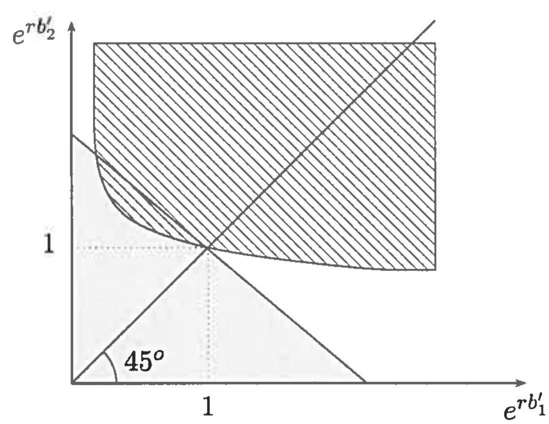

FIG. 3.3 – Both Constraints

The grey and hatched region is the set of all contracts like δ'. To know if there exists such contracts, I must analyze the slope of the two constraints at the point $(1,1)$. Lets begin with the first constraint.

$$\frac{d(e^{rb'_2})}{d(e^{rb'_1})} = -\frac{\beta\rho e^{r(y_2-y_1)}}{1-\beta\rho}$$

For the second one, I obtain :

$$\frac{d(e^{rb'_2})}{d(e^{rb'_1})} = \frac{d}{d(e^{rb'_1})}\left(\frac{\beta-\beta\rho}{1-(1-\beta+\beta\rho)\,e^{-rb'_1}}\right)$$
$$\frac{d(e^{rb'_2})}{d(e^{rb'_1})} = -\frac{\beta-\beta\rho}{\left[1-(1-\beta+\beta\rho)\,e^{-rb'_1}\right]^2}(1-\beta+\beta\rho)\,e^{-rb'_1}$$

If I evaluate this slope at $(1,1)$, I obtain

$$\frac{d(e^{rb'_2})}{d(e^{rb'_1})} = -\frac{(1-\beta+\beta\rho)}{\beta-\beta\rho}$$

In order for self-enforcing contracts other than the TSEC to exist, the slope of the second constraint must be larger than the slope of the first constraint.

$$-\frac{(1-\beta+\beta\rho)}{\beta-\beta\rho} \geq -\frac{\beta\rho e^{r(y_2-y_1)}}{1-\beta\rho}$$
$$\frac{(1-\beta\rho)(1-\beta+\beta\rho)}{\beta\rho(\beta-\beta\rho)} \leq e^{r(y_2-y_1)}$$
$$1+\frac{1-\beta}{\beta\rho(\beta-\beta\rho)} \leq e^{r(y_2-y_1)}$$

Then, the slope of the first constraint is lower than the slope of the second if $e^{r(y_2-y_1)} > 1+\frac{1-\beta}{\beta\rho*(\beta-\beta\rho)}$. ■

For the moment, I do not know if the optimal contract is first-best. Proposition 3.1 tells us only under which conditions a non-trivial solution to (3.4) exists. Proposition 3.2 gives the condition to have a self-enforcing first-best contract.

Proposition 3.2 *Let ρ be the probability of being in the state of nature 1 and $y_2 > y_1$. If $e^{r(y_2-y_1)} \geq \left[1+\frac{1-\beta}{\beta^2\rho*(1-\rho)}\right]^2$, then there is some value of $\overline{V}$ such that the optimal contract is first-best.*

Proof. Suppose that the optimal first-best contract is b_1^{fb}, b_2^{fb} and let U^{fb} and V^{fb} be the gain in utility for agent 1 and 2 with the contract δ^{fb}. The first-best contract is self-enforcing if it fulfills the self-enforcing constraints. In the proof of Proposition (3.1), I state that only two self-enforcing constraints are relevant.

$$\begin{aligned}-e^{-r(y_2-b_2^{fb})}+e^{-ry_2}+\beta E_s\left[U^{fb}\right] &\geq 0\\ -e^{-r(\overline{w}+b_1^{fb})}+e^{-r\overline{w}}+\beta E_s\left[V^{fb}\right] &\geq 0\end{aligned}$$

By (3.5), I know that the first-best contract is given by the following relation :

$$e^{rb_2^{fb}}=e^{\frac{r}{2}(y_2-y_1)}e^{rb_1^{fb}}$$

Let A be the NTSEC that fulfills both self-enforcing constraints with equality. Then, some first-best contracts are self-enforcing if A is on the left side of the first-best contract line. To proceed, I must find the solutions to the equations for the constraints. Since the TSEC satisfies the constraints, I must focus on the other solution (point A). Let $\left(b_1^A,b_2^A\right)$ be the values of the transfers at point A and let U^A and V^A be the gain in utility of agent 1 and 2 with the contract δ^A. Then, point A represents the non-trivial solution of

$$\begin{aligned}-e^{-r(y_2-b_2^{A})}+e^{-ry_2}+\beta E_s\left[U^{A}\right] &= 0\\ -e^{-r(\overline{w}+b_1^{A})}+e^{-r\overline{w}}+\beta E_s\left[V^{A}\right] &= 0\end{aligned}$$

In the previous proof, I have found that those equations can be written as :

$$\begin{aligned}\frac{\beta\rho}{1-\beta\rho}\left(-e^{r(y_2-y_1)}e^{rb_1^A}+e^{r(y_2-y_1)}\right)+1 &= e^{rb_2^A}\\ \frac{\beta-\beta\rho}{1-(1-\beta+\beta\rho)\left(e^{-rb_1^A}\right)} &= e^{rb_2^A}\end{aligned}$$

By solving this system of equations, I find that the non-trivial solution is :

$$\begin{aligned}e^{rb_2^A} &= \left(\frac{\beta\rho e^{r(y_2-y_1)}}{1-\beta\rho}\right)(\beta-\beta\rho)+(\beta-\beta\rho)\\ e^{rb_1^A} &= 1+\frac{1-\beta\rho}{\beta\rho e^{r(y_2-y_1)}}-(\beta-\beta\rho)-\left(\frac{1-\beta\rho}{\beta\rho e^{r(y_2-y_1)}}\right)(\beta-\beta\rho)\end{aligned}$$

If I calculate the slope of the line which connects point A to the origin, $\frac{e^{rb_1^A}}{e^{rb_2^A}}$, I find :

$$\begin{aligned}
\frac{e^{rb_1^A}}{e^{rb_2^A}} &= \frac{1+\frac{1-\beta\rho}{\beta\rho e^{r(y_2-y_1)}}-(\beta-\beta\rho)-\left(\frac{1-\beta\rho}{\beta\rho e^{r(y_2-y_1)}}\right)(\beta-\beta\rho)}{\left(\frac{\beta\rho e^{r(y_2-y_1)}}{1-\beta\rho}\right)(\beta-\beta\rho)+(\beta-\beta\rho)} \\
&= \frac{1-\beta\rho}{\beta\rho e^{r(y_2-y_1)}}\frac{\frac{\beta\rho e^{r(y_2-y_1)}}{1-\beta\rho}+1-\frac{\beta\rho e^{r(y_2-y_1)}}{1-\beta\rho}(\beta-\beta\rho)-(\beta-\beta\rho)}{\left(\frac{\beta\rho e^{r(y_2-y_1)}}{1-\beta\rho}\right)(\beta-\beta\rho)+(\beta-\beta\rho)} \\
&= \frac{1-\beta\rho}{\beta\rho e^{r(y_2-y_1)}}\left(\frac{\frac{\beta\rho e^{r(y_2-y_1)}}{1-\beta\rho}+1}{\left(\frac{\beta\rho e^{r(y_2-y_1)}}{1-\beta\rho}\right)(\beta-\beta\rho)+(\beta-\beta\rho)}-1\right) \\
&= \frac{1-\beta\rho}{\beta\rho e^{r(y_2-y_1)}}\left(\frac{1}{\beta-\beta\rho}-1\right)
\end{aligned}$$

Now, I must compare this result with the slope of the line of first-best contracts. If the slope of the first-best contract line is lower than the slope I find above, then some first-best contracts are self-enforcing.

$$\begin{aligned}
\frac{e^{rb_1^{fb}}}{e^{rb_2^{fb}}} &\geq \frac{e^{rb_1^A}}{e^{rb_2^A}} \\
\frac{1}{e^{\frac{r}{2}(y_2-y_1)}} &\geq \frac{1-\beta\rho}{\beta\rho e^{r(y_2-y_1)}}\left(\frac{1}{\beta-\beta\rho}-1\right) \\
e^{\frac{r}{2}(y_2-y_1)} &\geq \frac{1-\beta\rho}{\beta\rho}\left(\frac{1-\beta+\beta\rho}{\beta-\beta\rho}\right) \\
e^{\frac{r}{2}(y_2-y_1)} &\geq \frac{1-\beta+\beta\rho-\beta\rho+\beta^2\rho-\beta^2\rho^2}{\beta\rho(\beta-\beta\rho)} \\
e^{\frac{r}{2}(y_2-y_1)} &\geq 1+\frac{1-\beta}{\beta\rho(\beta-\beta\rho)}
\end{aligned}$$

Then, if $e^{r(y_2-y_1)} \geq \left[1+\frac{1-\beta}{\beta\rho(\beta-\beta\rho)}\right]^2$, there exist some values of $\overline{V}$ such that the optimal contract is first-best. ■

The idea of the proof is the following : the first-best relation given by (3.5) must be compared with the non-trivial contract solving the two self-enforcing constraints. Precisely, I must compare ratios $\frac{e^{rb_1^{fb}}}{e^{rb_2^{fb}}}$ and $\frac{e^{rb_1^A}}{e^{rb_2^A}}$ where b_s^{fb} is the transfer in state s under a first best contract [8] and b_s^A is the transfer in state s when the contract is the non-trivial one solving self-enforcing constraints. Figure 3.4 illustrates the idea.

From the two preceding propositions, if y_2-y_1 increases, then the optimal contract will become non-trivial once $e^{r(y_2-y_1)} \geq \left[1+\frac{1-\beta}{\beta\rho(\beta-\beta\rho)}\right]$ and when $e^{r(y_2-y_1)} \geq \left[1+\frac{1-\beta}{\beta\rho(\beta-\beta\rho)}\right]^2$,

[8] Mathematically, I find that the ratio $\frac{e^{rb_1^{fb}}}{e^{rb_2^{fb}}}$ is constant for any first-best contract.

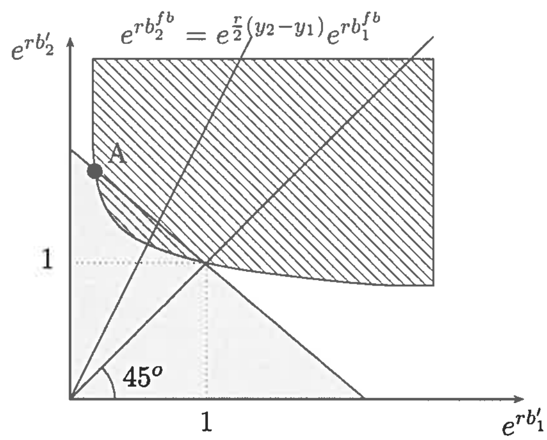

FIG. 3.4 – First-Best Contracts and Constraints

then the optimal contract will become first best. Those results can be viewed as the dual solution from Proposition 4 of Thomas and Worrall [19] which says that there is a discount factor β^* such that, for all $\beta > \beta^*$, some optimal contracts are first-best and there is a $\beta_* < \beta^*$ such that for all $\beta \in [\beta_*, \beta^*)$, the optimal contract is non-trivial but not first-best.[9]

I prove Proposition 3.2 by finding the condition such that a first-best contract satisfies all self-enforcing constraints. But, what can I say about the optimal contract ? Kocherlakota [12] proves that, when some optimal contracts are first-best, then the expected utility converges to a utility level given by a self-enforcing first-best contract.[10] I can rewrite this proposition in a equivalent way in term of contracts

Proposition 3.3 *Suppose that some first-best contract is optimal. Then, all optimal contracts converge to a first-best contract.*

Proof. See Proposition 4.1 of Kocherlakota. ■

I say that a contract δ is first-best convergent if it converges to a first-best contract. This definition will be very useful in Section 3.5.

3.3.2 Pareto Frontier

In the previous section, I derive the condition to have a NTSEC. Here, I want to show how the self-enforcing constraints affect the optimality of the contract. To do so, I use the Pareto frontier in either case where a first-best contract is or is not self-enforcing and I compare with the Pareto frontier when there is no self-enforcing constraint. I first begin with the Pareto frontier when there is no self-enforcing constraint.

[9] It is possible to write conditions to have a NTSEC or a first-best self-enforcing contract with beta on the left side but conditions become a bit messy

[10] See Proposition 4.1 of Kocherlakota [12].

Proposition 3.4 *Without self-enforcing constraints, the Pareto frontier is given by :*

$$U^U\left(\overline{V}\right) = \frac{1}{1-\beta}\left[E_s\left[e^{-ry_s}\right] - \frac{E_s\left[e^{-\frac{r}{2}y_s}\right]^2}{1-(1-\beta)\,e^{r\overline{w}}\overline{V}}\right]$$

with $\overline{V} \in \left[0, \frac{1}{(1-\beta)e^{r\overline{w}}}\left(1 - \frac{E_s\left[e^{-\frac{r}{2}y_s}\right]^2}{E_s\left[e^{-ry_s}\right]}\right)\right]$.

Proof. First, with the assumption of constant revenues for agent 2, I can rewrite the participation constraint. By Proposition 4.1 of Kocherlakota [12], I have that $V(\delta, h_t) = \overline{V}$. Then,

$$\begin{aligned} E_s\left[-e^{-r\left(\overline{w}_s+b_s^{fb}\right)} + e^{-r\overline{w}_s} + \beta\overline{V}\right] &\geq \overline{V} \\ E_s\left[-e^{-rb_s^{fb}} + 1\right] &\geq (1-\beta)\,e^{r\overline{w}}\overline{V} \\ E_s\left[e^{-rb_s^{fb}}\right] &= 1-(1-\beta)\,e^{r\overline{w}}\overline{V} \end{aligned}$$

I know from (3.5) that the relation between transfers is given by :

$$b_2^{fb} = b_1^{fb} + \frac{1}{2}(y_2 - y_1)$$

By introducing this result into the participation constraint for agent 2, I obtain :

$$\begin{aligned} \rho\left(e^{-rb_1^{fb}}\right) + (1-\rho)\left(e^{-rb_2^{fb}}\right) &= 1-(1-\beta)\,e^{r\overline{w}}\overline{V} \\ \rho\left(e^{-rb_1^{fb}}\right) + (1-\rho)\left(e^{-r\left(b_1^{fb}+\frac{1}{2}(y_2-y_1)\right)}\right) &= 1-(1-\beta)\,e^{r\overline{w}}\overline{V} \\ e^{-rb_1^{fb}} &= \frac{1-(1-\beta)\,e^{r\overline{w}}\overline{V}}{\rho + (1-\rho)\left(e^{\frac{r}{2}(y_1-y_2)}\right)} \end{aligned}$$

And $e^{-rb_2^{fb}}$ is given by :

$$e^{-rb_2^{fb}} = \frac{1-(1-\beta)\,e^{r\overline{w}}\overline{V}}{\rho\left(e^{\frac{r}{2}(y_2-y_1)}\right) + (1-\rho)}$$

Then, I am able to define the Pareto frontier explicitly by introducing b_1^{fb} and b_2^{fb} in the utility function of agent 1.

$$
\begin{aligned}
U^{fb} &= E_s\left[-e^{-r(y_s-b_s^{fb})}+e^{-ry_s}+\beta U^{fb}\right]\\
U^{fb} &= \frac{1}{1-\beta}\left[E_s\left[e^{-ry_s}\right]-\rho e^{-r(y_1-b_1^{fb})}-(1-\rho)\,e^{-r(y_2-b_2^{fb})}\right]\\
U^{fb} &= \frac{1}{1-\beta}\left[E_s\left[e^{-ry_s}\right]-\frac{E_s\left[e^{-\frac{r}{2}y_s}\right]^2}{1-(1-\beta)\,e^{r\overline{w}}\overline{V}}\right]
\end{aligned}
$$

The maximum value for $\overline{V}$ is reached when $U^{fb}=0$.

$$
\begin{aligned}
0 &= \frac{1}{1-\beta}\left[E_s\left[e^{-ry_s}\right]-\frac{E_s\left[e^{-\frac{r}{2}y_s}\right]^2}{1-(1-\beta)\,e^{r\overline{w}}\overline{V}_{MAX}}\right]\\
E_s\left[e^{-ry_s}\right] &= \frac{E_s\left[e^{-\frac{r}{2}y_s}\right]^2}{1-(1-\beta)\,e^{r\overline{w}}\overline{V}_{MAX}}\\
\overline{V}_{MAX} &= \frac{1}{(1-\beta)e^{r\overline{w}}}\left(1-\frac{E_s\left[e^{-\frac{r}{2}y_s}\right]^2}{E_s\left[e^{-ry_s}\right]}\right)
\end{aligned}
$$

■

Figure 3.5 represents the unconstrained Pareto frontier when there are no self-enforcing constraints.

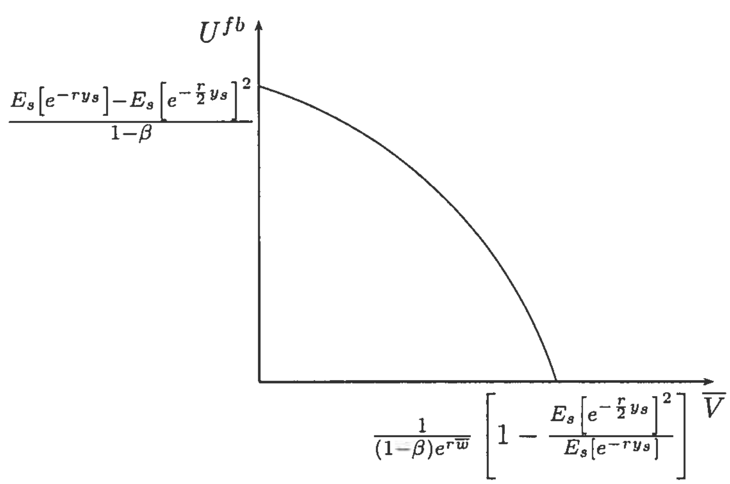

FIG. 3.5 – Unconstrained Pareto Frontier

Without self-enforcing constraints, this Pareto frontier is attainable everywhere. This is not the case when I add self-enforcing constraints. With self-enforcing constraints, as shown above, there are two possibilities : either some first-best contracts are self-enforcing or no first-best contract is. In the following proposition, I present the Pareto frontier if there is no self-enforcing first-best contracts.

Proposition 3.5 *Suppose that* $\left[1+\frac{1-\beta}{\beta^2\rho*(1-\rho)}\right]^2 > e^{r(y_2-y_1)} \geq \left[1+\frac{1-\beta}{\beta^2\rho*(1-\rho)}\right]$ *and let*

$$\begin{aligned} e^{rb_1^A} &= 1-\beta+\beta\rho+\frac{1-\beta\rho}{\beta\rho e^{r(y_2-y_1)}}(1-\beta+\beta\rho) \\ e^{rb_2^A} &= \left(\frac{\beta\rho e^{r(y_2-y_1)}}{1-\beta\rho}\right)(\beta-\beta\rho)+(\beta-\beta\rho) \\ V^A &= \frac{(1-\rho)\,e^{-r\overline{w}}}{1-\beta+\beta\rho}\left(1-e^{-rb_2^A}\right) \\ V^{MAX} &= \frac{(1-\rho)\,e^{-r\overline{w}}}{(1-\beta\rho)(1-\beta+\beta\rho)}\left(1-e^{-rb_2^A}\right) \end{aligned}$$

Then,

- *if* $\overline{V}\in\left[0,V^A\right]$, *then the optimal contract is given by :*
  - $b_t(h_t,s)=b_1^A$ *if the state of nature s is 1.*
  - $b_t(h_t,s)=\frac{(1-\rho)}{(1-\rho)-[1-(1-\rho)\beta]e^{r\overline{w}}\overline{V}}$ *if the history is* $h_t=(2,2,...,2)$.
  - $b_t(h_t,s)=b_2^A$ *otherwise.*
- *if* $\overline{V}\in\left[V^A,V^{MAX}\right]$, *then the optimal contract is given by :*
  - $b_t(h_t,s)=b_2^A$ *if the state of nature s is 2.*
  - $b_t(h_t,s)=\frac{\rho(1-\beta+\beta\rho)}{\rho(1-\beta+\beta\rho)+(1-\rho)\left(1-e^{-rb_2^A}\right)-(1-\rho\beta)(1-\beta+\beta\rho)*e^{rd}*\overline{V}}$ *if the history is* $h_t=(1,1,...,1)$.
  - $b_t(h_t,s)=b_1^A$ *otherwise.*

And

- *if* $\overline{V}\in\left[0,V^A\right]$, *then the Pareto frontier is given by :*

$$\begin{aligned} U^{SE}\left(\overline{V}\right) &= \frac{\rho\left(-e^{-r(y_1-b_1^A)}+e^{-ry_1}\right)}{(1-\beta+\beta\rho)(1-\rho\beta)} \\ &\quad+\frac{(1-\rho)*e^{-ry_2}}{(1-\beta+\beta\rho)}\left(1-\frac{1-\rho}{1-\rho-e^{r\overline{w}}*(1-\beta+\beta\rho)*\overline{V}}\right) \end{aligned}$$

- *if* $\overline{V}\in\left[V^A,V^{MAX}\right]$, *then the Pareto frontier is given by :*

$$U^{SE}\left(\overline{V}\right) = \frac{\rho e^{-ry_1}}{1-\beta\rho}\left(1-\frac{\rho(1-\beta+\beta\rho)}{\gamma}\right)$$

$$where\ \gamma = \rho\left(1-\beta+\beta\rho\right)+\left(1-\rho\right)\left(1-e^{-rb_2^A}\right)-\left(1-\rho\beta\right)\left(1-\beta+\beta\rho\right)*e^{r\overline{w}}\overline{V}.$$

Proof. By Proposition 3.1 and 3.2, I already know that there is no first-best self-enforcing contract. By Lemma 3.2, the optimal contract converges monotonically to the contract given by the non-trivial solution of the following 2 self-enforcing constraints :

$$\begin{aligned}
\frac{\beta\rho}{1-\beta\rho}\left(-e^{r(y_2-y_1)}e^{rb_1'}+e^{r(y_2-y_1)}\right)+1 &= e^{rb_2'} \\
\frac{\beta-\beta\rho}{1-\left(1-\beta+\beta\rho\right)\left(e^{-rb_1'}\right)} &= e^{rb_2'}
\end{aligned}$$

which is

$$\begin{aligned}
e^{rb_1^A} &= 1-\beta+\beta\rho+\frac{1-\beta\rho}{\beta\rho e^{r(y_2-y_1)}}\left(1-\beta+\beta\rho\right) \\
e^{rb_2^A} &= \left(\frac{\beta\rho e^{r(y_2-y_1)}}{1-\beta\rho}\right)\left(\beta-\beta\rho\right)+\left(\beta-\beta\rho\right)
\end{aligned}$$

Graphically,

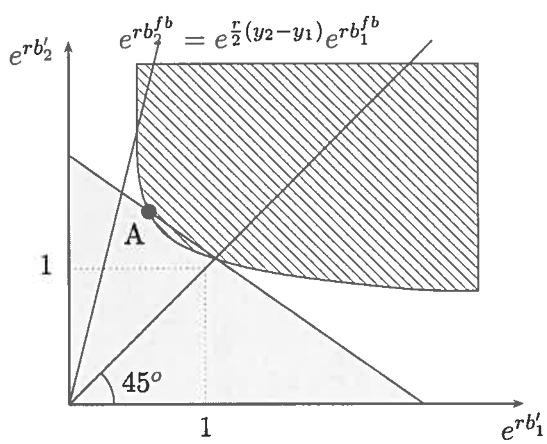

FIG. 3.6 – Stationary contract

Let V^A be the utility for agent 2 at point A. Then,

$$V^A=\rho\left(-e^{-r(\overline{w}+b_1^A)}+e^{-r\overline{w}}+\beta V^A\right)+\left(1-\rho\right)\left(-e^{-r(\overline{w}+b_2^A)}+e^{-r\overline{w}}+\beta V^A\right)$$

But, the stationary contract satisfies the relevant participation constraint with equality. Then :

$$-e^{-r(\overline{w}+b_1^A)}+e^{-r\overline{w}}+\beta V^A=0$$

Then, I have :

$$\begin{aligned} V^A &= (1-\rho)\left(-e^{-r(\overline{w}+b_2^A)} + e^{-r\overline{w}} + \beta V^A\right) \\ V^A &= \frac{e^{-r\overline{w}}(1-\rho)\left(1-e^{-rb_2^A}\right)}{1-\beta(1-\rho)} \end{aligned}$$

Then, if $\overline{V} = V^A$, the optimal contract is the contract represented by point A. If $\overline{V} \neq V^A$, then the optimal contract is different than the contract represented by point A, but must monotonically converge to the A-contract. In the proof of Lemma 3.2, I have seen that a contract can only differ from a stable contract at the beginning and until the state of nature switches. In other words, the transfer in state 1 at period t can be different from b_1^A if state 2 is not yet realized in the t first periods and the transfer in state 2 at period t can be different from b_2^A if state 1 is not yet realized in the t first periods.

This results in two types of contracts :

Type 1 :

- The transfer at period t is b_1^A if the state of nature is 1.
- The transfer at period t is $b_2^* \leq b_2^A$ if the state of nature is 2 at period t and the state of nature was not realized in the first $t-1$ periods.
- The transfer at period t is b_2^A if the state of nature is 2 at period t and the state of nature was realized in the first $t-1$ periods.

Type 2 :

- The transfer at period t is b_2^A if the state of nature is 2.
- The transfer at period t is $b_1^* \geq b_1^A$ if the state of nature is 1 at period t and the state of nature was not realized in the first $t-1$ periods.
- The transfer at period t is b_1^A if the state of nature is 1 at period t and the state of nature was realized in the first $t-1$ periods.

The type 1 contract gives more utility to agent 1 and less to agent 2 and the opposite is true for type 2 contract. Then, when $\overline{V} \leq V^A$, the optimal contract is type 1 and when $\overline{V} \geq V^A$, the optimal contract is type 2.

Now, I must calculate the transfer in the first t periods in term of $\overline{V}$. Let's begin with the case where $\overline{V} \leq V^A$. Then,

$$\overline{V} = \rho\left(-e^{-r(\overline{w}+b_1^A)} + e^{-r\overline{w}} + \beta V^A\right) + (1-\rho)\left(-e^{-r(\overline{w}+b_2^*)} + e^{-r\overline{w}} + \beta\overline{V}\right)$$

But, I find b_1^A by using the self-enforcing constraint :

$$-e^{-r(\overline{w}+b_1^A)} + e^{-r\overline{w}} + \beta V^A = 0$$

I obtain :

$$\begin{aligned}
\overline{V} &= (1-\rho)\left(-e^{-r(\overline{w}+b_2^*)}+e^{-r\overline{w}}+\beta\overline{V}\right)\\
\overline{V} &= (1-\rho)\left(-e^{-r(\overline{w}+b_2^*)}+e^{-r\overline{w}}\right)+(1-\rho)\,\beta\overline{V}\\
(1-(1-\rho)\,\beta)\,\overline{V} &= (1-\rho)\left(-e^{-r(\overline{w}+b_2^*)}+e^{-r\overline{w}}\right)\\
\left(\frac{1-(1-\rho)\,\beta}{(1-\rho)}\right)\overline{V} &= \left(-e^{-r(\overline{w}+b_2^*)}+e^{-r\overline{w}}\right)\\
\left(\frac{1-(1-\rho)\,\beta}{(1-\rho)}\right)e^{r\overline{w}}\overline{V} &= \left(1-e^{-rb_2^*}\right)
\end{aligned}$$

And

$$\begin{aligned}
e^{-rb_2^*} &= 1-\frac{1-(1-\rho)\,\beta}{(1-\rho)}e^{r\overline{w}}\overline{V}\\
e^{rb_2^*} &= \frac{(1-\rho)}{(1-\rho)-[1-(1-\rho)\,\beta]\,e^{r\overline{w}}\overline{V}}
\end{aligned}$$

If $\overline{V}=0$, then $b_2^*=0$. If $\overline{V}=V^A$, then :

$$\begin{aligned}
e^{rb_2^*} &= \frac{(1-\rho)}{(1-\rho)-[1-(1-\rho)\,\beta]\,e^{r\overline{w}}\frac{e^{-r\overline{w}}(1-\rho)\left(1-e^{-rb_2^A}\right)}{1-\beta(1-\rho)}}\\
e^{rb_2^*} &= \frac{(1-\rho)}{(1-\rho)-(1-\rho)\left(1-e^{-rb_2^A}\right)}\\
e^{rb_2^*} &= e^{rb_2^A}
\end{aligned}$$

Now, I examine the case where $\overline{V}>V^A$.

$$\begin{aligned}
\overline{V} &= \rho\left(-e^{-r(\overline{w}+b_1^*)}+e^{-r\overline{w}}+\beta\overline{V}\right)+(1-\rho)\left(-e^{-r(\overline{w}+b_2^A)}+e^{-r\overline{w}}+\beta V^A\right)\\
(1-\beta\rho)\,\overline{V} &= \rho\left(-e^{-r(\overline{w}+b_1^*)}+e^{-r\overline{w}}\right)+(1-\rho)\left(-e^{-r(\overline{w}+b_2^A)}+e^{-r\overline{w}}+\beta V^A\right)\\
(1-\beta\rho)\,e^{r\overline{w}}\overline{V} &= \rho\left(-e^{-rb_1^*}+1\right)+(1-\rho)\left(-e^{-rb_2^A}+1+\beta e^{r\overline{w}}V^A\right)
\end{aligned}$$

Then :

$$\begin{aligned}
(1-\beta\rho)\,e^{r\overline{w}}\overline{V}-(1-\rho)\left(-e^{-rb_2^A}+1+\beta e^{r\overline{w}}V^A\right) &= \rho\left(-e^{-rb_1^{sb}}+1\right)\\
1-\frac{(1-\beta\rho)}{\rho}e^{r\overline{w}}\overline{V}+\frac{(1-\rho)}{\rho}\left(-e^{-rb_2^A}+1+\beta e^{r\overline{w}}V^A\right) &= e^{-rb_1^{sb}}
\end{aligned}$$

Moreover, I have already found that $V^A = \frac{(1-\rho)\left(-e^{-r(\overline{w}+b_2^A)}+e^{-r\overline{w}}\right)}{1-\beta(1-\rho)}$. Hence,

$$
\begin{aligned}
e^{-rb_1^*} &= 1 - \frac{(1-\beta\rho)}{\rho} e^{r\overline{w}}\overline{V} + \frac{(1-\rho)}{\rho}\left(-e^{-rb_2^A} + 1 + \beta e^{r\overline{w}} \frac{(1-\rho)\left(-e^{-r(\overline{w}+b_2^A)} + e^{-r\overline{w}}\right)}{1-\beta(1-\rho)}\right) \\
e^{-rb_1^{sb}} &= 1 - \frac{(1-\beta\rho)}{\rho} e^{r\overline{w}}\overline{V} + \frac{(1-\rho)}{\rho}\left(-e^{-rb_2^A} + 1 + \beta \frac{(1-\rho)\left(-e^{-rb_2^A} + 1\right)}{1-\beta(1-\rho)}\right) \\
e^{-rb_1^{sb}} &= 1 - \frac{(1-\beta\rho)}{\rho} e^{r\overline{w}}\overline{V} + \frac{(1-\rho)}{\rho(1-\beta+\beta\rho)}\left(1 - e^{-rb_2^A}\right) \\
e^{rb_1^{sb}} &= \frac{\rho(1-\beta+\beta\rho)}{\rho(1-\beta+\beta\rho) + (1-\rho)\left(1-e^{-rb_2^A}\right) - (1-\rho\beta)(1-\beta+\beta\rho) * e^{rd} * \overline{V}}
\end{aligned}
$$

Let V^{MAX} be the maximal value for $\overline{V}$. Then, V^{MAX} is reached when $b_1^* = 0$. To have $b_1^* = 0$, I must have :

$$
\begin{aligned}
(1-\beta\rho)(1-\beta+\beta\rho) e^{r\overline{w}} V^{MAX} &= (1-\rho)\left(1 - e^{-rb_2^A}\right) \\
V^{MAX} &= \frac{(1-\rho) e^{-r\overline{w}}}{(1-\beta\rho)(1-\beta+\beta\rho)}\left(1 - e^{-rb_2^A}\right)
\end{aligned}
$$

The previous part of the proof gives the optimal contract relative to the value of $\overline{V}$. Then, if I replace those values in the utility function of agent 1, I obtain the Pareto frontier equation. Let's begin with the case where $\overline{V} \in \left[0, V^A\right]$. In this case, the utility function of agent 1 is given by :

$$
U^{OP}\left(\overline{V}\right) = \rho\left(-e^{-r(y_1-b_1^A)} + e^{-ry_1} + \beta U^A\right) + (1-\rho)\left(-e^{-ry_2}e^{rb_2^*} + e^{-ry_2} + \beta U^{OP}\left(\overline{V}\right)\right)
$$

With

$$
U^A = \rho\left(-e^{-r(y_1-b_1^A)} + e^{-ry_1} + \beta U^A\right) + (1-\rho)\left(-e^{-ry_2}e^{rb_2^A} + e^{-ry_2} + \beta U^A\right)
$$

Because $-e^{-r(y_2-b_2^A)} + e^{-ry_2} + \beta U^A = 0$ by the self-enforcing constrain, I obtain :

$$
\begin{aligned}
U^A &= \rho\left(-e^{-r(y_1-b_1^A)} + e^{-ry_1} + \beta U^A\right) \\
U^A &= \frac{\rho}{(1-\rho\beta)}\left(-e^{-r(y_1-b_1^A)} + e^{-ry_1}\right)
\end{aligned}
$$

Then,

$$
\begin{aligned}
U^{OP}\left(\overline{V}\right) &= \frac{\rho}{(1-\rho\beta)}\left(-e^{-r(y_1-b_1^A)}+e^{-ry_1}\right) \\
&\quad +(1-\rho)\left(-e^{-ry_2}e^{rb_2^*}+e^{-ry_2}+\beta U^{OP}\left(\overline{V}\right)\right) \\
(1-\beta+\beta\rho)\,U^{OP}\left(\overline{V}\right) &= \frac{\rho}{(1-\rho\beta)}\left(-e^{-r(y_1-b_1^A)}+e^{-ry_1}\right)+(1-\rho)\left(-e^{-ry_2}e^{rb_2^*}+e^{-ry_2}\right) \\
U^{OP}\left(\overline{V}\right) &= \frac{\rho}{(1-\beta+\beta\rho)(1-\rho\beta)}\left(-e^{-r(y_1-b_1^A)}+e^{-ry_1}\right) \\
&\quad +\frac{(1-\rho)}{(1-\beta+\beta\rho)}\left(-e^{-ry_2}e^{rb_2^*}+e^{-ry_2}\right)
\end{aligned}
$$

If I substitute $e^{rb_2^*}=\frac{1-\rho}{1-\rho-e^{r\overline{w}}(1-\beta+\beta\rho)\overline{V}}$, I obtain :

$$
U^{OP}\left(\overline{V}\right)=\frac{\rho\left(-e^{-r(y_1-b_1^A)}+e^{-ry_1}\right)}{(1-\beta+\beta\rho)(1-\rho\beta)}+\frac{(1-\rho)*e^{-ry_2}}{(1-\beta+\beta\rho)}\left(1-\frac{1-\rho}{1-\rho-e^{r\overline{w}}(1-\beta+\beta\rho)\overline{V}}\right)
$$

$$
U^{OP}\left(\overline{V}\right)=\frac{\rho\left(-e^{-r(y_1-b_1^A)}+e^{-ry_1}\right)}{(1-\beta+\beta\rho)(1-\rho\beta)}+\frac{(1-\rho)*e^{-ry_2}}{(1-\beta+\beta\rho)}\left(\frac{-e^{r\overline{w}}*(1-\beta+\beta\rho)*\overline{V}}{1-\rho-e^{r\overline{w}}(1-\beta+\beta\rho)\overline{V}}\right)
$$

Now, for the case where $\overline{V}\in\left[V^A,V^{MAX}\right]$. In this case, the utility function for agent 1 is given by :

$$
U^{OP}\left(\overline{V}\right)=\rho\left(-e^{-r(y_1-b_1^*)}+e^{-ry_1}+\beta U^{OP}\left(\overline{V}\right)\right)+(1-\rho)\left(-e^{-r(y_2-b_2^A)}+e^{-ry_2}+\beta f\left(V^A\right)\right)
$$

I already know that $-e^{-r(y_2-b_2^A)}+e^{-ry_2}+\beta U^A=0$. Then,

$$
\begin{aligned}
U^{OP}\left(\overline{V}\right) &= \frac{\rho}{1-\beta\rho}\left(-e^{-r(y_1-b_1^*)}+e^{-ry_1}\right) \\
U^{OP}\left(\overline{V}\right) &= \frac{\rho}{1-\beta\rho}e^{-ry_1} \\
&\left(1-\frac{\rho(1-\beta+\beta\rho)}{\rho(1-\beta+\beta\rho)+(1-\rho)\left(1-e^{-rb_2^A}\right)-(1-\rho\beta)(1-\beta+\beta\rho)e^{r\overline{w}}\overline{V}}\right)
\end{aligned}
$$

■

Since the unconstrained Pareto frontier represents the maximum agent's utilities under all first-best contracts, then the Pareto frontier when no first-best contract is self-enforcing is strictly lower. Another important point to underline is the discontinuity of the Pareto frontier. Kocherlakota [12] says that the Pareto frontier is differentiable everywhere. In fact, as corrected by Koeppl [13], the Pareto frontier is not differentiable everywhere (Proposition 3.1). If I examine the Pareto frontier where a non-trivial solution exists, I find that the Pareto frontier is continuous but not differentiable everywhere[11]. The problem of differentiability occurs at the intersection of 2 segments.

This problem of discontinuity occurs also when some first-best contracts are self-enforcing. The next proposition shows the Pareto frontier in this case.

Proposition 3.6 *Suppose that* $e^{r(y_2-y_1)} \geq \left[1+\frac{1-\beta}{\beta^2\rho*(1-\rho)}\right]^2$ *and let*

$$\begin{aligned}
e^{rb_1^B} &= 1-\beta+\beta\rho+(\beta-\beta\rho)\,e^{\frac{r}{2}(y_1-y_2)} \\
e^{rb_1^B} &= \beta-\beta\rho+(1-\beta+\beta\rho)\,e^{\frac{r}{2}(y_2-y_1)} \\
V^B &= \frac{(\beta-\beta\rho)\left(1-e^{\frac{r}{2}(y_1-y_2)}\right)}{e^{r\overline{w}}\left(1-(\beta-\beta\rho)\left(1-e^{\frac{r}{2}(y_1-y_2)}\right)\right)} \\
e^{rb_1^C} &= \frac{\beta\rho e^{-ry_1+(1-\beta\rho)e^{-ry_2}}}{\beta\rho e^{-ry_1}+(1-\beta\rho)\,e^{\frac{-r}{2}(y_1+y_2)}} \\
e^{rb_2^C} &= \frac{\beta\rho e^{-ry_1+(1-\beta\rho)e^{-ry_2}}}{\beta\rho e^{\frac{-r}{2}(y_1+y_2)}+(1-\beta\rho)\,e^{-ry_2}} \\
V^C &= \frac{\rho\left(1-e^{\frac{r}{2}(y_1-y_2)}\right)\left(\beta(1-\rho)\,e^{-ry_2}-(1-\beta\rho)\,e^{\frac{r}{2}(y_1+y_2)}\right)}{e^{r\overline{w}}(1-\beta)\left(\beta\rho e^{ry_2}+(1-\beta\rho)\,e^{ry_1}\right)} \\
V^{MAX} &= \frac{(1-\rho)\,e^{-r\overline{w}}}{(1-\beta\rho)(1-\beta+\beta\rho)}\left(1-e^{-rb_2^C}\right)
\end{aligned}$$

Then,

- *if* $\overline{V}\in\left[0,V^B\right]$, *the optimal contract is given by :*
  - $b_t(h_t,s)=b_1^B$ *if the state of nature s is 1.*
  - $b_t(h_t,s)=\frac{(1-\rho)}{(1-\rho)-[1-(1-\rho)\beta]e^{r\overline{w}}\overline{V}}$ *if the history* $h_t=(2,2,...,2)$.
  - $b_t(h_t,s)=b_2^B$ *otherwise.*
- *if* $\overline{V}\in\left[V^B,V^C\right]$, *the optimal contract is given by :*
  - $b_t(h_t,1)=\frac{\rho+(1-\rho)e^{\frac{r}{2}(y_1-y_2)}}{1-(1-\beta)e^{r\overline{w}}\overline{V}}$.
  - $b_t(h_t,2)=\frac{\rho e^{\frac{r}{2}(y_2-y_1)}+(1-\rho)}{1-(1-\beta)e^{r\overline{w}}\overline{V}}$.
- *if* $\overline{V}\in\left[V^C,V^{MAX}\right]$, *the optimal contract is given by :*
  - $b_t(h_t,s)=b_2^C$ *if the state of nature s is 2.*
  - $b_t(h_t,s)=\frac{\rho(1-\beta+\beta\rho)}{\rho(1-\beta+\beta\rho)+(1-\rho)\left(1-e^{-rb_2^C}\right)-(1-\rho\beta)(1-\beta+\beta\rho)*e^{r\overline{w}}*\overline{V}}$ *if the history* $h_t=(1,1,...,1)$.
  - $b_t(h_t,s)=b_1^C$ *otherwise.*

And

[11]The continuity is quite obvious because each segment is continuous and at intersection of two segments, the contract is defined evenly on both segments.

- *if* $\overline{V} \in \left[0, V^B\right]$, *then the Pareto frontier is given by :*

$$
\begin{aligned}
U^{OP}\left(\overline{V}\right) &= \frac{(1-\rho)\,e^{-ry_2}}{1-\beta+\rho}\left(e^{rb_2^B} - \frac{(1-\rho)}{(1-\rho)-[1-(1-\rho)\,\beta]\,e^{r\overline{w}}\overline{V}}\right) \\
&\quad + \frac{\left(\rho\left(-e^{-r\left(y_1-b_1^B\right)}+e^{-ry_1}\right)+(1-\rho)\left(-e^{-ry_2}e^{rb_2^B}+e^{-ry_2}\right)\right)}{1-\beta}
\end{aligned}
$$

- *if* $\overline{V} \in \left[V^B, V^C\right]$, *then the Pareto frontier is given by :*

$$
U^{sb} = \frac{1}{1-\beta}\left[E_s\left[e^{-ry_s}\right] - \frac{E_s\left[e^{-\frac{r}{2}y_s}\right]^2}{1-(1-\beta)\,e^{r\overline{w}}\overline{V}}\right]
$$

- *if* $\overline{V} \in \left[V^C, V^{MAX}\right]$, *then the Pareto frontier is given by :*

$$
U^{SE}\left(\overline{V}\right) = \frac{\rho e^{-ry_1}}{1-\beta\rho}\left(1 - \frac{\rho\,(1-\beta+\beta\rho)}{\gamma}\right)
$$

where $\gamma = \rho\,(1-\beta+\beta\rho) + (1-\rho)\left(1-e^{-rb_2^C}\right) - (1-\rho\beta)\,(1-\beta+\beta\rho) * e^{r\overline{w}} * \overline{V}$.

Proof. Since $e^{r(y_2-y_1)} \geq \left[1+\frac{1-\beta}{\beta^2\rho*(1-\rho)}\right]^2$, then there exists some $\overline{V}$'s such that the optimal contract is first-best. Let's find the set of those $\overline{V}$'s.

The first step is defining the transfer in a first-best contract in terms of $\overline{V}$.

$$
\begin{aligned}
\overline{V} &= \rho\left(-e^{-r\left(\overline{w}+b_1^{fb}\right)}+e^{-r\overline{w}}+\beta\overline{V}\right)+(1-\rho)\left(-e^{-r\left(\overline{w}+b_2^{fb}\right)}+e^{-r\overline{w}}+\beta\overline{V}\right) \\
(1-\beta)\,e^{r\overline{w}}\overline{V} &= \rho\left(1-e^{-rb_1^{fb}}\right)+(1-\rho)\left(1-e^{-rb_2^{fb}}\right) \\
1-(1-\beta)\,e^{r\overline{w}}\overline{V} &= \rho e^{-rb_1^{fb}}+(1-\rho)\,e^{-rb_2^{fb}}
\end{aligned}
$$

By (3.5), $e^{-rb_2^{fb}} = e^{-rb_1^{fb}}e^{-\frac{r}{2}(y_2-y_1)}$. Consequently

$$
\begin{aligned}
1-(1-\beta)\,e^{r\overline{w}}\overline{V} &= \rho e^{-rb_1^{fb}}+(1-\rho)\,e^{-rb_1^{fb}}e^{-\frac{r}{2}(y_2-y_1)} \\
\frac{\rho+(1-\rho)\,e^{\frac{r}{2}(y_1-y_2)}}{1-(1-\beta)\,e^{r\overline{w}}\overline{V}} &= e^{rb_1^{fb}}
\end{aligned}
$$

And

$$
e^{rb_2^{fb}} = \frac{\rho e^{\frac{r}{2}(y_2-y_1)}+(1-\rho)}{1-(1-\beta)\,e^{r\overline{w}}\overline{V}}
$$

Let V^B be the minimal utility of agent 2 when the contract is first best and self-enforcing. This contract is the first-best contract satisfying the self-enforcing constraint of agent 2. Let b_1^B and b_2^B be the transfers of the first-best contract for V^B. Then, I find that

$$\begin{aligned}
e^{rb_1^B} &= 1-\beta+\beta\rho+(\beta-\beta\rho)\,e^{\frac{r}{2}(y_1-y_2)} \\
e^{rb_1^B} &= \beta-\beta\rho+(1-\beta+\beta\rho)\,e^{\frac{r}{2}(y_2-y_1)} \\
V^B &= \frac{(\beta-\beta\rho)\left(1-e^{\frac{r}{2}(y_1-y_2)}\right)}{e^{r\overline{w}}\left(1-(\beta-\beta\rho)\left(1-e^{\frac{r}{2}(y_1-y_2)}\right)\right)}
\end{aligned}$$

Now, for the maximal $\overline{V}$, denoted V^C, given an optimal first-best contract, I must use the self-enforcing constraint of agent 1. Let b_1^C and b_2^C be the transfers of the first-best contract for V^C which is given by

$$\begin{aligned}
e^{rb_1^C} &= \frac{\beta\rho e^{-ry_1+(1-\beta\rho)e^{-ry_2}}}{\beta\rho e^{-ry_1}+(1-\beta\rho)\,e^{\frac{-r}{2}(y_1+y_2)}} \\
e^{rb_2^C} &= \frac{\beta\rho e^{-ry_1+(1-\beta\rho)e^{-ry_2}}}{\beta\rho e^{\frac{-r}{2}(y_1+y_2)}+(1-\beta\rho)\,e^{-ry_2}} \\
V^C &= \frac{\rho\left(1-e^{\frac{r}{2}(y_1-y_2)}\right)\left(\beta\,(1-\rho)\,e^{-ry_2}-(1-\beta\rho)\,e^{\frac{r}{2}(y_1+y_2)}\right)}{e^{r\overline{w}}\,(1-\beta)\left(\beta\rho e^{ry_2}+(1-\beta\rho)\,e^{ry_1}\right)}
\end{aligned}$$

If $\overline{V}\in\left[V^B,V^C\right]$, then the optimal contract is given by the first-best contract given by :

$$\begin{aligned}
e^{rb_1^{fb}} &= \frac{\rho+(1-\rho)\,e^{\frac{r}{2}(y_1-y_2)}}{1-(1-\beta)\,e^{r\overline{w}}\overline{V}} \\
e^{rb_2^{fb}} &= \frac{\rho e^{\frac{r}{2}(y_2-y_1)}+(1-\rho)}{1-(1-\beta)\,e^{r\overline{w}}\overline{V}}
\end{aligned}$$

Now, I study the case when $\overline{V}<V^B$. Equivalent to the proof of Proposition 3.5, the optimal contract in this case is given by :

- The transfer at period t is b_1^B if the state of nature is 1.
- The transfer at period t is $b_2^*\le b_2^B$ if the state of nature is 2 at period t and the other possible state of nature has not been realized at any moment during the first $t-1$ periods.
- The transfer at period t is b_2^B if the state of nature is 2 at period t and the other possible state of nature was realized at some point during the first $t-1$ periods.

To find b_2^*, I must solve

$$\overline{V}=\rho\left(-e^{-r(\overline{w}+b_1^B)}+e^{-r\overline{w}}+\beta V^B\right)+(1-\rho)\left(-e^{-r(\overline{w}+b_2^*)}+e^{-r\overline{w}}+\beta\overline{V}\right)$$

But, I have found that b_1^B by using the self-enforcing constraint :

$$-e^{-r(\overline{w}+b_1^B)}+e^{-r\overline{w}}+\beta V^B=0$$

I obtain :

$$e^{rb_2^*} = \frac{(1-\rho)}{(1-\rho)-[1-(1-\rho)\beta]\,e^{r\overline{w}}\overline{V}}$$

When $\overline{V} > V^C$, the optimal contract in this case is given by :
- The transfer at period t is b_2^C if the state of nature is 2.
- The transfer at period t is $b_1^* \geq b_1^C$ if the state of nature is 1 at period t and the state of nature has not been realized at any moment during the first $t-1$ periods.
- The transfer at period t is b_1^C if the state of nature is 1 at period t and the state of nature was realized at some point during the first $t-1$ periods.

To find b_1^*, I must isolate it in :

$$\overline{V} = \rho\left(-e^{-r(\overline{w}+b_1^*)}+e^{-r\overline{w}}+\beta\overline{V}\right)+(1-\rho)\left(-e^{-r(\overline{w}+b_2^C)}+e^{-r\overline{w}}+\beta V^C\right)$$

With some manipulations...

$$\begin{aligned}(1-\beta\rho)\,\overline{V} &= \rho\left(-e^{-r(\overline{w}+b_1^*)}+e^{-r\overline{w}}\right)+(1-\rho)\left(-e^{-r(\overline{w}+b_2^C)}+e^{-r\overline{w}}+\beta V^C\right)\\ (1-\beta\rho)\,e^{r\overline{w}}\overline{V} &= \rho\left(-e^{-rb_1^*}+1\right)+(1-\rho)\left(-e^{-rb_2^C}+1+\beta e^{r\overline{w}}V^C\right)\end{aligned}$$

$$\begin{aligned}(1-\beta\rho)\,e^{r\overline{w}}\overline{V}-(1-\rho)\left(-e^{-rb_2^A}+1+\beta e^{r\overline{w}}V^A\right) &= \rho\left(-e^{-rb_1^{sb}}+1\right)\\ 1-\frac{(1-\beta\rho)}{\rho}e^{r\overline{w}}\overline{V}+\frac{(1-\rho)}{\rho}\left(-e^{-rb_2^A}+1+\beta e^{r\overline{w}}V^A\right) &= e^{-rb_1^{sb}}\end{aligned}$$

By the self-enforcing constraint of agent 2, I have $V^C = \frac{(1-\rho)\left(-e^{-r(\overline{w}+b_2^C)}+e^{-r\overline{w}}\right)}{1-\beta(1-\rho)}$.

$$\begin{aligned}e^{-rb_1^*} &= 1-\frac{(1-\beta\rho)}{\rho}e^{r\overline{w}}\overline{V}+\frac{(1-\rho)}{\rho}\left(-e^{-rb_2^C}+1+\beta e^{r\overline{w}}\frac{(1-\rho)\left(-e^{-r(\overline{w}+b_2^A)}+e^{-r\overline{w}}\right)}{1-\beta(1-\rho)}\right)\\ e^{-rb_1^{sb}} &= 1-\frac{(1-\beta\rho)}{\rho}e^{r\overline{w}}\overline{V}+\frac{(1-\rho)}{\rho}\left(-e^{-rb_2^C}+1+\beta\frac{(1-\rho)\left(-e^{-rb_2^C}+1\right)}{1-\beta(1-\rho)}\right)\\ e^{-rb_1^{sb}} &= 1-\frac{(1-\beta\rho)}{\rho}e^{r\overline{w}}\overline{V}+\frac{(1-\rho)}{\rho(1-\beta+\beta\rho)}\left(1-e^{-rb_2^C}\right)\\ e^{rb_1^{sb}} &= \frac{\rho(1-\beta+\beta\rho)}{\rho(1-\beta+\beta\rho)+(1-\rho)\left(1-e^{-rb_2^C}\right)-(1-\rho\beta)(1-\beta+\beta\rho)*e^{rd}*\overline{V}}\end{aligned}$$

Let V^{MAX} be the maximal value for $\overline{V}$. Then, V^{MAX} is reached when $b_1^* = 0$. To have $b_1^* = 0$, I must have :

$$\begin{aligned}
(1-\beta\rho)(1-\beta+\beta\rho)e^{r\overline{w}}V^{MAX} &= (1-\rho)\left(1-e^{-rb_2^C}\right) \\
V^{MAX} &= \frac{(1-\rho)e^{-r\overline{w}}}{(1-\beta\rho)(1-\beta+\beta\rho)}\left(1-e^{-rb_2^C}\right)
\end{aligned}$$

I have already found that the Pareto frontier is composed of three parts. Let's begin with the second one, when the optimal contract is first-best. By Proposition 3.4, I know that the Pareto frontier is given by :

$$U^{fb} = \frac{1}{1-\beta}\left[E_s\left[e^{-ry_s}\right] - \frac{E_s\left[e^{-\frac{r}{2}y_s}\right]^2}{1-(1-\beta)e^{r\overline{w}}\overline{V}}\right]$$

Then, when $\overline{V} \in \left[V^B, V^C\right]$, the Pareto frontier is given by this relation.

For the first case, i.e. when $\overline{V} \in \left[0, V^B\right]$, I can use the same approach from the preceding proof.

$$U^{OP}\left(\overline{V}\right) = \rho\left(-e^{-r\left(y_1-b_1^B\right)} + e^{-ry_1} + \beta U^B\right) + (1-\rho)\left(-e^{-ry_2}e^{rb_2^*} + e^{-ry_2} + \beta U^{OP}\left(\overline{V}\right)\right)$$

With

$$U^B = \rho\left(-e^{-r\left(y_1-b_1^B\right)} + e^{-ry_1} + \beta U^B\right) + (1-\rho)\left(-e^{-ry_2}e^{rb_2^B} + e^{-ry_2} + \beta U^B\right)$$

If I compute $U^{OP} - U^B$, I find that

$$\begin{aligned}
U^{OP}\left(\overline{V}\right) - U^B &= (1-\rho)\left(-e^{-ry_2}\left(e^{rb_2^*} - e^{rb_2^B}\right) + \beta\left(U^{OP} - U^B\right)\left(\overline{V}\right)\right) \\
U^{OP}\left(\overline{V}\right) - U^B &= \frac{(1-\rho)e^{-ry_2}}{1-\beta+\rho}\left(e^{rb_2^B} - e^{rb_2^*}\right)
\end{aligned}$$

Then,

$$\begin{aligned}
U^{OP}\left(\overline{V}\right) &= \frac{(1-\rho)e^{-ry_2}}{1-\beta+\rho}\left(e^{rb_2^B} - e^{rb_2^*}\right) \\
&\quad + \frac{\left(\rho\left(-e^{-r\left(y_1-b_1^B\right)} + e^{-ry_1}\right) + (1-\rho)\left(-e^{-ry_2}e^{rb_2^B} + e^{-ry_2}\right)\right)}{1-\beta}
\end{aligned}$$

And if I replace $e^{rb_2^*}$ by $\frac{(1-\rho)}{(1-\rho)-[1-(1-\rho)\beta]e^{r\overline{w}}\overline{V}}$, I find that :

$$\begin{aligned}
U^{OP}\left(\overline{V}\right) &= \frac{(1-\rho)e^{-ry_2}}{1-\beta+\rho}\left(e^{rb_2^B} - \frac{(1-\rho)}{(1-\rho)-[1-(1-\rho)\beta]e^{r\overline{w}}\overline{V}}\right) \\
&\quad + \frac{\left(\rho\left(-e^{-r\left(y_1-b_1^B\right)} + e^{-ry_1}\right) + (1-\rho)\left(-e^{-ry_2}e^{rb_2^B} + e^{-ry_2}\right)\right)}{1-\beta}
\end{aligned}$$

Now, for the case where $\overline{V} \in \left[V^C, V^{MAX}\right]$. In this case, the utility function of agent 1 is given by :

$$U^{OP}\left(\overline{V}\right) = \rho\left(-e^{-r(y_1-b_1^*)} + e^{-ry_1} + \beta U^{OP}\left(\overline{V}\right)\right) + (1-\rho)\left(-e^{-r(y_2-b_2^C)} + e^{-ry_2} + \beta f\left(V^C\right)\right)$$

I already know that $-e^{-r(y_2-b_2^C)} + e^{-ry_2} + \beta U^C = 0$. Then,

$$\begin{aligned}
U^{OP}\left(\overline{V}\right) &= \frac{\rho}{1-\beta\rho}\left(-e^{-r(y_1-b_1^*)} + e^{-ry_1}\right) \\
U^{OP}\left(\overline{V}\right) &= \frac{\rho}{1-\beta\rho}e^{-ry_1} \\
&\left(1 - \frac{\rho(1-\beta+\beta\rho)}{\rho(1-\beta+\beta\rho) + (1-\rho)\left(1-e^{-rb_2^C}\right) - (1-\rho\beta)(1-\beta+\beta\rho)e^{r\overline{w}}\overline{V}}\right)
\end{aligned}$$

■

Of course, the Pareto frontier in each case is dominated by the Pareto frontier in the case without self-enforcing constraints.[12] Figures 3.7 and 3.8 illustrate this fact.

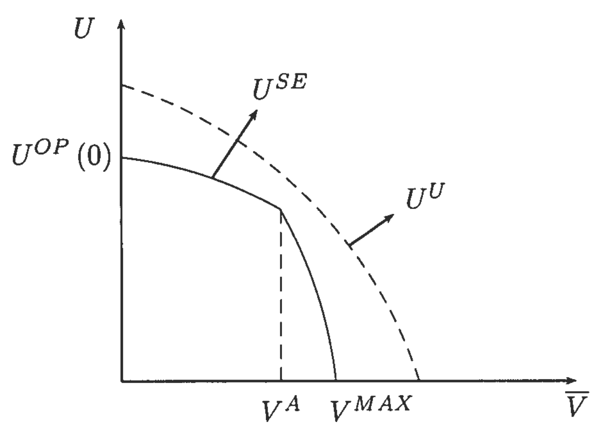

FIG. 3.7 – Pareto Frontier with no self-enforcing first-best contracts

At the opposite of the case where no first-best contracts are self-enforcing, a part of the unconstrained Pareto frontier may be reached when some first-best contracts are self-enforcing. This comes from the fact that, if a first-best contract is self-enforcing, then self-enforcing constraints do not apply and the problem is similar to the one without

[12] The Pareto frontier in case where some first-best contracts are self-enforcing is weakly dominated while the Pareto frontier in the other case is dominated everywhere.

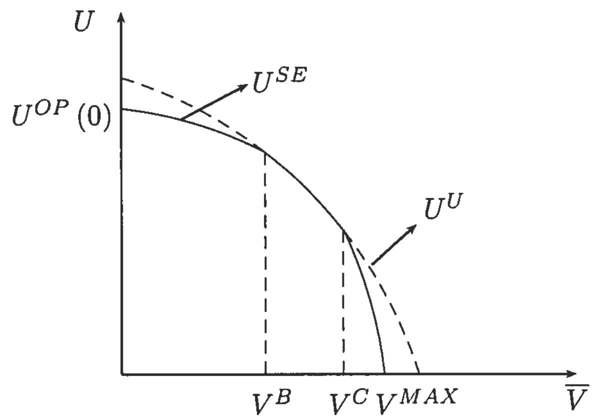

FIG. 3.8 – Pareto Frontier with self-enforcing first-best contracts

self-enforcing constraints.

If I take a look at Figure 3.8, I see that the Pareto frontier reaches the unconstrained Pareto frontier at the middle. At the extremities, self-enforcing constraints apply and no first-best contracts are possible. The gain to respect the contract is not high enough to compensate agents to accept a net transfer to the other. In extremities, a NTSEC exists but it cannot be first-best.

3.4 Variance

Thomas and Worrall [19] show that there exist 2 thresholds β_* and β^* with $0 < \beta_* < \beta^* < 1$ such that for any $\beta \in [0, \beta_*]$ the optimal contract is the TSEC; for any $\beta \in (\beta_*, \beta^*)$ the optimal contract is NTSEC but this contract is not first-best; and for $\beta \in [\beta^*, 1)$ some first-best contracts are self-enforcing. I now examine the effect of the variance on these thresholds.

To do so, I constrain our analysis to the case where agent 2 is risk-neutral. In this case, the problem can be written as

$$MAX \quad U(\delta, h_1) \tag{3.6}$$

subject to

$$
\begin{aligned}
V(\delta, h_1) &\geq \overline{V} \\
u(y_s - b_\tau(h_\tau, s)) - u(y_s) + \beta U(\delta, h_{\tau+1}) &\geq 0 \qquad \tau = 1, 2, \ldots \quad \forall s \in S, \quad \forall h_\tau \\
b_\tau(h_\tau, s) + \beta V(\delta, h_{\tau+1}) &\geq 0 \qquad \tau = 1, 2, \ldots \quad \forall s \in S, \quad \forall h_\tau
\end{aligned}
$$

Let $\mathbb{F}(F_1)$ be the set of all distribution functions for which the number of states of nature is equal to the number of states of nature of F_1 and the revenue in state s is given by $(y_s)_1 + \gamma((y_s)_1 - y)$ for $y = \mathrm{E}_s[(y_s)_1]$ and for all $\gamma > 0$. Note that for all distributions of revenue $F_2 \in \mathbb{F}(F_1)$, the expected revenue is equal to the expected revenue of F_1, in other words $\mathrm{E}_s[(y_s)_2] = \mathrm{E}_s[(y_s)_1]$. Distribution F_2 is a mean-preserving spread of distribution F_1.

Proposition 3.7 *Suppose I have two distributions of revenue, F_1 and $F_2 \in \mathbb{F}(F_1)$. Let $\overline{y}_1$ be the expected value of the revenue under F_1. Let $(\beta_*)_1$ and β_1^* be respectively the threshold to have a NTSEC and the threshold to have a first-best self-enforcing contract with the distribution of revenues F_1 and let $(\beta_*)_2$ and β_2^* be the thresholds with F_2. Then*

a) $(\beta_*)_1 > (\beta_*)_2$;

b) $\beta_1^* > \beta_2^*$.

Proof. a) : Let $\beta > (\beta_*)_1$ and δ_1 be the optimal contract. Then I have for $t = 1, 2, \ldots$, $\forall s \in S$ and $\forall h_t$,

$$
u\left((y_s)_1 - b_t^1(h_t, s)\right) - u((y_s)_1) + \beta E_s^t \left[\sum_{\tau=t}^{\infty} \beta^{\tau-t}\left[u\left(y_1^\tau - b_\tau(h_\tau, s)\right) - u\left(y_1^\tau\right)\right]\right] \geq 0
$$

By strict concavity of u, then

$$
u\left((y_s)_2 - b_t^1(h_t, s)\right) - u((y_s)_2) + \beta E_s^t \left[\sum_{\tau=t}^{\infty} \beta^{\tau-t}\left[u\left(y_2^\tau - b_\tau^1(h_\tau, s)\right) - u\left(y_2^\tau\right)\right]\right] > 0
$$

Let $\widetilde{\delta}^2$ be the contract such that $\widetilde{b}_\tau^2(h_\tau, s) = b_\tau^1(h_\tau, s) + \epsilon$ with $\epsilon > 0$. By continuity, I know there exists an ϵ such that

$$
\begin{aligned}
u\left((y_s)_2 - b_t^2(h_t, s)\right) - u((y_s)_2) + \beta U\left(\delta^2, h_{t+1}\right) &> 0 \\
b_t^2(h_t, s) + \beta V\left(\delta^2, h_{t+1}\right) &> 0
\end{aligned}
$$

Then, I can find a NTSEC for every $\beta > (\beta_*)_1$. Since u and v are strictly increasing, then $(\beta_*)_1 > (\beta_*)_2$.

b) Now, let δ_1 be the optimal first-best contract when the distribution of revenue is F_1 and $\beta = \beta_1^*$. Since δ_1 is first-best, then transfers are independent of the history. Let b_s^1 be the transfer in state s. By definition, if δ_1 is a first-best contract, the ratios of marginal utilities for agent 1 and agent 2 for each state must be equal.

$$\frac{u'\left((y_1)_2 - b_1^1\right)}{v'\left(\overline{w} + b_1^1\right)} = \frac{u'\left((y_2)_2 - b_2^1\right)}{v'\left(\overline{w} + b_2^1\right)} = \ldots = \frac{u'\left((y_S)_2 - b_S^1\right)}{v'\left(\overline{w} + b_S^1\right)}$$

If agent 2 is risk-neutral, then the first-best contract leaves agent 1 with a constant stream of net revenue,

$$(y_s)_1 - b_s^1 = (y_\sigma)_1 - b_\sigma^1 \quad \forall s, \sigma \in S \tag{3.7}$$

Let $\overline{b}_1$ be the expected value of the transfers under the distribution F_1. Consider the contract δ^2 where $b_s^2 = (1+\gamma)b_s^1$.

If I examine agent 2's self-enforcing constraints with the contract δ^2, I have that $\forall s \in S$,

$$b_s^2 + \beta E_s^t \left[\sum_{\tau=t}^{\infty} \beta^{\tau-t} b_s^2\right]$$

If I replace b_s^2 with their values, I find

$$(1+\gamma)b_s^1 + \beta E_s^t \left[\sum_{\tau=t}^{\infty} \beta^{\tau-t}(1+\gamma)b_s^1\right]$$
$$(1+\gamma)\left(b_s^1 + \beta E_s^t \left[\sum_{\tau=t}^{\infty} \beta^{\tau-t} b_s^1\right]\right)$$

Since δ_1 is self-enforcing, then .

$$(1+\gamma)\left(b_s^1 + \beta E_s^t \left[\sum_{\tau=t}^{\infty} \beta^{\tau-t} b_s^1\right]\right) \geq 0$$

If I examine agent 1's self-enforcing constraints under the distribution F_2, I have that $\forall s \in S$,

$$u\left((y_s)_2 - b_s^2\right) - u\left((y_s)_2\right) + \beta E_s^t \left[\sum_{\tau=t}^{\infty} \beta^{\tau-t}\left[u\left(y_2^\tau - b_s^2\right) - u\left(y_2^\tau\right)\right]\right]$$

If I replace $(y_s)_2$ and b_s^2 by their values, I find

$$u\left((y_s)_1 + \gamma\left((y_s)_1 - y\right) - (1+\gamma)b_s^1\right) - u\left((y_s)_1 + \gamma\left((y_s)_1 - y\right)\right)$$
$$+\beta E_s^t \left[\sum_{\tau=t}^{\infty} \beta^{\tau-t}\left[u\left(y_1^\tau + \gamma\left(y_1^\tau - y\right) - (1+\gamma)b_s^1\right) - u\left(y_1^\tau + \gamma\left(y_1^\tau - y\right)\right)\right]\right]$$

$$u\left((1+\gamma)((y_s)_1 - b_s^1) - \gamma y\right) - u\left((1+\gamma)(y_s)_1 - \gamma y\right)$$
$$+\beta E_s^t\left[\sum_{\tau=t}^{\infty}\beta^{\tau-t}\left[u\left((1+\gamma)(y_1^\tau - b_s^1) - \gamma y\right) - u\left((1+\gamma)y_1^\tau - \gamma y\right)\right]\right]$$

Because the self-enforcing constraints for agent 1 matter only when transfers are positive, which is the case when revenues are high, I concentrate my attention on those cases. Since $(1+\gamma)((y_s)_1 - b_s^1) - \gamma y < (y_s)_1 - b_s^1$ when $(y_s)_1 < y$ and $(1+\gamma)((y_s)_1 - b_s^1) - \gamma y > (y_s)_1 - b_s^1$ when $(y_s)_1 > y$, then, by the strictly concavity of u and since δ_1 is self-enforcing, I have that

$$u\left((1+\gamma)((y_s)_1 - b_s^1) - \gamma y\right) - u\left((1+\gamma)(y_s)_1 - \gamma y\right)$$
$$+\beta E_s^t\left[\sum_{\tau=t}^{\infty}\beta^{\tau-t}\left[u\left((1+\gamma)(y_1^\tau - b_s^1) - \gamma y\right) - u\left((1+\gamma)y_1^\tau - \gamma y\right)\right]\right] > 0$$

By the same argument I use in a), there exists a $\epsilon > 0$ such that the contract δ_ϵ with $b_s^\epsilon = b_s^2 + \epsilon$, which is first-best, respects the self-enforcing constraint with strict inequality. ■

When the variance increases, the gain for agent 1 to sign a contract increases since agent 1 is risk-averse. Then, the incentive is bigger for agent 1 to sign a contract. Without the assumption about the type of change in agent 1's revenue, an increase in the variance does not necessarily result in a lower threshold.[13] It could be that the increase in the tails are so large that they cannot be compensated by other states of nature. Take the following example : Suppose that there are two revenue distributions F_1 and F_2. Let p_y^i be the probability to get y under the distribution function F_i. Suppose F_1 is characterized by $p_5^1 = p_{10}^1 = 0.5$. Suppose also that $p_5^2 = p_{10}^2 = 0.495$, $p_0^2 = 0.009925$ and $p_{1000}^2 = 0.000075$. It is easy to show that the expected revenue is the same under F_1 and F_2 but the variance under F_2 is higher. The gain to break the contract when the revenue is 1000 could be positive for any possible contract and then, it is possible that, for a given discount factor β, there is a NTSEC for F_1 but not for F_2.

3.5 Merger

The question of mergers in the context of self-enforcing constraints is interesting. It has often been argued that conglomerates serve the purpose of providing insurance to shareholders. With the sophistication of financial markets, many have raised doubts about the ability of mergers for providing insurance beyond that which shareholders can get

[13] It is possible to get this kind of result for the case where agent 2 has a random revenue but the condition over the increase in the variance does not stay the same. To obtain a result in the case of random revenue for both agents, I must define some conditions on revenues of both agents.

by themselves. This is certainly true in the presence of perfect financial markets. When these markets are imperfect, however, conglomerates may play a role. A merger could potentially provide better insurance than imperfect financial markets. I examine this logic when financial imperfections are caused by commitment problems, meaning that financial contracts must be self-enforcing.

In the previous section, I show that an increase in the variance decreases the threshold beyond which it is possible to sign a NTSEC. Proposition 3.7 gives the possibility to discuss mergers of firms with perfectly correlated revenues. If two firms have perfectly correlated revenues, then the merged firm will have the same number of states of nature. By Proposition 3.7, if firm revenues are negatively correlated, then the merger decreases the variance and thresholds increase. But, since the merged firm has smoother post-merger revenue, the final effect is quite difficult to predict. In the case of perfect positive correlation, the merger increases the range of β's for which there exists a NTSEC. On the other hand, the variance of the revenues increases at the same time. Consequently, the ultimate impact of the merger on agent 1' utility is difficult to see. To get an idea about the possible outcomes, I use a numerical example.

I use a CARA function to model a risk-averse agent's utility and I suppose there are two symmetric risk-averse firms with random revenues. They have the possibility of signing a self-enforcing contract with a risk-neutral agent (the market). There are two states of nature with equal probability ($\frac{1}{2}$). In the bad state, firms get \$1 each and they get \$3 in the good state. Let the risk-aversion coefficient for both firms r equal to 1. Firm 1 has to choose between two possibilities : either stand alone to get financing, or to merge with another firm and then get financing.

3.5.1 Stand-alone case

Both firms are symmetric and thus I study the stand alone problem for one firm, say firm 1. Let x_1 and x_2 be firm 1's revenue in states 1 and 2 respectively and b_1 and b_2 the transfers. I assume that there are many risk-neutral agents. Consequently, the reservation value for them is 0 and I can write the stand-alone problem as follows :

$$MAX \quad E_s^1\left[-e^{-r(x_s-b_1(h_1,s))}+e^{-rx_s}+\beta U\left(\delta,h_2\right)\right] \tag{3.8}$$

subject to

$$\begin{aligned}
E_s^1\left[b_1\left(h_1,s\right)+\beta V\left(\delta,h_2\right)\right] &\geq 0 \\
-e^{-r(x_s-b_\tau(h_\tau,s))}+e^{-rx_s}+\beta E_s^\tau[U\left(\delta,h_{\tau+1}\right)] &\geq 0 \qquad \tau=1,2,\ldots \quad s=1,2 \quad \forall h_\tau \\
b_\tau\left(h_\tau,s\right)+\beta E_s^\tau[V\left(\delta,h_{\tau+1}\right)] &\geq 0 \qquad \tau=1,2,\ldots \quad s=1,2 \quad \forall h_\tau
\end{aligned}$$

Let U_{SA} be the expected utility for firm 1 in the stand-alone situation. I define the per period certainty equivalent (CE_{SA}) as the amount of money for which firm 1 is indifferent between this amount and its net revenue with the self-enforcing contract. In other words, the certainty equivalent in the stand-alone case is such that

$$-e^{-rCE_{SA}} = (1-\beta)\, U_{SA}$$

Table I gives U_{SA} for different values of β. The thresholds to have a NTSEC and to have a self-enforcing first-best contract are approximately $\beta_* = 0.52$ and $\beta^* = 0.76$ respectively.

TAB. I: Utility of firm 1 in the stand-alone case

β	U_{SA}	CE_{SA}	β	U_{SA}	CE_{SA}
0.20	-0.261042	1.5662	0.60	-0.460352	1.6921
0.22	-0.267735	1.5662	0.62	-0.466551	1.7300
0.24	-0.274781	1.5662	0.64	-0.475109	1.7659
0.26	-0.282207	1.5662	0.66	-0.486559	1.7992
0.28	-0.290046	1.5662	0.68	-0.501567	1.8295
0.30	-0.298333	1.5662	0.70	-0.520951	1.8561
0.32	-0.307108	1.5662	0.72	-0.545875	1.8783
0.34	-0.316414	1.5662	0.74	-0.577699	1.8958
0.36	-0.326302	1.5662	**0.76**	**-0.618338**	**1.9078**
0.38	-0.336828	1.5662	0.78	-0.669118	1.9159
0.40	-0.348055	1.5662	0.80	-0.730235	1.9238
0.42	-0.360057	1.5662	0.82	-0.805026	1.9317
0.44	-0.372917	1.5662	0.84	-0.898617	1.9395
0.46	-0.386728	1.5662	0.86	-1.019065	1.9472
0.48	-0.401602	1.5662	0.88	-1.179796	1.9549
0.50	-0.417667	1.5662	0.90	-1.404976	1.9626
0.52	**-0.435069**	**1.5662**	0.92	-1.742939	1.9702
0.54	-0.451988	1.5706	0.94	-2.306465	1.9777
0.56	-0.453389	1.6120	0.96	-3.433892	1.9852
0.58	-0.456081	1.6526	0.98	-6.816911	1.9926

Figure 3.9 graphs the certainty equivalent as a function of β. Note that there are two breakpoints. The first breakpoint is when β reaches 0.52. For all β lower than or equal to 0.52, there is no NTSEC. Agent 1 is unable to sign a contract which is non-

trivial. Consequently, the per period utility remains unchanged while β increases but the certainty equivalent for the stand-alone case does not change with the value of β.[14] For greater values, some non-trivial contracts become self-enforcing, so the value for the certainty equivalent increases. The other breakpoint arrives at $\beta = 0.76$. At this point, the optimal self-enforcing contract converges to a first-best contract.

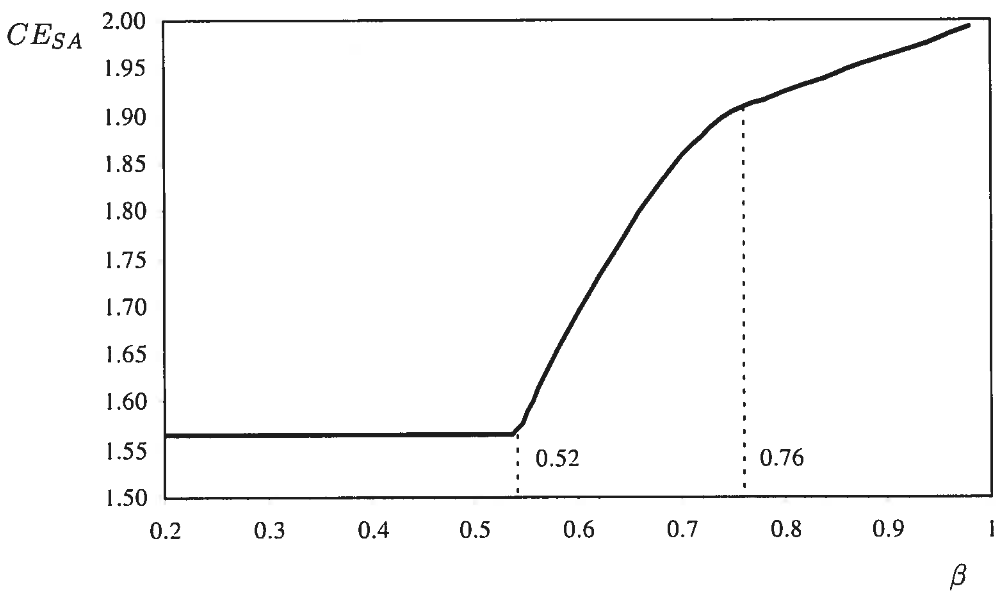

FIG. 3.9 – Certainty Equivalent

3.5.2 Merger case

The second possibility for firm 1 is to buy firm 2 by paying CE_{SA} in each period, and signing a self-enforcing contract considering that it gets the aggregate revenue. Since I have two states of nature for each firm, the merged firm will face four states of nature.

TAB. II: States of nature

	state 1	state 2	state 3	state 4
firm 1 revenue	1	3	1	3
firm 2 revenue	1	1	3	3

[14]U_{SA} changes since it's the weighted sum of present and future gains in utility.

To study the effect of correlation between firm revenues on the profitability of the merger, I need to define the coefficient of correlation ρ which is given by

$$\rho = \frac{COV(X,Y)}{\sigma_X \sigma_Y}$$

where σ_X and σ_Y are the standard error of revenues for firm 1 and firm 2 respectively.

Table III gives the probability of each state of nature for different coefficients of correlation.

TAB. III: Coefficient of correlation and states of nature

ρ	state 1	state 2	state 3	state 4
-1	0	0.5	0.5	0
-0.8	0.05	0.45	0.45	0.05
-0.5	0.125	0.375	0.375	0.125
-0.2	0.2	0.3	0.3	0.2
0	0.25	0.25	0.25	0.25
0.2	0.3	0.2	0.2	0.3
0.5	0.375	0.125	0.125	0.375
0.8	0.45	0.05	0.05	0.45
1	0.5	0	0	0.5

Since there are two states of nature for each firm and they have symmetric payoffs, the merged firm faces three different states of nature. Let $z_1 = 2$, $z_2 = 4$ and $z_3 = 6$ be the revenues in each state. Using this approach allows for a simple model in which I can analyze the effect of correlation between firm revenues.

Let $b_t\left(h_t, s\right))$ be the transfer for period t in state s.[15] I suppose that the per period cost of acquiring firm 2 is its certainty equivalent (CE_{SA}). Then, the problem of the merged firm[16] is

$$MAX \quad E_s^1\left[-e^{-r(z_s-b_1(h_1,s))-CE_{SA})}+e^{-r(z_s-CE_{SA})}+\beta U\left(\delta,h_2\right)\right] \tag{3.9}$$

[15]Because there are more than 2 states of nature, the stationary contract is dependent on the history.

[16]With CARA utility functions, the payment of CE_{SA} does not affect the resolution of the problem. It is possible to isolate $e^{rCE_{SA}}$ in the objective function and in the firm self-enforcing constraints. Then, $e^{rCE_{SA}}$ affects only the utility but not the optimal contract itself.

subject to

$$E_s^1\left[b_1\left(h_1,s\right)\right) + \beta V\left(\delta,h_2\right)] \;\geq\; \overline{V}$$

and $\tau = 1,2,...,s = 1,2,3$ and $\forall h_\tau$,

$$\begin{aligned} -e^{-r(z_s-b_\tau(h_\tau,s))-CE_{SA})} + e^{-r(z_s-CE_{SA})} + \beta E_s^\tau[U\left(\delta,h_{\tau+1}\right)] &\;\geq\; 0 \\ b_\tau\left(h_\tau,s\right)) + \beta E_s^\tau[V\left(\delta,h_{\tau+1}\right)] &\;\geq\; 0 \end{aligned}$$

The expected utility of the merged firm is given by U_M.

TAB. IV: Net gain of utility from the merger (positive value in bold)

β	$\rho=-1$	$\rho=-0.8$	$\rho=-0.5$	$\rho=-0.2$	$\rho=0$	$\rho=0.2$	$\rho=0.5$	$\rho=0.8$	$\rho=1$
0.20	**0.151**	**0.121**	**0.076**	**0.030**	**0**	-0.030	-0.076	-0.121	-0.151
0.22	**0.155**	**0.124**	**0.078**	**0.031**	**0**	-0.031	-0.078	-0.124	-0.155
0.24	**0.159**	**0.128**	**0.080**	**0.032**	**0**	-0.032	-0.080	-0.128	-0.158
0.26	**0.164**	**0.131**	**0.082**	**0.033**	**0**	-0.033	-0.082	-0.125	-0.143
0.28	**0.168**	**0.135**	**0.084**	**0.034**	**0**	-0.034	-0.084	-0.111	-0.127
0.30	**0.173**	**0.138**	**0.087**	**0.035**	**0**	-0.035	-0.073	-0.097	-0.110
0.32	**0.178**	**0.143**	**0.089**	**0.036**	**0**	-0.034	-0.060	-0.081	-0.093
0.34	**0.184**	**0.147**	**0.092**	**0.037**	**0**	-0.023	-0.046	-0.065	-0.076
0.36	**0.189**	**0.151**	**0.095**	**0.038**	**0.006**	-0.012	-0.032	-0.048	-0.058
0.38	**0.195**	**0.156**	**0.098**	**0.039**	**0.016**	**0.001**	-0.017	-0.031	-0.039
0.40	**0.202**	**0.162**	**0.101**	**0.045**	**0.027**	**0.014**	-0.001	-0.014	-0.021
0.42	**0.209**	**0.167**	**0.104**	**0.055**	**0.040**	**0.029**	**0.015**	**0.005**	-0.002
0.44	**0.216**	**0.173**	**0.108**	**0.066**	**0.053**	**0.044**	**0.032**	**0.023**	**0.018**
0.46	**0.224**	**0.180**	**0.112**	**0.079**	**0.067**	**0.059**	**0.049**	**0.038**	**0.037**
0.48	**0.233**	**0.186**	**0.117**	**0.092**	**0.083**	**0.076**	**0.068**	**0.055**	**0.056**
0.50	**0.242**	**0.194**	**0.127**	**0.107**	**0.099**	**0.093**	**0.078**	**0.072**	**0.075**
0.52	**0.252**	**0.202**	**0.139**	**0.122**	**0.116**	**0.111**	**0.092**	**0.088**	**0.094**
0.54	**0.261**	**0.208**	**0.149**	**0.136**	**0.131**	**0.116**	**0.102**	**0.101**	**0.109**
0.56	**0.245**	**0.187**	**0.132**	**0.121**	**0.117**	**0.094**	**0.080**	**0.082**	**0.093**
0.58	**0.228**	**0.166**	**0.115**	**0.106**	**0.087**	**0.070**	**0.057**	**0.061**	**0.075**
0.60	**0.212**	**0.143**	**0.098**	**0.092**	**0.062**	**0.044**	**0.029**	**0.037**	**0.055**
0.62	**0.195**	**0.120**	**0.082**	**0.061**	**0.034**	**0.015**	**0.001**	**0.045**	**0.035**
0.64	**0.178**	**0.096**	**0.067**	**0.034**	**0.004**	-0.019	-0.031	**0.024**	**0.015**
0.66	**0.161**	**0.071**	**0.053**	**0.004**	-0.029	-0.053	**0.012**	**0.005**	-0.004
0.68	**0.145**	**0.051**	**0.040**	-0.029	-0.066	-0.091	**0.001**	-0.014	-0.023

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β	$\rho=-1$	$\rho=-0.8$	$\rho=-0.5$	$\rho=-0.2$	$\rho=0$	$\rho=0.2$	$\rho=0.5$	$\rho=0.8$	$\rho=1$
0.70	**0.130**	**0.034**	**0.031**	-0.063	-0.104	**0.001**	-0.016	-0.031	-0.039
0.72	**0.118**	**0.021**	-0.011	-0.100	**0**	-0.014	-0.031	-0.044	-0.052
0.74	**0.109**	**0.013**	-0.037	-0.138	-0.012	-0.025	-0.041	-0.054	-0.062
0.76	**0.104**	**0.012**	-0.063	-0.003	-0.018	-0.031	-0.046	-0.058	-0.065
0.78	**0.104**	**0.016**	-0.090	-0.006	-0.020	-0.032	-0.046	-0.058	-0.065
0.80	**0.103**	**0.021**	-0.123	-0.008	-0.022	-0.033	-0.047	-0.058	-0.063
0.82	**0.103**	**0.027**	-0.165	-0.011	-0.024	-0.035	-0.047	-0.057	-0.062
0.84	**0.102**	**0.034**	**0.013**	-0.014	-0.026	-0.036	-0.047	-0.057	-0.061
0.86	**0.102**	-0.010	**0.009**	-0.017	-0.028	-0.038	-0.048	-0.056	-0.061
0.88	**0.102**	-0.050	**0.004**	-0.020	-0.031	-0.039	-0.049	-0.056	-0.060
0.90	**0.101**	-0.107	-0.002	-0.024	-0.034	-0.041	-0.049	-0.055	-0.057
0.92	**0.101**	**0.033**	-0.009	-0.029	-0.037	-0.043	-0.050	-0.055	-0.058
0.94	**0.101**	**0.023**	-0.017	-0.034	-0.041	-0.046	-0.051	-0.055	-0.057
0.96	**0.100**	**0.008**	-0.027	-0.040	-0.045	-0.048	-0.052	-0.055	-0.056
0.98	**0.100**	-0.015	-0.039	-0.046	-0.049	-0.051	-0.053	-0.054	-0.055

Table V gives the value of the thresholds for each value of ρ. In the previous section, I find that the thresholds β_* and β^* must decrease (increase) while variance increases (decreases). Since the variance increases with the correlation coefficient, I have that thresholds decrease with ρ. These findings confirm the results of Proposition 3.7.

TAB. V: Thresholds for NTSEC

β	$\rho=-0.8$	$\rho=-0.5$	$\rho=-0.2$	$\rho=0$	$\rho=0.2$	$\rho=0.5$	$\rho=0.8$	$\rho=1$	stand-alone
β_*	0.68	0.48	0.40	0.36	0.32	0.30	0.26	0.26	0.52
β^*	0.92	0.82	0.76	0.72	0.69	0.65	0.60	0.59	0.76

3.5.3 Results

Figures 3.10 and 3.11 show the differences in utility levels between the merger case with different correlations and the stand-alone case.

To analyze the effect of a merger, consider four cases : the perfect negative correlation case ($\rho=-1$), the negative (non perfect) correlation case ($\rho=-0.5$), the no correlation case ($\rho=0$) and the positive (non perfect) correlation case ($\rho=0.8$).

Case 1 : The case of perfect negative correlation is represented by $\rho=-1$. This situation

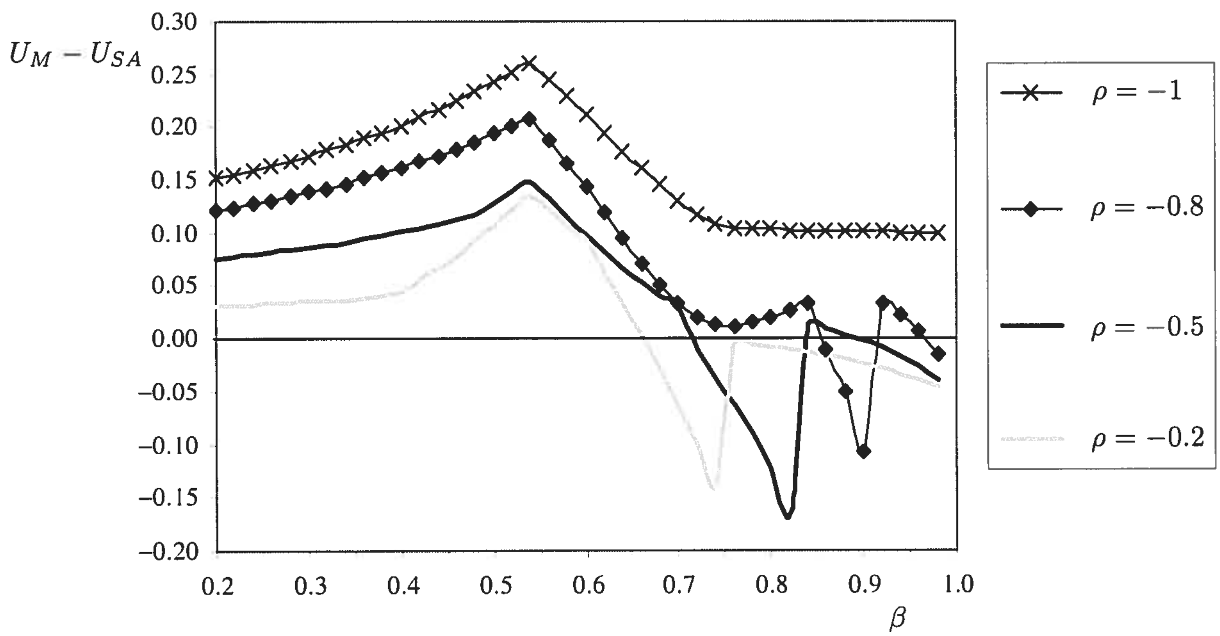

FIG. 3.10 – Negative Correlation and Merger

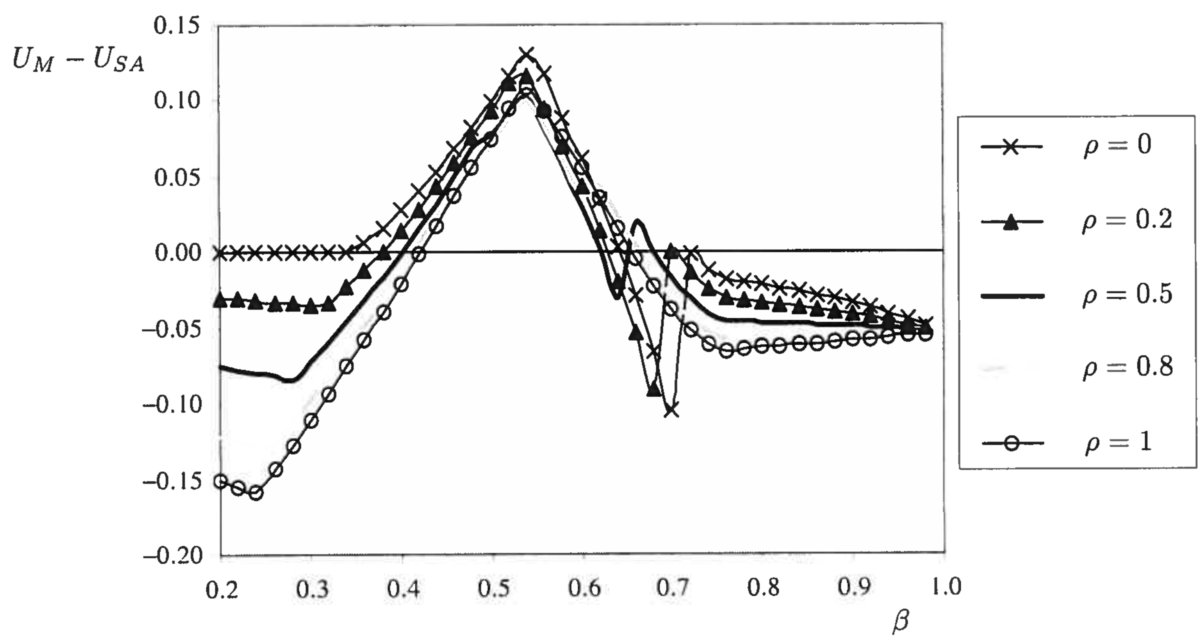

FIG. 3.11 – Positive Correlation and Merger

could arise when one firm has contracyclical revenues relative to the other one. Figure 3.12 shows the certainty equivalent in the stand-alone case CE_{SA} and in the merger case CE_M with $\rho = -1$.

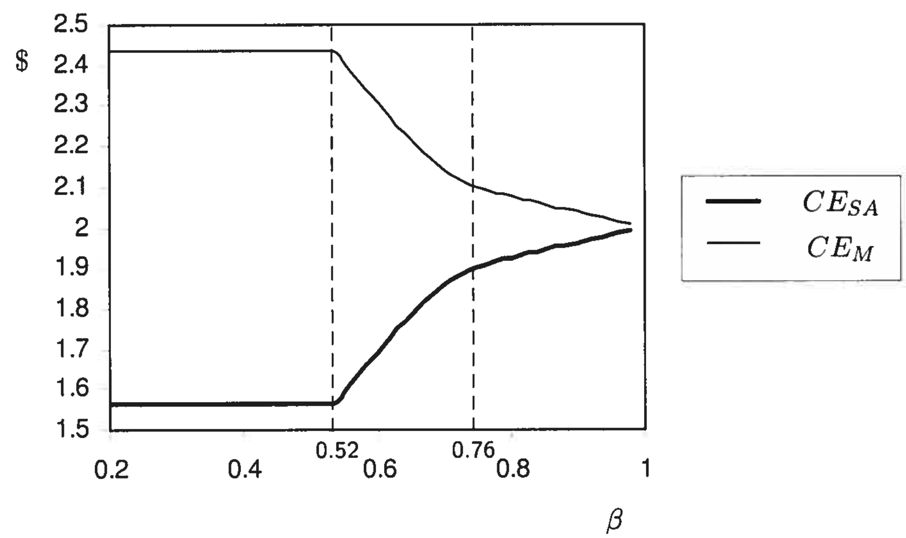

FIG. 3.12 – Certainty Equivalent for the stand-alone case and the merger case with $\rho = -1$

Note that the certainty equivalent in both cases have the same form but inverse. This particularity comes from the fact that, in the perfect negative correlation case, the firm revenue is constant for any given β. Consequently, there is no gain to sign a self-enforcing contract. However, CE_M is decreasing since firm 1 must pay CE_{SA} to firm 2. Since CE_{SA} depends on the value of β, the certainty equivalent for the merger case is decreasing with β but always greater than the certainty equivalent of the stand-alone case.

Case 2 : When revenues are negatively, but not perfectly, correlated ($\rho = -0.5$), the benefit associated with a merger can be positive or negative depending on the value of β.

If β is lower than 0.48, there is no NTSEC for either the merged firm or the stand-alone firm, as there is for the stand-alone firm. But, the merged firm has a smoother revenue stream which leaves the firm with a gain by merging (see Figure 3.13). When β is between 0.48 and 0.52, it becomes possible for the merged firm to sign a NTSEC. The relative gain in utility becomes more important. At $\beta = 0.52$, it is possible for the stand-alone firm to sign a NTSEC. So the gain resulting from merging decreases and becomes negative at $\beta = 0.72$. For $\beta > 0.82$, it becomes possible for the merged firm to sign a first-best convergent contract. So the gain increases again with β but

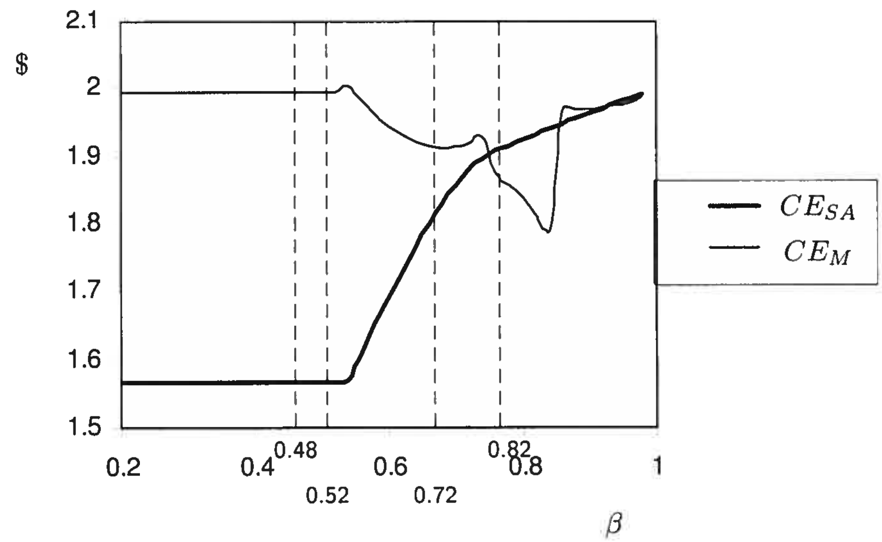

FIG. 3.13 – Certainty Equivalent for the stand-alone case and the merger case with $\rho = -0.5$

there is a threshold for which the gain cannot overcome the first-best convergent contract gain in the stand-alone case. After a small range of values for β (between 0.84 and 0.88) for which the merged firm gains, the net gain decreases and becomes negative.

What happens when β is close to 1 is another interesting case to study. When β is high enough, the merged firm and the stand-alone firm can sign a first-best convergent contract. Then, why does the merger appear non-profitable for β close to 1? First, by Proposition 3.3, if $\beta > \beta^*$, then the optimal contract converges monotonically to a first-best contract. Since I use the assumption that the reservation utility level for the market is equal to zero, the optimal contract, in both cases, converges to the first-best contract satisfying the self-enforcing constraints of the market. Let's suppose that δ^{SA} and δ^{M} are those first-best contracts. Then,

$$\begin{aligned} b_1^{SA} + \frac{\beta}{1-\beta} E_s[b_s^{SA}] &= 0 \\ b_1^{M} + \frac{\beta}{1-\beta} E_s[b_s^{M}] &= 0 \end{aligned}$$

where b_s^{SA} and b_s^{M} are respectively the transfer in state s for the stand-alone case and the merged case. I have already found (Equation (3.5)) that $b_z^i = b_s^i + y_z^\tau - y_s^t$ for $i = SA, M$. If I introduce these equations into the market self-enforcing constraints, I find :

$$\begin{aligned} b_1^{SA} + \frac{\beta}{1-\beta} E_s[b_1^{SA} + 10000 - x_s] &= 0 \\ b_1^{M} + \frac{\beta}{1-\beta} E_s[b_1^{SA} + 20000 - z_s] &= 0 \end{aligned}$$

and

$$\begin{aligned} b_1^{SA} + \frac{\beta}{1-\beta} (b_1^{SA} + 10000 - 20000]) &= 0 \\ b_1^{M} + \frac{\beta}{1-\beta} (b_1^{SA} + 20000 - 40000]) &= 0 \end{aligned}$$

I obtain that $b_1^M = 2b_1^{SA}$. This means that, once we subtract the CE_{SA}, the merged entity obtains the same level of utility than the stand-alone firm. Consequently, the optimal contracts of the merger case and the stand-alone case converge to first-best contracts that give the same level of utility.

Second, I know that optimal contracts are not first-best. They converge to some first-best contracts, but before state 1 is realized (see Section 3.3), transfers do not satisfy (3.5). Until then, the stand-alone firm gain more than the merged firm. Because of the concavity of CARA utility functions, the expected gain for being in the good state (state 2 for the stand-alone case and state 3 for the merger case) is higher in the stand-alone situation. It is therefore better for the firm to stand alone than to merge. This result applies to all cases where the correlation coefficient is not -1.[17]

Case 3 : The independent case ($\rho = 0$) characterizes firms involved in different markets which are neither complements nor substitutes. In this case, there is no gain from merging when β is lower than 0.36. At this point, the merged firm can sign a NTSEC which leaves the firm better off. As for other cases, when β reaches 0.52, the gain from merging decreases. When β reaches 0.66, the net gain to merge becomes negative and remains negative while β increase. At $\beta = 0.72$, the merged firm can sign a first-best contract and the gain from merging increases but it is counterbalanced by the stand-alone contracting gain (see Figure 3.14).

Case 4 : The case where firms produce complements is represented by a positive correlation. With positive correlation ($\rho = 0.8$), the net gain from merging is negative for $\beta < 0.26$ (see Figure 3.15). At $\beta = 0.26$, the merged firm signs a NTSEC and the gain starts to increase. For β between 0.52 and 0.60, the gain diminishes as the stand-alone firm signs a NTSEC. For $\beta > 0.60$, the merged firm can sign a first-best contract. Consequently, the gain from the optimal first-best contract increases but

[17] When the correlation coefficient goes to -1, then the value of β such that to stand alone is better increases. For example, when $\rho = -0.9$, to stand alone is better when β is higher than 0.99.

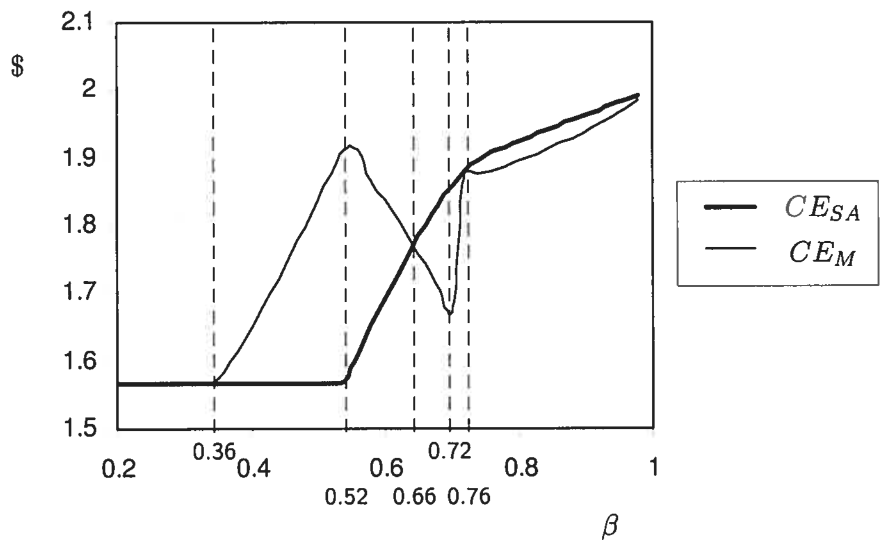

FIG. 3.14 – Certainty Equivalent for the stand-alone case and the merger case with $\rho = 0$

the benefit to sign a contract for the stand-alone firm becomes more important, so the merger leaves more profits. Even with positively correlated revenues, there is an interval of β (in this case between 0.42 and 0.66) for which a merger could be profitable for the merged firm.

I can use the analysis I have from these different cases to draw general conclusions for the question of merger in a self-enforcing environment. If revenues are nearly perfectly negatively correlated, then the merger allows the new owner to smooth its revenues across time without any contract. This situation leads to the agent always being better off merging.

What is interesting is the influence of the correlation on the gain of a merger. When revenues are negatively correlated, the merger creates a kind of internal insurance market. The smoother revenue schedule leads to a gain in utility by decreasing the variance of revenues but decreases the possible gain from signing an insurance contract with the market. If *beta* is high but not too close of 1, then the merger could be beneficial. Take the case where $\rho = -0.8$. The merger option leaves the merged firm with gain when β is greater than 0.92 but smaller than 0.98. For all $\rho > -1$, then there exists a $\tilde{\beta} < 1$ such that for all $\beta \in [\tilde{\beta}, 1)$, then to stand alone is better for shareholders.

With no correlation, the new owner has the possibility of signing a contract in the case where β is small. Since the variance has increased, the possibility to sign a NTSEC has increased. But, the agent may do better in the stand-alone case depending of the value of β. As ρ goes to 1 (positive correlation), the threshold for having a self-enforcing contract

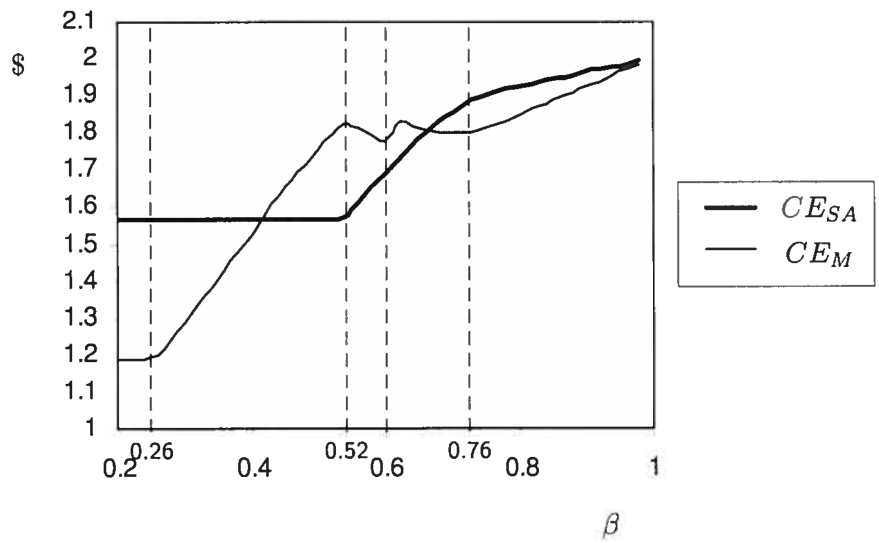

FIG. 3.15 – Certainty Equivalent for the stand-alone case and the merger case with $\rho = 0.8$

decreases but it is possible that the gain from the contract cannot compensate the cost stemming from the increase of variance. So in the end, the agent is worse over for the majority of values of β.

3.6 Conclusion

In the first part of the paper, I explicitly solve the contract design problem with self-enforcing constraints. To obtain this solution, I must impose additional constraints on the model. The most important one is on the number of states of nature. The two states of nature problem is relatively easy to solve since there are only two transfers in the stationary contract. With three states, the number of transfers increases to four, and with four states, the number of transfers in the stationary contract is eight. The number of transfers in the stationary contract increases more quickly than the number of states of nature.

In the second part, I find that variance affects the nature of the contract. If the variance increases, then the potential benefits with respect to the contract increases and the threshold to have a NTSEC decreases.

The most interesting finding is the effect of self-enforcing constraints on the effects of a merger. I find that, even with a very high positive correlation between firms' revenues, there is some discount value for which firms could gain by a merger. The most important parameter in the merger decision seems to be the discount factor. If owners are not really patient, then a merger could lead to an increase in utility. This could explain in part why firms in the same market merge together while their revenues are highly positively correlated.

One of the possible avenues for future research would be to test the sensibility of these results to a change in the risk-aversion coefficient. My guess is that it will not change the scheme of the results but the level of thresholds.

3.7 References

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Conclusion

Dans cette thèse, j'ai étudié trois problématiques reliées à la théorie microéconomique. J'ai abordé un problème de stratégie optimale pour une firme multiproduit, la question de rationalisation dans le cadre d'allocation de biens indivisibles et le design de contrats d'assurance en présence de contraintes auto-exécutoires.

Dans le premier chapitre, j'ai modélisé un monopoleur multiproduit ayant comme stratégies possibles le fait de monitorer ou non. Tout d'abord, avec des hypothèses relativement standard utilisées dans la littérature en organisation industrielle[18], je réussis à démonter l'existence d'une solution. Cependant, la question de l'unicité de cette solution n'est étudiée pas en profondeur. Il semble que la solution est unique pour presque toutes les fonctions et pour presque tout nombre de consommateurs. Le second résultat important est la présence d'au moins un contrat non-monitoré dans l'ensemble des contrats optimaux. Ce résultat tient pour n'importe quelle forme fonctionnelle de la fonction de coûts d'administration.

Dans de prochains travaux sur le sujet, l'impact sur le bien-être des consommateurs devrait être étudié. Ce point peut devenir très intéressant dans le cadre de monopoleur étatique ou régulé. Pour le moment, il me semble que l'impact est très difficile à prévoir. Une autre extension possible devrait se faire au niveau d'un marché oligopolistique. Cependant, l'étude de la question du monitoring pour des firmes en compétition semble très complexe puisque la multidimensionnalité des préférences, le nombre, le type de contrats et la composition des contrats rendent le problème très complexe. L'utilisation de simulations numériques pourrait rendre cette étude possible.

Pour le second chapitre, j'ai analysé la rationalisation des préférences des agents dans le cadre d'allocations de biens indivisibles. Puisque le nombre de sous-ensembles d'allocations et le nombre de profils de préférence est trop grand lorsque le nombre de biens est supérieur à 3, j'ai utilisé la notion de cycle pour étudier la question. Dans un premier temps, je trouve que l'existence d'un cycle dans l'ensemble des optimums de Pareto nous informe sur les préférences des agents qui composent le cycle. Ces derniers, pour chaque pair de biens qui sont des voisins immédiats dans le cycle, ont les mêmes préférences. De plus, si le nombre d'agents qui composent le cycle est un nombre premier, alors tous les agents de ce

[18]La seule hypothèse nouvelle que j'introduis est le concept de Δ_u-*monotonicity*.

cycle ont les mêmes préférences sur l'ensemble des biens du cycle. Dans un second temps, je détermine le nombre minimal de cycles qui implique la présence d'un cycle complet. Ce résultat peut être utile pour déterminer la présence nécessaire d'allocations dans un cycle. Dans un troisième temps, je trouve des contraintes sur le nombre d'allocations que l'ensemble des optimums de Pareto doit contenir.

Le troisième chapitre traite principalement de la question des fusions en présence de contraintes auto-exécutoires. Dans la première partie, j'utilise des fonctions d'utilité de type CARA pour solutionner explicitement le contrat optimal. Je peux également tracer les frontières de Pareto dans les différents cas où un contrat de type *first best* peut être auto-excécutoire ou non. La seconde partie s'attarde à l'impact d'une augmentation de la variance sur le contract optimal. Plus précisément, j'étudie le comportement des seuils pour avoir un contrat non-trivial, de type *first best* ou non, suite à un changement dans les revenus de la firme. Suite à un certain type de changement dans la distribution de revenus qui augmentent la variance, les seuils diminuent alors qu'il n'est pas possible de conclure pour d'autres types de changement. Finalement, la troisième partie s'intéresse à la question des fusionnements dans le contexte de contrats auto-excécutoires. Je trouve que les fusions peuvent augmenter le bien-être même lorsque les revenus des firmes sont positivement corrélés. Ce résultat provient du fait que l'augmentation de la variance fait en sorte qu'il devient possible pour la firme fusionnée, sous certaines valeurs de β, de signer un contrat non-trivial alors qu'il est impossible pour la firme non-fusionnée de le faire. Cependant, pour qu'une fusion soit bénéfique, il faut que le taux d'escompte soit relativement bas.

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