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Génération d'états cohérents et comprimés pour des algèbres et superalgèbres de symétrie de systèmes quantiques

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À mes parents,

mes enfants

Nibaldo et Vicente

et leur mère María

et toute ma famille

Sommaire

L'objet de ce travail est de générer des états cohérents et comprimés associés à des algèbres et superalgèbres de Lie de symétrie de systèmes quantiques caractérisés par une équation d'évolution temporelle et un Hamiltonien.

La méthode que nous suivons pour obtenir tels états consiste à déterminer, en premier lieu, l'ensemble des états propres d'algèbres ou de superalgèbres, définis comme l'ensemble d'états propres de combinaisons linéaires complexes des générateurs de ces algèbres ou superalgèbres. Ensuite, en ce qui concerne les algèbres, nous choisissons parmi ces combinaisons, des paires d'opérateurs hermitiens et nous les connectons avec le concept d'états minimaux d'incertitude, pour construire ainsi les états cohérents et comprimés generalisés désirés. Par ailleurs, autant pour les algèbres que les superalgèbres, nous choisissons parmi ces combinaisons, les opérateurs d'annihilation généralisés d'une classe determinée de systèmes physiques et nous construisons les états cohérents ou supercohérents associés.

Nous étendons les considérations précédentes à la génération d'états cohérents et comprimés associés à des groupes quantiques ou algèbres de Hopf déformées obtenus en appliquant la méthode de la matrice R, la matrice universelle vérifiant l'équation quantique de Yang-Baxter.

Parmi les systèmes quantiques dont nous nous servons pour illustrer les concepts précédents, se trouvent les oscillateurs harmoniques standard et supersymétrique, les systèmes de Pauli, de Jaynes-Cummings et une extension supersymétrique de ce dernier.

En considérant leurs algèbres de symétrie, nous contruisons de nouvelles classes d'états cohérents généralisant une gamme importante de tels états obtenus dans la littérature, comme les états cohérents standards de $su(2)$, les états super-cohérents d'Aragone et Zipmann ainsi que les états supercohérents associés à l'oscillateur harmonique supersymétrique. Parmi les classes d'états comprimés généralisés que nous obtenons, nous comptons une classe d'états minimaux d'incertitude associés à la superalgèbre orthosymplectique $osp(2/2)$.

D'autre part, en nous basant sur la construction d'opérateurs d'annihilation appropriés, nous définissons de nouveaux Hamiltoniens Hermitiens, η–pseudo-Hermitiens et η–pseudo-super-Hermitiens isospectraux avec l'Hamiltonien de l'oscillateur harmonique standard, en plus d'autres Hamiltoniens que nous avons appelés canoniques et non canoniques. Les états cohérents associés à ces Hamiltoniens ont aussi été calculés.

Finalement, nous appliquons les considérations précédentes pour obtenir les états propres d'algèbres associés à des algèbres quantiques déformées de Heisenberg. Nous obtenons ainsi de nouvelles classes d'états cohérents et comprimés déformés associés à l'oscillateur harmonique standard.

Mots Clés

Symétries et supersymétries des systèmes quantiques, algèbre et superalgèbre, états propres, états cohérents et comprimés généralisés, groupes quantiques déformés, Hamiltoniens η–pseudo-Hermitiens.

Abstract

The goal of this work is to generate coherent and squeezed states associated to Lie algebra and superalgebra of symmetries of quantum systems characterized by a temporal evolution equation and a Hamiltonian.

The method we follow to obtain such states consists to determine, firstly, the set of algebra or superalgebra eigenstates, defined as the set of eigenstates of complex linear combinations of generators of these algebras or superalgebras. Afterward, with respect to the algebras, we choose between these combinations, pairs of hermitian operators and we connect them with the minimum uncertainty states concept, to construct the desired coherent and squeezed states. On the other hand, for both the algebras and superalgebras, we choose between these combinations, the generalized annihilation operators of a determined class of physical systems and we construct the associated coherent or supercoherent states.

We extend these considerations to generate coherent and squeezed states associated to quantum groups or deformed Hopf algebras obtained by applying the R-matrix method, the universal matrix satisfying the quantum Yang-Baxter equation.

Among the quantum systems that help to illustrate the preceding concepts, we consider the standard and supersymmetric harmonic oscillators, the Pauli and the Jaynes-Cummings systems and a supersymmetric extension of this last.

Considering their symmetry algebras, we construct new classes of coherent states that generalize an important class of such states obtained in the literature, as the standard $su(2)$ coherent states, the Aragone and Zipmann's super-coherent states and also the supercoherent states associated to the supersymmetric harmonic oscillator. Among the classes of squeezed states that we obtain, we have a class of minimum uncertainty states associated to the ortho-symplectic superalgebra $osp(2/2)$.

On the other hand, based on the construction of suitable annihilation operators, we define the new Hermitian, η–pseudo-Hermitian and η–pseudo-super-Hermitian Hamiltonians,

isospectral to the standard harmonic oscillator in addition to the Hamiltonians that we have called canonical and non-canonical. The coherent states associated to these Hamiltonians are also computed.

Finally, we apply the preceding considerations to obtain the algebra eigenstates associated to the deformed quantum Heisenberg algebra. In this way, we get the new classes of deformed coherent and squeezed states associated to the standard harmonic oscillator.

Key words

Symmetries and supersymmetries of quantum systems, algebra and superalgebra eigenstates, generalized coherent and squeezed states, deformed quantum groups, η–pseudo-Hermitian Hamiltonians.

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Introduction

Le système d'états non orthogonaux introduits par Schrödinger[1] en 1926 pour décrire les paquets d'ondes non étendus associés aux oscillateurs quantiques a été repris comme objet d'étude, entre autres, par Klauder[2, 3] et Glauber[4, 5]. C'est justement ce dernier qui l'a dénommé système d'états cohérents pour son utilité, en théorie quantique, dans la description d'un faisceau cohérent de lumière laser. En mécanique quantique, les états cohérents associés au système formé d'une particule de masse m soumise à l'action d'un potentiel quadratique, appelé oscillateur harmonique standard, décrit par l'Hamiltonien $H_0 = (p^2/2m) + (kx^2/2)$, où x et p représentent les opérateurs de position et impulsion de la particule, respectivement, et k la constante de couplage, peuvent être définis de trois façons équivalentes:

a) Ils sont formés des états propres de l'opérateur d'annihilation a, défini par la relation $a = (mwx + ip)/\sqrt{2mw\hbar}$ où $\hbar$ est la constante de Planck et $w = \sqrt{k/m}$ est la fréquence angulaire. En fait, les opérateurs x and p agissent sur les fonctions $|\psi\rangle$ de l'espace d'Hilbert standard $\mathcal{H}_0$, qui représentent les états du système, et vérifient les relations de commutation d'Heisenberg-Weyl $[x,p] = i\hbar I$, où I est l'opérateur identité. En termes des opérateurs d'annihilation a et de création $a^\dagger$ (l'adjoint de a), l'Hamiltonien H_0 prend la forme $H_0 = \hbar w(a^\dagger a + 1/2)$ et les relations de commutation de Heisenberg-Weyl deviennent $[a,a^\dagger] = I$. En vertu de cette relation de commutation et du fait que $[H_0,a] = -\hbar w a$, si $\{|n\rangle\}_{n=0}^{\infty}$ dénote l'ensemble d'états propres orthonormaux d'énergie, dans la représentation de Fock, tels que $H_0|n\rangle = E_n|n\rangle$, avec $E_n = \hbar w(n + 1/2)$, alors l'action de l'annihilateur a sur les états propres d'énergie est donnée par $a|n\rangle = \sqrt{n}|n-1\rangle$, en particulier $a|0\rangle = 0$. De la même façon, on trouve que l'action de l'opérateur $a^\dagger$ sur ces états est donnée par $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$. C'est ainsi que, si l'on veut connaître les états propres de a, i.e., les solutions de l'équation aux valeurs propres $a|z\rangle = z|z\rangle$, on peut insérer la solution du type $|z\rangle = \sum_{n=0}^{\infty} c_n|n\rangle$, $c_n \in \mathbb{C}$, $n = 0,1,\ldots$, dans cette équation et obtenir, après quelques manipulations, les états cohérents normalisés $|z\rangle = \exp(-|z|^2/2)\exp(za^\dagger)|0\rangle$.

b) Ils sont aussi associés à l'orbite de l'état fondamental $|0\rangle$ sous l'action d'un opérateur unitaire de déplacement

$$D(z) = \exp(za^\dagger - \bar{z}a). \tag{1}$$

En effet, en utilisant la formule de Baker-Campbell-Hausdorff (BCH)[6], il est aisé de démontrer que $D(z)|0\rangle = \exp(-|z|^2/2)\exp(za^\dagger)\exp(\bar{z}a)|0\rangle$, et comme $a|0\rangle = 0$, on récupère le résultat obtenu en a).

c) Enfin, ils constituent les états minimaux d'incertitude, i.e., les états qui minimisent la relation d'incertitude de Heisenberg-Weyl (RIHW),

$$(\Delta x)^2(\Delta p)^2 \geq \frac{1}{4}[\langle(-i[x,p])\rangle]^2 = (\hbar^2/4). \tag{2}$$

La dispersion $(\Delta x)^2$ est définie par $(\Delta x)^2 = \langle\psi|x^2|\psi\rangle - \langle\psi|x|\psi\rangle^2$, où $\langle\psi|x|\psi\rangle$ représente la valeur moyenne de l'opérateur x dans l'état $|\psi\rangle$ et une définition similaire pour la valeur moyenne de l'opérateur x^2. On peut, en effet, facilement démontrer que les états $|\psi\rangle$ qui satisfont $a|\psi\rangle = z|\psi\rangle$, $z \in \mathbb{C}$, vérifient l'égalité dans la RIHW. De plus, on a $(\Delta x)^2 = (\Delta p)^2 = \hbar/2$.

Pour des systèmes quantiques arbitraires, caractérisés par l'Hamiltonien H et l'équation d'évolution temporelle du type Schrödinger $i\hbar\partial_t|\psi\rangle = H|\psi\rangle$, il existe des généralisations du concept d'états cohérents, non tout-à-fait équivalentes, qui consistent essentiellement en des extensions des trois définitions précédentes accompagnées d'interprétations physiques plausibles. Si $\mathcal{L}_0$ représente l'algèbre de Lie des symétries dynamiques de cette équation ou une sous-algèbre de celle-ci, et G le groupe local correspondant obtenu en exponentiant l'algèbre $\mathcal{L}_0$, alors on appelle système d'états cohérents généralisés, l'ensemble des états $\{|\psi\rangle\}$, dans l'espace d'Hilbert $\mathcal{H}$, qui s'ajuste à l'une des trois définitions suivantes:

d) Les états propres d'un opérateur d'annihilation $\mathcal{A}$ du système. Par exemple, lorsque H a un spectre discret et ses états propres sont denotés par $|E_n\rangle$, $n = 0,1,\ldots$, l'opérateur $\mathcal{A}$, à une transformation unitaire près, peut être défini[7] à travers la relation de commutation $[H,\mathcal{A}] = -\mathcal{A}\mathcal{N}$, où $\mathcal{N}$ est un opérateur tel que

$$\mathcal{N}|E_n\rangle = (E_n - E_{n-1})|E_n\rangle. \tag{3}$$

e) A la façon de Perelomov[8], on a la définition groupe théorique. L'ensemble des états $\{|\psi_g\rangle,\, g \in G\}$, tels que $|\psi_g\rangle = T(g)|\psi_0\rangle$, où $T(g)$ est une représentation unitaire et irréductible du groupe G, agissant sur les états de l'espace d'Hilbert $\mathcal{H}$, et $|\psi_0\rangle$ est un état fixé dans cet espace. Si $\tilde{H}$ est le sous-groupe (maximal) d'isotropie pour l'état $|\psi_0\rangle$, i.e., $T(h)|\psi_0\rangle = e^{i\alpha(h)}|\psi_0\rangle$, $\alpha(h) \in \mathbb{R}$, $\forall h \in \tilde{H}$, alors un état cohérent $|\psi_g\rangle$ est déterminé par un point $x = x(g)$ dans l'espace quotient $G/\tilde{H}$, correspondant à l'élément g : $|\psi_g\rangle = e^{i\alpha}|x\rangle, |\psi_0\rangle = |0\rangle$. Les états cohérents généralisés dépendent essentiellement du choix de l'état $|\psi_0\rangle$. Le problème revient donc à choisir l'état $|\psi_0\rangle$ pour engendrer des états de l'espace d'Hilbert $\mathcal{H}$, les plus proches des états classiques. Pour cela, on doit étendre l'algèbre de Lie $\mathcal{L}_0$ du groupe G à l'algèbre complexe $\mathcal{L}_0^c$ et considérer dans $\mathcal{L}_0^c$ la sous-algèbre d'isotropie $\mathcal{B}$ pour l'état $|\psi_0\rangle$, i.e., l'ensemble des éléments τ de $\mathcal{L}_0^c = \mathcal{L}_0 \oplus i\mathcal{L}_0$ qui satisfont $\tau|\psi_0\rangle = \lambda_\tau|\psi_0\rangle$, $\lambda_\tau \in \mathbb{C}$. Les états cohérents engendrés à partir des états $|\psi_0\rangle$ dont la sous-algèbre $\mathcal{B}$ est maximale sont les plus proches des états classiques ($\mathcal{B}$ est maximale lorsque $\mathcal{B} \oplus \bar{\mathcal{B}} = \mathcal{L}_0^c$, où $\bar{\mathcal{B}}$ est l'algèbre des éléments hermitiens conjugués des éléments appartenant à $\mathcal{B}$). Une des propriétés importante de ces états cohérents généralisés est qu'ils sont surcomplets. La complétude est une conséquence directe de l'irréductibilité de la représentation $T(g)$.

f) Ce sont des états qui minimisent la relation d'incertitude de Schrödinger-Robertson [6, 9, 10, 11, 12]. Soit $|\psi\rangle$ un état normalisé décrivant l'évolution d'un système quantique, la valeur moyenne et la dispersion dans cet état d'une observable physique représentée par un opérateur hermitien A sont données par $\langle A\rangle = \langle\psi|A|\psi\rangle$ et $(\Delta A)^2 = \langle A^2\rangle - \langle A\rangle^2$, respectivement. Pour deux opérateurs hermitiens A et B tels que

$$[A,B] = iC, \quad C \neq 0, \tag{4}$$

le produit de leurs dispersions satisfait la relation d'incertitude de Schrödinger-Robertson (RISR)

$$(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}(\langle C\rangle^2 + \langle F\rangle^2) \geq \frac{1}{4}\langle C\rangle^2, \tag{5}$$

où l'opérateur F est hermitien et donné par $F = \{A - \langle A\rangle I, B - \langle B\rangle I\}$, où $\{\ ,\ \}$ dénote l'anti-commutateur. $\langle F\rangle$ est une mesure de la corrélation entre A et B. Par conséquent, les états qui minimisent la RISR vérifient l'équation aux valeurs propres

$$[A + i\lambda B]|\psi\rangle = \beta|\psi\rangle, \quad \beta,\lambda \in \mathbf{C}, \lambda \neq 0. \tag{6}$$

Pour ces états, on a

$$(\Delta A)^2(\Delta B)^2 = \frac{1}{4}(\langle C\rangle^2 + \langle F\rangle^2) = \Delta^2, \tag{7}$$

avec

$$(\Delta A)^2 = |\lambda|\Delta, \quad (\Delta B)^2 = \frac{1}{|\lambda|}\Delta. \tag{8}$$

Il est clair que ces états dépendent du paramètre λ. On appelle états cohérents ceux qui minimisent la RISR pour des valeurs de λ telles que $|\lambda| = 1$ et états comprimés[13] ceux qui minimisent la RISR pour des valeurs de λ telles que $|\lambda| \neq 1$. On s'aperçoit que pour les états cohérents ainsi définis, la dispersion de l'opérateur A est égale à la dispersion de l'opérateur B et toutes les deux sont égales au facteur Δ. Pour les états dits A-comprimés tels que $|\lambda| < 1$, la dispersion de l'opérateur A est plus petite que Δ et la dispersion de l'opérateur B est plus grande que Δ, i.e., $(\Delta A)^2 < \Delta < (\Delta B)^2$. Ces inégalités sont renversées dans les cas des états B-comprimés.

Dans les définitions sur les états cohérents généralisés données en d) et f), on voit que pour obtenir ces états, il faut résoudre une équation aux valeurs propres. Dans certains cas d'intérêt, l'ensemble de ces états cohérents forme un sous-ensemble de l'ensemble des états propres d'un opérateur construit comme une combinaison linéaire, à coefficients complexes, des générateurs de l'algèbre dynamique du système quantique considéré. Les éléments de cet ensemble ont été appelés, par Brif[14], les états propres d'algèbres associés à l'algèbre de Lie $\mathcal{L}_0$. En général, si $\mathcal{L}_0$ est une algèbre de Lie engendrée par l'ensemble d'opérateurs $a_1, a_2, \ldots, a_m$, les états propres d'algèbres $|\psi\rangle$ associés à $\mathcal{L}_0$ sont déterminés par l'équation aux valeurs propres:

$$\left[\sum_{i=1}^{m} \alpha_i a_i\right] |\psi\rangle = z|\psi\rangle, \tag{9}$$

où $\alpha_j \in \mathbb{C}$, $\forall i = 1,2,\ldots,m$, $z \in \mathbb{C}$ et $|\psi\rangle$ est un état dans $\mathcal{W}_0$, l'espace d'Hilbert de représentation de l'algèbre.

Par exemple, dans le cas de l'oscillateur harmonique de dimension 1, l'algèbre de Lie dynamique standard[15] est isomorphe à l'algèbre $so(2,1) \ni h(2)$, où $so(2,1)$ est engendrée par l'Hamiltonien H et les générateurs $C_+(t) = \frac{i\omega}{2}e^{-2i\omega t}(a^\dagger)^2, C_-(t) = \frac{i\omega}{2}e^{2i\omega t}a^2$ et $h(2)$, l'algèbre de Heisenberg-Weyl, engendrée par les générateurs $\{A_-(t), A_+(t), I\}$, où $A_-(t) = e^{i\omega t}a$ and $A_+(t) = e^{-i\omega t}a^\dagger$.

Les états propres d'algèbres associés à cette algèbre s'obtiennent en résolvant l'équation aux valeurs propres[16]

$$[\alpha_- A_-(t) + \alpha_+ A_+(t) + \alpha_3 I + \beta_- C_-(t) + \beta_+ C_+(t) + \beta_3 H]|\psi\rangle = z|\psi\rangle, \qquad (10)$$

où $\alpha_\pm$,α_3,$\beta_\pm$,β_3,z $\in \mathbb{C}$. Il est aisé de démontrer que les états cohérents et comprimés standard associés à ce système sont un sous-ensemble des états propres d'algèbres associés à l'algèbre de Heisenberg-Weyl, obtenus en résolvant (10), pour les valeurs particulières des paramètres $\beta_\pm = \beta_3 = 0$ et $\alpha_- \neq 0$. Les états propres d'algèbres associés à d'autres algèbres ont aussi été calculés. Par exemple, dans le cas de l'algèbre $su(2)$, différentes approches ont été utilisées telles que le formalisme de la "constellation"[17], la méthode des équations différentielles ordinaires du premier ordre[14] ou encore la méthode des opérateurs ordonnés[18]. Il en est de même dans le cas de l'algèbre $su(1,1)$[14, 18]. Dans ces approches, il a été démontré que les états cohérents généralisés de Perelomov associés aux groupes $SU(2)$ et $SU(1,1)$ sont des sous-ensembles de l'ensemble des états propres d'algèbres associés aux algèbres $su(2)$ et $su(1,1)$, respectivement.

Il existe des systèmes quantiques[19, 20] dont l'algèbre dynamique associée n'est pas une algèbre de Lie mais plutôt une superalgèbre de Lie[21] ou encore dont le groupe d'invariance associé est un supergroupe de Lie. En fait, l'apparition des concepts de superalgèbres et de supergroupes, entre autres, vient du souci de traiter d'une façon unifiée les systèmes quantiques avec des degrés de liberté bosoniques et fermioniques. Les définitions des états cohérents généralisés associés à une algèbre de Lie ou un groupe de Lie ont alors été étendues au cas de superalgèbres de Lie ou supergroupes de Lie. En effet, une généralisation de la définition e) a été donnée[20, 22] pour le cas de supergroupes de Lie. Des applications de cette définition pour obtenir les états supercohérents associés au système de l'oscillateur harmonique supersymétrique ont été discutées[22, 23]. Elles tiennent compte de la possibilité d'obtenir ces superétats comme états propres d'un opérateur d'annihilation ou encore comme états minimaux d'incertitude. Les états supercohérents pour le modèle t–J, qui correspondent essentiellement aux états supercohérents associés à la superalgèbre de Lie $u(1/2;\mathbb{C})$, ont été calculés[20] en généralisant la définition de Perelomov[8]. La formule de BCH pour factoriser les éléments du supergroupe $U(1/2;\mathbb{C})$, exprimés comme des exponentielles de combinaisons linéaires d'éléments de la superalgèbre, a été appliquée dans le but de trouver le sous-supergroupe d'isotropie des états de plus haut poids, obtenus à l'aide de la structure de superalgèbre de Cartan de $u(1/2;\mathbb{C})$. Les formes explicites de ces états supercohérents peuvent aussi être obtenues en utilisant la méthode de factorisation des éléments du supergroupe au moyen de supermatrices[24].

Une définition des états supercohérents généralisés pour un supergroupe associé à une superalgèbre de Lie dynamique générale (avec emphase sur les superalgèbres pour lesquelles une base de Cartan-Weyl peut être définie) et les propriétés algébriques et geométriques de ces états a aussi été donnée[25] en suivant la méthode de Perelomov[8].

Par ailleurs, les états supercomprimés généralisés associés à une superalgèbre de Lie, n'ont été définis que pour certaines superalgèbres en imitant leur structure dans le cas de l'oscillateur harmonique standard, c'est-à-dire, en appliquant un opérateur unitaire de supercompression sur un état supercohérent du système[26].

De plus, des états cohérents associés à des algèbres de Lie déformées et à des groupes quantiques[27, 28] ont été déterminés. C'est le cas, par exemple, de la q–algèbre de l'oscillateur [29, 30], les algèbres de Hopf quantiques déformées $su_q(2)$ et $su_q(1,1)$[29]. En général, la construction de tels états se base sur la définition d'un opérateur d'annihilation déformé, sur l'application du concept d'états cohérents de Perelomov (dans le cas des groupes quantiques compacts[31]) ou encore sur les propiétés de sur-complétude et résolution de l'identité de ces états.

L'objectif de ce travail est d'utiliser et de généraliser le concept d'états propres d'algèbres pour construire de nouvelles classes d'états cohérents et comprimés généralisés associés à des algèbres et superalgèbres de symétries de systèmes quantiques. De plus, nous voulons établir une correspondance avec les conditions physiques régissant la définition de tels états, par exemple, les états propres d'un certain opérateur d'annihilation ou les états minimaux d'incertitude. Nous voulons également étendre ces concepts pour inclure les groupes quantiques. Plus précisément, des groupes quantiques construits à l'aide de la méthode de la matrice R[32, 33, 34, 35] qui fournissent des algèbres et superalgèbres déformées des structures usuelles, comme par exemple, l'algèbre et la superalgèbre de Heisenberg-Weyl.

Dans le chapitre 1, nous rappelons les concepts d'algèbres de symétries associées à des systèmes quantiques. D'abord, nous introduisons les notions d'algèbres de Lie d'invariance cinématique et dynamique maximales. Ensuite, nous décrivons l'algèbre de Lie d'invariance cinématique maximale[15] pour le système de l'oscillateur harmonique de dimension p, où

p est entier positif. Enfin, nous donnons l'algèbre de Lie d'invariance dynamique maximale standard[36] pour ce système. Ce chapitre ne contient pas d'éléments originaux mais permet de mettre en évidence les notations et définitions utilisées dans la suite de notre travail.

Dans le chapitre 2, nous implémentons une procédure basée sur la méthode de prolongation de champs de vecteurs[37, 38] et sur le concept de symétrie dynamique, pour obtenir les superalgèbres de Lie de symétries et supersymétries des systèmes quantiques caractérisés par les Hamiltoniens de l'oscillateur harmonique supersymétrique[39], de Pauli, de Jaynes-Cummings[40] et par un Hamiltonien supersymétrique représentant une généralisation de ce dernier. Ce chapitre constitue une contribution originale[41] qui prolonge les éléments du chapitre 1.

Le chapitre 3 consiste en un article original[42] qui fut le premier a être publié dans le cadre de cette recherche. Nous calculons les états propres d'algèbres pour l'algèbre de Lie $h(2) \oplus su(2)$ (que dans cet article nous dénotons $h(1) \oplus su(2)$), c'est-à-dire, la somme directe de l'algèbre de Heisenberg-Weyl, $h(2)$, et de l'algèbre de spin, $su(2)$. Nous nous servons de ce calcul et du concept d'états minimaux d'incertitude liés à la RISR, pour obtenir les états cohérents et comprimés généralisés associés à des couples d'opérateurs hermitiens formés d'une combinaison linéaire, à coefficients complexes, des générateurs de cette algèbre. Pour un choix particulier des paramètres et en considérant la représentation de spin $j = \frac{1}{2}$ de $su(2)$, nous retrouvons les états super-cohérents introduits par Aragone et Zypman[43], construits comme les états propres d'un opérateur d'annihilation supersymétrique. Nous généralisons ceci pour une représentation de spin j quelconque et nous montrons que cela équivaut au problème de connaître, pour tout j, les états super-cohérents associés aux opérateurs de super-position et super-impulsion. Nous étudions aussi les propriétés d'Hamiltoniens de la forme $\mathcal{H} = \mathcal{A}^\dagger \mathcal{A}$, où $\mathcal{A}$ est un élément de l'algèbre complexe associée à $h(2) \oplus su(2)$. Parmi ces Hamiltoniens, représentant des systèmes quantiques, nous comptons une version généralisée du Hamiltonien de l'oscillateur harmonique, dont les états cohérents associés généralisent ceux associés au Hamiltonien de l'oscillateur harmonique standard et dont les propriétés ressemblent à celles du Hamiltonien supersymétrique, mais aussi l'Hamiltonien de Jaynes-Cumming dans la limite de couplage fort et une classe d'Hamiltoniens que nous avons appelés non canoniques.

Le chapitre 4 constitue aussi en une contribution originale[44] à cette thèse. Nous introduisons le concept d'états propres de superalgèbres et l'appliquons à la superalgèbre de Heisenberg-Weyl $sh(2/2)$. Nous obtenons des classes d'états supercohérents qui généralisent celles obtenues en suivant d'autres approches [22, 43]. De plus, nous trouvons des classes d'états comprimés associées à la superalgèbre orthosymplectique $osp(2/2)$, engendrée par les huit opérateurs formés en prenant les produits quadratiques des générateurs de $sh(2/2)$. Nous comparons les caractéristiques de ces derniers états avec ceux du même genre obtenus en suivant la méthode groupe théorique[26, 45]. En outre, nous construisons des Hamiltoniens super-Hermitiens[46, 47] et η-pseudo-super-Hermitiens[48] sans parité de Grassmann définie, isospectraux avec l'oscillateur harmonique standard et dont l'annihilateur associé est un elément de $sh(2/2)$. Nous déterminons le spectre et les états cohérents associés à ces systèmes.

Finalement, dans le chapitre 5, nous appliquons le concept d'états propres d'algèbres aux algèbres de Hopf quantiques déformées. Plus précisément, nous étudions l'algèbre quantique de Heisenberg, déformée en utilisant la méthode de la matrice R. Nous proposons des représentations physiques de cette algèbre en termes des opérateurs de création $a^\dagger$, d'annihilation a et identité I. En calculant ces états, nous obtenons de nouvelles classes d'états cohérents et comprimés déformés, associés à l'oscillateur harmonique standard, paramétrés par des nombres réels régissant la déformation de l'algèbre. Nous étudions aussi le cas où ces paramètres sont considérés comme des nombres réels de paragrassmann. Nous obtenons ainsi de nouvelles classes d'états qui, d'une certaine façon, généralisent celles obtenues dans le chapitre 4. Nous étudions les propriétés des dispersions des opérateurs de position et d'impulsion d'une particule dans les états obtenus précédemment pour de petites valeurs des paramètres de déformation et nous les comparons avec celles des états non déformés, décrites dans le chapitre 3. Ce chapitre représente une approche nouvelle des états cohérents et comprimés associés à des déformations quantiques d'algèbres de Lie. Les résultats ont été récemment soumis pour publication[49].

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Chapitre 1

Algèbres de symétrie dynamiques associées à des systèmes quantiques

Lorsque nous étudions les systèmes d'équations différentielles ordinaires ou aux dérivées partielles décrivant le comportement d'un système physique, nous observons que, dans certains cas, nous pouvons nous servir des propriétés de symétrie de ces systèmes, soit pour simplifier leur résolution, soit pour tirer des conséquences a priori sur leur comportement. En général, les symétries d'un système physique sont reliées à des algèbres de Lie. Dans ce chapitre, nous discutons des symétries associées à des systèmes quantiques et nous décrivons les algèbres de Lie associées pour le cas de l'oscillateur harmonique standard en p dimensions, où p est un entier positif.

1.1 Algèbres de Lie cinématiques et dynamiques

Soit $W(t,x)$, avec $t \in \mathbb{R}$, $x \in \mathbb{R}^p$, un opérateur différentiel linéaire qui agit sur la fonction $\psi(t,x)$ pour donner l'équation

$$W(t,x)\psi(t,x) = 0. \tag{1.1}$$

On dit que l'équation (1.1) est invariante sous les transformations d'espace-temps inversibles

$$(t,x) \mapsto g(t,x) = (g^0(t,x),\vec{g}(t,x)), \tag{1.2}$$

s'il existe une transformation T_g de la fonction $\psi(t,x)$ et une fonction $f(t,x)$:

$$\psi(t,x) \mapsto (T_g\psi)(t,x) = f_g[g^{-1}(t,x)]\psi[g^{-1}(t,x)], \tag{1.3}$$

avec la propriété que

$$W(t,x)(T_g\psi)(t,x) = 0. \tag{1.4}$$

Cela signifie que si $\psi(t,x)$ est une solution de l'équation (1.1) alors $(T_g\psi)(t,x)$ l'est aussi.

Le groupe d'invariance cinématique maximal de l'équation (1.1) est le plus grand groupe de transformations d'espace-temps qui laisse invariante cette équation. Ainsi, trouver le groupe d'invariance cinématique maximal de l'équation (1.1) signifie encore de déterminer toutes les solutions possibles (g,f_g) de l'équation

$$W[g(t,x)][f_g(t,x)\psi(t,x)] = 0, \tag{1.5}$$

pour une solution arbitraire ψ de (1.1). Lorsqu'on a déterminé le groupe d'invariance maximal. i.e., les transformations $g(\alpha,t,x)$ et les fonctions $f_g(\alpha,t,x)$, en termes des constantes d'intégration α^μ, $\mu = 1,2,\ldots,m$, représentant une paramétrisation convenable, on peut calculer les générateurs de transformations infinitésimales de ce groupe. Ceux-ci sont donnés par

$$X_\mu(t,x) = i\Big[\frac{\partial f_g}{\partial \alpha^\mu}(0,t,x) - \frac{\partial g^0}{\partial \alpha^\mu}(0,t,x)\partial_t - \sum_{k=1}^{p}\frac{\partial g^k}{\partial \alpha^\mu}(0,t,x)\frac{\partial}{\partial x^k}\Big], \quad \mu = 1,2,\ldots,m. \tag{1.6}$$

Ces générateurs engendrent une algèbre de Lie, l'algèbre de Lie cinématique maximale de l'équation (1.1). On peut aussi obtenir cette algèbre de Lie en considérant les transformations infinitésimales

$$T_g = 1 + i\epsilon G(t,x), \quad \text{avec} \quad G(t,x) = a_0(t,x)\partial_t - \sum_{i=0}^{p} a_i(t,x)\partial_{x_i} - c(t,x), \tag{1.7}$$

où ϵ est un paramètre infinitésimal, les $a_j(t,x)$, $j = 0,1,2,\ldots,p$ et $c(t,x)$ sont des fonctions de (t,x) à déterminer. En insérant (1.7) dans (1.4), on obtient

$$W(t,x)[1 + i\epsilon G(t,x)]\psi(t,x) = 0. \tag{1.8}$$

Si l'on considère que $W(t,x)$ est le seul annihilateur pour une solution arbitraire $\psi(t,x)$ de l'équation (1.1), l'équation (1.8) est équivalente à

$$[W(t,x),G(t,x)] = i\rho(t,x)W(t,x), \tag{1.9}$$

où $\rho(t,x)$ est une fonction arbitraire. En comparant les deux membres de l'équation (1.9), on obtient un système d'équations différentielles aux dérivées partielles linéaires pour les fonctions $a_j(t,x)$, $j = 0,1,2,\ldots,p$, $c(t,x)$ et $\rho(t,x)$, servant à déterminer $G(t,x)$ et, en conséquence, les générateurs infinitésimaux de la transformation T_g. En général, $G(t,x)$ a la forme

$$G(t,x) = \sum_{\mu=1}^{m} b_\mu X_\mu(t,x), \tag{1.10}$$

où les $X_\mu(t,x)$, $\mu = 1,2,\ldots m$, sont donnés en (1.6) et et b_μ, $\mu = 1,2,\ldots m$, sont des constantes arbitraires. En vertu de l'équation (1.9) et de l'arbitrarité des constantes b_μ, on a que

$$[W(t,x),X_\mu(t,x)]\psi(t,x) = 0, \quad \mu = 1,2,\ldots m, \tag{1.11}$$

sur l'espace des solutions de l'équation (1.1). L'équation (1.11) peut être généralisée pour inclure d'autres types de symétries du système physique, c'est-à-dire, que l'on peut considérer une classe plus générale d'opérateurs différentiels satisfaisant cette équation. S'il existe des opérateurs X_i, $i = 1,2,\ldots,n$, engendrant une algèbre de Lie $\mathcal{L}_0$ et vérifiant, sur l'espace des solutions de l'équation (1.1),

$$[W,X_i]\psi(t,x) = 0, \qquad i = 1,2,\ldots n, \tag{1.12}$$

alors l'ensemble de toutes les solutions de l'équation (1.1) engendre un espace de représentation pour l'algèbre de Lie $\mathcal{L}_0$. On s'aperçoit que si $\psi(t,x)$ est une solution de (1.1), les $X_i\psi(t,x)$, $i = 1,2,\ldots,n$ sont aussi des solutions et on a

$$[W,X_i] = F(W), \tag{1.13}$$

où F est une fonction polynômiale (d'après l'équation (1.1), ceci évite de diviser par zéro) arbitraire avec des coefficients dépendants des coordonnées (t,x) et vérifiant $F(0) = 0$.

En particulier, si

$$W(t,x) = i\partial_t - H, \tag{1.14}$$

où H est l'Hamiltonien associé à un système quantique, alors l'algèbre de Lie $\mathcal{L}_0$ est appelée algèbre de Lie dynamique du système quantique. En général, l'algèbre de Lie dynamique contient des opérateurs dépendant du temps qui, sur l'espace des solutions $\psi(t,x)$ de l'équation (1.1), satisfont l'équation de Heisenberg

$$[i\partial_t,X_k(t)] = [H,X_k(t)], \quad k = 1,2,\ldots n. \tag{1.15}$$

Les solutions de l'équation de Heisenberg (1.15) sont données par

$$X_k(t) = U(t,t_0)X_k(t_0)U^\dagger(t,t_0), \quad \text{avec} \quad U(t,t_0) = \tau\left\{\exp\left(-i\int_{t_0}^{t} H(t')dt'\right)\right\}, \tag{1.16}$$

où $U(t,t_0)$ est l'opérateur unitaire d'évolution temporelle du système et τ, devant le facteur exponentiel, avertit que l'application des opérateurs $H(t')$, $t_0 \leq t' \leq t$, sur les états du système se fait en suivant l'ordre croissant du paramètre t'. La sous-algèbre $\mathcal{L}'_0 \subset \mathcal{L}_0$ formée des opérateurs qui commutent avec W est une définition plus restreinte de symétrie. $\mathcal{L}'_0$ est représentée sur le même espace de Hilbert que $\mathcal{L}_0$. Dans le cas où $[H(t),H(t')] = 0$, $\forall\, t,t'$, l'opérateur $H(t_0)$ commute avec l'opérateur d'évolution temporelle $U(t,t_0)$ et l'algèbre dynamique $\{X_k(t)\}_{k=1}^n$ est unitairement équivalente à l'algèbre dynamique $\{X_k(t_0)\}_{k=1}^n$. Ceci permet, lorsqu'on étudie un problème concret, de fixer la valeur initiale du paramètre temporel, par exemple $t_0 = 0$, et de restreindre alors les considérations à l'analyse des algèbres dynamiques indépendantes du temps. La sous-algèbre $\tilde{\mathcal{L}}_0 \subset \mathcal{L}'_0$ des opérateurs indépendants du temps satisfait

$$[H,\tilde{\mathcal{L}}_0] = 0, \tag{1.17}$$

i.e., elle correspond à l'algèbre de symétrie du Hamiltonien.

1.2 Symétries de l'oscillateur harmonique standard

L'oscillateur harmonique de dimension p est caractérisé par l'opérateur différentiel:

$$W(t,x) = i\partial_t - H_0, \quad \text{avec} \quad H_0 = -\frac{1}{2M}\sum_{k=1}^{p}\partial_{x_k}^2 + \frac{Mw^2}{2}\sum_{k=1}^{p}x_k^2, \tag{1.18}$$

où M et w sont des constantes réelles. Le groupe d'invariance cinématique maximal a été calculé par Niederer[15]. Les transformations $g(\alpha^\mu,t,x)$ sont données par

$$g(S,\vec{a},\vec{v},R;t,x) = \left(\frac{1}{w}\arctan\frac{\alpha\eta(t)+\beta}{\gamma\eta(t)+\delta}, u_g^{-1/2}(t)\,[Rx + \vec{v}\sin wt + \vec{a}\cos wt]\right), \tag{1.19}$$

où

$$u_g(t) = [1+\eta^2(t)]^{-1}[(\alpha\eta(t)+\beta)^2 + (\gamma\eta(t)+\delta)^2], \tag{1.20}$$

avec

$$\eta(t) = \tan wt, \qquad \alpha\delta - \beta\gamma = 1, \tag{1.21}$$

et

$$S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in Sl(2,\mathbb{R}), \quad \vec{a},\vec{v} \in \mathbb{R}^p, R \in O(p). \tag{1.22}$$

Ici, $O(p)$ dénote le groupe des matrices réelles orthogonales de dimension $p \times p$, i.e., $R \in O(p)$ implique que $R^{-1} = R^t$. La fonction $f_g(t,x)$ est donnée par

$$f_g(t,x) = (u_g(t))^{1/n} \exp\left[-i\frac{M}{4u_g(t)}h_g(t,x)\right], \tag{1.23}$$

avec

$$h_g(t,x) = \frac{du_g(t)}{dt}[R\vec{x} + \vec{y}_g(t)]^2 - 2u_g(t)\frac{d\vec{y}_g}{dt}(t) \cdot [2R\vec{x} + \vec{y}_g(t)] \tag{1.24}$$

et

$$\vec{y}_g(t) = \vec{v}\sin wt + \vec{a}\cos wt. \tag{1.25}$$

Une paramétrisation convenable pour les éléments de $Sl(2,\mathbb{R})$ est

$$\begin{aligned} \alpha &= \cosh\frac{s}{2} + \frac{s_1}{s}\sinh\frac{s}{2}, \qquad \beta = \frac{s_2+s_3}{s}\sinh\frac{s}{2}, \\ \gamma &= \frac{s_2-s_3}{s}\sinh\frac{s}{2} \quad \text{et} \quad \delta = \cosh\frac{s}{2} - \frac{s_1}{s}\sinh\frac{s}{2}, \end{aligned} \tag{1.26}$$

avec $\vec{s} = (s_1,s_2,s_3)$ et $s = (s_1^2 + s_2^2 + s_3^2)^{1/2}$. Dans le cas unidimensionel ($p = 1$), les générateurs engendrant l'algèbre de Lie cinématique maximale, en accord avec l'équation (1.6), sont donnés par

$$\begin{aligned} I_1 &= -\frac{i}{2w}\sin 2wt\,\partial_t - \frac{i}{2}\cos 2wt\,x\frac{\partial}{\partial x} - \frac{i}{4}\cos 2wt + \frac{1}{2}Mwx^2\sin 2wt, \\ I_2 &= -\frac{i}{2w}\cos 2wt\,\partial_t + \frac{i}{2}\sin 2wt\,x\partial_x + \frac{i}{4}\sin 2wt + \frac{1}{2}Mwx^2\cos 2wt, \\ I_3 &= -\frac{i}{2w}\partial_t, \\ P &= -i\cos wt\,\frac{\partial}{\partial x} + Mwx\sin wt, \qquad K = i\sin wt\,\frac{\partial}{\partial x} + Mwx\cos wt. \end{aligned} \tag{1.27}$$

Les transformations infinitésimales correspondantes sont données par

$$T_g(\vec{s},a,v) = 1 - i\vec{s}\cdot\vec{I} - iaP + ivK, \tag{1.28}$$

où $\vec{I} = (I_1,I_2,I_3)$. On peut combiner ces générateurs pour obtenir les générateurs:

$$C_+(t) = w(I_1(t) + iI_2(t)) = \frac{iw}{2}e^{-2iwt}\left[\left(a^\dagger(x)\right)^2 - \frac{1}{w}W(t,x)\right], \tag{1.29}$$

$$C_-(t) = iw(I_2(t) + iI_1(t)) = \frac{iw}{2}e^{2iwt}\left[\left(a(x)\right)^2 - \frac{1}{w}W(t,x)\right], \tag{1.30}$$

$$H_0 = -2wI_3(t) = i\partial_t, \tag{1.31}$$

$$A_+(t) = \sqrt{2Mw}(K - iP) = e^{-iwt}a^\dagger(x), \tag{1.32}$$

$$A_-(t) = \sqrt{2Mw}(K + \imath P) = e^{iwt}a(x), \tag{1.33}$$

où

$$a^\dagger(x) = \frac{(Mwx - \partial_x)}{\sqrt{2Mw}}, \quad \text{et} \quad a(x) = \frac{(Mwx + \partial_x)}{\sqrt{2Mw}}. \tag{1.34}$$

En termes des générateurs (1.34), l'Hamiltonien H_0 prend la forme

$$H_0(x) = w\left(a^\dagger(x)a(x) + \frac{1}{2}\right). \tag{1.35}$$

On sait que les générateurs $a(x)$, $a^\dagger(x)$ dans (1.34) et l'identité 1, satisfont l'algèbre de Heisenberg-Weyl $h(2)$ et on peut les considérer comme des opérateurs agissant sur un espace de représentation de Fock. Il est ici engendré par l'ensemble des états propres du Hamiltonien H_0, à savoir, l'ensemble d'états $\{|n\rangle\}_{n=0}^{\infty}$ tels que

$$H_0|n\rangle = w\left(n + \frac{1}{2}\right)|n\rangle, \tag{1.36}$$

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \qquad \text{et} \qquad a|n\rangle = \sqrt{n}|n-1\rangle. \tag{1.37}$$

Dans cette représentation, l'équation $W(t,x)\psi(t,x) = 0$, prend la forme

$$i\partial_t|\psi(t)\rangle = H_0|\psi(t)\rangle, \tag{1.38}$$

où, en général,

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} c_n(t)|n\rangle, \qquad c_n(t) \in \mathbb{C},\, n = 0,1,2,\ldots. \tag{1.39}$$

L'action des opérateurs correspondant aux générateurs de symétrie (1.29)-(1.33) sur les états solutions de l'équation de Schrödinger (1.38), à savoir

$$|\psi(t)\rangle = e^{-iH_0(t-t_0)}|\psi(t_0)\rangle, \tag{1.40}$$

où $|\psi(t_0)\rangle$ est un état initial du système, peut s'écrire:

$$C_+(t)|\psi(t)\rangle = \frac{iw}{2}e^{-2iwt}(a^\dagger)^2|\psi(t)\rangle, \quad C_-(t)|\psi(t)\rangle = \frac{iw}{2}e^{2iwt}a^2|\psi(t)\rangle, \tag{1.41}$$

$$H_0|\psi(t)\rangle = \frac{w}{2}(a^\dagger a + aa^\dagger)|\psi(t)\rangle, \tag{1.42}$$

$$A_+|\psi(t)\rangle = e^{-iwt}a^\dagger|\psi(t)\rangle \quad \text{et} \quad A_-|\psi(t)\rangle = e^{iwt}a|\psi(t)\rangle. \tag{1.43}$$

On observe donc qu'il existe des opérateurs qui apparaissent comme des combinations quadratiques des opérateurs du premier ordre $A_\pm(t)$ lorsqu'on considère leur action sur les solutions de l'équation de Schrödinger, à savoir, les opérateurs $C_+(t)$, $C_-(t)$ et H_0. En accord avec l'équation (1.16), lorsque $t_0 = 0$, l'algèbre de Lie engendrée par les opérateurs

$C_\pm(t)$,H_0,$A_\pm(t)$ et l'opérateur identité I, est unitairement équivalente à l'algèbre de Lie engendrée par les opérateurs $C_\pm(0)$,H_0,$A_\pm(0)$ et l'opérateur identité I. Cette algèbre de Lie correspond à $so(2,1) \ni\!\!\!\!\!\supset h(2)$, i.e., la somme semi-directe de l'algèbre $so(2,1) \sim su(1,1)$, engendrée par $C_\pm(0)$ et H_0, et de l'algèbre de Heisenberg-Weyl $h(2)$, engendrée par $A_\pm(0)$ et I. Ici, la notation utilisée pour la somme semi-directe indique que la sous-algèbre $so(2,1)$ agit sur la sous-algèbre $h(2)$. En fait, les relations de commutation non nulles correpondant à $so(2,1) \ni\!\!\!\!\!\supset h(2)$ sont données par

$$[H_0,C_\pm(0)] = \pm 2wC_\pm(0), \qquad [C_+(0),C_-(0)] = wH_0, \tag{1.44}$$

$$[A_-(0),A_+(0)] = I, \tag{1.45}$$

$$[C_\pm(0),A_\mp(0)] = \mp iwA_\pm(0), \qquad [H_0,A_\mp(0)] = \mp wA_\mp(0). \tag{1.46}$$

Dans le cas de l'oscillateur harmonique de dimension p, l'algèbre de Lie cinématique maximale est engendrée par l'ensemble des opérateurs dépendant du temps:

$$A_{+,k}(t) = e^{-iwt}a_k^\dagger, \qquad A_{-,k}(t) = e^{iwt}a_k, \tag{1.47}$$

lorsque $k = 1,2,\ldots,p$,

$$C_+(t) = \frac{iw}{2}\sum_{k=1}^{p}(A_{+,k}(t))^2 = \frac{iw}{2}e^{-2iwt}\sum_{k=1}^{p}(a_k^\dagger)^2, \tag{1.48}$$

$$C_-(t) = \frac{iw}{2}\sum_{k=1}^{p}(A_{-,k}(t))^2 = -\frac{i}{2}e^{2iwt}\sum_{k=1}^{p}{a_k}^2 \tag{1.49}$$

et l'ensemble des opérateurs indépendant du temps

$$H_0 = \frac{w}{2}\sum_{k=1}^{p}\{A_{-,k}(t),A_{+,k}(t)\} = \frac{w}{2}\sum_{k=1}^{p}\{a_k,a_k^\dagger\} \tag{1.50}$$

et

$$L_{kl} = i\,[A_{-,k}(t)A_{+,l}(t) - A_{-,l}(t)A_{+,k}(t)] = i(a_k a_l^\dagger - a_l a_k^\dagger), \tag{1.51}$$

lorsque $k,l = 1,2,\ldots p$, $k < l$. Ici, les opérateurs de création $a_k^\dagger$ et d'annihilation a_k sont définis par

$$a_k^\dagger = \frac{1}{\sqrt{2Mw}}\left(Mwx_k + \partial_{x_k}\right), \qquad a_k = \frac{1}{\sqrt{2Mw}}\left(Mwx_k - \partial_{x_k}\right) \tag{1.52}$$

et satisfont les relations de commutation

$$[a_k,a_l^\dagger] = \delta_{kl}, \quad [a_k,a_l] = [a_k^\dagger,a_l^\dagger] = 0. \tag{1.53}$$

En incluant l'identité, cette algèbre de Lie a la structure $[so(2,1) \oplus o(p)] \uplus\!\!\!\supset h(2p)$ et la dimension $(3 + \frac{1}{2}p(p-1) + 2p + 1)$. Les opérateurs dépendant du temps vérifient l'équation de Heisenberg (1.15) avec $H \equiv H_0$ donné par (1.50) tandis que les opérateurs indépendant du temps commutent avec H_0, i.e., ils font partie de l'algèbre de symétrie de ce Hamiltonien. Quant à l'algèbre de Lie dynamique standard maximale associée à l'oscillateur harmonique de dimension p [36], elle est engendrée par les opérateurs dépendant du temps (1.47) et toutes les combinations quadratiques

$$C_{+,kl}(t) = \frac{iw}{2}\{A_{+,k}(t),A_{+,l}(t)\} = \frac{iw}{2}e^{-2iwt}\{a_k^\dagger,a_l^\dagger\}, \tag{1.54}$$

$$C_{-,kl}(t) = \frac{iw}{2}\{A_{-,k}(t),A_{-,l}(t)\} = \frac{iw}{2}e^{2iwt}\{a_k,a_l\} \tag{1.55}$$

ainsi que les opérateurs indépendant du temps

$$T_{kl} = \frac{1}{2}\{A_{-,k}(t),A_{+,l}(t)\} = \frac{w}{2}\{a_k,a_l^\dagger\}. \tag{1.56}$$

En incluant l'identité, ces opérateurs engendrent l'algèbre de Lie $sp(2p) \uplus\!\!\!\supset h(2p)$ de dimension $(p+1)(2p+1)$. De nouveau, les opérateurs dépendant du temps vérifient l'équation de Heisenberg (1.15) avec $H \equiv H_0$ tandis que les opérateurs indépendant du temps commutent avec H_0. On note que, pour l'oscillateur harmonique unidimensionnel l'algèbre de Lie cinématique maximale coïncide avec l'algèbre de Lie dynamique maximale tandis que pour l'oscillateur harmonique de dimension p, $p > 1$, l'algèbre de Lie cinématique maximale est contenue dans l'algèbre de Lie dynamique maximale. Si l'on inclut l'identité, ceci peut être exprimé par la relation $[so(2,1) \oplus so(p)] \uplus\!\!\!\supset h(2p) \subset [sp(2p) \uplus\!\!\!\supset h(2p)]$.

Chapitre 2

Champs de vecteurs invariants et la méthode de prolongation pour des systèmes quantiques supersymétriques

Résumé

Les symétries cinématiques et dynamiques des équations décrivant l'évolution temporelle de systèmes quantiques tels que l'oscillateur harmonique supersymétrique à une dimension spatiale et l'interaction d'une particule de spin $1/2$ avec un champ magnétique constant sont revues du point de vue de la méthode de prolongation des champs de vecteurs. Les générateurs de supersymétries sont alors introduits afin d'obtenir les superalgèbres de Lie de symétries et supersymétries. Cette approche ne nécessite pas l'introduction de variables de Grassmann dans les équations différentielles mais une réalisation matricielle spécifique et le concept de symétrie dynamique. Le modèle de Jaynes-Cummings et des généralisations supersymétriques sont alors étudiés. Nous démontrons comment il est rélié aux modèles précédents. Les algèbres de Lie de symétries et supersymétries sont aussi obtenues.

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Invariant vector fields and the prolongation method for supersymmetric quantum systems

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Abstract
The kinematical and dynamical symmetries of equations describing the time evolution of quantum systems such as the supersymmetric harmonic oscillator in one space dimension and the interaction of a non-relativistic spin one-half particle in a constant magnetic field are reviewed from the point of view of the vector field prolongation method. Generators of supersymmetries are then introduced so that we get Lie superalgebras of symmetries and supersymmetries. This approach does not require the introduction of Grassmann-valued differential equations but a specific matrix realization and the concept of dynamical symmetry. The Jaynes–Cummings model and supersymmetric generalizations are then studied. We show how it is closely related to the preceding models. Lie algebras of symmetries and supersymmetries are also obtained.

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1. Introduction

The symmetries of a system of ordinary differential equations (ODEs) or partial differential equations (PDEs) are usually obtained by using the so-called prolongation method of vector fields [1, 2]. It consists of finding the infinitesimal generators which close the maximal invariant Lie algebra of the system of equations. The corresponding symmetry group is the Lie group of local transformations of independent and dependent variables which leaves invariant the system under consideration. Such a system may be associated with the wave equation of some quantum model. The independent variables are the usual space–time coordinates while the dependent ones are the components of the wavefunction. The symmetries may be related to the so-called kinematical Lie algebra [3] of the quantum system.

Now, if we have in mind supersymmetric (SUSY) quantum models [4, 5], the question is how to find them from this prolongation method. We answer this question by considering first some standard examples where the kinematical Lie superalgebras are known. This is the case of the SUSY harmonic oscillator in one space dimension (see [6, 7] and reference therein) and the Pauli equation, in two space dimensions, describing the motion of a non-relativistic spin one-half particle in a constant magnetic field [7]. An important part of this work is concerned by the study of symmetries of the Jaynes–Cummings (JC) model [8] based on the same approach. Let us recall that the JC model, which consists of an idealized description of the interaction of a quantized electromagnetic field and an atomic system with two levels, is closely related to the two models considered before. An interesting point is that it can be made SUSY in a non-trivial manner and our approach will clarify this point and will make the connection with different works on this subject [9, 10].

At the classical level, Grassmann-valued differential equations have been introduced [11–14] and the prolongation method has been extended to include Grassmann independent and dependent variables [15, 16]. For example, SUSY extensions of Korteweg–de Vries and other equations have been studied and maximal invariant Lie superalgebras have been obtained.

At the quantum level, the problem is somewhat different. The SUSY system is nothing but a set of PDEs with the usual independent and dependent variables. So it is really of the type where the usual prolongation method can be used and the vector fields obtained close a Lie algebra. The non-trivial question we ask is how to get the generators which are associated with supersymmetries from this method and which, together with the symmetry generators, close a Lie superalgebra.

To clarify the context we are working with, let us here recall the prolongation method [1, 2] for determining the symmetries of a system of m PDEs of order n of the type

$$\Delta^{(k)}\left[x; u_\alpha, u^{(1)}_{\alpha_{x_{j_1}}}, u^{(2)}_{\alpha_{x_{j_1}x_{j_2}}}, \dots, u^{(n)}_{\alpha_{x_{j_1}x_{j_2}\cdots x_{j_n}}}\right] = 0 \qquad k = 1, 2, \dots, m \tag{1}$$

with p independent variables x_j ($j = 1, 2, \dots, p$), and q dependent variables $u_\alpha(x)$ ($\alpha = 1, 2, \dots, q$). The derivatives of the dependent variables are defined as

$$u^{(l)}_{\alpha_{x_{j_1}x_{j_2}\cdots x_{j_l}}} \equiv u^{(l)}_\alpha \equiv \frac{\partial^l u_\alpha(x)}{\partial x_{j_1}\partial x_{j_2}\cdots\partial x_{j_l}} \qquad 1 \leqslant l \leqslant n \tag{2}$$

where the integers j_r ($r = 1, 2, \dots, l$) are such that $0 \leqslant j_r \leqslant p$.

The Lie group of local transformations of independent and dependent variables which leave invariant such a system is obtained by performing the following infinitesimal transformation on the independent and dependent variables:

$$\tilde{x}_j = x_j + \epsilon \sum_j \xi_j(x, u_\alpha) + \mathcal{O}(\epsilon^2) \tag{3}$$

$$\tilde{u}_\alpha(\tilde{x}, \tilde{u}_\beta) = u_\alpha(x, u_\beta) + \epsilon\phi_\alpha(x, u_\beta) + \mathcal{O}(\epsilon^2). \tag{4}$$

Assuming that they satisfy, at first order in ϵ, the equation

$$\Delta^{(k)}\left[\tilde{x}; \tilde{u}_\alpha, \tilde{u}^{(1)}_{\alpha_{\tilde{x}_{j_1}}}, \tilde{u}^{(2)}_{\alpha_{\tilde{x}_{j_1}\tilde{x}_{j_2}}}, \dots, \tilde{u}^{(n)}_{\alpha_{\tilde{x}_{j_1}\tilde{x}_{j_2}\cdots\tilde{x}_{j_n}}}\right] = 0 \tag{5}$$

for $k = 1, 2, \dots, m$ and when the $u_\alpha(x)$ solve the system (1), we can find the functions ξ_j and ϕ_α. A practical way to do it is to introduce the vector field

$$v = \sum_{j=1}^{p} \xi_j(x, u_\beta)\partial_{x_j} + \sum_{\alpha=1}^{q} \phi_\alpha(x, u_\beta)\partial_{u_\alpha} \tag{6}$$

associated with the transformations (3) and (4) and define the nth order prolongation of v as

$$pr^{(n)}v = v + \sum_{\alpha=1}^{q} \sum_{J_l, l=1,2,\ldots,n} \phi_\alpha^{J_l}\left(x, u_\beta, u_\beta^{(1)}, u_\beta^{(2)}, \ldots, u_\beta^{(n)}\right)\partial_{u_\alpha^{J_l}} \tag{7}$$

where $J_l = \left(x_{j_1}, x_{j_2}, \ldots, x_{j_l}\right)$ is the multi-index notation for the differentiation with respect to the x_j and $\partial_{u_\alpha^{J_l}} \equiv \partial_{u_\alpha^{(l)}}$. Note that the coefficients $\phi_\alpha^{J_l}$ satisfy the following recurrence relation

$$\phi_\alpha^{J_l, x_k} = D_{x_k}\phi_\alpha^{J_l} - \sum_{j=1}^{p} \left(D_{x_k}\xi_j\right)\frac{\partial u_\alpha^{J_l}}{\partial x_j} \tag{8}$$

where D_{x_k} is the total derivative with respect to x_k. The infinitesimal criterion for invariance (5) may then be written as

$$pr^{(n)}v\left\{\Delta^{(k)}\left[x; u_\alpha, u^{(1)}_{\alpha_{x_{j_1}}}, u^{(2)}_{\alpha_{x_{j_1}x_{j_2}}}, \ldots, u^{(n)}_{\alpha_{x_{j_1}x_{j_2}\cdots x_{j_n}}}\right]\right\} = 0 \qquad k = 1, 2, \ldots, m \tag{9}$$

when the $u_\alpha(x)$ satisfy (1). Condition (9) gives a set of PDEs called the determining equations which can be solved to get the explicit form of the functions ξ_j and ϕ_α in (6). The resolution may lead to different possibilities: no nontrivial solutions, a finite number of integration constants or that the general solution depends on arbitrary functions. Let us also mention that we have the following properties of the vector field prolongations:

$$pr^{(n)}(c_1v_1 + c_2v_2 + \cdots + c_mv_m) = pr^{(n)}c_1v_1 + pr^{(n)}c_2v_2 + \cdots + pr^{(n)}c_mv_m \tag{10}$$

and

$$pr^{(n)}[v_1, v_2] = [pr^{(n)}v_1, pr^{(n)}v_2]. \tag{11}$$

The contents of the paper are thus described as follows. Section 2 is devoted to the construction of invariant vector fields for the SUSY harmonic oscillator in one dimension. It admits a large set of symmetries and the integration of vector fields gives a matrix realization of the symmetry generators which is essential in order to find the generators of supersymmetries. The corresponding kinematical and dynamical invariance superalgebras will be recovered in this context. In section 3, the model of a non-relativistic spin-$\frac{1}{2}$ particle in a constant magnetic field is studied. It can be reduced to a two-dimensional model and shows a similar behaviour to the SUSY harmonic oscillator. The symmetry algebra and superalgebra are obtained from the prolongation of the vector fields method and connected to the preceding case. In section 4, we start with a quantum evolution equation which is a realization of the JC model and determine the invariant vector fields and the associated invariant algebra. The connection with the preceding models is very helpful to get a Lie superalgebra of symmetries for a generalized JC model. In section 5, we propose a SUSY version of this model and give the corresponding symmetries and supersymmetries. We also make the connection with preceding attempts to get SUSY JC models.

2. The SUSY harmonic oscillator

The first set of equations we are considering is the one associated with the SUSY harmonic oscillator in one space dimension. The corresponding Schrödinger evolution equation is

$$(\mathrm{i}\partial_t - H_{\mathrm{SUSY}})\Psi(t, x) = 0. \tag{12}$$

Let us mention that throughout this work we use the convention that $\hbar = 1$. The SUSY Hamiltonian [17] is given by

$$H_{\mathrm{SUSY}} = \left(-\frac{1}{2M}\frac{\partial^2}{\partial x^2} + \frac{1}{2}M\omega^2x^2\right)\sigma_0 - \frac{\omega}{2}\sigma_3 \tag{13}$$

where σ_0 is the identity matrix and $\sigma_3 = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$. The wavefunction takes the form

$$\Psi(t,x) = \begin{pmatrix}\psi_1(t,x)\\ \psi_2(t,x)\end{pmatrix} \qquad \psi_1, \psi_2 \in L^2(\mathbb{R}). \tag{14}$$

It is convenient to write equation (12) as a set of two equations

$$\mathrm{i}\frac{\partial\psi_\alpha}{\partial t} + \frac{1}{2M}\frac{\partial^2\psi_\alpha}{\partial x^2} - \frac{1}{2}M\omega^2x^2\psi_\alpha + \frac{\omega_\alpha}{2}\psi_\alpha = 0 \qquad \alpha = 1,2 \tag{15}$$

where we have set $\omega_1 = \omega$ and $\omega_2 = -\omega$.

The kinematical and dynamical symmetries and supersymmetries have been largely studied [3, 5–7, 18] but these approaches were different from that we want to apply. Indeed for the usual harmonic oscillator, Niederer [3] has first shown that the maximal kinematical algebra is the semi-direct sum $so(2,1) \ni\!\!\!\!\supset h(2)$, where $h(2)$ is the usual Heisenberg–Weyl algebra. The maximal dynamical algebra [18], defined as that associated with the degeneracy group of the model, is given by $sp(2) \ni\!\!\!\!\supset h(2)$ and includes the preceding kinematical algebra. The dynamical and kinematical superalgebras of the SUSY version coincide in this one-dimensional case and are given by $osp(2/2) \ni\!\!\!\!\supset sh(2/2)$ [6, 7]. We will show how to recover these structures starting from the prolongation method of vector fields applied to the system (15).

2.1. *Prolongation method and invariant vector fields*

A standard way of applying the prolongation method to a system containing complex-valued functions is to express the components of the wavefunction (14) as

$$\psi_1(t,x) = u_1(t,x)\,\mathrm{e}^{\mathrm{i}\nu_1(t,x)} \qquad \psi_2(t,x) = u_2(t,x)\,\mathrm{e}^{\mathrm{i}\nu_2(t,x)} \tag{16}$$

where u_1, u_2, ν_1 and ν_2 are real functions of t and x. Inserting (16) into (15) and separating the real and complex parts of the resulting equations, we are led to a set of four coupled equations in u_1, u_2, ν_1 and ν_2. The vector field (6) may be written explicitly as

$$v = \xi_1\partial_t + \xi_2\partial_x + \phi_1\partial_{u_1} + \phi_2\partial_{u_2} + \varphi_1\partial_{\nu_1} + \varphi_2\partial_{\nu_2} \tag{17}$$

where ξ_j ($j = 1,2$), ϕ_α and φ_α ($\alpha = 1,2$) are real functions which depend on t, x, u_1, u_2, ν_1 and ν_2.

A simpler way of solving the problem is to consider the set (15) together with its complex conjugate

$$-\mathrm{i}\frac{\partial\bar\psi_\alpha}{\partial t} + \frac{1}{2M}\frac{\partial^2\bar\psi_\alpha}{\partial x^2} - \frac{1}{2}M\omega^2x^2\bar\psi_\alpha + \frac{\omega_\alpha}{2}\bar\psi_\alpha = 0 \qquad \alpha = 1,2. \tag{18}$$

Now the corresponding vector field takes the form

$$v = \xi_1\partial_t + \xi_2\partial_x + \Phi_1\partial_{\psi_1} + \bar\Phi_1\partial_{\bar\psi_1} + \Phi_2\partial_{\psi_2} + \bar\Phi_2\partial_{\bar\psi_2} \tag{19}$$

where now ξ_j ($j = 1,2$) are real functions of the variables $t, x, \psi_1, \psi_2, \bar\psi_1$ and $\bar\psi_2$ and $\Phi_\alpha, \bar\Phi_\alpha$ ($\alpha = 1,2$) are possible complex-valued functions of these variables. In terms of these variables, the second-order prolongation of v takes the form

$$pr^{(2)}v = v + \sum_{\alpha=1}^{2}\left(\Phi_\alpha^t\partial_{\psi_{\alpha,t}} + \Phi_\alpha^x\partial_{\psi_{\alpha,x}} + \Phi_\alpha^{tt}\partial_{\psi_{\alpha,tt}} + \Phi_\alpha^{tx}\partial_{\psi_{\alpha,tx}} + \Phi_\alpha^{xx}\partial_{\psi_{\alpha,xx}}\right) + [\mathrm{c.c.}] \tag{20}$$

where, for example, $\psi_{\alpha,t}$ is the usual partial derivative of ψ_α with respect to t. Applying this prolongation to the system consisting of equations (15) and (18), we get

$$\mathrm{i}\Phi_\alpha^t + \frac{1}{2M}\Phi_\alpha^{xx} - \frac{1}{2M}\omega^2x^2\Phi_\alpha + \frac{\omega_\alpha}{2}\Phi_\alpha - M\omega^2x\xi_1\psi_\alpha = 0 \tag{21}$$

$$-\mathrm{i}\bar{\Phi}_\alpha^t + \frac{1}{2M}\bar{\Phi}_\alpha^{xx} - \frac{1}{2M}\omega^2 x^2 \bar{\Phi}_\alpha + \frac{\omega_\alpha}{2}\bar{\Phi}_\alpha - M\omega^2 x \xi_1 \bar{\psi}_\alpha = 0 \tag{22}$$

where we have

$$\Phi_\alpha^t = D_t \Phi_\alpha - (D_t\xi_1)\psi_{\alpha,t} - (D_t\xi_2)\psi_{\alpha,x} \tag{23}$$

$$\Phi_\alpha^{xx} = \left(D_x \Phi_\alpha^x\right) - (D_x\xi_1)\psi_{\alpha,xt} - (D_x\xi_2)\psi_{\alpha,xx} \tag{24}$$

with

$$\Phi_\alpha^x = D_x \Phi_\alpha - (D_x\xi_1)\psi_{\alpha,t} - (D_x\xi_2)\psi_{\alpha,x} \tag{25}$$

together with their complex conjugate and for $\alpha = 1, 2$. Inserting these expressions into system (21), (22), taking into account equations (15) and (18) and identifying to zero the coefficients of the partial derivatives, we get a set of determining equations which will give the functions ξ_j, Φ_α and $\bar{\Phi}_\alpha$. Solving these equations, we get

$$\xi_1(t) = \frac{1}{2\omega}(\delta_1 \sin 2\omega t - \delta_2 \cos 2\omega t) + \delta_3 \tag{26}$$

$$\xi_2(t,x) = \frac{1}{2}(\delta_1 \cos 2\omega t + \delta_2 \sin 2\omega t)x + \delta_4 \cos \omega t + \delta_5 \sin \omega t \tag{27}$$

which are effectively real functions depending only on the coordinates t and x. We also have

$$\Phi_1(t, x, \psi_1, \psi_2) = A_0(t,x) + A_1(t,x)\psi_1 + A_2(t)\psi_2 \tag{28}$$

$$\Phi_2(t, x, \psi_1, \psi_2) = B_0(t,x) + B_1(t)\psi_1 + B_2(t,x)\psi_2 \tag{29}$$

where

$$\begin{aligned} A_1(t,x) = &-\frac{1}{4}(\mathrm{e}^{-2\mathrm{i}\omega t} + 2\mathrm{i}M\omega x^2 \sin 2\omega t)\delta_1 - \frac{\mathrm{i}}{4}(\mathrm{e}^{-2\mathrm{i}\omega t} - 2M\omega x^2 \cos 2\omega t)\delta_2 \\ &- \mathrm{i}M\omega x(\delta_4 \sin \omega t - \delta_5 \cos \omega t) + \delta_{13} + \mathrm{i}\delta_6 \end{aligned} \tag{30}$$

$$A_2(t) = (\delta_7 - \mathrm{i}\delta_{10})\,\mathrm{e}^{\mathrm{i}\omega t} \tag{31}$$

$$B_1(t) = (\delta_8 - \mathrm{i}\delta_{11})\,\mathrm{e}^{-\mathrm{i}\omega t} \tag{32}$$

$$\begin{aligned} B_2(t,x) = &-\frac{1}{4}(\mathrm{e}^{2\mathrm{i}\omega t} + 2\mathrm{i}M\omega x^2 \sin 2\omega t)\delta_1 + \frac{\mathrm{i}}{4}(\mathrm{e}^{2\mathrm{i}\omega t} + 2M\omega x^2 \cos 2\omega t)\delta_2 \\ &- \mathrm{i}M\omega x(\delta_4 \sin \omega t - \delta_5 \cos \omega t) + \delta_9 + \mathrm{i}\delta_{12}. \end{aligned} \tag{33}$$

The parameters δ_i $(i = 1, 2, \ldots, 13)$ are all real and the functions $\bar{\Phi}_1, \bar{\Phi}_2$ are in fact the complex conjugates of Φ_1, Φ_2. The functions $A_0(t,x)$, $B_0(t,x)$ and their conjugates $\bar{A}_0(t,x)$, $\bar{B}_0(t,x)$ are such that they satisfy respectively (15) and (18) for $\psi_1 = A_0$ and $\psi_2 = B_0$.

The infinitesimal generators of the invariance finite-dimensional Lie algebra are thus easily obtained using the preceding equations and (19). We get

$$\begin{aligned} \tilde{X}_1 = &\frac{1}{2\omega}\sin 2\omega t\,\partial_t + \frac{x}{2}\cos 2\omega t\,\partial_x - \frac{1}{4}\cos 2\omega t\left(\psi_1\partial_{\psi_1} + \bar{\psi}_1\partial_{\bar{\psi}_1} + \psi_2\partial_{\psi_2} + \bar{\psi}_2\partial_{\bar{\psi}_2}\right) \\ &- \mathrm{i}\frac{M\omega x^2}{2}\sin 2\omega t\left(\left(\psi_1\partial_{\psi_1} - \bar{\psi}_1\partial_{\bar{\psi}_1}\right) + \left(\psi_2\partial_{\psi_2} - \bar{\psi}_2\partial_{\bar{\psi}_2}\right)\right) \\ &+ \frac{\mathrm{i}}{4}\sin 2\omega t\left(\left(\psi_1\partial_{\psi_1} - \bar{\psi}_1\partial_{\bar{\psi}_1}\right) - \left(\psi_2\partial_{\psi_2} - \bar{\psi}_2\partial_{\bar{\psi}_2}\right)\right) \end{aligned}$$

$$\tilde{X}_2 = -\frac{1}{2\omega}\cos 2\omega t\,\partial_t + \frac{x}{2}\sin 2\omega t\,\partial_x - \frac{1}{4}\sin 2\omega t\left(\psi_1\partial_{\psi_1} + \bar{\psi}_1\partial_{\bar{\psi}_1} + \psi_2\partial_{\psi_2} + \bar{\psi}_2\partial_{\bar{\psi}_2}\right)$$
$$+\mathrm{i}\frac{M\omega x^2}{2}\cos 2\omega t\left(\left(\psi_1\partial_{\psi_1} - \bar{\psi}_1\partial_{\bar{\psi}_1}\right) + \left(\psi_2\partial_{\psi_2} - \bar{\psi}_2\partial_{\bar{\psi}_2}\right)\right)$$
$$-\frac{\mathrm{i}}{4}\cos 2\omega t\left(\left(\psi_1\partial_{\psi_1} - \bar{\psi}_1\partial_{\bar{\psi}_1}\right) - \left(\psi_2\partial_{\psi_2} - \bar{\psi}_2\partial_{\bar{\psi}_2}\right)\right)$$
$$\tilde{X}_3 = \partial_t$$
$$\tilde{X}_4 = \cos\omega t\,\partial_x - \mathrm{i}M\omega x\sin\omega t\left(\left(\psi_1\partial_{\psi_1} - \bar{\psi}_1\partial_{\bar{\psi}_1}\right) + \left(\psi_2\partial_{\psi_2} - \bar{\psi}_2\partial_{\bar{\psi}_2}\right)\right)$$
$$\tilde{X}_5 = \sin\omega t\,\partial_x + \mathrm{i}M\omega x\cos\omega t\left(\left(\psi_1\partial_{\psi_1} - \bar{\psi}_1\partial_{\bar{\psi}_1}\right) + \left(\psi_2\partial_{\psi_2} - \bar{\psi}_2\partial_{\bar{\psi}_2}\right)\right)$$
$$\tilde{X}_6 = \mathrm{i}\left(\psi_1\partial_{\psi_1} - \bar{\psi}_1\partial_{\bar{\psi}_1}\right)$$
$$\tilde{X}_7 = \mathrm{e}^{\mathrm{i}\omega t}\psi_2\partial_{\psi_1} + \mathrm{e}^{-\mathrm{i}\omega t}\bar{\psi}_2\partial_{\bar{\psi}_1}$$
$$\tilde{X}_8 = \mathrm{e}^{-\mathrm{i}\omega t}\psi_1\partial_{\psi_2} + \mathrm{e}^{\mathrm{i}\omega t}\bar{\psi}_1\partial_{\bar{\psi}_2}$$
$$\tilde{X}_9 = \psi_2\partial_{\psi_2} + \bar{\psi}_2\partial_{\bar{\psi}_2}$$
$$\tilde{X}_{10} = \mathrm{i}\left(\mathrm{e}^{-\mathrm{i}\omega t}\bar{\psi}_2\partial_{\bar{\psi}_1} - \mathrm{e}^{\mathrm{i}\omega t}\psi_2\partial_{\psi_1}\right)$$
$$\tilde{X}_{11} = \mathrm{i}\left(\mathrm{e}^{\mathrm{i}\omega t}\bar{\psi}_1\partial_{\bar{\psi}_2} - \mathrm{e}^{-\mathrm{i}\omega t}\psi_1\partial_{\psi_2}\right)$$
$$\tilde{X}_{12} = \mathrm{i}\left(\psi_2\partial_{\psi_2} - \bar{\psi}_2\partial_{\bar{\psi}_2}\right)$$
$$\tilde{X}_{13} = \left(\psi_1\partial_{\psi_1} + \bar{\psi}_1\partial_{\bar{\psi}_1}\right).$$

If we come back to the real variables u_α and v_α ($\alpha = 1, 2$) introduced in (16), we have the following correspondence:

$$\partial_{\psi_\alpha} = \frac{\mathrm{e}^{-\mathrm{i}v_\alpha}}{2}\left(\partial_{u_\alpha} - \frac{\mathrm{i}}{u_\alpha}\partial_{v_\alpha}\right) \qquad \partial_{\bar{\psi}_\alpha} = \frac{\mathrm{e}^{\mathrm{i}v_\alpha}}{2}\left(\partial_{u_\alpha} + \frac{\mathrm{i}}{u_\alpha}\partial_{v_\alpha}\right) \qquad \alpha = 1, 2. \tag{34}$$

For example, we can write

$$\left(\psi_\alpha\partial_{\psi_\alpha} + \bar{\psi}_\alpha\partial_{\bar{\psi}_\alpha}\right) = u_\alpha\partial_{u_\alpha} \qquad \mathrm{i}\left(\psi_\alpha\partial_{\psi_\alpha} - \bar{\psi}_\alpha\partial_{\bar{\psi}_\alpha}\right) = \partial_{v_\alpha} \qquad \alpha = 1, 2. \tag{35}$$

So from equations (16), (34) and after a slight change of basis, we get the following generators:

$$X_1 = \frac{1}{2\omega}\sin 2\omega t\,\partial_t + \frac{x}{2}\cos 2\omega t\,\partial_x - \frac{1}{4}\cos 2\omega t\left(u_1\partial_{u_1} + u_2\partial_{u_2}\right)$$
$$-\frac{M\omega x^2}{2}\sin 2\omega t\left(\partial_{v_1} + \partial_{v_2}\right) + \frac{1}{4}\sin 2\omega t\left(\partial_{v_1} - \partial_{v_2}\right)$$
$$X_2 = -\frac{1}{2\omega}\cos 2\omega t\,\partial_t + \frac{x}{2}\sin 2\omega t\,\partial_x - \frac{1}{4}\sin 2\omega t\left(u_1\partial_{u_1} + u_2\partial_{u_2}\right)$$
$$+\frac{M\omega x^2}{2}\cos 2\omega t\left(\partial_{v_1} + \partial_{v_2}\right) - \frac{1}{4}\cos 2\omega t\left(\partial_{v_1} - \partial_{v_2}\right)$$
$$X_3 = \partial_t + \frac{\omega}{2}\left(\partial_{v_1} - \partial_{v_2}\right)$$
$$X_4 = \cos\omega t\,\partial_x - M\omega x\sin\omega t\left(\partial_{v_1} + \partial_{v_2}\right)$$
$$X_5 = \sin\omega t\,\partial_x + M\omega x\cos\omega t\left(\partial_{v_1} + \partial_{v_2}\right)$$
$$X_6 = \left(\partial_{v_1} + \partial_{v_2}\right)$$
$$X_7 = \cos(\omega t + v_2 - v_1)u_2\partial_{u_1} + \frac{u_2}{u_1}\sin(\omega t + v_2 - v_1)\partial_{v_1}$$

Table 1. Commutation relations of a $sl(2,\mathbb{R}) \ni h(2)$ algebra.

	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	$\frac{1}{2\omega}X_3$	$2\omega X_2$	$-\frac{1}{2}X_4$	$\frac{1}{2}X_5$	0
X_2	$-\frac{1}{2\omega}X_3$	0	$-2\omega X_1$	$-\frac{1}{2}X_5$	$-\frac{1}{2}X_4$	0
X_3	$-2\omega X_2$	$2\omega X_1$	0	$-\omega X_5$	ωX_4	0
X_4	$\frac{1}{2}X_4$	$\frac{1}{2}X_5$	ωX_5	0	$M\omega X_6$	0
X_5	$-\frac{1}{2}X_5$	$\frac{1}{2}X_4$	$-\omega X_4$	$-M\omega X_6$	0	0
X_6	0	0	0	0	0	0

Table 2. Commutation relations of a complex extension of $su(2)$.

	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}
X_7	0	X_9	$-2X_7$	0	X_{12}	$-2X_{10}$
X_8	$-X_9$	0	$2X_8$	$-X_{12}$	0	$2X_{11}$
X_9	$2X_7$	$-2X_8$	0	$2X_{10}$	$-2X_{11}$	0
X_{10}	0	X_{12}	$-2X_{10}$	0	$-X_9$	$2X_7$
X_{11}	$-X_{12}$	0	$2X_{11}$	X_9	0	$-2X_8$
X_{12}	$2X_{10}$	$-2X_{11}$	0	$-2X_7$	$2X_8$	0

$$X_8 = \cos(\omega t + \nu_2 - \nu_1)u_1\partial_{u_2} - \frac{u_1}{u_2}\sin(\omega t + \nu_2 - \nu_1)\partial_{\nu_2}$$
$$X_9 = u_2\partial_{u_2} - u_1\partial_{u_1}$$
$$X_{10} = \sin(\omega t + \nu_2 - \nu_1)u_2\partial_{u_1} - \frac{u_2}{u_1}\cos(\omega t + \nu_2 - \nu_1)\partial_{\nu_1}$$
$$X_{11} = -\sin(\omega t + \nu_2 - \nu_1)u_1\partial_{u_2} - \frac{u_1}{u_2}\cos(\omega t + \nu_2 - \nu_1)\partial_{\nu_2}$$
$$X_{12} = \partial_{\nu_1} - \partial_{\nu_2}$$
$$X_{13} = u_1\partial_{u_1} + u_2\partial_{u_2}.$$

Table 1 shows the commutation relations between the generators X_j, $j = 1, 2, \ldots, 6$. They form a Lie algebra isomorphic to $sl(2,\mathbb{R}) \ni h(2) = \{X_1, X_2, X_3\} \ni \{X_4, X_5, X_6\}$. Table 2 shows the commutation relations between the generators X_j, $j = 7, \ldots, 12$, which form a Lie algebra isomorphic to the complex extension of $su(2)$ denoted by $su(2)^{\mathbb{C}}$. The generator X_{13} is a central element in this complete algebra. Since the generators of table 1 commute with those of table 2, we get the symmetry Lie algebra of the set (15) as $\{sl(2,\mathbb{R}) \ni h(2)\} \oplus su(2)^{\mathbb{C}} \oplus \{X_{13}\}$. The interpretation of these symmetries with respect to other approaches requires us to compute the finite symmetry transformations of the independent and dependent variables and also a specific realization of the preceding generators. That is what we propose to do in the following subsection.

2.2. *Integration of vector fields and realization of the generators*

Once we integrate the vector fields, we get the one-parameter groups of transformations which leave equation (15) invariant. To the generator X_1, corresponds the following transformation (with the integration parameter λ_1) on time and space coordinates

$$\tilde{t} = \frac{1}{\omega}\arctan(e^{\lambda_1}\tan\omega t) \qquad \tilde{x} = e^{(\lambda_1/2)}x\left(\frac{1+\tan^2\omega t}{1+e^{2\lambda_1}\tan^2\omega t}\right)^{1/2} \tag{36}$$

and on the wavefunction

$$\tilde{\Psi}(\tilde{t},\tilde{x}) = e^{\frac{\lambda_1}{4}}\left(\frac{1+\tan^2\omega\tilde{t}}{1+e^{-2\lambda_1}\tan^2\omega\tilde{t}}\right)^{1/4}\exp\left[i\frac{M\omega\tilde{x}^2}{2\tan\omega\tilde{t}}\left(1-\frac{1+\tan^2\omega\tilde{t}}{1+e^{-2\lambda_1}\tan^2\omega\tilde{t}}\right)\right]$$
$$\times\begin{pmatrix} e^{i\omega(\tilde{t}-t)} & 0 \\ 0 & e^{-i\omega(\tilde{t}-t)}\end{pmatrix}\Psi(t,x) \tag{37}$$

where t and x in the expression of $\Psi(t,x)$ given before have to be evaluated using the inverse of (36). To the generator X_2, corresponds

$$\tilde{t} = \frac{1}{\omega}\arctan\left(e^{-\lambda_2}\tan\left(\frac{\pi}{4}+\omega t\right)\right) - \frac{\pi}{4\omega} \qquad \tilde{x} = e^{-\frac{\lambda_2}{2}}x\left(\frac{1+\tan^2\left(\frac{\pi}{4}+\omega t\right)}{1+e^{-2\lambda_2}\tan^2\left(\frac{\pi}{4}+\omega t\right)}\right)^{1/2} \tag{38}$$

and

$$\tilde{\Psi}(\tilde{t},\tilde{x}) = e^{\lambda_2/4}\left(\frac{1+\tan^2\left(\frac{\pi}{4}+\omega\tilde{t}\right)}{1+e^{2\lambda_2}\tan^2\left(\frac{\pi}{4}+\omega\tilde{t}\right)}\right)^{1/4}$$
$$\times\exp\left[i\frac{M\omega\tilde{x}^2}{2\tan\left(\frac{\pi}{4}+\omega\tilde{t}\right)}\left(1-\frac{1+\tan^2\left(\frac{\pi}{4}+\omega\tilde{t}\right)}{1+e^{2\lambda_2}\tan^2\left(\frac{\pi}{4}+\omega\tilde{t}\right)}\right)\right]$$
$$\times\begin{pmatrix} e^{i\omega(\tilde{t}-t)} & 0 \\ 0 & e^{-i\omega(\tilde{t}-t)}\end{pmatrix}\Psi(t,x) \tag{39}$$

where t and x in the expression of $\Psi(t,x)$ in this equation have to be evaluated using the inverse of (38). The generator X_3 corresponds to a time translation and the following transformation of the wavefunction

$$\tilde{\Psi}(\tilde{t},\tilde{x}) = \begin{pmatrix} e^{i\omega\lambda_3/2} & 0 \\ 0 & e^{-i\omega\lambda_3/2}\end{pmatrix}\Psi(\tilde{t}-\lambda_3,\tilde{x}). \tag{40}$$

With the generator X_4, we associate the transformation

$$\tilde{t} = t \qquad \tilde{x} = x + \lambda_4\cos\omega t \tag{41}$$

and

$$\tilde{\Psi}(\tilde{t},\tilde{x}) = \exp\left[-iM\omega\left(\lambda_4\tilde{x} - \frac{\lambda_4^2}{2}\cos\omega\tilde{t}\right)\sin\omega\tilde{t}\right]\Psi(\tilde{t},\tilde{x}-\lambda_4\cos\omega\tilde{t}). \tag{42}$$

The transformation associated with the generator X_5 is similar and gives

$$\tilde{t} = t \qquad \tilde{x} = x + \lambda_5\sin\omega t \tag{43}$$

together with

$$\tilde{\Psi}(\tilde{t},\tilde{x}) = \exp\left[iM\omega\left(\lambda_5\tilde{x} - \frac{\lambda_5^2}{2}\sin\omega\tilde{t}\right)\cos\omega\tilde{t}\right]\Psi(\tilde{t},\tilde{x}-\lambda_5\sin\omega\tilde{t}). \tag{44}$$

The generators X_j, $j = 6, 7, \ldots, 13$ are not associated with space–time transformations but with transformations of the wavefunction which leave invariant the original set of equations. The integration of these vector fields leads to the following transformations:

$$\tilde{\Psi}(t,x) = e^{i\lambda_6}\Psi(t,x) \qquad \tilde{\Psi}(t,x) = \begin{pmatrix} 1 & \lambda_7 e^{i\omega t} \\ 0 & 1\end{pmatrix}\Psi(t,x) \tag{45}$$

$$\tilde{\Psi}(t,x) = \begin{pmatrix} 1 & 0 \\ \lambda_8 \mathrm{e}^{-\mathrm{i}\omega t} & 1 \end{pmatrix} \Psi(t,x) \qquad \tilde{\Psi}(t,x) = \begin{pmatrix} \mathrm{e}^{-\lambda_9} & 0 \\ 0 & \mathrm{e}^{\lambda_9} \end{pmatrix} \Psi(t,x) \tag{46}$$

$$\tilde{\Psi}(t,x) = \begin{pmatrix} 1 & -\mathrm{i}\lambda_{10} \mathrm{e}^{\mathrm{i}\omega t} \\ 0 & 1 \end{pmatrix} \Psi(t,x) \qquad \tilde{\Psi}(t,x) = \begin{pmatrix} 1 & 0 \\ -\mathrm{i}\lambda_{11} \mathrm{e}^{-\mathrm{i}\omega t} & 1 \end{pmatrix} \Psi(t,x) \tag{47}$$

$$\tilde{\Psi}(t,x) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\lambda_{12}} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\lambda_{12}} \end{pmatrix} \Psi(t,x) \qquad \tilde{\Psi}(t,x) = \mathrm{e}^{\lambda_{13}} \Psi(t,x). \tag{48}$$

Now from these finite transformations we can find a matrix realization of the infinitesimal generators of the invariance Lie algebra. It is easy to show that we get

$$\begin{aligned} C_-(t) &= 2\omega(\mathrm{i}X_1 - X_2) \\ &= \mathrm{e}^{2\mathrm{i}\omega t}\left(\left(\partial_t + \mathrm{i}\omega x \partial_x + \mathrm{i}M\omega^2 x^2 + \mathrm{i}\frac{\omega}{2}\right)\sigma_0 - \mathrm{i}\frac{\omega}{2}\sigma_3\right) \end{aligned} \tag{49}$$

$$\begin{aligned} C_+(t) &= -2\omega(\mathrm{i}X_1 + X_2) \\ &= \mathrm{e}^{-2\mathrm{i}\omega t}\left(\left(\partial_t - \mathrm{i}\omega x \partial_x + \mathrm{i}M\omega^2 x^2 - \mathrm{i}\frac{\omega}{2}\right)\sigma_0 - \mathrm{i}\frac{\omega}{2}\sigma_3\right) \end{aligned} \tag{50}$$

$$H_0 = \mathrm{i}X_3 = \mathrm{i}\sigma_0\partial_t + \frac{\omega}{2}\sigma_3 \tag{51}$$

$$A_{x,-}(t) = \frac{1}{\sqrt{2M\omega}}(X_4 + \mathrm{i}X_5) = \frac{1}{\sqrt{2M\omega}}\mathrm{e}^{\mathrm{i}\omega t}(M\omega x + \partial_x)\sigma_0 \tag{52}$$

$$A_{x,+}(t) = \frac{-1}{\sqrt{2M\omega}}(X_4 - \mathrm{i}X_5) = \frac{1}{\sqrt{2M\omega}}\mathrm{e}^{-\mathrm{i}\omega t}(M\omega x - \partial_x)\sigma_0 \tag{53}$$

$$I = \mathrm{i}X_6 = X_{13} = \sigma_0 \tag{54}$$

$$T_+(t) = X_7 = -\mathrm{i}X_{10} = \mathrm{e}^{\mathrm{i}\omega t}\sigma_+ \tag{55}$$

$$T_-(t) = X_8 = -\mathrm{i}X_{11} = \mathrm{e}^{-\mathrm{i}\omega t}\sigma_- \tag{56}$$

$$2Y = X_9 = -\mathrm{i}X_{12} = \sigma_3 \tag{57}$$

where $\sigma_+ = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = \frac{1}{2}(\sigma_1 + \mathrm{i}\sigma_2)$ and $\sigma_- = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) = \frac{1}{2}(\sigma_1 - \mathrm{i}\sigma_2)$, with $\sigma_1, \sigma_2, \sigma_3$ the standard Pauli matrices. The generators C_-, C_+, H_0, $A_{x,-}(t)$, $A_{x,+}(t)$ and I correspond exactly to the maximal kinematical algebra $sl(2,\mathbb{R}) \ni\!\!\!\!\supset h(2)$. The generators T_+, T_- and Y correspond to the algebra $su(2)$ and are associated with the fermionic symmetries of the SUSY harmonic oscillator.

Since we expect for the SUSY harmonic oscillator the presence of bosonic (even) and fermionic (odd) symmetries, we can associate with these generators a parity, i.e., those represented in terms of diagonal matrices are called even and those represented by anti-diagonal matrices are called odd. So they now close a Lie superalgebra

$$(sl(2, \mathbb{R} \ni\!\!\!\!\supset sh(2/2)) \oplus \{Y\}.$$

The Lie superalgebra $sh(2/2)$ is given by the set $\{A_{x,-}(t), A_{x,+}(t), I; T_+(t), T_-(t)\}$ and the associated non-zero super-commutation relations are

$$[A_{x,-}(t), A_{x,+}(t)] = \{T_-(t), T_+(t)\} = I. \tag{58}$$

The SUSY generators are not obtained by these procedures. Since they play the role of exchanging bosonic and fermionic fields, they are known to be associated with a composition of even and odd generators. Indeed, they may be written as the products

$$Q_+ = \sqrt{\omega}T_+(t)A_{x,+}(t) \qquad Q_- = \sqrt{\omega}T_-(t)A_{x,-}(t) \tag{59}$$

$$S_+(t) = \sqrt{\omega} T_+(t) A_{x,-}(t) \qquad S_-(t) = \sqrt{\omega} T_-(t) A_{x,+}(t) \tag{60}$$

and they close together with the original ones, given in equations (49)–(57), the superalgebra $osp(2/2) \supset sh(2/2)$. So the prolongation method, using such matrix realization, has given bosonic and fermionic symmetries which close a superalgebra. The SUSY generators have been included by hand by taking suitable products of some basic even ($A_{x,\pm}(t)$) and odd ($T_\pm(t)$) generators. To explain why these products appear, it is convenient to relate the prolongation method of searching for symmetries to the general concept of symmetry of a quantum system.

We consider the general transformation

$$\tilde{\Psi}(t, x) = X\Psi(t, x) \tag{61}$$

on the wavefunction $\Psi(t, x)$ of our quantum system (12), where X is a operator such that

$$(\mathrm{i}\partial_t - H_{\mathrm{SUSY}})X\Psi(t, x) = 0 \tag{62}$$

i.e., X transforms solutions of our system into solutions. The operator X of (62) is called a symmetry operator of the model under study. In this more general context it is clear that, if two operators X and Y satisfy (62), the product XY does also. Moreover, if equation (12) satisfies the superposition principle of solutions, the linear combination $\alpha X + \beta Y$, where $\alpha, \beta \in \mathbb{C}$, also satisfies (62). But the complete set of operators obtained by this procedure does not necessarily close a Lie algebra or superalgebra.

Let us take the operator X in (62) to be on the differential form [6]

$$X = \sum_{\mu=0}^{4} \left[\sigma_\mu (\phi_\mu^0(t, x) + \phi_\mu^1(t, x)\partial_t + \phi_\mu^2(t, x)\partial_x)\right] \tag{63}$$

where the $\phi_\mu^k(t, x)$, $k = 0, 1, 2$; $\mu = 1, \ldots, 4$, are real functions of t and x. Comparing the coefficients of several independent products of derivatives we get a system of PDEs which can be solved to determine $\phi_\mu^0(t, x)$, $\phi_\mu^1(t, x)$ and $\phi_\mu^2(t, x)$, up to a finite number of arbitrary integration constants. Solving this system, inserting the results in (63), identifying the different operators according to each integration constant, and finally, taking suitable combinations of these operators, we obtain the generators (49)–(57) and also other ones. These last are the second-order products of the original ones, i.e.,

$$YC_+(t) \quad YC_-(t) \quad YA_{x,+}(t) \quad YA_{x,-}(t) \quad YX_3 \tag{64}$$

$$T_+(t)C_+(t) \quad T_+(t)C_-(t) \quad T_+(t)A_{x,+}(t) \quad T_+A_{x,-}(t) \quad T_+(t)X_3 \tag{65}$$

$$T_-(t)C_+(t) \quad T_-(t)C_-(t) \quad T_-(t)A_{x,+}(t) \quad T_-A_{x,-}(t) \quad T_-(t)X_3. \tag{66}$$

It is easy to show that the whole set of symmetries does not close a Lie algebra or a Lie superalgebra and the only way to close the structure under both commutation and anti-commutation relations is to select among all the preceding products the four ones given in (59), (60).

Let us finally mention that on the space of solutions of equation (12), the superalgebra $osp(2/2) \supset sh(2/2)$ may be expressed as

$$C_-(t) = \mathrm{i}\omega\, \mathrm{e}^{2\mathrm{i}\omega t}\frac{a_x^2}{2}\sigma_0 \qquad C_+(t) = \mathrm{i}\omega\, \mathrm{e}^{-2\mathrm{i}\omega t}\frac{(a_x^\dagger)^2}{2}\sigma_0 \qquad H_0 = \omega\left(a_x^\dagger a_x + \frac{1}{2}\right) \tag{67}$$

$$A_{x,-}(t) = \mathrm{e}^{\mathrm{i}\omega t} a_x \sigma_0 \qquad A_{x,+}(t) = \mathrm{e}^{-\mathrm{i}\omega t} a_x^\dagger \sigma_0 \qquad I = \sigma_0 \tag{68}$$

$$T_+(t) = \mathrm{e}^{\mathrm{i}\omega t}\sigma_+ \qquad T_-(t) = \mathrm{e}^{-\mathrm{i}\omega t}\sigma_- \qquad Y = \frac{\sigma_3}{2} \tag{69}$$

Table 3. Super-commutation relations of a $osp(2/2)$ superalgebra.

	H_0	$C_-(t)$	$C_+(t)$	Y	Q_-	Q_+	$S_-(t)$	$S_+(t)$
H_0	0	$-2\omega C_-$	$2\omega C_+$	0	$-\omega Q_-$	ωQ_+	ωS_-	$-\omega S_+$
$C_-(t)$	$2\omega C_-(t)$	0	$-\omega H_0$	0	0	$\mathrm{i}\omega S_+$	$\mathrm{i}\omega Q_-$	0
$C_+(t)$	$-2\omega C_+$	ωH_0	0	0	$-\mathrm{i}\omega S_-$	0	0	$-\mathrm{i}\omega Q_+$
Y	0	0	0	0	$-Q_-$	Q_+	$-S_-$	S_+
Q_-	ωQ_-	0	$\mathrm{i}\omega S_-$	Q_-	0	$H_0 - \omega Y$	0	$-2\mathrm{i}C_-$
Q_+	ωQ_+	$\mathrm{i}\omega S_+$	0	$-Q_+$	$H_0 - \omega Y$	0	$-2\mathrm{i}C_+$	0
$S_-(t)$	$-\omega S_-$	$-\mathrm{i}\omega Q_-$	0	$2S_-$	0	$-2\mathrm{i}C_+$	0	$H_0 + \omega Y$
$S_+(t)$	ωS_+	0	$\mathrm{i}\omega Q_+$	$-2S_-$	$-2\mathrm{i}C_-$	0	$H_0 + \omega Y$	0

Table 4. Super-commutation relations between the generators of $osp(2/2)$ and $sh(2/2)$.

	H_0	$C_-(t)$	$C_+(t)$	Y	Q_-	Q_+	$S_-(t)$	$S_+(t)$
$A_{x,-}(t)$	$\omega A_{x,-}$	0	$\mathrm{i}\omega A_{x,+}$	0	0	$\sqrt{\omega}T_+$	$\sqrt{\omega}T_-$	0
$A_{x,+}(t)$	$-\omega A_{x,+}$	$-\mathrm{i}\omega A_{x,-}$	0	0	$-\sqrt{\omega}T_-$	0	0	$-\sqrt{\omega}T_+$
I	0	0	0	0	0	0	0	0
$T_-(t)$	0	0	0	T_-	0	$\sqrt{\omega}A_{x,+}$	0	$\sqrt{\omega}A_{x,-}$
$T_+(t)$	0	0	0	$-T_+$	$\sqrt{\omega}A_{x,-}$	0	$\sqrt{\omega}A_{x,+}$	0

$$Q_+ = \sqrt{\omega}a_x^\dagger\sigma_+ \qquad Q_- = \sqrt{\omega}a_x\sigma_- \tag{70}$$
$$S_+(t) = \sqrt{\omega}\,\mathrm{e}^{2\mathrm{i}\omega t}a_x\sigma_+ \qquad S_-(t) = \sqrt{\omega}\,\mathrm{e}^{-2\mathrm{i}\omega t}a_x^\dagger\sigma_- \tag{71}$$

where

$$a_x = \frac{1}{\sqrt{2M\omega}}(M\omega x + \partial_x) \qquad a_x^\dagger = \frac{1}{\sqrt{2M\omega}}(M\omega x - \partial_x) \tag{72}$$

are the usual annihilation and creation operators, respectively. They satisfy the commutation relation

$$\left[a_x, a_x^\dagger\right] = 1. \tag{73}$$

Table 3 shows the commutation and anticommutation relations between the generators of the orthosymplectic superalgebra and, as expected, the generators $Q_\pm$ are the supercharges of the system. This means that they satisfy

$$\{Q_+, Q_-\} = H_0 - \omega Y = H_{\mathrm{SUSY}} \qquad (Q_\pm)^2 = 0 \tag{74}$$

and

$$[H_{\mathrm{SUSY}}, Q_\pm] = 0. \tag{75}$$

Table 4 shows the structure relations between the generators of $osp(2/2)$ and $sh(2/2)$.

3. A non-relativistic spin-$\frac{1}{2}$ particle in a constant magnetic field

A problem which is related to the preceding one is the search for symmetries and supersymmetries of the Schrödinger–Pauli equation describing the motion in the plane of a non-relativistic spin-$\frac{1}{2}$ particle of electric charge e in a constant magnetic field $\vec{B} = (0, 0, B)$ orthogonal to the plane. We thus have the equation

$$(\mathrm{i}\partial_t - H_P)\Psi(t, x, y) = 0 \tag{76}$$

where $\Psi(t, x, y)$ is as given in (14) except that ψ_1 and ψ_2 depend on t, x and y. The Hamiltonian is explicitly given by

$$H_P = \frac{(\vec{\sigma}\cdot(\vec{p} - e\vec{A}))^2}{2M} = \frac{(\vec{p} - e\vec{A})^2}{2M} + \mathrm{i}\vec{\sigma}\cdot((\vec{p} - e\vec{A}) \times (\vec{p} - e\vec{A})). \tag{77}$$

The vector $\vec{\sigma}$ is given by $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, where the σ_i are the usual Pauli matrices, $\vec{p} = (p_x, p_y, 0)$ is the linear momentum and

$$\vec{A} = \left(-\tfrac{1}{2}By, \tfrac{1}{2}Bx, 0\right) \tag{78}$$

is the vector potential in the symmetric gauge. We can thus write equation (76) as the following set of equations:

$$\left\{\mathrm{i}\partial_t + \frac{1}{2M}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \mathrm{i}eB\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) - \frac{e^2B^2}{4}(x^2 + y^2) + eB_\alpha\right)\right\}\psi_\alpha(t, x, y) = 0 \tag{79}$$

with $\alpha = 1, 2$ and where we have set $B_1 = B$, $B_2 = -B$.

To get the infinitesimal generators corresponding to the symmetries of this set, we can again apply the prolongation method where the wavefunction $\Psi(t, x, y)$ may be written now as

$$\psi_1(t, x, y) = u_1(t, x, y)\,\mathrm{e}^{\mathrm{i}v_1(t,x,y)} \qquad \psi_2(t, x, y) = u_2(t, x, y)\,\mathrm{e}^{\mathrm{i}v_2(t,x,y)} \tag{80}$$

where u_1, u_2, v_1 and v_2 are real functions. The corresponding vector field is

$$v = \xi_1\partial_t + \xi_2\partial_x + \xi_3\partial_y + \phi_1\partial_{u_1} + \phi_2\partial_{u_2} + \varphi_1\partial_{v_1} + \varphi_2\partial_{v_2} \tag{81}$$

where ξ_j $(j = 1, 2, 3)$, ϕ_α and φ_α $(\alpha = 1, 2)$ are real functions of $t, x, y, u_1, u_2, v_1, v_2$. As before, it is easier to make the calculation using the complex form of the vector field

$$v = \xi_1\partial_t + \xi_2\partial_x + \xi_3\partial_y + \Phi_1\partial_{\psi_1} + \bar{\Phi}_1\partial_{\bar{\psi}_1} + \Phi_2\partial_{\psi_2} + \bar{\Phi}_2\partial_{\bar{\psi}_2} \tag{82}$$

and to go back to the real form after. We finally get the following generators, where we have introduced $\omega = \frac{eB}{2M}$:

$$X_0 = \partial_t - \omega(x\partial_y - y\partial_x) + \omega\left(\partial_{v_1} - \partial_{v_2}\right)$$

$$\begin{aligned}X_1 = {} & \cos 2\omega t\,\partial_t - \omega(x\sin 2\omega t - y\cos 2\omega t)\partial_x - \omega(x\cos 2\omega t + y\sin 2\omega t)\partial_y \\ & - M\omega^2(x^2 + y^2)\cos 2\omega t\left(\partial_{v_1} + \partial_{v_2}\right) + \omega\left(\sin 2\omega t\left(u_1\partial_{u_1} + u_2\partial u_2\right)\right. \\ & \left. + \cos 2\omega t\left(\partial_{v_1} - \partial_{v_2}\right)\right)\end{aligned}$$

$$\begin{aligned}X_2 = {} & -\sin 2\omega t\,\partial_t - \omega(x\cos 2\omega t + y\sin 2\omega t)\partial_x + \omega(x\sin 2\omega t - y\cos 2\omega t)\partial_y \\ & + M\omega^2(x^2 + y^2)\sin 2\omega t\left(\partial_{v_1} + \partial_{v_2}\right) + \omega\left(\cos 2\omega t\left(u_1\partial_{u_1} + u_2\partial u_2\right)\right. \\ & \left. - \sin 2\omega t\left(\partial_{v_1} - \partial_{v_2}\right)\right)\end{aligned}$$

$$X_3 = x\partial_y - y\partial_x$$

$$X_4 = -\frac{1}{2\omega}(\cos 2\omega t\,\partial_x - \sin 2\omega t\,\partial_y) + \frac{M}{2}(x\sin 2\omega t + y\cos 2\omega t)\left(\partial_{v_1} + \partial_{v_2}\right)$$

$$X_5 = \frac{1}{2\omega}(\sin 2\omega t\,\partial_x + \cos 2\omega t\,\partial_y) + \frac{M}{2}(x\cos 2\omega t - y\sin 2\omega t)\left(\partial_{v_1} + \partial_{v_2}\right)$$

$$X_6 = \left(\partial_{v_1} + \partial_{v_2}\right)$$

Table 5. Commutation relations for a $\{sl(2,\mathbb{R}) \oplus so(2)\} \ni\!\!\!\!\supset h(4)$ algebra.

	X_0	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_0	0	$2X_2$	$-2\omega X_1$	0	ωX_5	$-\omega X_4$	0	ωX_8	$-\omega X_7$
X_1	$-2\omega X_2$	0	$-2\omega X_0$	0	$\frac{1}{2}X_8$	$\frac{1}{2}X_7$	0	$2\omega^2 X_5$	$2\omega^2 X_4$
X_2	$2\omega X_1$	$2\omega X_0$	0	0	$-\frac{1}{2}X_7$	$\frac{1}{2}X_8$	0	$-2\omega^2 X_4$	$2\omega^2 X_5$
X_3	0	0	0	0	X_5	$-X_4$	0	$-X_8$	X_7
X_4	$-\omega X_5$	$-\frac{1}{2}X_8$	$\frac{1}{2}X_7$	$-X_5$	0	$-\frac{M}{2\omega}X_6$	0	0	0
X_5	ωX_4	$-\frac{1}{2}X_7$	$-\frac{1}{2}X_8$	X_4	$\frac{M}{2\omega}X_6$	0	0	0	0
X_6	0	0	0	0	0	0	0	0	0
X_7	$-\omega X_8$	$-2\omega^2 X_5$	$2\omega^2 X_4$	X_8	0	0	0	0	$-2M\omega X_6$
X_8	ωX_7	$-2\omega^2 X_4$	$-2\omega^2 X_5$	$-X_7$	0	0	0	$2M\omega X_6$	0

$$
\begin{aligned}
X_7 &= \partial_x + M\omega y\left(\partial_{v_1} + \partial_{v_2}\right) \\
X_8 &= \partial_y - M\omega x\left(\partial_{v_1} + \partial_{v_2}\right) \\
X_9 &= \cos(2\omega t + v_2 - v_1)u_2\partial_{u_1} + \frac{u_2}{u_1}\sin(2\omega t + v_2 - v_1)\partial_{v_1} \\
X_{10} &= \cos(2\omega t + v_2 - v_1)u_1\partial_{u_2} - \frac{u_1}{u_2}\sin(2\omega t + v_2 - v_1)\partial_{v_2} \\
X_{11} &= u_2\partial_{u_2} - u_1\partial_{u_1} \\
X_{12} &= \sin(2\omega t + v_2 - v_1)u_2\partial_{u_1} - \frac{u_2}{u_1}\cos(2\omega t + v_2 - v_1)\partial_{v_1} \\
X_{13} &= -\sin(2\omega t + v_2 - v_1)u_1\partial_{u_2} - \frac{u_1}{u_2}\cos(2\omega t + v_2 - v_1)\partial_{v_1} \\
X_{14} &= \partial_{v_1} - \partial_{v_2} \\
X_{15} &= u_1\partial_{u_1} + u_2\partial_{u_2}
\end{aligned}
$$

The commutation relations between the generators X_μ ($\mu = 0, 1, 2, \ldots, 8$) are given in table 5 and give an algebra isomorphic to $\{sl(2,\mathbb{R}) \oplus so(2)\} \ni\!\!\!\!\supset h(4)$. It is easy to show that the generators X_μ ($\mu = 9, 10, \ldots, 14$) form an algebra isomorphic to $su(2)^{\mathbb{C}}$ and equivalent to that given in table 2 for the SUSY harmonic oscillator. These two sets commute with each other and X_{15} is a central element. So we have an algebra isomorphic to $\{(sl(2,\mathbb{R}) \oplus so(2)) \ni\!\!\!\!\supset h(4)\} \oplus su(2)^{\mathbb{C}} \oplus \{X_{15}\}$.

Once again, we can get a specific matrix realization proceeding as in the case of the SUSY harmonic oscillator. It is easy to show that we have the following form for the generators of symmetries of the equation (76):

$$H_0 = \mathrm{i}X_0 = (\mathrm{i}\partial_t + \omega(xp_y - yp_x))\sigma_0 + \omega\sigma_3 \tag{83}$$

$$
\begin{aligned}
C_-(t) = \frac{(X_1 - \mathrm{i}X_2)}{2} = \frac{\mathrm{e}^{2\mathrm{i}\omega t}}{2}\{(\partial_t - \mathrm{i}\omega(xp_y - yp_x) \\
- \omega(xp_x + yp_y) + \mathrm{i}\omega + \mathrm{i}M\omega^2(x^2 + y^2))\sigma_0 - \mathrm{i}\omega\sigma_3\}
\end{aligned} \tag{84}
$$

$$
\begin{aligned}
C_+(t) = \frac{(X_1 + \mathrm{i}X_2)}{2} = \frac{\mathrm{e}^{-2\mathrm{i}\omega t}}{2}\{(\partial_t - \mathrm{i}\omega(xp_y - yp_x) \\
+ \omega(xp_x + yp_y) - \mathrm{i}\omega + \mathrm{i}M\omega^2(x^2 + y^2))\sigma_0 - \mathrm{i}\omega\sigma_3\}
\end{aligned} \tag{85}
$$

$$L = -\mathrm{i}X_3 = xp_y - yp_x \tag{86}$$

$$\mathcal{A}(t) = \sqrt{\frac{\omega}{M}}(\mathrm{i}X_4 + X_5) = \frac{\mathrm{e}^{2\mathrm{i}\omega t}}{2\sqrt{\omega M}}[(p_x + \mathrm{i}p_y) - \mathrm{i}M\omega(x + \mathrm{i}y)]\sigma_0 \tag{87}$$

$$\mathcal{A}^\dagger(t) = \sqrt{\frac{\omega}{M}}(\mathrm{i}X_4 - X_5) = \frac{\mathrm{e}^{-2\mathrm{i}\omega t}}{2\sqrt{\omega M}}[(p_x - \mathrm{i}p_y) + \mathrm{i}M\omega(x - \mathrm{i}y)]\sigma_0 \tag{88}$$

$$I = X_6 = X_{15} = \sigma_0 \tag{89}$$

$$A_- = \frac{-1}{2\sqrt{M\omega}}(X_8 + \mathrm{i}X_7) = \frac{1}{2\sqrt{M\omega}}[(p_x - \mathrm{i}p_y) - \mathrm{i}M\omega(x - \mathrm{i}y)]\sigma_0 \tag{90}$$

$$A_+ = \frac{1}{2\sqrt{M\omega}}(X_8 - \mathrm{i}X_7) = \frac{1}{2\sqrt{M\omega}}[(p_x + \mathrm{i}p_y) + \mathrm{i}M\omega(x + \mathrm{i}y)]\sigma_0 \tag{91}$$

$$T_+(t) = X_9 = -\mathrm{i}X_{12} = \mathrm{e}^{2\mathrm{i}\omega t}\sigma_+ \qquad T_-(t) = X_{10} = -\mathrm{i}X_{13} = \mathrm{e}^{-2\mathrm{i}\omega t}\sigma_- \tag{92}$$

$$Y = X_{11} = -\mathrm{i}X_{14} = \sigma_3. \tag{93}$$

We see that H_0 is essentially the Hamiltonian of the harmonic oscillator in two dimensions, $C_\pm(t)$ correspond to the so-called conformal transformations and L is the angular momentum. The two sets $\{\mathcal{A}(t), \mathcal{A}^\dagger(t)\}$ and $\{A_-, A_+\}$ may be associated with pairs of annihilation and creation operators and I represents the identity generator. Indeed, they can be written as

$$\mathcal{A}(t) = \frac{1}{\sqrt{2}}\mathrm{e}^{2\mathrm{i}\omega t}(a_y - \mathrm{i}a_x)\sigma_0 \qquad \mathcal{A}^\dagger(t) = \frac{1}{\sqrt{2}}\mathrm{e}^{-2\mathrm{i}\omega t}(a_y^\dagger + \mathrm{i}a_x^\dagger)\sigma_0 \tag{94}$$

$$A_- = -\frac{1}{\sqrt{2}}(a_y + \mathrm{i}a_x)\sigma_0 \qquad A_+ = -\frac{1}{\sqrt{2}}(a_y^\dagger - \mathrm{i}a_x^\dagger)\sigma_0 \tag{95}$$

where a_x, $a_x^\dagger$, a_y and $a_y^\dagger$ are defined as in (72). The set $\{T_+(t), T_-(t), Y\}$ corresponds to the Lie algebra $su(2)$. These generators form the maximal kinematical algebra of the Pauli equation (76) which is $\{(sl(2, \mathbb{R}) \oplus so(2)) \ni h(4)\} \oplus su(2)$.

Now the products of the generators which will lead to the invariance superalgebras are obtained from the sets $\{\mathcal{A}(t), \mathcal{A}^\dagger(t)\}$, $\{A_-, A_+\}$ and $\{T_+(t), T_-(t)\}$. There are eight possible products. If we first take

$$Q_- = \sqrt{2\omega}\mathcal{A}(t)T_-(t) \qquad Q_+ = \sqrt{2\omega}\mathcal{A}^\dagger(t)T_+(t) \tag{96}$$

we see that they are independent of time as it was the case with the SUSY harmonic oscillator. Moreover they satisfy

$$Q_-^2 = Q_+^2 = 0 \tag{97}$$

and

$$\{Q_-, Q_+\} = H_P = H_0 - \omega L - \omega Y \qquad [H_P, Q_\mp] = 0. \tag{98}$$

This means that $Q_\mp$ are the supercharges for the Pauli Hamiltonian H_P and a class of supersymmetries of our system. Another set of supersymmetries is found to be

$$S_-(t) = \sqrt{2\omega}A_+T_-(t) \qquad S_+(t) = \sqrt{2\omega}A_-T_+(t). \tag{99}$$

Let us insist on the fact that the two sets of annihilation and creation operators $\{\mathcal{A}(t), \mathcal{A}^\dagger(t)\}$ and $\{A_-, A_+\}$ thus appear in these supersymmetry generators. The operators $S_\pm(t)$, which are now time dependent, satisfy

$$(S_-(t))^2 = (S_+(t))^2 = 0 \tag{100}$$

and

$$\{S_-(t), S_+(t)\} = H_0 + \omega L + \omega Y \qquad [\mathrm{i}\partial_t - H_P, S_\pm(t)] = 0. \tag{101}$$

The other structure relations are computed and the non-zero ones are

$$[H_0, Q_\pm] = \pm\omega Q_\pm \qquad [H_0, S_\pm] = \mp\omega S_\pm(t) \tag{102}$$

$$[C_+(t), Q_-] = \mathrm{i}\omega S_-(t) \qquad [C_-(t), Q_+] = -\mathrm{i}\omega S_+(t) \tag{103}$$

$$[Y, Q_\pm] = \pm 2Q_\pm \qquad [Y, S_\pm(t)] = \pm 2S_\pm(t) \tag{104}$$

$$\{Q_-, S_+(t)\} = 2\mathrm{i}C_-(t) \qquad \{Q_+, S_-\} = 2\mathrm{i}C_+(t). \tag{105}$$

This means that the generators $\{H_0, C_\pm(t), Y, Q_\pm, S_\pm(t)\}$ close the orthosymplectic superalgebra $osp(2/2)$. Together with the other generators of the maximal kinematical algebra, we get the so-called maximal kinematical superalgebra of the Pauli equation defined as $\{osp(2/2)\oplus so(2)\} \ni sh(2/4)$, where $sh(2/4)$ is the Heisenberg–Weyl superalgebra generated by the fermionic generators $T_\pm(t)$, the bosonic generators $\mathcal{A}(t), \mathcal{A}^\dagger(t), A_-, A_+$ and the identity I [7]. Let us finally mention that the remaining products of $\{\mathcal{A}(t), \mathcal{A}^\dagger(t)\}$ and $\{A_-, A_+\}$ with $\{T_+(t), T_-(t)\}$ give rise to the following generators:

$$U_+(t) = \sqrt{2\omega}A_+T_+(t) \qquad U_-(t) = \sqrt{2\omega}A_-T_-(t) \tag{106}$$

$$V_+(t) = \sqrt{2\omega}\mathcal{A}(t)T_+(t) \qquad V_-(t) = \sqrt{2\omega}\mathcal{A}^\dagger(t)T_-(t). \tag{107}$$

They have been introduced in [7] and are contained in the maximal dynamical superalgebra of the system under consideration which is $osp(2/4) \ni sh(2/4)$. Indeed to close the structure with these additional supersymmetries, it is necessary to include new even generators of the dynamical algebra of our model.

4. The Jaynes–Cummings model

Now we consider the system described by a particle of electric charge e, spin $\frac{1}{2}$ and mass M moving in the plane in the presence of constant electric $\vec{E}$ and magnetic $\vec{B}$ fields which are both perpendicular to the plane. It has been shown to be related to the Jaynes–Cummings model [19]. Indeed, the Hamiltonian characterizing such a system is given by

$$H = \frac{(\vec{p} - e\vec{A})^2}{2M} - \frac{e}{2M}\vec{B}\cdot\vec{\sigma} + \frac{e}{4M^2}\vec{E}\cdot(\vec{\sigma}\times(\vec{p} - e\vec{A})) + e\vec{E}\cdot\vec{x} \tag{108}$$

where $\vec{x}$ is the position vector of the particle and $\vec{A}$ is the potential vector given in (78). We are again interested in the motion in the xy-plane for which the contribution of the preceding Hamiltonian is

$$\begin{aligned} H_{\mathrm{JC}} = {} & \frac{1}{2M}\left(p_x^2 + p_y^2 + \frac{e^2B^2}{4}(x^2+y^2) + eB(yp_x - xp_y)\right)\sigma_0 - \frac{eB}{2M}\sigma_3 \\ & + \frac{\mathrm{i}eE}{4M^2}\left((p_x - \mathrm{i}p_y) + \mathrm{i}\frac{eB}{2}(x - \mathrm{i}y)\right)\sigma_+ \\ & - \frac{\mathrm{i}eE}{4M^2}\left((p_x + \mathrm{i}p_y) - \mathrm{i}\frac{eB}{2}(x + \mathrm{i}y)\right)\sigma_- \end{aligned} \tag{109}$$

or again,

$$H_{\mathrm{JC}} = H_P + \frac{\mathrm{i}eE}{4M^2}\left((p_x - \mathrm{i}p_y) + \mathrm{i}\frac{eB}{2}(x - \mathrm{i}y)\right)\sigma_+ - \frac{\mathrm{i}eE}{4M^2}\left((p_x + \mathrm{i}p_y) - \mathrm{i}\frac{eB}{2}(x + \mathrm{i}y)\right)\sigma_- \tag{110}$$

where H_P is the Pauli Hamiltonian (77). Let us assume without loss of generality that $e > 0$ and introduce the operators

$$\mathcal{A} = \mathcal{A}(0) = \frac{1}{\sqrt{2eB}}\left((p_x + \mathrm{i}p_y) - \mathrm{i}\frac{eB}{2}(x + \mathrm{i}y)\right)\sigma_0 \tag{111}$$

$$\mathcal{A}^\dagger = \mathcal{A}^\dagger(0) = \frac{1}{\sqrt{2eB}}\left((p_x - \mathrm{i}p_y) + \mathrm{i}\frac{eB}{2}(x - \mathrm{i}y)\right)\sigma_0 \tag{112}$$

which satisfy the commutation relation

$$[\mathcal{A}, \mathcal{A}^\dagger] = I. \tag{113}$$

These operators are nothing other than the generators given by (87) and (88) at $t = 0$ when we take again $\omega = \frac{eB}{2M}$. Let us recall that they correspond to symmetries of the Pauli system. Here we will see that they are not symmetries of the JC model. The Hamiltonian H_{JC} can thus be written in the form

$$H_{\mathrm{JC}} = \tilde{\omega}\left(\mathcal{A}^\dagger\mathcal{A} + \frac{1}{2}\right)\sigma_0 - \frac{\tilde{\omega}}{2}\sigma_3 + \kappa\mathcal{A}^\dagger\sigma_+ + \bar{\kappa}\mathcal{A}\sigma_- \tag{114}$$

where $\tilde{\omega} = 2\omega = \frac{eB}{M}$ and $\kappa = \frac{\mathrm{i}eE\sqrt{2eB}}{4M^2}$. This means that the Hamiltonian (114) is a realization of the JC Hamiltonian [8, 19, 20] in the special case where the detuning between the frequency of the cavity mode and the atom transition frequency is zero. We also see a close connection with the SUSY harmonic oscillator Hamiltonian described in terms of new annihilation and creation operators $\mathcal{A}$ and $\mathcal{A}^\dagger$ as given in (111), (112).

4.1. Lie algebra of symmetries

We are interested in determining the symmetries of the corresponding evolution equation

$$(\mathrm{i}\partial_t - H_{\mathrm{JC}})\Psi(t, x, y) = 0 \tag{115}$$

where $\Psi(t, x, y)$ is again a two-component wavefunction as in the Pauli equation considered in the preceding section. It can be written explicitly as

$$\mathrm{i}\psi_{1,t} + \frac{1}{2M}\left[\psi_{1,xx} + \psi_{1,yy} - \mathrm{i}eB(x\psi_{1,y} - y\psi_{1,x}) - \frac{e^2B^2}{4}(x^2 + y^2)\psi_1 + eB_1\psi_1\right] + \frac{eE_1}{4M^2}\left(\psi_{2,x} + \mathrm{i}\frac{eB}{2}y\psi_2\right) + \frac{eE}{4M^2}\left(\mathrm{i}\psi_{2,y} + \frac{eB}{2}x\psi_2\right) = 0 \tag{116}$$

$$\mathrm{i}\psi_{2,t} + \frac{1}{2M}\left[\psi_{2,xx} + \psi_{2,yy} - \mathrm{i}eB(x\psi_{2,y} - y\psi_{2,x}) - \frac{e^2B^2}{4}(x^2 + y^2)\psi_2 + eB_2\psi_2\right] + \frac{eE_2}{4M^2}\left(\psi_{1,x} + \mathrm{i}\frac{eB}{2}y\psi_1\right) + \frac{eE}{4M^2}\left(\mathrm{i}\psi_{1,y} + \frac{eB}{2}x\psi_1\right) = 0 \tag{117}$$

where we have set $B_1 = B$, $B_2 = -B$, $E_1 = -E$ and $E_2 = E$.

As in the preceding sections, to get the symmetries of the system (116), (117), we apply the prolongation method to it and the conjugate system. Once again the vector field has the form (82) with

$$\xi_1 = \delta_1 \qquad \xi_2(t, y) = -\delta_2 y + \delta_3 \qquad \xi_3(t, x) = \delta_2 x + \delta_4 \tag{118}$$

$$\Phi_1(t, x, y) = A_0(t, x, y) + A_1(x, y)\psi_1 \qquad \Phi_2(t, x, y) = C_0(t, x, y) + C_2(x, y)\psi_2 \tag{119}$$

where

$$A_1(x,y) = -\mathrm{i}\frac{eB}{2}(\delta_4 x - \delta_3 y) - \mathrm{i}\frac{\delta_2}{2} + (\delta_6 + \mathrm{i}\delta_5) \tag{120}$$

$$C_2(x,y) = -\mathrm{i}\frac{eB}{2}(\delta_4 x - \delta_3 y) + \mathrm{i}\frac{\delta_2}{2} + (\delta_6 + \mathrm{i}\delta_5). \tag{121}$$

The δ_j $(j = 1, \ldots, 6)$ are arbitrary real constants and $A_0(t,x,y)$ and $C_0(t,x,y)$ are arbitrary functions that satisfy the system (116), (117) for $\psi_1 = A_0$ and $\psi_2 = C_0$. The finite-dimensional Lie algebra of symmetries is thus formed by the following infinitesimal generators:

$$X_1 = \partial_t$$
$$X_2 = (x\partial_y - y\partial_x) - \frac{\mathrm{i}}{2}(\psi_1\partial_{\psi_1} - \bar{\psi}_1\partial_{\bar{\psi}_1}) + \frac{\mathrm{i}}{2}(\psi_2\partial_{\psi_2} - \bar{\psi}_2\partial_{\bar{\psi}_2})$$
$$X_3 = \partial_x + \mathrm{i}\frac{eB}{2}y(\psi_1\partial_{\psi_1} - \bar{\psi}_1\partial_{\bar{\psi}_1} + \psi_2\partial_{\psi_2} - \bar{\psi}_2\partial_{\bar{\psi}_2})$$
$$X_4 = \partial_y - \mathrm{i}\frac{eB}{2}x(\psi_1\partial_{\psi_1} - \bar{\psi}_1\partial_{\bar{\psi}_1} + \psi_2\partial_{\psi_2} - \bar{\psi}_2\partial_{\bar{\psi}_2})$$
$$X_5 = \mathrm{i}(\psi_1\partial_{\psi_1} - \bar{\psi}_1\partial_{\bar{\psi}_1}) + \mathrm{i}(\psi_2\partial_{\psi_2} - \bar{\psi}_2\partial_{\bar{\psi}_2})$$
$$X_6 = (\psi_1\partial_{\psi_1} + \bar{\psi}_1\partial_{\bar{\psi}_1}) + (\psi_2\partial_{\psi_2} + \bar{\psi}_2\partial_{\bar{\psi}_2}).$$

Using the real components of the wavefunction $\Psi(t,x,y)$ given by (80), the infinitesimal generators take the form

$$X_1 = \partial_t \tag{122}$$

$$X_2 = (x\partial_y - y\partial_x) - \frac{1}{2}(\partial_{v_1} - \partial_{v_2}) \tag{123}$$

$$X_3 = \partial_x + \frac{eB}{2}y(\partial_{v_1} + \partial_{v_2}) \tag{124}$$

$$X_4 = \partial_y - \frac{eB}{2}x(\partial_{v_1} + \partial_{v_2}) \tag{125}$$

$$X_5 = (\partial_{v_1} + \partial_{v_2}) \tag{126}$$

$$X_6 = u_1\partial_{u_1} + u_2\partial_{u_2}. \tag{127}$$

It is easy to see that X_1 and X_6 are both central elements. The generator X_2 corresponds to a $so(2)$ algebra and the set $\{X_3, X_4, X_5\}$ generates $h(2)$. These last generators are thus associated with a Lie algebra isomorphic to $so(2) \ni\!\!\!\supset h(2)$ and satisfy the following non-zero commutation relations:

$$[X_2, X_3] = -X_4 \qquad [X_2, X_4] = X_3 \tag{128}$$

$$[X_3, X_4] = -eBX_5. \tag{129}$$

Integration of the vector fields gives rise to finite transformations of independent and dependent variables which leave the equation (115) invariant. Explicitly, with X_1 corresponds the invariance under time translation such that

$$\tilde{\Psi}(\tilde{t},\tilde{x},\tilde{y}) = \Psi(\tilde{t} - \lambda_1, \tilde{x}, \tilde{y}). \tag{130}$$

The vector field X_2 corresponds to the invariance under rotation in the xy-plane. The transformation is

$$(\tilde{t},\tilde{x},\tilde{y}) = (t, x\cos\lambda_2 - y\sin\lambda_2, x\sin\lambda_2 + y\cos\lambda_2) \tag{131}$$

and

$$\tilde{\Psi}(\tilde{t},\tilde{x},\tilde{y}) = \begin{pmatrix} e^{-\frac{i}{2}\lambda_2} & 0 \\ 0 & e^{\frac{i}{2}\lambda_2} \end{pmatrix} \Psi(\tilde{t}, \tilde{x}\cos\lambda_2 + \tilde{y}\sin\lambda_2, -\tilde{x}\sin\lambda_2 + \tilde{y}\cos\lambda_2) \tag{132}$$

The vector fields X_3 and X_4 correspond to the invariance under space translations. We have

$$\tilde{\Psi}(\tilde{t},\tilde{x},\tilde{y}) = e^{i\frac{eB}{2}\lambda_3\tilde{y}}\Psi(\tilde{t},\tilde{x}-\lambda_3,\tilde{y}) \tag{133}$$

and

$$\tilde{\Psi}(\tilde{t},\tilde{x},\tilde{y}) = e^{-i\frac{eB}{2}\lambda_4\tilde{x}}\Psi(\tilde{t},\tilde{x},\tilde{y}-\lambda_4) \tag{134}$$

respectively. The vector fields X_5 and X_6 are not related to space–time transformations but to the following phase and scale transformations of the wavefunction respectively:

$$\tilde{\Psi}(t,x,y) = e^{i\lambda_5}\Psi(t,x,y) \tag{135}$$

and

$$\tilde{\Psi}(t,x,y) = e^{\lambda_6}\Psi(t,x,y). \tag{136}$$

From these finite transformations, we easily get a matrix realization of the Lie algebra of symmetries of the equation (115),

$$X_1 = \sigma_0\partial_t \qquad X_2 = (x\partial_y - y\partial_x)\sigma_0 + \frac{i}{2}\sigma_3 \tag{137}$$

$$X_3 = \left(\partial_x - i\frac{eB}{2}y\right)\sigma_0 \qquad X_4 = \left(\partial_y + i\frac{eB}{2}x\right)\sigma_0 \tag{138}$$

$$X_5 = -iX_6 = -iI = -i\sigma_0. \tag{139}$$

Now X_1, while acting on the space of solution of (115), is the Hamiltonian H_{JC} as expected and $J = -iX_2$ is the total angular momentum. Complex linear combinations of X_3 and X_4 give

$$A_- = \frac{1}{\sqrt{2eB}}\left[(p_x - ip_y) - i\frac{eB}{2}(x - iy)\right]\sigma_0 \tag{140}$$

and

$$A_+ = \frac{1}{\sqrt{2eB}}\left[(p_x + ip_y) + i\frac{eB}{2}(x + iy)\right]\sigma_0 \tag{141}$$

which satisfy

$$[A_-, A_+] = I. \tag{142}$$

They are exactly the symmetries of the Pauli Hamiltonian as given in (90) and (91) and close together with the identity I the algebra $h(2)$. The operators $\{H_{JC}, J, A_\mp, I\}$ are thus associated with the maximal kinematical invariance algebra of the JC model and are isomorphic to $(so(2) \ni h(2)) \oplus u(1)$. They are all time independent and thus commute with H_{JC}. Moreover they are all diagonal matrices and correspond necessarily to even generators.

It is neither possible from the prolongation method to produce a Lie superalgebra of symmetries nor a set of supersymmetries as was the case for the preceding models. We know in fact [9, 10] that the standard JC model does not admit a $N = 2$ supersymmetry. It is due to the presence in the Hamiltonian (114) of the additional term $(\kappa\mathcal{A}^\dagger\sigma_+ + \bar{\kappa}\mathcal{A}\sigma_-)$ which can be written as

$$\mathcal{Q} = \frac{1}{\sqrt{2\omega}}(\kappa Q_+ + \bar{\kappa}Q_-) = \kappa\mathcal{A}^\dagger\sigma_+ + \bar{\kappa}\mathcal{A}\sigma_- \tag{143}$$

where Q_+ and Q_- are given in (96). Due to the value of κ, such a term is essentially a multiple of the combination $Q_+ - Q_-$. It is an additional symmetry of H_{JC} which cannot be obtained from the prolongation method. It commutes with $A_\pm, \sigma_0$ and J so that the set $\{H_{JC}, A_\pm, I, J, Q_+ - Q_-\}$ closes a Lie algebra but not a Lie superalgebra. Indeed, $(Q_+ - Q_-)^2$ is not a linear combination of the preceding generators.

5. A generalized Jaynes–Cummings model

As already mentioned in the approach by Andreev and Lerner [9], to be able to get a supersymmetry for the JC model, it is necessary to consider a 4 × 4 matrix version of the JC Hamiltonian. Let us show here how the prolongation method may be adapted to such a model and will lead to the presence of supersymmetry generators as for the SUSY harmonic oscillator and the Pauli Hamiltonians. We first construct a generalized JC model for which the prolongation method will produce symmetries associated with a Lie superalgebra. Next we examine the possibility of getting supersymmetries for a symmetrized JC model and compare the results with the ones associated with the so-called standard SUSY JC model [21].

Let us start with a version of the JC Hamiltonian of the following type:

$$H_T = \begin{pmatrix} H_{JC} - \frac{eB}{2M}\alpha\sigma_0 & 0 \\ 0 & H_{JC} - \frac{eB}{2M}\beta\sigma_0 \end{pmatrix} \tag{144}$$

where H_{JC} is given in (109) and α, β are real parameters. The Schrödinger-type equation is

$$(\mathrm{i}\partial_t - H_T)\Psi(t, x, y) = 0 \tag{145}$$

where $\Psi(t, x, y)$ is a four-component wavefunction whose entries are complex-valued functions equal to $\psi_\rho(t, x, y)(\rho = 1, \dots, 4)$. Since (144) is diagonal we get a system of four equations which are firstly given by (116), (117), where now we have set $B_1 = (\alpha + 1)B$ and $B_2 = (\alpha - 1)B$ while E_1, E_2 are still given by $E_1 = -E_2 = -E$. The second set of equations is similar and we have

$$\mathrm{i}\psi_{3,t} + \frac{1}{2M}\left[\psi_{3,xx} + \psi_{3,yy} - \mathrm{i}eB(x\psi_{3,y} - y\psi_{3,x}) - \frac{e^2B^2}{4}(x^2 + y^2)\psi_3 + eB_3\psi_3\right] + \frac{eE_3}{4M^2}\left(\psi_{4,x} + \mathrm{i}\frac{eB}{2}y\psi_4\right) + \frac{e\tilde{E}_3}{4M^2}\left(\mathrm{i}\psi_{4,y} + \frac{eB}{2}x\psi_4\right) = 0 \tag{146}$$

$$\mathrm{i}\psi_{4,t} + \frac{1}{2M}\left[\psi_{4,xx} + \psi_{4,yy} - \mathrm{i}eB(x\psi_{4,y} - y\psi_{4,x}) - \frac{e^2B^2}{4}(x^2 + y^2)\psi_4 + eB_4\psi_4\right] + \frac{eE_4}{4M^2}\left(\psi_{3,x} + \mathrm{i}\frac{eB}{2}y\psi_3\right) + \frac{e\tilde{E}_4}{4M^2}\left(\mathrm{i}\psi_{3,y} + \frac{eB}{2}x\psi_3\right) = 0 \tag{147}$$

where we have set $B_3 = (\beta + 1)B$, $B_4 = (\beta - 1)B$, $E_3 = -E_4 = -E$ and $\tilde{E}_3 = \tilde{E}_4 = E$. The prolongation method applied to this system and the associated complex conjugated equations leads to a vector field of the form

$$v = \xi_1\partial_t + \xi_2\partial_x + \xi_3\partial_y + \sum_{\rho=1}^{4}\Phi_\rho\partial_{\psi_\rho} + \sum_{\rho=1}^{4}\bar{\Phi}_\rho\partial_{\bar{\psi}_\rho} \tag{148}$$

where ξ_j $(j = 1, 2, 3)$ and Φ_ρ $(\rho = 1, \dots, 4)$ are functions dependent on t, x, y, ψ_ρ and $\bar{\psi}_\rho$. We get the solutions

$$\xi_1 = \delta_1 \qquad \xi_2(t, y) = -\delta_2 y + \delta_3 \qquad \xi_3(t, x) = \delta_2 x + \delta_4 \tag{149}$$

$$\Phi_1(t,x,y) = A_0(t,x,y) + A_1(x,y)\psi_1 + f(t)\psi_3 \tag{150}$$

$$\Phi_2(t,x,y) = C_0(t,x,y) + C_2(x,y)\psi_2 + f(t)\psi_4 \tag{151}$$

$$\Phi_3(t,x,y) = D_0(t,x,y) + g(t)\psi_1 + D_3(x,y)\psi_3 \tag{152}$$

$$\Phi_4(t,x,y) = F_0(t,x,y) + g(t)\psi_2 + F_4(x,y)\psi_4. \tag{153}$$

The functions A_0, C_0, D_0 and F_0 are again arbitrary and such that they satisfy the equation (145) with $\Psi = (A_0, C_0, D_0, F_0)^t$. The other functions in (150) are given by

$$A_1(x,y) = -\mathrm{i}\frac{eB}{2}(\delta_4 x - \delta_3 y) - \mathrm{i}\frac{\delta_2}{2} + (\delta_7 + \mathrm{i}\delta_5) \tag{154}$$

$$C_2(x,y) = -\mathrm{i}\frac{eB}{2}(\delta_4 x - \delta_3 y) + \mathrm{i}\frac{\delta_2}{2} + (\delta_7 + \mathrm{i}\delta_5) \tag{155}$$

$$D_3(x,y) = -\mathrm{i}\frac{eB}{2}(\delta_4 x - \delta_3 y) - \mathrm{i}\frac{\delta_2}{2} + (\delta_8 + \mathrm{i}\delta_6) \tag{156}$$

$$F_4(x,y) = -\mathrm{i}\frac{eB}{2}(\delta_4 x - \delta_3 y) + \mathrm{i}\frac{\delta_2}{2} + (\delta_8 + \mathrm{i}\delta_6) \tag{157}$$

and the functions $f(t)$ and $g(t)$ are obtained as

$$f(t) = (\delta_9 - \mathrm{i}\delta_{11})\,\mathrm{e}^{\mathrm{i}\omega_{\alpha\beta}t} \qquad g(t) = (\delta_{10} - \mathrm{i}\delta_{12})\,\mathrm{e}^{-\mathrm{i}\omega_{\alpha\beta}t}. \tag{158}$$

with $\omega_{\alpha\beta} = \omega(\alpha - \beta) = \frac{eB}{2M}(\alpha - \beta)$.

Let us here comment on this last solution. The prolongation method has been applied to the set of equations (116), (117) and (146), (147) for arbitrary values of B_1, B_2, B_3 and B_4, and leads to the following sets of equations for f and g:

$$\mathrm{i}\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{e}{2M}(B_3 - B_1)f = \frac{e}{2M}(B_4 - B_2)f \tag{159}$$

and

$$\mathrm{i}\frac{\mathrm{d}g}{\mathrm{d}t} = \frac{e}{2M}(B_1 - B_3)g = \frac{e}{2M}(B_2 - B_4)g. \tag{160}$$

They are always compatible in the case under consideration, i.e. the one associated with the Hamiltonian (144), and we get the explicit solution (158). In this context, a particular constant solution is obtained when $\omega_{\alpha\beta} = 0$ or equivalently when $\alpha = \beta$. But if $(B_3 - B_1) \neq (B_4 - B_2)$, equations (159) and (160) admit the trivial solution $f(t) = g(t) = 0$ which is a case that will be considered later.

Let us insist on the fact that since we are here in the case where f and g are given by (158), we will get symmetries expressed by odd generators that will satisfy structure relations corresponding to a superalgebra.

The infinitesimal generators of the finite-dimensional Lie algebra of symmetries may be directly obtained from (148) with the preceding values (149)–(153) but once again to be able to get the finite transformations of symmetries, we have to express the vector fields in terms of the real variables u_ρ, ν_ρ, such that $\psi_\rho = u_\rho\,\mathrm{e}^{\mathrm{i}\nu_\rho}$ $(\rho = 1, \ldots, 4)$. We thus get the following basis of generators:

$$X_1 = \partial_t - \frac{\omega_{\alpha\beta}}{2}\left[\left(\partial_{\nu_3} - \partial_{\nu_1}\right) + \left(\partial_{\nu_4} - \partial_{\nu_2}\right)\right] \tag{161}$$

$$X_2 = (x\partial_y - y\partial_x) - \tfrac{1}{2}[(\partial_{v_1} + \partial_{v_3}) - (\partial_{v_2} + \partial_{v_4})] \tag{162}$$

$$X_3 = \partial_x + \frac{eB}{2}y\sum_{\rho=1}^{4}\partial_{v_\rho} \tag{163}$$

$$X_4 = \partial_y - \frac{eB}{2}x\sum_{\rho=1}^{4}\partial_{v_\rho} \tag{164}$$

$$X_5 = \sum_{\rho=1}^{4}\partial_{v_\rho} \tag{165}$$

$$X_6 = \sum_{\rho=1}^{4}u_\rho\partial_{u_\rho} \tag{166}$$

$$\begin{aligned} X_7 = {} & \cos(\omega_{\alpha\beta}t - v_3 - v_1)u_3\partial_{u_1} + \frac{u_3}{u_1}\sin(\omega_{\alpha\beta}t - v_3 - v_1)\partial_{v_1} \\ & + \cos(\omega_{\alpha\beta}t - v_4 - v_2)u_4\partial_{u_2} + \frac{u_4}{u_2}\sin(\omega_{\alpha\beta}t - v_3 - v_1)\partial_{v_2} \end{aligned} \tag{167}$$

$$\begin{aligned} X_8 = {} & \cos(\omega_{\alpha\beta}t - v_3 - v_1)u_1\partial_{u_3} - \frac{u_1}{u_3}\sin(\omega_{\alpha\beta}t - v_3 - v_1)\partial_{v_3} \\ & + \cos(\omega_{\alpha\beta}t - v_4 - v_2)u_2\partial_{u_4} - \frac{u_2}{u_4}\sin(\omega_{\alpha\beta}t - v_3 - v_1)\partial_{v_4} \end{aligned} \tag{168}$$

$$X_9 = (u_3\partial_{u_3} - u_1\partial_{u_1}) + (u_4\partial_{u_4} - u_2\partial_{u_2}) \tag{169}$$

$$\begin{aligned} X_{10} = {} & \sin(\omega_{\alpha\beta}t - v_3 - v_1)u_3\partial_{u_1} - \frac{u_3}{u_1}\cos(\omega_{\alpha\beta}t - v_3 - v_1)\partial_{v_1} \\ & + \sin(\omega_{\alpha\beta}t - v_4 - v_2)u_4\partial_{u_2} - \frac{u_4}{u_2}\cos(\omega_{\alpha\beta}t - v_3 - v_1)\partial_{v_2} \end{aligned} \tag{170}$$

$$\begin{aligned} X_{11} = {} & -\sin(\omega_{\alpha\beta}t - v_3 - v_1)u_1\partial_{u_3} - \frac{u_1}{u_3}\sin(\omega_{\alpha\beta}t - v_3 - v_1)\partial_{v_3} \\ & - \sin(\omega_{\alpha\beta}t - v_4 - v_2)u_2\partial_{u_4} - \frac{u_2}{u_4}\cos(\omega_{\alpha\beta}t - v_3 - v_1)\partial_{v_4} \end{aligned} \tag{171}$$

$$X_{12} = (\partial_{v_3} - \partial_{v_1}) + (\partial_{v_4} - \partial_{v_2}). \tag{172}$$

The generators X_j ($j = 1, \ldots, 6$) satisfy the same commutation relations as the ones satisfied by the generators (122)–(127). The other generators X_j ($j = 7, \ldots, 12$) form an algebra isomorphic to $su(2)^{\mathbf{C}}$ as in the Pauli case. The corresponding Lie algebra of symmetries of (145) is thus isomorphic to $(so(2) \supset\!\!\!\!+ h(2)) \oplus su(2)^{\mathbf{C}} \oplus \{X_1, X_6\}$.

5.1. A Lie superalgebra of symmetries

The integration of the preceding vector fields leads to the corresponding finite transformations on the space–time coordinates and wavefunctions. We get, for X_1, the invariance under time translation such that

$$\tilde{\Psi}(\tilde{t}, \tilde{x}, \tilde{y}) = \begin{pmatrix} e^{\frac{i}{2}\omega_{\alpha\beta}\lambda_1}\sigma_0 & 0 \\ 0 & e^{-\frac{i}{2}\omega_{\alpha\beta}\lambda_1}\sigma_0 \end{pmatrix} \Psi(\tilde{t} - \lambda_1, \tilde{x}, \tilde{y}). \tag{173}$$

The integration of the vector field X_2 implies

$$\tilde{t} = t \qquad \tilde{x} = x\cos\lambda_2 - y\sin\lambda_2 \qquad \tilde{y} = x\sin\lambda_2 + y\cos\lambda_2 \tag{174}$$

and

$$\tilde{\Psi}(\tilde{t},\tilde{x},\tilde{y}) = \begin{pmatrix} e^{-\frac{i}{2}\lambda_2} & 0 & 0 & 0 \\ 0 & e^{\frac{i}{2}\lambda_2} & 0 & 0 \\ 0 & 0 & e^{-\frac{i}{2}\lambda_2} & 0 \\ 0 & 0 & 0 & e^{\frac{i}{2}\lambda_2} \end{pmatrix} \Psi(\tilde{t}, \tilde{x}\cos\lambda_2 + \tilde{y}\sin\lambda_2, -\tilde{x}\sin\lambda_2 + \tilde{y}\cos\lambda_2). \tag{175}$$

The integration of the vector fields X_3 and X_4 implies the invariance under space translations such that

$$\tilde{\Psi}(\tilde{t},\tilde{x},\tilde{y}) = e^{\frac{i}{2}eB\tilde{y}\lambda_3}\Psi(\tilde{t},\tilde{x}-\lambda_3,\tilde{y}) \tag{176}$$

and

$$\tilde{\Psi}(\tilde{t},\tilde{x},\tilde{y}) = e^{-\frac{i}{2}eB\tilde{x}\lambda_4}\Psi(\tilde{t},\tilde{x},\tilde{y}-\lambda_4). \tag{177}$$

The integration of the vector fields X_5 and X_6 leads to phase and scale transformations of the wavefunctions

$$\tilde{\Psi}(t,x,y) = e^{i\lambda_5}\Psi(t,x,y) \tag{178}$$

and

$$\tilde{\Psi}(t,x,y) = e^{\lambda_6}\Psi(t,x,y). \tag{179}$$

The remaining vector fields X_j $(j = 7, \ldots, 12)$ lead to the transformations

$$\tilde{\Psi}(t,x,y) = \begin{pmatrix} \sigma_0 & \lambda_7 e^{i\omega_{\alpha\beta}t}\sigma_0 \\ 0 & \sigma_0 \end{pmatrix}\Psi(t,x,y) \tag{180}$$

$$\tilde{\Psi}(t,x,y) = \begin{pmatrix} \sigma_0 & 0 \\ \lambda_8 e^{-i\omega_{\alpha\beta}t}\sigma_0 & \sigma_0 \end{pmatrix}\Psi(t,x,y) \tag{181}$$

$$\tilde{\Psi}(t,x,y) = \begin{pmatrix} e^{-\lambda_9}\sigma_0 & 0 \\ 0 & e^{\lambda_9}\sigma_0 \end{pmatrix}\Psi(t,x,y) \tag{182}$$

$$\tilde{\Psi}(t,x,y) = \begin{pmatrix} \sigma_0 & -i\lambda_{10} e^{i\omega_{\alpha\beta}t}\sigma_0 \\ 0 & \sigma_0 \end{pmatrix}\Psi(t,x,y) \tag{183}$$

$$\tilde{\Psi}(t,x,y) = \begin{pmatrix} \sigma_0 & 0 \\ -i\lambda_{11} e^{-i\omega_{\alpha\beta}t}\sigma_0 & \sigma_0 \end{pmatrix}\Psi(t,x,y) \tag{184}$$

$$\tilde{\Psi}(t,x,y) = \begin{pmatrix} e^{-i\lambda_{12}}\sigma_0 & 0 \\ 0 & e^{i\lambda_{12}}\sigma_0 \end{pmatrix}\Psi(t,x,y). \tag{185}$$

A specific matrix realization of the symmetry generators is obtained from these transformations, when they are developed at first order in the parameter λ_i. After some linear combinations and redefinitions, we get the following generators

$$X_1 = \partial_t \mathbb{I} - i\frac{\omega_{\alpha\beta}}{2}\begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix} \qquad X_2 = (x\partial_y - y\partial_x)\mathbb{I} + \frac{i}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \tag{186}$$

$$X_3 = \left(\partial_x - i\frac{eB}{2}y\right)\mathbb{I} \qquad X_4 = \left(\partial_y + i\frac{eB}{2}x\right)\mathbb{I} \qquad X_5 = -iX_6 = -i\mathbb{I} \tag{187}$$

$$\mathbb{T}_+(t) = X_7 = -iX_{10} = e^{i\omega_{\alpha\beta}t}\begin{pmatrix} 0 & \sigma_0 \\ 0 & 0 \end{pmatrix} \tag{188}$$

$$\mathbb{T}_-(t) = X_8 = -\mathrm{i}X_{11} = \mathrm{e}^{-\mathrm{i}\omega_{\alpha\beta}t}\begin{pmatrix}0 & 0\\ \sigma_0 & 0\end{pmatrix} \tag{189}$$

$$\mathbb{Y} = X_9 = -\mathrm{i}X_{12} = \begin{pmatrix}\sigma_0 & 0\\ 0 & -\sigma_0\end{pmatrix} \tag{190}$$

where $\mathbb{I}$ is the 4×4 identity matrix. We see that with X_1, we can associate the generator

$$\mathbb{H} = \mathrm{i}\partial_t \mathbb{I} + \frac{\omega_{\alpha\beta}}{2}\begin{pmatrix}\sigma_0 & 0\\ 0 & -\sigma_0\end{pmatrix} \tag{191}$$

which will be related later to a new Hamiltonian refering to a symmetrized version of the JC Hamiltonian. The generator X_2 corresponds to the total angular momentum

$$\mathbb{J} = \begin{pmatrix}J & 0\\ 0 & J\end{pmatrix} \tag{192}$$

and X_3 and X_4 may be combined to give

$$\mathbb{A}_- = \begin{pmatrix}A_- & 0\\ 0 & A_-\end{pmatrix} \qquad \mathbb{A}_+ = \begin{pmatrix}A_+ & 0\\ 0 & A_+\end{pmatrix} \tag{193}$$

where A_- and A_+ are given in (140), (141).

The generators $\mathbb{T}_+(t)$, $\mathbb{T}_-(t)$ and $\mathbb{Y}$ may now be associated with odd generators and, together with $\mathbb{A}_-$, $\mathbb{A}_+$ and $\mathbb{I}$, form a $sh(2/2)$ superalgebra. As in the cases of the SUSY harmonic oscillator and the Pauli systems, odd products may be formed between $\{\mathbb{A}_-, \mathbb{A}_+\}$ and $\{\mathbb{T}_+(t), \mathbb{T}_-(t)\}$ and among the possible ones we get the following generators:

$$\mathbb{S}_-(t) = \sqrt{\tilde{\omega}}\mathbb{A}_+\mathbb{T}_-(t) = \sqrt{\omega}\,\mathrm{e}^{-\mathrm{i}\omega_{\alpha\beta}t}\begin{pmatrix}0 & 0\\ A_+ & 0\end{pmatrix} \tag{194}$$

$$\mathbb{S}_+(t) = \sqrt{\tilde{\omega}}\mathbb{A}_-\mathbb{T}_+(t) = \sqrt{\tilde{\omega}}\,\mathrm{e}^{\mathrm{i}\omega_{\alpha\beta}t}\begin{pmatrix}0 & A_-\\ 0 & 0\end{pmatrix} \tag{195}$$

and

$$\mathbb{U}_-(t) = \sqrt{\tilde{\omega}}\mathbb{A}_-\mathbb{T}_-(t) = \sqrt{\tilde{\omega}}\,\mathrm{e}^{-\mathrm{i}\omega_{\alpha\beta}t}\begin{pmatrix}0 & 0\\ A_- & 0\end{pmatrix} \tag{196}$$

$$\mathbb{U}_+(t) = \sqrt{\tilde{\omega}}\mathbb{A}_+\mathbb{T}_+(t) = \sqrt{\tilde{\omega}}\,\mathrm{e}^{\mathrm{i}\omega_{\alpha\beta}t}\begin{pmatrix}0 & A_+\\ 0 & 0\end{pmatrix} \tag{197}$$

which satisfy the anticommutation relations

$$\{\mathbb{S}_-(t), \mathbb{S}_+(t)\} = \mathbb{H}_0 + \frac{\tilde{\omega}}{2}\mathbb{Y} = \tilde{\omega}\begin{pmatrix}A_-A_+ & 0\\ 0 & A_+A_-\end{pmatrix} \tag{198}$$

and

$$\{\mathbb{U}_-(t), \mathbb{U}_+(t)\} = \mathbb{H}_0 - \frac{\tilde{\omega}}{2}\mathbb{Y} = \tilde{\omega}\begin{pmatrix}A_+A_- & 0\\ 0 & A_-A_+\end{pmatrix} \tag{199}$$

where we have defined

$$\mathbb{H}_0 = \tilde{\omega}\begin{pmatrix}(A_+A_- + \frac{1}{2}) & 0\\ 0 & (A_+A_- + \frac{1}{2})\end{pmatrix}. \tag{200}$$

The two components of $\mathbb{H}_0$ are not related to $\mathbb{H}_{\mathrm{JC}}$ since they have only diagonal terms. The non-zero commutation relations between the generators $\mathbb{S}_\pm(t)$ and $\mathbb{U}_\pm(t)$ are given by

$$\{\mathbb{S}_-, \mathbb{U}_+\} = -2\,\mathrm{i}\mathbb{C}_+ = \tilde{\omega}(\mathbb{A}_+)^2 \qquad \{\mathbb{S}_+, \mathbb{U}_-\} = -2\,\mathrm{i}\mathbb{C}_- = \tilde{\omega}(\mathbb{A}_-)^2. \tag{201}$$

Table 6. Super-commutation relations of an $osp(2/2)$ superalgebra.

	$\mathbb{H}_0$	$\mathbb{C}_-$	$\mathbb{C}_+$	$\mathbb{Y}$	$\mathbb{S}_-$	$\mathbb{S}_+$	$\mathbb{U}_-$	$\mathbb{U}_+$
$\mathbb{H}_0$	0	$-2\tilde{\omega}\mathbb{C}_-$	$2\tilde{\omega}\mathbb{C}_+$	0	$\tilde{\omega}\mathbb{S}_-$	$-\tilde{\omega}\mathbb{S}_+$	$-\tilde{\omega}\mathbb{U}_-$	$\tilde{\omega}\mathbb{U}_+$
$\mathbb{C}_-$	$2\tilde{\omega}\mathbb{C}_-$	0	$-\tilde{\omega}\mathbb{H}_0$	0	$i\tilde{\omega}\mathbb{U}_-$	0	0	$i\tilde{\omega}\mathbb{S}_+$
$\mathbb{C}_+$	$-2\tilde{\omega}\mathbb{C}_+$	$\tilde{\omega}\mathbb{H}_0$	0	0	0	$-i\tilde{\omega}\mathbb{U}_+$	$-i\tilde{\omega}\mathbb{S}_-$	0
$\mathbb{Y}$	0	0	0	0	$-2\mathbb{S}_-$	$2\mathbb{S}_+$	$-2\mathbb{U}_-$	$2\mathbb{U}_+$
$\mathbb{S}_-$	$-\tilde{\omega}\mathbb{S}_-$	$-i\tilde{\omega}\mathbb{U}_-$	0	$2\mathbb{S}_-$	0	$\mathbb{H}_0+\tilde{\omega}\mathbb{Y}/2$	0	$-2i\mathbb{C}_+$
$\mathbb{S}_+$	$\tilde{\omega}\mathbb{S}_+$	0	$i\tilde{\omega}\mathbb{U}_+$	$-2\mathbb{S}_+$	$\mathbb{H}_0+\tilde{\omega}\mathbb{Y}/2$	0	$-2i\mathbb{C}_-$	0
$\mathbb{U}_-$	$\tilde{\omega}\mathbb{U}_-$	0	$i\tilde{\omega}\mathbb{S}_-$	$2\mathbb{U}_-$	0	$-2i\mathbb{C}_-$	0	$\mathbb{H}_0-\tilde{\omega}\mathbb{Y}/2$
$\mathbb{U}_+$	$-\tilde{\omega}\mathbb{U}_+$	$-i\tilde{\omega}\mathbb{S}_+$	0	$-2\mathbb{U}_+$	$-2i\mathbb{C}_+$	0	$\mathbb{H}_0-\tilde{\omega}\mathbb{Y}/2$	0

Table 7. Commutation relations between $(so(2) \ni osp(2/2))$ and $sh(2/2)$.

	$\mathbb{H}_0$	$\mathbb{C}_-$	$\mathbb{C}_+$	$\mathbb{Y}$	$\mathbb{S}_-$	$\mathbb{S}_+$	$\mathbb{U}_-$	$\mathbb{U}_+$
$\mathbb{J}$	0	$-2\mathbb{C}_-$	$2\mathbb{C}_+$	0	$\mathbb{S}_-$	$-\mathbb{S}_+$	$-\mathbb{U}_-$	$\mathbb{U}_+$
$\mathbb{A}_-$	$\tilde{\omega}\mathbb{A}_-$	0	$i\tilde{\omega}\mathbb{A}_+$	0	$\sqrt{\tilde{\omega}}\mathbb{T}_-$	0	0	$\sqrt{\tilde{\omega}}\mathbb{T}_+$
$\mathbb{A}_+$	$-\tilde{\omega}\mathbb{A}_+$	$-i\tilde{\omega}\mathbb{A}_-$	0	0	0	$-\sqrt{\tilde{\omega}}\mathbb{T}_+$	$-\sqrt{\tilde{\omega}}\mathbb{T}_-$	0
$\mathbb{I}$	0	0	0	0	0	0	0	0
$\mathbb{T}_-$	0	0	0	$2\mathbb{T}_-$	0	$\sqrt{\tilde{\omega}}\mathbb{A}_-$	0	$\sqrt{\tilde{\omega}}\mathbb{A}_+$
$\mathbb{T}_+$	0	0	0	$-2\mathbb{T}_+$	$\sqrt{\tilde{\omega}}\mathbb{A}_+$	0	$\sqrt{\tilde{\omega}}\mathbb{A}_-$	0

Now the set $\{\mathbb{H}, \mathbb{H}_0, \mathbb{C}_\pm, \mathbb{Y}, \mathbb{S}_\pm(t), \mathbb{U}_\pm(t), \mathbb{J}, \mathbb{A}_\pm, \mathbb{I}, \mathbb{T}_\pm(t)\}$ closes a superalgebra. We see that $\mathbb{H}$ commutes with all the generators. The commutation relations between the generators $\mathbb{H}_0, \mathbb{C}_\pm, \mathbb{Y}, \mathbb{S}_\pm(t)$ and $\mathbb{U}_\pm(t)$ are given in table 6. These last generators form a superalgebra isomorphic to $osp(2/2)$. The non-zero super-commutation relations between the generators $\mathbb{J}, \mathbb{A}_\pm, \mathbb{I}, \mathbb{T}_\pm(t)$ are now given by

$$[\mathbb{J}, \mathbb{A}_+] = \mathbb{A}_+, \qquad [\mathbb{J}, \mathbb{A}_-] = -\mathbb{A}_- \tag{202}$$

and

$$[\mathbb{A}_-, \mathbb{A}_+] = \mathbb{I} = \{\mathbb{T}_-(t), \mathbb{T}_+(t)\}, \tag{203}$$

leading to the superalgebra $so(2) \ni sh(2/2)$. Finally the super-commutation relations between the two sets are presented in table 7. So we find a structure isomorphic to the superalgebra $(so(2) \ni osp(2/2)) \ni sh(2/2)$. Let us insist on the fact that the existence of such a superalgebra does not imply the presence of supersymmetries for the original Hamiltonian (144). Indeed, no Q-type supercharges may be constructed from the preceding symmetries. In the last subsections, SUSY JC models will be constructed and the symmetries and supersymmetries will be given.

5.2. A supersymmetric JC model

The generator (191) when acting on the space of solutions of (145), corresponds to a symmetrized version of the Hamiltonian (144) given by

$$\mathbb{H} = H_T + \frac{\omega_{\alpha\beta}}{2}\mathbb{Y} = \begin{pmatrix} H_{\mathrm{JC}} - \frac{eB}{4M}(\alpha+\beta)\sigma_0 & 0 \\ 0 & H_{\mathrm{JC}} - \frac{eB}{4M}(\alpha+\beta)\sigma_0 \end{pmatrix}. \tag{204}$$

From the preceding results, it is easy to show that the symmetries of this new Hamiltonian are given by the set $\{\mathbb{H}, \mathbb{H}_0, \mathbb{C}_\pm, \mathbb{Y}, \mathbb{J}, \mathbb{A}_\pm, \mathbb{I}, \mathbb{T}_\pm(0)\}$, all of these generators being time independent. Indeed, $\mathbb{H}$ can be seen as a particular H_T as given in (144) where $\alpha = \beta$ and thus $\omega_{\alpha\beta} = 0$.

It also admits the supersymmetries $\mathbb{S}_\pm(0)$ and $\mathbb{U}_\pm(0)$ which are now time independent and satisfy again (198) and (199). None of them are the supercharges of $\mathbb{H}$.

Let us now show that, for a specific value of $\alpha + \beta$, the symmetrized Hamiltonian $\mathbb{H}$ can be made supersymmetric. Indeed, if we take $(\alpha + \beta) = -\frac{eE}{8M^2B}$, we can define [9]

$$\mathbb{Q}_+ = \frac{\bar{\kappa}}{2\sqrt{\bar{\omega}}}\mathbb{T}_+(0) + \sqrt{\bar{\omega}}\begin{pmatrix} 0 & (Q_+ - Q_-) \\ 0 & 0 \end{pmatrix} \tag{205}$$

and

$$\mathbb{Q}_- = \frac{\kappa}{2\sqrt{\bar{\omega}}}\mathbb{T}_-(0) + \sqrt{\bar{\omega}}\begin{pmatrix} 0 & 0 \\ (Q_- - Q_+) & 0 \end{pmatrix}. \tag{206}$$

We thus have

$$\{\mathbb{Q}_+, \mathbb{Q}_-\} = \mathbb{H} \qquad [\mathbb{H}, \mathbb{Q}_\pm] = 0. \tag{207}$$

The time-independent generators $\mathbb{H}, \mathbb{Y}, \mathbb{J}, \mathbb{A}_\pm, \mathbb{I}$ and $\mathbb{Q}_\pm$ form a superalgebra of supersymmetries of $\mathbb{H}$. The additional super-commutation relations are

$$[\mathbb{Y}, \mathbb{Q}_\pm] = \pm 2\mathbb{Q}_\pm. \tag{208}$$

If we include the generators $\mathbb{T}_\pm(0)$ as symmetries of $\mathbb{H}$, we get the following superalgebra $\{\mathbb{H}, \mathbb{Y}, \mathbb{J}, \mathbb{A}_\pm, \mathbb{I}, \mathbb{Q}_\pm, \mathbb{Q}_0, \mathbb{T}_\pm(0)\}$, where

$$\mathbb{Q}_0 = \begin{pmatrix} (Q_+ - Q_-) & 0 \\ 0 & (Q_+ - Q_-) \end{pmatrix}. \tag{209}$$

Indeed we have

$$\{\mathbb{T}_+(0), \mathbb{Q}_-\} = \frac{\kappa}{2\sqrt{\bar{\omega}}}\mathbb{I} - \sqrt{\bar{\omega}}\mathbb{Q}_0 \qquad \{\mathbb{T}_-(0), \mathbb{Q}_+\} = \frac{\bar{\kappa}}{2\sqrt{\bar{\omega}}}\mathbb{I} + \sqrt{\bar{\omega}}\mathbb{Q}_0. \tag{210}$$

This superalgebra may be written as $(\{\mathbb{H}, \mathbb{Y}, \mathbb{Q}_\pm\} \oplus \{\mathbb{J}\}) \ni \{\mathbb{A}_\pm, \mathbb{I}, \mathbb{T}_\pm(0), \mathbb{Q}_0\}$.

In the approach of Andreev and Lerner [9], the preceding Hamiltonian $\mathbb{H}$ has been generalized to $\mathbb{H}(\varphi)$ where φ is an arbitrary phase. Indeed, $\mathbb{H}(\varphi)$ is block diagonal where, up to the addition of a multiple of the identity, the first block is the JC Hamiltonian (114) and the second one is obtained from it by changing $\mathcal{A} \mapsto e^{-i\varphi}\mathcal{A}$ and $\mathcal{A}^\dagger \mapsto e^{i\varphi}\mathcal{A}^\dagger$. With respect to our approach, it is associated with the original set of equations (116), (117) and the new set (146), (147) where $E_3 = -\bar{E}_3 = Ee^{i\varphi}$ and $E_4 = \bar{E}_4 = E\,e^{-i\varphi}$. The algebra of symmetries is the same as for the case $\varphi = 0$ studied before, so all the results on the existence of supersymmetry transformations remain valid. The only changes are in the following generators:

$$\mathbb{T}_+(\varphi, t) = e^{i\omega_{\alpha\beta}t}\begin{pmatrix} 0 & \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix} \\ 0 & 0 \end{pmatrix} \qquad \mathbb{T}_-(\varphi, t) = e^{-i\omega_{\alpha\beta}t}\begin{pmatrix} 0 & 0 \\ \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} & 0 \end{pmatrix}. \tag{211}$$

It follows that the generators $\mathbb{S}_\mp(t)$ and $\mathbb{U}_\mp(t)$ given in (194), (195) and (196), (197) are now written as $\mathbb{S}_\mp(\varphi, t) = \sqrt{\bar{\omega}}\mathbb{A}_\pm\mathbb{T}_\mp(\varphi, t)$ and $\mathbb{U}_\mp(\varphi, t) = \sqrt{\bar{\omega}}\mathbb{A}_\mp\mathbb{T}_\mp(\varphi, t)$.

The supercharges are found to be

$$\mathbb{Q}_+(\varphi) = \frac{\bar{\kappa}}{2\sqrt{\bar{\omega}}}\mathbb{T}_+(\varphi, 0) + \sqrt{\bar{\omega}}\begin{pmatrix} 0 & (e^{i\varphi}Q_+ - Q_-) \\ 0 & 0 \end{pmatrix} \tag{212}$$

$$\mathbb{Q}_-(\varphi) = \frac{\kappa}{2\sqrt{\tilde{\omega}}}\mathbb{T}_-(\varphi, 0) + \sqrt{\tilde{\omega}}\begin{pmatrix} 0 & 0 \\ (\mathrm{e}^{-\mathrm{i}\varphi}Q_- - Q_+) & 0 \end{pmatrix} \tag{213}$$

and satisfy

$$\{\mathbb{Q}_+(\varphi), \mathbb{Q}_-(\varphi)\} = \mathbb{H}(\varphi) \qquad [\mathbb{H}(\varphi), \mathbb{Q}_\pm(\varphi)] = 0. \tag{214}$$

The last generator that is modified is

$$\mathbb{Q}_0(\varphi) = \begin{pmatrix} (Q_+ - Q_-) & 0 \\ 0 & (\mathrm{e}^{\mathrm{i}\varphi}Q_+ - \mathrm{e}^{-\mathrm{i}\varphi}Q_-) \end{pmatrix}. \tag{215}$$

5.3. The usual supersymmetric structure

Another SUSY version of the JC model may be deduced from our preceding considerations. Let us refer it as the standard or strong coupling limit one in reference to the literature [20–22]. If we start again with the system of equations (116), (117) and (146), (147) for which the symmetries have been determined for arbitrary values of the parameters B_1, B_2, B_3, B_4, we can in particular take $B_1 = -B_2 = B$, while $B_3 = -3B$ and $B_4 = -B$. This is the case where equations (159) and (160) admit the trivial solution $f(t) = g(t) = 0$ and such that no symmetries associated with odd generators appear. This means that, by the prolongation method, it will be impossible to get a superalgebra of symmetries.

Meanwhile, if we choose $E_2 = -E_1 = E_4 = -E_3 = \tilde{E}_3 = \tilde{E}_4 = E = M/\sqrt{2eB}$, it is possible to write the evolution equations (116), (117) and (146), (147) as

$$(\mathrm{i}\partial_t - \mathbb{H}_{\mathrm{JC}})\Psi(t, x, y) = 0 \tag{216}$$

where $\mathbb{H}_{\mathrm{JC}}$ has the standard SUSY form [23]

$$\mathbb{H}_{\mathrm{JC}} = \tilde{\omega}\begin{pmatrix} \tilde{\mathcal{A}}^\dagger\tilde{\mathcal{A}} & 0 \\ 0 & \tilde{\mathcal{A}}\tilde{\mathcal{A}}^\dagger \end{pmatrix} \tag{217}$$

with

$$\tilde{\mathcal{A}} = \mathcal{A} + \mathrm{i}\sigma_+ \qquad \tilde{\mathcal{A}}^\dagger = \mathcal{A}^\dagger - \mathrm{i}\sigma_-. \tag{218}$$

These last operators satisfy

$$[\tilde{\mathcal{A}}, \tilde{\mathcal{A}}^\dagger] = \sigma_0 + \sigma_3. \tag{219}$$

Note that the two components of the Hamiltonian $\mathbb{H}_{\mathrm{JC}}$ are closely related to the Hamiltonian (114). Indeed, we have

$$\mathbb{H}_{\mathrm{JC}} = \begin{pmatrix} H_{\mathrm{JC}} & 0 \\ 0 & H_{\mathrm{JC}} + \tilde{\omega}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \tag{220}$$

when $\kappa = \mathrm{i}\tilde{\omega}$. The standard supercharges are given by

$$\tilde{\mathcal{Q}}_+ = \tilde{\omega}^{1/2}\begin{pmatrix} 0 & \tilde{\mathcal{A}}^\dagger \\ 0 & 0 \end{pmatrix} \qquad \tilde{\mathcal{Q}}_- = \tilde{\omega}^{1/2}\begin{pmatrix} 0 & 0 \\ \tilde{\mathcal{A}} & 0 \end{pmatrix} \tag{221}$$

satisfying the following relations:

$$\mathbb{H}_{\mathrm{JC}} = \{\tilde{\mathcal{Q}}_+, \tilde{\mathcal{Q}}_-\} \qquad (\tilde{\mathcal{Q}}_\pm)^2 = 0 \qquad [\mathbb{H}_{\mathrm{JC}}, \tilde{\mathcal{Q}}_\pm] = 0. \tag{222}$$

Again these supercharges cannot be obtained from the product of symmetries determined by the prolongation method.

Let us finally mention that such a Hamiltonian is a particular case of a matrix SUSY one where the quantities $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{A}}^\dagger$ would correspond to elements of the algebra $h(2) \oplus su(2)$, that is linear combinations of the generators of this algebra. Different assumptions may thus be imposed on the commutator $[\mathcal{A}, \mathcal{A}^{\dagger}]$ [22]. In the canonical case, that is the case where the commutator is a multiple of the identity, the prolongation method reproduces all the dynamical supersymmetries. This was the case for the SUSY harmonic oscillator and the Pauli Hamiltonians. In the non-canonical case, such as the JC model, the supercharges are not obtained from the prolongation method but may be constructed by the standard structure of the Hamiltonian as in (217) for the JC model.

6. Conclusion

We have shown that the prolongation method used for finding symmetries of classical as well as quantum-mechanical systems may be useful to determine the supersymmetries of SUSY quantum-mechanical systems. We took simple examples, such as the SUSY harmonic oscillator and the Pauli equations to improve the method. Indeed, we already knew the kind of kinematical and dynamical superalgebras we were searching for. This was very helpful to be able to get new results on the JC model. First we determined the Lie algebra of symmetries for the usual 2×2 matrix model. Second, we gave the symmetry superalgebra for a generalized version which is an amplification of the usual JC model to a 4×4 matrix representation. Finally, two ways of getting SUSY versions were given. In the first case the supersymmetry was present only if we admitted a specific shifting in the JC Hamiltonian. In the second case, the supersymmetry appeared due to the fact that the amplification of the JC model is similar to that of the SUSY harmonic oscillator but to do that it was necessary to take the coupling constant between the electromagnetic field and the atom as a linear function of the frequency of these fields. In all these cases, the detuning between the electromagnetic field and atom frequencies has been assumed to be equal to zero.

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Chapitre 3

États cohérents et comprimés basés sur l'algèbre $h(1) \oplus \mathrm{su}(2)$

Résumé

Des états qui minimisent la relation d'incertitude de Schrödinger–Robertson sont construits comme états propres d'un opérateur qui est un élément de l'algèbre $h(1) \oplus \mathrm{su}(2)$. Les liens avec les états supercohérents and supercomprimés de l'oscillateur harmonique supersymétrique sont donnés. De plus, nous construisons des Hamiltoniens généraux dont le comportement ressemble à celui de l'oscillateur harmonique ou encore est relié à celui de Jaynes–Cummings.

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Generalized coherent and squeezed states based on the $h(1)\oplus su(2)$ algebra

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States which minimize the Schrödinger–Robertson uncertainty relation are constructed as eigenstates of an operator which is an element of the $h(1)\oplus su(2)$ algebra. The relations with supercoherent and supersqueezed states of the supersymmetric harmonic oscillator are given. Moreover, we are able to compute general Hamiltonians which behave like the harmonic oscillator Hamiltonian or are related to the Jaynes–Cummings Hamiltonian.
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I. INTRODUCTION

Minimum uncertainty states (MUSs) are usually understood through the minimization of the Heisenberg uncertainty relation (HUR). These states are well-known[1] since they are long associated with the so-called coherent states (CSs)[2] and squeezed states (SSs)[3]. But, it has been observed[4–6] that a more accurate uncertainty relation may be used to construct generalized CSs and SSs. Indeed, this relation known as the Schrödinger–Robertson uncertainty relation (SRUR)[7] can be minimized and gives rise to new classes of CSs and SSs which have received different names in the literature, such as correlated states[4] or intelligent states.[5] There are two main reasons to consider such last states. First, when the two Hermitian operators entering in the SRUR are noncanonical operators, i.e., their commutator is not a multiple of the identity, the HUR could be redundant while the SRUR not. Second, the MUSs that minimize the SRUR are shown to be eigenstates of a linear combination of the two Hermitian operators entering in the SRUR.

Recently[8] a connection has been made with the CS and SS based on group theoretic approaches[9] and the concept of algebra eigenstates (AESs). In particular, AESs have been constructed for the algebras su(2) and su(1,1). This concept constitutes a unification of different definitions of CS and SS.

In this article, we give a general construction of AESs based on the direct sum $h(1)\oplus su(2)$. The Heisenberg algebra $h(1)$ being relevant for the problem of the harmonic oscillator and the algebra su(2) for particles with spin, we have a procedure to find general CSs and SSs for supersymmetric systems, for example. These are clearly MUSs for which the dispersions of corresponding operators may be calculated easily. We show finally how to use these states in the construction of particularly relevant Hamiltonians and in the calculation of their dispersions.

In Sec. II, we put the emphasis on the SRUR and its relevancy with respect to the determination of MUSs. The application to the position and momentum operators MUS leads to the well-known CS and SS of the harmonic oscillator while when the angular momentum operators MUS are considered we have in mind the su(2) CS and SS. These particular applications are given to bring a new light on these states and also to facilitate the treatement of the $h(1)\oplus su(2)$ CS and SS. In Sec. III, we construct the AES based on the $h(1)\oplus su(2)$ algebra and show how this gives CSs and SSs which generalize the supercoherent and supersqueezed states obtained in other approaches.[10,11] Finally, in Sec. IV, we construct general Hamiltonians similar to the one of the

[redacted]

harmonic oscillator but where the so-called annihilation operator is now an element of the algebra $h(1)\oplus su(2)$. This permits us to use our CS and SS to compute the mean value and the dispersions of the corresponding energies. We show also how the well-known Jaynes–Cummings Hamiltonian enters in this scheme.

II. COHERENT AND SQUEEZED STATES AS MINIMUM UNCERTAINTY STATES

This section will be concerned by the general definition and properties of MUS (Sec. II A). They are explicitly constructed when the usual position and momentum operators are considered (Sec. II B) as well as when the angular momentum operators are taken (Sec. II C). The connection is made with already known results.

A. Minimum uncertainty relation

It is well-known[7] that, for two Hermitian operators A and B such that the commutator is

$$[A,B]=iC, \quad C\neq 0, \tag{2.1}$$

the HUR

$$(\Delta A)^2(\Delta B)^2 \geqslant \frac{\langle C\rangle^2}{4} \tag{2.2}$$

is satisfied. The mean value and dispersion of a given operator X are defined, as usual, by

$$\langle X\rangle = \langle\psi|X|\psi\rangle, \quad (\Delta X)^2 = \langle X^2\rangle - \langle X\rangle^2, \tag{2.3}$$

for a normalized state $|\psi\rangle$ describing the evolution of a quantum system. As observed by Puri,[6] for noncanonical operators, i.e., such that C is not a multiple of the identity I, we can have $\langle C\rangle=0$ and the relation (2.2) is then redundant. The SRUR[1,7] is never redundant and writes

$$(\Delta A)^2(\Delta B)^2 \geqslant \tfrac{1}{4}(\langle C\rangle^2+\langle F\rangle^2), \tag{2.4}$$

where $\langle F\rangle$ is a measure of the correlation between A and B. The operator F is Hermitian and given by

$$F=\{A-\langle A\rangle I, B-\langle B\rangle I\}, \tag{2.5}$$

where $\{\ ,\ \}$ denotes the anticommutator. If there is no correlation between the operators A and B, i.e., if $\langle F\rangle=0$, the SRUR reduces to the usual HUR.

We are interested here in the description of states which minimize the SRUR (2.4). A necessary and sufficient condition to get them is to solve the eigenvalues equation:

$$[A+i\lambda B]|\psi\rangle = \beta|\psi\rangle, \tag{2.6}$$

where

$$\beta=[\langle A\rangle + i\lambda\langle B\rangle], \quad \lambda\in\mathbb{C}, \quad \lambda\neq 0. \tag{2.7}$$

Note that, if $\operatorname{Re}\lambda\neq 0$, once we know the value of β, this last relation may be inverted to give the mean values

$$\langle A\rangle = \operatorname{Re}\beta + \frac{\operatorname{Im}\lambda}{\operatorname{Re}\lambda}\operatorname{Im}\beta, \quad \langle B\rangle = \frac{\operatorname{Im}\beta}{\operatorname{Re}\lambda} \tag{2.8}$$

and, if $\operatorname{Re}\lambda=0$, we get

$$\langle A\rangle = \operatorname{Re}\beta + \operatorname{Im}\lambda\langle B\rangle. \tag{2.9}$$

As a consequence of (2.6), one has

$$(\Delta A)^2=|\lambda|\Delta, \quad (\Delta B)^2=\frac{1}{|\lambda|}\Delta, \tag{2.10}$$

with

$$\Delta=\tfrac{1}{2}\sqrt{\langle C\rangle^2+\langle F\rangle^2}. \tag{2.11}$$

So the states $|\psi\rangle$ satisfying (2.6) with $|\lambda|=1$ will be called **coherent** because they satisfy

$$(\Delta A)^2=(\Delta B)^2=\Delta, \tag{2.12}$$

i.e., the dispersions in A and B are the same and minimized in the sense of SRUR. The states $|\psi\rangle$ satisfying (2.6) with $|\lambda|\neq 1$ will be called **squeezed** because if $|\lambda|<1$, we have $(\Delta A)^2<\Delta<(\Delta B)^2$ and if $|\lambda|>1$, we have $(\Delta B)^2<\Delta<(\Delta A)^2$.

Some other relations are also useful for our considerations. The direct computation of $(\Delta A)^2$ and $(\Delta B)^2$ is usually complicated but in the MUSs that satisfy (2.6), we can write

$$(\Delta A)^2=\tfrac{1}{2}|\mathrm{Re}\,\lambda\langle C\rangle+\mathrm{Im}\,\lambda\langle F\rangle|, \tag{2.13}$$

$$(\Delta B)^2=\frac{1}{2|\lambda|^2}|\mathrm{Re}\,\lambda\langle C\rangle+\mathrm{Im}\,\lambda\langle F\rangle|, \tag{2.14}$$

with

$$\mathrm{Im}\,\lambda\langle C\rangle=\mathrm{Re}\,\lambda\langle F\rangle. \tag{2.15}$$

For $\mathrm{Re}\,\lambda=0$, we have $\langle C\rangle=0$, which corresponds to the case where the HUR is redundant. The MUSs satisfy the minimum SRUR (MSRUR)

$$(\Delta A)^2(\Delta B)^2=\Delta^2, \tag{2.16}$$

with

$$(\Delta A)^2=\frac{1}{2}|\mathrm{Im}\,\lambda\langle F\rangle|, \quad (\Delta B)^2=\frac{1}{2}\left|\frac{\langle F\rangle}{\mathrm{Im}\,\lambda}\right| \tag{2.17}$$

and

$$\Delta=\tfrac{1}{2}|\langle F\rangle|. \tag{2.18}$$

For $\mathrm{Re}\,\lambda\neq 0$, from (2.15), we have

$$\langle F\rangle=\frac{\mathrm{Im}\,\lambda}{\mathrm{Re}\,\lambda}\langle C\rangle. \tag{2.19}$$

Moreover, from (2.13) and (2.14), we get

$$(\Delta A)^2=\left|\frac{|\lambda|^2}{2\,\mathrm{Re}\,\lambda}\langle C\rangle\right|, \quad (\Delta B)^2=\left|\frac{1}{2\,\mathrm{Re}\,\lambda}\langle C\rangle\right|, \tag{2.20}$$

and, then,

$$\Delta=\left|\frac{|\lambda|}{2\,\mathrm{Re}\,\lambda}\langle C\rangle\right|. \tag{2.21}$$

In this case, it is sufficient to compute the mean value of C to deduce that of F and the dispersions. The particular case where $\mathrm{Im}\,\lambda=0$ corresponds to the fact that the MSUR coïncides with the minimum HUR (MHUR).

B. Position and momentum coherent and squeezed states

Let us apply the preceding considerations to the special case of the usual position x and momentum p operators of a given quantum system. The canonical commutation relation (if $\hbar=1$) being

$$[x,p]=iI, \tag{2.22}$$

the SRUR writes

$$(\Delta x)^2(\Delta p)^2 \geq \tfrac{1}{4}(1+\langle F\rangle^2). \tag{2.23}$$

The MUSs $|\psi,\lambda,\beta\rangle$ satisfy the eigenvalues equation:

$$[x+i\lambda p]|\psi,\lambda,\beta\rangle=\beta|\psi,\lambda,\beta\rangle. \tag{2.24}$$

If we introduce the usual creation $a^\dagger$ and annihilation a operators,

$$a^\dagger=\frac{x-ip}{\sqrt{2}}, \quad a=\frac{x+ip}{\sqrt{2}}, \tag{2.25}$$

such that $[a,a^\dagger]=I$, the equation (2.24) becomes

$$\frac{1}{\sqrt{2}}[(1-\lambda)a^\dagger+(1+\lambda)a]|\psi,\lambda,\beta\rangle=\beta|\psi,\lambda,\beta\rangle. \tag{2.26}$$

The general resolution of Eq. (2.26) is obtained by expressing the state $|\psi,\lambda,\beta\rangle$ as a superposition of the energy eigenstates $\{|n\rangle, n=0,1,2,\dots\}$ of the usual harmonic oscillator Hamiltonian

$$H_0=w(a^\dagger a+\tfrac{1}{2}). \tag{2.27}$$

Let us recall that these eigenstates satisfy

$$a|n\rangle=\sqrt{n}|n-1\rangle, \quad a^\dagger|n\rangle=\sqrt{n+1}|n+1\rangle \tag{2.28}$$

and we can write them as

$$|n\rangle=\frac{a^{\dagger^n}}{\sqrt{n!}}|0\rangle, \quad n=0,1,2,\dots\,. \tag{2.29}$$

So, if we insert

$$|\psi,\lambda,\beta\rangle=\sum_{n=0}^{\infty} C_{\lambda,\beta,n}|n\rangle, \quad C_{\lambda,\beta,n}\in\mathbb{C}, \tag{2.30}$$

in Eq. (2.26), using the expressions (2.28), we get the recurrence system

$$\frac{1}{\sqrt{2}}[\sqrt{n}(1-\lambda)C_{\lambda,\beta,n-1}+\sqrt{n+1}(1+\lambda)C_{\lambda,\beta,n+1}]=\beta C_{\lambda,\beta,n}, \quad n=1,2,3,\ldots,$$
$$\frac{(1+\lambda)}{\sqrt{2}}C_{\lambda,\beta,1}=\beta C_{\lambda,\beta,0}. \tag{2.31}$$

The case $\lambda=-1$ does not give any solution and must be eliminated. If we set

$$\left(\frac{1-\lambda}{1+\lambda}\right)=\delta e^{i\phi}, \quad \delta\in R_+, \phi\in\left[-\frac{\pi}{2},\frac{3\pi}{2}\right[, \tag{2.32}$$

the resolution of the recurrence system (2.31) leads to the general solution of Eq. (2.26):

$$|\psi,\lambda,\beta\rangle=C_{\lambda,\beta,0}\exp\left(-\delta e^{i\phi}\frac{a^{\dagger 2}}{2}\right)\exp\left(\frac{\beta}{\sqrt{2}}(1+\delta e^{i\phi})a^{\dagger}\right)|0\rangle. \tag{2.33}$$

The special case $\lambda=1$ corresponds to $\delta=0$ and gives rise to the usual expression of the CS of the harmonic oscillator. These states (2.33) can also be obtained as the action of two unitary operators on the fundamental state. The first one[9] is the usual displacement operator D associated with an irreducible representation of the Heisenberg–Weyl group $H(1)$ with algebra $h(1)=\{a,a^{\dagger},I\}$. The second one is the squeezed operator S associated with an irreducible representation of SU(1,1) with algebra $su(1,1)=\{a^2,(a^{\dagger})^2,aa^{\dagger}+a^{\dagger}a\}$. This is a known fact[12] when squeezed states of the harmonic oscillator are studied. We have explicitly

$$|\psi,\lambda,\beta\rangle=S(\chi(\delta,\phi))D(\eta)|0\rangle, \tag{2.34}$$

where

$$D(\eta)=\exp(\eta a^{\dagger}-\bar{\eta}a) \quad \text{and} \quad S(\chi)=\exp\left(\chi\frac{a^{\dagger 2}}{2}-\bar{\chi}\frac{a^2}{2}\right) \tag{2.35}$$

with

$$\eta=\frac{\beta}{\sqrt{2}}\frac{(1+\delta e^{i\phi})}{\sqrt{1-\delta^2}} \quad \text{and} \quad \chi(\delta,\phi)=-\tanh^{-1}(\delta)e^{i\phi}. \tag{2.36}$$

The condition for having normalizable states is that $0\leqslant\delta<1$. Let us insist here on the fact that these SSs already obtained in the literature as eigenstates of a linear combination of a and $a^{\dagger}$ are also MUSs such that $(\Delta x)^2(\Delta p)^2=\Delta^2=(1+\langle F\rangle^2)/4$. From Eq. (2.19) and the fact that $\langle C\rangle=1$, we get

$$\langle F\rangle=\frac{\mathrm{Im}\,\lambda}{\mathrm{Re}\,\lambda}=\frac{-2\delta\sin\phi}{(1-\delta^2)} \tag{2.37}$$

and the factor Δ is

$$\Delta(\delta,\phi)=\sqrt{\frac{1}{4}(1+\langle F\rangle^2)}=\sqrt{\frac{1}{4}+\frac{\delta^2\sin^2\phi}{(1-\delta^2)^2}}. \tag{2.38}$$

Moreover, from (2.13) and (2.14), the dispersions are

$$(\Delta x)^2=\frac{|\lambda|^2}{2|\mathrm{Re}\,\lambda|}=\frac{(1-2\delta\cos\phi+\delta^2)}{2(1-\delta^2)} \tag{2.39}$$

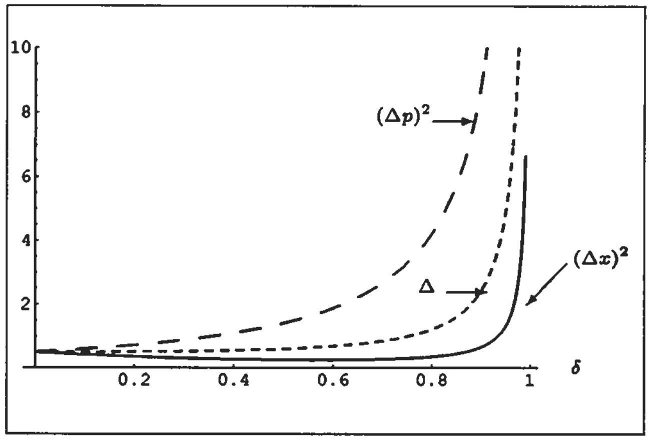

FIG. 1. Graphs of the dispersions $(\Delta x)^2$, $(\Delta p)^2$ and the Δ factor as functions of δ for $\phi=\pi/6$.

and

$$(\Delta p)^2=\frac{1}{2|\mathrm{Re}\,\lambda|}=\frac{(1+2\delta\cos\phi+\delta^2)}{2(1-\delta^2)}. \tag{2.40}$$

Let us recall now that the CSs are not only the one for $\lambda=1$ but also all the states where $|\lambda|=1$. From the relation (2.32), we deduce that

$$\lambda=\frac{1-\delta e^{i\phi}}{1+\delta e^{i\phi}}=\frac{(1-\delta^2)-2i\delta\sin\phi}{(1+2\delta\cos\phi+\delta^2)} \tag{2.41}$$

and then

$$|\lambda|^2=\frac{1-2\delta\cos\phi+\delta^2}{1+2\delta\cos\phi+\delta^2}. \tag{2.42}$$

This means that CSs occur also for $\phi=-\pi/2$ or $\phi=\pi/2$ and $\delta\neq 0$. The other values of λ describe x-squeezed states when $\phi\in]-\pi/2,\pi/2[$ and p-squeezed states when $\phi\in]\pi/2,3\pi/2[$. On the other hand, for fixed values of ϕ the expression (2.38) attains its minimum value $\frac{1}{2}$ when $\delta=0$ and when $\phi=0$ and $\phi=\pi$ for fixed values of δ. In the first of these cases, we have $\lambda=1$ and we are in the standard CSs of the harmonic oscillator, i.e., eigenstates of the a operator. In the second case, λ is a positive real quantity equal to $(1-\delta)/(1+\delta)\leqslant 1$ if $\phi=0$ and to $(1+\delta)/(1-\delta)\geqslant 1$ if $\phi=\pi$. We are in the special SSs that are eigenstates of the $(a+\delta a^\dagger)$ and $(a-\delta a^\dagger)$ operators, respectively.

Figure 1 shows the behavior of $(\Delta x)^2$, $(\Delta p)^2$ and Δ as functions of δ for $\phi=\pi/6$. In this region $(\Delta x)^2[(\Delta p)^2]$ is always less (greater) than Δ, as expected. For $\delta=0$, the three curves coïncide, and the intersection point corresponds to the CS $|\psi,1,\beta\rangle$. The value of $\Delta=(2.38)$ when $\delta=0$ is also the minimum value $\frac{1}{2}$ which corresponds to the MHUR. Figure 2 shows the behavior of the same quantities as functions of ϕ for $\delta=0.5$. The points where the three curves intersect are the CS.

C. Angular momentum coherent and squeezed states

Let us now take the angular momentum operators J_k for $k=1,2,3$, which satisfy the usual su(2) commutations relations

$$[J_k,J_l]=i\varepsilon_{klm}J_m, \quad k,l,m=1,2,3. \tag{2.43}$$

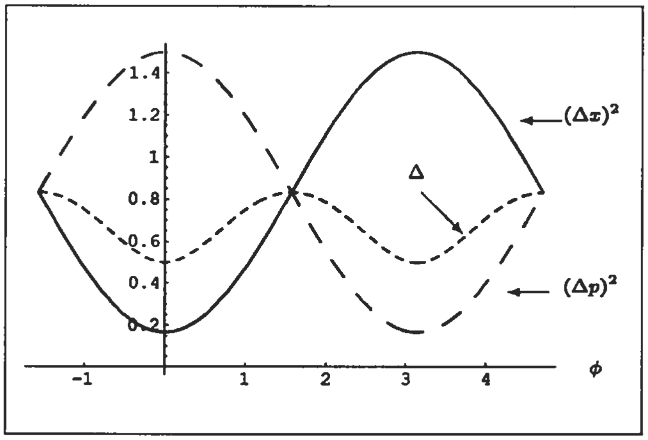

FIG. 2. Graphs of the dispersions $(\Delta x)^2$, $(\Delta p)^2$ and the Δ factor as functions of ϕ for $\delta=0.5$.

Here we want to solve the eigenvalues equation

$$(J_1+i\lambda J_2)|\psi,\lambda,\beta\rangle=\beta|\psi,\lambda,\beta\rangle, \tag{2.44}$$

where $\beta=[\langle J_1\rangle+i\lambda\langle J_2\rangle]$. On the contrary of the preceding example where the HUR is never redundant (because x and p are canonical), here the commutator of J_1 and J_2 is not a multiple of the identity and then $\langle J_3\rangle$ may be equal to zero for some special cases. Some of these cases have been discussed elsewhere.[6,13–15] Here we give the general solution of the equation (2.44), for all possible values of λ and β.

It would be better to work with the operators $J_\pm=J_1\pm iJ_2$ instead of J_1 and J_2, so that the equation (2.44) becomes

$$\tfrac{1}{2}[(1+\lambda)J_+ + (1-\lambda)J_-]|\psi,\lambda,\beta\rangle=\beta|\psi,\lambda,\beta\rangle. \tag{2.45}$$

Using the usual complete set of angular momentum states $\{|j,r\rangle\}$, j integer or half-odd integer and $r\in\{-j,-(j-1),\dots,j-1,j\}$, we know that

$$J^2|j,r\rangle=(J_1^2+J_2^2+J_3^2)|j,r\rangle=j(j+1)|j,r\rangle, \tag{2.46}$$

$$J_3|j,r\rangle=r|j,r\rangle \tag{2.47}$$

and

$$J_\pm|j,r\rangle=\sqrt{(j\mp r)(j\pm r+1)}|j,r\pm 1\rangle. \tag{2.48}$$

This means that for each j fixed, the eigenstates $|\psi,\lambda,\beta\rangle^j$ of Eq. (2.45) may be written as

$$|\psi,\lambda,\beta\rangle^j=\sum_{r=-j}^{j} C^j_{\lambda,\beta,r}|j,r\rangle, \quad C^j_{\lambda,\beta,r}\in\mathbb{C}, \tag{2.49}$$

where the coefficients $C^j_{\lambda,\beta,r}$ satisfy a recurrence system of the form

$$(1+\lambda)\sqrt{(j+r)(j-r+1)}C^j_{\lambda,\beta,r-1}+(1-\lambda)\sqrt{(j-r)(j+r+1)}C^j_{\lambda,\beta,r+1}=2\beta C^j_{\lambda,\beta,r}, \tag{2.50}$$

for $r=-j,\dots,j$ and $C^j_{\lambda,\beta,j+1}=C^j_{\lambda,\beta,-(j+1)}=0$.

For $\lambda=\pm 1$, the unique eigenstates are $|\psi,\pm 1,0\rangle^j=|j,\pm j\rangle$. For $\lambda\neq\pm 1$ and $\beta=0$, the recurrence relation (2.50) is solved to give

$$|\psi,\lambda,0\rangle^j = C^j_{\lambda,0,j} e^{i(j\phi/2)} \sum_{k=0}^{j} (-1)^k \frac{\binom{j}{k}}{\sqrt{\binom{2j}{2k}}} \delta^k e^{-i(j-2k)\phi/2} |j,j-2k\rangle, \quad j \text{ integer}, \tag{2.51}$$

where we have used the formula (2.32) to express λ in terms of the δ and ϕ. It is again possible to express such a state from the action of unitary operators associated with an irreducible representation of a group which is here SU(2). Indeed, we have

$$|\psi,\lambda,0\rangle^j = C^j_{\lambda,0} \exp[-\tfrac{1}{2}\ln(\delta)J_3] U |j,0\rangle, \tag{2.52}$$

where

$$U = \exp\left(-\frac{\pi}{4}(e^{-i\phi/2}J_+ - e^{i\phi/2}J_-)\right). \tag{2.53}$$

For the general case $\lambda \neq \pm 1$, the analysis of the system (2.50) shows that for each j, there exist $(2j+1)$ possible values for the eigenvalue β, which are

$$\beta^j_m = m\sqrt{1-\lambda^2}, \quad m = -j,\dots,j. \tag{2.54}$$

If we use the relation

$$[J_1 + i\lambda J_2][\exp(-\tfrac{1}{2}\ln(\delta)J_3)U] = [\exp(-\tfrac{1}{2}\ln(\delta)J_3)U][\sqrt{1-\lambda^2}J_3], \tag{2.55}$$

we see immediately that the corresponding eigenstate $|\psi,\lambda,\beta^j_m\rangle^j$ is

$$|\psi,\lambda,\beta^j_m\rangle^j \equiv |\psi,\lambda,m\rangle^j = C^j_{\lambda,m} \exp[-\tfrac{1}{2}\ln(\delta)J_3] U |j,m\rangle, \quad m = -j,\dots,j, \tag{2.56}$$

where $U \equiv (2.53)$. They can be written in terms of the Jacobi polynomials as

$$|\psi,\lambda,m\rangle^j = C^j_{\lambda,m} \times \exp\left(-\frac{1}{2}\ln(\delta)J_3\right) e^{im\phi/2} e^{-i(\phi/2)J_3} \times \sum_{r=-j}^{j} 2^r \sqrt{\frac{(j+r)!(j-r)!}{(j-m)!(j+m)!}} P_{j+r}^{-r+m,-r-m}(0)|j,r\rangle. \tag{2.57}$$

In these last states, we want to compute now the mean values and dispersions of some operators in order to exhibit their behavior in the CS and SS.

If $\mathrm{Re}\,\lambda \neq 0$, the mean values of J_1 and J_2 in the states (2.57) are obtained using (2.8) and (2.54). In terms of δ and ϕ as defined by (2.32), we get

$$\langle J_1\rangle^j_m = 2m\frac{\delta^{1/2}}{(\delta+1)}\cos\left(\frac{\phi}{2}\right), \quad \langle J_2\rangle^j_m = 2m\frac{\delta^{1/2}}{(\delta+1)}\sin\left(\frac{\phi}{2}\right). \tag{2.58}$$

The relations (2.19)–(2.21) applied to our case tell us that $(\Delta J_1)^2$, $(\Delta J_2)^2$, Δ and $\langle F\rangle$ are all obtained from the mean value of J_3, i.e.,

$$((\Delta J_1)^2)^j_m = \frac{|\lambda|^2}{2\,\mathrm{Re}\,\lambda}\langle J_3\rangle^j_m, \quad ((\Delta J_2)^2)^j_m = \frac{1}{2\,\mathrm{Re}\,\lambda}\langle J_3\rangle^j_m,$$
$$\Delta^j_m = \frac{|\lambda|}{2\,\mathrm{Re}\,\lambda}\langle J_3\rangle^j_m, \quad \langle F\rangle^j_m = \frac{\mathrm{Im}\,\lambda}{\mathrm{Re}\,\lambda}\langle J_3\rangle^j_m. \tag{2.59}$$

The mean values of J_3 in the states (2.57) or equivalently in the states (2.56) are given by

$$\langle J_3\rangle^j_m = -\frac{\partial}{\partial q}\ln(\langle j,m|U^\dagger e^{-qJ_3}U|j,m\rangle), \tag{2.60}$$

where $q=\ln\delta$. After some computations, we get

$$\langle J_3\rangle^j_m = -|m|\tanh\left(\frac{q}{2}\right) - \frac{1}{2}\sinh(q)(j+|m|+1)\frac{P^{1,1+2|m|}_{j-|m|-1}(\cosh q)}{P^{0,2|m|}_{j-|m|}(\cosh q)}. \tag{2.61}$$

Inserting (2.61) into the expression (2.59), we get

$$((\Delta J_1)^2)^j_m = (1-2\delta\cos\phi+\delta^2)\Lambda^j_m(\delta), \quad ((\Delta J_2)^2)^j_m = (1+2\delta\cos\phi+\delta^2)\Lambda^j_m(\delta), \tag{2.62a}$$

$$(\Delta)^j_m = \sqrt{1-2\delta^2\cos(2\phi)+\delta^4}\,\Lambda^j_m(\delta), \quad \langle F\rangle^j_m = -4\delta\sin\phi\,\Lambda^j_m(\delta), \tag{2.62b}$$

where

$$\Lambda^j_m(\delta) = \left[\frac{|m|}{2(1+\delta)^2} + \frac{(j+|m|+1)}{8\delta}\,\frac{P^{1,1+2|m|}_{j-|m|-1}((1+\delta^2)/2\delta)}{P^{0,2|m|}_{j-|m|}((1+\delta^2)/2\delta)}\right]. \tag{2.63}$$

The case $\mathrm{Re}\,\lambda=0$ may be obtained as the limit case of the preceding one by taking $\delta=1$ in the expressions (2.62a), (2.62b) and (2.63). Let us recall that it corresponds to $\langle J_3\rangle=0$ and $\lambda=-i\tan\phi/2$. We get

$$((\Delta J_1)^2)^j_m = \frac{1}{2}[j(j+1)-m^2]\sin^2\left(\frac{\phi}{2}\right), \quad ((\Delta J_2)^2)^j_m = \frac{1}{2}[j(j+1)-m^2]\cos^2\left(\frac{\phi}{2}\right), \tag{2.64a}$$

$$(\Delta)^j_m = \frac{1}{4}[j(j+1)-m^2]|\sin\phi| \text{ and } \langle F\rangle^j_m = -\frac{1}{2}[j(j+1)-m^2]\sin\phi, \tag{2.64b}$$

using the fact that

$$P^{\alpha,\beta}_n(1) = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!}. \tag{2.65}$$

These are exactly the results given by Puri.[6]

To illustrate these considerations by a concrete example, let us take the "spin-$\frac{1}{2}$" case, i.e., $j=\frac{1}{2}$. The expressions (2.62a) and (2.62b) thus reduce to

$$((\Delta J_1)^2)_\pm = \frac{(1-2\delta\cos\phi+\delta^2)}{4(1+\delta)^2}, \quad ((\Delta J_2)^2)_\pm = \frac{(1+2\delta\cos\phi+\delta^2)}{4(1+\delta)^2} \tag{2.66}$$

and

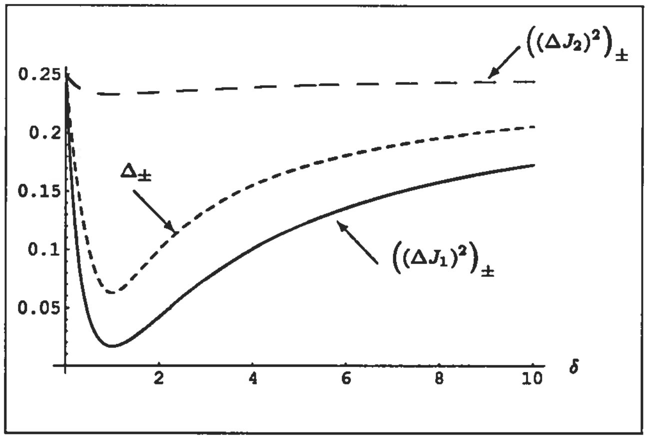

FIG. 3. Graphs of the dispersions $((\Delta J_1)^2)_\pm$, $((\Delta J_2)^2)_\pm$ and the $\Delta_\pm$ factor as functions of δ for $\phi=\pi/6$ and $j=\frac{1}{2}$.

$$\Delta_\pm(\delta,\phi)=\frac{1}{4}\sqrt{1+4\left(\frac{\delta^2\sin^2\phi-\delta(1+\delta)^2}{(1+\delta)^4}\right)}, \tag{2.67}$$

where we have used the $\pm$ sign for the values of $m=\pm\frac{1}{2}$. The MSRUR thus writes

$$((\Delta J_1)^2)_\pm((\Delta J_2)^2)_\pm=(\Delta_\pm)^2(\delta,\phi)=\frac{1}{16}\left[1+4\left(\frac{\delta^2\sin^2\phi-\delta(1+\delta)^2}{(1+\delta)^4}\right)\right]. \tag{2.68}$$

For fixed values of $\phi\neq 0$ and π, the expression (2.67) attains its minimum value $|\sin\phi|/8$ when $\delta=1$. On the other hand, for fixed values of δ such that $\delta\in[0,1[\cup]1,\infty]$, the minimum of (2.67) is $(\frac{1}{4})\sqrt{[1-(4\delta)/(1+\delta)^2]}$ when $\phi=0$ or $\phi=\pi$. In the first case we have $\lambda=-i(\sin\phi)/(1+\cos\phi)$, which means that we have some special classes of SSs from which we recognize CSs with $\lambda=-i$ (eigenstates of the J_1+J_2 operator) and with $\lambda=i$ (eigenstates of the J_1-J_2 operator). In the second case, we have $\lambda=(1-\delta)/(1+\delta)\leq 1$ if $\phi=0$ and $\lambda=(1+\delta)/(1-\delta)\geq 1$ if $\phi=\pi$, i.e., the minimum $\Delta_\pm(\delta,0)=\Delta_\pm(\delta,\pi)$ values occur for the special states which are eigenstates of the operators $(J_+ +\delta J_-)$ and $(J_+ -\delta J_-)$, respectively. Let us recall that the CSs with $\lambda=1$ occur when $\delta=0$ and those with $\lambda=-1$ when $\delta\mapsto\infty$. They correspond to the eigenstates of J_+ and J_- operators, respectively. For such states, according to Eq. (2.68), we have $((\Delta J_1)^2)_\pm=((\Delta J_2)^2)_\pm=(\Delta_\pm(0,\phi))^2=\lim_{\delta\mapsto\infty}(\Delta_\pm(\delta,\phi))^2=\frac{1}{4}$.

Figure 3 shows the behavior of the dispersions $((\Delta J_1)^2)$, $((\Delta J_2)^2)$ and Δ as functions of δ for $\phi=\pi/6$ and $j=\frac{1}{2}$. The minimum value of $\Delta_\pm$ is here 0,0625. In Fig. 4, we see that the graphs as a function of ϕ are very similar to ones for the preceding example of x and p.

III. ALGEBRA EIGENSTATES ASSOCIATED TO $h(1)\oplus su(2)$

This section begins (Sec. III A) with a review of the SUSY harmonic oscillator and its supercoherent states (SCSs) studied by Aragone and Zypman.[10] We follow (Sec. III B) by the general construction of AES based on the algebra $h(1)\oplus su(2)$. These states are defined as eigenstates of an arbitrary linear combination of the generators of the considered algebra.[8] Then we consider special solutions to CSs and SSs for the so-called super-position and super-momentum operators (Sec. III C).

A. The SUSY harmonic oscillator and its super-coherent states

Let us recall that the quantum SUSY harmonic oscillator is defined as a combination of a bosonic and a fermionic oscillators. Its Hamiltonian is given by

$$H_{SUSY}=w(a^\dagger a-f^\dagger f), \tag{3.1}$$

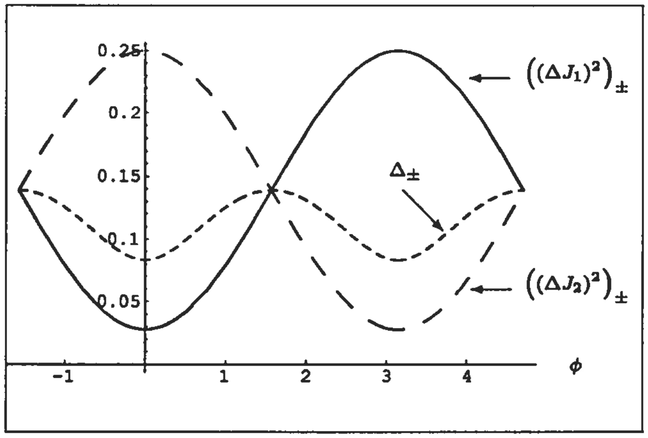

FIG. 4. Graphs of the dispersions $((\Delta J_1)^2)_\pm$, $((\Delta J_2)^2)_\pm$ and the $\Delta_\pm$ factor as functions of ϕ for $\delta=0.5$ and $j=\frac{1}{2}$.

where the bosonic creation and annihilation operators $a^\dagger$ and a are defined as in (2.25) and the corresponding fermionic operators $f^\dagger$ and f are defined as

$$f^\dagger=\sigma_+=\tfrac{1}{2}(\sigma_1+i\sigma_2),\quad f=\sigma_-=\tfrac{1}{2}(\sigma_1-i\sigma_2), \tag{3.2}$$

(the σ_i, $i=1,2$, being the usual Pauli matrices) for the spin $\frac{1}{2}$ fermion. We can thus write

$$H_{\text{SUSY}}=w\left(a^\dagger a-\frac{1}{2}\right)-\frac{w}{2}\sigma_3. \tag{3.3}$$

The representation space we are working with in this context is nothing else than the direct product

$$\mathcal{F}=\mathcal{F}_b\otimes\mathcal{F}_f=\{|n\rangle,n=0,1,2,\dots\}\otimes\left\{\left|\frac{1}{2},\frac{1}{2}\right\rangle=|+\rangle,\left|\frac{1}{2},\frac{-1}{2}\right\rangle=|-\rangle\right\}$$
$$=\{|n,+\rangle,|n,-\rangle,n=0,1,2,\dots\}. \tag{3.4}$$

Following Aragone and Zypman,[10] SCSs may be constructed as eigenstates of a SUSY annihilation operator $[\sqrt{2}(a+\sigma_+)]$. They are shown to be given as a linear combination of the following normalized pure states:

$$|\psi\rangle_+=D\left(\frac{z}{\sqrt{2}}\right)|0,+\rangle \tag{3.5}$$

and

$$|\psi\rangle_-=D\left(\frac{z}{\sqrt{2}}\right)\frac{[a^\dagger|0,+\rangle-|0,-\rangle]}{\sqrt{2}}, \tag{3.6}$$

in terms of the displacement operator D given in (2.35) and where we recognize in (3.5) the usual CS of the harmonic oscillator. A discussion[10,11] of the properties of such states has led to the observation that, except for the state $|\psi_+\rangle\equiv$ (3.5), no other linear combination of (3.5) and (3.6) will minimize the usual HUR. This means that these states satisfy $(\Delta x)^2(\Delta p)^2\geq\frac{1}{4}$, the equality between the position x and the momentum p being realized only for $|\psi_+\rangle\equiv(3.5)$.

Such a fact can be clarified from our discussion of Sec. II A. The SCSs (3.5) and (3.6) are in fact MUS for the SRUR (2.4) with

$$A=\frac{1}{\sqrt{2}}[(a^{\dagger}+a)+\sigma_1]=\left[x+\frac{\sigma_1}{\sqrt{2}}\right] \quad \text{and} \quad B=\frac{1}{\sqrt{2}}[i(a^{\dagger}-a)+\sigma_2]=\left[p+\frac{\sigma_2}{\sqrt{2}}\right], \tag{3.7}$$

these operators being different from x and p. The SCSs are coherent in the sense that they satisfy Eq. (2.6) with $\lambda=1$.

Clearly, in such a context, through the group theory level, we are combining the information coming from both the Heisenberg–Weyl $h(1)$ and the su(2) algebras realized in terms of the Pauli matrices in the spin $\frac{1}{2}$ case. It is then natural to ask the questions of determining the general CS and SS for the direct sum $h(1)\oplus\mathrm{su}(2)$ which will indeed include the special SCS we just discussed.

B. Algebra eigenstates

We are working with the $h(1)\oplus\mathrm{su}(2)$ algebra generated by $\{a,a^{\dagger},I;J_+,J_-,J_3\}$ as defined in the preceding sections. The AESs[8] for this algebra are defined as eigenstates corresponding to a complex combination of the associated generators. A general Hermitian operator A constructed from a combination of these generators is

$$A=A_1a+\bar{A}_1a^{\dagger}+A_2I+A_3J_++\bar{A}_3J_-+A_4J_3, \quad A_2,A_4\in\mathbb{R}, \; A_1,A_3\in\mathbb{C}. \tag{3.8}$$

Two such operators, called A and B, satisfy the commutation relation (2.1) with

$$C=[i(\bar{A}_1B_1-A_1\bar{B}_1)I+2i(B_3\bar{A}_3-\bar{B}_3A_3)J_3+i(A_3B_4-A_4B_3)J_++i(A_4\bar{B}_3-\bar{A}_3B_4)J_-]. \tag{3.9}$$

Once we search for states satisfying (2.6), i.e., for eigenstates of $A+i\lambda B$ ($\lambda\in\mathbb{C},\lambda\neq 0$), we are in fact considering AESs and we know from Sec. II A that they minimize the SRUR (2.4). Let us then study the solutions of such a general eigenstate equation (2.6) for A and B on the form (3.8).

It is convenient to rewrite this equation as

$$[\alpha_-a+\alpha_+a^{\dagger}+\alpha_3I+\beta_-J_++\beta_+J_-+\beta_3J_3]|\psi\rangle=z|\psi\rangle, \tag{3.10}$$

where

$$\begin{aligned}&\alpha_-=A_1+i\lambda B_1, \quad \alpha_+=\bar{A}_1+i\lambda\bar{B}_1, \quad \alpha_3=A_2+i\lambda B_2,\\&\beta_-=A_3+i\lambda B_3, \quad \beta_+=\bar{A}_3+i\lambda\bar{B}_3, \quad \beta_3=A_4+i\lambda B_4.\end{aligned} \tag{3.11}$$

To solve (3.10), we express $|\psi\rangle$ as a superposition of fundamental states $|n;j,m\rangle$ which constitute a generalization of the Fock space (3.4) for spin j. We write

$$|\psi\rangle^j=\sum_{m=-j}^{j}\sum_{n=0}^{\infty}C^j_{n,m}|n;j,m\rangle, \tag{3.12}$$

for fixed j, integer or half-odd integer. Let us recall that we have

$$\begin{aligned}a|n;j,m\rangle&=\sqrt{n}|n-1;j,m\rangle,\\a^{\dagger}|n;j,m\rangle&=\sqrt{n+1}|n+1;j,m\rangle,\\J_{\pm}|n;j,m\rangle&=\sqrt{(j\mp m)(j\pm m+1)}|n;j,m\pm 1\rangle,\end{aligned} \tag{3.13}$$

with

$$\langle n;j,m|l;j,r\rangle=\delta_{nl}\delta_{mr}. \tag{3.14}$$

Inserting (3.12) into (3.10) and taking into account the relations (3.13) and (3.14), we get a recurrence system which becomes more and more complicated as j increases. We also notice that the case where $\alpha_-=0$ with $\alpha_+\neq 0$ does not give any solution and must be eliminated. Here two ways of solving it completely are presented. The first one uses the results obtained in Sec. II B and Appendix A where AESs of su(2) are explicitly constructed. It is described explicitly in this section using operators acting on a fundamental state. The second one is based on the method of resolution of a first order system of linear differential equations and is described in Appendix B.

With respect to the discussion in Appendix A, we have mainly two types of eigenvalues for z. The first type is given by

$$z=\rho_m^j+\alpha_3+mb, \quad \rho_m^j\in \mathrm{C}, \tag{3.15}$$

for fixed j and where $m=-j,\ldots,j$ and

$$b=\sqrt{4\beta_+\beta_-+\beta_3^2}\neq 0. \tag{3.16}$$

If we compare equations (2.26) and (A5) and their respective solutions (2.33) and (A15), we find the set of solutions

$$|\psi\rangle_m^j=(C_m^j)^{-1/2}\exp\left[-\frac{\alpha_+}{2\alpha_-}a^{\dagger 2}+\frac{\rho_m^j}{\alpha_-}a^\dagger\right]T_{\mathrm{eff}}|0;j,m\rangle, \tag{3.17}$$

when $\alpha_-\neq 0$. Here T_{eff} is given by (A14) when $\{\beta_+\neq 0,\beta_-\neq 0\}$, (A18) when $\{\beta_+=0,\beta_3\neq 0\}$, (A20) when $\{\beta_-=0,\beta_3\neq 0\}$ and finally the identity when $\{\beta_-=\beta_+=0,\beta_3\neq 0\}$.

The second type corresponds to the so-called degenerate case ($b=0$) where $z=\rho+\alpha_3$. The sets of independent solutions are now given by

$$|\psi\rangle_m^j=(C_m^j)^{-1/2}\exp\left[-\frac{\alpha_+}{2\alpha_-}a^{\dagger 2}+\frac{\rho}{\alpha_-}a^\dagger\right] \times\sum_{k=0}^{j-m}(-1)^k\binom{j-m}{k}\frac{(2j-k)!}{(2j)!}(a^\dagger)^{j-m-k}\left(\frac{\alpha_-J_-}{\beta_-}\right)^k|0;j,j\rangle \tag{3.18}$$

when $\beta_+=\beta_3=0$,

$$|\psi\rangle_m^j=(C_m^j)^{-1/2}\exp\left[-\frac{\alpha_+}{2\alpha_-}a^{\dagger 2}+\frac{\rho}{\alpha_-}a^\dagger\right] \times\sum_{k=0}^{j-m}(-1)^k\binom{j-m}{k}\frac{(2j-k)!}{(2j)!}(a^\dagger)^{j-m-k}\left(\frac{\alpha_-J_+}{\beta_+}\right)^k|0;j,-j\rangle \tag{3.19}$$

when $\beta_-=\beta_3=0$, and

$$|\psi\rangle_m^j=(C_m^j)^{-1/2}\exp\left[-\frac{\alpha_+}{2\alpha_-}a^{\dagger 2}+\frac{\rho}{\alpha_-}a^\dagger\right] \times\left[\sum_{k=0}^{j-m}(-1)^k\binom{j-m}{k}\frac{(2j-k)!}{(2j)!}(a^\dagger)^{j-m-k}\left(\frac{\alpha_-}{\beta_+}\right)^k\frac{d^k e^{\vartheta J_+}}{d\vartheta^k}\right]|0;j,-j\rangle \tag{3.20}$$

when β_+,β_- and β_3 are different from zero and for $\vartheta=\beta_3/(2\beta_+)=-2\beta_-/\beta_3$.

C. Coherent and squeezed states for the super-position and super-momentum operators

Let us consider the eigenstates of Eq. (3.10) corresponding to the following special values of the parameters

$$A_4=B_4=A_2=B_2=0, \quad A_1=iB_1=\frac{\mu}{\sqrt{2}}, \quad (\mu\neq 0), \quad A_3=iB_3=\frac{\tau}{\sqrt{2}}, \tag{3.21}$$

so that A will be called the super-position operator denoted by X and B the super-momentum operator denoted by P. We have

$$X=\frac{1}{\sqrt{2}}[(\mu a+\bar{\mu}a^\dagger)+(\tau J_+ +\bar{\tau}J_-)], \quad P=\frac{i}{\sqrt{2}}[(\bar{\mu}a^\dagger-\mu a)+(\bar{\tau}J_- -\tau J_+)]. \tag{3.22}$$

We see that the operators (3.7) associated to the SCS are then a special case where $\mu=\bar{\mu}=\tau=\bar{\tau}=1$ in the spin-$\frac{1}{2}$ case.

The eigenstate equation (3.10) now writes

$$[X+i\lambda P]|\psi\rangle=z|\psi\rangle \tag{3.23}$$

and the operator C in (3.9) is diagonal and takes the form

$$C=|\mu|^2 I+2|\tau|^2 J_3. \tag{3.24}$$

Since we have

$$\alpha_-=\frac{\mu(1+\lambda)}{\sqrt{2}}, \quad \alpha_+=\frac{\bar{\mu}(1-\lambda)}{\sqrt{2}}, \quad \alpha_3=0,$$
$$\beta_-=\frac{\tau(1+\lambda)}{\sqrt{2}}, \quad \beta_+=\frac{\bar{\tau}(1-\lambda)}{\sqrt{2}}, \quad \beta_3=0, \tag{3.25}$$

and finally

$$b=\sqrt{2}|\tau|\sqrt{1-\lambda^2}, \tag{3.26}$$

we can use the preceding solutions to give all the solutions of Eq. (3.23).

For $\lambda=1$, we have $\alpha_+=\beta_+=b=0$ and the eigenstate equation is

$$[\mu a+\tau J_+]|\psi\rangle=\frac{z}{\sqrt{2}}|\psi\rangle. \tag{3.27}$$

The normalized solutions are obtained from (3.18) and take the form

$$|\psi\rangle_m^j=(C_m^j(\mu,\tau))^{-1/2}D\left(\frac{z}{\mu\sqrt{2}}\right)\left[\sum_{k=0}^{j-m}(-1)^k\binom{j-m}{k}\frac{(2j-k)!}{(2j)!}(a^\dagger)^{j-m-k}\left(\frac{\mu J_-}{\tau}\right)^k\right]|0;j,j\rangle, \tag{3.28}$$

where the normalization constant is given by

$$C_m^j(\mu,\tau)=(j-m)!\left[\sum_{k=0}^{j-m}\binom{j-m}{k}\frac{(2j-k)!}{(2j)!}\left(\frac{|\mu|^2}{|\tau|^2}\right)^k\right]. \tag{3.29}$$

Let us recall that in this case we have CSs for which

$$(\Delta X)=(\Delta P)=\Delta=\tfrac{1}{2}\langle C\rangle. \tag{3.30}$$

The mean value of C is easy to compute and we have

$$\langle C\rangle_m^j=|\mu|^2+2|\tau|^2\left[j+|\tau|^2\frac{\partial}{\partial|\tau|^2}\ln(C_m^j(\mu,\tau))\right]. \tag{3.31}$$

In the special case $j=\frac{1}{2}$, we find the normalized and orthogonal states

$$|\psi\rangle^+=D\left(\frac{z}{\mu\sqrt{2}}\right)|0;+\rangle,\quad |\psi\rangle^-=D\left(\frac{z}{\mu\sqrt{2}}\right)\frac{|\tau|}{\sqrt{|\mu|^2+|\tau|^2}}\left[a^\dagger|0;+\rangle-\frac{\mu}{\tau}|0;-\rangle\right], \tag{3.32}$$

where D is again given by (2.35). In those states, we have

$$\langle C\rangle^+=|\mu|^2+|\tau|^2,\quad \langle C\rangle^-=\left[(|\mu|^2+|\tau|^2)-\frac{2|\mu|^2|\tau|^2}{(|\mu|^2+|\tau|^2)}\right]. \tag{3.33}$$

This is clearly a generalization of SCSs considered by Aragone and Zypman[10] and recalled in (3.5) and (3.6).

From (3.33), we see that the dispersions of ΔX and ΔP given by (3.30) computed in the CS $|\psi\rangle^-$ are smaller than in the states $|\psi\rangle^+$. The states $|\psi\rangle^-$ thus are the closest to classical states for the SUSY harmonic oscillators (this means with respect to the super-position and the super-momentum) while $|\psi\rangle^+$ are indeed the ones closest to classical states of the standard harmonic oscillator (i.e., they minimize the HUR for X and P). Let us mention that if we take $\mu=1$, we see that $\langle C\rangle^+$ has its minimum value equal to 1 for $\tau\mapsto 0$ and in this case $X=x$ and $P=p$. For the same value of μ, we see that $\langle C\rangle^-$ takes the form

$$\langle C\rangle^-=\frac{1+|\tau|^4}{1+|\tau|^2}, \tag{3.34}$$

which has a minimum value $\langle C\rangle_{\min}^-=2(\sqrt{2}-1)<1$ for $|\tau|^2=\sqrt{2}-1$.

For $\lambda\neq\pm1$, from Eq. (3.17) and $T_{\text{eff}}\equiv$ (A13), using also (2.35) and (2.36), we get the states

$$|\psi\rangle_m^j=(C_m^j)^{-1/2}S(\chi(\delta,\phi-2\phi_u))D(\eta_m(z,\delta,\phi,\mu,\tau))\times\exp\left(\frac{-\tau\delta^{-1/2}e^{-i\phi/2}}{|\tau|}J_+\right)\exp\left(\frac{\bar{\tau}\delta^{1/2}e^{i\phi/2}}{2|\tau|}J_-\right)|0;j,m\rangle, \tag{3.35}$$

where

$$\eta_m(z,\delta,\phi,\mu,\tau)=\frac{1}{\mu}\left\{\frac{z(1+\delta e^{i\phi})}{\sqrt{2}}-2m|\tau|\delta^{1/2}e^{i\phi/2}\right\},\quad \mu=|\mu|e^{i\phi_u} \tag{3.36}$$

and where we have used instead of λ the parameters δ and ϕ as given in (2.32). Let us mention that this general expression (3.35) clearly shows the presence of the unitary operators D and S associated with $h(1)$ and su(1,1), respectively, which is the contribution of the bosonic part of our SUSY model. Moreover, the fermionic contribution appears through the action of a unitary operator associated with su(2).

Now these states satisfy the MUR

$$(\Delta X)_m^j(\Delta P)_m^j=\Delta_m^j=\frac{1}{2}\sqrt{1+\frac{4\delta^2\sin^2\phi}{(1-\delta^2)^2}}|\langle C\rangle_m^j|. \tag{3.37}$$

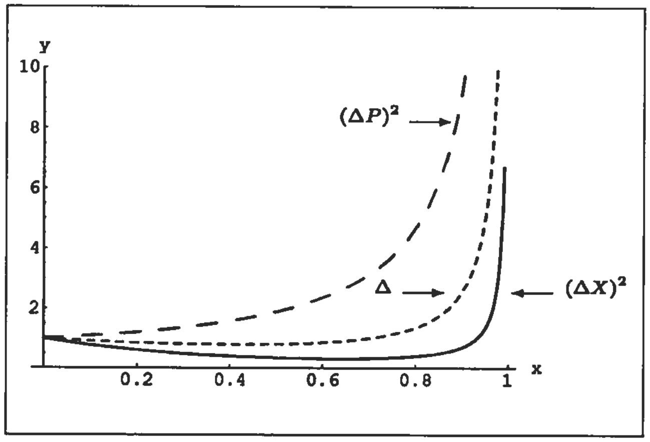

FIG. 5. Graphs of the dispersions $(\Delta X)^2$, $(\Delta P)^2$ and the factor Δ as functions of $x \equiv \delta$ for $\phi=\pi/6$. $|\tau|=|\mu|=1$, $j=\frac{1}{2}$.

The mean value of C is

$$\langle C\rangle_m^j=|\mu|^2+2|\tau|^2\frac{(1-\delta)}{(1+\delta)}\left(j-\frac{4(j+|m|)\delta}{(1+\delta)^2}\Omega\right), \tag{3.38}$$

where Ω is expressed in terms of Jacobi polynomials (see Appendix A),

$$\Omega=\frac{P_{j-|m|-1}^{(-2j,1)}(1-(8\delta/(1+\delta)^2))}{P_{j-|m|}^{(-2j-1,0)}(1-(8\delta/(1+\delta)^2))}, \tag{3.39}$$

for $m=-j+1,\dots,j-1$ and $\Omega=0$ for $m=\pm j$. In fact, we see that in these last cases, we have

$$\langle C\rangle_{\pm j}^j=|\mu|^2+2j|\tau|^2\frac{(1-\delta)}{(1+\delta)}. \tag{3.40}$$

It is now interesting to examine the behavior of the dispersions ΔX and ΔP in these states for the spin $\frac{1}{2}$ case. Using (2.20) with (3.40) for $j=\frac{1}{2}$, we get

$$((\Delta X)^2)_\pm=\frac{(1-2\delta\cos\phi+\delta^2)}{2(1-\delta^2)}\left[|\mu|^2+|\tau|^2\frac{(1-\delta)}{(1+\delta)}\right],$$
$$((\Delta P)^2)_\pm=\frac{(1+2\delta\cos\phi+\delta^2)}{2(1-\delta^2)}\left[|\mu|^2+|\tau|^2\frac{(1-\delta)}{(1+\delta)}\right], \tag{3.41}$$

with

$$\Delta_\pm=\frac{\sqrt{(1-\delta^2)^2+4\delta^2\sin^2\phi}}{2(1-\delta^2)}\left[|\mu|^2+|\tau|^2\frac{(1-\delta)}{(1+\delta)}\right]. \tag{3.42}$$

If we take $\delta=0$ (i.e., $\lambda=1$) in these last expressions, we find only the values of the dispersions of X and P in the usual coherent states $|\psi\rangle^+$ as given by (3.32) and not the ones in the CS $|\psi\rangle^-$, which is the reason why that case has been treated separately.

Figures 5 and 6 show the behavior of $((\Delta X)^2)_\pm$ and $((\Delta P)^2)_\pm$ and $\Delta_\pm$ as functions of δ for $\phi=\pi/6$ and as functions of ϕ for $\delta=0.5$, respectively. We notice a similar behavior as for the position and momentum operators.

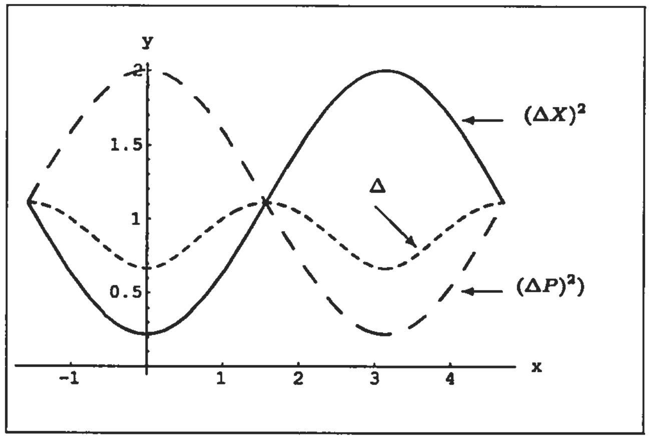

FIG. 6. Graphs of the dispersions $(\Delta X)^2$, $(\Delta P)^2$ and the factor Δ as functions of $x\equiv\phi$, $\delta=0.5$, $|\tau|=|\mu|=1$, $j=\frac{1}{2}$.

IV. CONSTRUCTION OF $h(1)\oplus su(2)$ HAMILTONIANS

An application of our CS and SS based on the algebra $h(1)\oplus su(2)$ will be the study of possible Hamiltonians which can be written as $\mathcal{H}=w\mathcal{A}^\dagger\mathcal{A}$, where $\mathcal{A}$ is a linear combination of the generators of $h(1)\oplus su(2)$. It is clear that the usual harmonic oscillator Hamiltonian will enter in the scheme as a special case (Sec. IV A) but also the Jaynes–Cummings[16] one in the strong coupling limit (Sec. IV B and C).

Moreover, since the CSs and SSs already constructed in the preceding section are in fact eigenstates of the operator $\mathcal{A}$, we would be able to find easily some properties of the mean value and the dispersion of the associated energies in those states.

A. Isospectral $h(1)\oplus su(2)$ harmonic oscillator Hamiltonians

We are interested in systems for which the Hamiltonian is expressed in the form

$$\mathcal{H}=w\mathcal{A}^\dagger\mathcal{A}, \tag{4.1}$$

where

$$\mathcal{A}=\alpha_-a+\alpha_+a^\dagger+\alpha_3 I+\beta_-J_++\beta_+J_-+\beta_3J_3, \quad \alpha_-\neq 0, \tag{4.2}$$

is an element of the $h(1)\oplus su(2)$ algebra. The commutator of the operators $\mathcal{A}$ and $\mathcal{A}^\dagger$ is

$$[\mathcal{A},\mathcal{A}^\dagger]=(|\alpha_-|^2-|\alpha_+|^2)I+(|\beta_-|^2-|\beta_+|^2)J_3+(\beta_3\bar{\beta}_+-\bar{\beta}_3\beta_-)J_++(\bar{\beta}_3\beta_+-\beta_3\bar{\beta}_-)J_-. \tag{4.3}$$

If $|Z\rangle$ is an eigenstate of the operator $\mathcal{A}$ with eigenvalue z, i.e.,

$$\mathcal{A}|Z\rangle=z|Z\rangle, \tag{4.4}$$

then the mean value of the energy in this state will always be given by

$$\langle Z|\mathcal{H}|Z\rangle=w|z|^2 \tag{4.5}$$

and the dispersion by

$$(\Delta\mathcal{H})^2=w^2|z|^2\langle Z|[\mathcal{A},\mathcal{A}^\dagger]|Z\rangle. \tag{4.6}$$

First, let us consider the special case where

$$[\mathcal{A},\mathcal{A}^\dagger]=I. \tag{4.7}$$

This imposes the following conditions on the parameters:

$$|\alpha_-|^2-|\alpha_+|^2=1, \quad |\beta_-|=|\beta_+| \quad \text{and} \quad \beta_3\bar{\beta}_+-\bar{\beta}_3\beta_-=0, \tag{4.8}$$

i.e.,

$$\alpha_-=\cosh\alpha e^{i\theta_-}, \quad \alpha_+=\sinh\alpha e^{i\theta_+}, \quad \beta_\pm=\beta e^{i\varphi_\pm}, \tag{4.9}$$

and

$$\beta_3=\begin{cases} re^{i(\varphi_++\varphi_-)/2}, r\in\mathbb{R}_+\cup\{0\}, & \text{if } \beta\neq 0,\\ re^{i\varphi_3}, r\in\mathbb{R}_+\cup\{0\}, & \text{if } \beta=0. \end{cases} \tag{4.10}$$

When $\beta\neq 0$, the operator $\mathcal{A}$ then takes the form

$$\mathcal{A}=\cosh\alpha e^{i\theta_-}a+\sinh\alpha e^{i\theta_+}a^\dagger+\alpha_3 I+\beta(e^{i\varphi_-}J_++e^{i\varphi_+}J_-)+re^{i(\varphi_++\varphi_-)/2}J_3. \tag{4.11}$$

The parameter b given in (3.16) becomes $b=\sqrt{4\beta^2+r^2}e^{i(\varphi_++\varphi_-)/2}$ and is different from zero. Therefore in this case, according to the equation (3.17), the normalized solutions of the eigenstate equation (4.4) are given by

$$|Z\rangle_m^j=S(\Lambda)D(\zeta_m(\alpha_3,1))TD(ze^{-i\theta_-})|0;j,m\rangle, \tag{4.12}$$

where

$$\Lambda=-\alpha e^{i(\theta_+-\theta_-)}, \quad \zeta_m(\alpha_3,\epsilon)=-[\alpha_3+\epsilon m\sqrt{4\beta^2+r^2}e^{i(\varphi_++\varphi_-)/2}]e^{-i\theta_-}, \tag{4.13}$$

and

$$T=\exp\left(-\frac{\tilde{\theta}}{2}[e^{-i(\varphi_+-\varphi_-)/2}J_+-e^{i(\varphi_+-\varphi_-)/2}J_-]\right), \tag{4.14}$$

with

$$\frac{\tilde{\theta}}{2}=\tan^{-1}\left(\sqrt{1-\frac{r}{2\beta^2}(\sqrt{4\beta^2+r^2}-r)}\right). \tag{4.15}$$

This means that T is a unitary operator.

We remark that, if we define the new operator

$$\begin{aligned}\mathcal{A}_0&=D^\dagger(-\alpha_3e^{-i\theta_-})S^\dagger(\Lambda)\mathcal{A}S(\Lambda)D(-\alpha_3e^{-i\theta_-})\\ &=e^{i\theta_-}a+\beta(e^{i\varphi_-}J_++e^{i\varphi_+}J_-)+re^{i(\varphi_++\varphi_-)/2}J_3,\end{aligned} \tag{4.16}$$

which is simpler than the original $\mathcal{A}$, then the new Hamiltonian $\mathcal{H}_0=w\mathcal{A}_0^\dagger\mathcal{A}_0$ is isospectral to the Hamiltonian $\mathcal{H}\equiv$ (4.1).

The dispersion of $\mathcal{H}$ calculated on the states (4.12) is, from (4.6) and (4.7), given by $(\Delta\mathcal{H})^2=w^2|z|^2$ and is the same as the one of $\mathcal{H}_0$ calculated on the states $D(\zeta_m(-z,1))T|0;j,m\rangle$. This value is exactly the dispersion of the harmonic oscillator in the usual CS.

On the other hand, due to (4.7) we have $[\mathcal{H},\mathcal{A}]=-w\mathcal{A}$, so we have a complete analogy with the harmonic oscillator. The CSs associated to the Hamiltonian $\mathcal{H}$, called generalized harmonic oscillator, are those given by the equation (4.12) and, thus, one can write them in the form

$$|Z\rangle_m^j = \mathcal{D}(z)|\tilde{0}\rangle_m^j, \quad \text{where } \mathcal{D}(z) = \exp(z\mathcal{A}^\dagger - \bar{z}\mathcal{A}) \tag{4.17}$$

and $|\tilde{0}\rangle_m^j$, $m=-j,\ldots,j$, are the fundamental states of the system $\mathcal{H}$, that is, the eigenstates of $\mathcal{H}$ corresponding to the $(2j+1)$ degenerate eigenvalue 0. They are also eigenstates of $\mathcal{A}$ corresponding to the eigenvalue 0. So, they can be written

$$|\tilde{0}\rangle_m^j = S(\Lambda)D(\zeta_m(\alpha_3,1))T|0;j,m\rangle. \tag{4.18}$$

Furthermore, the SSs associated with $\mathcal{H}$ are given by

$$|\tilde{\psi}\rangle_m^j = \mathcal{S}(\chi)\mathcal{D}(z)|\tilde{0}\rangle_m^j, \tag{4.19}$$

where the super-squeezed operator $\mathcal{S}(\chi)$ is given by $\exp(\chi\mathcal{A}^{\dagger 2}/2 - \bar{\chi}\mathcal{A}^2/2)$ and the super-displacement operator $\mathcal{D}(z)$ is given in (4.17). If we define $\mathcal{X}=(\mathcal{A}+\mathcal{A}^\dagger)/\sqrt{2}$ and $\mathcal{P}=i(\mathcal{A}^\dagger - \mathcal{A})/\sqrt{2}$, these states (4.19) minimize the SRUR $(\Delta\mathcal{X})^2(\Delta\mathcal{P})^2=(1+\langle F\rangle^2)/4$, i.e., they are solutions of the eigenstate equation $[(1-\lambda)\mathcal{A}^\dagger + (1+\lambda)\mathcal{A}]|\psi\rangle = \sqrt{2}|\psi\rangle$.

The eigenstates of $\mathcal{H}$ corresponding to the $(2j+1)$ degenerate energy eigenvalue $E_n = nw$ are now given by

$$|\tilde{n}\rangle_m^j = \frac{\mathcal{A}^{\dagger n}}{\sqrt{n!}}|\tilde{0}\rangle_m^j. \tag{4.20}$$

These states may be obtained as the action of a unitary operator on the states $|n;j,m\rangle$. Indeed, if we introduce the unitary operator

$$U_n^m = e^{-in\theta_-}S(\Lambda)D(\zeta_m(\alpha_3,1))T, \tag{4.21}$$

we see that, from (4.20), we have

$$\begin{aligned} |\tilde{n}\rangle_m^j &= \frac{e^{in\theta_-}}{\sqrt{n!}}(\mathcal{A}^\dagger)^n U_n^m|0;j,m\rangle, \\ &= \frac{e^{in\theta_-}}{\sqrt{n!}}U_n^m((U_n^m)^\dagger\mathcal{A}^\dagger U_n^m)^n|0;j,m\rangle, \\ &= \frac{e^{in\theta_-}}{\sqrt{n!}}U_n^m(e^{-i\theta_-}a^\dagger + \sqrt{4\beta^2+r^2}e^{-i(\varphi_+ + \varphi_-)/2}(J_3-m))^n|0;j,m\rangle. \end{aligned} \tag{4.22}$$

Since we have $(J_3-m)|0;j,m\rangle=0$, we finally find

$$|\tilde{n}\rangle_m^j = U_n^m|n;j,m\rangle. \tag{4.23}$$

In the case $\beta=0$, the operator $\mathcal{A}$ is given by

$$\mathcal{A} = \cosh\alpha e^{i\theta_-}a + \sinh\alpha e^{i\theta_+}a^\dagger + \alpha_3 I + re^{i\varphi_3}J_3. \tag{4.24}$$

Then, if $r\neq 0$, one has the same results as above, except that it is necessary to replace T by I and b by $\beta_3 = re^{i\varphi_3}$. If $r=0$, $\mathcal{A}$ is an element of the algebra $h(1)$ and then the results are the ones obtained in Sec. II A for the standard harmonic oscillator after applying the unitary transformation $S(\Lambda)D(-\alpha_3 e^{-i\theta_-})$.

B. Strong-coupling limit of the Jaynes–Cummings Hamiltonian as limit of $h(1)\oplus su(2)$ Hamiltonians

We are going to consider now the case where

$$[\mathcal{A},\mathcal{A}^\dagger]=I+2xJ_3, \quad x\in\mathbb{R}. \tag{4.25}$$

This imposes the following conditions on the parameters:

$$|\alpha_-|^2-|\alpha_+|^2=1, \quad |\beta_-|^2-|\beta_+|^2=x \quad \text{and} \quad \beta_3\bar{\beta}_+-\bar{\beta}_3\beta_-=0. \tag{4.26}$$

We already know the results when $x=0$. When $x\neq 0$, the conditions (4.26) imply

$$\alpha_-=\cosh\alpha e^{i\theta_-}, \quad \alpha_+=\sinh\alpha e^{i\theta_+}, \quad \beta_3=0, \tag{4.27}$$

and

$$\beta_-=\begin{cases} x^{1/2}\cosh\beta e^{i\varphi_-}, & \text{if } x>0,\\ |x|^{1/2}\sinh\beta e^{i\varphi_-}, & \text{if } x<0,\end{cases} \tag{4.28}$$

$$\beta_+=\begin{cases} x^{1/2}\sinh\beta e^{i\varphi_+}, & \text{if } x>0,\\ |x|^{1/2}\cosh\beta e^{i\varphi_+}, & \text{if } x<0.\end{cases} \tag{4.29}$$

The parameter $b\equiv$ (3.16) becomes $b=|x|^{1/2}\sqrt{2\sinh(2\beta)}e^{i(\varphi_++\varphi_-)/2}$. This means that $b=0$ if and only if $\beta=0$.

In the case $\beta\neq 0$, according to the equations (3.17), (A7), (A11), and (A12), the normalized eigenstates of the operator $\mathcal{A}$ are given by

$$|Z(x)\rangle_m^j=(C_m^j(x))^{-1/2}S(\Lambda)D(-\alpha_3e^{-i\theta_-})D(\eta_m(z,x))\exp\left[-\frac{x}{2|x|}\ln(\tanh\beta)J_3\right]U|0;j,m\rangle, \tag{4.30}$$

where

$$\eta_m(z,x)=[z-m|x|^{1/2}\sqrt{2\sinh(2\beta)}e^{i(\varphi_++\varphi_-)/2}]e^{-i\theta_-}, \tag{4.31}$$

$$U=\exp\left[-\frac{\pi}{4}(e^{-i(\varphi_+-\varphi_-)/2}J_+-e^{i(\varphi_+-\varphi_-)/2}J_-)\right] \tag{4.32}$$

and

$$\begin{aligned} C_m^j(x)&=\langle j,m|U^\dagger\exp\left[-\frac{x}{|x|}\ln(\tanh\beta)J_3\right]U|j,m\rangle\\ &=\left(\frac{1+\tanh\beta}{2\sqrt{\tanh\beta}}\right)^{\mp 2m}P_{j\pm m}^{0;\mp 2m}\left(\frac{1+\tanh^2\beta}{2\tanh\beta}\right). \end{aligned} \tag{4.33}$$

From (4.6) and (4.25), the dispersion of the $\mathcal{H}\equiv$ (4.1) in the states (4.30) can be calculated explicitly. We get

$$(\Delta\mathcal{H})^2)_m^j=w^2|z|^2(1+2x_m^j\langle Z(x)|J_3|Z(x)\rangle_m^j). \tag{4.34}$$

In the last expression, the mean value of J_3 is obtained in a similar way then to get (2.61). The result is

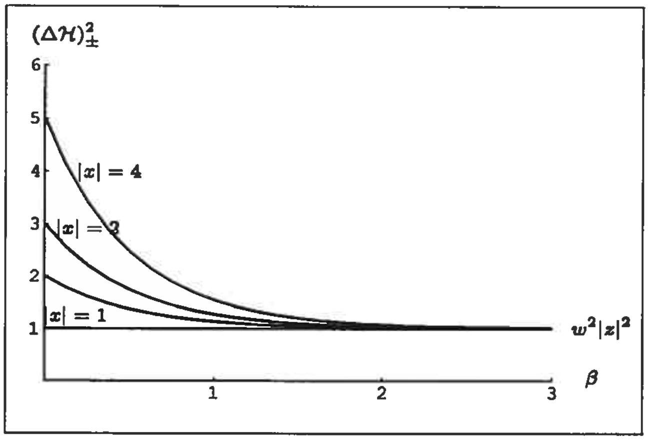

FIG. 7. Graphs of the dispersions $((\Delta\mathcal{H})^2)_\pm \equiv (4.37)$ as functions of $\beta>0$ for $|x|=0,1,2,4$.

$$
{}^{j}_{m}\langle Z(x)|J_3|Z(x)\rangle^{j}_{m} = \frac{x}{|x|}\left\{ |m|e^{-2\beta} + \frac{(j+|m|+1)}{2\sinh(2\beta)} \frac{P^{1;2|m|}_{j-|m|-1}(\coth(2\beta))}{P^{0;2|m|}_{j-|m|}(\coth(2\beta))} \right\}. \tag{4.35}
$$

If we take $m=\pm j$, the dispersion of $\mathcal{H}$ is

$$
((\Delta\mathcal{H})^2)^{j}_{\pm j} = w^2|z|^2(1+2j|x|e^{-2\beta}), \tag{4.36}
$$

and, in particular, when $j=\frac{1}{2}$, we get

$$
((\Delta\mathcal{H})^2)_{\pm} = w^2|z|^2(1+|x|e^{-2\beta}). \tag{4.37}
$$

Figure 7 shows the graphs of $((\Delta\mathcal{H})^2)_\pm$ as functions of β for different values of $|x|$ when $w^2|z|^2$ is taken equal to 1.

Let us compute the new operator $\mathcal{A}_0$ defined as (4.16). We get

$$
\mathcal{A}_0 = \begin{cases} e^{i\theta_-}a + x^{1/2}\cosh\beta e^{i\varphi_-}J_+ + x^{1/2}\sinh\beta e^{i\varphi_+}J_-, & \text{if } x>0, \\ e^{i\theta_-}a + |x|^{1/2}\sinh\beta e^{i\varphi_-}J_+ + |x|^{1/2}\cosh\beta e^{i\varphi_+}J_-, & \text{if } x<0, \end{cases} \tag{4.38}
$$

and a new Hamiltonian $\mathcal{H}_0 = w\mathcal{A}_0^\dagger\mathcal{A}_0$ isospectral to the Hamiltonian $\mathcal{H}$ which takes the form

$$
\begin{aligned}
\mathcal{H}_0 = w\{ & a^\dagger a + |x|[\sinh^2(\beta)J_-J_+ + \cosh^2(\beta)J_+J_-] + |x|^{1/2}\cosh\beta[e^{i(\varphi_+-\theta_-)}a^\dagger J_- \\
& + e^{-i(\varphi_+-\theta_-)}aJ_+] + |x|^{1/2}\sinh\beta[e^{i(\varphi_--\theta_-)}a^\dagger J_+ + e^{-i(\varphi_--\theta_-)}aJ_-] \\
& + |x|\sinh\beta\cosh\beta[e^{i(\varphi_+-\varphi_-)}J_-^2 + e^{-i(\varphi_+-\varphi_-)}J_+^2]\},
\end{aligned} \tag{4.39}
$$

if $x<0$. If $x>0$, we get a similar expression except that we must make the change $\sinh\beta \leftrightarrow \cosh\beta$.

In the spin-$\frac{1}{2}$ representation, we have

$$
J_-^2 = J_+^2 = 0, \quad J_+J_- = \frac{I}{2} + J_3 \quad \text{and} \quad J_-J_+ = \frac{I}{2} - J_3, \tag{4.40}
$$

hence (4.39) becomes

$$\mathcal{H}_0 = w\left\{\left(a^\dagger a + \frac{I}{2}\right) - xJ_3 + |x|^{1/2}\cosh\beta[e^{i(\varphi_+ - \theta_-)}a^\dagger J_- + e^{-i(\varphi_+ - \theta_-)}aJ_+]\right.$$
$$\left. + |x|^{1/2}\sinh\beta[e^{i(\varphi_- - \theta_-)}a^\dagger J_+ + e^{-i(\varphi_- - \theta_-)}aJ_-] + (|x|\cosh(2\beta) - 1)\frac{I}{2}\right\} \quad (4.41)$$

and a similar expression when $x>0$, making the literal change $\sinh\beta \leftrightarrow \cosh\beta$. If we take $x = -w_0/w$, $\varphi_+ = \theta_-$ and the limit $\beta \mapsto 0$, then $\mathcal{H}_0 \equiv$ (4.41) becomes

$$\mathcal{H}_0 = w\left(a^\dagger a + \frac{1}{2}\right) + w_0 J_3 + \sqrt{ww_0}(a^\dagger J_- + aJ_+) + \frac{w - w_0}{2} I, \quad (4.42)$$

which is the Jaynes–Cummings Hamiltonian[16] up to a constant term and for a coupling constant given by $\kappa = \sqrt{ww_0}$. Let us recall that this Hamiltonian describes the interaction of a cavity mode (with frequency w) with a two level-system (w_0 being the atomic frequency). When $x = -1$, i.e., for $w = w_0$, (4.42) becomes the strong-coupling limit of the Jaynes–Cummings Hamiltonian.

In the case $\beta = 0$, the new operator $\mathcal{A}_0 \equiv$ (4.16) reduces now to

$$\mathcal{A}_0(x) = \begin{cases} e^{i\theta_-}a + |x|^{1/2}e^{i\varphi_+}J_-, & \text{if } x<0, \\ e^{i\theta_-}a + |x|^{1/2}e^{i\varphi_-}J_+, & \text{if } x>0. \end{cases} \quad (4.43)$$

As we have here $b=0$, according to the expressions (3.18) and (3.19), the orthonormalized eigenstates of $\mathcal{A}_0$ are given by

$$|Z(x)\rangle_m^j = (\tilde{C}_m^j(x))^{1/2} D(ze^{-i\theta_-}) \times \sum_{k=0}^{j-m} (-1)^k \binom{j-m}{k} \frac{(2j-k)!}{(2j)!} (e^{-i\theta_-}a^\dagger)^{j-m-k} \left(J_\mp \frac{e^{-i\varphi_\mp}}{\sqrt{|x|}}\right)^k \left|0; j, \frac{x}{|x|} j\right\rangle, \quad (4.44)$$

where the $-$ sign refers to $x>0$ and the sign $+$ to $x<0$ and

$$\tilde{C}_m^j(x) = (j-m)! \sum_{k=0}^{j-m} \binom{j-m}{k} \frac{(2j-k)!}{(2j)!} \left(\frac{1}{|x|}\right)^k. \quad (4.45)$$

Since, in this case, we have

$${}_m^j\langle Z(x)|J_3|Z(x)\rangle_m^j = \frac{x}{|x|}\left[j + |x| \frac{\partial}{\partial |x|} \ln(\tilde{C}_m^j(x))\right], \quad (4.46)$$

the dispersion of $\mathcal{H}_0 = w\mathcal{A}_0^\dagger \mathcal{A}_0$ in the states (4.44) is given by

$$((\Delta\mathcal{H}_0)^2)_m^j = w^2|z|^2\left[1 + 2|x|j + 2|x|^2 \frac{\partial}{\partial |x|} \ln(\tilde{C}_m^j(x))\right]. \quad (4.47)$$

When $m=j$, we have $\tilde{C}_m^j(x) = 1$, so that we get

$$((\Delta\mathcal{H}_0)^2)_j^j = w^2|z|^2(1 + 2|x|j). \quad (4.48)$$

For example, when $j = \frac{1}{2}$, the dispersion corresponding to $m = \frac{1}{2}$ is given by

$$((\Delta\mathcal{H}_0)^2)_+ = w^2|z|^2(1 + |x|) \quad (4.49)$$

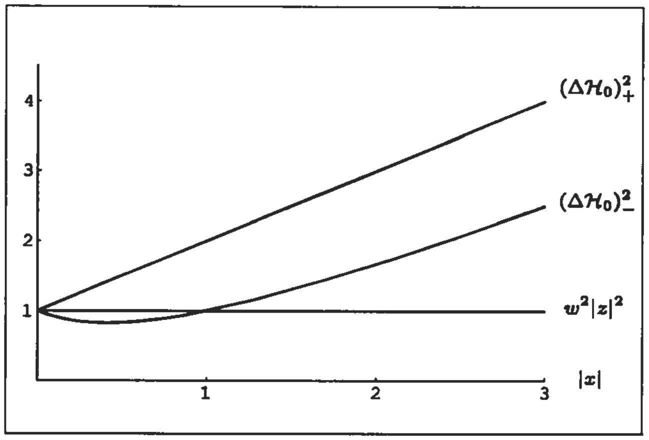

FIG. 8. Graphs of the dispersions $((\Delta\mathcal{H}_0)^2)_\pm$ as given by (4.49) and (4.50) as functions of $|x|$.

and one obtains the same result as in the preceding case when we take the limit $\beta\mapsto 0$. On the other hand, for $m=-1/2$, we get

$$((\Delta\mathcal{H}_0)^2)_- = w^2|z|^2\left[1+|x|\frac{(|x|-1)}{(|x|+1)}\right] \tag{4.50}$$

and it is always smaller than $((\Delta\mathcal{H}_0)^2)_+$. In this last case, we see that if $|x|>1$, the dispersion is bigger than $w^2|z|^2$, while if $|x|<1$ it is smaller than $w^2|z|^2$, and if $|x|=1$, it is equal to $w^2|z|^2$. Furthermore, the dispersion reaches its minimum $0.83w^2|z|^2$ when $|x|=(\sqrt{2}-1)$. Figure 8 shows the behavior of dispersions $((\Delta\mathcal{H}_0)^2)_\pm$ as function of $|x|$.

Let us finally mention that the Hamiltonian $\mathcal{H}_0$ in this case and for $j=\frac{1}{2}$ corresponds to (4.41) when $\beta=0$. A special case is again the Jaynes–Cummings Hamiltonian (4.42) so we get eigenstates of $\mathcal{A}_0\equiv$ (4.43) such that the dispersion of this Hamiltonian is minimized and lower than $w^2|z|^2$.

C. Generalized $h(1)\oplus su(2)$ noncanonical commutation relation

In the case where we have

$$[\mathcal{A},\mathcal{A}^\dagger]=I+\gamma J_+ + \bar{\gamma}J_-, \quad \gamma\in\mathbb{C}, \quad \gamma\neq 0. \tag{4.51}$$

According to (4.3), the necessary conditions on the original parameters are

$$|\alpha_-|^2-|\alpha|_+{}^2=1, \quad |\beta_-|=\beta_+, \quad \beta_3\bar{\beta}_+-\bar{\beta}_3\beta_-=\gamma=\rho e^{i\nu}, \tag{4.52}$$

where $\rho\in\mathbb{R}_+$. A suitable choice of the parameters is

$$\alpha_-=\cosh\alpha e^{i\theta_-}, \quad \alpha_+=\sinh\alpha e^{i\theta_+}, \quad \beta_\pm=\beta e^{i\varphi_\pm}, \quad \beta_3=re^{i\varphi_3}, \quad \beta\neq 0, \ r\neq 0, \tag{4.53}$$

such that

$$r\beta[e^{i(\varphi_3-\varphi_+)}-e^{-i(\varphi_3-\varphi_+)}]=\rho e^{i\nu}. \tag{4.54}$$

Equation (4.54) implies that

$$\rho=2r\beta\left|\sin\left(\varphi_3-\frac{(\varphi_++\varphi_-)}{2}\right)\right| \tag{4.55}$$

and the following conditions on the phases: $\varphi_3 \neq (\varphi_+ + \varphi_-)/2$, $\varphi_3 \neq (\varphi_+ + \varphi_-)/2 + \pi$ and $\varphi_+ - \varphi_- = \pi - 2\nu, \nu \in [0, 3\pi/2]$ or $\varphi_+ - \varphi_- = 3\pi - 2\nu$, $\nu \in [\pi/2, 2\pi]$. Thus, the operator $\mathcal{A}$ compatible with all the previous conditions is

$$\mathcal{A} = \cosh \alpha e^{i\theta_-} a + \sinh \alpha e^{i\theta_+} a^\dagger + \alpha_3 I + e^{i(\varphi_- - \nu)} \left[\beta (e^{i\nu} J_+ - e^{-i\nu} J_-) + \frac{\rho}{2\beta |\cos \theta|} e^{i\theta} J_3 \right], \tag{4.56}$$

where

$$\theta = \varphi_3 - (\varphi_- - \nu), \quad -\frac{\pi}{2} < \theta < 3\frac{\pi}{2}. \tag{4.57}$$

The new operator $\mathcal{A}_0$ defined in (4.16) is then given by

$$\mathcal{A}_0 = e^{i\theta_-} a + e^{i(\varphi_- - \nu)} \left[-\beta (e^{i\nu} J_+ - e^{-i\nu} J_-) + \frac{\rho}{2\beta |\cos \theta|} e^{i\theta} J_3 \right]. \tag{4.58}$$

The parameter $b \equiv$ (3.16) is now $b = i\sqrt{16\beta^2 \cos^2(\theta) - \rho^2 e^{2i\theta}}\, e^{i(\varphi_- - \nu)}/(2\beta |\cos \theta|)$, i.e., $b = 0$ if and only if $\beta = \sqrt{\rho}/2$ and $\theta = \pi$.

Here we can proceed as before, that is, when $b = 0$, find, by means of the equation (3.20) the eigenstates of $\mathcal{A}_0$ and, when $b \neq 0$, find the solutions by means of the equation (3.17) and then calculate the dispersions of $\mathcal{H}_0$.

But, we will follow another treatment which teaches us about the similarities between the canonical and the noncanonical cases. Indeed, seen in another perspective, the commutation relation (4.51) can be expressed in the form

$$[\mathcal{A}_0, \mathcal{A}_0^\dagger] = I + 2\rho \mathbb{J}_3, \tag{4.59}$$

where we have set

$$\mathbb{J}_3 = \frac{(e^{i\nu} J_+ + e^{-i\nu} J_-)}{2}. \tag{4.60}$$

Thus, when $b = 0$, $\mathcal{A}_0$ becomes

$$\mathcal{A}_0 = e^{i\theta_-} a + \sqrt{\rho} e^{i(\varphi_- - \nu)} \mathbb{J}_+, \tag{4.61}$$

with

$$\mathbb{J}_\pm = \pm \frac{(e^{i\nu} J_+ - e^{-i\nu} J_-)}{2} - J_3. \tag{4.62}$$

The operators $\mathbb{J}_3$, $\mathbb{J}_\pm$ satisfy the su(2) algebra and let us denote by $|J, M\rangle$ the eigenstates of both $\mathbb{J}^2$ and $\mathbb{J}_3$. We have again

$$\mathbb{J}_3 |J, M\rangle = M |J, M\rangle, \quad \mathbb{J}_\pm |J, M\rangle = \sqrt{(J \mp M)(J \pm M + 1)} |J, M\rangle. \tag{4.63}$$

Now, it is clear that the resolution of the problem to find the eigenstates of $\mathcal{A}_0$ is similar to the canonical case. Indeed, the normalized eigenstates of $\mathcal{A}_0$ are given by

$$|Z(\rho)\rangle_M^J=(\tilde{C}_M^J(\rho))^{1/2}D(ze^{-i\theta_-})$$
$$\times\sum_{k=0}^{J-M}(-1)^k\binom{J-M}{k}\frac{(2J-k)!}{(2J)!}(e^{-i\theta_-}a^\dagger)^{J-M-k}\left(\frac{\mathbb{J}_-e^{-i(\varphi_--\nu)}}{\sqrt{\rho}}\right)^k|0;J,J\rangle, \tag{4.64}$$

where $\tilde{C}_M^J(\rho)$ is given as in (4.45).

As before, the dispersion of $\mathcal{H}_0$ in the states (4.64) is given by

$$((\Delta\mathcal{H}_0)^2)_M^J=w^2|z|^2\left[1+2J\rho+2\rho^2\frac{\partial}{\partial\rho}\ln(\tilde{C}_M^J(\rho))\right]. \tag{4.65}$$

For example, when $J=\frac{1}{2}$, we have

$$((\Delta\mathcal{H}_0)^2)_+=w^2|z|^2(1+\rho),\quad ((\Delta\mathcal{H}_0)^2)_-=w^2|z|^2\left[1+\rho\frac{(\rho-1)}{(\rho+1)}\right]. \tag{4.66}$$

Evidently, the behavior of these dispersions as functions of ρ is identical to that described in the last paragraph of the previous section.

In the general case where $b\neq 0$, $\mathcal{A}_0$ can be expressed in the form

$$\mathcal{A}_0=e^{i\theta_-}a+e^{i(\varphi_--\nu)}\left\{\left[\frac{4\beta^2|\cos\theta|-\rho e^{i\theta}}{4\beta|\cos\theta|}\right]\mathbb{J}_+-\left[\frac{4\beta^2|\cos\theta|+\rho e^{i\theta}}{4\beta|\cos\theta|}\right]\mathbb{J}_-\right\}. \tag{4.67}$$

From (3.17), we see that the eigenstates of $\mathcal{A}_0$ are

$$|Z\rangle_M^J=(C_m^j)^{-1/2}D(ze^{-i\theta_-})T_{\text{eff}}|0;J,M\rangle, \tag{4.68}$$

where

$$T_{\text{eff}}=e^{\Phi_-\mathbb{J}_+}e^{\Phi_+\mathbb{J}_-}, \tag{4.69}$$

with

$$\Phi_-=i\frac{[4\beta^2|\cos\theta|-\rho e^{i\theta}]}{R^{1/2}e^{i\bar{\varphi}/2}},\quad \Phi_+=i\frac{[4\beta^2|\cos\theta|+\rho e^{i\theta}]}{2R^{1/2}e^{i\bar{\varphi}}/2}. \tag{4.70}$$

The dispersion of $\mathcal{H}_0$ in these states is

$$((\Delta\mathcal{H}_0)^2)_M^J=w^2|z|^2[1+2\rho_M^J\langle Z|\mathbb{J}_3|Z\rangle_M^J], \tag{4.71}$$

where[17]

$${}_M^J\langle Z|\mathbb{J}_3|Z\rangle_M^J=M\left(\frac{1-|\Phi_-|^2}{1+|\Phi_-|^2}\right)+\frac{(J-M+1)}{2}\frac{P_{J+M-1}^{1,-2M+1}(\Lambda)}{P_{J+M}^{0,-2M}(\Lambda)}\tilde{\Lambda}, \tag{4.72}$$

with

$$\Lambda=1+2|\Phi_-+\bar{\Phi}_+(1+|\Phi_-|^2)|^2 \tag{4.73}$$

and

$$\tilde{\Lambda}=2[|\Phi_-|^2(1+\Phi_-\Phi_++\bar{\Phi}_-\bar{\Phi}_+)+|\Phi_+|^2(|\Phi_-|^4-1)]. \tag{4.74}$$

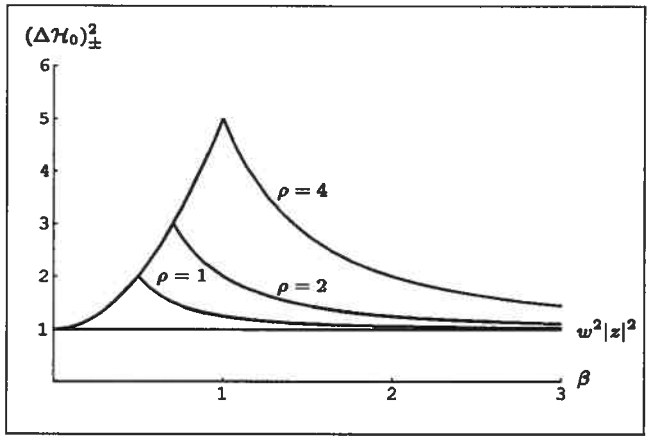

FIG. 9. Graphs of the dispersions $((\Delta\mathcal{H}_0)^2)_\pm \equiv$ (4.76) as functions of $\beta>0$, $\theta=\pi$ and $\rho=1,2,4$.

Thus, in the spin-$\frac{1}{2}$ representation, we get

$$_\pm\langle Z|\mathbb{J}_3|Z\rangle_\pm=\frac{1}{2}\left(\frac{|\Phi_-|^2-1}{1+|\Phi_-|^2}\right). \tag{4.75}$$

Finally, by direct computation, we find

$$((\Delta\mathcal{H}_0)^2)_\pm=w^2|z|^2\left[1+\rho\frac{[16\beta^4\cos^2(\theta)+\rho^2-8\rho\beta^2\cos\theta|\cos\theta|]-R}{[16\beta^4\cos^2(\theta)+\rho^2-8\rho\beta^2\cos\theta|\cos\theta|]+R}\right], \tag{4.76}$$

where

$$R=\sqrt{[16\beta^4\cos^2(\theta)-\rho^2\cos(2\theta)]^2+\rho^4\sin^2(2\theta)}. \tag{4.77}$$

We see that, for fixed value of ρ, Eq. (4.76) as a function of β is symmetric around $\theta=\pi$.

Figure 9 shows the behavior of the dispersions (4.76) as functions of $\beta>0$ when $\theta=\pi$ and for different values of parameter ρ. Let us notice the similarity between these curves starting from a certain value of β and the curves for the canonical case showed in Fig. 7.

Figure 10 shows the behavior of the same functions as functions of $\beta>0$, for different values

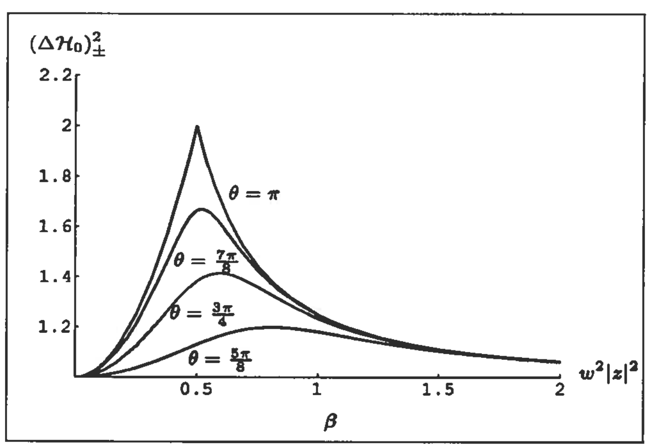

FIG. 10. Graphs of the dispersions $((\Delta\mathcal{H}_0)^2)_\pm \equiv$ (4.76) as functions of $\beta>0$ for $\rho=1$. $\theta=5\pi/8$, $3\pi/4$, $7\pi/8$ and π.

of θ when $\rho=1$. We observe that when the angle θ is different from π the curves have a continuous derivative with respect to β but, when the angle $\theta=\pi$, the derivative of the curve at the point $\beta=0.5=\sqrt{\rho}/2$ is not continuous.

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APPENDIX A: ALGEBRA EIGENSTATES ASSOCIATED TO su(2)

In this appendix we want to solve the eigenvalue equation

$$[\vec{\beta}\cdot\vec{J}]|\psi\rangle=[\beta_1 J_1+\beta_2 J_2+\beta_3 J_3]|\psi\rangle=\Gamma|\psi\rangle, \quad \beta_1,\beta_2,\beta_3\in\mathbb{C}, \tag{A1}$$

where J_1, J_2 and J_3 are the su(2) generators which have already been given in Sec. II C. The eigenvalue equation (A1) can also be written as

$$[\beta_- J_+ +\beta_+ J_- +\beta_3 J_3]|\psi\rangle=\Gamma|\psi\rangle, \tag{A2}$$

where J_1 and J_2 have been expressed in terms of the usual operators $J_\pm$ and

$$\beta_\pm=\frac{\beta_1\pm i\beta_2}{2}. \tag{A3}$$

We see that Eq. (2.45) is just a particular case of Eq. (A2). The eigenvalue equation (A2) has already been solved by Brif[8] by expanding the state $|\psi\rangle$ in the standard coherent-state basis[9], introducing in this way analytic functions and asking for solving a first order differential equation. Here, we consider a different method based on the operator algebra technique.

For j fixed, we can show that (A2) admits the eigenvalues

$$\Gamma_m^j=mb, \tag{A4}$$

with $m=-j,\ldots,j$ and $b=\sqrt{\beta_1^2+\beta 0_2^2+\beta_3^2}=\sqrt{4\beta_+\beta_- +\beta_3^2}$. We then solve

$$[\beta_- J_+ +\beta_+ J_- +\beta_3 J_3]|\psi\rangle_m^j=\Gamma_m^j|\psi\rangle_m^j, \tag{A5}$$

by using

$$|\psi\rangle_m^j=(N_m^j)^{-1/2}T|j,m\rangle, \tag{A6}$$

where the N_m^j are normalization constants and T is an operator that has to be determined. We take it as

$$T=\exp\left(-\frac{\tilde{\theta}}{2}[e^{-i\tilde{\phi}}J_+ - e^{i\tilde{\phi}}J_-]\right), \quad \tilde{\phi},\tilde{\theta}\in\mathbb{C}. \tag{A7}$$

Inserting (A6) with (A7) into (A5) leads to

$$[\vec{\beta}\cdot\vec{J}]T|j,m\rangle=mbT|j,m\rangle. \tag{A8}$$

Using the usual decomposition

$$T=\exp\left(-e^{-i\tilde{\phi}}\tan\left(\frac{\tilde{\theta}}{2}\right)J_+\right)\ \exp\left(\ln\sec^2\left(\frac{\tilde{\theta}}{2}\right)J_3\right)\ \exp\left(e^{i\tilde{\phi}}\tan\left(\frac{\tilde{\theta}}{2}\right)J_-\right) \tag{A9}$$

and the relations

$$e^{\eta J_3}J_\pm e^{-\eta J_3}=e^{\pm\eta}J_\pm,\quad e^{\eta J_\pm}J_3e^{-\eta J_\pm}=J_3\mp\eta J_\pm,\quad e^{\eta J_\pm}J_\mp e^{-\eta J_\pm}=J_\mp\pm 2\eta J_3-\eta^2 J_\pm, \tag{A10}$$

we can show that, for $\beta_+\neq 0$, $\beta_-\neq 0$ and $b\neq 0$, we have

$$e^{i\tilde{\phi}}=\sqrt{\frac{\beta_+}{\beta_-}}, \tag{A11}$$

and

$$\frac{\tilde{\theta}}{2}=\arctan\left(\sqrt{\frac{b-\beta_3}{b+\beta_3}}\right). \tag{A12}$$

Inserting the results (A11) and (A12) in (A9), we obtain

$$T=\exp\left(-\frac{2\beta_-}{b+\beta_3}J_+\right)\exp\left(\ln\left(\frac{2b}{b+\beta_3}\right)J_3\right)\exp\left(\frac{2\beta_+}{b+\beta_3}J_-\right). \tag{A13}$$

The original form (A7) of the T operator allows us to look easily for the special cases studied in Refs. 6, and 9 and in the preceding sections while the form (A13) allows us to calculate directly the explicit form of the eigenstates (A6). Indeed, the first relation (A10) allows us to pass the exponential term $\exp(\ln(2b/(b+\beta_3))J_3)$ to the right in (A13) and this without changing essentially the operator action on the pure states $|j,m\rangle$ because $|j,m\rangle$ is an eigenstate of the operator J_3. Thus, in Eq. (A6), we can replace the operator T by the operator

$$T_{\text{eff}}=\left(\frac{b}{\beta_+}\right)^{j+m}\sqrt{\frac{(j+m)!(j-m)!}{(2j)!}}\exp\left(-\frac{2\beta_-}{b+\beta_3}J_+\right)\exp\left(\frac{\beta_+}{b}J_-\right), \tag{A14}$$

such that

$$|\psi\rangle_m^j=(\tilde{N}_m^j)^{-1/2}T_{\text{eff}}|j,m\rangle, \tag{A15}$$

where $\tilde{N}_m^j$ are new normalization constants. Redefining the summation indices, we get

$$|\psi\rangle_m^j=(\tilde{N}_m^j)^{-1/2}\sum_{u=-j}^{j}\sqrt{\frac{(j+u)!(j-u)!}{(2j)!}}\left(\frac{b}{\beta_+}\right)^{j+u}\frac{(j+m)!}{(j-u)!}$$
$$\times\sum_{n=0}^{j+u}(-1)^n\frac{(j-u+n)!}{n!(m-u+n)!(j+u-n)!}\left(\frac{(1-\beta_3/b)}{2}\right)^n|j,u\rangle. \tag{A16}$$

We also have an expression in terms of the Jacobi polynomials (see Ref. 18):

$$|\psi\rangle_m^j=(\tilde{N}_m^j)^{-1/2}\sum_{u=-j}^{j}\sqrt{\frac{(j+u)!(j-u)!}{(2j)!}}\left(\frac{b}{\beta_+}\right)^{j+u}P_{j+u}^{-u+m,-u-m}\left(\frac{\beta_3}{b}\right)|j,u\rangle, \tag{A17}$$

which is the result obtained by Brif.[8]

For the special case where $\beta_+=0$, $\beta_3\neq 0$ so that, in connection with (A4), we have $b=\beta_3$, we find the operator

$$T_{\text{eff}}=\exp\left(-\frac{\beta_-}{\beta_3}J_+\right). \tag{A18}$$

The eigenstates are

$$|\psi\rangle_m^j=(C_m^j)^{-1/2}\sum_{u=m}^{j}\sqrt{\frac{(j+u)!}{(j-u)!}}\frac{1}{(u-m)!}\left(-\frac{\beta_-}{\beta_3}\right)^{u-m}|j,u\rangle, \tag{A19}$$

and become the standard CS of SU(2) (Ref. 9) when $m=-j$.

For the special case where $\beta_-=0$, $\beta_3\neq 0$, we have similar results. Indeed, the new operator T_{eff} is

$$T_{\text{eff}}=\exp\left(\frac{\beta_+}{\beta_3}J_-\right) \tag{A20}$$

and the eigenstates write

$$|\psi\rangle_m^j=(C_m^j)^{-1/2}\sum_{u=-j}^{m}\sqrt{\frac{(j-u)!}{(j+u)!}}\frac{1}{(m-u)!}\left(\frac{\beta_+}{\beta_3}\right)^{u-m}|j,u\rangle, \tag{A21}$$

which become the standard CS of SU(2) (Ref. 9) when $m=j$.

Now, for the case $\beta_+=0$ and $\beta_3=0$ ($\beta_-=0$ and $\beta_3=0$), the only normalizable solution is $|j,-j\rangle$ ($|j,j\rangle$). For $\beta_+=\beta_-=0$ and $\beta_3\neq 0$, the AES are evidently the pure states $|j,m\rangle$.

Finally, the degenerate case $b=0$ leads to the solution $|\psi\rangle_{-j}^j=(C_{-j}^j)^{-1/2}T_{\text{eff}}|j,-j\rangle$ with $T_{\text{eff}}=\exp(-2(\beta_-/\beta_3)J_+)$, that is the standard CS of SU(2).

The mean value of J_3 in the states (A17) has already been calculated by Brif.[8] We have

$$\langle J_3\rangle_m^j=\frac{jY+m(S_+-S_-)}{S_+S_-}-\frac{(j+|m|)Yt}{S_+^2S_-^2}\Omega, \tag{A22}$$

where

$$S_\pm=1+\left|\frac{2\beta_-}{\beta_3\mp b}\right|^2,\quad t=\left|\frac{b}{\beta_+}\right|^2,\quad Y=S_+S_--S_+-S_- \tag{A23}$$

and

$$\Omega=\frac{P_{j-|m|-1}^{(-2j,1)}(1-(2t/S_+S_-))}{P_{j-|m|}^{(-2j-1,0)}(1-(2t/S_+S_-))},\quad \text{if } |m|<j;\quad \Omega=0,\quad \text{if } |m|=j. \tag{A24}$$

APPENDIX B: RESOLUTION OF A FIRST ORDER SYSTEM OF DIFFERENTIAL EQUATIONS

Let us recall that a realization[9] of the Fock space $\mathcal{F}_b=\{|n\rangle,n=0,1,2,\ldots\}$ of energy eigenstates of the harmonic oscillator as a space $\mathcal{H}$ of analytic functions $f(\zeta)$ is obtained by expanding this function in the basis of analytic functions $\{\varphi_n(\zeta)=\zeta^n/\sqrt{n!},n=0,1,2,\ldots\}$, that is,

$$f(\zeta)=\sum_{n=0}^{\infty}c_n\varphi_n(\zeta)=\sum_{n=0}^{\infty}c_n\frac{\zeta^n}{\sqrt{n!}},\quad \zeta\in\mathbb{C}. \tag{B1}$$

The scalar product is

$$(f_1,f_2)=\int_C\bar{f}_1(\zeta)f_2(\zeta)e^{-|\zeta|^2}\frac{d\zeta\, d\bar{\zeta}}{2\pi i},\quad \forall f_1,f_2\in\mathcal{H}, \tag{B2}$$

the integral being extended to the complex plane. The action of the creation $a^\dagger$ and annihilation a operators on the $\mathcal{H}$ space is then given by

$$a^\dagger \equiv \zeta, \quad a \equiv \frac{d}{d\zeta}. \tag{B3}$$

The eigenvalue equation (2.26) thus becomes a first order differential equation

$$\frac{1}{\sqrt{2}}\left((1+\lambda)\frac{d}{d\zeta}+(1-\lambda)\zeta\right)f(\zeta)=\beta f(\zeta), \tag{B4}$$

for which normalized solutions are obtained for $\lambda \neq -1$. The general solution of (B4) is

$$f(\zeta)=f(0)\exp\left(\frac{2\sqrt{2}\beta\zeta-(1-\lambda)\zeta^2}{2(1+\lambda)}\right). \tag{B5}$$

With respect to the scalar product (B2), the normalization constant $f(0)$ is computed by imposing

$$\int_C |f(\zeta)|^2 e^{-|\zeta|^2}\frac{d\zeta\, d\bar{\zeta}}{2\pi i}=1, \tag{B6}$$

and we find the normalized solution of (B4) as

$$f(\zeta)=(1-|\eta_1|^2)^{1/4}\exp\left(-\frac{1}{2}\left[\frac{|\eta_2|^2-\mathrm{Re}(\bar{\eta}_1\eta_2^2)}{1-|\eta_1|^2}\right]\right)\exp\left(\eta_2\zeta-\frac{\eta_1}{2}\zeta^2\right), \tag{B7}$$

with

$$\eta_1=\frac{(1-\lambda)}{(1+\lambda)}=\delta e^{i\phi} \quad \text{and} \quad \eta_2=\frac{\sqrt{2}\beta}{(1+\lambda)}=\frac{\beta}{\sqrt{2}}(1+\delta e^{i\phi}). \tag{B8}$$

This corresponds to the states (2.33) after normalization.

Now we are concerned with the algebra eigenstates satisfying the equation (3.10) in the Fock space $\mathcal{F}\equiv$ (3.4). A realization of $\mathcal{F}$ can be easily given from the preceding considerations and the expression (3.12) of a state $|\psi\rangle$ for a fixed j. Indeed, we have

$$\psi_m^j(\zeta)=\langle\zeta;j,m|\psi\rangle \tag{B9}$$

and the eigenvalue equation (3.10) then becomes a system of first order differential equations

$$\begin{aligned}\left(\alpha_-\frac{d}{d\zeta}+\alpha_+\zeta+\alpha_3\right)\psi_m^j(\zeta)+[\beta_-\sqrt{(j-m+1)(j+m)}\psi_{m-1}^j(\zeta)\\ +\beta_+\sqrt{(j+m+1)(j-m)}\psi_{m+1}^j(\zeta)+\beta_3 m\psi_m^j(\zeta)]=\beta\psi_m^j(\zeta),\end{aligned} \tag{B10}$$

where j is fixed but m takes the values $-j,\ldots,j$. Let us now solve this system by first introducing the differential operator

$$L=\alpha_-\frac{d}{d\zeta}+\alpha_+\zeta+\alpha_3-\beta \tag{B11}$$

and, second, defining the vector

$$\Psi=\begin{pmatrix}\psi^j_{-j}\\ \psi^j_{-j+1}\\ \vdots\\ \psi^j_{j-1}\\ \psi^j_j\end{pmatrix}. \tag{B12}$$

The system (B10) thus becomes a matrix differential system

$$L\Psi=-A\Psi, \tag{B13}$$

with A a $(2j+1)\times(2j+1)$ matrix given by

$$A=\begin{pmatrix} -j\beta_3 & \sqrt{2j}\beta_+ & 0 & 0 & \cdots & 0\\ \sqrt{2j}\beta_- & (-j+1)\beta_3 & \sqrt{(2j-1)2}\beta_+ & 0 & \cdots & 0\\ 0 & \sqrt{(2j-1)2}\beta_- & (-j+2)\beta_3 & \sqrt{(2j-2)3}\beta_+ & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots\\ 0 & 0 & \sqrt{(2j-2)3}\beta_- & (j-2)\beta_3 & \sqrt{(2j-1)2}\beta_+ & 0\\ 0 & 0 & 0 & \sqrt{(2j-1)2}\beta_- & (j-1)\beta_3 & \sqrt{2j}\beta_+\\ 0 & 0 & 0 & 0 & \sqrt{2j}\beta_- & j\beta_3 \end{pmatrix}. \tag{B14}$$

If we can find a nonsingular matrix S that diagonalizes A on the form $D=S^{-1}AS$ where

$$D=\mathrm{diag}(\lambda^j_{-j},\lambda^j_{-j+1},\cdots,\lambda^j_j), \tag{B15}$$

the system (B13) will reduce to

$$L\tilde{\Psi}=-D\tilde{\Psi},\quad \tilde{\Psi}=S^{-1}\Psi. \tag{B16}$$

Thus, for $\alpha_-\neq 0$, the direct integration of (B16) will lead to

$$\tilde{\psi}^j_m=\tilde{\psi}^j_m(0)\exp\left(\frac{\beta-\alpha_3-\lambda^j_m}{\alpha_-}\zeta-\frac{\alpha_+}{2\alpha_-}\zeta^2\right) \tag{B17}$$

and the general solution Ψ will be obtained as

$$\begin{pmatrix}\psi^j_{-j}\\ \psi^j_{-j+1}\\ \vdots\\ \psi^j_{j-1}\\ \psi^j_j\end{pmatrix}=S\begin{pmatrix}\tilde{\psi}^j_{-j}\\ \tilde{\psi}^j_{-j+1}\\ \vdots\\ \tilde{\psi}^j_{j-1}\\ \tilde{\psi}^j_j\end{pmatrix}=\sum_{m=-j}^{j}\tilde{\psi}^j_m(0)\exp\left(\frac{\beta-\alpha_3-\lambda^j_m}{\alpha_-}\zeta-\frac{\alpha_+}{2\alpha_-}\zeta^2\right)\begin{pmatrix}S_{-j,m}\\ S_{-j+1,m}\\ \vdots\\ S_{j-1,m}\\ S_{j,m}\end{pmatrix}, \tag{B18}$$

where S is assumed to be of the form

$$S=\begin{pmatrix} S_{-j,-j} & S_{-j,-j+1} & \cdots & S_{-j,j-1} & S_{-j,j} \\ S_{-j+1,-j} & S_{-j+1,-j+1} & \cdots & S_{-j+1,j-1} & S_{-j+1,j} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ S_{j-1,-j} & S_{j-1,-j+1} & \cdots & S_{j-1,j-1} & S_{j-1,j} \\ S_{j,-j} & S_{j,-j+1} & \cdots & S_{j,j-1} & S_{j,j} \end{pmatrix}. \tag{B19}$$

Computing the eigenvalues of A, we find that we have to distinguish two cases, i.e., the one with $b=\sqrt{4\beta_+\beta_-+\beta_3^2}\neq 0$ and the one with $b=0$. For the first case $b\neq 0$, all eigenvalues are different and given by

$$\lambda_m^j=mb, \quad m=-j,\cdots,j. \tag{B20}$$

The system is diagonalizable and the general solution is given by (B18) with

$$S_{u,m}=\sqrt{\frac{(j+u)!(j-u)!}{(2j)!}}\left(\frac{b}{\beta_+}\right)^{j+u} P_{j+u}^{-u+m,-u-m}\left(\frac{\beta_3}{b}\right), \quad u=-j,\cdots,j, \tag{B21}$$

when $\beta_-\neq 0$, $\beta_+\neq 0$ and $\beta_3\neq 0$,

$$S_{u,m}=\sqrt{\frac{(j-u)!}{(j+u)!}}\frac{1}{(m-u)!}\left(\frac{\beta_+}{\beta_3}\right)^{u-m}, \quad -j\leqslant u\leqslant m, \quad S_{u,m}=0, \quad m<u\leqslant j, \tag{B22}$$

when $\beta_-=0$, $\beta_+\neq 0$ and $\beta_3\neq 0$ and

$$S_{u,m}=\sqrt{\frac{(j+u)!}{(j-u)!}}\frac{1}{(u-m)!}\left(-\frac{\beta_-}{\beta_3}\right)^{u-m}, \quad m\leqslant u\leqslant j, \quad S_{u,m}=0, \quad -j\leqslant u<m, \tag{B23}$$

when $\beta_-\neq 0$, $\beta_+=0$ and $\beta_3\neq 0$.

In the Fock space representation, the solutions (B18) with (B21), (B22) and (B23) correspond, apart from a superfluous change of notation, exactly to the states (3.17) with $T_{\rm eff}$ given by (A14), (A18), and (A20), respectively.

For the second case $b=0$, the matrix A can not be diagonalized. We could use the Jordan form or start from the differential equation system again and include this condition. Taking the second way, we can express the $\psi_m^j(\zeta)$ components in the form

$$\psi_m^j(\zeta)=\exp\left[-\frac{\alpha_+}{2\alpha_-}\zeta^2+\frac{(\beta-\alpha_3-m\beta_3)}{\alpha_-}\zeta\right]\tilde{\psi}_m^j(\zeta), \tag{B24}$$

and insert these in Eq. (B10). We get to the following system:

$$\alpha_-\frac{d}{d\zeta}\tilde{\psi}_m^j(\zeta)+\beta_-\sqrt{(j-m+1)(j+m)}e^{\beta_3\zeta/\alpha_-}\tilde{\psi}_{m-1}^j(\zeta)$$

$$+\beta_+\sqrt{(j+m+1)(j-m)}e^{-\beta_3\zeta/\alpha_-}\tilde{\psi}_{m+1}^j(\zeta)=0, \tag{B25}$$

when $m=-j,\ldots,j$. By handling these equations suitably we can, for example, obtain an ordinary differential equation of the $2j+1$ order for $\tilde{\psi}_{-j}^j(\zeta)$, namely

$$\left[\prod_{-j}^{j}\left(\frac{d}{d\zeta}-\mu_m^j\right)\right]\tilde{\psi}_{-j}^j(\zeta)=0, \tag{B26}$$

where

$$\mu_m^j = -j\frac{\beta_3}{\alpha_-} + m\frac{b}{\alpha_-}. \tag{B27}$$

When $b=0$, we have $2j+1$ equal roots. This means that the solutions for $\tilde{\psi}_{-j}^j(\zeta)$ take the form:

$$\tilde{\psi}_{-j}^j(\zeta) = \exp\left(\frac{-j\beta_3\zeta}{\alpha_-}\right)\sum_{q=0}^{2j} A_q \zeta^q. \tag{B28}$$

Then, we can insert (B28) in (B25) and thus obtain, in an iterative way, all solutions $\tilde{\psi}_m^j(\zeta)$ and, thereafter, using (B24), all solutions $\psi_m^j(\zeta)$.

For example, in the case $\beta_+ = \beta_3 = 0$ and $\beta_- \neq 0$, we have

$$\tilde{\psi}_{-j}^j(\zeta) = \psi_{-j}^j(0), \tag{B29}$$

i.e., a constant and, consequently, by integrating one by one the equations of the system (B25), we obtain

$$\tilde{\psi}_m^j(\zeta) = \sum_{k=0}^{j+m}\left(-\frac{\beta_-}{\alpha_-}\right)^k \frac{\zeta^k}{k!}\sqrt{\frac{(j+m)!(j-m+k)!}{(j-m)!(j+m-k)!}}\,\psi_{m-k}^j(0), \tag{B30}$$

when $m=-j,\ldots,j$. The general solution (B12) is then given by

$$\Psi = \exp\left[-\frac{\alpha_+}{2\alpha_-}\zeta^2 + \frac{(\beta-\alpha_3)}{\alpha_-}\zeta\right]\sum_{m=-j}^{j}\psi_m^j(0)$$
$$\times\left[\sum_{k=0}^{j-m}\frac{(-1)^k}{k!}\sqrt{\frac{(j-m)!(j+m+k)!}{(j-m-k)!}}\left(\frac{\beta_-}{\alpha_-}\right)^k \zeta^k \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right], \tag{B31}$$

where, in each sum, the 1 in the vector column is placed in the $(j+m+k+1)$ row. We thus obtain the $(2j+1)$ independent solutions of the system of differential equations.

In the Fock space representation, we can show that the independent solutions given by Eq. (B31) correspond, apart from a superfluous change of notation, to the states (3.18). In the case $\beta_- = \beta_3 = 0$ with $\beta_+ \neq 0$, following a similar procedure, one finds the expression (3.19).

Finally, when β_+, β_-, $\beta_3 \neq 0$, by inserting (B28) in (B25) and ordering the independent solutions with respect to the arbitrary constants A_q, one finds

$$\Psi(\zeta) = \exp\left[-\frac{\alpha_+}{2\alpha_-}\zeta^2 + \frac{(\beta-\alpha_3)}{\alpha_-}\zeta\right]\sum_{q=0}^{2j} A_q$$
$$\times\left[\sum_{k=0}^{q}(-1)^k\binom{q}{k}\frac{(2j-k)!}{(2j)!}\zeta^{q-k}\left(\frac{\alpha_-}{\beta_+}\right)^k\left[\frac{d^k}{d\vartheta^k}\sum_{r=0}^{2j}\sqrt{\frac{(2j)!}{(2j-r)!r!}}\vartheta^r\right]\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}\right], \tag{B32}$$

where $\vartheta=\beta_3/2\beta_+=-2\beta_-/\beta_3$ and, in each sum, the 1 in the vector column is placed in the $r+1$ row. In the Fock space representation, these solutions, with a slight change of notation, correspond to Eq. (3.20).

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Chapitre 4

États propres de la superalgèbre $sh(2/2)$, états supercohérents et supercomprimés généralisés

Résumé

Le concept d'états propres de superalgèbre est introduit et appliqué pour trouver les états propres associés à la superalgèbre $sh(2/2)$, aussi connue comme la superalgèbre de Lie de Heisenberg–Weyl. Ceci implique de résoudre une super-équation aux valeurs propres à valeurs dans une algèbre de Grassmann. Ainsi, les états propres de $sh(2/2)$ contiennent une classe d'états supercohérents associés à l'oscillateur harmonique supersymétrique ainsi qu'une classe d'états supercomprimés associés à la superalgèbre $osp(2/2) \supset\!\!\!\!\!+ \; sh(2/2)$. La superalgèbre de Lie orthosymplectique $osp(2/2)$ est engendrée par l'ensemble d'opérateurs formés à partir des produits quadratiques des générateurs de la superalgèbre de Heisenberg–Weyl. Les propriétés de tous ces états sont étudiées et comparées avec celles des états obtenus en appliquant les techniques de la théorie des groupes. De plus, de nouvelles classes d'états supercohérents et supercomprimés généralisés sont obtenues. Cela permet de construire des Hamiltoniens super–Hermitiens et η–pseudo–super–Hermitiens sans parité de Grassmann définie et isospectraux avec l'oscillateur harmonique. Les états propres et états supercohérents associés sont calculés.

$sh(2/2)$ superalgebra eigenstates and generalized supercoherent and supersqueezed states

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Abstract

The superalgebra eigenstates (SAES) concept is introduced and then applied to find the SAES associated to the $sh(2/2)$ superalgebra, also known as Heisenberg–Weyl Lie superalgebra. This implies to solve a Grassmannian eigenvalue superequation. Thus, the $sh(2/2)$ SAES contain the class of supercoherent states associated to the supersymmetric harmonic oscillator and also a class of supersqueezed states associated to the $osp(2/2) \niplus sh(2/2)$ superalgebra, where $osp(2/2)$ denotes the orthosymplectic Lie superalgebra generated by the set of operators formed from the quadratic products of the Heisenberg–Weyl Lie superalgebra generators. The properties of these states are investigated and compared with those of the states obtained by applying the group-theoretical technics. Moreover, new classes of generalized supercoherent and supersqueezed states are also obtained. As an application, the superHermitian and η–pseudo–superHermitian Hamiltonians without a defined Grassmann parity and isospectral to the harmonic oscillator are constructed. Their eigenstates and associated supercoherent states are calculated.

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1 Introduction

The algebra eigenstates (AES) associated to a real Lie algebra have been defined as the set of eigenstates of an arbitrary complex linear combination of the generators of the considered algebra[7, 8]. According to the particular realization of the Lie algebra generators, the determination of the AES implies, for instance, to solve an ordinary or a partial differential equation, to apply the operator technics, etc. For example, in the case of the $su(2)$ Lie algebra, different approaches have been used such as the constellation formalism[4], the ordinary first order differential equations[8] or the operator method[1]. The same methods have also been applied to find the AES for the $su(1,1)$ Lie algebra[1, 8]. In the case of the two-photon AES, associated to the $su(1,1) \oplus h(2)$ Lie algebra, used have been done of ordinary second order differential equation[7]. More recently, the AES associated to the $h(2) \oplus su(2)$ Lie algebra have been obtained using these types of methods[2]. In particular[8] it has been demonstrated that the generalized coherent states (GCS) associated to the $SU(2)$ and $SU(1,1)$ Lie groups, based on group-theoretical approach[21], are subsets of the sets of AES associated to their corresponding Lie algebras. Moreover, the super coherent states of the supersymmetric harmonic oscillator[10] as defined by Aragone and Zypmann[3] and a new class of supercoherent and supersqueezed states regarded as minimum uncertainty states have been obtained[2]. Generalized supercoherent states (GSCS) associated to Lie supergroups have also been calculated following a generalized group-theoretical approach. This is the case, for example, of the supercoherent states associated to the following supergroups: Heisenberg–Weyl (H–W) and $OSp(1/2)$[14], $U(1/2)$[15, 25], $U(1/1)$[20] and $OSp(2/2)$ [13].

In the view of these approaches we ask the question of how we can generalize the AES concept valid for Lie algebras to Lie superalgebras. In general, as the even subspace of a Lie superalgebra is an ordinary Lie algebra, it is clair that the new concept must generalize in an appropriate form the AES concept. Indeed, the set of superalgebra eigenstates (SAES) associated to linear combinations of even generators of the Lie superalgebra must contain the AES associated to the Lie algebra generated by these generators. Moreover, we expect that the SAES associated to a certain class of superalgebras contain the GSCS of the related Lie supergroups. Another criterion to define the SAES concept start from the utility that we can give to this concept when we study a particular quantum system, more precisely when we want to know the eigenstates of a physical observable represented by a superHermitian operator formed by a linear combination of the superalgebra generators or by a suitable product of these generators. According with these requirements, we propose the following definition of the SAES concept.

Definition 1.1 *The SAES associated to a Lie superalgebra correspond to the set of eigenstates of an arbitrary linear combination, with coefficients in the Grassmann algebra $\mathbb{C}B_L$, of the superalgebra generators. This means that if $\mathcal{L}$ is a superalgebra generated by the set of even operators $\Phi(a_1), \Phi(a_2), \ldots \Phi(a_m)$ and the set of odd operators $\Phi(a_{m+1}), \Phi(a_{m+2}), \ldots, \Phi(a_{m+n})$, the SAES associated to $\mathcal{L}$ are determined by the eigenvalue equation*

$$\left[\sum_{i=1}^{m+n} B^i \Phi(a_i)\right] |\psi\rangle = Z|\psi\rangle, \tag{1}$$

where $B^i \in \mathbb{C}B_L$, $\forall i = 1, 2, \ldots, m+n$ and $Z \in \mathbb{C}B_L$.

In general, the superstate $|\psi\rangle$ is a linear combination, with coefficients in $\mathbb{C}B_L$, of the basis vectors of a graded superHilbert space $\mathcal{W}$, the representation space of the superalgebra on which it acts.

Let us here mention that the Appendix A contains the notations and conventions used in the context of Grassmann algebras, Lie superalgebras and supergroups. This will help for a good understanding of this work.

From the preceding definition, we see that to know explicitly the SAES associated to a given Lie superalgebra, we must analyze case by case the different possible solutions of the Grassmannian eigenvalue equation (1) taking into account both the domain of definition of the Grassmann coefficients and the parity of them. In general, the calculations can be long and fastidious, but in physical applications, some simplifications appear due to some constrains on the coefficients like assuming a certain type of parity.

A natural generalization of the concept of AES to SAES starts with H–W superalgebra $sh(2/2)$ generated by the bosonic operators $a, a^\dagger$ and I and the fermionic ones b and $b^\dagger$. We expect to recover the usual algebra eigenstates[2, 3, 19] but also supercoherent and supersqueezed states based on a group theoretical approach[16, 18].

Let us remind that the well-known bosonic algebra is generated by the even operators $a, a^\dagger$ and I, that satisfy the usual non-zero commutation relation

$$[a, a^\dagger] = I, \tag{2}$$

and act on the usual Fock space $\mathcal{F}_b = \{|n\rangle, \ n \in \mathbb{N}\}$, as follows

$$a|n\rangle = \sqrt{n}|n-1\rangle, \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad n \in \mathbb{N}. \tag{3}$$

The operators $a, a^\dagger$ are the usual annihilation and creation operators of the harmonic oscillator, and I acts as the identity operator. The corresponding fermionic superalgebra is generated by the odd operators $b, b^\dagger$ and the even operator I, which satisfy the non-zero super commutation relation

$$\{b, b^\dagger\} = I. \tag{4}$$

These operators act on the graded space $\mathcal{F}_f = \{|+\rangle, |-\rangle\}$ as follows

$$b|+\rangle = |-\rangle, \quad b|-\rangle = 0, \quad b^\dagger|+\rangle = 0, \quad b^\dagger|-\rangle = |+\rangle. \tag{5}$$

Taking the all set $\{a, a^\dagger, I, b, b^\dagger\}$ satisfying the non-zero supercommutation relations (2) and (4), we get the H–W superalgebra $sh(2/2)$. Its acts naturally on the graded Fock space $\mathcal{F}_b \otimes \mathcal{F}_f = \{|n, \pm\rangle, \ n \in \mathbb{N}\}$. In order to compute the SAES of this superalgebra we will consider linear combinations over the field of Grassmann numbers. This means that, in general, we will deal with linear combinations of the bosonic (even) and fermionic (odd) operators with the coefficients taking values in the set $\mathbb{C}B_L$.

The paper will be thus distributed as follows. In section 2, we will determine the SAES associated to the bosonic H–W Lie algebra. A significant difference with respect to the other approaches is now that linear combinations of generators is considered over the field of Grassmann numbers. Connections with preceding approaches will be made. In section 3, fermionic H–W Lie superalgebra will be considered. These special SAES cases will give a good understanding of the specificities induced by working with Grassmann valued variables and will help us to give a complete description of the SAES associated to the H–W Lie superalgebra in section 4. Finally, in section 5, Hamiltonians which are isospectral to the harmonic oscillator one will be constructed and their associated supercoherent states will be described. The notations and conventions used in this work will be revised in the Appendix A whereas the details of calculus of the SAES of section 4 will be presented in the Appendix B.

2 SAES associated to the Heisenberg–Weyl Lie algebra, generalized supercoherent and supersqueezed states

The SAES associated to the H–W Lie algebra will be obtained as the states $|\psi\rangle$ that verify the eigenvalue equation

$$[A_- a + A_+ a^\dagger + A_3 I]|\psi\rangle = Z|\psi\rangle, \tag{6}$$

where $A_\pm, A_3$ and $Z \in \mathbb{C}B_L$. From the structure of this equation, we expect to recover the usual results concerning, in particular, the eigenstates of a, i.e., the standard coherent states of the harmonic oscillator[21]. That is the reason why we begin our considerations by taking first $A_+ = A_3 = 0$. In this context, we will distinguish between the cases where $(A_-)_\phi$ is zero and not zero. Next, the general combination (6) will be considered with $(A_-)_\phi \neq 0$. This means that A_- is an invertible Grassmann number and the relation (6) thus reduces to

$$[a + \beta a^\dagger]|\psi\rangle = z|\psi\rangle, \qquad \beta,\ z \in \mathbb{C}B_L. \tag{7}$$

2.1 Generalized coherent states

If we take $A_+ = A_3 = 0$, the eigenvalue equation (6) thus writes

$$A_- a|\psi\rangle = Z|\psi\rangle. \tag{8}$$

Let us assume a solution of the type

$$|\psi\rangle = \sum_{n=0}^{\infty} C_n|n\rangle, \qquad C_n \in \mathbb{C}B_L. \tag{9}$$

By inserting (9) in (8), applying (3) and using the orthogonality property of states $\{|n\rangle\}_{n=0}^{\infty}$, we get to the following recurrence relation

$$A_- C_{n+1} = \frac{Z C_n}{\sqrt{n+1}}, \qquad n = 0, 1, \ldots. \tag{10}$$

Here we must consider two cases: the cases $(A_-)_\phi \neq 0$ and $(A_-)_\phi = 0$.

In the first case, $(A_-)_\phi \neq 0$ is thus an invertible quantity and we can isolate the coefficient C_{n+1} in (10). It is easy to show that we get:

$$C_n = \frac{\left((A_-)^{-1} Z\right)^n}{\sqrt{n!}} C_0, \qquad n = 1, 2, \ldots. \tag{11}$$

The SAES associated to the operator A_-a with eigenvalue Z are then given by

$$|z\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} C_0 |n\rangle = \sum_{n=0}^{\infty} \frac{(za^\dagger)^n}{n!} C_0 |0\rangle = e^{za^\dagger} C_0 |0\rangle, \tag{12}$$

where $z = (A_-)^{-1} Z$. As we are interested in normalized eigenstates, we take $(C_0)_\phi \neq 0$ and the eigenstates can be written as

$$|z\rangle = \mathbb{D}(z_0)\mathbb{D}(z_1)|0\rangle, \tag{13}$$

where

$$\mathbb{D}(z_0) = \exp\left(z_0 a^\dagger - z_0^\ddagger a\right), \qquad \mathbb{D}(z_1) = \exp\left(z_1 a^\dagger - z_1^\ddagger a\right), \tag{14}$$

$z_0 = \left((A_-)^{-1} Z\right)_0$ and $z_1 = \left((A_-)^{-1} Z\right)_1$.

We notice that the generalized coherent states associated to the harmonic oscillator system, considered as eigenstates of the annihilation operator a, are here given by (13) when $A_- = \epsilon_\phi$, i.e., when $z_0 = Z_0$ and $z_1 = Z_1$. This states are obtained by applying successively the superunitary operators $\mathbb{D}(Z_1)$ and $\mathbb{D}(Z_0)$ to the fundamental state $|0\rangle$.

In the second case, that is when $(A_-)_\phi = 0$, we can not obtain a simple closed expression to describe all the algebra eigenstates. A class of solution is:

$$C_n = \frac{\left(C_1 (C_0)^{-1}\right)^n}{\sqrt{n!}} C_0, \qquad n = 2, 3, \ldots, \tag{15}$$

together with

$$A_- C_1 = Z C_0 \tag{16}$$

and C_0 is an arbitrary coefficient such that $(C_0)_\phi \neq 0$. The condition (16) implies that $Z_\phi = 0$ and is equivalent to the following system of superequations

$$(A_-)_0 (C_1)_0 + (A_-)_1 (C_1)_1 = Z_0 (C_0)_0 + Z_1 (C_0)_1, \tag{17}$$

$$(A_-)_0 (C_1)_1 + (A_-)_1 (C_1)_0 = Z_0 (C_0)_1 + Z_1 (C_0)_0, \tag{18}$$

where we have decomposed A_-, C_0, C_1 and Z into their even and odd parts. This system can be solved to give C_1 in terms of C_0. A set of normalized eigenstates corresponding to the eigenvalue $Z = \alpha A_-$, $\alpha \in \mathbb{C}$, is given by the standard coherent states

$$|\alpha\epsilon_\phi\rangle = \exp\left(\alpha\epsilon_\phi a^\dagger - \bar{\alpha}\epsilon_\phi a\right)|0\rangle = \mathbb{D}(\alpha\epsilon_\phi)|0\rangle. \tag{19}$$

So, in the special case when $(A_-)_0 = 0$, the algebra eigenstates of the odd operator $(A_-)_1 a$ contain the set of coherent states of the standard harmonic oscillator.

2.1.1 Density of algebra

It is interesting to mention that we can interpret this last result in terms of the concept of density of algebra. Indeed, let us define the odd operators

$$\mathbb{A}_- = z_1^{\ddagger} a, \qquad \mathbb{A}_+ = -z_1 a, \quad z_1 \in \mathbb{C}B_{L_1}. \tag{20}$$

By integrating these operators with respect to the corresponding odd variable, we get

$$a = \int \mathbb{A}_- dz_1^{\ddagger}, \qquad a^{\dagger} = \int dz_1 \mathbb{A}_+, \tag{21}$$

i.e., $\mathbb{A}_-$ and $\mathbb{A}_+$ fulfill the role of a linear density of the annihilation a and the creation $a^{\dagger}$, respectively. We notice that

$$[a, a^{\dagger}] = \int \{\mathbb{A}_-, \mathbb{A}_+\} dz_1^{\ddagger} dz_1, \qquad \{a, a^{\dagger}\} = \int [\mathbb{A}_-, \mathbb{A}_+] dz_1^{\ddagger} dz_1, \tag{22}$$

i.e., the commutator and anticommutator of the even operators a and $a^{\dagger}$ are obtained by integrating, on the entire odd Grassmann space, the anticommutator and commutator of the odd operators $\mathbb{A}_-$ and $\mathbb{A}_+$, respectively. This suggests the following definitions of the densiy of identity $\mathbb{I}$ and of an energy type density $\mathbb{H}$:

$$\mathbb{I} = \{\mathbb{A}_-, \mathbb{A}_+\} = z_1 z_1^{\ddagger}, \qquad \mathbb{H} = [\mathbb{A}_-, \mathbb{A}_+] = \frac{w}{2} z_1 z_1^{\ddagger} \{a, a^{\dagger}\}. \tag{23}$$

As we know, the eigenstates of the annihilation operator correponding to the complex eigenvalue α are given by the standard harmonic oscillator coherent states $|\alpha\rangle = D(\alpha)|0\rangle$. They verify the eigenvalue equation

$$a|\alpha\rangle = \alpha|\alpha\rangle. \tag{24}$$

Multipliying both sides of this equation by $z_1^{\ddagger}$, then integrating with respect to this Grassmann variable and finally using (21), we get

$$\int \mathbb{A}_- |\alpha\rangle dz_1^{\ddagger} = \int \alpha z_1^{\ddagger} |\alpha\rangle dz_1^{\ddagger}, \tag{25}$$

i.e., by comparing both sides of this last equation we conclude that a class of eigenstates of the odd operator $\mathbb{A}_-$ corresponding to the $\alpha z_1^{\ddagger}$ eigenvalue are given by the standard harmonic oscillator coherent states $\epsilon_{\phi}|\alpha\rangle$.

2.2 Generalized supersqueezed states

Let us now solve the eigenvalue equation (7). A class of solutions can be constructed firstly, by expressing $|\psi\rangle$ in terms of a generalized $su(1,1)$ squeeze operator (the normaliser of the H–W algebra), following this way the construction of the standard squeezed states associated to the simple harmonic oscillator system[19]. Indeed, let us write

$$|\psi\rangle = S(\mathcal{X}_0)|\varphi\rangle, \tag{26}$$

where the squeeze operator $S(\mathcal{X}_0)$ is given by

$$S(\mathcal{X}_0) = \exp\left(\mathcal{X}_0 \frac{(a^\dagger)^2}{2} - \mathcal{X}_0{}^\ddagger \frac{a^2}{2}\right), \tag{27}$$

with $\mathcal{X}_0$ an even invertible Grassmann number, $\mathcal{X}_0^\ddagger$ its adjoint (see Appendix A).

Inserting (26) in (7), using the relation

$$S^\ddagger(\mathcal{X}_0) a S(\mathcal{X}_0) = \cosh(\|\mathcal{X}_0\|)\, a + \sqrt{\mathcal{X}_0}\left(\sqrt{\mathcal{X}_0^\ddagger}\right)^{-1} \sinh(\|\mathcal{X}_0\|)\, a^\dagger, \tag{28}$$

where $\|\mathcal{X}_0\| = \sqrt{\mathcal{X}_0 \mathcal{X}_0^\ddagger}$, and choosing $\mathcal{X}_0$ in such a way that it satisfies

$$\sqrt{\mathcal{X}_0}\left(\sqrt{\mathcal{X}_0^\ddagger}\right)^{-1} \sinh(\|\mathcal{X}_0\|) + \beta_0 \cosh(\|\mathcal{X}_0\|) = 0, \tag{29}$$

we get the following eigenvalue equation for $|\varphi\rangle$:

$$\left[\mathcal{G}(\mathcal{X}_0, \beta)\, a + \beta_1 \cosh(\|\mathcal{X}_0\|)\, a^\dagger\right]|\varphi\rangle = z|\varphi\rangle, \tag{30}$$

where

$$\mathcal{G}(\mathcal{X}_0, \beta) = \cosh(\|\mathcal{X}_0\|) + \beta\sqrt{\mathcal{X}_0^\ddagger}\left(\sqrt{\mathcal{X}_0}\right)^{-1} \sinh(\|\mathcal{X}_0\|). \tag{31}$$

Let us notice that this last coefficient can be written on the form

$$\mathcal{G}(\mathcal{X}_0, \beta) = \mathcal{G}(\mathcal{X}_0, \beta_0)\left(\epsilon_\phi + \beta_1 \left(\mathcal{G}(\mathcal{X}_0, \beta_0)\right)^{-1}\sqrt{\mathcal{X}_0^\ddagger}\left(\sqrt{\mathcal{X}_0}\right)^{-1} \sinh(\|\mathcal{X}_0\|)\right) \tag{32}$$

where, taking into account (29),

$$\mathcal{G}(\mathcal{X}_0, \beta_0) = \left[\epsilon_\phi - \beta_0^2\, \mathcal{X}_0^\ddagger (\mathcal{X}_0)^{-1}\right] \cosh(\|\mathcal{X}_0\|). \tag{33}$$

Multiplying both sides of the equation (30) by the inverse of $\mathcal{G}(\mathcal{X}_0, \beta)$ and taking into account (32), we get

$$[a + \hat{\beta}_1 a^\dagger]|\varphi\rangle = \hat{z}|\varphi\rangle, \tag{34}$$

where

$$\hat{\beta}_1 = \beta_1(\mathcal{G}(\mathcal{X}_0,\beta_0))^{-1}\cosh(\|\mathcal{X}_0\|) \quad \in \mathbb{C}B_{L_1}, \tag{35}$$

and

$$\hat{z} = \left[(\mathcal{G}(\mathcal{X}_0,\beta_0))^{-1} - \beta_1\sqrt{(\mathcal{X}_0{}^{\dagger})}\left(\sqrt{\mathcal{X}}_0\right)^{-1}\sinh(\|\mathcal{X}_0\|)\right] z. \tag{36}$$

The equation (34) is thus simpler to solve than (7). Indeed, we can again try a solution of the type

$$|\varphi\rangle = \sum_{n=0}^{\infty} C_n|n\rangle, \qquad C_n \in \mathbb{C}B_L. \tag{37}$$

Inserting it in (34), using the raising and lowering properties of the operators $a^{\dagger}$ and a, and the orthogonality conditions of the states $\{|n\rangle\}$, we get the recurrence relation

$$C_{n+1} = \frac{[\hat{z}C_n - \sqrt{n}\hat{\beta}_1 C_{n-1}]}{\sqrt{n+1}}, \qquad n = 2,\ldots, \tag{38}$$

with

$$C_1 = \hat{z}C_0, \tag{39}$$

and C_0 is an arbitrary constant. Proceeding by iteration we get

$$C_n = \frac{1}{\sqrt{n!}}\left(\hat{z}^n - \sum_{k=0}^{n-2}(k+1)\hat{z}^{(n-2-k)}(\hat{z}^*)^k\,\hat{\beta}_1\right) C_0, \qquad n = 2,3,\ldots. \tag{40}$$

This expression may be written in a closed form. Indeed, as we can show that

$$\sum_{k=0}^{n-2}(k+1)\hat{z}^{(n-2-k)}(\hat{z}^*)^k = \frac{n(n-1)}{2!}(\hat{z}_0)^{n-2} - \frac{n(n-1)(n-2)}{3!}(\hat{z}_0)^{n-3}\hat{z}_1 \tag{41}$$

$$= \frac{1}{2!}\frac{\partial^2}{\partial\hat{z}_0^2}(\hat{z}_0)^n - \frac{1}{3!}\frac{\partial^3}{\partial\hat{z}_0^3}(\hat{z}_0)^n\,\hat{z}_1, \tag{42}$$

the relation (40) becomes

$$C_n = \frac{1}{\sqrt{n!}}\left(\hat{z}^n - \left[\frac{1}{2!}\frac{\partial^2}{\partial\hat{z}_0^2}(\hat{z}_0)^n - \frac{1}{3!}\frac{\partial^3}{\partial\hat{z}_0^3}(\hat{z}_0)^n\,\hat{z}_1\right]\hat{\beta}_1\right) C_0, \qquad n = 2,3,\ldots, \tag{43}$$

which is also valid for $n = 1$. Finally, inserting this result into (37), and after some manipulations we obtain a general solution of (34), which is

$$|\varphi\rangle = \left[e^{\hat{z}_1 a^{\dagger}} - \hat{\beta}_1\frac{(a^{\dagger})^2}{2!} + \hat{z}_1\hat{\beta}_1\frac{(a^{\dagger})^3}{3!}\right] e^{\hat{z}_0 a^{\dagger}}|0\rangle C_0. \tag{44}$$

A normalized version of (44) is given by

$$|\varphi\rangle = \exp\left[-\hat{\beta}_1\frac{(a^{\dagger})^2}{2} - \hat{z}_1\hat{\beta}_1\frac{(a^{\dagger})^3}{3}\right]\mathbb{D}(\hat{z}_0)\mathbb{D}(\hat{z}_1)|0\rangle\,\hat{C}(\hat{z},\hat{\beta}_1), \tag{45}$$

where the operator $\mathbb{D}$ has been defined in (14). The normalization constant $\hat{C}$ is given by

$$\hat{C}(\hat{z},\hat{\beta}_1) = \left(\sqrt{\Gamma}\right)^{-1}\left[\epsilon_\phi + \frac{1}{2}\left(\sqrt{\Gamma}\right)^{-1}\Omega\left(\sqrt{\Gamma}\right)^{-1}\right], \tag{46}$$

with

$$\Gamma(\hat{z},\hat{\beta}_1) = \epsilon_\phi - \frac{1}{2}\left(\left(\hat{z}^\dagger\right)^2\hat{\beta}_1 + \left(\hat{\beta}_1\right)^\dagger\hat{z}^2\right) - \frac{1}{3}\left(\left(\hat{z}^\dagger\right)^3\hat{z}_1\hat{\beta}_1 + \left(\hat{\beta}_1\right)^\dagger(\hat{z}_1)^\dagger\hat{z}^3\right) \tag{47}$$

and

$$\begin{aligned}
\Omega(\hat{z},\hat{\beta}_1) &= \Bigg[\frac{1}{6}\left(\left(\hat{z}^\dagger\right)^3\hat{z}_1\hat{z}^2 + \left(\hat{z}^\dagger\right)^2(\hat{z}_1)^\dagger\hat{z}^3\right) \\
&+ \left(\left(\hat{z}^\dagger\right)^2\hat{z}_1\hat{z} + (\hat{z}^\dagger)\hat{z}_1 + (\hat{z}^\dagger)(\hat{z}_1)^\dagger\hat{z}^2 + (\hat{z}_1)^\dagger\hat{z}\right) - \frac{1}{4}\left(\left(\hat{z}^\dagger\right)^2\hat{z}^2 + 4\hat{z}^\dagger\hat{z} + 2\right)\Bigg]\left(\hat{\beta}_1\right)^\dagger\hat{\beta}_1 \\
&- \frac{1}{9}\left(\left(\hat{z}^\dagger\right)^3\hat{z}^3 + 9\left(\hat{z}^\dagger\right)^2\hat{z}^2 + 24\hat{z}^\dagger\hat{z} + 6\right)(\hat{z}_1)^\dagger\hat{z}_1\left(\hat{\beta}_1\right)^\dagger\hat{\beta}_1 \\
&- \left(\left(\hat{z}_0^\dagger\right)^2\hat{\beta}_1 + \left(\hat{\beta}_1\right)^\dagger(\hat{z}_0)^2\right)(\hat{z}_1)^\dagger\hat{z}_1.
\end{aligned}$$

From (26) and (45) we conclude that a class of normalized solutions of the eigenvalue equation (7), corresponding to the eigenvalue z, is given by the generalized supersqueezed states

$$|\psi\rangle = S(\mathcal{X}_0)\exp\left[-\hat{\beta}_1\frac{\left(a^\dagger\right)^2}{2} - \hat{z}_1\hat{\beta}_1\frac{\left(a^\dagger\right)^3}{3}\right]\mathbb{D}(\hat{z}_0)\mathbb{D}(\hat{z}_1)|0\rangle\,\hat{C}(\hat{z},\hat{\beta}_1). \tag{48}$$

Let us now give some examples of such states.

2.2.1 Standard supersqueezed states

The standard supersqueezed states are obtained from (48) when $\beta_1 = 0$ and $z_1 = 0$, i.e., when $\hat{\beta}_1 = 0$, $\hat{z}_1 = 0$ and $\hat{z}_0 = \left(\mathcal{G}(\mathcal{X}_0,\beta_0)\right)^{-1}z_0$. They are given by

$$|\psi\rangle = S(\mathcal{X}_0)\mathbb{D}(\hat{z}_0)|0\rangle, \tag{49}$$

where $\mathcal{X}_0$ and $\hat{z}_0$ remain even Grassmann valued numbers.

2.2.2 A new class of supersqueezed states

Another class of supersqueezed states appears in (48), because of the possibility to choose in (7) a non zero odd component of the variable β. For example, if we choose $\beta_0 = 0$, i.e., $\mathcal{X}_0 = 0$, $\hat{\beta}_1 = \beta_1$ and $\hat{z} = z$, then from (48) we obtain the following class of states

$$|\psi\rangle = \exp\left[-\beta_1\frac{\left(a^\dagger\right)^2}{2} - z_1\beta_1\frac{\left(a^\dagger\right)^3}{3}\right]\mathbb{D}(z_0)\mathbb{D}(z_1)|0\rangle\hat{C}(z,\beta_1). \tag{50}$$

They are obtained by applying the operator

$$\exp\left[-\beta_1\frac{\left(a^\dagger\right)^2}{2} - z_1\beta_1\frac{\left(a^\dagger\right)^3}{3}\right] \tag{51}$$

to the generalized coherent states (13) of a. In the special case where $z_1 = 0$, we get to the normalized supersqueezed states

$$\begin{aligned} |\psi\rangle &= \left[\epsilon_\phi + \frac{1}{4}\beta_1^\ddagger\beta_1\left(z_0^2(z_0^\ddagger)^2 + 4z_0z_0^\ddagger + 2\right)\right]\exp\left[-\frac{1}{8}\beta_1^\ddagger\beta_1\left(a^2(a^\dagger)^2 + (a^\dagger)^2a^2\right)\right] \\ &\quad \exp\left[-\left(\beta_1\frac{\left(a^\dagger\right)^2}{2} - \beta_1^\ddagger\frac{a^2}{2}\right)\right]\mathbb{D}(z_0)|0\rangle, \end{aligned} \tag{52}$$

which are written in terms of the superunitary operator $S(-\beta_1)$ as defined in (27). Moreover, in the case where $\beta_1 \in \mathbb{R}B_{L_1}$, this last equation becomes

$$|\psi\rangle = S(-\beta_1)\mathbb{D}(z_0)|0\rangle, \tag{53}$$

i.e., we are in the presence of a class of supersqueezed states which are constructed by applying the superunitary supersqueeze operator $S(-\beta_1)$ to the standard harmonic oscillator coherent states.

3 SAES associated to the fermionic superalgebra

In this section, we will construct the SAES associated with the fermionic superalgebra generated by $\{b, b^\dagger, I\}$ which satisfy the non-zero supercommutation relation (4) . The general eigenvalue equation writes as

$$[B_-b + B_+b^\dagger + B_3I]|\psi\rangle = Z|\psi\rangle, \qquad B_\pm,\ Z \in \mathbb{C}B_L. \tag{54}$$

Here we will distinguish again two cases: firstly when $B_+ = B_3 = 0$ and secondly when B_- is invertible so that the equation (54) reduces to

$$(b + \delta b^\dagger)|\psi\rangle = z|\psi\rangle, \qquad \delta, z\ \in \mathbb{C}B_L. \tag{55}$$

3.1 The b-fermionic eigenstates

Let us solve

$$B\ b|\psi\rangle = Z|\psi\rangle, \qquad B, Z\ \in \mathbb{C}B_L. \tag{56}$$

Since the fermionic graded Fock space is reduced to the vectors $|-\rangle$ (even) and $|+\rangle$ (odd) which act as in (5), a solution of (56) writes as

$$|\psi\rangle = C|-\rangle + D|+\rangle, \qquad C, D \in \mathbb{C}B_L. \tag{57}$$

Inserting (57) into (56) and using (5), we get

$$BD^*|-\rangle = ZC|-\rangle + ZD|+\rangle. \tag{58}$$

The orthogonality of the states $|-\rangle$ and $|+\rangle$ leads to the following set of algebraic equations

$$\begin{aligned} BD^* &= ZC \\ ZD &= 0, \end{aligned} \tag{59}$$

or by conjugation of the first one,

$$\begin{aligned} B^*D &= Z^*C^* \\ ZD &= 0. \end{aligned} \tag{60}$$

Let us mention that, when $B_\phi \neq 0$, we have evidently the normalized solution $|\psi\rangle = |-\rangle$ when the eigenvalue Z is zero, but due to the presence of Grassmann value quantities, when $B_\phi = 0$, we have a larger set of solutions. For instance, for $B = B_1$, we find, a solution of the form

$$|\psi\rangle = C|-\rangle \pm B_1|+\rangle. \tag{61}$$

Normalized eigenstates are given by

$$|\psi\rangle = \exp\left[\pm\left(B_1 b^\dagger + B_1^\dagger b\right)\right]|-\rangle. \tag{62}$$

When $Z \neq 0$, non-trivial solutions appears if and only if $Z_\phi = 0$. From (60), we have $D_\phi = 0$. To solve completely the system (60) we have to distinguish two cases.

If $B_\phi \neq 0$, we can solve D from the first equation of (60)

$$D = (B^*)^{-1}Z^*C^* = \left(B^{-1}Z\right)^*C^* = z^*C^*, \tag{63}$$

where $z = z_0 + z_1 = \left(B^{-1}Z\right)$. Now inserting (63) into the second equation of (60), we get

$$Zz^*C^* = 0. \tag{64}$$

Normalized solutions will be obtained if $C_\phi \neq 0$ and we thus get

$$Zz^* = 0, \tag{65}$$

which can be written explicitly

$$z_0^2 = 0, \qquad z_0 Z_1 = z_1 Z_0. \tag{66}$$

The normalized eigenstates of Bb with the eigenvalue Z satisfying (66) are given by

$$|\psi\rangle = \Big(|-\rangle + z^*|+\rangle\Big)C, \tag{67}$$

where C is an arbitrary Grassmann number such that $C_\phi \neq 0$. They can be written as

$$|z_0; z_1\rangle = \mathbb{T}(z_1)\mathbb{T}(z_0)|-\rangle, \tag{68}$$

where the superunitary operators $\mathbb{T}$ are given by

$$\mathbb{T}(z_1) = \exp\left(b^\dagger z_1 - z_1^\ddagger b\right), \qquad \mathbb{T}(z_0) = \exp\left(z_0 b^\dagger - z_0^\ddagger b\right). \tag{69}$$

The b–SAES are obtained from (68) when $B = \epsilon_\phi$, so that $z_0 = Z_0$ and $z_1 = Z_1$. We notice that when $z_0 = 0$, they reduces to the standard supercoherent states associated to the system characterized by the fermionic Hamiltonian $H = b^\dagger b - \frac{1}{2}$.

If $B_\phi = 0$, the problem is a little more tricky. We can write (59) explicitly as

$$B_0 d_0 - B_1 d_1 = Z_0 c_0 + Z_1 c_1 \tag{70}$$

$$B_1 d_0 - B_0 d_1 = Z_1 c_0 + Z_0 c_1 \tag{71}$$

$$Z_0 d_0 + Z_1 d_1 = 0 \tag{72}$$

$$Z_1 d_0 + Z_0 d_1 = 0, \tag{73}$$

where we have taken $C = c_0 + c_1$ and $D = d_0 + d_1$. In this way, for instance, when $B_0 \neq 0$ and $(B_0)^2 \neq 0$, we can combine (70) and (71) to obtain

$$(B_0)^2 d_0 = (B_0 Z_0 - B_1 Z_1)c_0 + (B_0 Z_1 - B_1 Z_0)c_1, \tag{74}$$

$$(B_0)^2 d_1 = (B_1 Z_0 - B_0 Z_1)c_0 + (B_1 Z_1 - B_0 Z_0)c_1 \tag{75}$$

and then combine this last system of equations with (72) and (73) to get

$$Z_0\left(2B_1 Z_1 - B_0 Z_0\right)c_0 + B_1 (Z_0)^2 c_1 = 0, \tag{76}$$

$$Z_0\left(2B_1 Z_1 - B_0 Z_0\right)c_1 + B_1 (Z_0)^2 c_0 = 0. \tag{77}$$

The systems (74-75) and (76-77) are equivalent to

$$(B_0)^2 D = BZ^* C^* \tag{78}$$

and

$$Z_0 \left(2B_1 Z_1 - B_0 Z_0 + B_1 Z_0\right) C = 0, \tag{79}$$

respectively. As we search for normalized solutions, we must take $C_\phi \neq 0$. This implies the following condition for the Z eigenvalue:

$$\begin{aligned} Z_0 \left(2B_1 Z_1 - B_0 Z_0\right) &= 0 & (80)\\ B_1 (Z_0)^2 &= 0. & (81)\end{aligned}$$

Then, the normalized eigenstates of (56) corresponding to the Z eigenvalue satisfying (80-81) are given by (57), with C an arbitrary Grassmann number such that $C_\phi \neq 0$, and D verifying (78).

Following a similar procedure, when $B_0 = 0$ and $B_1 \neq 0$, the normalized solutions of (56) corresponding to the Z eigenvalue satisfying the conditions

$$(Z_0)^2 = 0, \qquad Z_0 Z_1 = 0, \tag{82}$$

are given by (57), with $C_\phi \neq 0$, and D verifying

$$B_1 D = -Z^* C^*. \tag{83}$$

When $B_0 \neq 0$ et $B_1 = 0$, the solutions corresponding to the Z eigenvalue satisfying the conditions

$$(Z_0)^2 = 0, \tag{84}$$

are given by (57), with $C_\phi \neq 0$, and D verifying

$$B_0 D = Z^* C^*. \tag{85}$$

Other classes of solutions can be reached by imposing other conditions on the coefficient B.

3.2 Supersqueezed states

Let us now solve the eigenvalue (55). If we assume again a solution of the type (57), then by inserting it in (55), using the raising and lowering properties (5) and the orthogonality between the sates $|-\rangle$ and $|+\rangle$, we get the following algebraic Grassmann equations for determining C and D :

$$D^* = zC, \tag{86}$$

$$\delta C^* = zD. \tag{87}$$

By conjugating the equation (86) and then by inserting it in (87), we get

$$(zz^* - \delta)C^* = 0. \tag{88}$$

As we are interested in normalized solutions, we must take $C_\phi \neq 0$, then (88) implies:

$$z_0^2 = \delta, \tag{89}$$

that is, δ is an even Grassmann number. Inserting (86) in (57) and considering the conditions (89), we conclude that a set of normalized eigentates of the operator $(b + \delta_0 b^\dagger)$ corresponding to the eigenvaleue $z = \pm\sqrt{\delta_0} + z_1$ is given by

$$|\delta_0, z_1\rangle^\pm = \Big(|-\rangle - (z_1 \mp \sqrt{\delta_0})|+\rangle\Big)C. \tag{90}$$

It is not too hard to show that the corresponding normalized supersqueezed states are given by

$$|\delta_0, z_1\rangle^\pm = \exp\left(b^\dagger z_1 - z_1^\ddagger b\right) \exp\left[\pm\sqrt{\delta_0}\left(b^\dagger + z_1^\ddagger\right)\right] |-\rangle\, N^\pm(\delta_0, z_1), \tag{91}$$

where the normalization constant $N^\pm$ is given by

$$N^\pm(\delta_0, z_1) = \mathcal{F}^{-1}\left[\epsilon_\phi \mp \frac{1}{2}\mathcal{F}^{-1}\left(\sqrt{\delta_0} z_1^\ddagger + (\sqrt{\delta_0})^\ddagger z_1 \mp \sqrt{\delta_0}(\sqrt{\delta_0})^\ddagger z_1^\ddagger z_1\right)\mathcal{F}^{-1}\right], \tag{92}$$

with

$$\mathcal{F}(\delta_0) = \sqrt{1 + \sqrt{\delta_0}(\sqrt{\delta_0})^\ddagger}. \tag{93}$$

We notice that in the limit $\delta_0 \mapsto 0$ the supersqueezed states (91) becomes the eigenstates of the operator b corresponding to the eigenvalue $z = z_1$.

4 SAES associated to the Heisenberg–Weyl Lie superalgebra

Let us now compute the SAES associated to the H-W Lie superalgebra generated by the set of generators $\{a, a^\dagger, I, b, b^\dagger\}$ whose non zero super-commutation relations are given by the relations (2) and (4). The eigenvalue equation is written as

$$[A_- a + A_+ a^\dagger + A_3 I + B_- b + B_+ b^\dagger]|\psi\rangle = Z|\psi\rangle, \qquad A_\pm, A_3, B_\pm, Z \in \mathbb{C}B_L. \tag{94}$$

Here we concentrate in the case where $(A_-)_\phi \neq 0$, i.e., A_- is an invertible Grassmann number. In this case, we can express (94) in the form

$$[a + \beta a^\dagger + \gamma b + \delta b^\dagger]|\psi\rangle = z|\psi\rangle, \qquad \beta, \gamma, \delta, z \in \mathbb{C}B_L. \tag{95}$$

Special cases of this problem have been considered in sections 2 and 3. Here we consider the cases where we have the presence of both bosonic and fermionic operators in the eigenvalue equation (95).

4.1 Generalized supercoherent states

First, we take the particular eigenvalue equation

$$[a + \gamma b]|\psi\rangle = z|\psi\rangle, \qquad \gamma, z \in \mathbb{C}B_L. \tag{96}$$

Let us assume a solution of the type

$$|\psi\rangle = \sum_{n=0}^{\infty}\Big(C_n|n;-\rangle + D_n|n;+\rangle\Big), \tag{97}$$

where C_n, $D_n \in \mathbb{C}B_L$. By inserting (97) in (96), using the lowering properties of operators a and b, Eqs. (3) and (5), and the orthogonality properties of the graded Fock space basis $\{|n;-\rangle, |n;+\rangle, n \in \mathbb{N}\}$, we get the recurrence relations

$$\sqrt{n+1}C_{n+1} + \gamma D_n^* = zC_n, \tag{98}$$

$$\sqrt{n+1}D_{n+1} = zD_n. \tag{99}$$

From (99), it is easy to find the expression of the coefficients D_n in terms of an arbitrary constant D_0 :

$$D_n = \frac{z^n}{\sqrt{n!}}D_0, \quad n = 1, 2, \ldots. \tag{100}$$

Then, by inserting (100) in (98), we get the following recurrence relation for the coefficients C_n :

$$C_{n+1} = \frac{1}{\sqrt{n+1}}\left[zC_n - \gamma\frac{(z^*)^n}{\sqrt{n!}}D_0^*\right], \qquad n = 0, 1, 2, \ldots. \tag{101}$$

Finally, proceeding by iteration we get

$$C_n = \frac{1}{\sqrt{n!}}\left[z^n C_0 - \left(\sum_{k=0}^{n-1} z^{(n-1-k)}\gamma(z^*)^k\right) D_0^*\right], \qquad n = 1, 2, \ldots, \tag{102}$$

where C_0 is an arbitrary constant. Since C_0 and D_0 are arbitrary constants, the equation (97) gives two independent solutions. The first one consists of the standard coherent states

$$|z;-\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} C_0 |n;-\rangle. \tag{103}$$

To find the second one, we use the formula

$$\frac{1}{n+1} \sum_{k=0}^{n} z^{(n-k)} \gamma (z^*)^k = (\gamma_0 {z_0}^n + z^n \gamma_1). \tag{104}$$

We thus get the generalized coherent states on the form

$$\begin{aligned} \widetilde{|z,\gamma;+\rangle} = \widetilde{|z,\gamma_0,\gamma_1;+\rangle} &= \left[\sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n;+\rangle - a^\dagger \sum_{n=0}^{\infty} \frac{(\gamma_0 {z_0}^n + z^n \gamma_1)}{\sqrt{n!}} |n;-\rangle \right] D_0^* \\ &= \exp\left[-\left(\gamma_0 (1 + z_1 a^\dagger) + \gamma_1 \right) a^\dagger b \right] e^{z a^\dagger} |0;+\rangle\, D_0^*. \end{aligned} \tag{105}$$

The normalized version of the states (103) is given by

$$|z;-\rangle = |z_0, z_1;-\rangle = \mathbb{D}(z_0) \mathbb{D}(z_1) |0;-\rangle. \tag{106}$$

It is similar to the one obtained in (13). A set of normalized generalized supercoherent states, orthogonal to (106) is given by the formula

$$|z,\gamma,+\rangle = |z_0, z_1, \gamma_0, \gamma_1;+\rangle = \frac{\widetilde{|z,\gamma_0,\gamma_1,+\rangle} - |z;-\rangle \langle -;z\widetilde{|z,\gamma_0,\gamma_1,+\rangle}}{\| \widetilde{|z,\gamma_0,\gamma_1,+\rangle} - |z;-\rangle \langle -;z\widetilde{|z,\gamma_0,\gamma_1,+\rangle} \|}. \tag{107}$$

After some calculations, we get the set of generalized supercoherent states

$$\begin{aligned} |z_0, z_1, \gamma_0, \gamma_1;+\rangle &= \mathbb{D}(z_0)\mathbb{D}(z_1) \Big\{ |0;+\rangle \\ &- \Big[\left(1 - \frac{1}{2} z_1^\ddagger z_1 \right) \mathbb{D}(-z_1)(a^\dagger + z_0^\ddagger) \gamma_0 e^{z_1 z_0^\ddagger} + (1 + z_1^\ddagger z_1) a^\dagger \gamma_1 \\ &- (1 - z_1^\ddagger z_1) z^\ddagger \gamma_0 e^{z_1 z_0^\ddagger} \Big] |0;-\rangle \Big\} N(z_0, z_1, \gamma_0, \gamma_1), \end{aligned} \tag{108}$$

where the normalization constant N is given by

$$N(z_0, z_1, \gamma_0, \gamma_1) = \mathcal{B}^{-1} \left[1 - \mathcal{B}^{-1} \left(\gamma_1^\ddagger \gamma_1 - \gamma_0^\ddagger \gamma_0 (z_0^\ddagger z_0)^2 \right) z_1^\ddagger z_1 \mathcal{B}^{-1} \right], \tag{109}$$

with

$$\mathcal{B}(\gamma_0, \gamma_1) = \sqrt{1 + \gamma^\ddagger \gamma} = \sqrt{1 + \gamma_0^\ddagger \gamma_0 + \gamma_0^\ddagger \gamma_1 + \gamma_1^\ddagger \gamma_0 + \gamma_1^\ddagger \gamma_1}. \tag{110}$$

4.1.1 Super coherent states

The supercoherent states (108) constitute a generalization of the super coherent states found by Aragone and Zypman[3]. Indeed, from equations (108-110) we see that, in the case where $\gamma_1 = 0$ and $z_1 = 0$, we have

$$|z_0, 0, \gamma_0, 0; +\rangle = \left(\sqrt{1+\gamma_0{}^{\ddagger}\gamma_0}\right)^{-1} \mathbb{D}(z_0) \left(|0;+\rangle - \gamma_0 a^{\dagger}|0;-\rangle\right). \tag{111}$$

4.1.2 Other classes of supercoherent states

Now if in (108-110), we take $\gamma_0 = 0$ and $z_0 = 0$, we get

$$|0, z_1, 0, \gamma_1; +\rangle = \left(1 - \frac{1}{2}\gamma_1^{\ddagger}\gamma_1 - \gamma_1^{\ddagger}\gamma_1 z_1^{\ddagger} z_1\right) \mathbb{D}(z_1) \left(|0;+\rangle - (1 + z_1^{\ddagger} z_1) a^{\dagger}\gamma_1|0;-\right). \tag{112}$$

We can also distinguish the case where $\gamma_1 = 0$ and $z_0 = 0$. We get

$$\begin{aligned} |0, z_1, \gamma_0, 0; +\rangle &= \left(\sqrt{1+\gamma_0{}^{\ddagger}\gamma_0}\right)^{-1} \mathbb{D}(z_1) \\ & \left\{|0;+\rangle + \gamma_0 \left[\left(\frac{z_1^{\ddagger} z_1}{2} - 1\right) \mathbb{D}(-z_1) a^{\dagger} + z_1^{\ddagger}\right] |0;-\rangle\right\}. \end{aligned} \tag{113}$$

4.1.3 Standard supercoherent states

In the case where $\gamma = 0$, (108) becomes the standard coherent states

$$|z;+\rangle = |z_0, z_1; +\rangle = \mathbb{D}(z_0)\mathbb{D}(z_1)|0;+\rangle. \tag{114}$$

By combining the two independent solutions (106) and (114), we can construct a solution of the type

$$|z; \rho, \tau\rangle = \rho|z;-\rangle + \tau|z;+\rangle, \tag{115}$$

where ρ and τ are Grassmann numbers such that $\rho_1 z_1 = \tau_1 z_1 = 0$. Thus the states (115) are eigenstates of a corresponding to the eigenvalue z. In particular, if we take for example $\rho = 1 - \frac{z_1^{\ddagger} z_1}{2}$ and $\tau = -z_1$, then we obtain the supercoherent states

$$|z\rangle = \mathbb{D}(z_0)\mathbb{D}(z_1)\mathbb{T}(z_1)|0;-\rangle. \tag{116}$$

Moreover, if we take $z_1 = 0$, $\rho = 1 - \frac{\theta_1^{\ddagger}\theta_1}{2}$ and $\tau = -\theta_1$, we get the standard supercoherent states associated to the supersymmetric harmonic oscillator [6, 14]

$$|z_0, \theta_1\rangle = \mathbb{D}(z_0)\mathbb{T}(\theta_1)|0;-\rangle. \tag{117}$$

4.2 Generalized supersqueezed states

Let us now find the SAES associated to the sub-superalgebra $\{a, b, b^\dagger, I\}$. If the coefficient of a in the linear combination is invertible the problem reduces to solve the eigenvalue equation:

$$[a + \gamma b + \delta b^\dagger]|\psi\rangle = z|\psi\rangle, \qquad \gamma, \delta \in \mathbb{C}B_L. \tag{118}$$

We can show, see Appendix B section B.1, that two classes of independent solutions of the eigenvalue equation (118) exist and are given by

$$|\psi; -\rangle = \Big[\sum_{\ell \text{ even}}^{\infty} \mathcal{O}_{a^\dagger}(\ell, \gamma, \delta^*, z_1) e^{za^\dagger}|0; -\rangle - \sum_{\ell \text{ odd}}^{\infty} \mathcal{O}_{a^\dagger}(\ell, \delta, \gamma^*, z_1) e^{za^\dagger}|0; +\rangle\Big] C_0 \tag{119}$$

and

$$|\psi; +\rangle = \Big[\sum_{\ell \text{ even}}^{\infty} \mathcal{O}_{a^\dagger}(\ell, \delta, \gamma^*, z_1) e^{za^\dagger}|0; +\rangle - \sum_{\ell \text{ odd}}^{\infty} \mathcal{O}_{a^\dagger}(\ell, \gamma, \delta^*, z_1) e^{za^\dagger}|0; -\rangle\Big] D_0^*, \tag{120}$$

where C_0 and D_0^* are arbitrary and invertible Grassmann constants and

$$\begin{aligned} \mathcal{O}_{a^\dagger}(\ell, \gamma, \delta^*, z_1) &= \frac{1}{\ell!}\Bigg\{\overbrace{(\gamma\delta^*\gamma\delta^*\cdots)}^{\ell \text{ factors}} \left((a^\dagger)^\ell - z_1 (a^\dagger)^{\ell+1}\right) \\ &+ \frac{1}{\ell+1}\sum_{j=0}^{\ell}(-1)^{j+\ell}\, \overbrace{(\gamma\delta^*\gamma\delta^*\cdots)}^{(\ell-j) \text{ factors}}\, z_1\, \overbrace{(\cdots\gamma\delta^*\gamma\cdots)}^{j \text{ factors}}\, (a^\dagger)^{\ell+1}\Bigg\}, \end{aligned} \tag{121}$$

where $\ell = 0, 1, 2, \ldots$.

The superstates (119) and (120) can be written in the form of a supersqueeze operator acting on the supercoherent state, that is

$$\begin{aligned} |\psi; -\rangle &= \mathcal{O}_{\text{even}}(a^\dagger, \gamma, \delta^*, z_1) \exp\Big[-\Big(\mathcal{O}_{\text{even}}(a^\dagger, \gamma, \delta^*, z_1)\Big)^{-1} \\ &\quad \Big(\mathcal{O}_{\text{odd}}(a^\dagger, \delta, \gamma^*, z_1)\Big) e^{2z_1 a^\dagger} b^\dagger\Big] \mathbb{D}(z_0)\mathbb{D}(z_1)\, |0; -\rangle\, \tilde{C}_0, \end{aligned} \tag{122}$$

$$\begin{aligned} |\psi; +\rangle &= \mathcal{O}_{\text{even}}(a^\dagger, \delta, \gamma^*, z_1) \exp\Big[-\Big(\mathcal{O}_{\text{even}}(a^\dagger, \delta, \gamma^*, z_1)\Big)^{-1} \\ &\quad \Big(\mathcal{O}_{\text{odd}}(a^\dagger, \gamma, \delta^*, z_1)\Big) e^{2z_1 a^\dagger} b\Big] \mathbb{D}(z_0)\mathbb{D}(z_1)\, |0; +\rangle\, \tilde{D}_0^*, \end{aligned} \tag{123}$$

where

$$\mathcal{O}_{\text{even}}(a^\dagger, \gamma, \delta^*, z_1) = \sum_{\ell \text{ even}}^{\infty} \mathcal{O}_{a^\dagger}(\ell, \gamma, \delta^*, z_1) \tag{124}$$

and

$$\mathcal{O}_{\text{odd}}(a^\dagger, \gamma, \delta^*, z_1) = \sum_{\ell \text{ odd}}^{\infty} \mathcal{O}_{a^\dagger}(\ell, \gamma, \delta^*, z_1). \tag{125}$$

4.2.1 Standard superqueezed states

In the case where γ and δ are odd Grassmann numbers, that is when $\gamma = \gamma_1$ and $\delta = \delta_1$, it is easy to see from (121) that, the non zero $\mathcal{O}_{a^\dagger}$ operators in (119) and (120) corresponds to

$$\begin{aligned} \mathcal{O}_{a^\dagger}(0,\gamma_1,-\delta_1,z_1) &= 1, \quad \mathcal{O}_{a^\dagger}(1,\delta_1,-\gamma_1,z_1) = \delta_1 a^\dagger - 2\delta_1 z_1 (a^\dagger)^2, \\ \mathcal{O}_{a^\dagger}(2,\gamma_1,-\delta_1,z_1) &= -\frac{1}{2!}\gamma_1\delta_1 (a^\dagger)^2, \end{aligned} \tag{126}$$

and

$$\begin{aligned} \mathcal{O}_{a^\dagger}(0,\delta_1,-\gamma_1,z_1) &= 1, \quad \mathcal{O}_{a^\dagger}(1,\gamma_1,-\delta_1,z_1) = \gamma_1 a^\dagger - 2\gamma_1 z_1 (a^\dagger)^2, \\ \mathcal{O}_{a^\dagger}(2,\delta_1,-\gamma_1,z_1) &= -\frac{1}{2!}\delta_1\gamma_1 (a^\dagger)^2, \end{aligned} \tag{127}$$

respectively. By inserting this results in (119) and (120), and after some simple manipulations, we get the supersqueezed states

$$|\psi;-\rangle = \exp\left[-\frac{1}{2}\gamma_1\delta_1 (a^\dagger)^2\right] e^{-\delta_1 a^\dagger b^\dagger} e^{z a^\dagger} |0;-\rangle\, C_0, \tag{128}$$

and

$$|\psi;+\rangle = \exp\left[-\frac{1}{2}\delta_1\gamma_1 (a^\dagger)^2\right] e^{-\gamma_1 a^\dagger b} e^{z a^\dagger} |0;+\rangle\, D_0^*, \tag{129}$$

which are eigenstates of $a + \gamma_1 b + \delta_1 b^\dagger$. In these last expressions, we notice the action of an normalizer operator acting on the corresponding supercoherent states. The normalizer in equation (128) transforms the algebra element $a+\gamma_1 b+\delta_1 b^\dagger$ into $a+\gamma_1 b$ whereas the normalizer in equation (129) transforms it into $a+\delta_1 b^\dagger$. In fact, a complete reduction into the element a only can be obtained. For instance, that is the case if we multiply the normalizer in equation (128) by the corresponding normalizer of the equation (105) in the special case where $\gamma_0 = 0$, that is, by $e^{-\gamma_1 a^\dagger b}$. Moreover, if we consider the algebra element $a + \beta_0 a^\dagger + \gamma_1 b + \delta_1 b^\dagger$, a normalizer operator transforming it into the element a is given by the standard supersqueeze operator[9]

$$\mathbf{G}(\beta_0,\gamma_1,\delta_1) = \exp\left[-(\beta_0 + \gamma_1\delta_1)\frac{(a^\dagger)^2}{2}\right] \exp\left(-\delta_1 a^\dagger b^\dagger\right) \exp\left(-\gamma_1 a^\dagger b\right). \tag{130}$$

In this way, using the algebra eigenstates (117) of the a annihilator, we observe that a class of superalgebra eigenstates of $a + \beta_0 a^\dagger + \gamma_1 b + \delta_1 b^\dagger$, corresponding to the eigenvalue z_0, is given by

$$\mathbf{G}(\beta_0,\gamma_1,\delta_1)\mathbb{D}(z_0)\mathbb{T}(\theta_1)|0;-\rangle C_0. \tag{131}$$

We notice that, these supersqueezed states are obtained by acting with a supersqueeze operator that is an element of the $OSP(2/2)$ supergroup on the supercoherent states associated to the supersymmetric harmonic oscillator. In this way, these SAES of the algebra element $a + \beta_0 a^\dagger + \gamma_1 b + \delta_1 b^\dagger$, are comparable to the supersqueezed states for the supersymmetric harmonic oscillator [16, 18].

4.2.2 Spin $\frac{1}{2}$ representation AES structure

Let us consider now the special case where both γ and δ are even invertible Grassmann numbers. Let us write $\gamma = \gamma_0$ and $\delta = \delta_0$. In this case, from (121), we obtain

$$\mathcal{O}_{a^\dagger}(\ell, \gamma_0, \delta_0, z_1) = \begin{cases} \frac{(a^\dagger)^\ell}{\ell!}(\gamma_0\delta_0)^{\ell/2} \exp\left(-\frac{\ell}{\ell+1} z_1 a^\dagger\right), & \text{if } \ell \text{ is even} \\ \frac{(a^\dagger)^\ell}{\ell!}(\gamma_0\delta_0)^{(\ell-1)/2}\gamma_0 \exp\left(-z_1 a^\dagger\right), & \text{if } \ell \text{ is odd} \end{cases}. \tag{132}$$

Thus, by inserting these results in (124) and (125), we get

$$\begin{aligned} \mathcal{O}_{\text{even}}(a^\dagger, \gamma_0, \delta_0, z_1) &= \sum_{\ell \text{ even}}^{\infty} \frac{(\sqrt{\gamma_0\delta_0}\, a^\dagger)^\ell}{\ell!} \exp\left(-\frac{\ell}{\ell+1} z_1 a^\dagger\right) \\ &= \cosh(\sqrt{\gamma_0\delta_0}\, a^\dagger) e^{-z_1 a^\dagger} \\ &\quad \exp\left[z_1 (\sqrt{\gamma_0\delta_0})^{-1} \left(\cosh(\sqrt{\gamma_0\delta_0}\, a^\dagger)\right)^{-1} \sinh(\sqrt{\gamma_0\delta_0}\, a^\dagger)\right] \end{aligned} \tag{133}$$

and

$$\begin{aligned} \mathcal{O}_{\text{odd}}(a^\dagger, \gamma_0, \delta_0, z_1) &= (\sqrt{\delta_0})^{-1} \sqrt{\gamma_0} \sum_{\ell \text{ odd}}^{\infty} \frac{(\sqrt{\gamma_0\delta_0}\, a^\dagger)^\ell}{\ell!} \exp\left(-z_1 a^\dagger\right) \\ &= (\sqrt{\delta_0})^{-1} \sqrt{\gamma_0} \sinh(\sqrt{\gamma_0\delta_0}\, a^\dagger) \exp\left(-z_1 a^\dagger\right). \end{aligned} \tag{134}$$

By inserting these results in (119) and (120) and after some manipulations, we get the set of independent eigenstates of $a + \gamma_0 b + \delta_0 b^\dagger$:

$$\begin{aligned} |\psi; -\rangle &= \exp\left[-z_1 \left(a^\dagger - (\sqrt{\gamma_0\delta_0})^{-1} T_h(\gamma_0, \delta_0, a^\dagger)\right)\right] \cosh\Big\{ \sqrt{\gamma_0\delta_0}\, a^\dagger - \\ &\quad (\sqrt{\gamma_0})^{-1} \sqrt{\delta_0} \left[1 + z_1 \left(2a^\dagger - (\sqrt{\gamma_0\delta_0})^{-1} T_h(\gamma_0, \delta_0, a^\dagger)\right)\right] b^\dagger \Big\} e^{z a^\dagger} |0; -\rangle\, C_0 \end{aligned} \tag{135}$$

and

$$\begin{aligned} |\psi; +\rangle &= \exp\left[-z_1 \left(a^\dagger - (\sqrt{\gamma_0\delta_0})^{-1} T_h(\gamma_0, \delta_0, a^\dagger)\right)\right] \cosh\Big\{ \sqrt{\gamma_0\delta_0}\, a^\dagger - \\ &\quad (\sqrt{\delta_0})^{-1} \sqrt{\gamma_0} \left[1 + z_1 \left(2a^\dagger - (\sqrt{\gamma_0\delta_0})^{-1} T_h(\gamma_0, \delta_0, a^\dagger)\right)\right] b \Big\} e^{z a^\dagger} |0; +\rangle\, D_0^*, \end{aligned} \tag{136}$$

where

$$T_h(\gamma_0, \delta_0, a^\dagger) = \left(\cosh(\sqrt{\gamma_0\delta_0}\, a^\dagger)\right)^{-1} \sinh(\sqrt{\gamma_0\delta_0}\, a^\dagger). \tag{137}$$

In the special case where $z_1 = 0$, (135) and (136) reduces to

$$\begin{aligned} |\psi; -\rangle &= \cosh\left[\sqrt{\gamma_0\delta_0}\, a^\dagger - (\sqrt{\gamma_0})^{-1}\sqrt{\delta_0}\, b^\dagger\right] e^{z_0 a^\dagger}|0; -\rangle\, C_0 \\ &= -(\sqrt{\gamma_0})^{-1}\sqrt{\delta_0}\sinh\left[\sqrt{\gamma_0\delta_0}\, a^\dagger - (\sqrt{\delta_0})^{-1}\sqrt{\gamma_0}\, b\right] e^{z_0 a^\dagger}|0; +\rangle\, C_0 \end{aligned} \tag{138}$$

and

$$\begin{aligned} |\psi; +\rangle &= \cosh\left[\sqrt{\gamma_0\delta_0}\, a^\dagger - (\sqrt{\delta_0})^{-1}\sqrt{\gamma_0}\, b\right] e^{z_0 a^\dagger}|0; +\rangle\, D_0^* \\ &= -(\sqrt{\delta_0})^{-1}\sqrt{\gamma_0}\sinh\left[\sqrt{\gamma_0\delta_0}\, a^\dagger - (\sqrt{\gamma_0})^{-1}\sqrt{\delta_0}\, b^\dagger\right] e^{z_0 a^\dagger}|0; -\rangle\, D_0^*, \end{aligned} \tag{139}$$

respectively. By combining both equations (138) and (139), we can express the set of independent solutions in the form

$$\widetilde{|\psi; -\rangle} = \exp\left(\sqrt{\gamma_0\delta_0}\, a^\dagger - (\sqrt{\gamma_0})^{-1}\sqrt{\delta_0}\, b^\dagger\right) e^{z_0 a^\dagger}|0; -\rangle\, \tilde{C}_0 \tag{140}$$

and

$$\widetilde{|\psi; +\rangle} = \exp\left(\sqrt{\gamma_0\delta_0}\, a^\dagger - (\sqrt{\delta_0})^{-1}\sqrt{\gamma_0}\, b\right) e^{z_0 a^\dagger}|0; +\rangle\, \tilde{D}_0. \tag{141}$$

Thus, we recover the structure of the spin $\frac{1}{2}$ representation algebra eigenstates associated to the subalgebra $\{a, J_+, J_-\}$ of the $h(2) \oplus su(2)$ Lie algebra [2].

4.3 The general case

Let us solve now the eigenvalue equation (95). The discussion at the end of section 4.2.1 shows that it can be reduced to a simpler one by expressing the eigenstate $|\psi\rangle$ as:

$$|\psi\rangle = \mathbf{G}(\beta_0, \gamma_1, \delta_1)|\varphi\rangle. \tag{142}$$

Indeed, inserting (142) into (95) and multiplying by the inverse of the supersqueeze operator $\mathbf{G}(\beta_0, \gamma_1, \delta_1)$, we get

$$[a + \hat{\beta}_1 a^\dagger + \gamma_0 b + \delta_0 b^\dagger]|\varphi\rangle = z|\varphi\rangle, \tag{143}$$

where

$$\hat{\beta}_1 = \beta_1 + \delta_0\gamma_1 + \gamma_0\delta_1 \qquad \in \mathbb{C}B_{L_1}. \tag{144}$$

We can show that, see Appendix B section B.2, two classes of independent solutions of the eigenvalue equation (143) exit and are given by

$$
\begin{aligned}
|\varphi;-\rangle &= \Big[\sum_{\ell \text{ even}}^{\infty} \exp\left(-\frac{\hat{\beta}_1(\gamma_0\delta_0)^{-1}}{2}\,\ell\right) \mathcal{O}_{a^\dagger}(\ell,\gamma_0,\delta_0,z_1)e^{za^\dagger}|0;-\rangle \\
&- \sum_{\ell \text{ odd}}^{\infty} \exp\left(-\frac{\hat{\beta}_1(\gamma_0\delta_0)^{-1}}{2}\,(\ell-1)\right) \mathcal{O}_{a^\dagger}(\ell,\delta_0,\gamma_0,z_1)e^{za^\dagger}|0;+\rangle\Big] C_0 \qquad (145)
\end{aligned}
$$

and

$$
\begin{aligned}
|\varphi,+\rangle &= \Big[\sum_{\ell \text{ even}}^{\infty} \exp\left(-\frac{\hat{\beta}_1(\gamma_0\delta_0)^{-1}}{2}\,\ell\right) \mathcal{O}_{a^\dagger}(\ell,\delta_0,\gamma_0,z_1)e^{za^\dagger}|0;+\rangle \\
&- \sum_{\ell \text{ odd}}^{\infty} \exp\left(-\frac{\hat{\beta}_1(\gamma_0\delta_0)^{-1}}{2}\,(\ell-1)\right) \mathcal{O}_{a^\dagger}(\ell,\gamma_0,\delta_0,z_1)e^{za^\dagger}|0;-\rangle\Big] D_0^*, \qquad (146)
\end{aligned}
$$

where C_0 and D_0^* are arbitrary and invertible Grassmann constants.

Using the results (132) for the $\mathcal{O}_{a^\dagger}(\ell,\gamma_0,\delta_0,z_1)$ operator, we get

$$
\begin{aligned}
|\varphi;-\rangle &= \Big[\cosh(\sqrt{\gamma_0\delta_0-\hat{\beta}_1}\,a^\dagger)\left(1+T_h(\gamma_0,\delta_0,\hat{\beta}_1,a^\dagger)\sqrt{\gamma_0\delta_0-\hat{\beta}_1}z_1\right)e^{-z_1a^\dagger}e^{za^\dagger}|0;-\rangle \\
&- (\gamma_0)^{-1}\sinh(\sqrt{\gamma_0\delta_0-\hat{\beta}_1}\,a^\dagger)\sqrt{\gamma_0\delta_0+\hat{\beta}_1}e^{-z_1a^\dagger}e^{za^\dagger}|0;+\rangle\Big] C_0 \qquad (147)
\end{aligned}
$$

and

$$
\begin{aligned}
|\varphi;+\rangle &= \Big[\cosh(\sqrt{\gamma_0\delta_0-\hat{\beta}_1}\,a^\dagger)\left(1+T_h(\gamma_0,\delta_0,\hat{\beta}_1,a^\dagger)\sqrt{\gamma_0\delta_0-\hat{\beta}_1}z_1\right)e^{-z_1a^\dagger}e^{za^\dagger}|0;+\rangle \\
&- (\delta_0)^{-1}\sinh(\sqrt{\gamma_0\delta_0-\hat{\beta}_1}\,a^\dagger)\sqrt{\gamma_0\delta_0+\hat{\beta}_1}e^{-z_1a^\dagger}e^{za^\dagger}|0;-\rangle\Big] D_0^*, \qquad (148)
\end{aligned}
$$

where

$$
T_h(\gamma_0,\delta_0,\hat{\beta}_1,a^\dagger) = \left(\cosh(\sqrt{\gamma_0\delta_0-\hat{\beta}_1}\,a^\dagger)\right)^{-1}\sinh(\sqrt{\gamma_0\delta_0-\hat{\beta}_1}\,a^\dagger). \qquad (149)
$$

4.3.1 Generalized Spin $\frac{1}{2}$ representation AES structure

In the special case where $z_1 = 0$, (147) and (148) reduces to

$$
\begin{aligned}
|\varphi;-\rangle &= \exp\left(-\frac{1}{2}(\gamma_0)^{-1}\hat{\beta}_1 a^\dagger b^\dagger\right) \\
&\cosh\left[\sqrt{\gamma_0\delta_0-\hat{\beta}_1}\,a^\dagger-(\gamma_0)^{-1}\sqrt{\gamma_0\delta_0+\hat{\beta}_1}\,b^\dagger\right]e^{z_0a^\dagger}|0;-\rangle\, C_0 \qquad (150)
\end{aligned}
$$

and

$$|\varphi;+\rangle = \exp\left(-\frac{1}{2}(\delta_0)^{-1}\hat{\beta}_1 a^\dagger b\right)$$
$$\cosh\left[\sqrt{\gamma_0\delta_0 - \hat{\beta}_1}\, a^\dagger - (\delta_0)^{-1}\sqrt{\gamma_0\delta_0 + \hat{\beta}_1}\, b\right] e^{z_0 a^\dagger}|0;+\rangle\, D_0^* \qquad (151)$$

respectively. Thus, we get a set of generalized SAES that contains the set of AES associated to the spin $\frac{1}{2}$ representation that we have studied in the section 4.2.2.

5 Isospectral harmonic oscillator Hamiltonians having odd interaction terms

In this section we search for some isospectral harmonic oscillator systems which are characterized by a Hamiltonian admitting an annihilation operator which is a Grassmannian linear combination of the generators of the H-W Lie superalgebra, i.e., of the form

$$\mathcal{A} = a + \beta a^\dagger + \gamma b + \delta b^\dagger, \qquad \beta, \gamma, \delta, \in \mathbb{C}B_L. \qquad (152)$$

A family of non-equivalent such Hamiltonians $\mathcal{H}$ can be contructed if first we consider a superHermitian Hamiltonian $\mathcal{H}_0$ such that the commutator is given by

$$[\mathcal{H}_0, \mathcal{A}_0] = -\mathcal{A}_0, \quad \text{and} \quad \mathcal{A}_0|E_0;\pm\rangle = 0, \qquad (153)$$

where

$$\mathcal{A}_0 = a + \hat{\beta}_1 a^\dagger + \gamma_0 b + \delta_0 b^\dagger, \qquad \gamma_0, \delta_0 \in \mathbb{C}B_{L_0}, \qquad (154)$$

$\hat{\beta}_1$ is given by (144) and $|E_0;\pm\rangle$ are the zero eigenvalue eigenstates of $\mathcal{H}_0$. In this way, $\mathcal{A}_0$ is effectively an annihilation operator and its associated superalgebra eigenstates a class of supercoherent states for the system characterized by the Hamiltonian $\mathcal{H}_0$. Second, according to the analysis of section B.2, it is possible to construct $\mathcal{H}$ satisfying

$$[\mathcal{H}, \mathcal{A}] = -\mathcal{A} \qquad (155)$$

by taking

$$\mathcal{A} = \mathbf{G}(\beta_0, \gamma_1, \delta_1)\mathcal{A}_0(\mathbf{G}(\beta_0, \gamma_1, \delta_1))^{-1} \quad \text{and} \quad \mathcal{H} = \mathbf{G}(\beta_0, \gamma_1, \delta_1)\mathcal{H}_0(\mathbf{G}(\beta_0, \gamma_1, \delta_1))^{-1}, \qquad (156)$$

where $\mathbf{G}(\beta_0, \gamma_1, \delta_1)$ is the standard supersqueeze operator defined in (130). We see that our original problem thus reduce to one of finding $\mathcal{H}_0$. We observe that, the Hamiltonian $\mathcal{H}$ in (156) is not superHermitian but it belongs to a class of Hamiltonians that generalize the one of η–pseudo–Hermitian Hamiltonians[17]. Indeed, it satisfies the relation

$$\mathcal{H}^{\ddagger} = \eta \mathcal{H} \eta^{-1}, \tag{157}$$

where η is the superHermitian operator

$$\eta = \left(\mathbf{G}^{-1}(\beta_0, \gamma_1, \delta_1)\right)^{\ddagger} \mathbf{G}^{-1}(\beta_0, \gamma_1, \delta_1). \tag{158}$$

Let us mention that a family of $\mathcal{H}_0$–equivalent Hamiltonians can be obtained if we replace $\mathbf{G}(\beta_0, \gamma_1, \delta_1)$ in (156) by a suitable $OSp(2/2)$ superunitary operator[9]

$$\mathbf{U}(\mathcal{X}_0, \Gamma_1, \Delta_1) = \exp\left(\mathcal{X}_0 \frac{(a^\dagger)^2}{2} - \mathcal{X}_0^{\ddagger} \frac{a^2}{2} + \Gamma_1 a^\dagger b^\dagger + \Gamma_1^{\ddagger} ab + \Delta_1 a^\dagger b + \Delta_1^{\ddagger} a b^\dagger\right), \tag{159}$$

where $\mathcal{X}_0 \in \mathbb{C}B_{L_0}$ and $\Gamma_1, \Delta_1, \in \mathbb{C}B_{L_1}$.

Let us also mention that if we denote $\mathcal{A}_0^{\ddagger}$ the adjoint of $\mathcal{A}_0$, then, the usual commutator leads to

$$\begin{aligned} [\mathcal{A}_0, \mathcal{A}_0^{\ddagger}] &= 1 - \hat{\beta}_1^{\ddagger}\hat{\beta}_1\{a, a^\dagger\} + (\delta_0^{\ddagger}\delta_0 - \gamma_0^{\ddagger}\gamma_0)[b^\dagger, b] \\ &+ 2\hat{\beta}_1\delta_0^{\ddagger} a^\dagger b - 2\delta_0\hat{\beta}_1^{\ddagger} a b^\dagger + 2\hat{\beta}_1\gamma_0^{\ddagger} a^\dagger b^\dagger - 2\gamma_0\hat{\beta}_1^{\ddagger} ab \end{aligned} \tag{160}$$

and we notice that, under the conditions $\gamma_0 = \delta_0 = 0$ or $\hat{\beta}_1 = 0$, the commutator (160) becomes a diagonal operator in the Fock vector basis $\{|n, \pm\rangle,\ n \in \mathbb{N}\}$.

5.1 $h(2)$ generalized isospectral oscillator system

Let us here consider the particular case where $\gamma_0 = \delta_0 = 0$. In this case, the operator $\mathcal{A}_0$ takes the simple form

$$\mathcal{A}_0 = a + \hat{\beta}_1 a^\dagger \tag{161}$$

and the commutator (160) writes

$$[\mathcal{A}_0, \mathcal{A}_0^{\ddagger}] = 1 - \hat{\beta}_1^{\ddagger}\hat{\beta}_1\{a, a^\dagger\}. \tag{162}$$

A class of Hamiltonian $\mathcal{H}_0$ satisfying (153) is given by

$$\begin{aligned} \mathcal{H}_0 &= (1 + \hat{\beta}_1^{\ddagger}\hat{\beta}_1)\left[\mathcal{A}_0^{\ddagger}\mathcal{A}_0 + \hat{\beta}_1^{\ddagger}\hat{\beta}_1 (a^\dagger)^2 a^2\right] \\ &= a^\dagger a + \hat{\beta}_1 (a^\dagger)^2 + \hat{\beta}_1^{\ddagger} a^2 + \hat{\beta}_1^{\ddagger}\hat{\beta}_1 (a^\dagger a + a a^\dagger) + \hat{\beta}_1^{\ddagger}\hat{\beta}_1 (a^\dagger)^2 a^2. \end{aligned} \tag{163}$$

We notice that we are in presence of a superHermitian Hamiltonian of the harmonic oscillator type with nilpotent interaction terms which contain odd contributions. We also notice that, this hamiltonian can be expressed in the form

$$\mathcal{H}_0 = \frac{\mathcal{N}}{2} + \mathcal{M} + \mathcal{Q}_+ + \mathcal{Q}_-, \tag{164}$$

where

$$\mathcal{N} = 2\hat{\beta}_1^\ddagger \hat{\beta}_1 (a^\dagger a + a a^\dagger), \quad \mathcal{Q}_+ = \hat{\beta}_1 (a^\dagger)^2, \quad \mathcal{Q}_- = \hat{\beta}_1^\ddagger a^2, \quad \mathcal{M} = a^\dagger a - \mathcal{Q}_+ \mathcal{Q}_-. \tag{165}$$

The non-zero super-commutation relations between these operators are given by

$$[\mathcal{M}, \mathcal{Q}_\pm] = \pm 2\mathcal{Q}_\pm, \qquad \{\mathcal{Q}_+, \mathcal{Q}_-\} = \mathcal{N}, \tag{166}$$

i.e., they have almost the structure of $u(1/1)$ superalgebra. Indeed, here $\mathcal{N}$ is an even nilpotent operator such that $\mathcal{N}^2 = 0$.

According to (153) and (163), a class of superalgebra eigenstates of $\mathcal{H}_0$ can be obtained by applying n times ($n = 0, 1, 2, \ldots$) the raising operator $\mathcal{A}_0^\ddagger$ on the zero eigenvalue eigenstates of $\mathcal{A}_0$. From (45), we deduce that these latter are given by

$$|E_0; j\rangle = \left(1 - \frac{1}{4}\hat{\beta}_1^\ddagger \hat{\beta}_1\right) \left[|0; j\rangle - \frac{\hat{\beta}_1}{\sqrt{2}}|2; j\rangle\right], \tag{167}$$

where j corresponds to the set $\{-, +\}$.

Then, as $\mathcal{H}_0 |E_0; j\rangle = 0$, the generated energy eigenstates are given by

$$|E_n; j\rangle \propto (\mathcal{A}_0^\ddagger)^n |E_0; j\rangle = \left((a^\dagger)^n + \hat{\beta}_1^\ddagger \sum_{k=0}^{n-1} (a^\dagger)^{(n-1-k)}\, a\, (a^\dagger)^k\right) |E_0; j\rangle \tag{168}$$

and the corresponding energy eigenvalues are $E_n^j = n$. An orthonormalized version of these states is given by

$$\begin{aligned} |E_n; j\rangle \;=\; & \left(1 - \frac{1}{4}\hat{\beta}_1^\ddagger \hat{\beta}_1 (2n+1)\right) \\ & \left[|n; j\rangle + \frac{\hat{\beta}_1^\ddagger}{2}\sqrt{n(n-1)}|n-2; j\rangle - \frac{\hat{\beta}_1}{2}\sqrt{(n+1)(n+2)}|n+2; j\rangle\right], \end{aligned} \tag{169}$$

where $n \in \mathbb{N}$. From (169), it is easy to calulate the action of $\mathcal{A}_0^\ddagger$ and $\mathcal{A}_0$ on the $|E_n; j\rangle$ eigenstates, we get

$$\mathcal{A}_0^\ddagger |E_n; j\rangle = \left(1 - \frac{1}{2}\hat{\beta}_1^\ddagger \hat{\beta}_1 (n+1)\right) \sqrt{n+1}|E_{n+1}; j\rangle \tag{170}$$

and

$$\mathcal{A}_0|E_n;j\rangle = \left(1-\frac{1}{2}\hat{\beta}_1^\dagger\hat{\beta}_1 n\right)\sqrt{n}|E_{n-1};j\rangle. \tag{171}$$

Thus, the orthonormalized energy eigenstates $|E_n;j\rangle$ can be written in the standard form

$$|E_n;j\rangle = \left(1+\frac{1}{4}\hat{\beta}_1^\dagger\hat{\beta}_1 n(n+1)\right)\frac{(\mathcal{A}_0^\dagger)^n}{\sqrt{n!}}|E_0;j\rangle. \tag{172}$$

This is a complete set of states. Indeed, using (169), we can demonstrate the completeness property

$$\sum_j\sum_{n=0}^{\infty}|E_n;j\rangle\langle E_n;j| = I\otimes I = \sum_j\sum_{n=0}^{\infty}|n;j\rangle\langle n;j|. \tag{173}$$

On the other hand, we can express the $|n;j\rangle$ states in the form

$$\begin{aligned}|n;j\rangle &= \left(1-\frac{1}{4}\hat{\beta}_1^\dagger\hat{\beta}_1(2n+1)\right)\left[|E_n;j\rangle - j\sqrt{(n+1)(n+2)}|E_{n+2};j\rangle\frac{\hat{\beta}_1}{2}\right.\\ &+ \left. j\sqrt{n(n-1)}|E_{n-2};j\rangle\frac{\hat{\beta}_1^\dagger}{2}\right],\end{aligned} \tag{174}$$

then, from (172) and after some manipulations, we get

$$|0;j\rangle = \left(1-\frac{1}{4}\hat{\beta}_1^\dagger\hat{\beta}_1\right)\exp\left(\frac{(\mathcal{A}_0^\dagger)^2}{2}\hat{\beta}_1\right)|E_0;j\rangle. \tag{175}$$

According to (45), the coherent states associated to a physical system characterized by the hamiltonian (163) can be written as:

$$\begin{aligned}|\varphi;j\rangle &= \exp\left[-\hat{\beta}_1\frac{(a^\dagger)^2}{2} - \hat{z}_1\hat{\beta}_1\frac{(a^\dagger)^3}{3}\right]\mathbb{D}(\hat{z}_0)\mathbb{D}(\hat{z}_1)\\ &\quad\left(1-\frac{1}{4}\hat{\beta}_1^\dagger\hat{\beta}_1\right)\exp\left(\frac{(\mathcal{A}_0^\dagger)^2}{2}\hat{\beta}_1\right)|E_0;j\rangle\,\hat{C}(\hat{z},\hat{\beta}_1).\end{aligned} \tag{176}$$

5.2 Spin $\frac{1}{2}$ generalized isospectral oscillator system

In the case where $\hat{\beta}_1 = 0$ and $\gamma_0^\dagger\gamma_0 = \delta_0^\dagger\delta_0$, the operator $\mathcal{A}_0$ takes the form

$$\mathcal{A}_0 = a + \gamma_0 b + \delta_0 b^\dagger \tag{177}$$

and the commutator (160) writes

$$[\mathcal{A}_0, \mathcal{A}_0^\dagger] = 1. \tag{178}$$

A class of Hamiltonian $\mathcal{H}_0$ satisfying (153) is given by

$$\mathcal{H}_0 = \mathcal{A}_0^\ddagger \mathcal{A}_0 = a^\dagger a + \gamma_0^\ddagger \gamma_0 + \gamma_0 a^\dagger b + \gamma_0^\ddagger a b^\dagger + \delta_0 a^\dagger b^\dagger + \delta_0^\ddagger ab. \tag{179}$$

We notice that this is a superHermitian Hamiltonian, without defined parity, which is a linear Grassmann combination of generators of the $osp(2/2) \oplus sh(2/2)$ Lie superalgebra. Then, in this aspect, the corresponding Hamiltonian $\mathcal{H}$ defined in (156), complement the classes of Hamiltonians considered by Buzano et al.[9].

By construction, the eigenstates of $\mathcal{A}_0$ corresponding to the eigenvalue $z = 0$ are eigenstates of $\mathcal{H}_0$ corresponding to the eigenvalue $E_0 = 0$. Let us to take these states to be the normalized version of states (140-141), when $z_0 = 0$, that is

$$\begin{aligned} |E_0, -\rangle &= \left(\sqrt{1 + (\sqrt{\gamma_0})^{-1}((\sqrt{\gamma_0})^{-1})^\ddagger \sqrt{\delta_0}(\sqrt{\delta_0})^\ddagger}\right)^{-1} \\ &\quad \mathbb{D}(\sqrt{\gamma_0 \delta_0}) \left[|0; -\rangle - (\sqrt{\gamma_0})^{-1} \sqrt{\delta_0}\, |0; +\rangle\right] \end{aligned} \tag{180}$$

and

$$\begin{aligned} |E_0, +\rangle &= \left(\sqrt{1 + (\sqrt{\delta_0})^{-1}((\sqrt{\delta_0})^{-1})^\ddagger \sqrt{\gamma_0}(\sqrt{\gamma_0})^\ddagger}\right)^{-1} \\ &\quad \mathbb{D}(\sqrt{\gamma_0 \delta_0}) \left[|0; +\rangle - (\sqrt{\delta_0})^{-1} \sqrt{\gamma_0}\, |0; -\rangle\right]. \end{aligned} \tag{181}$$

Thus, from (153) and (178), we deduce that a class of orthonormalized eigenstates of $\mathcal{H}_0$ corresponding to the eigenvalue $E_n^j = n$ is given by ($n = 0, 1, 2, \ldots; j = -, +$)

$$|E_n, j\rangle = \frac{(\mathcal{A}_0^\ddagger)^n}{\sqrt{n!}} |E_0, j\rangle. \tag{182}$$

Moreover, a class of normalized coherent states for this generalized harmonic system which are eigenstates of $\mathcal{A}_0$ corresponding to the eigenvalue $z = z_0$ is easily constructed as[2]

$$|z_0, j\rangle = \exp\left(z_0 \mathcal{A}_0^\ddagger - z_0^\ddagger \mathcal{A}_0\right) |E_0, j\rangle. \tag{183}$$

These coherent states are obtained from those of equations (140-141) by acting with the following superunitary transformation

$$\mathcal{U}(z_0; \gamma_0, \delta_0) = \exp\left[z_0(\gamma_0^\ddagger b^\dagger + \delta_0^\ddagger b) - z_0^\ddagger(\gamma_0 b + \delta_0 b^\dagger)\right]. \tag{184}$$

6 Conclusions

In this paper we have generalized the AES[8] concept to the one of SAES. We have demonstrate that the SAES associated to the H–W Lie superalgebra contain the sets of standard coherent and supercoherent states associated to the usual and supersymmetric harmonic oscillator systems, respectively[2, 3, 14, 21]. Also, these SAES contain both the standard squeezed and supersqueezed states[18, 19] and the supersqueezed states associated to the spin–$\frac{1}{2}$ representation of the AES of the $h(2) \oplus su(2)$ algebra[2]. Let us mention that the introduction of Grassmann coefficients in the linear combination of the superalgebra generators helps us to understand the role played by the c-numbers (even Grassmann numbers) and d-numbers (odd Grassmann numbers) interaction coefficients, in the mentioned literature. Moreover, from the idea of giving to the SAES the interpretation of an operator associated to a physical system, we have constructed some classes of superHermitian and η–pseudo–superHermitian Hamiltonians[12, 17], isospectral to the standard harmonic oscillator hamiltonian. We have found their physical eigenstates and their associated supercoherent states. In this respect, we see that the SAES concept constitute an alternative and unified approach for the construction of generalized coherent and supercoherent and also squeezed and supersqueezed states for a given quantum system.

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A Notations and conventions

In this appendix we want to fix the notations and conventions used in this work. They concern principally the concepts of Grassmann algebra, Lie superalgebra and their representations, superHermitian and superunitary operators, super Lie algebra and linear Lie supergroup.

Let us remind that a complex **Grassmann algebra**, $\mathbb{C}B_L$, is a linear vector space over the field of complex numbers, associative and Z_2 graded. It may thus be decomposed into $\mathbb{C}B_{L_0} + \mathbb{C}B_{L_1}$, where the even space $\mathbb{C}B_{L_0}$ is generated by the set of 2^{L-1} linearly independent generators $\mathcal{E}_\mu$ of even level and the odd space $\mathbb{C}B_{L_1}$ is generated by the set of 2^{L-1} linearly independent generators $\mathcal{E}_\mu$ of odd level. Here, the index μ represents either the empty set ϕ or the set $(j_1, j_2, \ldots, j_{N(\mu)})$ of $N(\mu)$ integer numbers such that $1 \leq j_1 < j_2 \cdots < j_{N(\mu)} \leq L$. $N(\mu)$ is the level of the generator $\mathcal{E}_\mu$. The identity of the algebra is $\mathcal{E}_\phi = \mathbf{1}$ and $\mathcal{E}_\mu = \mathcal{E}_{j_1}\mathcal{E}_{j_2}\cdots\mathcal{E}_{j_{N(\mu)}}$ is the ordered product of $N(\mu)$ odd generators of level 1 taken among the set of basic generators $\{\mathcal{E}_j, j = 1, 2, \ldots, L\}$. The product of these generators is associative and antisymmetric. Moreover, any non zero product of the type $\mathcal{E}_{j_1}\mathcal{E}_{j_2}\ldots\mathcal{E}_{j_r}$ of r generators is linearly independent of the products containing less than r generators and we have $\mathcal{E}_\phi\mathcal{E}_j = \mathcal{E}_j\mathcal{E}_\phi = \mathcal{E}_j$, $\forall j = 1, 2, \ldots, L$. The graduation is introduced by defining the degree of $\mathcal{E}_\mu$, that is

$$\deg \mathcal{E}_\mu = (-1)^{N(\mu)}, \tag{185}$$

with $N(\phi) = 0$.

Any element $B \in \mathbb{C}B_L$ can be written either in the form

$$B = \sum_\mu B_\mu \mathcal{E}_\mu, \qquad B_\mu \in \mathbb{C}, \tag{186}$$

or as the sum of its even part B_0 and its odd part B_1, i.e., $B = B_0 + B_1$ with

$$B_0 = \sum_{\text{even } N(\mu)} B_\mu \mathcal{E}_\mu, \qquad B_1 = \sum_{\text{odd } N(\mu)} B_\mu \mathcal{E}_\mu. \tag{187}$$

We also deduce the graded operations for the Grassmman algebra, i.e., for all $B_0, Z_0 \in \mathbb{C}B_{L_0}$, $B_1, Z_1 \in \mathbb{C}B_{L_1}$, we have

$$B_0 Z_0 = Z_0 B_0 \in \mathbb{C}B_{L_0}, \qquad B_0 Z_1 = Z_1 B_0 \in \mathbb{C}B_{L_1}, \qquad B_1 Z_1 = -Z_1 B_1 \in \mathbb{C}B_{L_0}. \tag{188}$$

In particular, for all $B = B_0 + B_1 \in \mathbb{C}B_L$ and $Z_1 \in \mathbb{C}B_{L_1}$,

$$BZ_1 = Z_1 B^*, \qquad Z_1 B = B^* Z_1, \tag{189}$$

where

$$B^* = B_0 - B_1, \tag{190}$$

is the **conjugate** of B. The product of any two elements of the algebra, B and B', corresponds to

$$BB' = \sum_\mu \sum_{\mu'} B_\mu B'_{\mu'} (\mathcal{E}_\mu \mathcal{E}_{\mu'}), \tag{191}$$

with

$$\mathcal{E}_\mu \mathcal{E}_{\mu'} = \pm \mathcal{E}_\nu, \quad \text{where} \quad N(\nu) = N(\mu) + N(\mu'), \tag{192}$$

when neither of the indices in the sets represented by μ and μ' are repeated, and $\mathcal{E}_\mu \mathcal{E}_{\mu'} = 0$, when at least one of the index in the set represented by μ and μ' is repeated. The sign $\pm$ in (192) is determined by using the antisymmetric property of the basic generators $\mathcal{E}_j$ when reordering the their product.

The identity component of the element B, usually called the body, is denoted by $\epsilon(B) = B_\phi \in \mathbb{C}$, whereas the nilpotent quantity $s(B) = B - B_\phi \mathcal{E}_\phi$, defines the soul of B.

With respect to the **complex conjugate** of the element $B \in \mathbb{C}B_L$, we follow the conventions of Cornwell[11] and thus write

$$\bar{B} = \sum_\mu \bar{B}_\mu \mathcal{E}_\mu, \tag{193}$$

i.e., the basis elements $\mathcal{E}_\mu$ are considered as the real Grassmann numbers. Also, the **adjoint** of B is defined by the relation

$$B^\ddagger = \sum_\mu \bar{B}_\mu \mathcal{E}_\mu^\ddagger, \tag{194}$$

where

$$\mathcal{E}_\mu^\ddagger = \begin{cases} \mathcal{E}_\mu, & \text{if } N(\mu) \text{ is even} \\ -i\mathcal{E}_\mu, & \text{if } N(\mu) \text{ is odd.} \end{cases} \tag{195}$$

This adjoint operation have the same properties than the ones of the usual adjoint operation for complex matrices.

The inverse of a Grassmann number B, denoted by $(B)^{-1}$ is defined as

$$B(B)^{-1} = (B)^{-1}B = \epsilon_\phi = 1. \tag{196}$$

It is important to mention that B is invertible if and only if $B_\phi \neq 0$.

The integration with respect to an odd Grassmann variable, must be considered in the Berezin sense[5], i.e., if $\eta \in \mathbb{C}B_{L_1}$, then

$$\int d\eta = 0, \qquad \int \eta d\eta = 1, \tag{197}$$

where the integration is taken over all the domain of definition of η.

Let us now recall some useful definitions and properties of Lie superalgebras, supergroups and associated representations.

Definition A.1 *A (m/n) dimensional complex Lie superalgebra $\mathcal{L}_s$, is a complex vector space, Z_2 graded with respect to a generalized Lie product, formed from the direct sum of two subspaces, the even subspace of dimension $m \geq 0$, which we denotes by $\mathcal{L}_0$, and the odd subspace* *of dimension* $n \geq 0$ *(*$m+n \geq 1$*), which we denotes by* $\mathcal{L}_1$*, such that, for all* $a, b \in \mathcal{L}_s$*, there exists a generalized Lie product (supercommutator)* $[a, b]$ *with the following properties:*

1) $[a, b] \in \mathcal{L}_s$*, for all* $a, b \in \mathcal{L}_s$*;*

2) for all $a, b, c \in \mathcal{L}_s$ *and any complex (real) numbers* α *and* β*,*

$$[\alpha a + \beta b, c] = \alpha[a, c] + \beta[b, c]; \tag{198}$$

3) if a *and* b *are homogeneous elements of* $\mathcal{L}_s$ *then* $[a, b]$ *is also a homogeneous element of* $\mathcal{L}_s$ *whose degree is* $(\deg\, a + \deg\, b)$ *mod 2; that is,* $[a, b]$ *is odd if either* a *or* b *is odd, but* $[a, b]$ *is even if* a *and* b *are both even or if* a *and* b *are both odd;*

4) for any homogeneous elements a *and* b *of* $\mathcal{L}_s$

$$[b, a] = -(-1)^{(dega)(debb)}[a, b]; \tag{199}$$

5) for any three homogeneous elements a, b *and* c *of* $\mathcal{L}_s$*, we have the generalized Jacobi identity:*

$$[a, [b, c]](-1)^{(\deg\, a)(\deg\, c)} + [b, [c, a]](-1)^{(\deg\, b)(\deg\, a)} + [a, [b, c]](-1)^{(\deg\, c)(\deg\, b)} = 0. \tag{200}$$

We notice that the even subspace $\mathcal{L}_0$, is an ordinary complex Lie algebra whereas the odd subspace, $\mathcal{L}_1$, is a carrier space for a representation of a Lie algebra $\mathcal{L}_0$.

Just as an ordinary Lie algebra can, in general, be represented by a set of complex matrices a Lie superalgebra can also be represented, in general, by a set of complex matrices. Nevertheless, the graded character of a superalgebra implies certain special conditions for the structure of these matrices.

Definition A.2 *Suppose that for every* $a \in \mathcal{L}_s$*, there exists a matrix* $\Gamma(a)$ *from the set of complex matrices partitioned in the form* $(d_0/d_1) \times (d_0/d_1)$*, that we denotes by* $M(d_0/d_1; \mathbb{C})$*, such that*

1) for all $a, b \in \mathcal{L}_s$ *and* α, β *of the field of* $\mathcal{L}_s$*,*

$$\Gamma(\alpha a + \beta b) = \alpha\Gamma(a) + \beta\Gamma(b); \tag{201}$$

2) for all $a, b \in \mathcal{L}_s$,

$$\Gamma([a,b]) = [\Gamma(a), \Gamma(b)]; \tag{202}$$

3) if $a \in \mathcal{L}_0$, the even subspace of $\mathcal{L}_s$, then $\Gamma(a)$ a la forme

$$\Gamma(a) = \begin{pmatrix} \Gamma_{00}(a) & 0 \\ 0 & \Gamma_{11}(a) \end{pmatrix}, \tag{203}$$

where $\Gamma_{00}(a)$ and $\Gamma_{11}(a)$ are $d_0 \times d_0$ and $d_1 \times d_1$ dimensional submatrices respectively; and if $a \in \mathcal{L}_1$, the odd subspace of $\mathcal{L}_s$, then $\Gamma(a)$ has the form

$$\Gamma(a) = \begin{pmatrix} 0 & \Gamma_{01}(a) \\ \Gamma_{10}(a) & 0 \end{pmatrix}, \tag{204}$$

where $\Gamma_{01}(a)$ and $\Gamma_{10}(a)$ are $d_0 \times d_1$ and $d_1 \times d_0$ dimensional submatrices respectively. Then these matrices $\Gamma(a)$ are said to form a (d_0/d_1)–dimensional **graded representation** *of $\mathcal{L}_s$.*

Let $\mathcal{L}_s$ be a (m/n) dimensional complex Lie superalgebra with even basis elements $a_1, a_2, \ldots, a_m$ and odd basis elements $a_{m+1}, a_{m+2}, \ldots, a_{m+n}$, represented by the set of matrices $\Gamma(a_k), k = 1, 2, \ldots, m+n$. To each matrix $\Phi(a_k)$, we can associate a linear operator $\Phi(a_k)$ acting on the carrier space $\mathcal{W}$ of the representation. This space is a $(d_0 + d_1)$ inner product vector space expanded by a basis formed by the set of even vectors $\{|w_j\rangle\}_{j=0}^{d_0}$ and the set of odd vectors $\{|w_j\rangle\}_{j=d_0+1}^{d_0+d_1}$ and this action is defined by the relation

$$\Phi(a_k)|w_j\rangle = \sum_{i=1}^{d_0+d_1} (\Gamma(a_k))_{ij}|w_i\rangle. \tag{205}$$

Then $\mathcal{L}_s$ can also be represented by set of even operators $\Phi(a_k)$ $(k = 1, 2, \ldots, m)$ and the set of odd operators $\Phi(a_k)$ $(k = m+1, m+2, \ldots, m+n)$, verifying the same super-commutation relations as the basis elements a_k $(k = 1, 2, \ldots, m+n)$.

Let $\mathcal{X}$ to be a polynomial function of the $\mathcal{L}_s$ superalgebra generators, with complex Grassmannian coefficients. We say that $\mathcal{X}$ is a **superHermitian** (anti–superHermitian) operator if $\mathcal{X} = \mathcal{X}^\ddagger$ ($\mathcal{X} = -\mathcal{X}^\ddagger$). In particular, if $\mathcal{X}$ is a complex Grassmannian linear combination of the $\mathcal{L}_s$ superalgebra generators, i.e.,

$$\mathcal{X} = \sum_{j=1}^{m} C^j \Phi(a_j) + \sum_{k=1}^{n} D^k \Phi(a_{m+k}), \tag{206}$$

where $C^j \in \mathbb{C}B_L$ $(j = 1, 2 \ldots, m)$ and $D^k \in \mathbb{C}B_L$ $(k = 1, 2, \ldots, n)$ then

$$\mathcal{X}^\ddagger = \sum_{j=1}^{m} (\Phi(a_j))^\dagger (C^j)^\ddagger + \sum_{k=1}^{n} (\Phi(a_{m+k}))^\dagger (D^k)^\ddagger, \tag{207}$$

where the $\dagger$ symbol is reserved for the usual adjoint operation. We say that a general $\mathcal{U}$ operator is **superunitary** if $\mathcal{U}\mathcal{U}^\ddagger = \mathcal{U}^\ddagger \mathcal{U} = I$, where I is the identity operator. In particular, if $\mathcal{X}$ is an anti–superHermitian operator, then $\mathcal{U} = e^{\mathcal{X}}$ is a superunitary operator.

If for $j = 1, 2, \ldots, m$ and every element $\mathcal{E}_\mu$ of $\mathbb{C}B_L$, we define the even operators

$$M_\mu^j = \mathcal{E}_\mu \Phi(a_j) \tag{208}$$

and for $k = 1, 2, \ldots, n$ and every odd element $\mathcal{E}_\nu$ of $\mathbb{C}B_L$, we define the even operators

$$N_\nu^k = \mathcal{E}_\nu \Phi(a_{m+k}), \tag{209}$$

then the set of $(m+n)2^{L-1}$ operators defined by the equation (208) and (209) form a basis of a $(m+n)2^{L-1}$ dimensional real Lie algebra, whose Lie product is given by the usual commutator induced by the generalized Lie product of $\mathcal{L}_s$. This real Lie algebra is denoted by $\mathcal{L}_s(\mathbb{C}B_L)$ and is called a **super Lie algebra**. A general element M of this super Lie algebra writes

$$M = \sum_{j=1}^{m} \sum_{\text{even } \mu} X_\mu^j M_\mu^j + \sum_{k=1}^{n} \sum_{\text{odd } \nu} \Theta_\nu^k N_\nu^k, \tag{210}$$

where X_μ^j and Θ_ν^k are real parameters. Also we can write this element in the form

$$M = \sum_{j=1}^{m} X^j M^j + \sum_{k=1}^{n} \Theta^k N^k, \tag{211}$$

where $X^j = \sum_{\text{even } \mu} X_\mu^j \mathcal{E}_\mu \in \mathbb{R}B_{L_0}$, $\Theta^k = \sum_{\text{odd } \nu} \Theta_\nu^k \mathcal{E}_\nu \in \mathbb{R}B_{L_1}$ and

$$M^j = \mathcal{E}_\phi \Phi(a_j), \qquad N^k = \mathcal{E}_\phi \Phi(a_{m+k}). \tag{212}$$

Let us end this Appendix by giving a method of construction of a linear Lie **supergroup**[22]. If $\mathcal{L}_s(\mathbb{C}B_L)$ is a real super Lie algebra whose basis elements are defined by (208) and (209), then every linear Lie group whose associated real Lie super algebra is given by $\mathcal{L}_s(\mathbb{C}B_L)$ is a (m/n) linear Lie supergroup, which we denote by $\mathcal{G}_s(\mathbb{C}B_L)$. The elements near the identity can be parametrized by

$$\mathbf{G}(\mathbf{X}; \boldsymbol{\Theta}) = \exp\{M\} = \exp\left\{\sum_{j=1}^{m} X^j M^j + \sum_{k=1}^{n} \Theta^k N^k\right\}. \tag{213}$$

B Solving $[a + \beta a^\dagger + \gamma b + \delta b^\dagger]|\psi\rangle = z|\psi\rangle$

In this appendix we will solve the eigenvalue equation (95). We will do it in two steps. Firstly, we will solve the eigenvalue equation (118) and express its solutions in terms of a generalized supersqueeze operator acting on the supercoherent states $e^z|0;\pm\rangle$. This supersqueeze operator is used to reduce the eigenvalue equation (95) to a simpler one, see section 4.3, that is to the eigenvalue equation (143). Finally, we will solve the eigenvalue equation (143).

B.1 The SAES of $a + \gamma b + \delta b^\dagger$

Let us solve the eigenvalue equation (118). The solution is assumed on the type (97) and by inserting it into (118), then using the usual properties of the operators and the states $\{|n;\pm\rangle\}$, we get the system ($n = 0, 1, 2\ldots$)

$$\sqrt{n+1}C_{n+1} + \gamma D_n^* = zC_n, \tag{214}$$

$$\sqrt{n+1}D_{n+1} + \delta C_n^* = zD_n. \tag{215}$$

Let us notice the symmetric form of this system. Proceeding by iteration we can express the C_n and D_n coefficients in terms of the arbitrary Grassmann constants C_0 and D_0, that is ($n = 1, 2, \ldots$)

$$\begin{aligned} C_n &= \frac{1}{\sqrt{n!}}\Big\{z^n C_0 - \sum_{k_1=0}^{(n-1)} z^{(n-1-k_1)}\gamma(z^*)^{k_1} D_0^* + \sum_{k_1=0}^{(n-2)}\sum_{k_2=0}^{(n-2-k_1)} z^{(n-2-k_1-k_2)}\gamma(z^*)^{k_2}\delta^* z^{k_1} C_0 \\ &- \sum_{k_1=0}^{(n-3)}\sum_{k_2=0}^{(n-3-k_1)}\sum_{k_3=0}^{(n-3-k_1-k_2)} z^{(n-3-k_1-k_2-k_3)}\gamma(z^*)^{k_3}\delta^* z^{k_2}\gamma(z^*)^{k_1} D_0^* + \ldots \\ &+ (-1)^n(\gamma\delta^*)^{[\frac{n}{2}]}\gamma^{(n-2[\frac{n}{2}])}F_{n-2[\frac{n}{2}]}\Big\}, \end{aligned} \tag{216}$$

and

$$\begin{aligned} D_n &= \frac{1}{\sqrt{n!}}\Big\{z^n D_0 - \sum_{k_1=0}^{(n-1)} z^{(n-1-k_1)}\delta(z^*)^{k_1} C_0^* + \sum_{k_1=0}^{(n-2)}\sum_{k_2=0}^{(n-2-k_1)} z^{(n-2-k_1-k_2)}\delta(z^*)^{k_2}\gamma^* z^{k_1} D_0 \\ &- \sum_{k_1=0}^{(n-3)}\sum_{k_2=0}^{(n-3-k_1)}\sum_{k_3=0}^{(n-3-k_1-k_2)} z^{(n-3-k_1-k_2-k_3)}\delta(z^*)^{k_3}\gamma^* z^{k_2}\delta(z^*)^{k_1} C_0^* + \ldots \\ &+ (-1)^n(\delta\gamma^*)^{[\frac{n}{2}]}\delta^{(n-2[\frac{n}{2}])}G_{n-2[\frac{n}{2}]}\Big\}, \end{aligned} \tag{217}$$

where $\left[\frac{n}{2}\right]$ represents the entire part of $\frac{n}{2}$ and $F_0 = C_0$, $F_1 = D_0^*$, $G_0 = D_0$, $G_1 = C_0^*$. Here we need to calculate the multiple summation. By expressing z as a sum of their even and odd parts, $z = z_0 + z_1$, we get for example, $(\ell = 1, 2, \ldots, n)$

$$\begin{aligned}
&\sum_{k_1=0}^{(n-\ell)} \sum_{k_2=0}^{(n-\ell-k_1)} \cdots \sum_{k_\ell=0}^{(n-\ell-k_1-k_2-\ldots-k_{\ell-1})} z^{(n-\ell-k_1-k_2-\ldots-k_\ell)} \gamma (z^*)^{k_\ell} \delta^* z^{k_{\ell-1}} \gamma (z^*)^{k_{\ell-2}} \delta^* \cdots \\
&= \frac{n!}{(n-\ell)!\ell!} \Bigg\{ \overbrace{(\gamma\delta^*\gamma\delta^*\cdots)}^{\ell \text{ factors}} z_0 \\
&+ \frac{(n-\ell)}{\ell+1} \sum_{j=0}^{\ell} (-1)^{j+\ell} \overbrace{(\gamma\delta^*\gamma\delta^*\cdots)}^{(\ell-j) \text{ factors}} z_1 \overbrace{(\cdots\gamma\delta^*\gamma\cdots)}^{j \text{ factors}} \Bigg\} z_0^{(n-\ell-1)} \\
&= \mathcal{O}_{z_0}(\ell, \gamma, \delta^*, z_1) z^n, \qquad (218)
\end{aligned}$$

where $\mathcal{O}_{z_0}$ is the differential operator

$$\begin{aligned}
\mathcal{O}_{z_0}(\ell, \gamma, \delta^*, z_1) &= \frac{1}{\ell!} \Bigg\{ \overbrace{(\gamma\delta^*\gamma\delta^*\cdots)}^{\ell \text{ factors}} \left(\frac{\partial^\ell}{\partial z_0^\ell} - z_1 \frac{\partial^{\ell+1}}{\partial z_0^{\ell+1}} \right) \\
&+ \frac{1}{\ell+1} \sum_{j=0}^{\ell} (-1)^{j+\ell} \overbrace{(\gamma\delta^*\gamma\delta^*\cdots)}^{(\ell-j) \text{ factors}} z_1 \overbrace{(\cdots\gamma\delta^*\gamma\cdots)}^{j \text{ factors}} \frac{\partial^{\ell+1}}{\partial z_0^{\ell+1}} \Bigg\}, \qquad (219)
\end{aligned}$$

which is also defined for $\ell = 0$, in fact $\mathcal{O}_{z_0}(0, \gamma, \delta^*, z_1) = 1$. By inserting (218) into (216) and (217), we get the compact form of C_n and D_n coefficients, that is

$$C_n = \sum_{\ell=0}^{n} (-1)^\ell \mathcal{O}_{z_0}(\ell, \gamma, \delta^*, z_1) \frac{z^n}{\sqrt{n!}} F_{\ell-2[\frac{\ell}{2}]} \qquad (220)$$

and

$$D_n = \sum_{\ell=0}^{n} (-1)^\ell \mathcal{O}_{z_0}(\ell, \delta, \gamma^*, z_1) \frac{z^n}{\sqrt{n!}} G_{\ell-2[\frac{\ell}{2}]}. \qquad (221)$$

By inserting (220) and (221) into (97) and then separating the terms to multiply arbitrary constants C_0 and D_0, we obtain two independent solutions for the eigenvalue equation (118):

$$|\psi; -\rangle = \Big[\sum_{n=0}^{\infty} \sum_{\ell \text{ even}}^{2[n/2]} \mathcal{O}_{z_0}(\ell, \gamma, \delta^*, z_1) \frac{z^n}{\sqrt{n!}} |n; -\rangle - \sum_{n=1}^{\infty} \sum_{\ell \text{ odd}}^{2[(n+1)/2]-1} \mathcal{O}_{z_0}(\ell, \delta, \gamma^*, z_1) \frac{z^n}{\sqrt{n!}} |n; +\rangle \Big] C_0 \qquad (222)$$

and

$$|\psi; +\rangle = \Big[\sum_{n=0}^{\infty} \sum_{\ell \text{ even}}^{2[n/2]} \mathcal{O}_{z_0}(\ell, \delta, \gamma^*, z_1) \frac{z^n}{\sqrt{n!}} |n; +\rangle - \sum_{n=1}^{\infty} \sum_{\ell \text{ odd}}^{2[(n+1)/2]-1} \mathcal{O}_{z_0}(\ell, \gamma, \delta^*, z_1) \frac{z^n}{\sqrt{n!}} |n; -\rangle \Big] D_0^*. \qquad (223)$$

As $\mathcal{O}_{z_0}(\ell, \gamma, \delta^*, z_1) z^n = 0$, when $\ell > n$, we can spread out the sum on ℓ index up to infinity and then place it out of the sum corresponding to the n index. In this way, we can add up on the n

index and express (222) and (223) on the form

$$|\psi;-\rangle = \Big[\sum_{\ell \text{ even}}^{\infty} \mathcal{O}_{z_0}(\ell,\gamma,\delta^*,z_1)e^{za^\dagger}|0;-\rangle - \sum_{\ell \text{ odd}}^{\infty} \mathcal{O}_{z_0}(\ell,\delta,\gamma^*,z_1)e^{za^\dagger}|0;+\rangle\Big]C_0 \tag{224}$$

and

$$|\psi;+\rangle = \Big[\sum_{\ell \text{ even}}^{\infty} \mathcal{O}_{z_0}(\ell,\delta,\gamma^*,z_1)e^{za^\dagger}|0;+\rangle - \sum_{\ell \text{ odd}}^{\infty} \mathcal{O}_{z_0}(\ell,\gamma,\delta^*,z_1)e^{za^\dagger}|0;-\rangle\Big]D_0^*, \tag{225}$$

respectively. Finally, using the fact that $\frac{\partial^\ell}{\partial z_0^\ell}e^{za^\dagger} = (a^\dagger)^\ell e^{za^\dagger}$, we get the generalized super-squeezed states (119) and (120).

B.2 The SAES of $a + \hat{\beta}_1 a^\dagger + \gamma_0 b + \delta_0 b^\dagger$

Let us solve the eigenvalue equation (143) by taking $|\varphi\rangle$ again on the form (97). By inserting it into (143), and proceeding as in the above sections, we get the algebraic system ($n = 1, 2\ldots$)

$$\sqrt{n+1}C_{n+1} + \gamma_0 D_n^* + \sqrt{n}\hat{\hat{\beta}}_1 C_{n-1} = zC_n, \tag{226}$$

$$\sqrt{n+1}D_{n+1} + \delta_0 C_n^* + \sqrt{n}\hat{\beta}_1 D_{n-1} = zD_n, \tag{227}$$

together with

$$C_1 = zC_0 - \gamma_0 D_0^*, \tag{228}$$

$$D_1 = zD_0 - \delta_0 C_0^*. \tag{229}$$

Again, we notice the symmetric form of this algebraic system. Proceeding by iteration, we can express the C_n and D_n coefficients in terms of the arbitrary Grassmann constants C_0 and D_0, we get ($n = 2, 3, \ldots$)

$$\begin{aligned} C_n \;=\; & \tilde{C}_n - \frac{1}{\sqrt{n!}}\Big[\sum_{\text{even } \ell=2}^{2[\frac{n}{2}]}\sum_{k_1=0}^{(n-\ell)}\sum_{k_2=0}^{(n-\ell-r_1)}\sum_{k_3=0}^{(n-\ell-r_2)}\cdots\sum_{k_{\ell-1}=0}^{(n-\ell-r_{\ell-2})} \\ & \sum_{j=1}^{\frac{\ell}{2}}(k_{2j-1}+1)z^{(n-\ell-r_{\ell-1})}(z^*)^{k_{\ell-1}}z^{k_{\ell-2}}\cdots(z^*)^{k_1}\left(\sqrt{\gamma_0\delta_0}\right)^{\ell-2}\hat{\beta}_1\Big]C_0 \\ & + \frac{1}{\sqrt{n!}}\Big[\sum_{\text{odd } \ell=3}^{2[\frac{n+1}{2}]-1}\sum_{k_1=0}^{(n-\ell)}\sum_{k_2=0}^{(n-\ell-r_1)}\sum_{k_3=0}^{(n-\ell-r_2)}\cdots\sum_{k_{\ell-1}=0}^{(n-\ell-r_{\ell-2})} \\ & \sum_{j=1}^{[\frac{\ell}{2}]}(k_{2j}+1)z^{(n-\ell-r_{\ell-1})}(z^*)^{k_{\ell-1}}z^{k_{\ell-2}}\cdots z^{k_1}\left(\sqrt{\gamma_0\delta_0}\right)^{\ell-3}\gamma_0\hat{\beta}_1\Big]D_0^*, \end{aligned} \tag{230}$$

$$D_n = \tilde{D}_n - \frac{1}{\sqrt{n!}}\Bigg[\sum_{\text{even } \ell=2}^{2[\frac{n}{2}]}\sum_{k_1=0}^{(n-\ell)}\sum_{k_2=0}^{(n-\ell-r_1)}\sum_{k_3=0}^{(n-\ell-r_2)}\cdots\sum_{k_{\ell-1}=0}^{(n-\ell-r_{\ell-2})}$$
$$\sum_{j=1}^{\frac{\ell}{2}}(k_{2j-1}+1)z^{(n-\ell-r_{\ell-1})}(z^*)^{k_{\ell-1}}z^{k_{\ell-2}}\cdots(z^*)^{k_1}\left(\sqrt{\gamma_0\delta_0}\right)^{\ell-2}\hat{\beta}_1\Bigg]D_0$$
$$+\frac{1}{\sqrt{n!}}\Bigg[\sum_{\text{odd } \ell=3}^{2[\frac{n+1}{2}]-1}\sum_{k_1=0}^{(n-\ell)}\sum_{k_2=0}^{(n-\ell-r_1)}\sum_{k_3=0}^{(n-\ell-r_2)}\cdots\sum_{k_{\ell-1}=0}^{(n-\ell-r_{\ell-2})}$$
$$\sum_{j=1}^{[\frac{\ell}{2}]}(k_{2j}+1)z^{(n-\ell-r_{\ell-1})}(z^*)^{k_{\ell-1}}z^{k_{\ell-2}}\cdots z^{k_1}\left(\sqrt{\gamma_0\delta_0}\right)^{\ell-3}\delta_0\hat{\beta}_1\Bigg]C_0^*, \tag{231}$$

where

$$r_\ell = \sum_{j=1}^{\ell} k_j \tag{232}$$

and, in accordance with Eqs. (220) and (221),

$$\tilde{C}_n = \sum_{\ell=0}^{n}(-1)^\ell \mathcal{O}_{z_0}(\ell,\gamma_0,\delta_0,z_1)\frac{z^n}{\sqrt{n!}}F_{\ell-2[\frac{\ell}{2}]} \tag{233}$$

and

$$\tilde{D}_n = \sum_{\ell=0}^{n}(-1)^\ell \mathcal{O}_{z_0}(\ell,\delta_0,\gamma_0,z_1)\frac{z^n}{\sqrt{n!}}G_{\ell-2[\frac{\ell}{2}]}. \tag{234}$$

Using the fact that for ℓ even, we have

$$z^{(n-\ell-r_{\ell-1})}(z^*)^{k_{\ell-1}}z^{k_{\ell-2}}\cdots(z^*)^{k_1} = z_0^{(n-\ell)} + [(n-\ell)-2(k_1+k_3+\ldots k_{l-1})]z_0^{(n-\ell-1)}z_1, \tag{235}$$

for ℓ odd, we have

$$z^{(n-\ell-r_{\ell-1})}(z^*)^{k_{\ell-1}}z^{k_{\ell-2}}\cdots z^{k_1} = z_0^{(n-\ell)} + [(n-\ell)-2(k_2+k_4+\ldots+k_{l-1})]z_0^{(n-\ell-1)}z_1, \tag{236}$$

and that

$$\sum_{k_1=0}^{(n-\ell)}\sum_{k_2=0}^{(n-\ell-r_1)}\sum_{k_2=0}^{(n-\ell-r_2)}\cdots\sum_{k_{\ell-1}=0}^{(n-\ell-r_{\ell-2})}\Lambda_\ell(k) \tag{237}$$

is equal to

$$\begin{cases} \frac{(n-1)!}{(n-\ell)!(\ell-1)!}, & \text{if } \Lambda_\ell(k)=1 \text{ and } \ell\geq 2, \\ \frac{(n-1)!}{2(n-\ell-1)!(\ell-1)!}, & \text{if } \Lambda_\ell(k)=(k_1+k_3+\ldots+k_{\ell-1}) \\ & \text{and } \ell=2,4,\ldots, \\ \frac{\ell(n-1)!}{2(n-\ell-1)!(\ell+1)!}\left[(n-\ell)+\frac{\ell}{2}(n-\ell+1)\right], & \text{if } \Lambda_\ell(k)=(k_1+k_3+\ldots+k_{\ell-1})^2 \\ & \text{and } \ell=2,4,\ldots, \\ \frac{(\ell-1)(n-1)!}{2(n-\ell-1)!\ell!}, & \text{if } \Lambda_\ell(k)=(k_2+k_4+\ldots+k_{\ell-1}) \\ & \text{and } \ell=3,5,\ldots, \\ \frac{(\ell-1)(n-\ell+1)(n-1)!}{4(n-\ell-1)!\ell!}, & \text{if } \Lambda_\ell(k)=(k_2+k_4+\ldots+k_{\ell-1})^2 \\ & \text{and } \ell=3,5,\ldots, \end{cases} \tag{238}$$

and after some manipulations, we can reduce (230) and (231) to

$$\begin{aligned} C_n &= \tilde{C}_n - \frac{\hat{\beta}_1}{2\sqrt{n!}}\Bigg[\sum_{\text{even } \ell=2}^{2[\frac{n}{2}]} \frac{n!}{(n-\ell)!(\ell-1)!}\left(z_0^{(n-\ell)} + \frac{(n-\ell)}{(\ell+1)} z_0^{(n-\ell-1)} z_1\right)\left(\sqrt{\gamma_0\delta_0}\right)^{\ell-2}\Bigg] C_0 \\ &+ \frac{\hat{\beta}_1}{2\sqrt{n!}}\Bigg[\sum_{\text{odd } \ell=3}^{2[\frac{n+1}{2}]-1} \frac{(\ell-1)n!}{(n-\ell)!\ell!} z_0^{(n-\ell)}\left(\sqrt{\gamma_0\delta_0}\right)^{\ell-3}\gamma_0\Bigg] D_0^* \end{aligned} \tag{239}$$

and

$$\begin{aligned} D_n &= \tilde{D}_n - \frac{\hat{\beta}_1}{2\sqrt{n!}}\Bigg[\sum_{\text{even } \ell=2}^{2[\frac{n}{2}]} \frac{n!}{(n-\ell)!(\ell-1)!}\left(z_0^{(n-\ell)} + \frac{(n-\ell)}{(\ell+1)} z_0^{(n-\ell-1)} z_1\right)\left(\sqrt{\gamma_0\delta_0}\right)^{\ell-2}\Bigg] D_0 \\ &+ \frac{\hat{\beta}_1}{2\sqrt{n!}}\Bigg[\sum_{\text{odd } \ell=3}^{2[\frac{n+1}{2}]-1} \frac{(\ell-1)n!}{(n-\ell)!\ell!} z_0^{(n-\ell)}\left(\sqrt{\gamma_0\delta_0}\right)^{\ell-3}\delta_0\Bigg] C_0^*, \end{aligned} \tag{240}$$

respectively. Then, using the fact that

$$\frac{n!}{(n-\ell)!} z_0^{n-l} = \left(\frac{\partial^\ell}{\partial z_0^\ell} - z_1 \frac{\partial^{\ell+1}}{\partial z_0^{\ell+1}}\right) z^n, \qquad \frac{n!}{(n-\ell-1)!} z_0^{n-\ell-l} z_1 = z_1 \frac{\partial^{\ell+1}}{\partial z_0^{\ell+1}} z^n, \tag{241}$$

we can write (239) and (240) in the form

$$\begin{aligned} C_n &= \tilde{C}_n - \frac{\hat{\beta}_1}{2\sqrt{n!}}\Bigg[\sum_{\text{even } \ell=2}^{2[\frac{n}{2}]} \frac{1}{(\ell-1)!}\left(\left(\frac{\partial^\ell}{\partial z_0^\ell} - z_1\frac{\partial^{\ell+1}}{\partial z_0^{\ell+1}}\right) + \frac{z_1}{(\ell+1)}\frac{\partial^{\ell+1}}{\partial z_0^{\ell+1}}\right) z^n \left(\sqrt{\gamma_0\delta_0}\right)^{\ell-2}\Bigg] C_0 \\ &+ \frac{\hat{\beta}_1}{2\sqrt{n!}}\Bigg[\sum_{\text{odd } \ell=3}^{2[\frac{n+1}{2}]-1} \frac{(\ell-1)}{\ell!}\left(\frac{\partial^\ell}{\partial z_0^\ell} - z_1\frac{\partial^{\ell+1}}{\partial z_0^{\ell+1}}\right)\left(\sqrt{\gamma_0\delta_0}\right)^{\ell-3}\gamma_0\Bigg] D_0^* \end{aligned} \tag{242}$$

$$\begin{aligned} D_n &= \tilde{D}_n - \frac{\hat{\beta}_1}{2\sqrt{n!}}\Bigg[\sum_{\text{even } \ell=2}^{2[\frac{n}{2}]} \frac{1}{(\ell-1)!}\left(\left(\frac{\partial^\ell}{\partial z_0^\ell} - z_1\frac{\partial^{\ell+1}}{\partial z_0^{\ell+1}}\right) + \frac{z_1}{(\ell+1)}\frac{\partial^{\ell+1}}{\partial z_0^{\ell+1}}\right) z^n \left(\sqrt{\gamma_0\delta_0}\right)^{\ell-2}\Bigg] D_0 \\ &+ \frac{\hat{\beta}_1}{2\sqrt{n!}}\Bigg[\sum_{\text{odd } \ell=3}^{2[\frac{n+1}{2}]-1} \frac{(\ell-1)}{\ell!}\left(\frac{\partial^\ell}{\partial z_0^\ell} - z_1\frac{\partial^{\ell+1}}{\partial z_0^{\ell+1}}\right)\left(\sqrt{\gamma_0\delta_0}\right)^{\ell-3}\delta_0\Bigg] C_0^*, \end{aligned} \tag{243}$$

respectively. We notice that, when the inverse of the product $\gamma_0\delta_0$ exist, or even if it does not exist, we can write formally these last equations in the compact form

$$\begin{aligned} C_n &= \tilde{C}_n - \frac{\hat{\beta}_1(\gamma_0\delta_0)^{-1}}{2}\Bigg[\sum_{\text{even } \ell=2}^{2[\frac{n}{2}]} \ell\, \mathcal{O}_{z_0}(\ell,\gamma_0,\delta_0,z_1)\frac{z^n}{\sqrt{n!}}\Bigg] C_0 \\ &+ \frac{\hat{\beta}_1(\gamma_0\delta_0)^{-1}}{2}\Bigg[\sum_{\text{odd } \ell=3}^{2[\frac{n+1}{2}]-1} (\ell-1)\, \mathcal{O}_{z_0}(\ell,\gamma_0,\delta_0,z_1)\frac{z^n}{\sqrt{n!}}\Bigg] D_0^* \end{aligned} \tag{244}$$

and

$$D_n = \tilde{D}_n - \frac{\hat{\beta}_1(\gamma_0\delta_0)^{-1}}{2}\left[\sum_{\text{even } \ell=2}^{2[\frac{n}{2}]} \ell\, \mathcal{O}_{z_0}(\ell,\delta_0,\gamma_0,z_1)\frac{z^n}{\sqrt{n!}}\right] D_0$$

$$+ \frac{\hat{\beta}_1(\gamma_0\delta_0)^{-1}}{2}\left[\sum_{\text{odd } \ell=3}^{2[\frac{n+1}{2}]-1} (\ell-1)\, \mathcal{O}_{z_0}(\ell,\delta_0,\gamma_0,z_1)\frac{z^n}{\sqrt{n!}}\right] C_0^*. \qquad (245)$$

Now, by inserting (244) and (245) into (97) and proceeding exactly as in section B.1, we get the two independent solutions (145) and (146).

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Chapitre 5

États propres d'algèbres pour les algèbres de Lie quantiques déformées de Heisenberg

Résumé

En partant des algèbres deformées quantiques de Heisenberg, quelques réalisations sont données en termes des opérateurs de création et d'annhilation habituels de l'oscillateur harmonique standard. Les états propres d'algèbres sont alors calculés donnant de nouvelles classes d'états cohérents et comprimés déformés. Ceux-ci sont libellés en termes des paramètres de déformation de l'algèbre et également de redéfinitions convenables de ces derniers en termes de nombres de paragrassmann. Comme application physique, le comportement du produit des dispersions des opérateurs de position et d'impulsion linéaire d'une particule sont calculés sur les états obtenus lorsque les paramètres de déformation sont petits.

Algebra eigenstates of deformed quantum Heisenberg Lie algebras

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Abstract

Starting from deformed quantum Heisenberg Lie algebras some realizations are given in terms of the usual creation and annihilation operators of the standard harmonic oscillator. Then the associated algebra eigenstates are computed and give rise to new classes of deformed coherent and squeezed states. They are parametrized by deformed algebra parameters and suitable redefinitions of them as paragrassmann numbers. As a physical application, the behavior of the product of the dispersions of position and linear momentum operators of a particle is computed in the obtained states when the parameters of deformation are small.

[redacted]

1 Introduction

The algebra eigenstates (AES) associated to a real Lie algebra have been defined as the set of eigenstates of an arbitrary complex linear combination of generators of the considered algebra [1]. The AES associated to a quantum real deformed Lie algebra can be defined in a similar way. Thus, if $A_k(q),\ k = 1, 2, \ldots, n$ denote the generators of this deformed algebra in a given representation, parametrized by the set of deformation parameters q, then the AES associated to this deformed algebra are given by the set of solutions of the eigenvalue equation

$$\sum_{k=1}^{n} \alpha_k A_k(q)|\psi\rangle = \lambda|\psi\rangle, \qquad \alpha_k, \lambda \in \mathbb{C}. \tag{1}$$

The purpose of this work is to compute the AES of the deformed quantum Heisenberg Lie algebras [2] , obtained by applying the R-matrix methods [3], and find new classes of deformed harmonic oscillator coherent and squeezed states. We will see that, these states will be new deformations of the standard coherent and squeezed states of the harmonic oscillator system and we will recover them in the limit when the deformation parameters go to zero. Let us observe that the deformed coherent states obtained by this method differ from the q–deformed coherent states [4, 5] associated to a q-deformed oscillator algebra which is not a Hopf algebra.

The paper is organized as follows. In section 2, we present a review [2] of the deformed quantum Heisenberg algebras obtained by applying systematically the R-matrix approach to three–dimensional representation of the standard Heisenberg group. We also give a physical representation of these deformed algebras in terms of the usual creation and annihilation operators associated to the standard harmonic oscillator system. In section 3, we compute the AES associated to these algebras and obtain new classes of deformed coherent and squeezed states that are true deformations of the standard coherent and squeezed states associated to the harmonic oscillator system. These states are parametrized by the deformation parameters, considered as real numbers, but also as real paragrassmann numbers. As a physical application, we compute the product of the dispersions of the position and linear momentum operators of a particle in these states when the parameters of deformation are small. We compare with the corresponding results obtained in the minimum uncertainty states [6]. Some details of calculations are presented in the Appendices A and B.

2 Deformed quantum Heisenberg algebras in the Fock representation space

In this section, we give the two types of deformed Heisenberg quantum algebras obtained by V. Hussin and A. Lauzon [2]. We also give a realization of these deformed algebras in terms of the usual creation and annihilation operators of the standard harmonic oscillator system.

2.1 Deformed Heisenberg quantum Lie algebras

The starting point is the three-dimensional matrix representation of the Heisenberg group

$$T = \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix}, \tag{2}$$

where the parameters α, β, γ and the unity 1 are considered as the generators of a commutative algebra $\mathcal{A}$, the space of linear functions of these generators, provided with a structure of Hopf algebra by tensorial multiplication:

$$\begin{aligned} \Delta 1 &= 1 \otimes 1, \\ \Delta \alpha &= 1 \otimes \alpha + \alpha \otimes 1, \\ \Delta \beta &= 1 \otimes \beta + \beta \otimes 1 + \alpha \otimes \gamma, \\ \Delta \gamma &= 1 \otimes \gamma + \gamma \otimes 1. \end{aligned} \tag{3}$$

According to this co-multiplication law, the generators of the corresponding Lie algebra, in the representation space $\mathcal{A}$, are given by

$$X_1 = \frac{\partial}{\partial \alpha}, \qquad X_2 = \frac{\partial}{\partial \beta}, \qquad X_3 = \alpha \frac{\partial}{\partial \beta} + \frac{\partial}{\partial \gamma}. \tag{4}$$

They verify the well-known commutation relations of the Heisenberg-Weyl Lie algebra, $h(2)$:

$$[X_1, X_2] = [X_2, X_3] = 0, \qquad [X_1, X_3] = X_2. \tag{5}$$

The non-deformed quantum Heisenberg Lie algebra corresponds to the dual space of $\mathcal{A}$. The action of their generators, A, B, C, on an arbitrary element $\mathcal{P} = \alpha^l \beta^m \gamma^n$ of $\mathcal{A}$, is given by

$$(A, \alpha^l \beta^m \gamma^n) = \Big(X_1 \mathcal{P}\Big)_{\{\alpha=\beta=\gamma=0\}} = \delta_{l1} \delta_{m0} \delta_{n0}, \tag{6}$$

$$(B, \alpha^l\beta^m\gamma^n) = \Big(X_2\mathcal{P}\Big)_{\{\alpha=\beta=\gamma=0\}} = \delta_{l0}\delta_{m1}\delta_{n0}, \tag{7}$$

$$(C, \alpha^l\beta^m\gamma^n) = \Big(X_3\mathcal{P}\Big)_{\{\alpha=\beta=\gamma=0\}} = \delta_{l0}\delta_{m0}\delta_{n1}. \tag{8}$$

The action of the product of two of these generators, on an arbitrary element $\mathcal{P}$, is computed with the help of the homomorphism property of the co-multiplication. For example,

$$(AB, \mathcal{P}) = (A \otimes B, \Delta\mathcal{P}) = \sum_{(c)}(A \otimes B, \mathcal{P}^{(c)} \otimes \mathcal{P}^{(c)}) = \sum_{(c)}(A, \mathcal{P}^{(c)})(B, \mathcal{P}^{(c)}), \tag{9}$$

where, in the usual notation, $\mathcal{P}^{(c)}$ represents the generic elements of $\mathcal{A}$ generated on the co-multiplication action. Then, the commutator of this generators is computed from

$$([A, B], \mathcal{P}) = (A \otimes B - B \otimes A, \Delta\mathcal{P}). \tag{10}$$

They satisfy the same commutation relations as (5), i.e.,

$$[A, B] = [B, C] = 0, \qquad [A, C] = B. \tag{11}$$

To obtain the possible deformed Heisenberg Hopf algebras, the well-known R–matrix method is applied [3]. It has been show that, in such a case, R satisfies a weak version of the Quantum Yang-Baxter equation [7]. We thus get the non-commutative algebra $\tilde{\mathcal{A}}_{x,z,p,q}$:

$$(\alpha\gamma - \gamma\alpha) = 2(x\alpha + z\gamma), \tag{12}$$

$$(\beta\alpha - \alpha\beta) = -x\alpha^2 - 2z\beta + p\alpha, \tag{13}$$

$$(\beta\gamma - \gamma\beta), = z\gamma^2 + 2x\beta + q\gamma, \tag{14}$$

where x, z, p and q are real deformation parameters. The corresponding deformed quantum Heisenberg Lie algebras are obtained by duality, according the relations (6–10).

In the case when $x = 0$ and $z \neq 0$ (the case $x \neq 0$ and $z = 0$ is similar), we have $p = q \neq 0$ and the two parameters deformed quantum Heisenberg Lie algebra, denoted by $\mathcal{U}_{z,p}(h(2))$, is given by

$$[A, B] = 0, \qquad [B, C] = -\frac{2z}{p^2}(\cosh(pB) - 1), \qquad [A, C] = \frac{1}{p}\sinh(pB). \tag{15}$$

When z goes to zero, we find the quantum Heisenberg algebra obtained in Celeghini et al. [8], i.e.,

$$[A, B] = [B, C] = 0, \qquad [A, C] = \frac{1}{p}\sinh(pB). \tag{16}$$

In the case when $x = z = 0$, the two parameters deformed quantum Heisenberg Lie algebra, denoted by $\mathcal{U}_{p,q}(h(2))$, is now given by

$$[A, B] = [B, C] = 0, \qquad [A, C] = \frac{e^{pB} - e^{-qB}}{p+q}. \tag{17}$$

When $p = q$, we find again (16).

2.2 Some realizations of the deformed Heisenberg quantum algebras

Starting from $\mathcal{U}_{z,p}(h(2))$ as given in (15), we define the new generators

$$\tilde{A} = zA, \qquad \tilde{B} = \frac{2}{p}\sinh\left(\frac{pB}{2}\right), \qquad \tilde{C} = \frac{pC}{z}\frac{1}{\sinh(pB)}, \tag{18}$$

so that we get

$$\left[\tilde{A}, \tilde{B}\right] = 0, \qquad \left[\tilde{A}, \tilde{C}\right] = I, \qquad \left[\tilde{B}, \tilde{C}\right] = -\tilde{B}. \tag{19}$$

A realization of this last Lie algebra, in terms of the usual creation operator, $a^\dagger$, and annihilation operator, a, associated to the standard quantum harmonic oscillator system is given by

$$\tilde{A} = -za^\dagger, \qquad \tilde{B} = e^{za^\dagger}, \qquad \tilde{C} = \frac{a}{z}. \tag{20}$$

If we combine equation (20) with (18), we obtain a realization of $\mathcal{U}_{z,p}(h(2))$ as

$$A = -a^\dagger, \qquad B = \frac{2}{p}\sinh^{-1}\left(\frac{p}{2}e^{za^\dagger}\right) \qquad C = e^{za^\dagger}\sqrt{1 + \left(\frac{p}{2}e^{za^\dagger}\right)^2}\, a. \tag{21}$$

When p goes to zero, the generators become

$$A = -a^\dagger, \qquad B = e^{za^\dagger}, \qquad C = e^{za^\dagger}a, \tag{22}$$

and satisfy the commutations relations

$$[A, B] = 0, \qquad [A, C] = B, \qquad [B, C] = -zB^2, \tag{23}$$

which corresponds to the correct limit of the deformed algebra (15), and will be denoted $\mathcal{U}_{z,0}(h(2))$ in the following.

Another realization for the Lie algebra (19) is given by

$$\tilde{A} = za, \qquad \tilde{B} = e^{-za}, \qquad \tilde{C} = \frac{a^\dagger}{z}. \tag{24}$$

If we combine equation (24) with (18), we obtain now a realization of $\mathcal{U}_{z,p}(h(2))$ as

$$A = a, \qquad B = \frac{2}{p}\sinh^{-1}\left(\frac{p}{2}e^{-za}\right), \qquad C = a^{\dagger}e^{-za}\sqrt{1+\left(\frac{p}{2}e^{-za}\right)^2}. \tag{25}$$

When p goes to zero, we get

$$A = a, \qquad B = e^{-za}, \qquad C = a^{\dagger}e^{-za}, \tag{26}$$

which also leads to $\mathcal{U}_{z,0}(h(2))$.

On the other hand, when z goes to zero, the operators (21) becomes

$$A = -a^{\dagger}, \qquad B = \frac{2}{p}\sinh^{-1}\left(\frac{p}{2}I\right), \qquad C = \sqrt{1+\frac{p^2}{4}}a, \tag{27}$$

while the operators (25) becomes

$$A = a, \qquad B = \frac{2}{p}\sinh^{-1}\left(\frac{p}{2}I\right), \qquad C = \sqrt{1+\frac{p^2}{4}}a^{\dagger}. \tag{28}$$

The operators given in (27) or (28), constitute a realization of deformed Heisenberg algebra (16). It is clear that when p goes to zero, we regain $h(2)$.

Finally, the algebra (17) is isomorphic to $h(2)$ if we introduce

$$\tilde{A} = A, \qquad \tilde{C} = C, \qquad \tilde{B} = \frac{e^{pB} - e^{-qB}}{p+q}. \tag{29}$$

A realization of this algebra is given by $A = a$, $C = a^{\dagger}$ and $B = \ln\left(-\frac{q}{p}\right)^{1/p+q} I$, when this last expression have a meaning.

Let us remark that, a realization of these deformed quantum Heisenberg Lie algebras, on the space of linear functions $\mathcal{A}$, is obtained by changing, in all the preceding equations, the operators $a, a^{\dagger}$ and I by the differential operators X_1, X_3 and X_2, given in equation (4), respectively.

3 AES and deformed coherent and squeezed states

In this section, we use the representations obtained in the preceding section to compute the AES associated to the deformed quantum Heisenberg Lie algebras. Thus, we obtain the new classes of deformed coherent and squeezed states associated to the harmonic oscillator system.

3.1 Deformed algebra eigenstates for $\mathcal{U}_{z,0}\left(h(2)\right)$

We start with $\mathcal{U}_{z,0}\left(h(2)\right)$ as given by (23) using the realizations (22) and (26). The AES are thus defined as the set of solutions of the eigenvalue equation

$$[\alpha_+ A + \alpha_0 B + \alpha_- C]|\psi\rangle = \alpha|\psi\rangle, \qquad \alpha_-, \alpha_0, \alpha_+, \alpha \in \mathbb{C}. \tag{30}$$

3.1.1 Deformed harmonic oscillator coherent and squeezed states

Let us take first the realization (22). Thus, if $\alpha_- \neq 0$, equation (30) can be written in the form

$$[e^{za^\dagger} a + \mu a^\dagger + \nu e^{za^\dagger}]|\psi\rangle = \lambda|\psi\rangle, \qquad \mu, \nu, \lambda \in \mathbb{C}. \tag{31}$$

By defining

$$|\psi\rangle = e^{-\nu a^\dagger}|\varphi\rangle \tag{32}$$

and using $e^{-\nu a^\dagger}\, a\, e^{\nu a^\dagger} = a + \nu$, equation (31) can be reduced to

$$[e^{za^\dagger} a + \mu a^\dagger]|\varphi\rangle = \lambda|\varphi\rangle, \qquad \mu, \lambda \in \mathbb{C}. \tag{33}$$

To solve this eigenvalue equation, let us consider the Bargmann space $\mathcal{F}$ of analytic functions $f(\xi)$ ($\xi \in \mathbb{C}$), provided with the scalar product

$$(f_1, f_2) = \int_{\mathbb{C}} \overline{f_1(\xi)} f_2(\xi) e^{-\bar{\xi}\xi} \frac{d\bar{\xi} d\xi}{2\pi i}, \qquad \forall\, f_1, f_2 \in \mathcal{F}. \tag{34}$$

It is well-know that any function $f \in \mathcal{F}$ can be expressed as a linear combination of orthonormalized functions $u_n(\xi) = \frac{\xi^n}{\sqrt{n!}}$, $n = 0, 1, 2, \ldots$, verifying

$$(u_m, u_n) = \int_{\mathbb{C}} \overline{u_m(\xi)} u_n(\xi) e^{-\bar{\xi}\xi} \frac{d\bar{\xi} d\xi}{2\pi i} = \delta_{mn}, \tag{35}$$

that is

$$f(\xi) = \sum_{n=0}^{\infty} c_n u_n(\xi), \tag{36}$$

with

$$c_n = \int_{\mathbb{C}} \overline{u_n(\xi)} f(\xi) e^{-\bar{\xi}\xi} \frac{d\bar{\xi} d\xi}{2\pi i}. \tag{37}$$

Let us assume a solution of (33) of the type

$$|\varphi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle, \tag{38}$$

where the set of states $\{|n\rangle\}_{n=0}^{\infty}$ form the basis of the standard Fock oscillator space, verifying the orthogonality relation

$$\langle m|n\rangle = \delta_{mn}. \tag{39}$$

As usually, the action of the operators a and $a^\dagger$ on these states is given by

$$a|n\rangle = \sqrt{n}|n-1\rangle, \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{40}$$

Let us take $|\bar{\xi}\rangle$ to be the standard coherent states associated to the harmonic oscillator system, that is

$$|\bar{\xi}\rangle = e^{\bar{\xi}a^\dagger}|0\rangle = \sum_{n=0}^{\infty} \frac{(\bar{\xi})^n}{\sqrt{n!}}|n\rangle. \tag{41}$$

Then, according to the orthogonality property (39), the projection of $|\varphi\rangle$ on the coherent state $|\bar{\xi}\rangle$ is given by the analytic function

$$\varphi(\xi) = \langle\bar{\xi}|\varphi\rangle = \sum_{n=0}^{\infty} c_n u_n(\xi). \tag{42}$$

The action of the operators $a^\dagger$ and a in this representation corresponds to

$$\langle\bar{\xi}|a^\dagger|\varphi\rangle = \xi\varphi(\xi), \qquad \langle\bar{\xi}|a|\varphi\rangle = \frac{d\varphi}{d\xi}(\xi). \tag{43}$$

respectively. Thus, by projecting both sides of the eigenvalue equation (33) on the coherent states $|\bar{\xi}\rangle$ and then using (43), we can write it as

$$\left(e^{z\xi}\frac{d}{d\xi} + \mu\xi\right)\varphi(\xi) = \lambda\varphi(\xi). \tag{44}$$

The general solution of this differential equation is given by

$$\varphi(\xi) = C_0(\lambda,\mu,z)\,\exp\left(\sum_{k=0}^{\infty}\frac{(-z\xi)^k}{(k+1)!}\left(\lambda\xi - \frac{k+1}{k+2}\mu\xi^2\right)\right), \tag{45}$$

where C_0 is an arbitrary constant which can be fixed from the normalization condition

$$(\varphi,\varphi) = \int_{\mathbb{C}} \overline{\varphi(\xi)}\varphi(\xi)e^{-\bar{\xi}\xi}\frac{d\bar{\xi}d\xi}{2\pi i} = 1. \tag{46}$$

Let us notice that in the particular limit when z goes to zero, the solution (45), becomes the symbol for the squeezed states [9] associated to the standard harmonic oscillator, that is

$$\varphi(\xi) = C_0'(\lambda,\mu,0)\,\exp\left(\lambda\xi - \frac{\mu}{2}\xi^2\right). \tag{47}$$

This quantity is normalizable only if $|\mu| < 1$ [10].

When $z \neq 0$, the solution (45) can be written in the form

$$\varphi(\xi) = C_0(\lambda, \mu, z)\ \exp\left(\frac{\lambda}{z} - \frac{\mu}{z^2}\right)\ \exp\left(e^{-z\xi}\frac{(\mu - \lambda z + \mu z\xi)}{z^2}\right). \tag{48}$$

Going back to the expression (42), we get the coefficients c_n, $n = 0, 1, \ldots,$ as

$$c_n = \int_{\mathbb{C}} \overline{u_n(\xi)}\varphi(\xi)e^{-\bar{\xi}\xi}\frac{d\bar{\xi}d\xi}{2\pi i} \;=\; C_0(\lambda, \mu, z)\ \exp\left(\frac{\lambda}{z} - \frac{\mu}{z^2}\right) \int_{\mathbb{C}} \frac{\bar{\xi}^n}{\sqrt{n!}} \exp\left(e^{-z\xi}\frac{(\mu - \lambda z + \mu z\xi)}{z^2}\right) e^{-\bar{\xi}\xi}\frac{d\bar{\xi}d\xi}{2\pi i}. \tag{49}$$

By using the polar change of variables $\xi = \rho e^{i\vartheta}$, this last equation can be written in the form

$$c_n = C_0(\lambda, \mu, z)\ \exp\left(\frac{\lambda}{z} - \frac{\mu}{z^2}\right) \int_0^\infty \int_0^{2\pi} \frac{\rho^{n+1}e^{-\rho^2}}{\sqrt{n!}}e^{-in\vartheta}\exp\left(\frac{e^{-z\rho e^{i\vartheta}}}{z^2}(\mu - \lambda z + \mu z\rho e^{i\vartheta})\right)\frac{d\rho d\vartheta}{\pi}. \tag{50}$$

Let us write the exponential factor in the form

$$\begin{aligned}
&\exp\left(\frac{e^{-z\rho e^{i\vartheta}}}{z^2}(\mu - \lambda z + \mu z\rho e^{i\vartheta})\right) \\
&= \sum_{k=0}^{\infty} \frac{\exp\left(-zk\rho e^{i\vartheta}\right)}{k!}\left(\frac{\mu - \lambda z + uz\rho e^{i\vartheta}}{z^2}\right)^k \\
&= \sum_{k,l=0}^{\infty}\sum_{m=0}^{k}\binom{k}{m}\rho^{l+m}e^{i(l+m)\vartheta}\frac{(-zk)^l(\mu z)^m(\mu - \lambda z)^{k-m}}{k!\,l!\,z^{2k}}
\end{aligned} \tag{51}$$

to get

$$c_n = C_0(\lambda, \mu, z)\ \exp\left(\frac{\lambda}{z} - \frac{\mu}{z^2}\right) \sum_{k,l=0}^{\infty}\sum_{m=0}^{k}\binom{k}{m}\frac{(-zk)^l(\mu z)^m(\mu - \lambda z)^{k-m}}{\sqrt{n!}\,k!\,l!\,z^{2k}} \left(\int_0^\infty \rho^{m+l+n+1}e^{-\rho^2}d\rho\right)\left(\int_0^{2\pi} e^{i(l+m-n)\vartheta}\frac{d\vartheta}{\pi}\right). \tag{52}$$

Using the known results

$$\int_0^{2\pi} e^{i(l+m-n)\vartheta}\frac{d\vartheta}{\pi} = 2\delta_{l+m-n,0}, \tag{53}$$

$$\int_0^\infty \rho^{m+l+n+1}e^{-\rho^2}d\rho = \frac{1}{2}\Gamma\left(\frac{m+l+n}{2} + 1\right), \tag{54}$$

and performing the sum over the index l, the expression for the coefficients c_n reduces to

$$c_n = C_0(\lambda, \mu, z)\ \exp\left(\frac{\lambda}{z} - \frac{\mu}{z^2}\right)\frac{z^n}{\sqrt{n!}}\sum_{k=0}^{\infty}\sum_{m=0}^{k_<}\binom{n}{m}\frac{(-k)^{n-m}}{(k-m)!}\left(\frac{\mu}{z^2}\right)^m\left(\frac{\mu}{z^2} - \frac{\lambda}{z}\right)^{k-m}, \tag{55}$$

where $k_<$ denotes the minimum between k and n. This last expression can be written in the form

$$c_n = C_0(\lambda,\mu,z)\frac{z^n}{\sqrt{n!}}\sum_{m=0}^{n}\sum_{j=0}^{n-m}\binom{n}{m}(-1)^{n-m}\upsilon_{mj}\left(\frac{\mu}{z^2}\right)^m\left(\frac{\mu}{z^2}-\frac{\lambda}{z}\right)^j, \tag{56}$$

where the coefficients υ_{mj} are obtained from

$$\frac{k^{n-m}}{(k-m)!} = \sum_{j=0}^{n-m}\frac{\upsilon_{mj}}{(k-m-j)!}. \tag{57}$$

Thus the coefficients c_n, $n = 1, 2, \ldots$, represent polynomials of degree $n-1$ in the z variable. For example, $c_1 = \lambda C_0$,

$$c_2 = C_0\sqrt{2!}\left[\left(\frac{\lambda^2}{2!}-\frac{\mu}{2}\right)-\frac{\lambda}{2}z\right],\quad c_3 = C_0\sqrt{3!}\left[\left(\frac{\lambda^3}{3!}-\frac{\mu\lambda}{2}\right)+\left(\frac{\mu}{3}-\frac{\lambda^2}{2}\right)z+\frac{\lambda}{6}z^2\right]. \tag{58}$$

The normalization constant C_0 can be now computed. Indeed, inserting (56) into (42) and the resulting expression into the normalization condition (46), using the orthogonality relation (35), we get

$$\begin{aligned} C_0(\lambda,\mu,z) &= \left[\sum_{n=0}^{\infty}\frac{z^{2n}}{n!}\sum_{m=0}^{n}\sum_{r=0}^{n}\sum_{j=0}^{n-m}\sum_{l=0}^{n-r}\binom{n}{m}\binom{n}{r}(-1)^{m+r}\upsilon_{mj}\upsilon_{rl}\right. \\ &\quad \left.\left(\frac{\mu}{z^2}\right)^m\left(\frac{\bar{\mu}}{z^2}\right)^r\left(\frac{\mu}{z^2}-\frac{\lambda}{z}\right)^j\left(\frac{\bar{\mu}}{z^2}-\frac{\bar{\lambda}}{z}\right)^l\right]^{-\frac{1}{2}}, \end{aligned} \tag{59}$$

which has been chosen real. The convergence of these series it not easy to determine. In the case where $z = 0$, as we have already mentioned, the series $\sum_{n=0}^{\infty}|c_n|^2$ converges for all λ provided that $|\mu| < 1$. In the case $\mu = 0$, this series becomes

$$\sum_{n=0}^{\infty}|c_n|^2 = |C_0(\lambda,z)|^2\exp\left(\frac{\lambda}{z}\right)\sum_{n=0}^{\infty}\frac{\left(-\frac{\lambda}{z}\right)^n}{n!}\exp\left(-\frac{\bar{\lambda}}{z}\sum_{k=1}^{\infty}\frac{\left(z^2n\right)^k}{k!}\right). \tag{60}$$

It converges for all $z > 0$ provided that the phase θ in $\lambda = \beta e^{i\theta}$ satisfies $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, whereas for all $z < 0$, it converges if $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$.

Finally, we can show that the normalized algebra eigenstates $|\varphi\rangle$, solving (33) , can be expressed in terms of a deformed squeezed operator acting on the ground state of the standard harmonic oscillator, that is

$$|\varphi\rangle = C_0(\lambda,\mu,z)\exp\left(\sum_{k=0}^{\infty}\frac{\left(-za^\dagger\right)^k}{(k+1)!}\left(\lambda a^\dagger - \frac{k+1}{k+2}\mu(a^\dagger)^2\right)\right)|0\rangle. \tag{61}$$

Also, combining this last equation with equation (32), we get the algebra eigenstates solving (31) to be the deformed coherent states

$$|\psi\rangle = N_0(\lambda,\mu,\nu,z)\ \exp\left(\sum_{k=0}^{\infty}\frac{(-za^\dagger)^k}{(k+1)!}\left(\lambda a^\dagger - \frac{k+1}{k+2}\mu(a^\dagger)^2\right)\right)e^{-\nu a^\dagger}|0\rangle, \qquad (62)$$

where $N_0\left(\lambda,\mu,\nu,z\right)$ is a normalization constant which can computed in the same way as $C_0\left(\lambda,\mu,z\right)$.

3.1.2 Perturbed squeezed states

Let us now assume that z is a small perturbation parameter of order $k_0 - 1$, where k_0 is a positive integer. From (61), neglecting the terms containing the power of z greater than $k_0 - 1$, we can write

$$\begin{aligned} |\varphi\rangle &\approx C_0(\lambda,\mu,z,k_0)\left[1+\sum_{k=1}^{k_0-1}\frac{(-za^\dagger)^k}{(k+1)!}\left(\lambda a^\dagger - \frac{k+1}{k+2}\mu(a^\dagger)^2\right)\right. \\ &+ \left.\cdots+\frac{1}{(k_0-1)!}\left(\frac{-za^\dagger}{2!}\left(\lambda a^\dagger - \frac{2}{3}\mu(a^\dagger)^2\right)\right)^{k_0-1}\right]\exp\left(\lambda a^\dagger - \frac{\mu}{2}(a^\dagger)^2\right)|0\rangle. (63) \end{aligned}$$

These states can be normalized in the standard form. For instance, when $k_0 = 2$, $\mu = \delta e^{i\phi}$, $\lambda = \beta e^{i\theta}$, where ϕ and θ are real phases, $0 \le \delta < 1$, and $\beta \ge 0$, a normalized version of the deformed squeezed states (63), is given by

$$\begin{aligned} |\varphi\rangle &\approx \Omega(\delta,\phi,\beta,\theta)\left[1+z\left(\frac{\delta e^{i\phi}}{3}(a^\dagger)^3 - \frac{\beta e^{i\theta}}{2}(a^\dagger)^2\right)\right] \\ & S\left(-\arctan(\delta)e^{i\phi}\right)D\left(\frac{\beta e^{i\theta}}{\sqrt{1-\delta^2}}\right)|0\rangle, \end{aligned} \qquad (64)$$

where

$$\begin{aligned} \Omega(\delta,\phi,\beta,\theta) &= 1+\frac{z\beta}{2(1-\delta^2)^2}\left[\left(2\delta^2+\beta^2\left(\frac{1+\delta^2}{1-\delta^2}\right)\right)\cos\theta\right. \\ &- \delta\left(1+\delta^2+\frac{2\beta^2}{1-\delta^2}\right)\cos(\phi-\theta) \\ &+ \left.\delta^2\beta^2\left(1+\frac{2\delta^2}{3(1-\delta^2)}\right)\cos(2\phi-3\theta)-\frac{2\delta\beta^2}{3(1-\delta^2)}\cos(\phi-3\theta)\right]. (65) \end{aligned}$$

Here $S(\chi) = \exp\left[-\left(\chi\frac{(a^\dagger)^2}{2} - \bar{\chi}\frac{a^2}{2}\right)\right]$ is the standard unitary squeezed operator [11] and $D(\lambda) = \exp\left(\lambda a^\dagger - \bar{\lambda}a\right)$ the standard displacement operator [12].

3.1.3 Deformed squeezed and coherent states parametrized by paragrassmann numbers

Let us now use the realization (26) of $\mathcal{U}_{z,0}\,(h(2))$. In the case $\alpha_+ \neq 0$, equation (30) can be now written in the form

$$[a + \mu a^\dagger e^{-za} + \nu e^{-za}]|\psi\rangle = \lambda|\psi\rangle, \qquad \mu, \nu, \lambda \in \mathbb{C}. \tag{66}$$

There are two types of equations to solve. The first type is obtained when $\mu \neq 0$ and $\nu \neq 0$. We can take

$$|\psi\rangle = \exp\left(\frac{\nu}{\mu}a\right)|\varphi\rangle \tag{67}$$

and use the relation, $\exp\left(-\frac{\nu}{\mu}a\right) a^\dagger \exp\left(\frac{\nu}{\mu}a\right) = a^\dagger - \frac{\nu}{\mu}$, to reduce (66) to the form

$$[a + \mu a^\dagger e^{-za}]|\varphi\rangle = \lambda|\varphi\rangle, \qquad \mu, \lambda \in \mathbb{C}. \tag{68}$$

If $\nu = 0$ and $\mu \neq 0$, we see from (66) that the same type of eigenvalue equation must be solved. The second type is obtained when $\mu = 0$. The eigenvalue equation is

$$[a + \nu e^{-za}]|\psi\rangle = \lambda|\psi\rangle, \qquad \nu, \lambda \in \mathbb{C}. \tag{69}$$

We begin with the resolution of Equation (68). Let us assume $|\varphi\rangle$ to be again a solution of the type (38). Thus, proceeding as in the preceding section, the eigenvalue equation satisfied by the symbol $\varphi(\xi)$, in the Bargmann representation, is given by

$$\left(\frac{d}{d\xi} + \mu\,\xi\, e^{-z\frac{d}{d\xi}}\right)\varphi(\xi) = \lambda\varphi(\xi), \qquad \mu, \lambda \in \mathbb{C}. \tag{70}$$

To solve this equation, let us assume that z is a real paragrassmann number [13, 14], that is $z^{k_0} = 0$, for some integer $k_0 \geq 1$. A detailed procedure of resolution of this equation is given in the Appendix A. Let us notice that the case $k_0 = 1$, i.e., $z = 0$, is somewhat trivial since the eigenfunctions $\varphi(\xi)$ solving (70), are given by the standard squeezed symbol (47). When $k_0 = 2$, or $z^2 = 0$, i.e., when z is a odd Grassmann number [15, 16], the eigenvalue equation (70) becomes

$$\left((1 - \mu z\xi)\frac{d}{d\xi} + \mu\,\xi\right)\varphi(\xi) = \lambda\varphi(\xi), \qquad \mu, \lambda \in \mathbb{C}. \tag{71}$$

There are two independent solutions (see Appendix A). The normalizable solution of this eigenvalue equation, is given by the deformed squeezed symbol

$$\varphi(\lambda, \mu, z)(\xi) = C_0(\lambda, \mu, z)\left[1 + z\mu\left(\lambda\frac{\xi^2}{2} - \mu\frac{\xi^3}{3}\right)\right]\exp\left(\lambda\xi - \frac{\mu}{2}\xi^2\right). \tag{72}$$

A normalized version of these states, in the Fock space representation, is given by

$$\begin{aligned} |\varphi\rangle &= \tilde{\Omega}(\delta,\phi,\beta,\theta)\left[1+z\delta\left(\frac{\delta e^{2i\phi}}{3}(a^\dagger)^3-\frac{\beta e^{i(\theta+\phi)}}{2}(a^\dagger)^2\right)\right] \\ & S\left(-\arctan(\delta)e^{i\phi}\right)D\left(\frac{\beta e^{i\theta}}{\sqrt{1-\delta^2}}\right)|0\rangle, \end{aligned} \tag{73}$$

where λ and μ have been chosen as before and

$$\begin{aligned} \tilde{\Omega}(\delta,\phi,\beta,\theta) &= 1-\frac{z\delta\beta}{2(1-\delta^2)^2}\left[\left(2\delta^2+\beta^2\left(\frac{1+\delta^2}{1-\delta^2}\right)\right)\cos(\theta-\phi)\right. \\ &- \delta\left(1+\delta^2+\frac{2\beta^2}{1-\delta^2}\right)\cos\theta \\ &+ \left.\delta^2\beta^2\left(1+\frac{2\delta^2}{3(1-\delta^2)}\right)\cos(\phi-3\theta)-\frac{2\delta\beta^2}{3(1-\delta^2)}\cos(2\phi-3\theta)\right]. \end{aligned} \tag{74}$$

When $k_0 = 3$, or $z^3 = 0$, the eigenvalue equation (70) becomes the second order differential equation

$$\left(\frac{1}{2}\mu z^2\xi\frac{d^2}{d\xi^2}+(1-\mu z\xi)\frac{d}{d\xi}\right)\varphi(\xi)=(\lambda-\mu\xi)\varphi(\xi), \qquad \mu,\lambda,\in\mathbb{C}. \tag{75}$$

According to the results obtained in Appendix A, the general solution of this equation can be expanded in the form

$$\varphi(\xi)=\varphi_0(\xi)+z\varphi_1(\xi)+z^2\varphi_2(\xi), \tag{76}$$

with

$$\varphi_0(\xi) = C_0\exp\left(\lambda\xi-\mu\frac{\xi^2}{2}\right), \tag{77}$$

$$\varphi_1(\xi) = \left[\mu\left(\lambda\frac{\xi^2}{2}-\mu\frac{\xi^3}{3}\right)C_0+C_1\right]\exp\left(\lambda\xi-\mu\frac{\xi^2}{2}\right), \tag{78}$$

$$\begin{aligned} \varphi_2(\xi) &= \left[\left(\mu(\mu-\lambda^2)\frac{\xi^2}{4}+\frac{2}{3}\mu^2\lambda\xi^3+\mu^2(\lambda^2-3\mu)\frac{\xi^4}{8}-\lambda\mu^3\frac{\xi^5}{6}+\mu^4\frac{\xi^6}{18}\right)C_0\right. \\ &+ \left.\mu\left(\lambda\frac{\xi^2}{2}-\mu\frac{\xi^3}{3}\right)C_1+C_2\right]\exp\left(\lambda\xi-\mu\frac{\xi^2}{2}\right), \end{aligned} \tag{79}$$

where C_0, C_1 and C_2 are arbitrary integration constants. Three independent solutions may thus be obtained. The first one is obtained by taking $C_1 = C_2 = 0$. We get

$$\begin{aligned} \varphi(\xi) &= C_0\left[1+z\mu\left(\lambda\frac{\xi^2}{2}-\mu\frac{\xi^3}{3}\right)+z^2\left(\mu(\mu-\lambda^2)\frac{\xi^2}{4}+\frac{2}{3}\mu^2\lambda\xi^3\right.\right. \\ &+ \left.\left.\mu^2(\lambda^2-3\mu)\frac{\xi^4}{8}-\lambda\mu^3\frac{\xi^5}{6}+\mu^4\frac{\xi^6}{18}\right)\right]\exp\left(\lambda\xi-\mu\frac{\xi^2}{2}\right) \end{aligned} \tag{80}$$

$$= C_0\exp\left[z\mu\left(\lambda\frac{\xi^2}{2}-\mu\frac{\xi^3}{3}\right)+z^2f(\xi)\right]\exp\left(\lambda\xi-\mu\frac{\xi^2}{2}\right), \tag{81}$$

where

$$f(\xi) = \left(\mu(\mu - \lambda^2)\frac{\xi^2}{4} + \frac{2}{3}\mu^2\lambda\xi^3 - 3\mu^3\frac{\xi^4}{8}\right). \tag{82}$$

This solution can be normalized and represents a second order paragrassmann deformation of squeezed states associated to the standard harmonic oscillator.

The other independent solutions are given respectively by

$$\varphi(\xi) = C_1\, z\left[1 + z\mu\left(\lambda\frac{\xi^2}{2} - \mu\frac{\xi^3}{3}\right)\right]\exp\left(\lambda\xi - \mu\frac{\xi^2}{2}\right) \tag{83}$$

and

$$\varphi(\xi) = C_2\, z^2 \exp\left(\lambda\xi - \mu\frac{\xi^2}{2}\right). \tag{84}$$

These solutions can not be normalized since z^k, $k = 1, 2$, are not invertible paragrassmann numbers and $z^k = 0$, $k = 3, 4, \ldots$.

The higher order paragrassmann deformations of the squeezed states associated to the standard harmonic oscillator can be obtained following a similar procedure (see Appendix A).

In the case of eigenvalue equation (69), the differential equation to solve is given by

$$\left(\frac{d}{d\xi} + \nu e^{-z\frac{d}{d\xi}}\right)\varphi(\xi) = \lambda\varphi(\xi), \qquad \nu, \lambda, \in \mathbb{C}. \tag{85}$$

Proceedings as before and considering the results of Appendix A, the normalizable solutions of this last equation, when $k_0 = 1, 2, 3$, are given respectively by the deformed coherent symbols

$$\varphi^{(1)}(\xi) = C_0 \exp\Big((\lambda - \nu)\xi\Big), \tag{86}$$

$$\varphi^{(2)}(\xi) = C_0\,[1 + z(\lambda - \nu)\nu\xi]\exp\Big((\lambda - \nu)\xi\Big) \tag{87}$$

and

$$\begin{aligned}\varphi^{(3)}(\xi) &= C_0\Big\{1 + z(\lambda - \nu)\nu\xi + z^2\Big[\left(\frac{\lambda^2\nu}{2} + 2\lambda\nu^2 - \frac{3\nu^3}{2}\right)\xi \\ &+ \left(\frac{\lambda^2\nu^2}{2} - \lambda\nu^3 + \frac{\nu^4}{2}\right)\xi^2\Big]\Big\}\exp\Big((\lambda - \nu)\xi\Big).\end{aligned} \tag{88}$$

Theses solutions can be normalized and represent zero, first and second order paragrassmann deformations, respectively, of coherent states associated to the standard harmonic oscillator. For higher values of k_0, we must proceed as in Appendix A.

3.2 Deformed algebra eigenstates for $\mathcal{U}_{z,p}(h(2))$

In this section, we consider the two parameter deformed algebra $\mathcal{U}_{z,p}(h(2))$ as given by (15), and compute the AES using the particular realization (21). More precisely, we have to solve again an eigenvalue equation of the type (30). Assuming $\alpha_- \neq 0$, we can reduce it to

$$\left[e^{za^\dagger}\sqrt{1+\left(\frac{p}{2}e^{za^\dagger}\right)^2}a+\mu a^\dagger+\frac{2\nu}{p}\sinh^{-1}\left(\frac{p}{2}e^{za^\dagger}\right)\right]|\psi\rangle=\lambda|\psi\rangle, \qquad \mu,\nu,\lambda\in\mathbb{C}. \quad (89)$$

In the Bargmann representation, this equation becomes the first order differential equation

$$\left[e^{z\xi}\sqrt{1+\left(\frac{p}{2}e^{z\xi}\right)^2}\frac{d}{d\xi}+\mu\xi+\frac{2\nu}{p}\sinh^{-1}\left(\frac{p}{2}e^{z\xi}\right)\right]\psi(\xi)=\lambda\psi(\xi), \qquad \mu,\nu,\lambda\in\mathbb{C}. \quad (90)$$

When $z=0$, we easily get the standard squeezed symbols

$$\psi_{o,p}(\xi)=C_0(p,\lambda,\mu,\nu)\exp\left[\left(\lambda-\frac{2\nu}{p}\sinh^{-1}(p/2)\right)\xi-\mu\frac{\xi^2}{2}\right]. \quad (91)$$

These symbols correspond to the Bargmann representation of the AES associated to the deformed quantum Heisenberg algebra realization (27). Moreover, when p goes to zero, these symbols becomes the standard squeezed symbols associated to $h(2)$.

When $z\neq 0$, making the change of variable $\zeta=e^{z\xi}$, rearranging the terms and using the method of characteristics curves to separate the differentials, we get

$$\frac{d\psi}{\psi}(\zeta)=\frac{\left[\lambda-\frac{\mu}{z}\ln\zeta-\frac{2\nu}{p}\sinh^{-1}\left(\frac{p\zeta}{2}\right)\right]}{z\,\zeta^2\sqrt{1+\frac{p^2\zeta^2}{4}}}d\zeta. \quad (92)$$

Integrating both sides of this equation and then exponentiating, we get

$$\begin{aligned}\psi_{z,p}(\zeta) &= C_0(\lambda,\mu,\nu;z,p)\,\exp\left[\frac{\sqrt{1+\frac{p^2\zeta^2}{4}}}{z^2\zeta}\left((1+\ln\zeta)\mu-\lambda z+\frac{2\nu z}{p}\sinh^{-1}(\frac{p\zeta}{2})\right)\right.\\ &- \left.\frac{\mu p}{2z^2}\sinh^{-1}(\frac{p\zeta}{2})-\frac{\nu}{z}\left(\ln\zeta+\frac{1}{2}\left(\sinh^{-1}(\frac{p\zeta}{2})\right)^2\right)\right]. \end{aligned} \quad (93)$$

This result includes the ones obtained for (31) when p goes to zero. Indeed, when we set also $\nu=0$, we regain (45).

3.2.1 Perturbed two parameters deformation coherent and squeezed states

Up to first order of approximation in z and p^2, the deformed symbol (93) writes

$$\begin{aligned}\psi_{z,p}(\xi) &\approx \tilde{C}_0(\lambda,\mu,\nu;z,p)\left[1+z\left(\frac{\mu\xi^3}{3}-\frac{\lambda\xi^2}{2}\right)\right.\\ &+ \left.\frac{p^2}{4}\left(\frac{\mu\xi^2}{4}-\left(\frac{\lambda}{2}-\frac{\nu}{3}\right)\xi\right)\right]\exp\left((\lambda-\nu)\xi-\frac{1}{2}\mu\xi^2\right). \end{aligned} \quad (94)$$

In the case $\mu = \delta e^{i\phi}$, $\lambda = \beta e^{i\theta}$ and $\nu = -\gamma e^{i\eta}$, where $\gamma \geq 0$, a normalized version of these states, in the Fock representation, is given by

$$\begin{aligned} |\psi\rangle \;\approx\; & \tilde{\Omega}(\delta,\phi,\beta,\theta,\gamma,\eta)\Big\{1 + \left[z\left(\frac{\delta e^{i\phi}}{3}(a^\dagger)^3 - \frac{\beta e^{i\theta}}{2}(a^\dagger)^2\right)\right] \\ & + \frac{p^2}{4}\left[\frac{\delta e^{i\phi}}{4}(a^\dagger)^2 - \left(\frac{\beta e^{i\theta}}{2} + \frac{\gamma e^{i\eta}}{3}\right)a^\dagger\right]\Big\} \\ & S\left(-\arctan(\delta)e^{i\phi}\right) D\left(\frac{\tilde{\beta}e^{i\tilde{\theta}}}{\sqrt{1-\delta^2}}\right)|0\rangle, \end{aligned} \tag{95}$$

where

$$\begin{aligned} \tilde{\Omega}(\delta,\phi,\beta,\theta,\gamma,\eta) \;=\; & 1 + \frac{z}{2(1-\delta^2)^2}\Big\{\tilde{\beta}\Big[\left(2\delta^2 + \tilde{\beta}^2\left(\frac{1+\delta^2}{1-\delta^2}\right)\right)\cos\tilde{\theta} \\ & - \delta\left(1 + \delta^2 + \frac{2\tilde{\beta}^2}{1-\delta^2}\right)\cos(\phi - \tilde{\theta}) \\ & + \delta^2\tilde{\beta}^2\left(1 + \frac{2\delta^2}{3(1-\delta^2)}\right)\cos(2\phi - 3\tilde{\theta}) - \frac{2\delta\tilde{\beta}^2}{3(1-\delta^2)}\cos(\phi - 3\tilde{\theta})\Big] \\ & - \gamma\Big[\tilde{\beta}^2\cos(\eta - 2\tilde{\theta}) - \delta(2\tilde{\beta}^2 + 1 - \delta^2)\cos(\eta - \tilde{\theta}) \\ & + \delta^2\tilde{\beta}^2\cos(2\phi - \eta - 2\tilde{\theta})\Big]\Big\} - \frac{p^2}{16(1-\delta^2)^2}\Big\{\delta\tilde{\beta}^2(3\cos(\phi - 2\tilde{\theta}) \\ & + \frac{2\gamma}{3}\tilde{\beta}(1-\delta^2)\left(\cos(\eta - \tilde{\theta}) + \delta\cos(\phi - \eta - \tilde{\theta})\right) - 2\tilde{\beta}^2 - \delta^2 + \delta^4\Big\}, \end{aligned} \tag{96}$$

where

$$\tilde{\beta} = \sqrt{\beta^2 + \gamma^2 + 2\beta\gamma\cos(\eta - \theta)}, \qquad \tilde{\theta} = \tan^{-1}\left(\frac{\beta\sin\theta + \gamma\sin\eta}{\beta\cos\theta + \gamma\cos\eta}\right). \tag{97}$$

We notice that, in the case $\gamma = 0$ and $p = 0$, these normalized states become the normalized states given in equation (64).

3.2.2 Squeezing properties

Let us now study the squeezing properties of physical quantities X and P, representing the position and linear momentum of a particle, respectively. Let us recall that, in the Fock space representation, these quantities are given by the hermitian operators (we have assumed that the mass, angular frequency and Planck's constant all equal to 1)

$$X = \frac{(a + a^\dagger)}{\sqrt{2}}, \qquad P = i\frac{(a^\dagger - a)}{\sqrt{2}}. \tag{98}$$

They verify the canonical commutation relation

$$[X,P] = iI. \tag{99}$$

The dispersion of these quantities, computed on a specific normalized particle state $|\psi\rangle$, is defined as

$$(\Delta X)^2 = \langle\psi|X^2|\psi\rangle - (\langle\psi|X|\psi\rangle)^2 \tag{100}$$

and

$$(\Delta P)^2 = \langle\psi|P^2|\psi\rangle - (\langle\psi|P|\psi\rangle)^2. \tag{101}$$

The product of these dispersions satisfy the Schrödinger-Robertson uncertainty relation (SRUR) [17, 18]

$$(\Delta X)^2 (\Delta P)^2 \geq \frac{1}{4}\Big(\langle I\rangle^2 + \langle F\rangle^2\Big) = \frac{1}{4}\Big(1 + \langle F\rangle^2\Big), \tag{102}$$

where F is the anti-commutator $F = \{X - \langle X\rangle I, P - \langle P\rangle I\}$. The mean value of F is a correlation measure between X and P. When $\langle F\rangle = 0$, we regain the standard Heisenberg uncertainty principle.

The minimum uncertainty states (MUS) are states that satisfy the equality in (102). They are called coherent states when the dispersions of both X and P are the same and squeezed states when these dispersions are different to each other. The states for which the dispersion of X is greater than the one of P are called X-squeezed whereas the states for which the dispersion of P is greater than the one of X are called P-squeezed.

We are interested to compute the dispersions of X and P, in the deformed squeezed states (95), when $\nu = 0$, or $\gamma = 0$. More precisely, we want to study the effect of the deformation parameters on the squeezed properties of these quantities. As we have seen, when z and p go to zero, the states (95) becomes the standard harmonic oscillator squeezed states. In such a case, we know that the dispersions of X and P are independent of $\lambda = \beta e^{i\theta}$, and given by [6]

$$(\Delta X)_0{}^2 = \frac{1 - 2\delta\cos\phi + \delta^2}{2(1-\delta^2)} \quad \text{and} \quad (\Delta P)_0{}^2 = \frac{1 + 2\delta\cos\phi + \delta^2}{2(1-\delta^2)}. \tag{103}$$

All these states are MUS, that is, they satisfy the equality in (102).

When $\gamma = 0$, the square of the mean value of X, in the states (95), to first order of approximation in z and p^2, is given by

$$\langle\psi|X|\psi\rangle^2 \approx 2\Big(\mathrm{Re}\,\Gamma_{01}\Big)\,\mathrm{Re}\Big\{\Big(1 + 4\epsilon(z,p)\Big)\Gamma_{01}$$

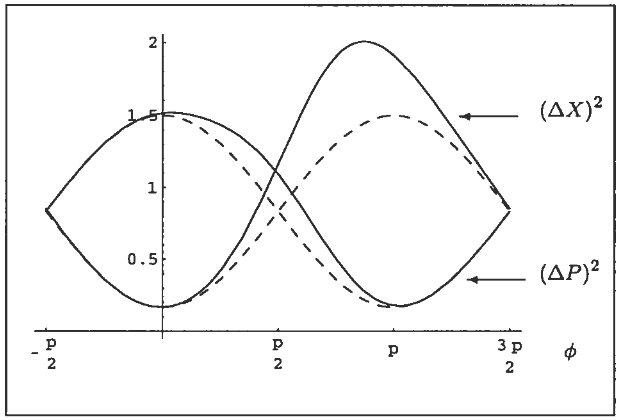

Figure 1: Graphs of the dispersions of X and P as functions of ϕ for $z = 0.0010$.

$$\begin{aligned} &+ \quad 2z\left(\frac{\delta e^{-i\phi}}{3}\Gamma_{04} - \frac{\beta e^{-i\theta}}{2}\Gamma_{03} + \frac{\delta e^{i\phi}}{3}\Lambda_{13} - \frac{\beta e^{i\theta}}{2}\Lambda_{12}\right) \\ &+ \quad \frac{p^2}{2}\left(\frac{\delta e^{-i\phi}}{4}\Gamma_{03} - \frac{\beta e^{-i\theta}}{2}\Gamma_{02} + \frac{\delta e^{i\phi}}{4}\Lambda_{12} - \frac{\beta e^{i\theta}}{2}\Lambda_{11}\right)\Bigg\}, \end{aligned} \tag{104}$$

where $\epsilon(z,p) = \tilde{\Omega}(\delta,\phi,\beta,\theta,0,0) - 1$ and Γ_{kl} and Λ_{kl}, $k,l = 1,2,\ldots$, are matrices elements defined in Appendix B. According to (98), we have the same expression for the square of the mean value of P, but taking the imaginary part in place of the real part.

On the other hand, the mean value of X^2 in the states (95), to first order of approximation in z and p^2, is given by

$$\begin{aligned} \langle\psi|X^2|\psi\rangle \quad &\approx \quad \frac{1}{2} + \Big(1 + 2\epsilon(z,p)\Big)(\Gamma_{11} + \operatorname{Re}\Gamma_{02}) \\ &+ \quad z\operatorname{Re}\left(\frac{\delta e^{-i\phi}}{3}\Gamma_{05} - \frac{\beta e^{-i\theta}}{2}\Gamma_{04} + \frac{\delta e^{i\phi}}{3}\Lambda_{23} - \frac{\beta e^{i\theta}}{2}\Lambda_{22}\right) \\ &+ \quad \frac{p^2}{4}\operatorname{Re}\left(\frac{\delta e^{-i\phi}}{4}\Gamma_{04} - \frac{\beta e^{-i\theta}}{2}\Gamma_{03} + \frac{\delta e^{i\phi}}{4}\Lambda_{22} - \frac{\beta e^{i\theta}}{2}\Lambda_{21}\right) \\ &+ \quad z\bigg(\frac{\delta e^{-i\phi}}{3}(\Lambda_{41} - \Gamma_{03}) - \frac{\beta e^{-i\theta}}{2}(\Lambda_{31} - \Gamma_{02}) + \frac{\delta e^{i\phi}}{3}(\Lambda_{14} - \Lambda_{03}) \\ &- \quad \frac{\beta e^{i\theta}}{2}(\Lambda_{13} - \Lambda_{02})\bigg) + \frac{p^2}{4}\bigg(\frac{\delta e^{-i\phi}}{4}(\Lambda_{31} - \Gamma_{02}) - \frac{\beta e^{-i\theta}}{2}(\Lambda_{21} - \Gamma_{01}) \\ &+ \quad \frac{\delta e^{i\phi}}{4}(\Lambda_{13} - \Lambda_{02}) - \frac{\beta e^{i\theta}}{2}(\Lambda_{12} - \Lambda_{01})\bigg). \end{aligned} \tag{105}$$

Again, according to (98), we have the same expression for the mean value of P^2, but taking the negative of the real part in place of the real part.

Combining (104) with (105), according to equation (100), we get the dispersion of X. In the same way, we can obtain the dispersion of P. Inserting the matrix elements Γ_{ij} and Λ_{ij},

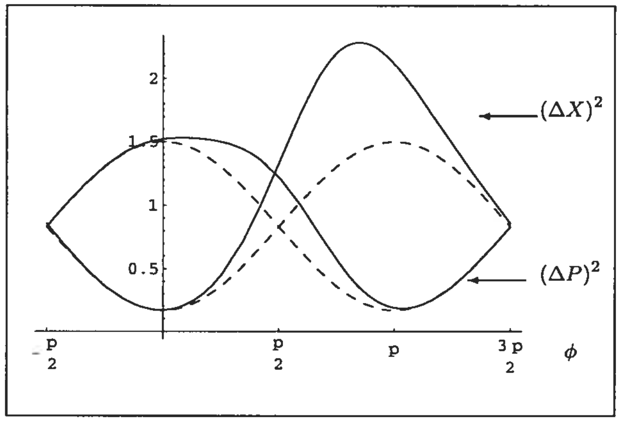

Figure 2: Graphs of the dispersions of X and P as functions of ϕ for $z = 0.0015$.

as given in the Appendix B, we can compute these dispersions explicitly.

Figures 1-3 show the dispersions of X and P in the minimum uncertainty squeezed states in dashed lines, and in the deformed squeezed states in solid lines, as a function of ϕ for fixed valued of the parameters δ, β, θ and p ($\delta = 0.5$, $\beta = 2.0$, $\theta = 0.8\,\pi, p = 0.001$) and for special values of $z = 0.0010, 0.0015, 0.0020$. We observe that, as a consequence of the small deformations in the parameters z and p, the squeezing properties of X and P have not been essentially changed. Thus, in all the cases, we have P–squeezed states when $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$, and X–squeezed states when $\frac{\pi}{2} < \phi < \frac{3\pi}{2}$. Also we observe that the product of the dispersions of X and P in the deformed squeezed states, for a given value of ϕ, is always greater than the product of the dispersions in the minimum uncertainty states, as required by the SRUR. These difference is more remarkable for values of ϕ in the range $\frac{\pi}{2} \leq \phi < \frac{3\pi}{2}$. Let us notice that when $\phi = \pm\frac{\pi}{2}$, the MUS are coherent states, in the sense of the SRUR, i,e., the dispersion of X and P, are the same. Indeed, in all these cases, $(\Delta X)_0{}^2 = (\Delta P)_0{}^2 = 0.83$. This value is conserved by the product of the dispersions of X and P in the deformed squeezed states when $\phi = -\frac{\pi}{2}$, but when $\phi = \frac{\pi}{2}$, it grows quickly as z increase.

Figure 4 shows the typical behavior of the dispersions of X and P in the minimum uncertainty squeezed states in dashed lines, and in the deformed squeezed states in solid lines, as a function of δ for $\phi = 0.5$, $\beta = 2.0$, $\theta = 0.8\,\pi, z = 0.0025$ and $p = 0, 001$. We observe again that, as a consequence of the small deformations in z and p, the squeezing properties of X and P have not been essentially changed. Thus, the figure shows the behavior of P–squeezed and P-deformed squeezed states. When $0 < \delta \lesssim 0.75$, the product of the dispersions of X and P, in the deformed squeezed states is always greater than the corresponding product in

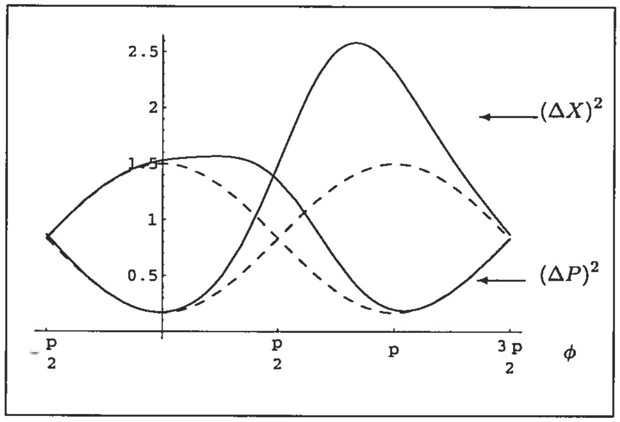

Figure 3: Graphs of the dispersions of X and P as functions of ϕ for $z = 0.0020$.

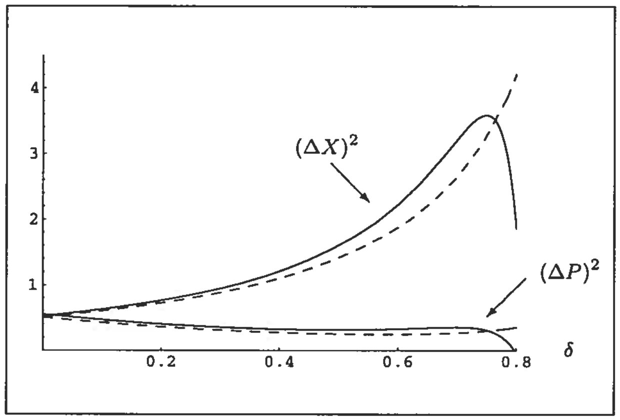

Figure 4: Graphs of the dispersions of X and P as functions of δ for $z = 0.0025, p = 0.01, \beta = 2.0, \theta = 0.8\pi$ and $\phi = \frac{\pi}{6}$.

the minimum uncertainty squeezed states, as required by the SRUR. For higher values of δ, only the dashed lines represent the true behavior of the dispersions of X and P. Indeed, the approximation for the deformed squeezed states, in this region, is not valid. These states are no longer normalizable.

4 Conclusions

In the present paper, we have found some representations of the deformed quantum Heisenberg Lie algebra, in terms of the usual creation and annihilation operators associated to the Fock space representation of the standard harmonic oscillator and also in terms of the differential generators associated to the three-dimensional matrix representation of the Heisenberg-Weyl group. The method used to get these representations can be easily applied to find the representations of other quantum Hopf algebras and super-algebras, such as the bosonic and fermionic oscillators Hopf algebras[19] or the quantum super-Heisenberg algebra, that can also be obtained by using the R-matrix approach.

We have computed the AES associated to the physical representation of the deformed quantum Heisenberg algebra $\mathcal{U}_{z,p}(h(2))$. We have seen that the set of AES contains the set of coherent and squeezed states associated to the standard harmonic oscillator system but also a new class of deformed coherent and squeezed states, parametrized by the deformation parameters. We have studied the behavior of the dispersions of the position and linear momentum operators of a particle in a class of perturbed squeezed states and we have compared them with the behavior of these dispersios in the minimum uncertainty squeezed states.

On the other hand, we have found new classes of deformed squeezed states, parametrized by a real paragrassmann number, i.e., a number z such that $z^{k_0} = 0$, for some $k_0 \in \mathbb{N}$. These states can be normalized, even if z is considered as a complex paragrassmann number. In this last case, when $k_0 = 2$, we can interpret z as an odd complex Grassmann number and obtain new classes of deformed squeezed states associated to the η-super-pseudo-Hermitian Hamiltonians [20, 21].

Acknowledgments

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A Solving a paragrassmann valued differential equation

In this appendix we are interested to solve the differential equation

$$\left[\frac{d}{d\xi} + (\mu\xi + \nu)\sum_{l=0}^{k_0-1}\frac{(-z)^l}{l!}\frac{d^l}{d\xi^l}\right]\varphi(\xi) = \lambda\varphi(\xi), \qquad \mu, \nu, \lambda \in \mathbb{C}, \tag{106}$$

where $k_0 \in \mathbb{N}$, $k_0 \geq 1$, and z is a paragrassmann generator such that $z^k = 0$, $\forall k \geq k_0$.

Let us assume a solution of the type

$$\varphi(\xi) = \sum_{k=0}^{k_0-1} z^k \varphi_k(\xi). \tag{107}$$

Inserting this solution into (106), we get

$$\left[\sum_{k=0}^{k_0-1} z^k \frac{d\varphi_k}{d\xi} + (\mu\xi+\nu)\sum_{l=0}^{k_0-1}\sum_{k=0}^{k_0-1}\frac{(-1)^l (z)^{k+l}}{l!}\frac{d^l\varphi_k}{d\xi^l}\right] = \lambda\sum_{k=0}^{k_0-1} z^k\varphi_k. \tag{108}$$

Identifying the coefficients of independent powers $z^k, k = 0, 1, 2, \ldots, k_0 - 1$, in this equality, we get the following system of differential equations ($k = 1, \ldots, k_0 - 1$)

$$\frac{d\varphi_k}{d\xi} + (\mu\xi+\nu)\sum_{l=1}^{k}\frac{(-1)^l}{l!}\frac{d^l\varphi_{k-l}}{d\xi^l} = [(\lambda-\nu)-\mu\xi]\,\varphi_k, \tag{109}$$

$$\frac{d\varphi_0}{d\xi} = [(\lambda-\nu)-\mu\xi]\,\varphi_0. \tag{110}$$

Let us notice that we can solve this system of differential equations proceeding by iteration. Indeed, from equation (110), we get

$$\varphi_0(\xi) = C_0\exp\left((\lambda-\nu)\xi - \frac{1}{2}\mu\xi^2\right), \tag{111}$$

where C_0 is an arbitrary integration constant. Also, from equation (109), for a given value of k, the general solution $\varphi_k(\xi)$ is of the type

$$\varphi_k(\xi) = [C_k + A_k(\xi)]\exp\left((\lambda-\nu)\xi - \frac{1}{2}\mu\xi^2\right), \qquad k = 1,\ldots,k_0-1, \tag{112}$$

where the C_k are arbitrary integration constants and $A_k(\xi)$ are functions of ξ which can be determined by solving the system of differential equations ($k = 1, 2, \ldots, k_0 - 1$)

$$\begin{aligned}\frac{dA_k}{d\xi} &= \exp\left(\frac{1}{2}\mu\xi^2 - (\lambda-\nu)\xi\right)\\ &\quad(\mu\xi+\nu)\sum_{l=1}^{k}\frac{(-1)^{l+1}}{l!}\frac{d^l}{d\xi^l}\left[(C_{k-l}+A_{k-l})\exp\left((\lambda-\nu)\xi-\frac{1}{2}\mu\xi^2\right)\right].\end{aligned} \tag{113}$$

Using the Leibnitz's derivation rule it is easy to prove that

$$\begin{aligned}&\exp\left(\frac{1}{2}\mu\xi^2-(\lambda-\nu)\xi\right)\frac{d^l}{d\xi^l}\left[(C_{k-l}+A_{k-l})\exp\left((\lambda-\nu)\xi-\frac{1}{2}\mu\xi^2\right)\right] =\\ &\sum_{m=0}^{l}\binom{l}{m}\left[C_{k-l}(\lambda-\nu)^{l-m}\left(\frac{\mu}{2}\right)^{m/2}(-1)^m H_m\left(\sqrt{\frac{\mu}{2}}\xi\right)\right.\\ &\left.+\frac{d^{l-m}A_{k-l}}{d\xi^{l-m}}\sum_{s=0}^{m}\binom{m}{s}(\lambda-\nu)^{m-s}\left(\frac{\mu}{2}\right)^{s/2}(-1)^s H_s\left(\sqrt{\frac{\mu}{2}}\xi\right)\right],\end{aligned} \tag{114}$$

where

$$H_m(x) = e^{x^2}\frac{d^m}{dx^m}e^{-x^2}, \qquad m = 0, 1, \ldots, \tag{115}$$

are the Hermite polynomials.

Inserting these results into (113) and integrating with respect to ξ, we get

$$\begin{aligned} A_k(\xi) &= \sum_{l=1}^{k}\sum_{m=0}^{l}\frac{(-1)^{l+1}}{l!}\binom{l}{m}\int(\mu\xi+\nu)\Big[C_{k-l}(\lambda-\nu)^{l-m}\left(\frac{\mu}{2}\right)^{m/2}(-1)^m H_m\left(\sqrt{\frac{\mu}{2}}\xi\right) \\ &+ \frac{d^{l-m}A_{k-l}}{d\xi^{l-m}}\sum_{s=0}^{m}\binom{m}{s}(\lambda-\nu)^{m-s}\left(\frac{\mu}{2}\right)^{s/2}(-1)^s H_s\left(\sqrt{\frac{\mu}{2}}\xi\right)\Big]d\xi, \end{aligned} \tag{116}$$

when $k = 1, 2, \ldots, k_0 - 1$. This system of integral equations can be solved by iteration using the initial condition $A_0(\xi) = 0$. For instance, when $k_0 \geq 2$, from equation (116), we get

$$A_1(\xi) = \left[(\lambda-\nu)\nu\xi + \mu(\lambda-2\nu)\frac{\xi^2}{2} - \mu^2\frac{\xi^3}{3}\right]C_0. \tag{117}$$

When $k_0 \geq 3$, from (116), we get

$$\begin{aligned} A_2(\xi) &= \Bigg[\left(\frac{\lambda^2\nu}{2}+\frac{\mu\nu}{2}+2\lambda\nu^2-\frac{3\nu^3}{2}\right)\xi \\ &- \left(\frac{\lambda^2\mu}{4}-\frac{\mu^2}{4}-2\lambda\mu\nu-\frac{\lambda^2\nu^2}{2}+\frac{9\mu\nu^2}{4}+\lambda\nu^3-\frac{\nu^4}{2}\right)\xi^2 \\ &+ \left(\frac{2\lambda\mu^2}{3}+\frac{\lambda^2\mu\nu}{2}-\frac{3\mu^2\nu}{2}-\frac{3\lambda\mu\nu^2}{2}+\mu\nu^3\right)\xi^3 \\ &+ \left(\frac{\lambda^2\mu^2}{8}-\frac{3\mu^3}{8}-\frac{5\lambda\mu^2\nu}{6}+\frac{5\mu^2\nu^2}{6}\right)\xi^4 \\ &- \left(\frac{\lambda\mu^3}{6}-\frac{\mu^3\nu}{3}\right)\xi^5+\frac{\mu^4\xi^6}{18}\Bigg]C_0 \\ &+ \left((\lambda-\nu)\nu\xi+\mu(\lambda-2\nu)\frac{\xi^2}{2}-\mu^2\frac{\xi^3}{3}\right)C_1. \end{aligned} \tag{118}$$

Finally, the general solution of the differential equation system (106), is obtained by inserting (112) into (107):

$$\varphi(\xi) = \left[\sum_{k=0}^{k_0-1} z^k(C_k + A_k(\xi))\right]\exp\left((\lambda-\nu)\xi - \frac{1}{2}\mu\xi^2\right), \tag{119}$$

with $A_k(\xi)$ given in equation (116). We notice that, there exists an independent solution for each integration constant C_k, $k = 0, 1, \ldots, k_0 - 1$.

In the case $\nu = 0$, equation (116) reduces to ($k = 1, 2, \ldots, k_0 - 1$)

$$\begin{aligned} A_k(\xi) &= \mu\sum_{l=1}^{k}\sum_{m=0}^{l}\frac{(-1)^{l+1}}{l!}\binom{l}{m}\int\xi\left[C_{k-l}\lambda^{l-m}\left(\frac{\mu}{2}\right)^{m/2}(-1)^m H_m\left(\sqrt{\frac{\mu}{2}}\xi\right)\right. \\ &+ \left.\frac{d^{l-m}A_{k-l}}{d\xi^{l-m}}\sum_{s=0}^{m}\binom{m}{s}\lambda^{m-s}\left(\frac{\mu}{2}\right)^{s/2}(-1)^s H_s\left(\sqrt{\frac{\mu}{2}}\xi\right)\right]d\xi. \end{aligned} \tag{120}$$

Thus, for instance, from equation (120), when $k_0 \geq 2$, we get

$$A_1(\xi) = \mu \left(\lambda \frac{\xi^2}{2} - \mu \frac{\xi^3}{3} \right) C_0. \tag{121}$$

When $k_0 \geq 3$, we get

$$\begin{aligned} A_2(\xi) &= \left[\left(\frac{\mu^2}{4} - \frac{\lambda^2 \mu}{4} \right) \xi^2 + \frac{2\lambda\mu^2}{3} \xi^3 + \left(\frac{\lambda^2 \mu^2}{8} - \frac{3\mu^3}{8} \right) \xi^4 - \frac{\lambda \mu^3}{6} \xi^5 + \frac{\mu^4 \xi^6}{18} \right] C_0 \\ &+ \left(\frac{\mu\lambda}{2} \xi^2 - \frac{\mu^2}{3} \xi^3 \right) C_1. \end{aligned} \tag{122}$$

In the case $\mu = 0$, equation (116) reduces to ($k = 1, 2, \ldots, k_0 - 1$)

$$\begin{aligned} A_k(\xi) &= \nu \sum_{l=1}^{k} \frac{(-1)^{l+1}}{l!} \left[(\lambda - \nu)^l \left(\xi C_{k-l} + \int A_{k-l}\, d\xi \right) \right. \\ &+ \left. \sum_{m=0}^{l-1} \binom{l}{m} (\lambda - \nu)^m \frac{d^{l-1-m}}{d\xi^{l-1-m}} A_{k-l} \right]. \end{aligned} \tag{123}$$

For instance, from this last equation, when $k_0 \geq 2$, we get

$$A_1(\xi) = (\lambda - \nu)\nu\xi C_0 \tag{124}$$

and when $k_0 \geq 3$, we get

$$A_2(\xi) = \left[\left(\frac{\lambda^2 \nu}{2} + 2\lambda\nu^2 - \frac{3\nu^3}{2} \right) \xi + \left(\frac{\lambda^2 \nu^2}{2} - \lambda\nu^3 + \frac{\nu^4}{2} \right) \xi^2 \right] C_0 + (\lambda - \nu)\nu\xi C_1. \tag{125}$$

B Matrix elements

In section 3.2.2, we need to compute the following matrix elements:

$$\Gamma_{kl} = \langle 0 | D^\dagger \left(\frac{\beta e^{i\theta}}{\sqrt{1-\delta^2}} \right) S^\dagger \left(-\tan^{-1}(\delta) e^{i\phi} \right) a^{\dagger k} a^l S \left(-\tan^{-1}(\delta) e^{i\phi} \right) D \left(\frac{\beta e^{i\theta}}{\sqrt{1-\delta^2}} \right) | 0 \rangle, \tag{126}$$

and

$$\Lambda_{kl} = \langle 0 | D^\dagger \left(\frac{\beta e^{i\theta}}{\sqrt{1-\delta^2}} \right) S^\dagger \left(-\tan^{-1}(\delta) e^{i\phi} \right) a^k a^{\dagger l} S \left(-\tan^{-1}(\delta) e^{i\phi} \right) D \left(\frac{\beta e^{i\theta}}{\sqrt{1-\delta^2}} \right) | 0 \rangle, \tag{127}$$

with $k, l = 0, 1, 2, \ldots$. Using the relation

$$S^\dagger \left(-\tan^{-1}(\delta) e^{i\phi} \right) a S \left(-\tan^{-1}(\delta) e^{i\phi} \right) = \frac{1}{\sqrt{1-\delta^2}} \left(a - \delta e^{i\phi} a^\dagger \right), \tag{128}$$

we can write them in the form

$$\Gamma_{kl} = \langle 0|D^\dagger \left(\frac{\beta e^{i\theta}}{\sqrt{1-\delta^2}}\right) \frac{\left(a^\dagger - \delta e^{-i\phi} a\right)^k \left(a - \delta e^{i\phi} a^\dagger\right)^l}{(1-\delta^2)^{\frac{k+l}{2}}} D\left(\frac{\beta e^{i\theta}}{\sqrt{1-\delta^2}}\right)|0\rangle \tag{129}$$

and

$$\Lambda_{kl} = \langle 0|D^\dagger \left(\frac{\beta e^{i\theta}}{\sqrt{1-\delta^2}}\right) \frac{\left(a - \delta e^{i\phi} a^\dagger\right)^k \left(a^\dagger - \delta e^{-i\phi} a\right)^l}{(1-\delta^2)^{\frac{k+l}{2}}} D\left(\frac{\beta e^{i\theta}}{\sqrt{1-\delta^2}}\right)|0\rangle, \tag{130}$$

respectively. From the above expressions, it is clear that

$$\Gamma_{0l} = \bar{\Gamma}_{l0} = \Lambda_{l0} = \bar{\Lambda}_{0l}, \quad \Gamma_{ll} = \bar{\Gamma}_{ll}, \quad \Lambda_{ll} = \bar{\Lambda}_{ll}, \qquad l = 0, 1, \ldots, \tag{131}$$

and

$$\Gamma_{kl} = \bar{\Gamma}_{lk}, \qquad \Lambda_{kl} = \bar{\Lambda}_{lk}, \qquad k, l = 0, 1, \ldots. \tag{132}$$

We notice that the Γ_{kl} matrix elements correspond to

$$\frac{\partial^k}{\partial\sigma^k}\frac{\partial^l}{\partial\tau^l}\langle 0|D^\dagger \left(\frac{\beta e^{i\theta}}{\sqrt{1-\delta^2}}\right) \frac{\exp\left[\sigma(a^\dagger - \delta e^{-i\phi}a)\right]}{(1-\delta^2)^{k/2}} \frac{\exp\left[\tau(a - \delta e^{i\phi}a^\dagger)\right]}{(1-\delta^2)^{l/2}} D\left(\frac{\beta e^{i\theta}}{\sqrt{1-\delta^2}}\right)|0\rangle, \tag{133}$$

when σ and τ go to zero. Applying the usual B.H.C. formula to disentangle the exponentials factors, we get

$$\exp\left[\sigma(a^\dagger - \delta e^{-i\phi}a)\right]\exp\left[\tau(a - \delta e^{i\phi}a^\dagger)\right] = \exp\left[\sigma\tau\delta^2 - \frac{1}{2}\sigma^2\delta e^{-i\phi} - \frac{1}{2}\tau^2\delta e^{i\phi}\right]$$
$$\exp\left[\left(\sigma - \tau\delta e^{i\phi}\right)a^\dagger\right]\exp\left[\left(\tau - \sigma\delta e^{-i\phi}\right)a\right]. \tag{134}$$

Inserting this result in (133), and acting with the exponential operators on the coherent states, we get

$$\Gamma_{kl} = \frac{1}{\left(\sqrt{1-\delta^2}\right)^{k+l}}\frac{\partial^k}{\partial\sigma^k}\frac{\partial^l}{\partial\tau^l}\left\{\exp\left[\sigma\tau\delta^2 - \frac{1}{2}\sigma^2\delta e^{-i\phi} - \frac{1}{2}\tau^2\delta e^{i\phi}\right]\right.$$
$$\left.\left.\exp\left[\left(\sigma - \tau\delta e^{i\phi}\right)\frac{\beta e^{-i\theta}}{\sqrt{1-\delta^2}}\right]\exp\left[\left(\tau - \sigma\delta e^{-i\phi}\right)\frac{\beta e^{i\theta}}{\sqrt{1-\delta^2}}\right]\right\}\right|_{\sigma=\tau=0}. \tag{135}$$

The matrix elements Λ_{kl} can be obtained in the same way, we get

$$\Lambda_{kl} = \frac{1}{\left(\sqrt{1-\delta^2}\right)^{k+l}}\frac{\partial^k}{\partial\sigma^k}\frac{\partial^l}{\partial\tau^l}\left\{\exp\left[\sigma\tau - \frac{1}{2}\sigma^2\delta e^{i\phi} - \frac{1}{2}\tau^2\delta e^{-i\phi}\right]\right.$$
$$\left.\left.\exp\left[\left(\sigma - \tau\delta e^{-i\phi}\right)\frac{\beta e^{i\theta}}{\sqrt{1-\delta^2}}\right]\exp\left[\left(\tau - \sigma\delta e^{i\phi}\right)\frac{\beta e^{-i\theta}}{\sqrt{1-\delta^2}}\right]\right\}\right|_{\sigma=\tau=0}. \tag{136}$$

For example,

$$\begin{aligned}
\Gamma_{00} &= \Lambda_{00} = 1, \qquad \Gamma_{01} = \bar{\Lambda}_{01} = \frac{\beta e^{i\theta} - \beta\delta e^{i(\phi-\theta)}}{(1-\delta^2)}, \\
\Gamma_{02} &= \bar{\Lambda}_{02} = \frac{\beta^2 e^{2i\theta} - \delta e^{i\phi}(2\beta^2 + 1 - \delta^2) + \beta^2\delta^2 e^{2i(\phi-\theta)}}{(1-\delta^2)^2} \\
\Gamma_{11} &= \Lambda_{11} - 1 = \frac{\beta^2(1+\delta^2) + \delta^2(1-\delta^2) - 2\beta^2\delta\cos(\phi-\delta)}{(1-\delta^2)^2}, \\
\Lambda_{12} &= \Big[\beta e^{-i\theta}\Big(\beta^2 + 2\beta^2\delta^2 + (2+\delta^2)(1-\delta^2)\Big) - \beta\delta e^{i(\theta-\phi)}\Big(2\beta^2 + \beta^2\delta^2 + 3(1-\delta^2)\Big) \\
&+ \beta^3\delta^2 e^{i(3\theta-2\phi)} - \beta^3\delta e^{i(\phi-3\theta)}\Big]/(1-\delta^2)^3. \qquad (137)
\end{aligned}$$

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Conclusion

Dans ce travail, nous avons proposé une approche originale pour générer de nouvelles classes d'états cohérents et comprimés généralisés associés à des algèbres et superalgèbres de Lie ainsi qu'à des algèbres de Lie déformées. Cette approche utilise et généralise le concept d'états propres d'algèbres associés à une algèbre de Lie.

D'abord, nous avons développé une nouvelle procédure, basée sur la méthode de prolongation des champs de vecteurs, pour obtenir les algèbres et superalgèbres de symétries dynamiques standards de quelques systèmes quantiques pertinents pour notre étude, tels que les systèmes de l'oscillateur harmonique standard et supersymétrique, de Pauli, de Jaynes-Cummings et une extension supersymétrique de ce dernier. Il s'agit d'une méthode systématique qui peut s'appliquer non seulement à des systèmes quantiques mais aussi à des systèmes classiques.

Ensuite, nous avons montré comment utiliser ces algèbres pour générer des états cohérents et comprimés généralisés associés à ces systèmes quantiques. Pour ce faire, nous avons relié les concepts d'états propres d'algèbres et états minimaux d'incertitude. En outre, nous avons construit, à partir des éléments de ces algèbres ou superalgèbres, des Hamiltoniens caractérisant ces systèmes ou encore de nouveaux systèmes quantiques. Nous avons calculé les états cohérents associés en nous servant des concepts d'états propres d'algèbres et d'opérateur d'annihilation. Pour de telles constructions, nous n'avons utilisé que les sous-algèbres fondamentales. L'utilisation d'algèbres de plus grande dimension, telles que $osp(2/2)$ et $osp(2/2) \ni\!\!\!\supset \; sh(2/2)$, est laissée pour de futures applications.

Finalement, nous avons abordé le problème de générer des états cohérents et comprimés généralisés associés à des algèbres de Lie quantiques déformées. Nous avons trouvé une représentation physique des générateurs des algèbres déformées de Heisenberg-Weyl, obtenues en utilisant la procédure de la matrice universelle R. Nous avons calculé les états propres d'algèbres et obtenu de nouvelles classes d'états cohérents et comprimés déformés associés à l'oscillateur harmonique standard. En suivant cette ligne de recherche, nous envisageons pour le futur, l'étude des états cohérents et comprimés associés à la superalgèbre déformée de Heisenberg.

Les concepts, que nous avons introduits tout au long de ce travail, sont de caractère général. Nous avons illustré ceux-ci en les appliquant à des algèbres bosoniques et fermioniques qui représentent des piliers de plusieurs théories physiques.

Le scénario naturel d'applications de résultats est le domaine de l'optique quantique ou de la mécanique quantique supersymétrique, mais ils peuvent aussi trouver des applications dans le domaine de la théorie quantique des champs.