Université de Montréal

The management of Innovation under Ambiguity

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CONTENTS

RÉSUMÉ

De récents développements en théorie de la decision ont largement enrichi notre connaissance de la notion d'incertitude knightienne, usuellement appelée ambiguïté. Néanmoins ces dévelopement tardent à être intégrés au coeur de la théorie économique. Nous suggérons que l'analyse de phénonèmes économiques tel que l'innovation et la Recherche et Développement gagnerait à intégrer les modèles de décision en situation d'ambiguïté. Nous étayons notre propos en analysant l'allocation des droits de propriété d'une découverte. Les deux premières parties de la présentation s'inspire d'un modèle d'Aghion et de Tirole, *The Management of Innovation*, portant sur l'allocation des droits de propriété entre une unité de recherche et un investisseur. Il est démontré qu'un désaccord entre les agents sur la technologie de recherche affecte leur niveau d'effort, l'allocation des droits de propriété et l'allocation des revenus subséquents. Finalement, nous examinons une situation où plusieurs chercheurs sont en compétition en s'inspirant du traitement de l'incertitude de Savage. La présence d'ambuïgité affecte le comportement des agents et l'allocation des droits de propriétés de manière qui n'est pas captée en assumant l'hypothèse de risque.

Mots clés: Incertitude, Risque, Ambiguité, Bayésianisme, néo-Bayésianisme, Savage, Innovation, Recherche et Développement, Droits de propriété, Organisation industrielle.

ABSTRACT

Recent developments in decision-theory have shed light on the concept of Knightian Uncertainty, or Ambiguity. However, this apparatus is still not fully integrated in economic theory. This presentation argue that the analysis of innovation and Research and Developments will gain substantial insights by modelling these activities as decision-making under Ambiguity. The main subject of of interest of this paper is the allocation of the property rights of a discovery. The first part of the analysis draws on a paper of Aghion and Tirole, *The Management of Innovation*, where they look at the optimal allocation of the property rights between a Research Unit and its financier. Allowing for heterogeneous beliefs affects the level of effort of the two agents, the sharing rule of the revenue of a discovery and the allocation of property rights. The second part follows Savage's framework to model an innovation competition between multiple researchers. The presence of Ambiguity impacts the behaviour of the agents and the allocation of the property rights in a way that could not be captured assuming Risk.

Keywords: Uncertainty, Risk, Ambiguity, Bayesianism, neo-Bayesianism, Savage, Innovation, Research and Development, Property rights, Industrial Organization.

LIST OF ABBREVIATIONS

αEU	Alpha-MaxMin Expected Utility
AA	Ambiguity Averse
AL	Ambiguity Loving
C	Customer
DM	Decision maker
EU	Expected Utility
Inf	Infimum
LHS	Left Hand Side (of an equation)
MaxEU	MaxMax Expected Utility
MinEU	MaxMin Expected Utility
NEU	Non Expected Utility
PH	Patent Holder
R&D	Research and Development
RHS	Right Hand Side (of an equation)
RU	Research Unit
SEU	Subjective Expected Utility
Sup	Supremum

CHAPTER 1

INTRODUCTION

Most economists agree that entrepreneurship and innovation are the backbone of capitalism. The preeminent feature of entrepreneurs is the high level of risk and Uncertainty to which they willingly expose themselves. They play the market at great personal risk, taking the chance to lose everything if their plans goes wrong. But the rewards are worth the risk, and the fortune of a successful entrepreneur far outreaches the initial investment. The growth of modern capitalist economies is dependent on the evolution of a specific type of innovation: technological innovation. Most technological innovation comes from sophisticated Research and Development (R&D) activities taking place in the private and public sector. But R& D shares many features of entrepreneurship: the risks and rewards are high, the initial investments are lost if the project fails, and they face a high level of Uncertainty. Moreover, both activities rely on a strong structure of property rights to make sure the innovator will receive the rewards for his efforts.

It is in this line of thought that I want to understand the impact of Uncertainty on the allocation of the property rights of an innovation. I leave aside entrepreneurship and focus on R&D. This choice is justified in the second chapter. R&D has been approached from different angles and a broad literature exists on the topic. I do not cover it extensively; a complete survey of the literature on R& D needs to range from Industrial Organisation to Financial economics, amongst others. A task too monumental for this essay. But numerous papers share one characteristic: they model the Uncertainty of R& D assuming Risk and the Expected Utility model. I depart from it by assuming that the Uncertainty surrounding R&D is Knightian Uncertainty, or Ambiguity.

The decision-theoretic literature about Ambiguity is still relatively new but it is also substantial. Again, it is not covered he extensively. Specific Utility functions are used. Those Utility functions are two special cases of Invariant Biseparable preferences called the MaxMin Expected Utility and the MaxMax Expected Utility. Chapter three is a short overview of the mechanics of Ambiguity where examples of Invariant Biseparable preferences are provided. A complete presentation of decision-making under Ambiguity could be found in [Gilboa and Marinacci, 2013]. A reader familiar with the topic and with Savage's framework of Subjective Expected Utility can safely go over chapter three.

The creative part of the essay is chapter four and five. Chapter four draws on a paper of Aghion and Tirole, *The management of innovation*, and builds a model of R&D where the researcher and the investor agree to disagree. I consider a situation where both agents maximise their Subjective Expected Utility but where I allow for heterogeneity of beliefs. The main result is simple: the more an agent is "optimistic" about the probability of a discovery, the more he wants to own the property rights of the innovation and the more he is willing to invest in the research project. This chapter does not build a model of decision-making under Ambiguity, although there is a direct correspondence under strict assumptions. These implicit assumptions are presented in chapter five.

Chapter five is the heart of the presentation. I model a R&D competition under Ambiguity with multiple research projects and investors. The model is based on Savage's framework of subjective probabilities but where the agents can exhibit Invariant Biseparable preferences. Building upon the insights of chapter four clear trends in the allocation of property rights are identified. Agents with the opposite Ambiguity Attitude tend to pair with each other while agreeing to allocate the property rights of the innovation to the most Ambiguity Loving of them. This is because the effectiveness of the incentive of ownership depends on the Ambiguity Attitude of the agents. This chapter concludes by looking at the impact of introducing a third agent, a patent holder. This patent holder has the ability to give access to a patent pool to a researcher at zero cost. This patent pool increases the probability of winning the R&D competition. The introduction of a patent holder affects both the allocation of resources between research projects and the ownership structure of some research projects.

CHAPTER 2

1- MOTIVATIONS

Frank Hyneman Knight, in his seminal book "Risk, Uncertainty and Profit" [Frank Hyneman Knight, 1948], makes the distinction between Risk and what is now called Knightian Uncertainty, or Ambiguity. He defines Risk as a form of Uncertainty that is measurable, such as flipping a coin or rolling a dice. It is a situation where the information is good enough to allow the decision-maker to describe all the possible states of the world with a unique probability distribution. On the other hand, Knightian Uncertainty is opaque. An agent facing Knightian Uncertainty is not only unable to predict the future, but he also has difficulty describing the relationships that may lead to this possible future. For Knight, the presence of this second type of Uncertainty presents a possibility for entrepreneurs to earn positive profits. Those profit opportunities exist even in the long run. When the entrepreneurs take risks [1] and innovate they identify profit opportunities in the economy that are not necessarily arbitrages. Entrepreneurs serve society as a whole because they open new realms of economic activity. There is now a large literature on the place of entrepreneurs in capitalism and Knight definitely had a large impact on it.

It is in the spirit of the Austrian tradition that I approach the topic of Research and Development (R&D). I focus on R&D in particular and not on innovations in general because R&D has attractive properties that makes it particularly amenable to analysis. Investing in R&D is an act of volition; the agents intends to make a discovery, to create something new that did not previously exist. Some of them are willing to take the risk of losing the money and the effort they invested. Furthermore, the R&D activity is oriented toward specific goals. Those goals take multiple forms such as inventing a new product, a new production process, etc. The goals can be ill-defined, but a least there is a direction toward which the agent wishes to progress. Those two characteris-

1. We define risk with a lower-case r as a situation where the exposition to Uncertainty can bring a bad outcome to the decision-maker. Both Risk and Knightian Uncertainty can contain risks.

tics make R&D different from other kinds of innovation that comes from some form of *tâtonnement*. Entrepreneurs can discover profits and arbitrage opportunities but they do not necessarily search for them consciously. In short, R&D needs the making of choices and judgement call before the innovation, and those choices are amenable to analysis. The last important feature of R&D is that discoveries can be protected by patents, but not all forms of entrepreneurial innovation have this privilege.

Which brings us to the main subject of the presentation. In the last decades an important body of literature has been written about Ambiguity. This literature gives the mathematical foundation and apparatus to model a higher order of Uncertainty. It could be argued that Ambiguity and Knightian Uncertainty are not the same. This distinction does not matter to me. To the extent of my knowledge few authors have used Ambiguity to analyse entrepreneurship and R&D. I intend to partially fill this gap by modelling the allocation of the property rights of a discovery. I look at the impact of Ambiguity on the incentives to invest in R&D and the allocation of the resulting patent. But the aim of this presentation is two-fold. The second goal is to show how decision-theoretic models can be used to understand economic problems such as R&D.

There are a few things I do not claim to do. I do not claim to provide a complete or exhaustive analysis of the allocation of the property rights under Ambiguity. I restrict my attention to special cases of Invariant Biseparable preferences that are readily suited to model R&D. But decision-theory can provide more than what can be covered here. I also do not claim to conduct any normative analysis of R&D and patent allocation. It will become clear that Ambiguity makes it challenging to discuss optimality, Pareto dominance, etc. This is because under Ambiguity one needs to look at the impact of *beliefs* of agents over the allocation of their resources. Finally, I do not claim to build an argument at the highest level of mathematical rigor. I use simplifying assumptions to shorten the exposition. Those assumptions are justified in the text, but I recognize that some readers can be dissatisfied by my treatment.

CHAPTER 3

DECISION-MAKING UNDER UNCERTAINTY

This chapter is divided into three sections. The first presents the ideas of Risk and Ambiguity in a neo-Bayesian framework following [Amarante, 2014]. The treatment is not rigorous but still presents some important properties of Ambiguity. The second section is an overview of Savage's framework of subjective probability. The impacts of imposing different axioms over the preference relation $\succsim$ is discussed. The third section explains some properties of decision-making under Ambiguity

3.1 An intuitive notion of Risk and Ambiguity

This section aims to present the main ideas of two papers of Massimiliano Amarante, "Foundations of neo-Bayesian statistics" [Amarante, 2009] and "What is Ambiguity?" [Amarante, 2014]. We use loose citations and only write the pages when we refer to "What is Ambiguity?". Our approach is limited but is sufficient for our purpose. We strongly invite the reader to refer to the original papers.

As mentionned, Frank Knight, in his seminal book "Risk, Uncertainty and Profit" [Frank Hyneman Knight, 1948], makes the distinction between Risk and what we now call Knightian Uncertainty, or Ambiguity. Risk is a form of Uncertainty that is measurable, such as flipping a coin or rolling a dice. It is a situation where the information is good enough to allow the decision-maker (DM) to describe all the possible states of the world with a unique probability distribution. This probability distribution does not need to be the "real" one; it could be purely subjective such as in a Bayesian setting.

On the other side, Knightian Uncertainty is a form of Uncertainty that is truly not measurable, plagued by opacity. This is a situation where the information is not good enough to allow the DM to describe all the possible states of the world with a unique probability distribution. In a neo-Bayesian settingm this Uncertainty is called Ambiguity. More exactly, the DM considers multiple probability distributions as plausible, but he is unable to pick only one of them. The three main ideas of "What is Ambiguity?" are the following:

1- Risk and Ambiguity arise because of different structures of the information available to the DM.

2- Where the Uncertainty is of the Risk type, the utility representation of the DM is additive. The family of additive utility functions is called SEU for Subjective Expected Utility. The Von Neumann-Morgenstern utility is one of them, and this family admit all possible Risk Attitudes.

3- Where the Uncertainty is of the Ambiguity type, the utility of the DM is non-additive. The broad class of non-additive utility is called NEU for non-Expected Utility. The SEU is a special case of these preferences. I refer to [Amarante, 2009] for a complete treatment. Non-additive utilities allow for the idea of Ambiguity Attitude.

Let us frame this in a simple mathematical way. Let p be a probabilistic description of all the states of the world and let P be the set of all p. Assume that P is non-empty, compact and convex. If the information is good enough, then P is a singleton and it is a situation of Risk. If the information is not good enough, then P contains two or more elements and it is a situation of Ambiguity. The Ellsberg's three-color urn experiment is the simplest example of the difference between the two concepts [Amarante, 2014, pp. 15-16].

Imagine there is an unobservable urn containing 90 balls distributed in a maximum of three colors: red, black or yellow. The DM knows the urn contains exactly 30 red balls, but he does not have any information about the distribution of the black or yellow balls. He can place a bet on one color to get picked and receive a payoff of x. The probability of picking a red ball is $p(r) = 30/90 = 1/3$. Here, $P(r)$ is a singleton and this Uncertainty is a situation of Risk. On the other hand, the probability of picking a black ball is not unique and can be described as the set $P(b) = [0, 1/90, 2/90, \dots, 60/90]$ [1]. Such a set of probability distributions characterises Ambiguity in its simplest form.

Let $u(p,x)$ be one SEU for a probability distribution p with finite payoff x. Observe that $u(p,x)$ is strictly increasing in p. Let U be the set of all $u(p,x)$ for all $p \in P$. Observe that U is closed. A NEU will be a decision rule such as the followings:

- Maxmin Expected Utility (MinEU): $MinEU = \inf U$ where inf stands for infimum. In the Ellsberg's urn example, the $MinEU$ of betting on a black ball is $\inf U = u(0 * X)$. The Maxmin Expected Utility comes from the preference relation that exhibits Ambiguity Aversion.

- Maxmax Expected Utility (MaxEU): $MaxEU = \sup U$ where sup stands for supremum. In the Ellsberg's urn example, the $MaxEU$ of betting on a black ball is $\sup U = u((2/3) * X)$. The Maxmax Expected Utility comes from the preference relation that exhibits Ambiguity Loving.

- Alpha-Max Expected Utility: $\alpha EU = \alpha * \inf U + (1 - \alpha) * \sup U$ where α stands for a coefficient of Ambiguity Aversion. The two Utility functions above are special cases of the αEU where $\alpha = 1$ and $\alpha = 0$ respectively. In the Ellsberg's urn example, the αEU of betting on a black ball is $\alpha EU = \alpha * u(0 * X) + (1 - \alpha) * u((2/3) * X)$.

One of the main characteristics of these representations is that Risk Aversion and Ambiguity Aversion are two different and complementary concepts. Risk Aversion relates to the inability of *knowing Ex-Ante which state of the world* one would have Ex-Post. Ambiguity Aversion relates to the inability to *have reliable knowledge* on the

1. I assume that $P(b)$ is equal to the superset of all possible $P(b)$.

process that picks the states of the world.

This relationship is very important. I personally consider Risk as a situation of first order Uncertainty. When a DM flips a coin he does not know in advance if he will get heads or tails. But he knows how to describe the relationship between the different states of the world and could safely assume that the probability of heads is one-half. But Ambiguity is a situation of deeper Uncertainty. The fact that the DM is unable to describe the exact relationship between the different states of the world could be understood as a form of second order Uncertainty. This is the reason why the Ambiguous Utility functions above are defined on a set of priors. This is also why behaviour under Risk is qualitatively and quantitatively different than behaviour under Ambiguity.

3.2 Invariant Biseparable preferences

Before going further I would like to comment the αEU presented above. This is a simplified version of those characterised in [Ghirardato et al., 2004] and [Amarante, 2009]. It is implicitly assumed that the coefficient α is a constant, but it is usually a function of the payoff. For example, a DM could be more Ambiguity Loving when he faces gains than losses, a situation not considered in the version above. I overlooked the preference relation $\succsim$ by assuming that the axioms of [Ghirardato et al., 2004] were respected and avoided any discussion of priors. My objective was to build a simple tool where one only needs to determine the best-case and worst-case scenario of a decision under Ambiguity. In other words, I only need to define the bounds of the set U to use the αEU. But the last model of this essay explicitly looks at the preference relation $\succsim$ and I need to expose the general idea. The presentation follows [Amarante, 2014] treatment and is almost identical to the pages 4 and 5.

In Savage's framework of [Savage, 1954], the alternatives available to the DM is a mapping $(S,\Sigma) \longmapsto X$ where (S,Σ) is a measurable space of states of the world and X

is a space of consequences.[2] Considers the set of all alternatives $\mathscr{A}$ with at least one constant alternative $\mathscr{A}_c$. Let denote some alternatives in this set as $f, g, h \in \mathscr{A}$, and one constant alternative as $i \in \mathscr{A}_c$. This constant alternative gives the same payoff x for all the possible states of the world. The preference relation $\succsim$ of the decision-maker is defined on the set of alternatives $\mathscr{A}$. In [Ghirardato et al., 2004], the authors identify a large class of preference relations that they call Invariant Biseparable preferences. These preferences share five axioms on the preference relation $\succsim$ and it is the imposition of the sixth axiom that determines which Utility function obtains. These 5 common axioms are the following:

A1 - Completion and transitivity The relation $\succsim$ is complete and transitive.

A2- C-independence For all $f, g \in \mathscr{A}$, for all $i \in \mathscr{A}_c$ and for all $\alpha \in (0, 1)$

$$f \succ g \Longleftrightarrow \alpha f + (1-\alpha)i \succ \alpha g + (1-\alpha)i$$

A3 - Archimedean property For all $f, g, h \in \mathscr{A}$, if $f \succ g$ and $g \succ h$ then there exist some $\alpha, \beta \in (0,1)$ such that $\alpha f + (1-\alpha)h \succ g$ and $\beta f + (1-\beta)h \prec g$.

A4 - Monotonicity For all $f, g \in \mathscr{A}$, if $f(s) \succsim g(s)$ for every states of the worlds $s \in S$, then $f \succsim g$.

A5 - Non-degeneracy There exist at least one set of consequences $x, y \in X$ such that $x \succ y$.

In his article [Amarante, 2014] Amarante shows that the imposition of the sixth axiom depends on the quality of the information available to the decision-maker. A decision-maker with Invariant Biseparable preferences intends to maximise his Subjective Expected Utility. If the information is good enough, it is a situation of Risk with the

2. For completeness, Σ is the metric of the space of states of the world.

following sixth Axiom:

A6 (a) - SEU For all $f, g \in \mathscr{A}$ such that $f \sim g$, $f/2 + g/2 \sim f$.

But when the information is not good enough it is a situation of Ambiguity and Axiom 6(a) cannot be imposed. The decision-maker can have different Ambiguity Attitudes that manifests in the preference relation $\succsim$. The sixth Axioms that gives the MaxMin Expected Utility (MinEU) and the MaxMax Expected Utility (MaxEU) are:

A6 (b) - MinEU For all $f, g \in \mathscr{A}$ such that $f \sim g$, $f/2 + g/2 \succsim f$.

A6 (c) - MaxEU For all $f, g \in \mathscr{A}$ such that $f \sim g$, $f/2 + g/2 \precsim f$.

Consider $\mathscr{I}_{AA}(f)$ the Ambiguity Averse Utility function that respects axioms A1 to A6 (b). It could be shown that $\mathscr{I}_{AA}(f)$ is concave in alternatives f. Furthermore, $\mathscr{I}_{AA}(f)$ exhibits superadditivity such that $\mathscr{I}_{AA}(f+g) \geq \mathscr{I}_{AA}(f) + \mathscr{I}_{AA}(g)$. Let $\mathscr{I}_{AL}(f)$ be the Ambiguity Loving Utility function that respect axioms A1 to A6(c). It could be shown that $\mathscr{I}_{AL}(f)$ is convex in alternative f and that it exhibits subadditivity such that $\mathscr{I}_{AL}(f+g) \leq \mathscr{I}_{AL}(f) + \mathscr{I}_{AL}(g)$.

3.3 Some characteristics of Ambiguity

By now it is possible to present more general properties of decision-making under Ambiguity. Let us consider two hypothetical scenarios. The first is an extension of Ellsberg's Urn experiment. Imagine that the DM can make a second bet, but this time he can choose a second ball to combine with the black ball. He can now bet either on black and yellow or on black and red. The probability of picking a black or a yellow ball is $p(b \cup y) = 1 - p(r) = 2/3$. On the other hand, the probability of picking a black or a red ball could be considered as the set $P(b \cup r) = [1/3, 31/90, 32/90, ..., 89/90, 90]$.[3]

In the second hypothetical scenario, the DM is an investor that faces a multiplicity of projects into which he can put his money. These projects are plagued by a high level of Uncertainty. He knows that if a project succeeds he will receive a very big return on his investment. But if the project fails he will lose his money. Assume that the DM is unable to define one subjective probability of success for each of the projects. That is, he has a multiplicity of priors over the probability of success for all the research projects. Therefore he is unable to make up his mind and compute a unique SEU for the return on investment. His outside (constant) alternative is to invest in a risk-free asset that pays a modest return. Four of the most fundamental properties of Ambiguity can now be stated.

1- Ambiguity is not a problem of compound lotteries. We cannot reasonably assume a probability distribution over the different priors to solve the conceptual problem that creates Ambiguity. In Ellsberg's Urn experiment, assuming that all the probability distributions of having a black ball are equally likely turns the problem into a compound lottery (remember that $P(b) = [0, 1/90, ..., 2/3]$). That is, assuming a "meta-probability" turns the initial problem into a complex situation of Risk and expels considerations about the second order Uncertainty. Assuming such "meta-probability" is arbitrary and does not give new insights on decision-making under Uncertainty. In the second thought experiments, a "meta-probability" transforms the portfolio choice into a simple situation of maximising the SEU of profit according to the investor's Risk Attitude and budget constraint.

2- An Ambiguity Averse (AA) decision-maker has a proclivity toward Risk and the constant alternative. In Ellsberg's Urn experiment the AA DM prefers to bet on the red ball if he can choose one ball. This could be deduced by comparing his SEU of a red ball to his $MinEU$ of a black ball: $EU(r) = u((1/3) * X) \geq u(0 * X) = MinEU(b)$. But

3. Again, I assume that the set $P(b \cup r)$ that the DM consider is equal to the superset of all possible $P(b \cup r)$.

when offered the possibility to bet on two balls he prefers the black and yellow because $EU(b \cup y) = u((2/3) * X) \geq u((1/3) * X) = MinEU(b \cup r)$. The AA DM *hedges* Ambiguity by mixing. Looking at axiom A6 (b), it is clear that the DM is indifferent between betting on a black or yellow ball, but prefers to bet on a mix of the two. In the second thought experiment, Ambiguity Aversion leads to two different results that depend on thestructure of his belief. The first possibility is that the DM chooses the risk-free asset and invest nothing in ambiguous projects. The second possibility is that the DM will hedge the Ambiguity by investing a small amount into a lot of projects. This happens when the structure of the beliefs allows the Utility function $\mathscr{I}_{AA}(f)$ to be concave in the alternative f of investing in a ambiguous asset. Therefore, $\mathscr{I}_{AA}(f)$ exhibits superadditivity such that $\mathscr{I}_{AA}(f+g) \geq \mathscr{I}_{AA}(f) + \mathscr{I}_{AA}(g)$ for all ambiguous assets f, g.

3- An Ambiguity Loving (AL) DM has a proclivity toward Ambiguity. In Ellsberg's Urn experiment the AL DM prefers to bet on the black ball if he can only choose one ball. This could be deduced by comparing his SEU of a red ball to his *MaxEU* of a black ball: $EU(r) = u((1/3) * X) < u((2/3) * X) = MaxEU(b)$. When offered to bet on two balls, he prefers black and red because $EU(b \cup y) = u((2/3) * X) < u(1 * X) = MaxEU(b \cup r)$. Looking at axiom A6 (c), it is clear that the DM is indifferent between betting on a black or yellow ball in combination with a red ball, but does not prefer to bet on a mix. This proclivity not to mix is reflected in the second thought experiment by a tendency to invest a large amount of money in a small number of projects. This comes from the fact that $\mathscr{I}_{AL}(f)$ is convex in the alternative f and therefore exhibits subadditivity for multiple investments.

4- A lot of Utility functions stemming from Invariant Biseparable preferences are not differentiable in the usual sense. It is not essential to present the mathematical properties of those functions but they may pose difficulties to standard analysis. In particular, non-differentiability may preclude the application of the equimarginal principle, where the marginal utility of an intrant should be equal to the marginal utility of all other intrants.

CHAPTER 4

THE BASIC MODEL

4.1 Aghion and Tirole's model under Risk

The basic model of Aghion and Tirole [Aghion and Tirole, 1994] [1] is as follows. There are two agents, a Research Unit (RU) and a Costumer (C). Only the Research Unit can conduct the research and it is cash constrained. The Research Unit input is its effort e, which is assumed not to be contractible. In the original paper the Customer is a monopoly in a good market but I will simply consider him as an investor. Its input E is assumed to be contractible. Aghion and Tirole suggest that E could be money, access to a patent pool, expertise of the Customer, etc. Here, it is assumed that E is money because the impacts of patents will be covered in chapter 5. The research technology is modelled as a probability of discovery $P(e,E)$ that is an increasing function of e and E. The research technology has standard concavity properties:

$$\partial_e P(e,E) > 0 \quad \partial_e^2 P(e,E) < 0 \quad \partial_E P(e,E) > 0 \quad \partial_E^2 P(e,E) < 0$$

$$\partial_e \hat{P}(0,E) = \infty \quad \partial_E \hat{P}(e,0) = \infty$$

I use $\hat{P}(0)$ to make explicit the fact that the productivity is infinite at a level of 0 input e or E. However, let standardize the *minimal level of* e *or* E at zero and henceforth write $P(0,0)$ as the probability of discovery at the minimum imaginable level of input. Finally, Aghion and Tirole consider that $P(e,E) < 1$ in the relevant range. Without loss of generality they assume that the cost of e and E is linear and that both agents are Risk-neutral. With V as the market value of the innovation they write the Social Planner's problem as [2]:

$$\underset{e,E}{\text{Maximise}}\; P(e,E) * V - e - E$$

1. This section is almost identical to the pp. 1185 to 1192 of the article. I slightly changed the notation.
2. They do not model the consumer surplus in the paper.

Which yields the first-best solution of :

$$\partial_e P(e,E) * V = \partial_E P(e,E) * V = 1 \tag{4.1}$$

To look at the market outcome of R&D they need to make a crucial assumption. They assume that both agents consider the sharing-rule of the profit to be renegotiable Ex-Post. The determination of the Ex-Post sharing rule will therefore follow a Rubinstein bargaining process [Rubinstein, 1982]. The equilibrium solution of this bargaining problem is to split the pot in half and gives each agent $V/2$. The agents anticipate this Ex-Post equilibrium and disregard any other Ex-Ante sharing rule. I do not agree with this hypothesis and try to propose an alternative approach in section 4.3. However, this assumption is useful because it makes clear that a property right is a discrete entity that pertains either to the Research Unit or to the Customer. When the Customer owns the property rights of the innovation he can keep the full revenue V and maximises its profits according to:

$$\underset{E}{\text{Maximise}}\ U_C^C = P(e,E) * V - E$$

Which yields:

$$\partial_E P(e,E) * V = 1 \tag{4.2}$$

This solution correspond to the first-best investment E of equation (4.1). The superscript of U stands for an index of the owners of the innovation and the subscript identifies the agents. On the other hand, the Research Unit receives no incentives at all and therefore puts its minimal level of effort 0. Writing $E^*(V)$ as the equilibrium investment as a function of V defined in equation (4.2), the Expected Utility of the two agents are:

$$U_{RU}^C = 0 \tag{4.3}$$

$$U_C^C = P(0, E^*(V)) * V - E^*(V) \tag{4.4}$$

When the Research Unit owns the innovation it maximises:

$$\underset{e}{\text{Maximise}}\, U_{RU}^{RU} = P(e,E) * V/2 - e$$

Which yields:

$$\partial_e P(e,E) * V/2 = 1 \tag{4.5}$$

That is written as $e^*(V/2)$. By the same argument the optimal investment of the Customer is $E^*(V/2)$. Both $e^*(V/2)$ and $E^*(V/2)$ are sub-optimal with regard to the first-best solution defined in equation (4.1). The Expected Utility of the two agents are the following:

$$U_{RU}^{RU} = P(e^*(V/2), E^*(V/2)) * V/2 - e^*(V/2) > 0 \tag{4.6}$$

$$U_{C}^{RU} = P(e^*(V/2), E^*(V/2)) * V/2 - E^*(V/2) \tag{4.7}$$

Equation (4.6) implies that the Research Unit *always prefers* owning the property rights. This fact will be useful later. But what does the Customer prefer? Clearly, if $U_C^{RU} > U_C^C$ then the Customer will let the property rights go to the Research Unit. But if $U_C^{RU} < U_C^C$ then the Customer wants to own the innovation. The property rights allocation will then be determined by the Ex-Ante bargaining power of the two parties. If the Research Unit have the bargaining power it will always keep the property rights to itself. In this case, if $U_{RU}^{RU} + U_C^{RU} > U_{RU}^C + U_C^C$, then the Research Unit allocates the property rights efficiently and it is a second-best equilibrium. If the Customer has the bargaining power and $U_C^{RU} < U_C^C$, he allocates the property rights to himself because the Research Unit is cash constrained. Therefore, if simultaneously $U_C^{RU} < U_C^C$ and $U_{RU}^{RU} + U_C^{RU} > U_{RU}^C + U_C^C$, then this allocation is an inefficient equilibrium that is sub-optimal, even compared to a second-best equilibrium. In the words of Aghion and Tirole, the allocation of the property rights does not only affect the distribution of the pie but also its size.

4.1.1 Critique of the model

I would like to give my interpretation of the assumption that the probability of discovery $P(e,E)$ is objective, given and known by the two agents. This assumption implies that there exists in the real world a unique probability that specific individuals in a specific period of time will have a specific idea. This idea, that nobody else in the economy already had, requires a certain amount of input as money and effort to come into the world. Furthermore, there exists a very specific relationship between the quantity of money and effort invested and the probability of discovery. Both agents know this relationship, know that the other knows, and they can agree on a contract that depends on the exact form it takes.

I believe that the conclusions of the model are somewhat embedded in this single assumption. If there exists a single objective function of discovery known by both agents their decisions will be almost exclusively determined by it. The Uncertainty here is of the same type as flipping a coin or rolling dice, and decision-makers only need to choose how much they are willing to bet on a pair of sixes. Considering an objective probability of discovery leaves little or no space for human decisions or interactions. In other words, there are no qualitative nor quantitative differences between the choices that venture capitalists or compulsive gamblers face.

The remainder of this chapter tries to lift the assumption that the technology of research $P(e,E)$ is given and objective. To be fair, I consider that Aghion and Tirole's model is very clear and gives us important insights. I intend to keep its essence. I will understand it as a benchmark case when the two agents maximise their SEU and share a common prior. However, I allow the agents to disagree on their subjective assessment of the probability of discovery. The next section aims to understand the impact of these disagreements on the allocation of the property rights.

4.2 Introducing "Ambiguity" in the model

As said, Aghion and Tirole frame their model in the context of an objective probability of discovery and assume standard Expected utility functions. The model could be translated in the context of Savage's framework by assuming that the two agents share a common prior. But the objective of this presentation is to introduce some "Ambiguity" in this model. Therefore, let us assume that the two agents are SEU maximisers, but let us also allow them to diverge on their assessment of the probability of discovery. The approach in this section is not a proper model of decision-making under Ambiguity; it is a model of heterogeneous beliefs where the agents agree to disagree. This reformulation of Aghion and Tirole's model allows us to look at the impact of divergences in beliefs on the allocation of the property rights. This, without the need to introduce a lot of new notation. There is a natural correspondence between what I present here and Ambiguity and I will make this relationship explicit throughout the text.

Let us consider three different functions $P(e,E)$ such that $P_L(e,E) < P(e,E) < P_H(e,E)$ for all e and E. Here, the subscript L stands for Low probability of discovery, the subscript H stands for High probability of discovery and the absence of subscript means that we use the same $P(e,E)$ as Aghion and Tirole. Therefore, we have a set of three different priors that the decision-makers can believe. An agent that is "pessimistic" adopts the Low probability prior, and an agent that his "optimistic" adopts the High probability. We have that $\partial_e P_L(e,E) < \partial_e P(e,E) < \partial_e P_H(e,E)$ for all e. It immediately follows that $e^*_L(V/2) < e^*(V/2) < e^*_H(V/2)$. We say that the more "optimistic" the Research Unit is, the more he is willing to invest in the project. The same result holds for the input E of the Customer.

The relation of this model to a situation of Ambiguity is already clear if we understand the set $\{P(e,E)\}$ as a set of priors. Assuming that there are no outside options (constant alternatives) and no possibilities for the agents to mix, the "pessimistic" type is an Ambiguity Averse decision-maker and the "optimistic" type is Ambiguity Loving .

We can also understand the agent that believes in $P(e,E)$ as a SEU maximiser, or somebody that does not perceive Ambiguity. For the remainder of this chapter I will therefore refer to the three types of agent using their corresponding Ambiguity Attitude. It is an abuse of nomenclature that will become useful in chapter 5.

4.2.1 Under Common Knowledge

To start let us assume that both agents *know* the Ambiguity Attitude (type) of their partner. That is, let us assume some form of Common Knowledge, where an agent knows the belief of the other. Recall that by equation (4.6) above the Research Unit *always prefers* to own the property rights of the innovation. Therefore, it is only necessary to look at the impact of a divergence of beliefs on the Customer's incentives to keep or let go the property rights. It is trivial to see that when both agents are of the same type they have a common prior in the set $\{P(e,E)\}$. In this situation we obtain the model of Aghion and Tirole but with a different probability distribution.

Let $a \in A$ be the Ambiguity Attitude of the Research Unit and $b \in B$ be the Ambiguity Attitude of the Customer. Here, $A = B := \{\text{Amb. Averse, Amb. "Neutral", Amb. Loving}\}$. By a second abuse of notation let us use the subscript of the probability of discovery to index the Ambiguity Attitude of the agents. Therefore, if C_b stands for the general Ambiguity Attitude of the Customer, C_L means that he is Ambiguity Averse, C_H means that he is Ambiguity Loving and C means that he considers $P(e,E)$. The same goes for the Research Unit. Therefore, $U_C^{RU}|_{C_L,RU}$ is understood as "the Utility of the Customer when the Customer is Ambiguity Averse and the Research Unit is Ambiguity Neutral".

Let us first consider the Customer's Subjective Expected Utility when he owns the property rights. The Research Unit put his minimal level of effort, standardised at 0. The Customer's SEU is therefore:

$$U_C^C|_{C_b,RU_a} = P_b(0, E_b^*(V)) * V - E_b^*(V)$$

This level of Utility needs to be compared to the Customer's Utility when the Research Unit owns the property rights, which is:

$$U_C^{RU}|_{C_b,RU_a} = P_b(e_a^*(V/2), E_b^*(V/2)) * V/2 - E_b^*(V/2)$$

When the agents disagree then one will be more optimistic than the other. This gives us two general situations that could be expressed in two propositions.

Proposition 1: The more the Customer is Ambiguity Averse in comparison to the Research Unit, the more incentives the Customer will have to let the property rights go. This comes from two facts: A- The Customer considers that the Research Unit will *over-invest* in the research project with regard to the Customer's belief about the probability of discovery. In other words, the Customer considers that allocating the property rights to the Research Unit gives him *more incentives* than in the standard model. B- A lower probability of discovery $P(e, E)$ implies a lower level of effort E for the Customer. This is true whoever owns the property rights. Therefore, the more Ambiguity Averse the Customer is, the less he wants to invest in the project. This gives him a strong incentive to surrender the property rights.

1- Case A: The customer is Ambiguity Averse. He will compare his Utility when he owns the property rights:

$$U_C^C|_{C_L,RU_{a\neq L}} = P_L(0, E_L^*(V)) * V - E_L^*(V) \tag{4.8}$$

with the utility he would have by letting the Research Unit own the property rights:

$$U_C^{RU}|_{C_L,RU} = P_L(e^*(V/2), E_L^*(V/2)) * V/2 - E_L^*(V/2) \tag{4.9}$$

$$U_C^{RU}|_{C_L,RU_H} = P_L(e_H^*(V/2), E_L^*(V/2)) * V/2 - E_L^*(V/2) \tag{4.10}$$

1- Case B: The customer is Ambiguity "Neutral" and maximises his Subjective Expected Utility with regard to $P(e, E)$. He will compare his Utility when he owns the property rights:

$$U_C^C|_{C,RU_H} = P(0, E^*(V)) * V - E^*(V) \tag{4.11}$$

with the Utility he would have by letting the Research Unit own the property rights:

$$U_C^{RU}|_{C,RU_H} = P(e_H^*(V/2), E^*(V/2)) * V/2 - E^*(V/2) \tag{4.12}$$

These equations show the importance of the Common Knowledge hypothesis. Equation (4.10) implies that the Customer will invest very little money in the project and still receive $V/2$ if there is a discovery. In other words, the Customer benefits a lot from the Ambiguity Loving attitude of the Research Unit. Without the Common Knowledge assumption this becomes a strategic situation. Let us come back on this possibility later.

Proposition 2: The more the Customer is Ambiguity Loving in comparison to the Research Unit, the more incentives the Customer will have to keep the property rights. This comes from two facts: A- The Customer considers that the Research Unit will *under-invest* in the research project with regard to the Customer's belief about the probability of discovery. In other words, the Customer considers that allocating the property rights to the Research Unit gives him *less incentives* than in the standard model. B- A higher probability of discovery implies a higher level of effort E for the Customer. Therefore, the more Ambiguity Loving the Customer is, the more he wants to invest in the project and the more incentives he has to keep the property rights.

2- Case C: The customer is Ambiguity Loving. He will compare his Utility when he owns the property rights:

$$U_C^C|_{C_H,RU_{a \neq H}} = P_H(0, E_H^*(V)) * V - E_H^*(V) \tag{4.13}$$

with the utility he would have by letting the Research Unit own the property rights:

$$U_C^{RU}|_{C_H,RU_L} = P_H(e_L^*(V/2), E_H^*(V/2)) * V/2 - E_H^*(V/2) \tag{4.14}$$

$$U_C^{RU}|_{C_H,RU} = P_H(e^*(V/2), E_H^*(V/2)) * V/2 - E_H^*(V/2) \tag{4.15}$$

2- Case D: The customer is Ambiguity "Neutral" and maximises his Subjective Expected Utility with regard to $P(e,E)$. He will compare his Utility when he owns the property rights:

$$U_C^C|_{C,RU_L} = P(0, E^*(V)) * V - E^*(V) \tag{4.16}$$

with the utility he would have by letting the Research Unit own the property rights:

$$U_C^{RU}|_{C,RU_L} = P(e_L^*(V/2), E^*(V/2)) * V/2 - E^*(V/2) \tag{4.17}$$

Finally, let us consider a fourth possible type of Customer. This new type is a mix between the "pessimistic" and the "optimistic". That is, let us consider the αMaxMin Expected Utility (αEU). Assume that the fourth type chooses a level of effort E_α in between the two opposite optimal level, E_L^* and E_H^*. There are two possible levels of effort that are a function of the sharing rule V. Thus:

$$E_{\alpha|C}^* = \alpha E_L^*(V) + (1-\alpha) E_H^*(V)$$

$$E_{\alpha|RU}^* = \alpha E_L^*(V/2) + (1-\alpha) E_H^*(V/2)$$

where $E_{\alpha|C}^*$ stands for "the level of effort E for an αEU Customer knowing that the Customer owns the property rights". Here, α represents the degree of "pessimism"/ Ambiguity Aversion. The Customer's Utility when he owns the property rights is:

$$\alpha EU_C^C = \alpha[P_L(0, E_{\alpha|C}^*) * V - E_{\alpha|C}^*] + (1-\alpha)[P_H(0, E_{\alpha|C}^*) * V - E_{\alpha|C}^*]$$

$$\alpha EU_C^C = \alpha[P_L(0, E_{\alpha|C}^*) * V] + (1-\alpha)[P_H(0, E_{\alpha|C}^*) * V] - E_{\alpha|C}^* \tag{4.18}$$

It is to be compared with the Customer's Utility when the Research Unit owns the property rights:

$$\alpha EU_C^{RU} = \alpha[P_L(e_a^*(V/2), E_{\alpha|RU}^*) * V/2 - E_{\alpha|RU}^*] + (1-\alpha)[P_H(e_a^*(V/2), E_{\alpha|RU}^*) * V/2 - E_{\alpha|RU}^*]$$
$$\alpha EU_C^{RU} = \alpha[P_L(e_a^*(V/2), E_{\alpha|RU}^*) * V/2] + (1-\alpha)[P_H(e_a^*(V/2), E_{\alpha|RU}^*) * V/2] - E_{\alpha|RU}^* \tag{4.19}$$

Taking equation (4.18) and equation (4.19) together show that for any Ambiguity Attitude α of the Research Unit there will be a "tipping point" where the Customer will either prefer to keep or to relinquish the property rights.

4.2.2 Without Common Knowledge

Let us now lift the assumption of Common Knowledge. Remember that in any situation the Research Unit prefers to have the property rights. Therefore the RU have a strong incentive to show that it is the optimistic/Ambiguity Loving type. This type is hard-working and invest the highest level of effort $e^*(V/2)$, which gives more incentives to the Customer to surrender the property rights. If the Research Unit has a signalling device this situation is identical to the analysis of the previous subsection. But if he does not have a signalling device then it is a situation of Cheap Talk where the Research Unit has no credibility. Therefore, the Customer will act according to his own beliefs over the type of the Research Unit. Let again $a \in A$ stands for the Ambiguity Attitude of the Research Unit such that $A := \{\text{Amb. Averse, Amb. "Neutral", Amb. Loving}\}$. Remember that if the Customer owns the property rights its Utility is:

$$U_C^C|_{C_b, RU_a} = P_b(0, E_b^*(V)) * V - E_b^*(V) \tag{4.20}$$

Case E: Assume that the Customer has a prior Π_A over the types A of the Research Unit. Therefore, π_a is the probability that the Customer accords to the Research Unit of being the type $a \in A$. The Utility of the Customer for letting go the property rights is :

$$U_C^{RU}(\Pi_A)|_{C_b} = \sum_{a \in A} \pi_a * [P_b(e_a^*(V/2), E_b^*(V/2)) * V/2] - E_b^*(V/2) \tag{4.21}$$

which is to be compared to equation (4.20). This is a standard situation of maximisation of Subjective Expected Utility.

Case F: Finally, consider that the Customer is unable to have a prior Π_A over the Research Unit type. That is, the Customer faces some form of "Ambiguity". At the exception of the αEU, the choice of the Customer will be "as if" the two agents where of the same type/Ambiguity Attitude. In other words, it is "as if" they shared a common prior. A pessimistic Customer will consider that the Research Unit is pessimistic, and vice-versa. The Ambiguity Averse Customer Utility is:

$$U_C^{RU}|_{C_L} = P_L(e_L^*(V/2), E_L^*(V/2))V/2 - E_L^*(V/2) \tag{4.22}$$

And the Ambiguity Loving Customer Utility:

$$U_C^{RU}|_{C_H} = P_H(e_H^*(V/2), E_H^*(V/2))V/2 - E_H^*(V/2) \tag{4.23}$$

Those two equations have to be compared to equation (4.20). Because in the belief of the Customer the two agents are of the same type, the analysis collapses in the same problem than the basic model of Aghion and Tirole. The only difference is that the Customer considers other probabilities of discovery than $P(e, E)$.

4.3 Endogenous sharing rule

In the section II.4 of their article (pp.1192-1195) Aghion and Tirole allow for co-financing by multiple investors. This transforms the choice of the allocation of the property from a discrete to a continuous one. I would like to emulate their idea but use a different approach. Their basic model could be understood as a game with the following form:

1- Nature - Determines the bargaining power of the two parties.
2- The agents choose the property rights allocation and the Ex-Ante sharing rule V.
3- The agents choose their respective efforts e, E.
4- Nature - Discovery or not.
5- Renegotiation of the sharing rule V.

In this game, Aghion and Tirole consider that the only re-bargaining solution of step five is $V/2$. They justify this stance using Rubinstein's paper on Perfect Equilibrium of a Bargaining Game [Rubinstein, 1982]. The agents therefore regard $V/2$ as the Ex-Ante sharing rule because it seems to be renegotiable Ex-Post.

I believe that this is an incomplete story. It is clear that the sharing rule *determines* the incentives to invest. In the words of Aghion and Tirole, the allocation of the property rights affects not only the distribution of the pie but also its size. Therefore, both agents should recognized that they will be better off if they can choose an optimal sharing rule Ex-Ante and stick to it. If they can commit to this Ex-Ante sharing rule then $V/2$ is not necessarily sub-game perfect. [3] Assuming they have an effective commitment device I want to look at the following game:

3. It should be noted that in Rubinstein's paper the size of the pie is given before the bargaining process.

1- Nature - Determines the bargaining power of the two parties.

2- The agents choose the property rights allocation and the Ex-Ante sharing rule V.

3- The agents choose their respective efforts e, E.

4- Nature - Discovery or not.

5- No renegotiation.

Which collapses in steps 1 to 4. Before continuing further I want to apologise to the reader. Due to my limited knowledge of Game Theory I proved unable to work out the Nash Equilibrium of this game. I do not know if there exists a unique equilibrium or if multiple equilibriums can arise. In particular, I am unable to determine whether the bargaining power of the parties is relevant or not. It seems possible that perfectly rational agents can find a solution to the game without the requirement of one of them imposing his will. But if the bargaining power reveals to determine the equilibrium its impact seem to be weakened when the agents can choose their business partners. I present this possibility in the next chapter.

4.3.1 Nash Equilibrium in the Bayesian game

Let the sharing rule $x \in X = [0,1]$ be the part of the revenue V that goes to the Customer[4]. When $x = 1$ it is the same situation as in the previous chapter; the Customer is the owner of the innovation and the Research Unit put $e = 0$ effort. Remember that $e = 0$ was standardised as the minimal level of effort possible. But remember also that the Research Unit was cash constrained and could not pay for a "minimal level of effort E". Therefore, let us exclude from the analysis the possibility that $x = 0$. For simplicity, let us assume that the agent with the biggest share of V owns the property rights. After all, intellectual property is always a discrete entity that pertains to only one agent. It should also be assumed that when $x = 0.5$ the agents have a mechanism to ascribe the

4. Let keep the term Customer to stay in line with the previous sections. However, the model could be extended to a situation of co-financing by m investors, with the investors $j \in J$ receiving $x_j * V$ and investing the level of effort $E_j^*(x_j * V)$.

property rights to one of them. In step 3 we have the Customer's problem:

$$\underset{E}{\text{Maximise}}\, U_C = x * P(e,E) * V - E$$

which yields:

$$x * \partial_E P(e,E) * V = 1 \tag{4.24}$$

and the Research Unit problem:

$$\underset{e}{\text{Maximise}}\, U_{RU} = (1-x) * P(e,E) * V - e$$

which yields:

$$(1-x) * \partial_E P(e,E) * V = 1 \tag{4.25}$$

From equation (4.24) and (4.25) let us write the optimal levels of effort as $E^*(x*V)$ and $e^*((1-x)*V)$ respectively. It is trivial to see that $\partial_x E^*(x*V) > 0$ and $\partial_x e^*((1-x)*V) < 0$. That is, the optimal investment/effort of the agents is an increasing function of their share of V. Now the problem is to find if there exists an $x^* \in X$ such that simultaneously $U_C(x^*) > U_C(x)$ and $U_{RU}(x^*) > U_{RU}(x)$ for all $x \in X$. The Utility function of the players are:

$$\begin{aligned}
U_C(x^*) &= x^* * P(e^*((1-x^*)*V), E^*(x^* * V)) - E^*(x^* * V)\\
U_C(x) &= x * P(e^*((1-x)*V), E^*(x*V)) - E*(x*V)\\
U_{RU}(x^*) &= (1-x^*) * P(e^*((1-x^*)*V), E^*(x^* * V)) - e^*((1-x^*)*V)\\
U_{RU}(x) &= (1-x) * P(e^*((1-x)*V), E^*(x*V)) - e^*((1-x)*V)
\end{aligned}$$

And this is where my limited knowledge becomes a vexation. For the two agents, a higher part of the revenue V means a higher profit, but only in a certain range. The agents need to recognise the incentives effect of giving more benefits to their partners. Let us solve analytically two specific cases. Without loss of generality assume that

$P(e,E) = \sqrt{e} + \sqrt{E}$. With this research technology the two optimal level of effort are $e^* = ((1-x)*V/2)^2$ and $E^* = (xV/2)*^2$. Substituting in the Customer's Utility function gives:

$$U_C(x) = V^2/2*(x - x^2/2)$$

a simple parabola that passes through 0 and reaches its maximum at $x = 1$. The Research Unit's Utility is:

$$U_{RU}(x) = V^2/4*(1-x^2)$$

which is maximised at $x = 0$. Total surplus $U_C(x) + U_{RU}(x)$ is maximised at $x = 0.5$. In this benchmark situation, each agent prefers to own the innovation, but total surplus is maximised at $V/2$. It is trivial to see that whenever the two agents are equally productive the total surplus is maximised at $x = 0.5$. Working the same calculation with $P(e,E) = e^{1/3} + E^{1/3}$ it is possible to show that $U_C(x)$ is maximised at $x = 0.88$ and $U_{RU}(x)$ is maximised at $x = 0.12$. The more productive the agents are, the smaller the range that the Nash Equilibrium x^* might be. In other words, higher productivity gives incentives to share more equally.

It is unfortunate that I, the writer, am unable to characterize the Nash Equilibrium of this game. I would like to model "Ambiguity" using different probabilities of discovery and look at their impact on equilibrium. Let's take for example a situation of asymmetric beliefs such as in the previous section. Assume that the Research Unit believes that $P(e,E) = \sqrt{e} + \sqrt{E}$ and chooses the optimal effort $e^* = ((1-x)V/2)^2$. But now assume that the Customer believes that $P(e,E) = e^{1/3} + E^{1/3}$ and chooses $E^* = (x*V/3)^{3/2}$. That is, the Customer is more optimistic about the probability of discovery than the Research Unit. Assuming Common Knowledge, the Utility of the two agents are:

$$U_C(x) = x*V*(((1-x)V/2)^{2/3} + (x*V/3)^{1/2}) - (x*V/3)^{3/2}$$

$$U_{RU}(x) = (1-x)*V(((1-x)V/2) + (x*V/3)^3) - ((1-x)V/2)^2$$

Drawing from the intuition of the previous sections, I conjecture that there will be at least two important considerations in the choice of x. To start, the agents will choose independently their level of effort according to their own beliefs. This means that for any sharing rule x, the Customer considers that the Research Unit will under-invest relative to the Customer's belief about the productivity of e. Alternatively, the Research Unit considers that the Customer will over-invest relative to the Research Unit's belief about the productivity of E. Logically, this should give a strong incentive to *both* of the agents to give a bigger share x of V to the Customer. However, I proved unable to solve this problem. Therefore I will assume for the rest of this essay that this intuition holds in any situation.

CHAPTER 5

A MARKET OF R&D

This final chapter will explore what happens to the allocation of property rights in a market with multiple researchers and investors. The incentives of the agents are described, but no equilibrium for the whole market is characterised. However, important elements that affects the dynamic of this market are exposed. Essentially, this section examines how three agents differs from each other in terms of bargaining power, incentives and opportunities to make money.

Let us consider a one time R&D competition for an innovation. Assume a winner-takes-all market, where only one firm wins the R&D competition and makes V in revenue from the innovation. The specific source of this revenue does not need to be modelled; it could comes from the possibility of establishing a monopoly in a good market or as a prize in a competition held by an external party. The last possibility is interesting because it allows a Social Planner to give incentives to firms to invest in R&D while taking into account the Consumer Surplus of the innovation. Substantial changes to the framework and notation of the previous chapter are required to fully capture the essence of the market. Consider that there is n researchers that follow different research paths or research projects. Assume that a researcher cannot pursue more than one research path at a time. Therefore, let R_i be the researcher of the project $i \in I$ such that $|I| = n$. Let use i interchangeably to describe either the research projects i or the researcher i. It is an abuse of notation but it simplifies the explanation. As in the previous section, a researcher i invest his effort e_i that is still assumed not to be contractible. But now let standardize *the minimal level of effort e* for the research project to be viable as e^0 instead of 0.

On the other side of the market there is the investors. Consider a pool of m possible investors j that are interested in the research projects such that $j \in J$, $|J| = m$. The input of these investors is *only* money, a lump-sum payment of E made up-front. Let standardise *the minimum level of effort E* as E^0. This E^0 is understood as the minimum amount that covers the fixed costs of a research project. These fixed costs do not include the wages of the researchers. Therefore, E^0 and e^0 are required to initiate the research and any amount above this threshold augment the "probability of discovery" in the same way than the previous sections. Every investor has a budget constraint b_j such that $b_j < \infty$. The problem of investor j is to build a portfolio $\mathscr{P}_j = \sum_{i \in I} E_{j,i}$ such that $\mathscr{P}_j \leq b_j$. Finally, the total amount of money available for investment is $B = \sum_{j \in J} b_j$.

The third agent is a firm with access to a patent pool that can help the research projects. Such a firm could be an incumbent monopoly in the good market toward which the innovation is oriented. For simplicity let consider only one firm, but the results are easy to generalise. This firm has to decide whether it will allow the researchers i to access the patent pool. Let write $q = 0$ or $q = 1$ the input of this firm, where $q = 1$ means that the access is allowed. This patent pool augments the probability of a research project to win the innovation race. Remember that in the model of Aghion and Tirole the effort E of the Customer could be input in the form of expertise, patents and money. It is important to distinguish between these last two possibilities for three reasons. First, they will have a different impact on the market. Second, I do not believe that giving access to a patent pool is costly and, therefore I assume that the cost of $q = 1$ is zero. Third, it is reasonable to model the investment of money as a continuous function, an assumption that cannot be made for the patents.

To model Ambiguity in this innovation competition it is necessary to recall Savage's framework. Assume that for each research project i the probability of discovery $P^i(e, E, q)$ is ill-defined, but that $\sum_{i \in I} P^i(e, E, q) \to 1$. For simplicity let assume that $\sum_{i \in I} P^i(e, E, q) = 1$, which means that there will be one winner with certainty. Our space of states of the world (S, Σ) contains n elements; this set (S, Σ) described each possibility that firm $i \in I$ wins. Of course, the space X of consequences described the allocation of the revenue V to the winner i. What is left for analysis is the set of alternatives $\mathscr{A}$ and the impact that different Ambiguity Attitudes will have on the allocation of property rights. Before going further let make an observation that will be useful in section 2. The assumption that $\sum_{i\in I} P^i(e,E,q) = 1$ implies that an investor who invests in all the research projects i for a fixed claim x will receive $x*V$ with certainty. That is, the ability to "cover" every research project $i \in I$ eliminates the Uncertainty. Therefore, if $\mathscr{P}_j = \sum_{i\in I} E_{j,i} \leq x*V$, the investors could want to buy all the research projects because there is an opportunity for profit without any risk.

The first two sections consider the relationship between only two types of agent, the researchers and the investors. Although the research technology is expressed as $P(e,E,q)$ for completeness, it is implicitly assumed that $q = 0$ until section three.

5.1 The researchers

A researcher i has three possibilities. First, he can opt-out of the innovation competition and go to work somewhere else. It is assumed that he does not need to make any effort in this outside option, so $e = 0$. He will definitely receive the utility of $U(e = 0)$, or $U(0)$. This level $U(0)$ is a constant alternative in the set $\mathscr{A}_i$. The participation constraint for entering the innovation competition is therefore $U(e > 0) > U(e = 0)$. Let consider a second possibility for the researcher where he can strike a deal with some investors and enter the competition as a simple worker. In other words, he can surrender the property rights and any incentives to invest more than the minimal level of effort e^0. In exchange he is guaranteed to receive an amount of money such that $U(e^0 > 0) > U(0)$. Let express this amount as the wage $w_i(e^0)$ of the researcher i. If it is assumed that the researcher can always take this kind of opportunity, then it gives a second constant alternative in the set $\mathscr{A}$. By construction, the outside option with $e = 0$ becomes irrelevant and the only possibility that matter is the choice of $e^*((1-x)*V) \geq e^0$. Remark however that the participation constraint gives the researcher some bargaining power.

Following the model of section 4.3, consider x the share that goes to the investor in case of a discovery. The incentive constraint of the researcher is $U(e((1-x)*V)) > U(e^0)$. But this alternative faces Uncertainty, and it is better to express the Utility function of the researcher as $\mathscr{I}^i(e((1-x)*V))$ to make clear that he can consider Ambiguity. Remember that the researcher *cannot work in more than one project* so there is no need to worry about mixing. This case of decision-making under Uncertainty becomes quite simple. The choice of the researcher will be determined by the exact structure of his beliefs and by his Ambiguity Attitude. The first possibility is that the researcher is a Subjective Expected Utility maximiser and considers only one prior over the probability of discovery $P(e,E,q)$. In this case, he will prefer the situation when he takes some risks if and only if $SEU(\pi, e^*((1-x^*)*V)) > U(e^0)$. [1] The left-hand side (LHS) of the inequality makes it clear that this researcher maximises his Subjective Expected Utility with regard to his prior π. This is a standard result. When the researcher faces Ambiguity he considers multiple priors $\pi \in \Pi$ over the probability of discovery $P(e,E,q)$. Because the researcher cannot mix, an Ambiguity Averse researcher has the following Utility:

$$\mathscr{I}^i_{AA}(e((1-x)*V)) = SEU(\inf \pi \in \Pi, e((1-x)*V)) \tag{5.1}$$

where the right-hand-side (RHS) of the equation means that the researcher only cares about the lowest probability of discovery he believes possible. On the other hand, the Ambiguity Averse researcher Utility function is:

$$\mathscr{I}^i_{AL}(e((1-x)*V)) = SEU(\sup \pi \in \Pi, e((1-x)*V)) \tag{5.2}$$

Equations (5.1) and (5.2) are in direct correspondence with the previous chapter, where $\inf \pi \in \Pi$ is interpreted as P_L and $\sup \pi \in \Pi$ as P_H. Using the insights gained earlier it is easy to look at the allocation of property rights and the incentives of the customer to invest his effort e.

1. Again, risk with a lower-case r denotes a situation when an agent expose himself to Uncertainty but where it is not specified whether it is Risk or Ambiguity.

To start, remark that an Ambiguity Loving researcher prefers to own a large share of the value $(1-x)*V$. This researcher is likely to put a high-level of effort e, a tendency that is magnified by the incentive effect of having a large share. The AL researcher wants to own the property rights of the innovation. If he meets the right partner he will get at least enough money to cover the fixed cost of the research. The AL researcher would like to have an investment higher than E^0, but he is reluctant to let go a big share of V. It will be clear in the next section that the natural partners of AL researchers are AA investors. I personally like to think about this type of researcher as an entrepreneur-innovator *à la* Silicon Valley.

At the other side of the spectrum is the Ambiguity Averse researchers. Introducing the option of working for a fixed wage $w_i(e^0)$ radically changes the model of the previous chapter: the AA DM *does not necessarily always prefer* to own the property rights of the innovation. Depending on the structure of their beliefs, some Ambiguity Averse researchers will prefer to be a worker that receives a lump-sum wage of $w_i(e^0)$. This happens when $SEU(\inf \pi \in \Pi, e((1-x)*V)) < U(e^0)$. But even the Ambiguity Averse researchers that do prefer to own a claim over the value V of the innovation tend to invest a low level of effort e. It takes a high level of incentive to make them do otherwise, and even then the effort will be lower than the Ambiguity Loving researchers. It is clear that the Ambiguity Averse researchers are more likely to relinquish the property rights of the innovation. This feature is interesting because it nuances the results of traditional Principal-Agent approach. Under Risk, the incentives given to an agent almost necessarily induces greater effort. This effort is discounted by the Risk Aversion of the decision-makers. But in this model it is possible to have Risk-neutral but Ambiguity Averse researchers for whom incentives are ineffective.

5.2 The investors

The analysis on the investors side is the reason why it was necessary to introduce Savage's framework in its entirety. Up to this point it could had been assumed that agents were maximising their Subjective Expected Utility but with different priors and yet obtain the same kind of results. It is not true for this section. Assume that there exists in the economy a risk-free asset that pays with certainty $(1+r)*E$, where r stand for the interest rate. [2] Investing in this risk-free asset is the constant alternative $\mathscr{A}_c$ of the set $\mathscr{A}$ of the investors. It defines the reservation Utility of the investors. The important difference between them and the researchers is their ability to invest in more than one research project. Assuming continuity, the investors' space of alternatives is a vector in $\mathbb{R}^{n+1}$ that is bounded by their budget constraint. Here, the alternative f is to invest into a research project i.

Now let us examine the preference relation $\succsim$ over these alternatives. To start, assume that the investors consider every research project i as "equally likely" to win the innovation race. [3] That is, assume that their preferences is such that $f \sim g$ for all $f, g \in \mathscr{A} \setminus \mathscr{A}_c$. In other words, $f \sim g$ for all the project $i \in I$. It is easy to check that the axioms A1 to A5 of chapter two are verified. The sixth axiom that determines the Ambiguity Attitude of the investors can now be imposed. By definition, an investor is a Subjective Expected Utility maximiser if simultaneously $f \sim g$ and $g/2 + h/2 \sim h$ holds. In this particular case, a SEU investor wants to construct an optimal portfolio according to a uniform prior. This is a standard problem, with the specificity that it is possible to make the revenue $x * V$ with certainty by adopting the strategy described at the start of this chapter.

At one extreme there are the Ambiguity Averse investors. By axiom A6(b), a decision-maker is Ambiguity Averse if simultaneously $f \sim g$ and $f/2 + g/2 \succsim f$ holds. Remember

2. It is possible to add to this model a Risky asset that pays $(1+r+z)*E$ in Expectation, where z stands for a Risk premium over the interest rate. However, I do not think that any insights can be gain from this assumption.

3. The words "equally likely" are in quotation mark because it should not be interpreted as a uniform prior.

that this axiom leads to a Utility function $\mathscr{I}^j_{AA}(f)$ that is concave in f and exhibits superadditivity such that $\mathscr{I}^j_{AA}(f+g) \geq \mathscr{I}^j_{AA}(f) + \mathscr{I}^j_{AA}(g)$. Of course, the choice of the Ambiguity Averse investor will depend on the exact structure of his beliefs. If he his pessimistic enough, he will strictly prefer to invest all his money in the risk-free asset and stay out of the innovation competition. This happens when $\mathscr{I}^j_{AA}(f') < U((1+r)E)$, where f' represents any combination of investment in the research projects. But the AA investors that stay in the market want to *hedge* Ambiguity by investing less money in more research projects. Those investors' portfolios will be highly diversified. It is possible that some of them will want to invest in every research project. However, by the intuition developed in the previous chapters, the AA investors will not be able to negotiate a high share x of the value V. Furthermore, those AA investors are unlikely to hire the AA researchers for a fixed wage $w_i(e^0)$. The wages augment the fixed costs of the research project and the AA investors want to keep it low. On the other hand, those investors like the Ambiguity Loving researchers; they consider that those researchers over-invest their effort e in the project. This gives a simple coupling of two partners, where both agents agree to give the property rights of the innovation to the AL researcher. The AA investors finance at least the fixed cost of the research project in exchange for a relatively small claim x of the revenue V.

At the other extreme are the Ambiguity Loving investors. By axiom A6(c), a decision-maker is Ambiguity Loving if simultaneously $g \sim h$ and $g/2 + h/2 \precsim h$ holds. Remember that the Utility function $\mathscr{I}^j_{AL}(f)$ is convex in alternative f and exhibits subadditivity such that $\mathscr{I}^j_{AL}(f+g) \leq \mathscr{I}^j_{AL}(f) + \mathscr{I}^j_{AL}(g)$. The Ambiguity Loving investors have a tendency to invest more money in fewer research projects. Depending on the exact structure of its beliefs, it is possible to have an AL investor that invests a lot in one research project and allocates the rest of his budget in the risk-free asset. But the Ambiguity Loving investors want a high share x of the revenue V, and they are more likely to want the property rights as well. The AL investors prefer the AA researchers because they are willing to give up the property rights if they can invest less effort into the research project. The AL investors are likely to hire Ambiguity Averse researchers who want to work for a fixed lump-sum wage $w_i(e^0)$ because then the investors can keep all the value V of a discovery ($x = 1$).

What emerges here is some form of "natural matching", where one agent has a tendency to reach an agreement with an agent that has the opposite Ambiguity Attitude. The Ambiguity Lover type wants to take greater risks, invest more, and keep the property rights of the innovation. Therefore, the AL agents are likely to strike a deal with the Ambiguity Averse agents who are looking for the exact opposite. It is in this sense that I suggested in chapter 4.3 that the bargaining power of the parties have only a small impact on the allocation of the property rights and the determination of the sharing rule x. The more agents in the innovation competition, the more likely there will be positive matches. As a personal inclination, I fancy thinking the AA investors as Venture capital firms that finance a lot of entrepreneur-innovators *à la* Silicon Valley. I also imagine the AL investors as big research units that hires the innovator-workers, a bit like the public sector or Universities' departments. Of course this is only a narrative; it should not be interpreted in a literal sense.

It is important to observe that this innovation competition is somewhat asymmetric. I do not want to endogenise the entry of the researchers and investors in the innovation competition, but it is reasonable to believe that the number of researchers n will be "relatively small" and its "supply" inelastic. After all, the ability to conduct a research project requires both skills and an idea of what to do. But on the investment side there are almost no barriers to entry; even Ambiguity Averse investors can make their Utility if they can hedge the Ambiguity. This leads me to think that A- the competition among investors will work in favour of more bargaining power to the researchers and patent holders. B- It is possible that a lot of money will be "wasted" on unfruitful research projects. C- A reduction of the returns r of the risk-free asset induces AA investors at the margin to invest in the innovation competition. This will have an impact on the total amount of investment, the risk level of portfolios and the bargaining power of the parties.

The relationship between the value V of the innovation and the total amount of investment $T = \sum_{j \in J} \sum_{i \in I} E^*_{j,i}$ is very thin. Of course, a higher V implies a higher level of investment T, so T is monotonically increasing in V. But because the marginal productivity of money is Ambiguous there is no reason to postulate any other relationship between T and V. In particular, there is no reason to consider that $T = V$. I am inclined to think that if the number of agents is high and $B > V$, then $T > V$. In words, I am inclined to think that if the number of agents is high their cumulative efforts will outreach the benefits available. This possibility is worth further investigation because it could be used by a Social Planner to induces firm to "over-invest" in R&D. This Social Planner could take into account the Consumer Surplus that cannot be captured by firms and choose the reward V accordingly.

To conclude this section it is necessary to look at the impacts of a change in the information available to the investors. Let us consider a situation where the probability of winning is a function of the ability of the researcher. This ability is not directly observable, but the investors believe that it is correlated with other characteristics such as the education or the track record of the researcher. Assuming that researchers are heterogeneous in their observable characteristics[4] induces a new ordering of the preferences of the investors about sets of alternatives $\mathscr{A}$. In other words, some research projects will rank higher in the preference relation $\succsim$ of the investors. It is easier to express this possibility in the framework of Ellsberg's three-colors urn experiment. Let's say that the DM knows that there is 90 balls of three colors, with exactly 30 red balls and *at least* one black ball. This situation may not change the Ambiguity Attitude of the person making a bet but an Ambiguity Loving DM will certainly prefer to bet on black. As a parallel, this means that the researcher(s) i at the top of the preference ranking may disproportionally be an attractive investment. And this is true without regard to its actual real ability. By the same token, this higher place in the preference ranking is likely to allow the researcher to negotiate a higher share $(1-x)$ of V. The impact of this information depends on the exact structure of the beliefs of the investors. It is possible that the few

4. They do not need to be heterogeneous in their ability.

researchers at the top of the preference ranking have a lot of resources, while the other researchers are not even considered for investment. This question of altering the ranking preference $\succsim$ leads us to the role of the last agent, the patent holder.

5.3 The Patent Holder

Remember that the patent holder (PH) has a simple choice: whether or not to give access to a patent pool to some researchers. This patent pool augments the probability of winning the innovation competition and giving access to it is free. However, the PH can and wants to make money out of it. He has three choices: 1- He can sell the access to the patent pool for a lump-sum payment. 2- He can sell the access for a claim x_q over the value V in case of a discovery. 3- He can become some form of middle-man, hiring the researcher on one side and looking for investment on the other. The patent holder may be cash constrained or not and it will affect his behaviour.

A researcher with access to the patent pool will have a higher probability of winning the innovation race. This reduces the Ambiguity surrounding the research project while making it more attractive for investment. In other words, a research project that has access ranks higher in the preferences of investors. Therefore, the compensation for the access needs to be either a lump-sum payment from the investor or a claim x_q over V because the researcher is cash constrained and cannot pay.[5] But the "natural matching" effect of the previous section offers a simple way to think about it. The AL investors / AA researcher duo is likely to prefer the lump-sum payment because the investor is willing to put more money up-front in exchange for a higher claim over the revenue. On the other hand, the AA investor / AL researcher duo is likely to prefer relinquishing a claim on the value V of a discovery. The AA investor wants to minimise his investment in one project (he wants to diversify) and the researcher is cash constrained.

5. It could also be a mix between the two solutions.

The third possibility is quite interesting. A patent holder that acts as a middle-man can extract money from Ambiguity Averse agents. Remember that on one side, the sufficiently AA researcher always prefers $U(e^0)$ to any other form of arrangement. Therefore, the PH can hire him at the price of $w_i(e^0)$ to do the research. The PH becomes himself a "research unit" that looks for outside investment. The AA investors are willing to invest in exchange of a relatively small claim x on V, but they are reluctant to pay for the wages of the researchers. Therefore, the PH can guarantee for himself the revenue $(1-x)P^i(e^0, E^0, q=1) * V$ for the cost $w_i(e^0)$ of hiring the researcher. Of course, the PH can invest more money if he deems it necessary and has access to the funds. This position between two AA agents also allows the PH to keep the property rights of the innovation: both AA agents are willing to let it go. Finally, the PH can even act as middle-man between other types of agents. But this possibility depends on the exact structure of the beliefs of the agents and cannot be explored further with our simple treatment.

Finally let us consider a succession of innovation competitions, where the patent registered at an early date has an impact on the probability of winning the next competition. In this case a patent holder is likely to stay at the head of the innovation competitions, especially if he can buy a lot of research projects cheaply. This is in essence similar to the idea of a Patent Gridlock, sometimes called an innovation Gridlock. Generally speaking, a Gridlock is a situation where an incumbent monopoly in a market is able to harness the competition and prevent entries because future innovations require access to the current technology, or patent. Which brings me to an embarrassing situation. I assumed that V was given, like a gift coming from the heaven. But now there is a situation where the next period can be influenced by the current period, and where the real value of holding the property rights may be higher than the simple value of the price V. In other words, the real future value of today's patent may affect the allocation of the property right in a way that is not captured by my treatment of the determination of the share x of V.

This presentation of the patent holder is limited but still gives insights regarding the impact of patents. In the last section it was suggested that a researcher who ranks higher in the preferences of the investors will benefit from it disproportionally. This is because all the investors will want to finance him, and they are more likely to overlook the other researchers. But now the PH can act as a research unit by hiring researchers on one side and capturing investments on the other. This means that he can "cover" multiple research paths while spending little of his own money and still keep both the property rights of a discovery and a share x_q of the revenue V. Furthermore, if the PH wins the innovation competition he will be in the same situation for the next competition related to his sector of activity. All the ingredients are in place for the PH to become a dominant player in his field.

CHAPTER 6

CONCLUSION

It is time to step back and think critically about this presentation. In the first chapter I mentioned that I am inspired by the Austrian approach to economics. This school of thought is very concerned by the impact of Uncertainty, incentives and property rights on economic outcomes. The meticulous attention they give to entrepreneurship and innovation is characteristic. The Austrians are also inclined to focus more on the processes that lead to an equilibrium than on the equilibrium itself. Finally, they recognize the crucial role of private property and property rights in modern economic development. But I disagree with the Austrians' relative distaste for a mathematical approach to economic problems.

The literature about decision-making under Ambiguity contains powerful tools that enable the modelling complex problems. I would argue that the analysis of R& D, entrepreneurship and innovation presents difficulties that could now be lifted. The first goal of this presentation was to give an example of how to approach ideas of the Austrian school through the lenses of decision-theory. By looking at the allocation of property rights under Ambiguity it was possible to turn a standard model of Industrial Organisation about R&D into a meditation on the impact of a higher order of Uncertainty. I would suggest that this first goal has been met and hope the reader agrees.

But the presentation lacks rigour. A complete analysis of the topic would need to use the real Ambiguous Utility functions of [Ghirardato et al., 2004], model every possible belief structure of the agents and characterise the resulting equilibrium. Furthermore, a complete analysis should also use other preference relations that are not Invariant Biseparable preferences. But all of this could only be done at the cost of extensive comparative statics. I do not know how much more insights this exercises would have yielded. Those insights may be summarised in four statements:

A- The effectiveness of the incentives given to an agent facing Ambiguity depends on his Ambiguity Attitude. Incentives given to an Ambiguity Loving agent will be magnified by his "optimism"; they are very effective. On the other hand, incentives given to an Ambiguity Averse agent will have less impact than in traditional Principal-Agent models. In extreme cases the incentives may even have no effect at all. The impact of the Ambiguity Attitude on incentives is independent of the Risk Attitude of the decision-makers.

B- The ability to mix between different alternatives will strongly impact the decision-making of Ambiguity Averse agents. AA agents that are unable to mix are more likely to choose the constant alternative than the one who can hedge Ambiguity. This manifests itself in our model by a tendency of the Ambiguity Averse researchers to enter the innovation competition as innovator-workers instead of innovator-entrepreneurs.

C- It seems better to allocate the property rights to the partner that is the more Ambiguity Loving. This leads to a natural partnership between AA and AL agents. Under Ambiguity the agents have difficulty in evaluating their productivity and allocate property rights accordingly. The allocation of property rights is therefore more influenced by the incentive effect of ownership on the level of effort than by the marginal productivity.

D- Changes in the information available to the decision-makers have a large impact on the allocation of resources. Specifically, a research project that has access to a patent pool faces less Ambiguity. This opens the possibility for a patent holder to make money with few risks. Furthermore, a patent holder can act as a middle-man between Ambiguity Averse agents and claim the property rights of a discovery at low cost.

I would like to invite the reader to think about the avenues of research this presentation opens. Even thought it is incomplete it still gives a basic framework to model the relationship between R&D and Ambiguity. Here are some topics worth of further investigation. One possibility is, of course, to enhance the model of chapter five with more structure by making explicit the beliefs of the agents. With specified beliefs it will also be possible to account for Ambiguous Utility functions that behave in different ways than $MinEU$ and $MaxEU$. In particular, I believe that the uncertainty averse preferences of [Cerreia-Vioglio et al., 2011] and the smooth ambiguity preferences of [Klibanoff et al., 2005] are great candidates. The Ambiguity Averse preferences used in our model has kinks in the Utility function $MinEU$. Following a presentation of professor Peter Klibanoff [Klibanoff, 2015], uncertainty averse and smooth ambiguity preferences have attractive monotonicity properties because they can be build with no kinks.

Another topic will be to characterise the different strategies available for a Social Planner to subsidise R&D. I suggested that a Social Planner could incentivise firms to invest in R&D by launching an innovation competition. This strategy is interesting because there is a possibility that the firms will "over-invest" with regard to the prize of the competition. Therefore, a Social Planner can take into account the consumer surplus while setting the value V of the prize. This feature could gives more "bang for the buck" than strategies such as direct subsidies to the researchers.

I see two way to infuse dynamics in the framework. A patent holder may stay at the frontier of innovation if there is a relationship between future and current technological development. This happens because the PH can exclude competition in the research phase. Modelling a succession of innovation competition under Ambiguity may shed light on this phenomenon. Finally, it is easy to transform this competition into an innovation race. It will be to develop the hypothesis that $\lim_{n\to\infty}\sum_{i\in n}=1$ instead of $\sum_{i\in n}=1$. That is, to consider a situation where as long as there is no discovery new researchers enter the race until somebody has the right approach and wins.

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