Embracing the Crisis in the Foundations of Mathematics

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Abstract

The crisis in the foundations of mathematics is a conceptual crisis. I suggest that we embrace the crisis and adopt a pluralist position towards foundations. There are many foundations in mathematics. However, 'many foundations' (for one building) is an oxymoron. Therefore, we shift vocabulary to say that mathematics, as one discipline, is composed of many different theories. This entails that there are no absolute mathematical truths, only truths within *a theory. There is no unified, consistent ontology, only ontology* within *a theory.*

Conceptual crises teach us that there is an instability in our preconceptions, theories or in our philosophical goals. The crisis in the foundations of mathematics is something we can regret, or something we can learn from. What I, and others, have learned is to abandon the goal of providing one formal mathematical theory as a foundation for the whole of mathematics.

In abandoning this goal, our first step is to embrace a moderate form of pluralism: 'Pluralism in Foundations', where we accept several foundations of mathematics. However, in seriously entertaining the idea of competing, or equally plausible, foundations in mathematics, we learn that the *very notion* of a formal theory being a 'foundation' for mathematics is unstable, and therefore that the very idea of 'foundation' (in mathematics) should be revisited.

This line of thinking takes us to the next step, where we open the door to a world of Pluralisms. From the doorway, we can now consider Pluralism in: epistemology, ontology, methodology and truth. In this paper, I shall take you to the doorway to this greater Pluralism.

Before starting on this path, let me fend off an easy confusion between Pluralism in Logic and Pluralism in Mathematics. *Logical* Pluralism[1] is a more established position than is Pluralism in Mathematics. A Logical Pluralist accepts different formal logical systems (usually propositional, modal, first-order or possibly second-order logic). See Haack (1978), Da Costa (2007), Batens (2002), for examples of logical Pluralists. See Beall and Restall (2006) for a fully articulated position called 'Logical Pluralism'.

In contrast, Pluralism in the philosophy of mathematics is a Pluralism about different *mathematical theories* as developed and practiced by recognised mathematicians.[2] What Pluralism in Theories amounts to is (at least) Pluralism in: epistemology, ontology, methodology and truth. There is no well worked-out philosophical position called Pluralism in Mathematics in print, to date. Sketches of positions can be found in Aberdein (2005), Bueno (2011) and Friend (2012), amongst others. Moreover, many present-day philosophers of mathematics work in a very pluralist way, but since there is no properly worked-out philosophical position, they are rarely specific about the delimitations of their Pluralism. Many philosophers, computer scientists, logicians and mathematicians tell me that they are pluralist, and, when they say that, they mean quite different things

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[1] I am using capitals to name a philosophical position, as opposed to naming an attribute of a philosophical position. For example, "pluralism is an attribute of some forms of Structuralism", but "Pluralism is a philosophical position".

[2] Here, 'recognition' is determined socially and institutionally. This is not to say that one has to be a university professor, in order to be considered to be a mathematician. Rather, one has to be engaged in the practice of mathematics, as it is recognised by said institutions. I think that this is the best the Pluralist philosopher can do (since he, or she, recognises no unique foundation for mathematics). The socio-institutional criterion is imperfect by its own standards, since, in retrospect, someone rejected by the institution (in the wide political sense) might later be recognised as an important contributor to mathematics. Similarly, someone judged as 'having an impact' might later fade into the background, as we discover better techniques or favour other theories.

by the term. For lack of a well worked-out, and in print, exposition of Pluralism as a philosophy of mathematics, this paper too is a sketch.

Nevertheless, I can say a few things to help us understand Pluralism better. Pluralism in mathematics is most starkly contrasted to the more traditional philosophies of mathematics which contributed to, and were developed in reaction to, the crisis in the foundations of mathematics. In these traditional philosophies, we propose one mathematical theory as a foundation for the whole of mathematics. With philosophical maturity, the founding theory becomes a norm for mathematical practice, or is intended to set a norm for future practice.[3] Examples include Realism, Logicism, and some versions of Constructivism, Naturalism, Formalism, Fictionalism and Structuralism. In contrast, Pluralism in Foundations (which is the first position we shall visit) rejects the idea of one foundation (in particular, in the form of a formal mathematical theory). The Pluralist in Foundations also rejects the normativity, nay prescriptivity, which accompanies traditional theories. Judgments about mathematical theories or theorems or hypotheses etc. are made from within a context, which is revisable, one standpoint amongst others (from which to make a judgment), and, therefore, not absolute. This is a lesson we can learn from the crisis, and it has taken us a while to learn it. The lesson was not learned in one step. Between our extreme Pluralism and the monism of the traditional positions, we can find some 'in-between' philosophies.

Philosophers of mathematics, who break from the more traditional positions are characterised, here, as favourably using the term 'pluralism' to name a characteristic of their position. We find the term used in some versions of: Structuralism, Formalism, Fictionalism or Naturalism.

In contrast to these in-between positions, the Pluralist places 'pluralism' as the chief virtue of his version of Pluralism. The Pluralist is inspired by the in-between positions, but goes beyond them. Forthwith, I restrict attention to the more traditional positions and

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[3] The founding theory is a norm in the following sense: purported mathematical work is not recognised as proper mathematics if it cannot, in principle, be reduced to the founding theory. There are strong philosophical objections to foundational theories. See, for example, (Vopeňka 1979), (Bueno 2011).

Pluralism, ignoring the in-between positions.[4] Traditional positions support one foundation to mathematics. The path to Pluralism starts by acknowledging several.

1. Preliminaries: Organisation, Argument and Approach

In this paper, I shall not give a strong direct argument for Pluralism. Instead, I shall discuss some motivating considerations, and then make an indirect argument, by giving a sense of the philosophical work which is done by a 'Pluralist' who takes on one version of Pluralism. To re-emphasise: I write 'one version' because there are many versions. The Pluralist is pluralist about Pluralism! The versions vary along at least two axes. One is: what it is one is pluralist about (this is not the same as ontology – it has to do with subject matter); the other is propositional: how one deals with the issues of truth and the conflict between claims, in and about mathematics.[5]

> *The Pluralism I shall start with (but revise) is a Pluralism about foundations in mathematics, the defender of which, insists that truth can only make sense within a mathematical theory.*

For such a Pluralist, a mathematical theory is not true in itself. It can only be true with reference to a meta-theory (usually by showing it to be equivalent to, or reducible to, that meta-theory). Such is familiar from Shapiro's Structuralism. This Pluralist arranges discussions in and about mathematics into a hierarchy of theories and meta-theories, distinguishing claims made *within* a theory from claims made *about* a theory, from the standpoint of a meta-theory.

The reason I start with this version of Pluralism is to meet a traditional philosopher of mathematics halfway. I can start with philosophical positions which are recognisable to her. As we shall see,

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[4] For detailed comparisons of the in-between positions and Pluralism, see (Friend 2012).

[5] Strictly speaking, the default Pluralist view is that 'truth' is handled by a formal logical theory, or at least, the formal theory is a guide to how we are to deal with questions of truth. Are the logical theories, themselves true or false? No. Similarly, the mathematical theories are neither true nor false.

Pluralism in Foundations is unstable. We shall modify it in due course.

I anticipate various reactions. If the presented Pluralist positions strike the reader as anathema to the philosophy of mathematics as it should be practiced, then more detailed, careful and convincing arguments are called for. But this is not the place for those. If the positions resonate with the reader, he might already be a Pluralist. If the positions and example of work done by the Pluralist appeal to the reader, he might be interested in looking further into Pluralism as a philosophy.

2. The Pluralist about 'Foundations' of Mathematics

We begin with a simplifying assumption, in order to present the initial position. The assumption will be revised when we set the Pluralist to work. The simplifying assumption is:

> *a proposed foundation (to mathematics) is a mathematical theory, and mathematical theories are individuated by a language, a set of axioms and some inference rules.*

Under this assumption, we can list different purported 'foundations'. Some foundations of mathematics include Zermelo-Fraenkel set theory with an axiom of choice and consistent extensions of ZFC. If we take the simplifying assumption seriously, then we can also count as foundations: non-well-founded set theories, type-theories, category theories and any other theory to which a lot of mathematics can, in principle, be reduced.[6] Finding such a foundation is the goal of the more traditional philosophies of

[6] If we do not take the assumption seriously, then we might think that all of these theories are much on a par, and are all aspects, or parts, of 'one big founding idea' of mathematics. Woolley language aside, in this case, the word 'theory' is used in a much looser sense than what is proposed in the simplifying assumption. To keep the discussion focused, we make the simplifying assumption, and we can look at a looser idea of theory or foundation later.

mathematics. In contrast, the Pluralist accepts that there are several big theories – many of which are plausible.[7]

> *The Pluralist in Foundations is agnostic as to which is the correct foundation, and over whether there is one unique foundation.*

Note that when we speak and write about several 'foundational theories' co-existing, the very term 'foundation' changes meaning. The building metaphor no longer works. It makes no sense to have several foundations for one building called 'mathematics'. Since the Pluralist is agnostic as to whether there are one or several 'foundations', and in order not to beg any questions, we speak of 'umbrella' theories.

3. Considerations Supporting the Pluralist's Agnosticism

With the shift in vocabulary from 'foundational theory' to 'umbrella theory', we draw attention to the distinction between our philosophical *aspirations* and the philosophical claims we can defend in argument. Pluralists might *hope* that mathematics will one day turn out to be unified in one umbrella theory, but, and this is important, such hope is simply a subjective private feeling, and is not supported on present evidence. So, similarly, a Pluralist might hope (maybe perversely) that there are several irreducible umbrella theories in mathematics, and that there will never, nor can ever, even in principle, be a way of unifying these. Again this is a hope and a private conviction. The Pluralist, as a (public) philosopher is simply agnostic on the issue of unification of mathematics; *mutatis mutandis* for the idea of 'truth of' a theory. No mathematical theory is true *tout court*. A mathematical theory can only be true relative to another theory (by being shown to be reducible to (or embeddable in) that theory). Whether the reducing theory is true, will depend again on another reducing theory. In fact, for the Pluralist, trying to determine the truth of a theory, in the absence of a meta-theory, or reducing theory is labour lost. It is more informative to frankly discuss embeddings, reductions, equi-consistency proofs and other limitative

[7] Keep track of the word 'plausible'. We shall revisit it.

results on a mathematical theory than the truth of the theory.[8] This agnostic attitude makes sense of the following remark, which I quote because I find it representative of some mathematicians' philosophical discomfort. Discussing the truth and independence of the continuum hypothesis, they write:

> *These logical results [of the independence of CH] do not settle the question originally asked by Cantor whether CH or GCH are true or false statements. However, it must be said that these seemingly obvious questions are not very clear: the concepts of truth and falsity (as opposed to the concept of the derivability from axioms) do not have a clear meaning in abstract set theory. Thus we cannot rule out the possibility that Cantor's original questions will turn out to be simply meaningless. (Kuratowski and Mostowski 1976, p. 290).*

The Pluralist's agnosticism is motivated by the following considerations:

(1) there are several, non-equivalent, umbrella theories – many of which have natural philosophical alliances (philosophies which accompany that umbrella theory). Call an 'umbrella theory plus its natural philosophy' a 'traditional position'.

(2) No traditional position is accepted by all mathematicians or by all philosophers.

(3) Moreover, there does not seem to be an immanent convergence of opinion (as we can see in a number of the papers in this volume).

[8] It is for this reason that 'Pluralism' is favoured as a name over 'Relativism' since the issues of truth and ontology are side-issues for the Pluralist. In contrast, the Relativist is first and foremost concerned with truth and ontology. Pluralism is related to Relativism in the following way: Pluralism is a mature Relativism. The Pluralist goes beyond Relativism because he is relativist along more dimensions than truth and ontology. The more mature Pluralist position is relativist about truth, ontology, mathematical theories, epistemology, logic and methodology.

Depending on how finely we want to distinguish traditional positions, we might even think that there are a large number of candidate potential positions not yet developed or explored.[9] It seems then, that rather than look for one foundation for mathematics, we should learn to accept the idea that there are several umbrella theories, several potential traditional positions, many of which are plausible; and therefore, no traditional position is correct in stating that their championed big theory represents 'mathematics'. This motivates a further modification of language. Rather than speak of *the true* mathematical theory (or foundation), we can speak of '*plausible* umbrella theories'.

When we replace 'truth of a theory' with 'plausibility of a theory' a few interesting things happen. First note that 'plausibility' is a relative term. Nevertheless, positions are rarely *equally* plausible. Sometimes, one position is more plausible than another on grounds of philosophy of logic. For example, trivial theories are implausible. We could also try to argue that one mathematical theory is implausible because it is illogical or unreasonable. The strongest way to make such an argument is by appeal to a particular formal logic. For example, we might decide that a mathematical theory is implausible because it endorses unrestricted choice, which is not allowed in a constructive logic. However, in the light of a Pluralism about formal logical systems, such an argument begs the question. Of course, not all of us are Logical Pluralists, but, today, logical monists have to defend their monism as well as their choice of logic. We could try to appeal to an informal notion of reasoning and logic, but this is usually unsatisfactory, since ambiguous (i.e., underdetermined) and therefore leading to disagreement (only solved by appeal to a formal system of logic). So the only sure lesson we can learn by appeal to logic or reasoning is to dismiss trivial theories as implausible. But this will not be very helpful in narrowing the field of plausible foundations!

Thankfully, the contrast between trivial and non-trivial mathematical theories is not the only recourse we have to rate the plausibility of a mathematical theory. Plausibility judgments can also

[9] We have to be careful about judgments about 'large numbers of positions', especially, if we consider not-in-print positions. First we have to individuate positions, next we have to decide what the parameters are on possible positions.

depend on background knowledge. Presumably, a budding mathematician with little experience and a very narrow area of specialisation is in a lesser position to make a judgment about plausibility, since what is plausible to her will depend on her experience, and resemblance with what she knows of mathematics. In contrast, a mathematician with a vast experience of mathematics and mathematical theories, might well decide that one umbrella theory is more plausible than another. An example is Gödel, although, strictly speaking, he would not have accepted our simplifying assumption. So, knowledge and experience can be used to partially evaluate plausibility. For this reason, attributing plausibility partly comes from an educated judgment, not a mere declaration of taste.

Alas, even amongst the most educated, we do not have consensus, nor do we seem to be heading towards a convergence of judgments. Future arguments and future information might lead to convergence, divergence, or convergence followed by divergence, divergence followed by convergence; we simply do not know. Under these considerations, we have no rational basis, nor authoritative basis (based on amount of education),[10] nor inductive basis, upon which to make a choice for one position as the true position.[11] It is for this reason that the Pluralist demurs from making a choice, and accepts all plausible umbrella theories as on a par, in the first instance (i.e., they will be subjected to further evaluation since we are aware that 'plausibility' is a relative term requiring qualification). The Pluralist is a principled agnostic (as opposed to a lazy agnostic).[12] For the

[10] 'Amount of education' is, of course, not to be confused with number of degrees or prestige of award granting institutions. Here 'education' is meant in the basic sense of pursued, sustained and critical enquiry. University degrees are a rough indicator.

[11] We can work to find the most plausible big theory, but given that to determine what is the most plausible theory, we have to make some choices which will seem arbitrary or questionable to some (respectable people), the purpose of so doing becomes unclear; unless we take it as a private project responding to our private convictions. To remain in the public sphere, we are rationally more secure in noting different notions of plausibility, and evaluating which theories fit which notions.

[12] A lazy agnostic is not interested in the debate (any more). So she gives up, and just says she is agnostic. A principled agnostic is interested in comparing, evaluating the comparison, changing his mind and being honest

principled agnostic, the traditional philosophies are not supportable on present evidence.

Some readers will be bristling, and I leave them in this state. Instead of giving a knock-down defence of Pluralism, or an exposition of the whole Pluralist position, I turn to what it is that the Pluralist does, as a philosopher of mathematics. The purpose is to fend off the *reductio* argument against the Pluralist that the Pluralist cannot practice philosophy. Rather, what is left to her is: sociology, psychology, history, historiography and so on – anything but philosophy. Actual, therefore possible, as the Mediaevals taught us. Since Pluralism in Foundations is unstable, I, instead, display the practice of the Pluralist in Mathematical Theories who is interested in the question of foundations. Details about his position, or rather, the family of positions he could occupy, need not concern us here.

4. Setting the Pluralist in Mathematical Theories to Work

To stay with the same theme of 'foundational mathematics', the Pluralist does not ignore and merely dismiss the foundational aspiration. *Au contraire*, the Pluralist concedes that there is something to the claim that ZFC is a 'foundation', even if we have learned to call it an 'umbrella' theory. After all, a number of very well respected mathematicians take ZFC, or something like it, to be very important. The observation is not lost on the Pluralist, who will now work with this observation. We can start by revising our simplifying assumption, and say something more accurate, namely that:

> *set theory (not ZFC, which is one formal representation of set theory, but whatever it is that ZFC+ is meant to represent) is really used as a 'lingua franca' or a 'reference point' by mathematicians (usually by appeal to the formal theory).*

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and industrious towards the question. He is also well aware that there might be no (definitive or even temporary) resolution to the issue under question. Maybe 'industrious agnostic' would be a better term. 'Principled' is meant to refer to the idea that there is a defence the agnostic can give for his agnosticism, and that he does care deeply about the outcome of the debate, but more than this, he cares about the honesty and strength of argument used in the debate.

We have modified our initial simplifying assumption in two respects. One is that we are not individuating theories by formally presented theories. The other is that we are talking of reference points rather than foundations. In some mathematical circles, set theory is 'central' to mathematics. Since the Pluralist is agnostic about the foundational claim, he wants to know what it is that is appealing in the claim that ZFC is a foundation, or why it is that set theory is a *lingua franca*, or a reference point for mathematicians. The claim about ZFC is descriptive. Because of this, answers can be sought by: sociologists, historians, historiographers, psychologists and so on. But the Pluralist can also ask a more philosophical question: "Is set theory really the *best* reference point for an arbitrary mathematical theory?" Or, similarly: "Should we continue to use ZFC as a *lingua franca*?" Both are hopelessly vague and ambiguous questions because of the words 'best' and 'should'. The Pluralist wants to give a rigorous and defendable answer, and we can do this. Let us begin by making some of the terms more precise: 'set theory' (most loyally represented by which formal theory?), 'the best' (according to what measure?), 'reference point' (what is the difference between reference point and foundation?), 'arbitrary mathematical theory' (what gets counted?). But we want to do more than this. We want to bear in mind that when we make these terms more precise, we set parameters on the question, and Pluralists are aware that these parameters can be revised. With that caution in mind, let us get to work and propose disambiguations of the assumption.

Working backwards, an 'arbitrary mathematical theory' is a theory, the inventor, discoverer or developer of which, believes that it is a mathematical theory. It has a formal language, which expresses all of the truths, or theorems, in the theory. It might be presented as a mathematical structure (as defined by model theory), as an axiomatic theory, as a rule-based theory, and possibly in some other way. But we cannot allow anything we wish. To give a precise answer to our question, within revisable but temporarily set parameters, we need 'arbitrary theory' to be comparable to set theory. To allow comparability, an arbitrary theory is either (i) a formally presented theory, in which case, let us modify 'set theory' to mean formal ZFC, or some appropriate version: ZF, ZFC+ etc. Or, (ii) our arbitrary theory is not formally presented (or is highly ambiguous between different formal presentations) so our comparison will be made with something underlying formally presented set theory, i.e., *that to which* ZFC+ is responsible (or, *against which* attempts at formal representation are judged).

'Reference point' is ambiguous between at least: 'can be reduced to ZFC', 'can be shown to be equi-consistent with ZFC', 'ZFC can be reduced to our theory' and 'shares some recognisable features with ZFC'. The last is the most nebulous. An example is the theory of semisets. The 'recognisable features' include sets and all of the axioms of set theory (some are a little modified), but our theory also includes the theory of semisets, which concerns something less definite than a set, that is, less definite in its membership (set members we can work with are fixed with respect to a perspective, as we change perspective, so the members we can manipulate and work with change). ZF is not entirely reducible to the theory of semisets, but there is a strong basis for comparison.

'The best' as it is used in the context of our question and given our agnosticism, is determined by appeal to the practice of mathematics. It means something of the form 'is recognised by, and is respected by, (a good enough number of) recognised and respected mathematicians today'. This is far from an absolute and pure judgment! Rather, it combines a statistic determined by some sense of the 'institution' of mathematical research and evaluation. This proposed disambiguation of 'the best' makes explicit the instability of a positive or negative answer to the questions, since it is the 'institution of mathematical research' which will be adding the normative force of 'should' or determining what counts as 'best'. Not only does the institution change over time, with the people and their attitudes, but, there will be different ways of working out what mathematicians consider to be the best reference point, depending on what we take to represent the 'institution'.

The last term is 'set theory'. 'Set theory' is what underlies the various formal representations.[13] It is a mathematical theory of sets and membership. To make precise, and fix, what 'set theory' means,

[13] Note that I write as though 'set theory' is one thing being formally represented. The only reason to write this way is for simplifying the conclusion. It is easier to keep some goal posts rigid, for a while. They can be moved later.

we give a formal representation, or several formal representations. Moreover, we often have a sense that the formal representation of set theory is not completed. We extend formal set theories by adding new axioms, and experimenting with these to see what follows when we add them, and try to determine whether or not they are 'fruitful' or 'natural'. Again, the latter are also hopelessly vague and ambiguous terms, which we can try to make more precise, maybe at the expense of our intuitions, in favour of an artificial precision.

5. Answers

Already in disambiguating, we have learned a lot about our question. The analysis of the very general philosophical question "Is set theory really the best reference point for an arbitrary mathematical theory? Or, should it be such?" gives some idea of the *modus operandi* of a Pluralist philosopher of mathematics. Having made his preliminary analysis, the Pluralist can now give an answer to the question.

The Pluralist will want to make a survey to confirm or disconfirm the following guesses.

(1) The first guess is that logicians (broadly construed to include set theorists, model theorists, logicians working in different traditions for propositional, first-order and second-order logics, proof theorists, some computer scientists, and maybe some others) tend to feel *more secure* if they can show that their theory is reducible to ZFC, or some formal theory which has ZFC as a core of the theory. They feel *secure* if their theory is equi-consistent with ZFC. That is, ZFC, as a reference point affords a sense of security for logicians.

(2) It is the *de facto practice* amongst logicians to use ZFC as a default reference point.

(3) Other logicians turn the tables. They *use* 'set theory' in the sense of 'what underlies the formal representation' as a core for recognising, developing and interpreting formal umbrella theories. They treat formal set theory as a starting point, not as a foundation. For example, set theory is what is used to compare what happens

when we make changes to ZFC, by adding new axioms. So 'set theory' in this sense, is assumed as a reference point *ab initio*. These logicians are set theorists in the broad sense of working in and around ZFC, and seeing mathematical theories in terms of ZFC.[14]

Indicators that these guesses are correct include the use of the notation we find in set theory textbooks, and the use of language, axioms and theorems familiar from set theory in the development of their own theory. These guesses concern logicians. What about other mathematicians? The speculation continues:

(4) some (respected *mathematicians*) *do not* use set theory as a reference point, and some *cannot*.

If the fourth guess is correct, then, I speculate that this is because amongst 'real working mathematicians' (*pace* the traditional philosopher of mathematics) very little attention is paid to set theory, or ZFC. They are usually not trained in set theory, and they are untroubled by foundational issues. They are also not concerned with finding 'a unique reference point', or default reference point for the theory (or theories) they are working in. Their sense of security is based in practice. It is enough if those judging, evaluating and correcting their work endorse it, and these endorsers will be people in the same or related fields of work in mathematics.

Is this a question of 'distribution of labour' (so all mathematical work is traceable, in some way, by more specialised mathematicians to set theory) or is it really that set theory simply does not act as a reference point? To answer this question, imagine if explicit traces of set theory were to disappear. All books about set theory and people working in set theory or with a good working knowledge or concern about set theory were to be shipped to another part of the galaxy, and be *incommunicado* with the mathematicians left behind. In this case I conjecture that there would be little pause in mathematical work, and there would be no urge to re-invent set theory, or find an overarching reference point or foundational theory. I make the conjecture based

[14] I should like to thank Brian Skyrms for pointing out this approach to me in conversation.

on some areas of mathematics where it makes no sense for ZFC to be the reference point for the claims of the theory, such as statistics or calculus.[15] Testing the conjecture is delicate future work.[16]

If my statistical guesses are confirmed by a survey, and my guesses are supported by further evidence and argument, then the Pluralist answer will be that:

> *a lot of logicians use ZFC as a reference point, but this is not representative of all mathematicians.*

6. Conclusion

Having gone through our careful exercise, we will also understand a number of subtleties attending the answer, and we can confirm or disconfirm and revise my guesses. We also see that we have to leave behind the territory familiar to the traditional philosopher of mathematics – leave ZFC, and look to the actual practices in mathematics.

More broadly, other Pluralist work includes: showing more precisely how Pluralism is different from the in-between positions. Recall that these are some versions of Structuralism, Formalism, Fictionalism and Naturalism. Most of these comparisons can be found elsewhere.

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[15] We can, of course, reduce, or interpret calculus or statistics in terms of set theory, but being able to, and having to do so, or needing to do so are quite distinct. Arguably, reducing calculus or statistics to set theory considerably distorts these areas of mathematics, and is quite unhelpful. This is not to say that a proposed reduction is completely unhelpful and uninformative. Rather, it is considerably inefficient if one is interested in calculus – at least on present evidence. To insist otherwise is to run the risk either of begging the question (so mathematics is just defined in terms of reference to set theory), or to make the term 'set theory' so broad as to be vacuous: so under the imagined scenario of losing trace of set theory, all of mathematics and mathematicians would be shipped. As I write, the issue is delicate, and the strength of conjecture rests on such work.

[16] We would have to navigate between Scylla and Charybdis. Scylla: if we take the term 'explicit mention' broadly, we will find that all, or almost all, mathematics (post 1870s) would disappear. Charybdis: we might think that set theory would still be 'there' underlying most mathematical thought.

Nevertheless, with the constrained exercise we performed here, we have indicated a way to embrace the crisis in the foundations of mathematics. We can now stand in the doorway to other forms of Pluralism in mathematics.

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