A non reductionist logicism with explicit definitions

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Abstract

This paper introduces and examines the logicist construction of Peano Arithmetic that can be performed into Leśniewski's logical calculus of names called Ontology. Against neo-Fregeans, it is argued that a logicist program cannot be based on implicit definitions of the mathematical concepts. Using only explicit definitions, the construction to be presented here constitutes a real reduction of arithmetic to Leśniewski's logic with the addition of an axiom of infinity. I argue however that such a program is not reductionist, for it only provides what I will call a picture of arithmetic, that is to say a specific interpretation of arithmetic in which purely logical entities play the role of natural numbers. The reduction does not show that arithmetic is simply a part of logic. The process is not of ontological significance, for numbers are not shown to be logical entities. This neo-logicist program nevertheless shows the existence of a purely analytical route to the knowledge of arithmetical laws.

1. Logicism and reductionism

Since Frege, Whitehead and Russell, logicism has been widely conceived as a program for the reduction of arithmetic to pure logic. If the goal of the intended reduction is clear and can be summarized by the short claim that arithmetic is nothing but logic, it is nevertheless far from easy to describe accurately what such a reduction is and what is its very significance. Technically speaking, the reducibility of a theory S_1 to a theory S_2 lies on the possibility to prove in S_2 the axioms of S_1 by means of explicit definitions of the primitive terms of S_1 in the language of S_2. But this does not mean that S_1 *is nothing but* S_2. It only means that S_1 can be *interpreted* in S_2. However, in this technical sense, the significance of a reduction's existence is not inconsiderable: first, it gives a consistency proof of S_1 relative to the consistency of S_2; second, it shows that certain S_2-entities can play the role of the objects of S_1, even if it does not guarantee that these two groups of entities are simply identical.

As we know, the kind of reduction logicists usually had in mind is much stronger than this technical process of reduction. For them, the definition of "number" was supposed to grasp the very notion of number. For original logicists, the aim of the search of a reduction of arithmetic to pure logic was actually to provide the most basic mathematical theory with a reliable epistemic foundation. Our mathematical knowledge would have been strongly secured if numbers had been shown to be logical entities, with properties depending only on basic logical laws.

Nevertheless, this form of logicism was a failure: Frege was faced with Russell's antinomy and the authors of the *Principia Mathematica* were forced to enlarge their logical basis with three non-logical axioms. After this historical impasse, the only way for logicism was the search of the weakest addition to pure logic allowing the reduction of Peano-Dedekind Arithmetic (PA), while preserving the epistemic component of the original program.

We know today, first from C. Parsons (1965), but also from C. Wright (1983) and G. Boolos (1987), that Russell's paradox was not actually the death sentence of Frege's work on the foundation of mathematics. What is now called *Frege's Theorem* – the proof that the fundamental laws of arithmetic can be derived from second-order logic through the addition of a single proper axiom – is considered as an extremely interesting result for the philosophy of mathematics. This axiom, usually called *Hume's Principle*, is the formula which is discussed by Frege in his *Grundlagen*, just before the unfortunate introduction of extensions. It can be formulated as

(HP) $$(\forall F)(\forall G)(N(F) = N(G) \equiv F \infty G)$$

where $N(F)$ stands for “the cardinal number of the concept *F*” and ∞ for the relation of equinumerosity between concepts.

But C. Wright’s and B. Hale’s claim that Frege’s Theorem is still a form of *logicism* is highly controversial (Hale & Wright 2001). Introducing “cardinal number” as a new primitive term, HP is clearly a proper axiom which cannot be said to be logical. According to Wright, though they are not purely logical truths, the laws of arithmetic are still shown to be analytic or purely conceptual by Frege’s Theorem. The neo-Fregeans from St Andrews[1] indeed consider HP as an analytical truth. But it is then quite difficult to understand what they intend to mean when they argue that stipulating HP as true is meaning-constitutive of the expression $N(-)$. If it is (analytically) true, HP is a proposition and its truth does not depend at all on our stipulation. And if it is not a proposition, it is an open formula, with $N(-)$ as a free variable. Actually, the only way I can understand “I stipulate HP as true” is as “let me consider $N(-)$ with one of *these* meaning*s* (if any!) which satisfy the open formula in question”.

HP is clearly considered as an *implicit definition* of $N(-)$. But the problems with implicit definitions abound. On the one hand, as any other additional axiom, they modify the whole system and can even lead to contradiction (like Frege’s Basic Law V, which is exactly shaped like HP). But even if HP is consistent, it is too strong as a *definition* of a *single* proper term, for it also modifies the logical constants with which the term to be defined is explained. For example, due to its impredicative character, HP excludes all finite models. In other terms, it involves an axiom of infinity. From a purely proof-theoretical point of view, the addition of HP makes provable formulas which contains only logical terms and which are not theorems of pure logic. For example:

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[1] See in particular: Wright (1983), Hale & Wright (2001), Ebert & Rossberg (2007).

$$(\exists x_1)(\exists x_2)(x_1 \neq x_2)$$

On the other hand, HP is too weak to be a *logicist* definition, for it does not warrant a definite and unique meaning for the concept to be introduced. The so-called "Caesar problem" is a consequence of this weakness. Saying, with neo-Fregeans, that open sentences of the form "$y = N(F)$" are only satisfied by objects falling under the identity conditions expressed by HP is not enough, for we cannot exclude that there is no (non-standard) "system of objects" (possibly including Julius Caesar) and satisfying HP however (even if Julius Caesar clearly does not fall under the identity conditions which follows from *our* intended interpretation of $N(-)$).

In spite of its elegance and very economical character, it is not clear whether Frege Arithmetic (FA = second-order logic + HP) is more *logicist* than Peano-Dedekind Arithmetic (PA). After all, Peano's axioms (as a whole) also constitute an implicit definition of "zero", "number" and "successor", explaining the related concepts in terms of pure logic. The unsolved Caesar problem shows that FA also invites the criticism Russell opposed to PA:

> We want our numbers to be such as can be used for counting common objects, and this requires that our numbers should have a *definite* meaning, not merely that they should have certain formal properties. (Russell 1919: 10).

But a reduction of a theory to another one does not signify that the reduced theory *itself* is derivable in (or a part of) the other one. It only means that the former is *interpretable* in the latter. Strictly speaking, the very mathematical notion of number cannot be defined in logic or in any other theory, for numbers would, then, have properties we are not ready to recognize as arithmetical ones. Mathematicians' positive integers cannot *be* such things as extensions or other logical objects abstracted from concepts; they cannot *be* classes of classes, or a certain kind of Zermelo-Fraenkel sets, neither – as suggested by P. Simons (2007) – properties of multitudes. In all these cases, numbers would have mathematically irrelevant properties, expressed by propositions which do not belong to arithmetic[2]. What I am ready to call "number" in mathematics is only one of these abstract and general entities which strictly satisfy all the theorems of arithmetic and no other.

In this perspective, what a logicist approach to a mathematical theory can provide is only what I will call a picture of this theory, that is to say a specific interpretation in which purely logical objects or constructions can play the role of mathematical notions or entities (numbers, in particular). Nevertheless, the existence of such a purely logical picture is far from pointless for the philosophy of mathematics. Naturally, it first gives a relative proof of consistency. But it especially provides an objective and conceptual path to arithmetical knowledge. A logicist picture gives such a secured epistemic justification, because it allows for a conceptual construction to replace the intuitive content and naïve notions which lead mathematicians, step by step, in the development of their practice, to the axiomatic characterization of their theory. The route thus constructed is epistemically secured, for the truth of the propositions it consists of depends only on logic.

The possibility of reinterpretations is today widely recognized by logicians and mathematicians as an essential advantage of axiomatic theories. Russell's above mentioned criticism was clearly overtaken by further developments of formal sciences. Nonetheless, his requirements – that our numbers can be used for counting common objects and that they have a definite meaning – are perfectly relevant relative to what I have called a picture of arithmetic. In order to provide the kind of justification I have just described, the picture must be materially adequate in Tarski's sense – it must present an adequate analysis of the naïve notion of number we use when counting concrete objects. On the other hand, it must also be definite in meaning. This requires the meaning of the defined terms to be fully determined by the meaning of logical constants. For this reason, the use of any implicit definition should be prohibited.

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2 As a basic example, if zero is defined as the empty set in ZF set theory, it receives the non-arithmetical property to be part of all the sets.

Frege's requirement that only logical constants occur in his Basic Laws is not followed by the neo-Fregeans. FA is undoubtedly a very nice theory to which arithmetic can be reduced. Nevertheless, it is neither arithmetic itself (as PA is), nor is it a good logicist picture of arithmetic, for it does not exclude reinterpretations of the proper term introduced by HP. For the latter condition to be satisfied, only explicit definitions must be used in the picture's construction.

In the following pages, I am going to show that a valuable logicist picture of arithmetic can be constructed from logic using only explicit definitions. This will be done without introducing extensions of concepts or classes – even as incomplete symbols or way of speaking – but on the ground of Stanisław Leśniewki's logical notion of name. Like plural terms do in natural languages, Leśniewki's names enable the expression of pluralities of things in the logical language.

2. A logic of names

When we assert a numerical statement like "there are five continents", according to Frege, we are speaking about a property of the concept *continent*. For Russell and Whitehead, it is to the class of continents that the property is asserted. Of course, neither the concept, nor the class can simply be said to be five. Before being analyzed by means of the logical relation of equinumerosity, the property in question can only be described as *having five objects falling under it* (for the concept), or *being a member of it* (for the class). On the other hand, *being five* is obviously not a property of the objects themselves: the continents *are* five, but none of them *is* five. According to P. Simons (2007), the property in question is a property of the "multitude" of continents (a notion he says to be akin to Husserl's "Vielheit" or Russell's "class as many"). But where is the expected solution? Like with "the concept of continent" or "the class of continents", "the multitude of continents" is obviously a *singular term*. The ordinary fact that we can use a single word or a single expression in order to refer to several objects seems to be mysterious for logicians as long as we do not postulate the existence of a single intermediate entity which has the (still mysterious?) virtue to gather together the things in question.

The idea underlying the logical picture of arithmetic to be presented hereafter is much more unsophisticated. Without trying to *explain* the one-to-many link between expressions and objects, one just *observes* that ordinary language involves expressions or words which are used to refer sometimes to a single thing (for example, "Cairo" or "the capital of Egypt"), sometimes to several things ("the African capitals", "horses") and also sometimes to nothing ("the capital of Africa", "Ulysses", "round circles"). The basic idea is to interpret numbers as *certain semantic properties of names*. Zero will then be depicted as the property of a name to be empty, one as the property of a name to be singular and three as the property of a name to refer to three things.

Leśniewski's calculus

Called *Ontology*, Leśniewski's system is grounded on such basic observations. It is an expansion of a quantified propositional calculus (called *Protothetic*). It has a single specific axiom introducing the constant copula *epsilon* and variables for names:

AxOnto:

$$\lfloor ab \rfloor \lceil a\varepsilon b \equiv (\lfloor \exists c \rfloor \lceil c\varepsilon a \rceil \wedge \lfloor cd \rfloor \lceil (c\varepsilon a \wedge d\varepsilon a) \supset c\varepsilon d \rceil \wedge \lfloor c \rfloor \lceil c\varepsilon a \supset c\varepsilon b \rceil) \rceil$$

The left-hand side of the equivalence '$a\varepsilon b$' is the general form of a singular proposition. It can be read as "*a* is *b*", or more precisely "the only object denoted by '*a*' is also denoted by '*b*' ". In other words, '$a\varepsilon b$' is truly asserted if and only if '*a*' stands for a singular name and '*b*' for a singular or plural name which denotes (possibly among others) the object denoted by '*a*'.

Definition rules

Among several peculiarities of Ontology, the system includes rules for stating explicit definitions of two kinds. Instead of stating definitions in the meta-language – like in the *Principia*, using the unspecified symbol '$=_{df}$' and introducing only convenient abbreviations – Leśniewski uses his primitive logical constants for expressing the equivalence relation between the *definiendum* (*Dum*) and the *definiens* (*Diens*)[3]. The first rule allows the introduction of propositional constants and functors while the second one allows the introduction of nominal constants or functors. The definitional equivalence is thus expressed by one of the two forms[4]:

$\lfloor v_1 \cdots v_n \rfloor \lceil Dum \equiv Diens \rceil$ Def$_S$ (propositional rule)

$\lfloor v_1 \cdots v_n \rfloor \lceil a\varepsilon Dum \equiv (a\varepsilon a \wedge Diens) \rceil$ Def$_N$ (nominal rule)

where 1. the left and right-hand sides of the equivalence involve the same (free) variables $v_1, \ldots, v_n$; 2. *Diens* is a formula with only primitive or already defined symbols; 3. *Dum* is of the following form, where # is the symbol to be defined and no symbol occur more than once:

Dum: $\#(v_1 \cdots v_i)(v_{i+1} \cdots v_j) \cdots (v_k \cdots v_n)$

As we will see, this general form of *Dum* relates to three possibilities. First, there can be no variable in *Dum*. The defined symbol is then either a constant proposition (with Def$_S$), or a constant name (with Def$_N$), like in the following examples:

D1. $\mathrm{T} \equiv \lfloor p \rfloor \lceil p \equiv p \rceil$ Def$_S$ (T : constant *true*)

D2. $\lfloor a \rfloor \lceil a\varepsilon \wedge \equiv (a\varepsilon a \wedge \sim (a\varepsilon a)) \rceil$ Def$_N$ ($\wedge$: empty name)

D3. $\lfloor a \rfloor \lceil a\varepsilon \vee \equiv (a\varepsilon a \wedge a\varepsilon a) \rceil$ Def$_N$ ($\vee$: universal name)

In the second case, the variables of *Dum* occur in a single pair of parentheses. The defined symbol is then a functor:

D4. $\lfloor ab \rfloor \lceil =\{ab\} \equiv (a\varepsilon b \wedge b\varepsilon a) \rceil$ Def$_S$

($=\{ab\}$: *a* is the same object as *b*)

D5. $\lfloor ab \rfloor \lceil \cong\{ab\} \equiv \lfloor c \rfloor \lceil c\varepsilon a \equiv c\varepsilon b \rceil \rceil$ Def$_S$

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[3] About Leśniewskian internal definitions vs classical external and meta-linguistic definitions, see Joray (2005), (2006) & (2011).

[4] Where, in Leśniewski's notation, the first pair of square brackets expresses the universal quantifiers and the second pair contains the formula on which it is applied.

($\cong\{ab\}$: *a* and *b* have the same reference(s))

D6. $\lfloor a \rfloor \lceil 0\{a\} \equiv \sim \lfloor \exists b \rfloor \lceil b\varepsilon a \rceil \rceil$ Def$_S$ ($0\{a\}$: *a* is empty)

D7. $\lfloor a \rfloor \lceil 1\{a\} \equiv a\varepsilon a \rceil$ Def$_S$ ($1\{a\}$: *a* is singular)

D8. $\lfloor ab \rfloor \lceil \approx[\varphi\psi] \equiv \lfloor a \rfloor \lceil \varphi\{a\} \equiv \psi\{a\} \rceil \rceil$ Def$_S$

($\approx[\varphi\psi]$: φ and ψ are co-extensive)

D9. $\lfloor ab \rfloor \lceil a\varepsilon(b{\cdot}c) \equiv (a\varepsilon b \wedge a\varepsilon c) \rceil$ Def$_N$

(nominal intersection)

D10. $\lfloor ab \rfloor \lceil a\varepsilon(b+c) \equiv (a\varepsilon b \vee a\varepsilon c) \rceil$ Def$_N$ (nominal union)

D11. $\lfloor ab \rfloor \lceil a\varepsilon(b-c) \equiv (a\varepsilon b \wedge \sim(a\varepsilon c)) \rceil$ Def$_N$

(nominal complement)

In the last case, the variables of *Dum* split up into more than one pair of parentheses. The symbol to be defined is thus a multi-link or parametric functor, i.e. a functor-forming functor:

D12. $\lfloor ab \rfloor \lceil \cong\langle a\rangle\{b\} \equiv \cong\{ba\} \rceil$ Def$_S$

(parametric co-reference; $\cong\langle a\rangle$: denoting like *a*)

D13. $\lfloor ab \rfloor \lceil \varepsilon\langle a\rangle\{b\} \equiv b\varepsilon a \rceil$ Def$_S$

(parametric *epsilon*; $\varepsilon\langle a\rangle$: being one of the *a*'s)

Without going into the details, I will just underline certain aspects of the definition rules which are central for the understanding of the definition of numbers in the next section.

First, it has to be noticed that the definition rules allow introducing symbols of categories which are not previously available in the language. This is quite obvious in the case of multi-link or parametric functors. In D12, for example, the parametric functor of co-reference is defined on the basis of the usual identity binary relation. The parametric functor is the result of a different analysis of the same content: first, it applies to '*a*' and the result '$\diamond\cong\langle a\rangle$' expresses the nominal property "denoting-(exactly)-the-*a*'s"; this property can, then, be applied to a name '*b*', obtaining '$\cong\langle a\rangle\{b\}$' which expresses that '*b*' denotes (exactly) the *a*'s. This process of definition is very similar to a λ-abstraction, and, in Leśniewski's language, '$\cong\langle a\rangle$' is exactly what would be expressed by '$\lambda b.(\cong\{ba\})$' in λ-notation.

Secondly, Leśniewski's system is such that the definition of a functor in a new category allows the use of variables of that category and the binding of these variables by quantifiers.

The power of definition rules makes Ontology a strong analytical tool. Every semantic category can be reached progressively and the order of the formal language depends on its specific definitional development. One of Leśniewski's main achievements was his ability to elaborate completely explicit semantic and syntactic constraints in order to impose extensionality to every category and to avoid ambiguity and contradiction in the potentially infinite process of definition[5].

3. A logicist construction[6]

In the following construction, natural numbers are going to be depicted as *cardinal properties of finite names* (names which denote only a finite quantity of objects). Before going into the definition of the general notion of natural number, let me first consider how any particular natural number can be defined. Zero and one have already been introduced by definitions D6 and D7:

D6. $\lfloor a \rfloor \lceil 0\{a\} \equiv \sim \lfloor \exists b \rfloor \lceil b\varepsilon a \rceil \rceil$ $\quad$ Def_S

D7. $\lfloor a \rfloor \lceil 1\{a\} \equiv a\varepsilon a \rceil$ $\quad$ Def_S

In order to define the successor *n*' of a previously defined number *n*, one has to state that a name *a* has the number *n*' if and only if a name which denotes exactly the *a*'s excepted one of them has the number *n* :

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[5] For a complete presentation of Leśniewski's Ontology, see Miéville (1984), (2004) and also the papers in Srzednicki & Rickey (1984).

[6] For the full presentation of the following logicist construction, with proofs and technical details, see Gessler, Joray, Degrange (2005: 73-137). The construction is partially inspired from Canty (1967).

D14. $\lfloor \varphi a \rfloor \lceil S(\varphi)\{a\} \equiv \lfloor b \rfloor \lceil b\varepsilon a \wedge \varphi\{a-b\} \rceil \rceil$ Def$_S$ (successor)

From this, it is obvious that a symbol $\boldsymbol{n}$ for any natural number $n > 0$ can be introduced with a definition of the following form, where '$S(\cdots)$' is iterated *n* times:

$$\lfloor a \rfloor \lceil \boldsymbol{n}\{a\} \equiv S(S(\ldots S(0)\ldots))\{a\} \rceil \qquad \text{Def}_S$$

Let me turn now to the definition of equinumerosity. Two names will be said to be equinumerous if and only if there is a one-to-one correspondence between the references of the first name and the references of the other one. In other terms, the one-to-one relation must have the first name as its *domain* and the second name as its *co-domain*. Let then consider the following preliminary definitions of *one-to-one relation*, *domain* and *co-domain* of a relation:

D15.
$$\lfloor R \rfloor \lceil 1\text{-}1[R] \equiv \lfloor abc \rfloor \lceil (R\{ac\} \wedge R\{bc\}) \vee (R\{ca\} \wedge R\{cb\}) \supset =\{ab\} \rceil \rceil$$
Def$_N$

D16. $\lfloor aR \rfloor \lceil a\varepsilon Dom\langle R \rangle \equiv (a\varepsilon a \wedge \lfloor \exists b \rfloor \lceil R\{ab\} \rceil) \rceil$ Def$_N$

D17. $\lfloor aR \rfloor \lceil a\varepsilon CoDom\langle R \rangle \equiv (a\varepsilon a \wedge \lfloor \exists b \rfloor \lceil R\{ba\} \rceil) \rceil$
Def$_N$

And finally the definition of *equinumerosity*:

D18.
$$\lfloor ab \rfloor \lceil \infty\{ab\} \equiv \lfloor \exists R \rfloor \lceil 1\text{-}1[R] \wedge \simeq\{Dom\langle R \rangle\, a\} \wedge \simeq\{CoDom\langle R \rangle\, b\} \rceil \rceil$$
Def$_S$

The cardinality of a name is, thus, simply defined as the parametric version of '∞':

D19. $\lfloor ab \rfloor \lceil \infty\langle a \rangle\{b\} \equiv \infty\{ba\} \rceil$ Def$_S$

By the abstraction of '*b*' in '$\infty\{ba\}$', one gets the complex functor '$\infty\langle a \rangle$', which expresses the nominal property "denoting as many objects as *a*" or "having the cardinality of *a*". '$\propto\langle\cdots\rangle$' is then a parametric functor. When it is applied to a name, the result is a functor expressing the cardinal property of this name. As numbers are depicted as properties of names, it is natural to read '$\propto\langle a\rangle$' as "the cardinal number of *a*"[7] and the following theorem, which is easy to prove from D19, as the Leśniewskian version of HP:

$\lfloor ab \rfloor \lceil \approx [\propto\langle a\rangle \propto\langle b\rangle] \equiv \propto\{ab\} \rceil$

Contrary to the Fregean version of HP, here, the left-hand side does not express an identity between singular names, but an identity between nominal functors. Leśniewskian versions of Fregean "abstraction principles" are strictly predicative. This has important consequences on the construction. First, Leśniewskian versions of the "abstraction principles" never lead to contradiction. In particular, the analogue of Frege's Basic Law V is perfectly harmless and can be easily inferred from D12:

$\lfloor ab \rfloor \lceil \approx [\simeq\langle a\rangle \simeq\langle b\rangle] \equiv \simeq\{ab\} \rceil$

Second, the fact that abstraction's results are not designated as objects preserves the ontological neutrality of logic. The theorems of Leśniewski's calculus are logically true in the sense that they are true in all domains, including the empty domain. A consequence of this is that there will be no way to avoid the addition of an axiom of infinity for the derivation of all Peano's propositions.

From D19, the general definition of *cardinal number* can then be stated as:

D20. $\lfloor \varphi \rfloor \lceil Cn[\varphi] \equiv \lfloor \exists a \rfloor \lceil \approx [\propto\langle a\rangle\, \varphi] \rceil \rceil$ Def$_S$

In order to specify which cardinal numbers are natural numbers the definition of finite names is required. Like in Frege's *Grundlagen*, this will be done using the notion of inductivity: a name is said to be *inductive* if it has all the properties of the empty name that are preserved by the addition of a single denotation:

[7] This is of course only a way of speaking, for '$\propto\langle a\rangle$' is not the name of an object, but a symbol for a function.

D21.

$$\lfloor a \rfloor \lceil Ind\{a\} \equiv \lfloor \varphi \rfloor \lceil (\varphi\{\wedge\} \wedge \lfloor bc \rfloor \lceil (\varphi\{c\} \wedge 1\{b\}) \supset \varphi\{c+b\} \rceil) \supset \varphi\{a\} \rceil \rceil$$

Def_N

From this, *natural numbers* can be characterized as the cardinal numbers of finite names:

D22. $\lfloor \varphi \rfloor \lceil Nn[\varphi] \equiv (Cn[\varphi] \wedge \lfloor a \rfloor \lceil \varphi\{a\} \supset Ind\{a\} \rceil \rceil$ Def_S

D6, D14 and D22 are the respective definitions in Leśniewski's Ontology of Peano's primitive terms *zero*, *successor* and *number*. Peano's propositions I, IV and V are derivable from these definitions in pure Ontology

P_I $Nn[0]$ (zero is a number)

P_{IV} $\lfloor \varphi \rfloor \lceil Nn[\varphi] \supset \sim (\approx [S(\varphi)0]) \rceil$

(zero is not the successor of a number)

P_V $\lfloor P \rfloor \lceil (P[0] \wedge \lfloor \varphi \rfloor \lceil (Nn[\varphi] \wedge P[\varphi]) \supset P[S(\varphi)] \rceil) \supset \lfloor \psi \rfloor \lceil Nn[\psi] \supset P[\psi] \rceil \rceil$

(mathematical induction)

the remaining two propositions being only derivable in infinite Ontology:

P_{II} $\lfloor \varphi \rfloor \lceil Nn[\varphi] \supset Nn[S(\varphi)] \rceil$

(the successor of a number is a number)

P_{III} $\lfloor \varphi\psi \rfloor \lceil (Nn[\varphi] \wedge Nn[\psi]) \supset (\approx [S(\varphi)S(\psi)] \supset \approx [\varphi\psi]) \rceil$

(different numbers have different successors)

The proofs, which are quite long, can be found by the reader in Gessler, Joray, Degrange 2005: 75-137.

The full picture of Peano Arithmetic is thus constructed in a third-order development of infinite Ontology: a system of pure logic with the addition of an axiom of infinity.

One can notice that the dependence of Peano's propositions vis-à-vis the single non-logical axiom is not in the same dependence one can read about in the *Principia*. Here, not only P_{III}, but also P_{II} ("the successor of a number is a number") requires the existence of infinitely many objects. This is due to the fact that P_{II} cannot be read here as "ambiguous as to type", avoiding the artificial meaning of P_{II} in the *Principia*: *for every number* n*, there is a type* t *in which the successor of* n *(in fact the analogue of* n *for* t*) is a number*.

As it has been shown by Nadine Gessler[8], type ambiguity is not required to warrant the unity of all the higher-degree arithmetics that can be developed in Ontology. Anyhow, since what is to be constructed is not arithmetic *itself*, but a logical *picture* of it – an interpretation of general arithmetic in a system of certain definite logical entities – the classical question of the unity of the type hierarchy of arithmetics becomes almost superfluous.

Conclusion

Considering the above logicist construction of Peano Arithmetic as a logical picture of the mathematical theory, I claim that the existence of the reduction of PA to Leśniewski's Ontology does not inform us about the nature or the essence of numbers. The reduction does not show that arithmetic is a part of logic. The kind of foundation obtained through this process is not of ontological significance. But the possibility to reach arithmetical laws in the realm of logic has an epistemic value. Using only explicit definitions, it shows a purely analytic route to the knowledge of arithmetic.

Of course the need of an axiom expressing the existence of infinitely many objects can certainly be considered as a limitation of this program. But neither in common counting, nor in any application of arithmetic, does the assumption that there will always be enough available objects for the successor of a given number to be different from the number in question imply any commitment concerning the nature of the real world. The axiom of infinity is not an empirical hypothesis, but a conceptual assumption specifying the kind of idealization through which we apply arithmetic to specific concrete or abstract situations.

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[8] Cf. Gessler, Joray, Degrange (2005: 9-36).

Hence, providing a purely conceptual content which guides us to Peano's axioms, the reduction to infinite Ontology gives an analytic justification for the adoption of these axioms as forming the basis for our coherent and applicable theory of pure mathematics.

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