Ernst Schroeder and Zermelo's Anticipation of Russell's Paradox

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Abstract

Ernst Zermelo presented an argument showing that there is no set of all sets that are members of themselves in a letter to Edmund Husserl on April 16th of 1902, and so just barely anticipated the same contradiction in Betrand Russell's letter to Frege from June 16th of that year. This paper traces the origins of Zermelo's paradox in Husserl's criticisms of a peculiar argument in Ernst Schroeder's 1890 Algebra der Logik. Frege had also criticized that argument in his 1985 "A Critical Elucidation of Some Points in E. Schroeder Vorlesungen über die Algebra der Logik", but did not see the paradox that Zermelo found. Alonzo Church, in "Schroeder's Anticipation of the Simple Theory of Types" from 1939, cricized Frege's treatment of Schroeder's views, but did not identify the connection with Russell's paradox.

Bertrand Russell wrote to Gottlob Frege on June 16th, 1902, saying that he had found "one point where I have encountered a difficulty" in reading over *Grundgesetze der Arithmetik* in preparation for publishing his own *Principles of Mathematics*. The difficulty is described as follows:

> Let *w* be the predicate: to be a predicate that cannot be predicated of itself. Can *w* be predicated of itself? From each answer its opposite follows. Therefore we must conclude that *w* is not a predicate.

Russell immediately follows this with what sounds like the different, and more familiar, problem of the class of all classes which do not belong to themselves, but is presented as another statement of the same point of difficulty:

> Likewise there is no class (as a totality) of those classes which, each taken as a totality, do not belong to themselves. From this I conclude that under certain circumstances a definable collection [Menge] does not form a totality.[1]

These are in fact two different problems. The first involves the notion of a predicate that is or is not predicated of itself. As Frege pointed out immediately in his response to Russell, if one makes a sharp distinction between predicates that are true of individuals, and the (higher order) predicates that are true of those predicates, as Frege himself did within his theory of concepts and objects, then it simply won't make sense to think of a predicate applying to itself. There will be no predicate "to be a predicate that is not predicated of itself," and so the paradox cannot arise. In fact, at the time, Russell did not accept the notion that concepts would be distinguished by type, and only acknowledged that something of the sort might be necessary in Appendix B of *Principles of Mathematics*. This first paradox was the most significant for Russell. He only formulated the second when he saw that related versions could be constructed for other theories, such as Frege's. Frege immediately acknowledged that there was a problem for his system in the second paradox.

Frege responded to Russell's letter quite quickly, with a reply dated June 22nd, just six days later. Frege concludes as follows:

> It seems, then, that transforming the generalization of an equality into an equality of courses-of-values (§9 of my *Grundgesetze*) is not always permitted, that my Rule V (§20, p.36) is false, and that my explanations in §31 are not sufficient to ensure that my combinations of signs have a meaning in all cases.

[1] Van Heijenoort (1967, pp. 127-128).

Russell did come to the paradox by considering Cantor's theorem that the set of subsets of a given set, its power set, cannot be put into a one-to-one correspondence with that set by considering the hypothesis that there is a universal set. If one follows the proof of Cantor's theorem the paradoxical set of all sets that are not members of themselves is the set that is added in the power set of the universe. Russell himself, however, did not see this as a problem for Cantor's set theory as much as for his own. Russell originally found the paradox for his own views in *Principles of Mathematics*, in which classes are represented by "class concepts", which in fact denote the members of a class. The members of the class can be taken "as one" or "as many" and it is when we speak of a class as one that it makes perfect sense for it to be an instance of a class, and so the concept "is not predicated of itself" can perfectly well be applied to itself (as one), thus leading to the contradiction.

In this paper, however, I will follow the earlier history of the second version of the paradox, which was anticipated by Zermelo. The paradox developed out of an argument of Ernst Schröder, which was subsequently discussed by Frege, Edmund Husserl, and it finally appeared in a letter from Zermelo to Husserl, although Hilbert also had his own version. Finding that this version of the paradox did in fact have a long history that preceded Russell's letter to Frege, if anything, will only add to the argument that it was that first version that most immediately interested Russell, the problem for his own theory, and that he wrote to Frege only to show how a similar problem arose in Frege's *Grundgesetze*.

Any discussion of anticipations of the second version of the paradox might look like an intervention in a mathematicians' *Prioritätstreit* (priority dispute), but it in fact leads to an interesting series of events in the history of logic, pieces of which are familiar, but which, it seems, has not been formed into a single story before.

The most prominent claim of priority is in Ernst Zermelo's famous 1908 paper "A new proof of the possibility of a well-ordering." Zermelo adds a footnote (numbered 9) to a mention of Russell's discussion of his "set-theoretic antinomy" in *Principles*:

> … I had, however, discovered this antinomy myself, independently of Russell, and had communicated it prior to 1903 to Professor Hilbert among others.

In a letter of 1903, thanking Frege for a copy of the second volume of his *Grundgesetze* with its discussion of Russell's paradox, Hilbert told Frege that he had heard of the paradox from Zermelo some years before, confirming Zermelo's claim. In fact, Hilbert had his own version of the contradiction, concerning the class of all sequences of "self-mappings'" of the numbers onto the numbers, using a diagonal argument to show that the set of such self-mappings does not exist.[2] In his biography of Zermelo, Ebbinghaus claims to just be following the "custom" at Göttingen, where the paradox was attributed to Zermelo alone. (Ebbinghaus actually more closely follows Abraham Fraenkel's terminology and calls it the "Zermelo Russell paradox" in the biography.[3])

"Hilbert's Paradox" is similar to Zermelo's in proving that some set does not exist, but in his case it is a set that one would expect given certain standard set theoretic operations. The paradox can be reconstructed from notes on Hilbert's lecture from 10 July 1905.[4] Hilbert derives a contradiction by taking the set of mappings of the numbers into the numbers M and then using two principles of set theory. The first principle allows one to unite "several sets and even infinitely many into a union." The other asserts that "in every case well-defined sets arise from well-defined sets by the self-mapping operation." Thus from the set M we can formulate the set M^M of all mappings from M into M. Hilbert then considers the set U derived by applying these "operations of addition and mapping an arbitrary number of times." Finally, he applies the mapping principle one more time, to U, so $F = U^U$. Now it would appear that F is already a subset of U, but, by a familiar diagonal argument based on Cantor's theorem, we can show that there is an element in F not in U.

[2] Ebbinghaus (2007, pp. 45-47) and Peckhaus (2004, pp. 505-506).

[3] Ebbinghaus (2007, pp.46-47) and Fraenkel (1927). Fraenkel, perhaps the first to use this term in print, puts "Zermelo" in parentheses, and calls it the "(Zermelo) Russell" paradox once in his presentation and not again in his discussion. He does not give any reason for his usage.

[4] Ebbinghaus (2007, pp. 45-47).

Hilbert's words are not precise; in particular it isn't clear what he means by applying the two operations an "arbitrary number of times." One guess is that he thinks of F as being the union of an infinite number of sets:

$$M^M,\ (M^M)^M,\ M^M \cup (M^M)^M,\ ((M^M)^M)^M),\ M^M \cup (M^M)^M \cup ((M^M)^M)^M),\ \ldots$$

It appears that, if we have continued this process an "arbitrary number of times", then anything in U^U must have been already included in this process and so the process yields nothing more than U. Of course we immediately sense that the "arbitrary" number of operations must pass through all the ordinals, taking a union after ω many iterations of M and beginning again with another process of either union or exponentiation for each ordinal number, and so on. Seen this way "Hilbert's Paradox" is a proof that there is no set of all ordinals. Hilbert was, reportedly, suspicious of the "philosophical" notions used in other versions of the paradox, such as the notion of a "set of all sets" or even a "set of all ordinal numbers", and so preferred to limit himself to more familiar mathematical notions, such as mapping, or the application of an operation an "arbitrary" number of times. Thus, the argument can be seen, in Hilbert's way, as a contradiction which can be derived from two apparently ordinary mathematical operations, considering function spaces and taking unions of arbitrary collections of things already constructed. It doesn't point to an internal contradiction in the very notion of set or class. It may well be an "antinomy", as Zermelo puts it, a real internal contradiction, but it is a contradiction in a theory, one which Hilbert held should be prevented by constructing a (provably) consistent axiomatic theory of sets.

Zermelo sent a letter on April 16th, 1902, to his former teacher, Edmund Husserl. Husserl's notes on the letter were found among the papers in the Husserl archives thirty years ago.[5] There, Zermelo reports a result that he had obtained some years before.[6] The letter was occasioned by a review of Schröder's *Algebra der Logik* (1890) that

[5] Rang and Thomas (1981) and Ebbinghaus (2007, p. 46).
[6] See Husserl (1979 p. 399) for Husserl's notes on this letter.

Husserl had written in 1891.[7] Schröder had presented a proof that a universal class, one that contains "everything conceivable", leads to a contradiction. (It thus appears that Schröder joined Cantor in being among the first to claim that there is a concept which does not have an extension.)

One primary goal of Schröder's lectures was to promote the work of Charles S. Peirce "and his school" among German logicians (Schröder 1890, p.iii). In Lecture IV, devoted to the theory of classes, Schröder presents the algebraic account of classes using one primitive notion of "subsumption" (*Subsumtion*). The assertion that a class *a* subsumes a class *b*, symbolized as *a* € *b* which is read as "*a* is *b*" or "all *a* are *b*", clearly means what we would express as "*a* is a subset of *b*", symbolized as $a \subseteq b$.[8] This logic interprets all predications to be of this form "*a* is *b*", relating one class to another.

One part of Schröder's account of sets is a rejection of Boole's notion of the *universe of discourse*, which in this framework will be the element 1 in the algebra of classes. The argument below is from page 245 of Schröder (1890) as quoted by Frege:

> As we have laid down, 0 would have to be contained in *every class* that can be got out of the manifold 1; … 0 would have to be the subject of *every* predicate. Now suppose we took *a* to be *the class of those classes of the manifold that are equal to* 1 (which would certainly be permissible if we could bring everything thinkable into the manifold 1), then this class of its very nature contains just one class, viz. the symbol 1 itself, or alternatively the whole of the manifold, which constitutes the reference of the symbol; *but therefore besides this it would contain "nothing,"* i.e. 0. Hence 1 and 0 would make up the class of the objects that are to be equal to 1; and so we should have to admit not only: 1 = 1 but *also*: 0 = 1. For a predicate that applies to a class -- in our case, the predicate: to be identically equal to 1 -- must also

[7] See Husserl (1979 p. 36).

[8] Actually the symbol is the identity sign = with a left parenthesis (overwritten, but it is close enough to the readily available Euro sign to justify that as the symbol for any revival of interest in Schröder's logic.

> apply to every individual in the class, by Principle II. (Frege, 1895, p.91)

For Schröder, all predications are assertions of subsumptions, and in particular, the predication 'is equal to 1', which we would express as '$x = 1$', becomes an assertion about subsumption. Given that the empty set, represented by 0, is subsumed by every class *a*, i.e. 0 € *a*, it is also subsumed by that of the things equal to 0, hence we derive 0 = 1, a contradiction.[9] This he takes to be a proof that there is no absolutely universal class 1. There will be sets that are not subsumed under 1, in particular, the empty class 0.

Husserl charged in his review that Schröder had ignored the distinction between subsets (the notion of a "subordinate class") and members. While it is true that the empty class 0 is a subset of every set, it is not a member of every set. In particular that 0 is a subset of the set of entities equal to 1 does not imply that it is equal to 1, the supposed contradiction which is derived from the assumption that 1 is a set of all sets. Zermelo then wrote to Husserl to point out that "on the issue, not the method of proof, Schröder is right…"[10]. Translated from the original German *Gabelsburger Stenographie* shorthand, the notes describe the argument as follows:

> A set *M*, which contains each of its subsets *m*, *m'*, … as elements, is an inconsistent set, i.e., such a set, if at all treated as a set, leads to contradictions.
>
> PROOF. We consider those subsets *m* which do not contain themselves as elements.

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[9] This is a charitable reconstruction. Notice that Schröder argues that the class of things equal to 1 includes 1 "and nothing else", i.e., also the empty class ("nothing"). This sounds like Carnap's later interpretation of what Heidegger says about "*Das Nichts*", or "Nothing", as something which is included in classes. Frege criticizes Schröder's argument on this point (p.98), anticipating Carnap's criticism of Heidegger: "If we say that the class *a* contains nothing besides the Moon, then we are denying the proposition that the class contains something besides the Moon; but we are not thereby asserting that the class contains, besides the Moon, an object with the name 'nothing'."

[10] Rang and Thomas (1981, p. 16).

> (*M* contains as elements each of its subsets; hence subsets of *M* will also contain certain subsets as elements, themselves [not] being elements, and now we consider just those subsets *m*, which may perhaps contain other subsets, but not themselves as elements.)
> These constitute in their totality a set M_0 (i.e., the set of all subsets of *M* which do not contain themselves as elements), and now I prove of M_0,
>
> (1) that it does not contain itself as an element,
> (2) that it contains itself as an element.
>
> Concerning (1): M_0, being a subset of *M*, is itself an element of *M*, but not an element of M_0. For otherwise, M_0 would contain as an element a subset of *M* (namely, M_0 itself) which contains itself as an element, and that would contradict the notion of M_0.
>
> Concerning (2): Hence M_0 itself is a subset of *M* which does not contain itself as an element. Thus it must be an element of M_0.[11]

So stated this is a proof that no set contains all of its own subsets as members. A universal set of all things, however, would certainly include all of its subsets as members, as those are all sets of things. The set M_0, then, the set of all subsets of the universal set that do not contain themselves, will thus be simply the set of all sets that do not contain themselves. M_0, in this case, is the Russell set. The proof that M_0 leads to contradiction is the same argument that Russell gives, if M_0 is a member of itself then it is not, and if it is not a member of itself then it is. We have the same contradiction as in Russell's letter, and both, it seems, come fairly directly from applying Cantor's theorem to a set of all sets (or in Zermelo's case, a set which contains at least all its subsets.)

But do we have Russell's paradox? In fact, what we have is a theorem to the effect that there is no set which contains all of its subsets as members. This is a theorem about sets. Indeed in another

[11] Rang and Thomas (1981, pp. 16-17).

paper from 1908, "Investigations in the foundations of set theory", we find it openly listed as a theorem with the following proof:

> 10. Theorem. Every set M possesses at least one subset M_0 that is not an element of M. Proof. It is definite for every element x of M whether $x \in x$ or not; the possibility that $x \in x$ is not in itself excluded by our axioms. If now M_0 is the subset of M that, in accordance with our Axiom III [Zermelo's Axiom of Separation], contains all those elements of M for which it is not the case that $x \in x$, then M_0 cannot be an element of M. For either $M_0 \in M_0$ or not. In the first case, M_0 would contain an element $x = M_0$ for which $x \in x$, and this would contradict the definition of M_0. Thus M_0 is surely not an element of M_0, and in consequence M_0, if it were an element of M, would also have to be an element of M_0, which was just excluded.

Zermelo concludes the proof with this remark:

> It follows from the theorem that not all objects x of the domain B can be elements of one and the same set; that is, *the domain B it is not itself a set*, and this disposes of the Russell antinomy so far as we are concerned.[12]

So Zermelo "disposes" of Russell's antinomy by presenting it as a theorem that a certain collection is not a set, a proof by a *reductio ad absurdum* argument.[13] Burali-Forti's proof from 1897 that the ordinals cannot be well-ordered was also taken as such an argument.[14] If every

[12] Van Heijenoort (1967, p. 203).

[13] In particular, there is no set that contains all of its subsets as members. This might sound a bit like the Axiom of Foundation, which says that there is no non-empty set that overlaps with every one of its members, and so shares a member with every one of its subsets. They are clearly very different claims, but do sound alike in the sense of saying that there is no set of a given sort.

[14] See Moore and Garciadiego (1981) for an account of how Burali-Forti's original theorem came to be seen as a "paradox". It started life as a claim that the set of all ordinals could not be well-ordered. After Burali-Forti's

set can be well ordered then it follows that there is no set of ordinals. Indeed Zermelo's proof would seem to be also a proof related to the idea of "absolute infinities," or of classes that are in some sense "too big" to be sets. A "set of all sets" would certainly contain all of its subsets as members, and so one immediate result from Zermelo's theorem is that there is no set of all sets.

Zermelo considers that he has "disposed" of Russell's paradox by showing it to be a simple non-existence proof, although a surprising one. But there is no "antinomy" in that. A proof that there is no set with a certain description is different from proving that nothing satisfies a given description, so that the set of things satisfying the description is empty. It is really a proof that some expression, which seems to refer to a set, in fact doesn't, because there is no such set. For every counter-example to the unrestricted comprehension principle, that is, a case of a predicate without a set as its extension, there will surely be a corresponding proof that there is no such set, that is, that no set that contains just the objects which satisfy that predicate. But there is no trivial reformulation of every non-existence theorem into a counter-example to the unrestricted comprehension principle. To think that would be to make a logical mistake. Suppose one thought that "there is no set *y* which contains all of its (*y*'s) own subsets" directly provides a predicate which does not correspond with a set, thus a counter-instance to the unrestricted comprehension principle:

$$\exists y \forall x\ (x \in y \equiv \ldots x \ldots)$$

Such an instance would have to say that there exists a *y* such that any *x* is an element of *y* if and only if … what? If and only if *x* is a subset of *y*? That would violate the clause in the comprehension principle stated above which bans free occurrences of *y* in the formula … *x* … that determines membership in the putative set. It is not possible to formulate the claim that no set contains all its own subsets as members as a claim that there is no set of all things

incorrect definition of the relevant notion of well-ordering was corrected, the result was not seen as a paradox until Russell first so presented it in Principles of Mathematics §301.

satisfying a certain formula.[15] So Zermelo does indeed propose that Russell should be seen as only proving a non-existence theorem rather than as proposing a counter-example to a comprehension principle that was presumably implausible from the beginning. There might be some surprise in discovering that there is no set of all sets and, in fact, that no set can contain all of its subsets as members, but it is not clear that that should count as a "paradox." In fact, Zermelo himself preferred the term "antinomy" for the result rather than paradox.[16] He says that "paradox" means "… a statement contradicting *the common opinion*; it doesn't contain anything of the *inner contradiction* as in the case for the paradoxes of Russell and Burali-Forti, and expressed by the term 'antinomy'." On the other hand, that an "antinomy", as Zermelo uses the term, can be derived within a formal theory is simply a proof that one of its axioms is incorrect and must be discarded. It doesn't mean necessarily that the very concepts or basic notions of a theory are at fault and lead to the contradiction. So far, then, it appears that while Zermelo had indeed anticipated the very mathematical argument that Russell later included in his letter, he did not see that it was a "paradox" which might affect several approaches to sets, including Frege's.

There is more to be said about Schröder's argument, however.[17] In fact, Frege discusses the argument in his essay "A critical elucidation of some points in E. Schröder's *Vorlesungen über die Algebra der Logik*" (1895). Frege begins his discussion with the quote above. He then continues as follows:

> On p. 246 the author shows that we can apply these considerations to any class *b* of the manifold, instead of 1, and thus reaches the conclusion $0 = b$.

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[15] It is possible to present the proof that there is no set of all things as a counter-example to an instance of the comprehension principle: $\exists y \forall x\, (x \in y \equiv x = x)$, but this can't be done for every non-existence theorem.

[16] See Peckhaus and Kahle (2002, p.158). He says this in correspondence with Leonard Nelson in response to a paper about the "paradoxes" of Russell and Burali-Forti. Peckhaus and Kahle cite a postcard to Nelson, postmarked 22 December 1907, in the Nelson Papers in Bonn.

[17] I am grateful to Allen Hazen who reminded me that the Church and Frege papers explicitly discuss Schröder's argument.

> This contradiction comes like a thunderbolt from a clear sky. How could we be prepared for anything like this in exact logic! Who can go surety for it that we shall not again suddenly encounter a contradiction as we go on? The possibility of such a thing points to a mistake in the original design. Herr Schröder derives from this the conclusion that the original manifold 1 must be so made up that, among the elements given as individuals within it, there are found no classes that, for their part, contain within themselves as individuals any elements of the same manifold. This expedient, as it were, belatedly gets the ship off the sandbank; but had she been properly steered, she could have kept off it altogether. It now becomes clear why at the very outset, in shrewd prevision of the imminent danger, a certain manifold was introduced as the theatre of operation, although there was no reason for this in the pure domain-calculus. The subsequent restriction of this field for our logical activities is by no means elegant. Whereas elsewhere logic may claim to have laws of unrestricted validity, we are here required to begin by delimiting a manifold with careful tests, and it is only then that we can move around inside it.

A few lines later:

> When Herr Schröder stipulates (p. 248), as regards the original manifold, that among the elements given as 'individuals' there shall be found no classes that, for their part, comprise within them as individuals any elements of the same manifold, he is obviously distinguishing the case where something is given as an individual belonging to a manifold or class, where something is comprised within a class as an individual, from the case where something is contained as a class within a manifold or class. Herr Husserl makes a similar distinction, in his review of Schröder's work, between the expressions 'a class contains something as an element' and 'a class contains something as a sub-class,' and by this he tries to remove the difficulty. (Frege 1895, p. 92)

Note that Zermelo's theorem 10, that "Every set M possesses at least one subset M_0 that is not an element of M", in effect *proves* that one must distinguish the notions of subset ($\subseteq$) and member ($\in$) for the subsets of a given set are distinct from its members. While "disposing" of Russell's paradox by showing that the argument proves a perfectly sound theorem about sets, Zermelo's proof also shows that Schröder's conception of sets is fundamentally flawed. While avoiding the paradox by relying on his own axiomatic formulation of set theory, Zermelo also finds an important lesson about the nature of sets in the argument that produces the paradox. Frege had also carefully studied Schröder's argument, but missed the consequence for his own theory. Zermelo's argument properly deserves to be called an "anticipation" of Russell's paradox and he discovered it by studying the very passage that Frege criticized in Schröder.

Schröder's contradictory conclusion that 0 = 1 is, of course, part of a *reductio ad absurdum* argument. Schröder does not draw the conclusion that every class must contain some classes that aren't members of that class, but rather that every class must contain only classes that in turn do not contain members of the original class. These will be the classes that serve as "elements" of the original class. Frege sees this as an *ad hoc* solution. It "gets the ship off the sandbank."[18] Frege sees laws of logic as having "unrestricted validity", and so presumably, the universal quantifier ranges over everything, without restriction.

In a paper delivered in 1939, but not published until 1976, Alonzo Church presents Schröder's proposal as an anticipation of the simple theory of types. While the "elements" of a given set *a* will not subsume any classes which are also members of *a*, they may well subsume members of some other class *b*. Church proposes that we might see *a* and *b* as belonging to different types, where *a* is one type higher than *b*, as the elements of *a* are all classes containing members from the type of *b*. Frege's opposition to these restrictions, and his insistence that the laws of logic are unrestricted is seen by Church as a repetition of his view that all objects, including the "courses of

[18] Frege (1895, pp. 339-340), discussed in Church (1976, p. 410).

values" which are Frege's sets, are of the same type, and fall within the range of the universal quantifier for objects.[19]

Frege cites the same review of Schröder that Zermelo corrects in his letter from 1902. There is no evidence that Zermelo had read Frege's paper. At the least it appears that both had read Husserl's review, and reacted to it. Frege and Husserl had both identified the conflation of membership and the subset relation in Schröder's notion of subsumption. Frege, however, also noticed, with Zermelo, that the point of the argument was to show that there is no universal set, or unrestricted "domain of discourse." Indeed, as Church notes, Frege was to fall for the very sort of paradox that he accuses Schröder of trying to avoid, namely one that follows from assuming that there is a class of all things. It is tempting to speculate that Zermelo was aware of this dispute among the logicians about the "universe of discourse" and saw Cantor's idea of "absolute" infinities and the non-existence of a set of all cardinal numbers, as a clearer account of these same issues. As a consequence his diagonal argument, mathematically the same as Russell's, would also have seemed to him to be an anticipation of the use to which Russell put his argument. But, to continue the speculation, of course, one would have to accuse Zermelo of not seeing the simple theory of types as an alternative resolution to the set theoretic "antinomy" he had discovered. But, in fact, Zermelo and Russell seem to have been working in different worlds in set theory; Zermelo within the tradition of Cantor's set theory at Göttingen, and Russell, as always, refining and abandoning his own earlier views.

There is no direct evidence that Russell studied Schröder's argument, or saw in it either the idea of a contradiction in the notion of a set of all sets, or an anticipation of the theory of types in its conclusion. Russell, did, however, study Frege's "A Critical Elucidation …" and made extensive notes on it as part of his preparation for adding "Appendix A: The Logical and Arithmetical Doctrines of Frege" to *The Principles of Mathematics* in the summer of 1902, the same preparation that led to his letter to Frege.[20] Nothing

[19] Church argues that Frege's theory of "Stufe" of concepts and concepts that apply to first level concepts, etc., is not a theory of types of *objects* in the sense in which a genuine theory of simple types of classes is.

[20] See Linsky (2004a and 2004b).

relevant to this argument appears in the notes or in the ultimate appendix.

However, there is evidence about Russell's general ideas about Schröder which comes from his interaction with Norbert Wiener in 1913.[21] In September 1913, Norbert Wiener, then just eighteen years old, visited Russell in Cambridge. He had just completed his PhD thesis at Harvard University, entitled *A comparison between the treatment of the algebra of relatives by Schröder and that by Whitehead and Russell.* Ivor Grattan-Guinness (1975) found several pages of comments by Russell and replies by Wiener, following a series of discussions the two had in September and October of that year. Wiener also attended Russell's lectures and some letters home report on the interchange.

Russell wrote about the discussions as well, including this in a letter to Lucy Donnelly from 19 October 1913:

> At the end of Sept. an infant prodigy named Wiener, Ph.D. (Harvard), aged 18, turned up with his father... The youth has been flattered, and thinks himself God Almighty – there is a perpetual contest between him and me as to which is to do the teaching. (Grattan-Guinness 1975, p.105)

Wiener brought a copy of his thesis, and he and Russell discussed it in a series of meetings, with an exchange of letters between meetings.[22] The thesis and letters include a number of points about Schröder's logic. The topic of the thesis was a defense of the merits of Schröder's logic in comparison with *Principia Mathematica*. Two passages singled out by Grattan-Guinness are particularly relevant to the issues introduced above:

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21 What follows is based on Grattan-Guinness (1975).

22 This correspondence clarifies Wiener's often cited remark that Russell later was not particularly impressed with the reduction of relations to sets of ordered pairs that was Wiener's first published contribution to mathematics. (Wiener 1953, p. 191). In fact Russell had told Wiener that he did not identify relations with sets of "couples", as was common in the tradition of Schröder and Peirce that Wiener was defending. See Grattan-Guinness, 1975, p. 122.

> A major point of contrast between Schröder's and Russell's systems is that Schröderian individuals correspond to unit classes of Russellian individuals. Thus Schröder has no analogue to Russell's relation of membership of an individual to a class. But it does not matter, since Schröder has no need of such a relation, contrary to the opinion of Padoa that he conflated membership and inclusion, and to the opinion of Russell that all predecessors of Peano and Frege regarded membership as a special case of inclusion. (Grattan-Guinness 1975, p. 124)

Russell does not accept Wiener's claim that Schröder didn't need the distinction between membership and subset. In one of his replies, he asks "Have you any evidence that Schröder knew that there was a difference between Peter and the class whose only member is Peter?" to which Wiener replies by repeating that Schröder is simply "not concerned" with that distinction. (Grattan-Guinness 1975, p. 128)

Wiener credits Schröder with an anticipation of Russell's theory of types in his discussion of the argument about the universal class under discussion above:

> No individual in a manifold can itself be composed of a collection of other individuals of that manifold. Instead, classes of individuals belong to the first 'derived' (*abgeleitete*) manifold of the 'original' (*ursprüngliche*) one. Classes of classes of individuals belong to the second derived manifold and so on. This creates a hierarchy of types corresponding to Russell's theory. The difference is that, while Russell can use more than one type at once, Schröder can speak only of one type at a time. (Grattan-Guinness 1975, p. 128)

(This is the very same as Church's analysis of Schröder's position, only expressed twenty-six years earlier.) There is no record of Russell's reaction to this part of the thesis, but it is clear that Russell had read in Wiener an expression of the view that Schröder had anticipated the theory of types in response to an argument that there cannot be a universal class to which everything belongs. It appears that Russell did not think that he alone had seen that a diagonal argument like that in Cantor’s proof would establish that there can be no universal set of all sets. Although he studied and cited Zermelo's papers from 1908 as well, there is also no record of any response to the claim that others had anticipated the paradox. Schröder’s logic was so alien to Russell’s that it is understandable that he did not find anticipations of his own theory of types in it. Russell does not cite any sources for the notion of types that appears first in Appendix A of *Principles of Mathematics*, although clearly it was Frege's notion of *Stufe* or the hierarchy of concepts, concepts of concepts, etc. that must have inspired it.

As with the anticipation of the theory of types, Russell was also close to Zermelo’s version of the paradox in Schröder's arguments, but clearly didn't arrive at it through that route. Russell’s own paradox was the first paradox of predicates. He saw that the paradox of the set of all sets that are members of themselves was a result of similar thinking. His interest in the first paradox made him blind to the possibility of the second paradox raised by Schröder’s argument, which Zermelo had identified. But then Frege, who also read Schröder’s argument carefully didn’t see the argument either. Their thoughts were somewhere else.

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