Ranking by Rating

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Abstract

Each item in a given collection is characterized by a set of possible performances. A (ranking) method is a function that assigns an ordering of the items to every performance profile. *Ranking by Rating* consists in evaluating each item's performance by using an exogenous rating function, and ranking items according to their performance ratings. Any such method is separable: the ordering of two items does not depend on the performances of the remaining items. We prove that every separable method must be of the ranking-by-rating type if (i) the set of possible performances is the same for all items and the method is anonymous, or (ii) the set of performances of each item is ordered and the method is monotonic. When performances are vectors in $\mathbb{R}_+^m$, a separable, continuous, anonymous, monotonic, and invariant method must rank items according to a weighted geometric mean of their performances along the m dimensions.

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1 Introduction

The issue. This note studies separable methods for constructing performance-based rankings. The problem under consideration is the following. Each of n items is characterized by a set of possible performances. A ranking method assigns an ordering of the items to every possible performance profile. Such a method is *separable* if the ordering of two items does not depend on the performances of the remaining items. The simplest separable methods work as follows: each item's performance is evaluated using an exogenous rating function defined over the set of its possible performances, and the items are then ranked according to their resulting performance ratings. We refer to this type of methods as *ranking by rating*. We ask whether all separable methods are of this type, and, if not, under what conditions that may be the case.

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Section 2 shows that there exist separable methods other than ranking by rating. Those methods need not be degenerate and can be quite flexible; their range may include all the linear orderings of the items.

Next, we identify two separate sets of conditions under which a separable method must be of the ranking-by-rating variety. Theorem 1 in Section 3 establishes that this is the case if the set of possible performances is the same for all items and the method is anonymous. Theorem 2 in Section 4 shows that separable methods must also be of the ranking-by-rating type if the set of performances of each item is (completely) ordered and the method is monotonic. Both of these results are rather elementary and perhaps folk knowledge, but were, to the best of our knowledge, in need of a proof.

Section 5 illustrates the usefulness of Theorem 1. We revisit the particular case of our model where the items' performances are evaluated according to different criteria: they are represented by vectors in $\mathbb{R}^m_+$. The recent literature emphasizes that when performances according to m criteria are measured in non-comparable units, a ranking method should be *invariant* under multiplication of the items' performances with respect to a given criterion by a constant. We show that a separable, continuous, anonymous, monotonic, and invariant method must rank items according to a weighted geometric mean of their performances according to the m criteria. We argue that, from an axiomatic viewpoint, these simple methods are serious competitors of the more sophisticated non-separable methods of the fixed-point type.

Related work. A sizable literature addresses the problem of characterizing separable orderings defined over a set of multidimensional alternatives such as a subset of $\mathbb{R}^m_+$. Separability, in that literature, means that the ordering of two alternatives whose coordinates coincide along one dimension does not change with the value of that coordinate. The seminal contribution is that of Gorman (1968), who showed that, under suitable (and important) topological assumptions, such an ordering can be represented by an additively separable function. Bradley, Hodge and Kilgour (2005) show that Gorman's result does not carry over to the finite case, and study properties of discrete separable orderings. Despite a formal similarity, our work is essentially unrelated to that literature. Even when the sets of possible performances of the n items are infinite, we are interested in ordering only the finite sets containing precisely n performances, one for each item. On the other hand, we want to order all such sets, and our separability condition is precisely a restriction on how these different rankings should be related: the ordering of two performances should not depend on what the remaining performances are.

Our separability condition *is* closely related to Arrow's (1963) axiom of Independence of Irrelevant Alternatives and its weakening by Hansson (1973). Arrow's aggregation problem, however, cannot be rephrased as a ranking problem of the type we analyze. If candidates (or social alternatives) are regarded as items, and each candidate's performance is defined as the list of ranks he occupies in the preferences of the voters, then the set of possible performance profiles is *not* a

Cartesian product: two candidates cannot both be ranked first by the same voter.

The sub-model discussed in Section 5 received a lot of attention. Our work differs from the bulk of the literature in two essential aspects: we study ranking methods that are ordinal and separable whereas the literature focuses on cardinal non-separable methods. A more detailed discussion is postponed to Section 5.

2 Separability

Let $N = \{1, ..., n\}$ be a finite set of items, $n \geq 2$. Each item $i \in N$ is characterized by a nonempty set of possible performances A_i. A *performance profile* is a list $a = (a_1, ..., a_n) \in A_N := \times_{i \in N} A_i$. Let $\mathcal{R}_N$ denote the set of (weak) orderings on N. A *(ranking) method* is a function $R : A_N \to \mathcal{R}_N$ that assigns to each performance profile a an ordering $R(a)$ of the items. The statement $(i, j) \in R(a)$, also written $iR(a)j$, means that the method R considers i at least as strong as j when the performance profile is a. Let $P(a)$ and $I(a)$ denote, respectively, the strict ordering and the equivalence relation associated with $R(a)$. If $R(a)$ is a linear ordering, it will sometimes be convenient to express it by listing the items according to their rank in that ordering: for instance, the linear ordering $iR(a)j \Leftrightarrow i \leq j$ will be written $R(a) = 1\ 2\ ...\ n$.

A method R is a *ranking-by-rating method* if there exist real-valued functions $v_1, ..., v_n$ defined, respectively, on $A_1, ..., A_n$, such that $iR(a)j \Leftrightarrow v_i(a_i) \geq v_j(a_j)$ for all $i, j \in N$ and all $a \in A_N$. We call $v_1, ..., v_n$ *rating functions.*

If R is a ranking-by-rating method, the relative ordering of two items depends only on the performances of these items. Formally, R satisfies the following property.

Separability. For all $i, j \in N$ and $a, a' \in A_N$, $\left[a_i = a'_i \text{ and } a_j = a'_j\right] \Rightarrow \left[iR(a)j \Leftrightarrow iR(a')j\right]$.

This property is vacuous if $n = 2$. We ask whether it characterizes the ranking-by-rating methods when $n \geq 3$. The following example shows that this is not the case.

Example 1. Let $N = \{1, 2, 3\}$, $A_i = \{0, 1\}$ for all $i \in N$, and

$$R(a) = \begin{cases} 1\ 2\ 3 \text{ if } a_1 = a_2, \\ 2\ 1\ 3 \text{ if } a_1 \neq a_2. \end{cases}$$

Since 3 is always ranked last, the relative ordering of 1 and 3 and the relative ordering of 2 and 3 are constant. The relative ordering of 1 and 2 varies, but it does not depend upon a_3. Thus, R is separable. If the rating functions v_1, v_2, v_3 represent R, we must have

$$
\begin{aligned}
1P(0,0,0)2 &\Rightarrow v_1(0) > v_2(0), \\
2P(1,0,0)1 &\Rightarrow v_2(0) > v_1(1), \\
1P(1,1,0)2 &\Rightarrow v_1(1) > v_2(1), \\
2P(0,1,0)1 &\Rightarrow v_2(1) > v_1(0).
\end{aligned}
$$

Since these inequalities are incompatible, R is not a ranking-by-rating method.

In this example, the range of R is very small. But there exist separable methods whose range contain all strict orderings on N that are not ranking-by-rating methods. For instance, let $N = \{1,2,3\}$, $A_i = \{0,1,2\}$ for all $i \in N$, and consider the method R depicted in Figure 1. It is tedious but straightforward to check that R is separable, and the same argument as above shows that it is not a ranking-by-rating method.

3 Anonymity

This section studies the case where the performance sets of all items coincide, that is, $A_1 = ... = A_n = A$, hence $A_N = A^N$, and the ranking method is anonymous in the sense of the following definition.

Anonymity. For all $i, j \in N$, all $a \in A^N$, and every bijection π from N to N, $iR(a)j \Leftrightarrow \pi(i)R(\pi a)\pi(j)$, where πa is the performance profile defined by $(\pi a)_{\pi(i)} = a_i$ for all $i \in N$.

Theorem 1. *Let $n \geq 3$ and let $A_1 = ... = A_n = A$ be a finite set. A ranking method $R : A^N \to \mathcal{R}_N$ is separable and anonymous if and only if there exists a function $v : A \to \mathbb{R}$ such that $iR(a)j \Leftrightarrow v(a_i) \geq v(a_j)$ for all $i, j \in N$ and all $a \in A^N$.*

Proof. The "if" statement requires no proof. To prove the "only if" statement, fix a separable and anonymous method R. Define the binary relations $\succ, \sim$, and $\succsim$ on A as follows.

$$
\begin{aligned}
\alpha &\succ \beta \Leftrightarrow \exists a_3, ..., a_n \in A \text{ such that } 1P(\alpha, \beta, a_3, ..., a_n)2, \\
\alpha &\sim \beta \Leftrightarrow \exists a_3, ..., a_n \in A \text{ such that } 1I(\alpha, \beta, a_3, ..., a_n)2, \\
\alpha &\succsim \beta \Leftrightarrow \alpha \succ \beta \text{ or } \alpha \sim \beta.
\end{aligned}
$$

We claim that $\succsim$ is an ordering.

Step 1. To see that $\succsim$ is reflexive, fix $\alpha \in A$, and note that Anonymity implies $1I(\alpha, \alpha, a_3, ..., a_n)2$ for all $a_3, ..., a_n \in A$, hence $\alpha \sim \alpha$.

Step 2. Observe that $\sim$ is symmetric. Indeed, let $\alpha, \beta \in A$ and suppose $\alpha \sim \beta$. Then there exist $a_3, ..., a_n \in A$ such that $1I(\alpha, \beta, a_3, ..., a_n)2$. By Anonymity, $2I(\beta, \alpha, a_3, ..., a_n)1$, hence $\beta \sim \alpha$.

Step 3. To prove that $\succsim$ is complete, let $\alpha, \beta \in A$, $\alpha \neq \beta$, and suppose $(\alpha, \beta) \notin \succsim$. Fix $a_3, ..., a_n \in A$. Then $(1,2) \notin P(\alpha, \beta, a_3, ..., a_n)$, hence $(2,1) \in R(\alpha, \beta, a_3, ..., a_n)$ by completeness of $R(\alpha, \beta, a_3, ..., a_n)$. Anonymity then implies $(1,2) \in R(\beta, \alpha, a_3, ..., a_n)$, which implies $\beta \succsim \alpha$.

Step 4. Finally, let us check that $\succsim$ is transitive. Fix $\alpha, \beta, \gamma \in A$ and suppose $(\alpha, \beta), (\beta, \gamma) \in \succsim$. Suppose, by way of contradiction, that $(\alpha, \gamma) \notin \succsim$. By Steps 2 and 3, $(\gamma, \alpha) \in \succ$. Now

$$
\begin{aligned}
\alpha &\succsim \beta \Rightarrow \exists a_3, ..., a_n \in A \text{ such that } 1R(\alpha, \beta, a_3, ..., a_n)2, \\
\beta &\succsim \gamma \Rightarrow \exists b_3, ..., b_n \in A \text{ such that } 1R(\beta, \gamma, b_3, ..., b_n)2, \\
\gamma &\succ \alpha \Rightarrow \exists c_3, ..., c_n \in A \text{ such that } 1P(\gamma, \alpha, c_3, ..., c_n)2.
\end{aligned}
$$

By Separability,

$$
\begin{aligned}
&1R(\alpha, \beta, \gamma, a_4, ..., a_n)2, \\
&1R(\beta, \gamma, \alpha, a_4, ..., a_n)2, \\
&1P(\gamma, \alpha, \beta, a_4, ..., a_n)2,
\end{aligned}
$$

and by Anonymity,

$$
\begin{aligned}
&1R(\alpha, \beta, \gamma, a_4, ..., a_n)2, \\
&2R(\alpha, \beta, \gamma, a_4, ..., a_n)3, \\
&3P(\alpha, \beta, \gamma, a_4, ..., a_n)1,
\end{aligned}
$$

violating the transitivity of $R(\alpha, \beta, \gamma, a_4, ..., a_n)$.

Since $\succsim$ is an ordering on the finite set A, it admits a numerical representation $v : A \longrightarrow \mathbb{R}$. It is straightforward to check that $iR(a)j \Leftrightarrow v(a_i) \geq v(a_j)$ for all $i, j \in N$ and all $a \in A^N$. Indeed, suppose $iR(a)j$. Let $\pi : N \longrightarrow N$ be a bijection such that $\pi(i) = 1$ and $\pi(j) = 2$. Then $1R(\pi a)2$ and, by definition of $\succsim$, $(\pi a)_1 \succsim (\pi a)_2$. But $\pi(i) = 1$ and $\pi(j) = 2$ imply $(\pi a)_1 = a_i$ and $(\pi a)_2 = a_j$, hence $a_i \succsim a_j$. Since v is a numerical representation of $\succsim$, $v(a_i) \geq v(a_j)$. The same argument shows that $iP(a)j$ implies $v(a_i) > v(a_j)$, completing the proof.■

The finiteness assumption in Theorem 1 is used to ensure the representability of the revealed dominance relation defined in the proof. Theorem 1 can be adapted to the infinite case if A is endowed with a topology and a suitable continuity condition is imposed on the method R. An example will be given in Section 5.

4 Monotonicity

This section considers the case where the performance sets need not coincide but each of them is endowed with an order structure. For simplicity, we assume that the performance sets are intervals: for each $i \in N$, there exist real numbers $\alpha_i < \beta_i$, such that $A_i = [\alpha_i, \beta_i]$, so that $A_N = \times_{i \in N} [\alpha_i, \beta_i]$.

We require that our ranking method be monotonic in the sense that a higher performance improves an item's position in the associated ranking. For all $i \in N$, $a \in A_N$, and $a_i' \in A_i$, let (a_i', a_{-i}) denote the performance profile obtained from a by replacing a_i with a_i'.

Monotonicity. For all distinct $i, j \in N$ and $a, a' \in A_N$, $[iR(a)j \text{ and } a_i' > a_i] \Rightarrow [iP(a_i', a_{-i})j]$.

Notice that this is a strict version of the monotonicity principle. Observe also that Monotonicity implies $\left[iR(a)j \text{ and } a_j' < a_j\right] \Rightarrow \left[iP(a_j', a_{-j})j\right]$ for all distinct $i, j \in N$ and $a, a' \in A_N$.

We further require that the method be continuous in the sense that any strict ordering of two items is robust to small changes in the performance profile.

Continuity. For all distinct $i, j \in N$, the set $\{a \in A_N \mid iP(a)j\}$ is relatively open in A_N.

For each $i \in N$, let $\mathcal{V}_i$ denote the set of increasing and continuous functions from A_i to $\mathbb{R}$.

Theorem 2. *Let $A_N = \times_{i \in N} [\alpha_i, \beta_i]$. A method $R : A_N \to \mathcal{R}_N$ is separable, monotonic, and continuous if and only if there exist $v_1 \in \mathcal{V}_1, ..., v_n \in \mathcal{V}_n$ such that $iR(a)j \Leftrightarrow v_i(a_i) \geq v_j(a_j)$ for all $i, j \in N$ and all $a \in A_N$.*

Contrary to Theorem 1, the assumption $n \geq 3$ is not needed. When $n = 2$, Separability is vacuous but Monotonicity and Continuity suffice to pin down the ranking-by-rating methods. This fact, recorded as Lemma 1 below, is the first step in the proof of Theorem 2.

Lemma 1. *Let $A_{\{1,2\}} = [\alpha_1, \beta_1] \times [\alpha_2, \beta_2]$. A method $R : A_{\{1,2\}} \to \mathcal{R}_{\{1,2\}}$ is monotonic and continuous if and only if there exist $v_1 \in \mathcal{V}_1$ and $v_2 \in \mathcal{V}_2$ such that $1R(a)2 \Leftrightarrow v_1(a_1) \geq v_2(a_2)$ for all $a \in A_{\{1,2\}}$.*

Proof. The "if" statement being obvious, we only prove the "only if" statement. Let $R : A_{\{1,2\}} \to \mathcal{R}_{\{1,2\}}$ be a monotonic and continuous method. For each $i \in \{1, 2\}$, define

$$A_i^{=} = \{a_i \in A_i \mid \exists a_j \in A_j \text{ such that } iI(a_i, a_j)j\},$$

where j denotes the item other than i in $\{1, 2\}$.

Either both $A_1^{=}$ and $A_2^{=}$ are empty, or both are nonempty. If both are empty, either $1P(a)2$ for all $a \in A_{\{1,2\}}$ or $2P(a)1$ for all $a \in A_{\{1,2\}}$. Without loss of generality, assume the first case. Defining $v_1(a_1) = a_1$ for all $a_1 \in A_1$ and $v_2(a_2) = a_2 - \beta_2 + \alpha_1 - 1$ for all $a_2 \in A_2$ proves the claim.

From now on, assume that both $A_1^{=}$ and $A_2^{=}$ are nonempty. For each $i \in \{1, 2\}$, Monotonicity implies that for each $a_i \in A_i^{=}$ there is a unique $a_j \in A_j$ such that $iI(a_i, a_j)j$: denote this unique a_j by $e(a_i)$; note that $e(A_i) \in A_j^{=}$.

For each $i \in \{1,2\}$, $A_i^=$ is a closed interval. That $A_i^=$ is a closed set follows from Continuity. To see that it is an interval, fix $a_i, a_i', a_i'' \in A_i$ such that $a_i < a_i' < a_i''$ and $a_i, a_i'' \in A_i^=$. By Monotonicity,

$$\begin{aligned} iI(a_i, e(a_i))j &\Rightarrow iP(a_i', e(a_i))j, \\ jI(a_i'', e(a_i''))i &\Rightarrow jP(a_i', e(a_i''))i. \end{aligned}$$

By Continuity, $iP(a_i', e(a_i))j$ and $jP(a_i', e(a_i''))i$ imply that there exists $a_j \in A_j$ such that $iI(a_i', a_j))j$, that is, $a_i' \in A_i^=$.

Let $A_i^= = \left[a_i^-, a_i^+\right]$. By Monotonicity, we have that for all $a \in A_{\{1,2\}}$ and all $i \in \{1,2\}$,

$$jP(a)i \text{ if } a_i < a_i^-, \tag{1}$$

$$iP(a)j \text{ if } a_i > a_i^+. \tag{2}$$

It follows that $a_1^- = \alpha_1$ or $a_2^- = \alpha_2$. Indeed, if $\alpha_i < a_i^-$ for $i = 1,2$, then $1P(a)2$ and $2P(a)1$ for all a such that $a_i < a_i^-$ for $i = 1,2$, which is impossible. From now on we assume, without loss of generality,

$$a_1^- = \alpha_1. \tag{3}$$

By the same argument as above,

$$a_1^+ = \beta_1 \text{ or } a_2^+ = \beta_2. \tag{4}$$

Define the functions $v_1 : A_1 \to \mathbb{R}$ and $v_2 : A_2 \to \mathbb{R}$ by

$$\begin{aligned} v_1(a_1) &= a_1, \\ v_2(a_2) &= \begin{cases} \alpha_1 + a_2 - a_2^- \text{ if } a_2 < a_2^-, \\ e(a_2) \text{ if } a_2^- \leq a_2 \leq a_2^+, \\ \beta_1 + a_2 - a_2^+ \text{ if } a_2^+ < a_2. \end{cases} \end{aligned}$$

We claim that $1R(a)2 \Leftrightarrow v_1(a_1) \geq v_2(a_2)$ for all $a \in A_{\{1,2\}}$.

If $1P(a)2$, then by (2) $a_2 \leq a_2^+$. If $a_2 < a_2^-$, then $v_1(a_1) = a_1 \geq \alpha_1 > \alpha_1 + a_2 - a_2^- = v_2(a_2)$. If $a_2^- \leq a_2 \leq a_2^+$, then $e(a_2)$ is well defined and $1I(e(a_2), a_2)2$ and $1P(a)2$ imply, by Monotonicity, $a_1 > e(a_2)$, that is, $v_1(a_1) > v_2(a_2)$.

If $2P(a)1$, then by (1) $a_2 \geq a_2^-$. If $a_2 > a_2^+$, then $v_2(a_2) = \beta_1 + a_2 - a_2^+ > \beta_1 \geq a_1 = v_1(a_1)$. If $a_2^- \leq a_2 \leq a_2^+$, then $e(a_2)$ is well defined and $1I(e(a_2), a_2)2$ and $2P(a)1$ imply, by Monotonicity, $a_1 < e(a_2)$, that is, $v_1(a_1) < v_2(a_2)$.

If $1I(a)2$, then $a_2^- \leq a_2 \leq a_2^+$ and $a_1 = e(a_2)$, hence, $v_1(a_1) = v_2(a_2)$.

It is straightforward to check that $e : A_2^= \to \mathbb{R}$ is increasing and continuous. To complete the proof of Lemma 1, we check that

$$e(a_2^-) = \alpha_1 \tag{5}$$

and

$$e(a_2^+) = \beta_1 \text{ if } a_2^+ < \beta_2. \tag{6}$$

By Monotonicity and the definition of $A_1^=, A_2^=$, we have $e(a_2^-) = a_1^-$, hence (5) follows from (3). Likewise, $e(a_2^+) = a_1^+$ and (6) follows from (4).■

Proof of Theorem 2. Again, we only prove the "only if" statement. Fix a continuous, monotonic, and separable method $R : A_N \to \mathcal{R}_N$. With some abuse of notation, we write $iR(a_i,a_j)j$ if and only if $iR(a_i,a_j,a_{-ij})j$ for all $a_{-ij} \in A_{N\setminus\{i,j\}}$. Because of Separability, $iR(a_i,a_j)j$ if and only if $iR(a_i,a_j,a_{-ij})j$ for some $a_{-ij} \in A_{N\setminus\{i,j\}}$. Similarly, we write $iR(a_i,a_j,a_k)j$ if and only if $iR(a_i,a_j,a_k,a_{-ijk})j$ for all $a_{-ijk} \in A_{N\setminus\{i,j,k\}}$.

By Lemma 1 and Separability, we know that for each pair of items $\{i,j\}$ there exist functions $v_i^{\{i,j\}} \in \mathcal{V}_i$ and $v_j^{\{i,j\}} \in \mathcal{V}_j$ such that

$$iR(a)j \Leftrightarrow v_i^{\{i,j\}}(a_i) \geq v_j^{\{i,j\}}(a_j) \text{ for all } a \in A. \tag{7}$$

From now on, we write ij instead of $\{i,j\}$.

We claim that there exist functions $v_1 \in \mathcal{V}_1, ..., v_n \in \mathcal{V}_n$ such that, for all $i,j \in N$,

$$iR(a)j \Leftrightarrow v_i(a_i) \geq v_j(a_j) \text{ for all } a \in A. \tag{8}$$

Define $v_1 := v_1^{12}$, $v_2 := v_2^{12}$, and observe that statement (8) is true for $i,j \in \{1,2\}$. Now fix $k \in \{3,...,n\}$ and make the induction hypothesis that there exist $v_1 \in \mathcal{V}_1, ..., v_{k-1} \in \mathcal{V}_{k-1}$ such that statement (8) is true for all $i,j \in \{1,...,k-1\}$. In order to prove our claim, it suffices to construct a function $v_k \in \mathcal{V}_k$ such that

$$iR(a)k \Leftrightarrow v_i(a_i) \geq v_k(a_k) \text{ for all } a \in A \text{ and all } i \in \{1,...,k-1\}. \tag{9}$$

The construction of v_k proceeds in two steps. We first construct the function on the subset of item k's performances where the method may tie k with some other item, then extend the function to the whole of A_k. For all $i,j \in N$, define

$$A_i(j) = \{a_i \in A_i \mid \exists a_j \in A_j \text{ such that } iI(a_i,a_j)j\}. \tag{10}$$

If this set is nonempty, Monotonicity and Continuity ensure that it is a closed interval, and we define $a_i^-(j) = \min A_i(j)$ and $a_i^+(j) = \max A_i(j)$. Define $K = \{i \in \{1,...,k-1\} \mid A_k(i) \neq \emptyset\}$.

For each $i \in K$, define $w_k^i : A_k(i) \to \mathbb{R}$ by

$$w_k^i(a_k) = (v_i \circ (v_i^{ik})^{-1} \circ v_k^{ik})(a_k). \tag{11}$$

This function is well defined because $v_k^{ik}(A_k(i)) = v_i^{ik}(A_i(k))$. To check this fact, let $z \in v_k^{ik}(A_k(i))$. Pick $a_k \in A_k(i)$ such that $v_k^{ik}(a_k) = z$. Since $a_k \in A_k(i)$, there exists $a_i \in A_i$ such that $kI(a_i, a_k)i$. Observe that $a_i \in A_i(k)$. By (7), $v_i^{ik}(a_i) = v_k^{ik}(a_k) = z$, hence $z \in A_i(k)$. This proves that $A_k(i) \subseteq A_i(k)$ and the reverse inclusion is proved similarly.

It is easy to check that w_k^i is continuous and increasing. Its range is therefore a closed interval which we denote $\Omega_k(i) = \left[\omega_k^-(i), \omega_k^+(i)\right]$. We are going to use the functions w_k^i, $i \in K$, to construct the required function v_k on $\cup_{i\in K} A_i(k)$. In order to do so, we establish two important properties of the functions w_k^i, $i \in K$. First, these functions are compatible.

Property 1. *For all $i, j \in K$ and all $a_k \in A_k(i) \cap A_k(j)$, $w_k^i(a_k) = w_k^j(a_k)$.*

To prove this, fix $i, j \in K$ and $a_k \in A_k(i) \cap A_k(j)$. By way of contradiction, suppose, say

$$w_k^i(a_k) > w_k^j(a_k). \tag{12}$$

By (7) and Separability, $a_k \in A_k(i)$ implies that there exists $a_i \in A_i$ such that $v_i^{ik}(a_i) = v_k^{ik}(a_k)$, hence $(v_i \circ (v_i^{ik})^{-1} \circ v_i^{ik})(a_i) = (v_i \circ (v_i^{ik})^{-1} \circ v_k^{ik})(a_k)$, that is,

$$v_i(a_i) = w_k^i(a_k). \tag{13}$$

By the same argument applied to j instead of i, there exists $a_j \in A_j$ such that $v_j^{jk}(a_j) = v_k^{jk}(a_k)$, hence

$$v_j(a_j) = w_k^j(a_k). \tag{14}$$

Since $v_i^{ik}(a_i) = v_k^{ik}(a_k)$ and $v_j^{jk}(a_j) = v_k^{jk}(a_k)$, (7) implies $iI(a_i, a_j, a_k)k$ and $jI(a_i, a_j, a_k)k$. But (12), (13), (14) imply $v_i(a_i) > v_j(a_j)$, which by the induction hypothesis implies $iP(a_i, a_j, a_k)j$, contradicting the transitivity of $R(a_i, a_j, a_k)$.□

The second property pertains to the ranges of two functions w_k^i, w_k^j whose domains are disjoint. For any two sets $X, Y \subseteq \mathbb{R}$, write $X < Y$ if $x < y$ for all $x \in X$ and all $y \in Y$.

Property 2. *For all $i, j \in K$, $A_k(i) < A_k(j) \Rightarrow \Omega_k(i) < \Omega_k(j)$.*

To prove this, fix $i, j \in K$, suppose $A_k(i) < A_k(j)$, and let $a_k \in A_k(i)$ and $b_k \in A_k(j)$. We must show that $w_k^i(a_k) < w_k^j(b_k)$. By definition of $A_k(i)$, $A_k(j)$ and by Separability, there exist $a_i \in A_i$ and $a_j \in A_j$ such that

$$kI(a_i, a_j, a_k)i \tag{15}$$

and

$$kI(a_i, a_j, b_k)j. \tag{16}$$

By (15) and (7), $v_k^{ik}(a_k) = v_i^{ik}(a_i)$. This implies $(v_i \circ (v_i^{ik})^{-1} \circ v_k^{ik})(a_k) = (v_i \circ (v_i^{ik})^{-1} \circ v_i^{ik})(a_i)$, that is,

$$w_k^i(a_k) = v_i(a_i). \tag{17}$$

Likewise, (16) and (7) imply $v_k^{jk}(b_k) = v_j^{jk}(a_j)$, hence $(v_j \circ (v_j^{jk})^{-1} \circ v_k^{jk})(b_k) = (v_j \circ (v_j^{jk})^{-1} \circ v_j^{jk})(a_j)$, that is,

$$w_k^j(b_k) = v_j(a_j). \tag{18}$$

Since $a_k < b_k$, Monotonicity and (16) imply $jP(a_i, a_j, a_k)k$. Combining this statement with (15) yields $jP(a_i, a_j, a_k)i$. Hence, by the induction hypothesis, $v_i(a_i) < v_j(a_j)$, and, by (17), (18), $w_k^i(a_k) < w_k^j(b_k)$.□

Let $B_k = \cup_{i \in K} A_k(i)$. Because of Property 1, there is a uniquely defined function $w_k : B_k \to \mathbb{R}$ such that

$$w_k(a_k) = w_k^i(a_k) \text{ if } i \in K \text{ and } a_k \in A_k(i). \tag{19}$$

Since the functions w_k^i, $i \in K$, are increasing and continuous, it follows from Property 2 that w_k too is increasing and continuous. Its range is $\cup_{i \in K} \Omega_k(i)$, which we denote by Ω_k. We now extend w_k to A_k by linear interpolation.

Let $a_k^- = \min B_k$ and $a_k^+ = \max B_k$. Let $\omega_k^- = \min \Omega_k$ and $\omega_k^+ = \max \Omega_k$. Observe that $A_k \setminus B_k$ is a finite union of intervals (relatively open in A). Define $v_k : A_k \to \mathbb{R}$ to be the unique continuous extension of w_k which is affine on each of these intervals and satisfies the normalization condition

$$v_k(a_k) = \begin{cases} \omega_k^- + (a_k - a_k^-) \text{ if } a_k < a_k^-, \\ \omega_k^+ + (a_k - a_k^+) \text{ if } a_k > a_k^+. \end{cases}$$

By construction, $v_k \in \mathcal{V}_k$.

Having constructed v_k, we now check that (9) is satisfied. Fix $a \in A$ and $i \in \{1, ..., k-1\}$.

Case 1. $a_k \in A_k(i)$.

This implies that $i \in K$ and $a_k \in B_k$. Then,

$$\begin{aligned} iR(a)k \;&\Leftrightarrow\; v_i^{ik}(a_i) \geq v_k^{ik}(a_k) \text{ by (7)} \\ &\Leftrightarrow\; v_i(a_i) \geq (v_i \circ (v_i^{ik})^{-1} \circ v_k^{ik})(a_k) \text{ because } v_i \circ (v_i^{ik})^{-1} \text{ is increasing} \\ &\Leftrightarrow\; v_i(a_i) \geq w_k^i(a_k) \text{ by (11)} \\ &\Leftrightarrow\; v_i(a_i) \geq w_k(a_k) \text{ by (19)} \\ &\Leftrightarrow\; v_i(a_i) \geq v_k(a_k) \text{ by definition of } v_k. \end{aligned}$$

Case 2. $a_k \in B_k \setminus A_k(i)$.

In this case there exists $j \in K \setminus \{k\}$ such that $a_k \in A_k(j)$ and either (i) $a_k < a_k^-(i)$ or (ii) $a_k > a_k^+(i)$. Assume (i); the argument is similar if (ii) holds. Since $a_k^-(i) = \min A_k(i)$,

$$iP(a_i, a_k)k \text{ for all } a_i \in A_i, \tag{20}$$

and we must show that $v_i(a_i) > v_k(a_k)$ for all $a_i \in A_i$.

Fix $a_i \in A_i$. From (20), $iP(a_i, a_k)k$. Since $a_k \in A_k(j)$, there exists $a_j \in A_j$ such that $kI(a_j, a_k)j$. By Separability, these two statements imply $iP(a_i, a_j, a_k)k$ and

$$kI(a_i, a_j, a_k)j, \tag{21}$$

hence also,

$$iP(a_i, a_j, a_k)j. \tag{22}$$

From (21) and (7), $v_k^{jk}(a_k) = v_j^{jk}(a_j)$, hence,

$$a_j = ((v_j^{jk})^{-1} \circ v_k^{jk})(a_k).$$

From (22) and the induction hypothesis, $v_i(a_i) > v_j(a_j)$, hence,

$$v_i(a_i) > (v_j \circ (v_j^{jk})^{-1} \circ v_k^{jk})(a_k) = w_k^j(a_k) = v_k(a_k).$$

Case 3. $a_k \in A_k \setminus B_k$.

Case 3.1. $a_k < a_k^-$ or $a_k > a_k^+$.

Assume the first inequality; the argument is similar if the second holds. In this case we know that $iP(a_i, a_k)k$ for all $a_i \in A_i$, and we must show that $v_i(a_i) > v_k(a_k)$ for all $a_i \in A_i$.

Fix $a_i \in A_i$. Since $a_k^-(i) = \min A_k(i)$, we have $iR(a_i, a_k^-(i))k$, hence by (7), $v_i^{ik}(a_i) \geq v_k^{ik}(a_k^-(i))$. It follows that

$$v_i(a_i) \geq (v_i \circ (v_i^{ik})^{-1} \circ v_k^{ik})(a_k^-(i)) = w_k^i(a_k^-(i)) \geq \omega_k^- > \omega_k^- + (a_k - a_k^-) = v_k(a_k).$$

Case 3.2. There exists $j \in K$ such that

$$\text{(i) } a_k^+(i) < a_k < a_k^-(j) \text{ and } v_k(a_k) = \left(\frac{a_k^-(j) - a_k}{a_k^-(j) - a_k^+(i)}\right)\omega_k^+(i) + \left(\frac{a_k - a_k^+(i)}{a_k^-(j) - a_k^+(i)}\right)\omega_k^-(j),$$

or (ii) the statement obtained by exchanging i and j in (i) is true.

Assume (i); the proof is similar if (ii) holds. Because of (i) we know that $kP(a_i, a_k)i$ for all $a_i \in A_i$ and we must prove that $v_i(a_i) < v_k(a_k)$ for all $a_i \in A_i$.

Fix $a_i \in A_i$. Since $a_k^+(i) = \max A_k(i)$, we have $kR(a_i, a_k^+(i))i$, hence by (7), $v_i^{ik}(a_i) \leq v_k^{ik}(a_k^+(i))$. It follows that

$$v_i(a_i) \leq (v_i \circ (v_i^{ik})^{-1} \circ v_k^{ik})(a_k^+(i)) = w_k^i(a_k^+(i)) = \omega_k^+(i) < v_k(a_k).\blacksquare$$

Theorem 2 assumes that each item's performance set is *completely* ordered. The result does not extend to the case where these sets are intervals in $\mathbb{R}^m$, $m \geq 2$, and the method is assumed to be monotonic with respect to the usual partial order of $\mathbb{R}^m$.

Example 2. Let $N = \{1, 2, 3\}$, $A_i = A = [0, 1]^2$ for all $i \in N$. A generic performance profile is a vector $a = (a_1, a_2, a_3) = ((a_1^1, a_1^2), (a_2^1, a_2^2), (a_3^1, a_3^2)) \in A^{\{1,2,3\}}$. Define the functions w_1, w_2, w_3 from $A^{\{1,2,3\}}$ to $\mathbb{R}$ by

$$\begin{aligned} w_1(a) &= (1 - a_2^2)a_1^1 + (1 - a_2^1)a_1^2, \\ w_2(a) &= \frac{1}{2}a_2^1 + \frac{1}{2}a_2^2, \\ w_3(a) &= -1 \end{aligned}$$

for all $a \in A^{\{1,2,3\}}$. Note that $w_1(a)$ varies with a_2. Define the method R by

$$iR(a)j \Leftrightarrow w_i(a) \geq w_j(a)$$

for all $a \in A^{\{1,2,3\}}$ and all $i, j \in \{1, 2, 3\}$. Since $w_1(a), w_2(a) \geq 0$ for all a, item 3 is ranked last at every performance profile. Moreover, the ranking of items 1 and 2 does not change with 3's performance. So R is separable. Since w_1, w_2, w_3 are continuous, R is also continuous. Furthermore, it is monotonic because w_1 is increasing in a_1 and w_2 is increasing in a_2.

This method is not a ranking-by-rating method. By definition of w_1, w_2, w_3 and R,

$$\begin{aligned} &1P((1,0),(1,0),(0,0))2, \\ &2P((0,1),(1,0),(0,0))1, \\ &1P((0,1),(0,1),(0,0))2, \\ &2P((1,0),(0,1),(0,0))1. \end{aligned}$$

If v_1, v_2, v_3 were rating functions from A to $\mathbb{R}$ such that $iR(a)j \Leftrightarrow v_i(a_i) \geq w_j(a_j)$ for all $a \in A^{\{1,2,3\}}$

and all $i, j \in \{1, 2, 3\}$, then

$$\begin{aligned} v_1(1,0) &> v_2(1,0), \\ v_2(1,0) &> v_1(0,1), \\ v_1(0,1) &> v_2(0,1), \\ v_2(0,1) &> v_1(1,0), \end{aligned}$$

which are incompatible inequalities.

5 Invariance

This section considers the case where the performance sets of the items coincide and are endowed with a partial order structure. More precisely, we assume that there is a finite set of criteria $M = \{1, ..., m\}$ and that $A_i = A = \mathbb{R}_+^M$ for each item $i \in N$. A generic performance for item i is a vector $a_i = (a_i^1, ..., a_i^m) \in A$. A performance profile is a matrix $a = (a_i^h) \in A^N$: rows correspond to items, columns to criteria, and the number a_i^h measures item i's performance according to criterion h. We write $b_i > a_i$ if $b_i^h \geq a_i^h$ for all $h \in M$ and $b_i^h > a_i^h$ for some $h \in M$. The monotonicity condition of the previous section is extended to the current setting: its formal definition is unchanged but $>$ is now interpreted as the partial order of $\mathbb{R}_+^M$.

This is a well known model. The particular case $M = N$ received considerable attention, with applications to the problem of ranking webpages (Kleinberg (1999)) or academic journals (see, e.g., Palacio-Huerta and Volij (2004) and the references therein). Demange (2014) studies the general model where M need not be equal to N, and contains more references. As mentioned in the Introduction, the bulk of that literature analyzes cardinal methods (with some exceptions such as Altman and Tennenholtz (2005)) and focuses on non separable methods: the cardinal scores of two items (and even their relative ranking) may depend on the entire performance profile. We briefly discuss cardinal methods at the end of this section.

Both Palacio-Huerta and Volij (2004) and Demange (2014) emphasize the importance of *(intensity) invariant* methods. In our ordinal formulation, the condition of invariance says that the ordering of the items remains unchanged if the performance of every item according to a given criterion is multiplied by the same positive number. This is compelling if the items' performances are measured on non-comparable scales across criteria.

To express the condition formally, we use the following notation. For every $\lambda = (\lambda^1, ..., \lambda^m) \in \mathbb{R}_{++}^M$, let $dg(\lambda)$ denote the $m \times m$ diagonal matrix whose hth diagonal entry is λ^h. With this notation, $a \cdot dg(\lambda)$ is the performance matrix obtained by multiplying each column h of a by λ^h.

Invariance. For all $a \in A^N$ and $\lambda \in \mathbb{R}_{++}^M$, $R(a \cdot dg(\lambda)) = R(a)$.

Let Δ_{++}^M denote the relative interior of the unit simplex of $\mathbb{R}^M$.

Theorem 3. *Let* $n \geq 3$ *and let* $A_1 = ... = A_n = A = \mathbb{R}^M_+$*. A ranking method* $R : A^N \to \mathcal{R}_N$ *is separable, continuous, anonymous, monotonic, and invariant if and only if there exists* $\beta = (\beta^1, ..., \beta^m) \in \Delta^M_{++}$ *such that*

$$iR(a)j \Leftrightarrow \prod_{h \in M} (a_i^h)^{\beta^h} \geq \prod_{h \in M} (a_j^h)^{\beta^h} \text{ for all } i, j \in N \text{ and all } a \in A^N. \tag{23}$$

Proof. The "if" statement is clear. The proof of the "only if" statement is a straightforward consequence of Theorem 1 and Osborne's (1976) characterization of the monotonic transformations of the weighted geometric means.

Fix a separable, continuous, anonymous, monotonic, and invariant method R. Separability, Anonymity and Continuity ensure the existence of a (continuous) function $w : \mathbb{R}^M_+ \to \mathbb{R}$ such that

$$iR(a)j \Leftrightarrow w(a_i) \geq w(a_j) \text{ for all } i, j \in N \text{ and all } a \in A^N. \tag{24}$$

To see this, define the binary relation $\succsim$ on $A = \mathbb{R}^M_+$ as in the proof of Theorem 1. The argument there shows that $\succsim$ is an ordering. Because R is continuous, so is $\succsim$. It therefore admits a (continuous) numerical representation w. The argument in the proof of Theorem 1 establishes (24).

Since R is monotonic, (24) implies that w is increasing: $a_i < a_j \Rightarrow w(a_i) < w(a_j)$. Because R is invariant, w is ordinally invariant in the sense that

$$w(a_i) \leq w(a_j) \Leftrightarrow w(\lambda^1 a_i^1, ..., \lambda^m a_i^m) \leq w(\lambda^1 a_j^1, ..., \lambda^m a_j^m)$$

for all $\lambda \in \mathbb{R}^M_{++}$. By Osborne (1976), there exist $\beta = (\beta^1, ..., \beta^m) \in \mathbb{R}^M_{++}$ and an increasing function $f : \mathbb{R} \to \mathbb{R}$ such that

$$w(a_i) = f\left(\prod_{h \in M} (a_i^h)^{\beta^h}\right) \tag{25}$$

for all $a_i \in A$. (In Osborne's theorem, w is nondecreasing and $\beta \in \mathbb{R}^M_+$. In our case, the fact that w is increasing guarantees that $\beta \in \mathbb{R}^M_{++}$. The normalization $\beta \in \Delta^M_{++}$ is innocuous.) Statement (23) now follows from (24) and (25).■

Of course, the weighted geometric mean numerical representation in Theorem 3 is unique only up to an increasing transformation.

We conclude with a digression on the use of separable methods in the context of *cardinal* evaluation of multidimensional performances. From now on, for comparability with existing work, we restrict our attention to the set A^N_+ of positive $n \times m$ matrices. A *grading method* is a function $G : A^N_+ \to \Delta^N$, where Δ^N denotes the unit simplex of $\mathbb{R}^N$. The vector $G(a) = (G_1(a), ..., G_n(a))$ is the grade distribution assigned by the method G to the performance matrix a. The grade of item

i, $G_i(a)$, is interpreted as a cardinal measure of its multidimensional performance. The grading method G clearly induces a ranking method R_G defined on A_+^N by $iR_G(a)j \Leftrightarrow G_i(a) \geq G_j(a)$, but the information contained in the grade distribution $G(a)$ is richer than that in the induced ranking $R_G(a)$.

When performances are cardinally measurable on non-comparable scales, two axioms appear to be unavoidable. The first is the cardinal version of the invariance axiom discussed earlier.

Cardinal Invariance. For all $a \in A_+^N$ and $\lambda \in \mathbb{R}_{++}^M$, $G(a \cdot dg(\lambda)) = G(a)$.

The second condition is *Homogeneity.* It requires that if an item's performance with respect to every criterion is multiplied by the same positive number, the ratio of that item's grade to any other item's grade is multiplied by the same number. This is compelling if performances with respect to each criterion are cardinally measurable. For every $\mu = (\mu_1, ..., \mu_n) \in \mathbb{R}_{++}^N$, let $dg(\mu)$ denote the $n \times n$ diagonal matrix whose ith diagonal entry is μ_i. With this notation, $dg(\mu) \cdot a$ is the performance matrix obtained by multiplying each row i of a by μ_i.

Homogeneity. For all $a \in A_+^N$ and $\mu \in \mathbb{R}_{++}^N$, $G(dg(\mu) \cdot a)$ is proportional to $dg(\mu) \cdot G(a)$.

Cardinally invariant and homogeneous grading methods admit a compact characterization. Call a performance matrix $a \in A_+^N$ *doubly balanced* if $\sum_{i \in N} a_i^h = 1$ for all $h \in M$ and $\sum_{h \in M} a_i^h = m/n$ for all $i \in N$. Let A_*^N denote the set of doubly balanced matrices. Sinkhorn (1967) proved that for every matrix $a \in A_+^N$ there exist a unique vector $\lambda(a) \in \mathbb{R}_{++}^M$ and a unique vector $\mu(a) \in \mathbb{R}_{++}^N$ such that $dg(\mu(a)) \cdot a \cdot dg(\lambda(a)) =: a^*$ is doubly balanced. This means that every positive matrix a can be reduced to a uniquely defined doubly balanced matrix a^* through a rescaling of its rows and columns. A cardinally invariant and homogeneous grading method G is therefore completely determined by its restriction to A_*^N, which itself is arbitrary. Formally, $G : A_+^N \to \Delta^N$ is cardinally invariant and homogeneous if and only if there exists a function $G^* : A_*^N \to \Delta^N$ such that $G(a)$ is proportional to $(dg(\mu(a)))^{-1} \cdot G^*(a^*)$ for all $a \in A_+^N$.

The *invariant grading method* proposed by Pinski and Narin (1976) and axiomatized by Palacio-Huerta and Volij (2004) is cardinally invariant but not homogeneous. Demange's (2014) *handicap-based grading method* is the unique cardinally invariant and homogeneous method that ties all items whenever the performance matrix is doubly balanced:

Uniformity. For all $a \in A_*^N$, $G(a) = (\frac{1}{n}, ..., \frac{1}{n})$.

The doubly balanced matrix a^* associated with a given a is computed through an iterative process of alternate rescaling of rows and columns. This makes the handicap-based method a rather sophisticated method whose behavior is somewhat difficult to apprehend. In particular, to the best of our knowledge, it is unknown whether the ranking method it induces is monotonic.

It may therefore be worth pointing out that for any $\beta \in \Delta_{++}^M$ the (relative and weighted) *geometric mean grading method*

$$G_i(a) = \frac{\prod_{h \in M} (a_i^h)^{\beta^h}}{\sum_{j \in N} \prod_{h \in M} (a_j^h)^{\beta^h}}$$

satisfies Cardinal Invariance and Homogeneity. Moreover, the ranking method induced by such a grading method is monotonic and separable. Neutrality, which requires that the grade distribution $G(a)$ should be unaffected by a permutation of the criteria, may be used to single out the uniform weight vector $\beta = (\frac{1}{m}, ..., \frac{1}{m})$. Overall, in spite of all its dullness, the geometric mean grading method seems to be a serious competitor of the non-separable methods proposed in the literature.

Of course, the method violates Uniformity. This need not be a weakness, however. Ranking item 1 above 2 and 3 in a doubly balanced matrix such as

$$\begin{pmatrix} 3/9 & 3/9 & 3/9 \\ 2/9 & 3/9 & 4/9 \\ 4/9 & 3/9 & 2/9 \end{pmatrix},$$

as the geometric mean does, seems to be reasonable. It is supported by a simple argument of variability aversion: the fact that the scores of item 1 coincide on all criteria gives them added value. Insisting on using the arithmetic mean on the doubly balanced matrices creates a tension with the requirements of Cardinal Invariance and Homogeneity that, remarkably, can only be resolved through the iterative procedure associated with the handicap method. If the geometric mean is regarded as an acceptable criterion on the doubly balanced matrices however, then there is no need to resort to a non-separable iterative method to guarantee Cardinal Invariance and Homogeneity.

To be sure, the geometric mean grading method does not endogenize the weights attached to the different criteria. Doing so is precisely the goal of any iterative method, but that goal does not seem to be captured by the Uniformity axiom.

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Figure 1. A full-range separable method not of the ranking-by-rating type