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Bounding The Hochschild Cohomological Dimension

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SOMMAIRE

Ce mémoire a deux objectifs principaux. Premièrement de développer et interpréter les groupes de cohomologie de Hochschild de basse dimension et deuxièmement de borner la dimension cohomologique des k-algèbres par dessous; montrant que presque aucune k-algèbre commutative est quasi-libre.

Mots-Clés: *Algèbre Homologique Relative, Théorie De La Dimension, Algèbre Non-Commutative, Cohomologie de Hochschild, Géométrie Non-Commutative*

SUMMARY

The aim of this master's thesis is two-fold. Firstly to develop and interpret the low dimensional Hochschild cohomology of a k-algebra and secondly to establish a lower bound for the Hochschild cohomological dimension of a k-algebra; showing that nearly no commutative k-algebra is quasi-free.

Keywords: Relative Homological Algebra, Dimension Theory, Noncommutative Algebra, Hochschild Cohomology, Noncommutative Geometry

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1

INTRODUCTION

Motivation

Noncommutative algebraic geometry is a rapidly developing area of contemporary mathematical research. Amongst the many topics studied therein, the proposed notions of a "noncommutative smoothness" such as Michel Van den Bergh's concept [PH] and Joachim Cuntz and Daniel Quillen's concept [AE] seemed particularly interesting to me.

This master's thesis found its beginnings in an attempt to understand the notion of noncommutative smoothness proposed by Joachim Cuntz and Daniel Quillen, called quasi-freeness. Defined similarly to the commutative notion of formal smoothness, quasi-freeness is defined as the lifting of all the square-zero extensions of a k-algebra. My driving question became "is this notion an analogue or a generalization of a classical notion of smoothness?"

In the case where k is an algebraically closed field, Joachim Cuntz and Daniel Quillen found that a k-algebra cannot be quasi-free if its Krull dimension is greater than 1 [AE]. Therefore it is possible for a k-algebra to be smooth and to not be quasi-free (for example $\mathbb{C}[x,y]$ is such a $\mathbb{C}$-algebra); whence over a field quasi-freeness is a noncommutative analogue of smoothness and not a generalization thereof.

Charles Weibel formulated an extension of the concept of a quasi-free k-algebra which no longer required k to be an algebraically closed field but only to be a commutative ring. This master's thesis's primary inspiration is to attempt to understand that notion of quasi-freeness and to relate it to commutative k-algebras. The summary of my findings is the content of theorem 7.

Organization Of This Master's Thesis

This master's thesis is organized around its two objectives. Firstly to prove that the smallness of a certain numerical invariant, the Hochschild cohomological dimension of a k-algebra A denoted $HCdim(A/k)$, has certain implications on A's properties:

1. **Result 1:** $HCdim(A/k) = 0$ if and only if all derivations of A in an (A,A)-bimodule M are inner derivations if and only if $\Omega^0(A/k)$ is a $\mathscr{E}^k_{A^e}$-projective A^e-module.
2. **Result 2:** $HCdim(A/k) \leq 1$ if and only if all square-zero extensions of A lift if and only if $\Omega^1(A/k)$ is a $\mathscr{E}^k_{A^e}$-projective A^e-module.

(The notation and concepts mentioned above will be clarified in this master's thesis).

In the case that k is a field results 1 and 2 were proven by Cuntz and Quillen in [AE] and was the starting point for the development of many of their outstanding results on quasi-free k-algebras.

The first central result in this master's thesis generalises their work to the case where k is an arbitrary commutative base ring and to attempt to characterise k-algebras for which $HCdim(A/k) \leq n$. That more general result is then interpreted for the cases where $n = 0,1$ and is presented here but is already known by [HI]. Moreover we extend the result to $n \geq 2$.

The second objective of this master's thesis is to understand what commutative k-algebras fail to be quasi-free, when k is no longer assumed to be field. This understanding comes from an original result describing a lower bound on the Hochschild cohomological dimension of a commutative k-algebra which we build in *chapter* 3 and then apply it to some concrete examples in *chapter* 4.

NOTATION AND CONVENTIONS Unless otherwise stated:

1. $\mathbb{N}$ is the set of non-negative integers.
2. All k-algebras are assumed to be unital and associative.
3. A *noncommutative k-algebra* is a unital k-algebra that may or may not be commutative.
4. The term *module* will always be short for *left-module*.
5. k and R are assumed to be non-zero commutative unital associative rings.
6. A denotes a k-algebra.
7. For any natural number n, $A^{\otimes n}$ will denote the n-fold tensor $- \otimes_k -$ power of A over k, $A^{\otimes 1}$ is defined to be A and $A^{\otimes 0}$ is defined to be k.

2

HOCHSCHILD THEORY

2.1 (A, A)-bimodules and enveloping k-algebras

The Hochschild cohomology of a k-algebra A is a cohomology theory of (A, A)-bimodules instead of A-modules In order to capture the relationship of the k-algebra A to its "modules" it seems appropriate to consider their left and right structures in a simultaneous and compatible way.

2.1.0.1 *General Definitions*

Definition 1. (A, B)***-bimodule***

*If A and B are k-algebras an (A, B)**-bimodule** is an k-module M which is both a left A-module and a right B-module and satisfies the following compatibility axiom:*

$$(\forall c \in k)(\forall m \in M)(\forall a \in A)(\forall b \in B)c \cdot ((a \cdot m) \cdot b) = (ca) \cdot (m \cdot b) = a \cdot (m \cdot cb) = a \cdot (c\dot{m}) \cdot b$$

where $a \cdot m$ denotes the left action of A on M and $m \cdot b$ denotes the right action of B on M.

Definition 2. ***Homomorphism of*** (A,B)***-bimodules***[1]

If A and B are k-algebras and M and N are (A,B)-bimodules then a homomorphism of k-modules $\phi : M \to N$ is said to be a ***homomorphism of*** (A,B)***-bimodules*** *if and only if it is both a left A-module homomorphism and a right B-module homomorphism.*

2.1.1 A^e-modules and (A,A)-bimodules

There is an occasionally more convenient way to view (A,A)-bimodules, by replacing A by a certain related k-algebra.

Definition 3. ***Opposite*** k***-Algebra***

If A is a k-algebra then the ***opposite*** k***-Algebra of*** *A denoted A^{op}, is defined as having the same underlying k-module structure as A but with its multiplication map $\mu_{A^{op}}$ being the k-module homomorphism $\mu_{A^{op}} : A^{op} \otimes_k A^{op} \to A^{op}$ defined as:*

$$(\forall a,b \in A^{op})\mu_{A^{op}}(a,b) := \mu_A(b,a) \tag{1}$$

where $\mu_A : A \otimes_k A \to A$ is the multiplication map on A.

Definition 4. ***Enveloping*** k***-Algebra***

If A is a k-algebra then the ***enveloping*** k***-Algebra of*** *A is defined as the k-algebra $A \otimes_k A^{op}$ and is denoted A^e.*

1. The category of (A,B)-bimodules and (A,B)-bimodule homomorphism is usually denoted by ${}_A Mod_B$.

For a k-algebra A its categories of (A,A)-bimodules and left A^e-modules are equivalent in the following way:

Proposition 1. *If A is a k-algebra then every A^e-module is an (A,A)-bimodule and visa-versa. Likewise every A^e-module morphism is an (A,A)-bimodule morphism and visa-versa.*

Proof.

— If M is a left A^e-module then for all $a,b,b' \in A$ and for all $m \in M$ define the left action of a on m as $a \cdot m := (a \otimes_k 1)m$ and the right action of b on m as $m \cdot b := (1 \otimes_k b)m$. This does indeed define an (A,A)-bimodule structure on M, since:

$$(a \cdot m) \cdot b \cdot b' = ((a \otimes_k 1)m) \cdot b \cdot b' \tag{2}$$

$$= (1 \otimes_k b)(a \otimes_k 1)m \cdot b' = (1 \otimes_k b')(1 \otimes_k b)(a \otimes_k 1)m = (a \otimes_k bb')m \tag{3}$$

$$= (a \otimes_k 1)(1 \otimes_k bb')m \tag{4}$$

$$= (a \otimes_k 1)(m \cdot bb') \tag{5}$$

$$= a \cdot (m \cdot bb') \tag{6}$$

M is a right A module and the right and left A-module structures of M are compatible.

Moreover if $c \in k$ and $m \in M$ then:

$$c \cdot m = c \otimes_k 1 \cdot m = 1 \otimes_k c \cdot m = m \cdot c. \tag{7}$$

Therefore the action of A^e on M is k-linear whence M is a k-module with left and right A-module actions satisfying (1); whence M is an (A,A)-bimodule.

— Conversely, if M is an (A,A)-bimodule then M may be made into a left A^e-module with left action defined (on elementary tensors) as: $(\forall a,b \in A)(\forall m \in M)(a \otimes_k b) \cdot$

$m := (am)b$. This action is associative if $a \otimes_k b, a' \otimes_k b' \in A^e$ and $m \in M$ then denoting by $\odot$ the multiplication in A^{op} and by $\bullet$ the multiplication in A^e:

$$(a \otimes_k b) \cdot ((a' \otimes_k b') \cdot m) = (a \otimes_k b) \cdot (a' m b') \tag{8}$$

$$= ((aa')m)(b'b) \tag{9}$$

$$= (aa' \otimes_k b'b) \cdot m \tag{10}$$

$$= (aa' \otimes_k b \odot b') \cdot m \tag{11}$$

$$= ((a \otimes_k b) \bullet (a' \otimes_k b')) \cdot m. \tag{12}$$

Moreover $1 \otimes_k 1 \cdot m = (1m)1 = m$. Therefore M is an A^e-module.

— Likewise for any (A, A)-bimodule homomorphism and any A^e-module homomorphisms.

□

Remark 1. *If A is a k-algebra and M is an A^e-module then M may be viewed as a right A^e module as:*

$$(\forall a, b \in A)(\forall m \in M) m \cdot_r (a \otimes_k b) := (b \otimes_k a) \cdot m \tag{13}$$

where $\cdot_r$ denoted the right action of A^e on M, indeed this action is associative and respects the unit.

In view of proposition 1, (A, A)-bimodules and A^e-modules will be viewed interchangeably, as is convenient based on the context.

Example 1. *If A is a k-algebra and $n \in \mathbb{N}$ then $A^{\otimes n+2}$ may be given the structure of an A^e-module with action on elementary tensors $a_0 \otimes_k ... \otimes_k a_{n+1}$ in $A^{\otimes n+2}$:*

$$(\forall a, b \in A)(a \otimes_k b) \cdot (a_0 \otimes_k ... \otimes_k a_{n+1}) := aa_0 \otimes_k ... \otimes_k a_{n+1} b. \tag{14}$$

Proof. If $a, b, a', b' \in A$ and $(a_0 \otimes_k ... \otimes_k a_{n+1})$ is an elementary tensor in $A^{\otimes n+2}$ then:

$$= (a' \otimes_k b') \cdot (a \otimes_k b) \cdot (a_0 \otimes_k ... \otimes_k a_{n+1}) \tag{15}$$

$$= (a' \otimes_k b') \cdot (aa_0 \otimes_k ... \otimes_k a_{n+1}b) \tag{16}$$

$$= (a'aa_0 \otimes_k ... \otimes_k a_{n+1}bb') \tag{17}$$

$$= ((a'a \otimes_k bb')) \cdot (a_0 \otimes_k ... \otimes_k a_{n+1}) \tag{18}$$

Therefore the action is associative; moreover it respects the unit since:

$$(1 \otimes_k 1) \cdot (a_0 \otimes_k ... \otimes_k a_{n+1}) = (1a_0 \otimes_k ... \otimes_k a_{n+1}1) \tag{19}$$

$$= (a_0 \otimes_k ... \otimes_k a_{n+1}) \tag{20}$$

□

Example 2. *If N and M are A-modules then $Hom_k(N, M)$ has the structure of a (A, A)-bimodule via the action:*

$$(\forall n \in N)\ (a, a') \cdot f(n) \mapsto af(a' \cdot n) \tag{21}$$

where $a, a' \in A$ and $f : N \to M$ is a k-module homomorphism.

Proof. Since $(af(n))a' = (af)(a' \cdot n) = af(a' \cdot n) = a(fa' \cdot n)$ the left and right A-module structures of $Hom_k(N, M)$ are compatible. Therefore $Hom_k(N, M)$ is indeed an (A, A)-bimodule. □

2.1.1.1 *A Note On The Tensor Product of A^e-modules*

If M and N are A^e-modules then by remark 1 M may be viewed as a right A^e-module, which we denote M_r whence the tensor product $M \otimes_{A^e} N$ may be defined as:

Definition 5. ***Tensor Product of A^e-modules***

*If M and N are A^e-modules then the **tensor product of** M **and** N **over** A^e is defined to be the k-module $M_r \otimes_{A^e} N$ and is denoted by $M \otimes_{A^e} N$.*

However we make use of a different tensor product of bimodules defined as usual as follows:

Definition 6. ***Tensor Product of bimodules***

Let A, B, C be rings, M be a (B, A)-bimodule and N be an (A, C)-bimodule.

The abelian group with basis the symbols $m \otimes_A n$, where $m \in M$ and $n \in N$ modulo its subgroup generated by all the elements of the set:

$$\{-(m+m') \otimes_A n + m' \otimes_A n + m \otimes_A n, \tag{22}$$

$$-m \otimes_A (n+n') + m \otimes_A n + m \otimes_A n', \tag{23}$$

$$m \otimes (a \cdot n) - (m \cdot a) \otimes_A n | m, m \in M \text{ and } n, n' \in N \text{ and } a \in A\} \tag{24}$$

*is called the **tensor product of** M **and** N **over** A and is denoted by $M \otimes_A N$.*

*For any $m \in M$ and $n \in N$ the coset of the symbol $m \otimes_A n$ is called an **elementary tensor** and is simply denoted by $m \otimes_A n$.*

2.1.2 *Hochschild Cohomology*

The entire theory reviewed and developed in this master's thesis revolves around a particular exact sequence related to the A^e-module A called the **Bar resolution of** A [2] .

2. The word *"Bar"* in the phrase *"Bar resolution of A"* arises from an notational convention that has generally fallen out of practice. Traditionally elementary tensors in $A^{\otimes n}$ were denoted by $a_1|....|a_n$ as can be

Example 3. ***The Bar Resolution of*** A

If A is a k-algebra then there is an acyclic chain complex of A^e-modules denoted $CB_\star(A)$, defined as:

$$(\forall n \in \mathbb{N}) CB_n(A) := A^{\otimes n+2} \tag{25}$$

With the A^e-module structure on $CB_n(A)$ taken to be the one described in example 1. With boundary operator:

$$(\forall n \in \mathbb{N}) b'_n(a_0 \otimes ... \otimes a_{n+1}) := \sum_{i=0,..,n} (-1)^i a_0 \otimes ... \otimes a_i a_{i+1} \otimes ... \otimes a_{n+1} \tag{26}$$

(By convention: b'_0 is the augmentation map $A \otimes_k A \to A$ and b'_{-1} is the zero map from A to 0).

The augmented Bar resolution of A will be denoted $\hat{CB_\star}(A)$.

Proof.

First for every $n \in \mathbb{N}$ we define a k-linear map, which we denote s_n and then we use those maps to show that $CB_\star$ is an acyclic chain complex.

For $n \in \mathbb{N}$ define the k-linear maps $s_n : \hat{CB_n}(A) \to \hat{CB_n}(A)$ on elementary tensors as

$$a_0 \otimes_k ... \otimes_k a_{n+1} \mapsto 1 \otimes_k a_0 \otimes_k ... \otimes_k a_{n+1} \tag{27}$$

If $a_0 \otimes_k ... \otimes_k a_{n+1} \in \hat{CB_\star}(A)$ then:

$$b'_{n+1}(s_n(a_0 \otimes_k ... \otimes_k a_{n+1})) + s_{n-1}(b'_n(a_0 \otimes_k ... \otimes_k a_{n+1})) \tag{28}$$

$$= b'_{n+1}(1 \otimes_k a_0 \otimes_k ... \otimes_k a_{n+1})) + s_{n-1}(\sum_{i=0}^{n} (-1)^i a_0 \otimes_k .. \otimes_k a_i a_{i+1} \otimes_k ... \otimes_k a_{n+1}) \tag{29}$$

$$= \sum_{i=-1}^{n} (-1)^i 1 \otimes_k a_0 \otimes_k .. \otimes_k a_i a_{i+1} \otimes_k ... \otimes_k a_{n+1}$$

seen on page 114 of [MH]. Furthermore the choice of the phrase *"resolution of A"* will be explored in later sections of this master's thesis.

$$+\sum_{i=0}^{n}(-1)^{i}1\otimes_k a_0\otimes_k ..\otimes_k a_ia_{i+1}\otimes_k ...\otimes_k a_{n+1} \tag{30}$$

$$= 1a_0\otimes_k \ldots\otimes_k a_{n+1}+\sum_{i=0}^{n}(-1)^{i}1\otimes_k a_0\otimes_k ..\otimes_k a_ia_{i+1}\otimes_k \ldots\otimes_k a_{n+1} \tag{31}$$

$$+\sum_{i=0}^{n}(-1)^{i}1\otimes_k a_0\otimes_k ..\otimes_k a_ia_{i+1}\otimes_k \ldots\otimes_k a_{n+1} \tag{32}$$

$$= a_0\otimes_k ...\otimes_k a_{n+1} \tag{33}$$

Therefore for every $n\in\mathbb{N}$:

$$b'_{n+1}\circ s_n + s_{n+1}\circ b'_n = 1_{C\hat{B_\star}(A)}. \tag{34}$$

Making use of (34) we first show that $C\hat{B}_\star$ is a chain complex and we show that the identity map $C\hat{B_\star}(A)$ is chain homotopic to the 0-map on $C\hat{B_\star}(A)$, therefore the homology of $C\hat{B_\star}(A)$ is trivial.

— We prove by induction on $\star$ that $CB_\star$ is a chain complex. If $n=1$ then:

$$(\forall a_0,a_1,a_2\in A)b'_0\circ b'_1(a_0\otimes_k a_1\otimes_k a_2) = b'_1(a_0a_1\otimes_k a_1 - a_0\otimes_k a_1a_2)$$
$$= a_0a_1a_2 - a_0a_1a_2 = 0.$$

Suppose for some $n>1$ $b'_n\circ b'_{n-1}=0$, then (34) implies:

$$b'_{n+1}\circ b'_n\circ s_n = b'_n\circ(1-s_{n-1}\circ b'_n) \tag{35}$$

$$= b'_n - b'_n\circ s_{n-1}\circ b_n \tag{36}$$

$$= b'_n - b'_n - s_{n-2}\circ b'_{n-1}\circ b'_n \tag{37}$$

$$= 0 + s_{n-2}\circ b'_{n-1}\circ b'_n \tag{38}$$

$$= 0 \text{ by the induction hypothesis.} \tag{39}$$

Therefore $b'_{n+1} \circ b'_n = 0$ which completes the induction, showing that $CB_\star$ is indeed a chain complex.

— Furthermore (34) says that the identity map $C\hat{B_\star}(A)$ is chain homotopic to the 0-map on $C\hat{B_\star}(A)$, therefore $C\hat{B_\star}(A)$ is chain homotopic to an acyclic complex.

□

Definition 7. ***Hochschild Cohomology***

The Hochschild cohomology of a k-algebra A with coefficients in an (A, A)-bimodule M, denoted $HH^\star(A, M)$ is defined as:

$$HH^\star(A, M) := H^\star(Hom_{A^e}(CB_\star(A), M), Hom_{A^e}(b'_\star, M)) \tag{40}$$

The coboundary map $Hom_{A^e}(b'_\star, M)$ is denoted by $b^\star$.

Proposition 2. *The Hochschild cohomology of a k-algebra A with coefficients in an A^e-module M may be computed as the cohomology of the following complex:*

$$0 \to M \xrightarrow{b^0} Hom_k(A, M) \xrightarrow{b^1} Hom_k(A^{\otimes 2}, M) \xrightarrow{b^2} .... \tag{41}$$

Where the coboundary map b^n is defined on $f \in Hom_k(A^{\otimes n}, M)$ and $a_0 \otimes_k .. \otimes_k a_n \in A^{\otimes n}$ as:

$$b^n(f(a_0 \otimes_k .. \otimes_k a_n)) = a_0 f(a_1 \otimes_k .. \otimes_k a_n) + \sum_{i=0}^{n-1}(-1)^i f(a_0 \otimes_k ... \otimes_k a_i a_{i+1} \otimes_k ... \otimes_k a_n) + (-1)^n f(a_0 \otimes_k .. \otimes_k a_{n-1}) a_n \tag{42}$$

Proof. We show that the complexes $Hom_{A^e}(CB_\star(A), M)$ and (41) are naturally isomorphic (whence their cohomology modules must be isomorphic).

— Viewing M as an (A, A)-bimodule as in proposition 1, if $f : A^{\otimes n} \to M$ is a k-module homomorphism then define the A^e-module map $\hat{f} : A^{\otimes n+2} \to M$ on elementary tensors as:

$$\hat{f}(a_0 \otimes_k .. \otimes_k a_{n+1}) := a_0 f(a_1 \otimes_k .. \otimes_k a_n) a_{n+1}. \tag{43}$$

We verify that $f \mapsto \hat{f}$ is indeed a k-isomorphism.

The k-linear map $\hat{f}$ is indeed an A^e-module map, since if $(a \otimes_k b)$ is an elementary tensor in A^e then:

$$(a \otimes_k b) \cdot \hat{f}(a_0 \otimes_k .. \otimes_k a_{n+1}) = (a \otimes_k b) \cdot a_0 f(a_1 \otimes_k .. \otimes_k a_n) a_{n+1} \tag{44}$$

$$= a a_0 f(a_1 \otimes_k .. \otimes_k a_n) a_{n+1} b \tag{45}$$

$$= (a a_0) f(a_1 \otimes_k .. \otimes_k a_n)(a_{n+1} b) \tag{46}$$

$$= \hat{f}((a a_0) \otimes_k a_1 \otimes_k .. \otimes_k a_n \otimes_k (a_{n+1} b)) \tag{47}$$

$$= \hat{f}((a \otimes_k b) \cdot (a_0 \otimes_k a_1 \otimes_k .. \otimes_k a_n \otimes_k a_{n+1})) \tag{48}$$

Since any A^e-module homomorphism $g : A^{\otimes n+2} \to M$ is k-linear, the map $\tilde{g} : A^{\otimes n} \to M$ defined on elementary tensors $a_0 \otimes_k .. \otimes_k a_n \in A^{\otimes n}$ as:

$$\tilde{g}(a_0 \otimes_k .. \otimes_k a_n) \mapsto g(1 \otimes_k a_0 \otimes_k .. \otimes_k a_n \otimes_k 1). \tag{49}$$

is a k-module homomorphism whose two-sided inverse is the map $f \mapsto \hat{f}$.

Denote this A^e-module isomorphism by $\Psi : Hom_{A^e}(A^{\otimes n+2}, M) \to Hom_k(A^{\otimes n}, M)$.

By definition $Hom_{A^e}(b'_n, M)$ is the pre-composition of any $f \in Hom_{A^e}(A^{\otimes n+2}, M)$ by b'_n. By furthermore pre-composing $f \circ b'_n$ by the A^e-module isomorphism Ψ the coboundary map:

$$f \circ b'_n \circ \Psi(a_0 \otimes_k .. \otimes_k a_n) = f \circ b'_n(1 \otimes_k a_0 \otimes_k .. \otimes_k a_n \otimes_k 1)$$

$$= f(a_0 a_1 \otimes_k .. \otimes_k a_n + \sum_{i=0}^{n-1} (-1)^i a_0 \otimes_k .. \otimes_k a_i a_{i+1} \otimes .. \otimes_k a_n + (-1)^{n+1} a_0 \otimes_k .. \otimes_k a_{n-1} a_n)$$

$$= a_0 f(a_1 \otimes_k .. \otimes_k a_n) + \sum_{i=0}^{n-1} (-1)^i f(a_0 \otimes_k .. \otimes_k a_i a_{i+1} \otimes .. \otimes_k a_n) + (-1)^{n+1} f(a_0 \otimes_k .. \otimes_k a_{n-1}) a_n)$$

$$= b^n(f)(a_0 \otimes_k .. \otimes_k a_n) \tag{50}$$

is obtained. □

Definition 8. ***Hochschild Cocomplex***

For any k-algebra A and any A^e-module M the cocomplex in proposition 2 is called the ***Hochschild cocomplex of*** A ***with respect to*** M *and is denoted by* $CH^\star(A, M)$.

2.1.3 *Computing the first few Hochschild Cohomology Groups*

To better interpret the Hochschild cohomology groups the first few are computed.[3]

2.1.3.1 HH^0

Definition 9. ***Center of an*** A***-bimodule***[4]

If M is an (A,A)-bimodule the collection of elements of M commuting with all the elements of A is called the A-centre of M and is denoted $Z_A(M)$. That is:

$$Z_A(M) := \{m \in M | (\forall a \in A) a \cdot m = m \cdot a\} \tag{51}$$

Proposition 3. *For any (A,A)-bimodule M, $Z_A(M)$ is an (A,A)-sub-bimodule of M.*

Proof. Let $a, b \in A$ and $n, m \in Z_A(M)$. Then:

— 1.

$$a \cdot (n + m) = a \cdot n + a \cdot m = n \cdot a + m \cdot a = (n + m) \cdot a. \tag{52}$$

therefore $Z_A(M)$ is closed under $+$.

2. Suppose there is some $a \in A$ and $n \in N$ such that $-n \notin Z_A(M)$ then: $a \cdot (-n) \neq -n \cdot a$ then by (52):

$$0 = a \cdot (-n + n)$$

3. For example $HH^0(A, M)$ is reminiscent of the 0^{th} group cohomology module of a G-module for a group G or the 0^{th} lie-algebra cohomology module of a $\mathfrak{g}$-module for some lie algebra $\mathfrak{g}$.

4. In particular if M is the (A,A)-bimodule A then $Z_A(A)$ is precisely the definition of the centre of A, hence it inherits a ring structure.

$$= a \cdot (-n) + a \cdot n \neq -n \cdot a + a \cdot n$$

$$= -n \cdot a + n \cdot a$$

$$= (-n + n) \cdot a = 0 \tag{53}$$

a contradiction, therefore $Z_A(M)$ is closed under $+$inversion.

Hence $Z_A(M)$ is an abelian subgroup of M.

— Let $a, b \in A^e$ then:

$$(ab) \cdot n = a \cdot (b \cdot n) = a \cdot (n \cdot b) = (a \cdot n) \cdot b = (n \cdot a) \cdot b = n \cdot (ab). \tag{54}$$

Therefore $Z_A(M)$ is a A^e-submodule of M.

□

The 0^{th} Hochschild cohomology group may be understood as describing $Z_A(M)$.

Proposition 4. ***Interpretation of*** HH^0

For a k-algebra A and any (A, A)-bimodule M there is an isomorphism of k-modules:

$$HH^0(A, M) \overset{\cong}{\to} Z_A(M). \tag{55}$$

Proof. By proposition 2: $HH^0(A, M) \cong Ker(b^0)/Im(0)$. Therefore:

$$HH^0(A, M) \cong Ker(b^0)$$

$$= \{m \in M | (\forall a \in A) b^0(m)(a) = 0\}$$

$$= \{m \in M | (\forall a \in A) ma - am = 0\}$$

$$= \{m \in M | (\forall a \in A) ma = am\} = Z_A(M). \tag{56}$$

□

2.1.3.2 HH^1

Definition 10. (A, A)***-Bimodule*** k***-Derivation*** [5]

*A k-linear map D from A to an (A, A)-bimodule M is called a (A, A)**-bimodule** k**-derivation** if and only if*

$$(\forall a, a' \in A)\ D(aa') = aD(a') + D(a)a'. \tag{57}$$

The k-module of all (A, A)-bimodule k-derivations of A into M is denoted $Der_k(A, M)$.

Definition 11. ***Inner*** (A, A)***-bimodule*** k***-Derivation*** [6]

*An (A, A)-bimodule k-Derivation $D : A \to M$ is said to be **inner** if and only if there exists some $m \in M$ such that:*

$$(\forall a \in A) D(a) = a \cdot m - m \cdot a. \tag{58}$$

The collection of all inner (A, A)-bimodule k-Derivations of A into M is denoted $Inn_k(A, M)$.

Note 1. *For legibility, when the context is clear (A, A)-bimodule k-derivations of A into M will simply be called* k***-derivations of*** A ***into*** M *or more plainly **derivations**.*

$HH^1(A, M)$ may be understood as classifying derivations of k-algebra A into an (A, A)-bimodule M.

Proposition 5. *For every k-algebra A and every (A, A)-bimodule M there is an isomorphism of k-modules:*

$$HH^1(A, M) \overset{\cong}{\to} Der_k(A, M)/Inn_k(A, M) \tag{59}$$

Proof. By proposition 2: $HH^1(A, M) \cong Ker(b^1)/Im(b^0)$. Therefore:

—

$$Ker(b^1) = \{f \in Hom_k(A, M) | (\forall \sum_{i=0}^{n} a_i \otimes_k b_i \in A^{\otimes 2})\ b^1(f)(\sum_{i=0}^{n} a_i \otimes_k b_i) = 0\} \tag{60}$$

5. These are similar to *crossed homomorphisms* of groups.
6. These are reminiscent of *principal crossed homomorphisms* between groups.

$$= \{f \in Hom_k(A,M) | (\forall a, a' \in A)\ b^1(f)(a \otimes_k a') = 0\} \tag{61}$$

$$= \{f \in Hom_k(A,M) | (\forall a, a' \in A)\ af(a') - f(aa') + f(a)a' = 0\} \tag{62}$$

$$= \{f \in Hom_k(A,M) | (\forall a, a' \in A)\ f(aa') = af(a') + f(a)a'\} \tag{63}$$

$$= Der_k(A,M) \tag{64}$$

— Similarly:

$$Im(b^0) = \{f \in Hom_k(A,M) | (\exists m \in M)(\forall a \in A)\ f(a) = ma - am\} \tag{65}$$

$$= Inn_k(A,M). \tag{66}$$

Therefore $HH^1(A,M) \cong Der_k(A,M)/Inn_k(A,M)$ (as k-modules). □

2.1.3.3 HH^2

Definition 12. ***k*-split Exact Sequence**

Let k be a ring. A short exact sequence of k-modules:

$$0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{\pi} M'' \longrightarrow 0 \tag{67}$$

*is said to be k-**split** (or k-**split-exact**) if and only if there exists a k-module homomorphism $s : M'' \to M$ such that $\pi \circ s = 1_{M''}$; the k-module homomorphism s is called a **section** of π.*

Definition 13. ***k*-Hochschild extension**

A k-split-exact sequence $\mathfrak{E}_\pi$ of k-modules where π is a k-algebra homomorphism:

$$\mathfrak{E}_\pi : 0 \longrightarrow M \xrightarrow{i} B \xrightarrow{\pi} A \longrightarrow 0 \tag{68}$$

*is called a k-**Hochschild extension** of A by M if both B and A are k-algebras and M is a two-sided ideal in B. In such a setting M is said to extend A (alternatively A is said be extended by M).*

*If $M^2 \cong 0$ then $\mathfrak{E}_\pi$ is said to be **square-zero**.*

Lemma 1. *If $\mathfrak{E}_\pi$ is a k-Hochschild extension of A by M then: $\mathfrak{E}_\pi$ is square-zero if and only if M is an (A, A)-bimodule with action described as:*

for all $a \in A$ and for all $m \in M$ the left action $a \cdot m$ (resp. right action $m \cdot a$) is defined as the multiplication $\mathfrak{b}m$ (resp. $m\mathfrak{b}$) in B, where $\mathfrak{b}$ is any element in the π-fibre above a.

Proof.

- Let $a \in A$ and $m \in M$.
- For any $m \in M$ and $a \in A$ the action $a \cdot m$ is well defined if and only if for any other elements $\mathfrak{b}'$ and $\mathfrak{b}$ in the π-fibre above a: $\mathfrak{b}m = \mathfrak{b}'m$. In other words the action is only well defined if and only if $(\mathfrak{b} - \mathfrak{b}')m = 0m = 0$. Therefore for every m in M there is some $m' := (\mathfrak{b} - \mathfrak{b}')$ in M such that $mm' = 0$. Hence in this way M is given a well defined left A-module structure if and only if M is a square zero-ideal in B.
- Mutatis mutandis, A may be given a right A-module with action $m \cdot a$ defined as the multiplication $m\mathfrak{b}$ in B, where $\mathfrak{b}$ is any element in the π-fibre above a *if and only if* M is a square zero-ideal in B.
- Let $a, a' \in A$ and $m \in M$ and choose some $b \in \pi^{-1}[a]$ and $b' \in \pi^{-1}[a']$ (by the above remarks this calculation will be independent of this choice). Then the associativity law of the k-algebra B implies:

$$a \cdot (m \cdot a') = b(mb') = (bm)b' = (a \cdot m) \cdot a'. \tag{69}$$

  Therefore the above left and right A-module structures are compatible. Hence M is an (A, A)-bimodule *if and only if* $\mathfrak{E}_\pi$ is square-zero.

□

Maintaining the notation of (68), since π splits $B \cong s(A) \oplus M$ as k-modules, where $s : A \to B$ is a section of π (that is s is a k-module homomorphism satisfying: $\pi \circ s = 1_A$). Moreover $s(A) \oplus M$'s multiplicative structure is dependent on the choice of the section s of π and may be understood as follows:

Proposition 6. *Maintaining the notation of* (68)*: if $\mathfrak{E}_\pi$ is a k-Hochschild extension of A by an (A, A)-bimodule M then for every section s of π, $s(A) \oplus M$'s multiplicative structure must be of the form:*

$$(\forall a, a' \in A)(\forall m, m' \in M)(a, m)(a', m') = (aa', am' + ma' + \mathfrak{B}_s(a, a')) \tag{70}$$

where $\mathfrak{B}_s$ is in $CH^2(A, M)$ and depends only on the choice of the section s.

Moreover $\mathfrak{B}_s$ must be a 2*-cocycle.*

Conversely, if M is an (A, A)-bimodule and $\mathfrak{B} : A \otimes_k A \to M$ is a 2*-cocycle then:*

$$\mathfrak{E} : 0 \longrightarrow M \longrightarrow M \oplus A \longrightarrow A \longrightarrow 0 \tag{71}$$

determines a Hochschild extension with $A \oplus M$'s multiplicative structure defined as:

$$(\forall a, a' \in A)(\forall m, m' \in M)(a, m)(a', m') = (aa', am' + ma' + \mathfrak{B}(a, a')) \tag{72}$$

Proof.

1. If $a, a', b, b' \in A$, $m, m' \in M$, $c, c' \in$ and $s : A \to B$ is k-section of π. Then s determines a map $\mathfrak{B}_s : A \otimes_k A \to B$ by $\mathfrak{B}_s(a \otimes_k a') := s(a)s(a') - s(aa')$. Since π is a morphism of k-algebras then:

$$\pi \circ \mathfrak{B}_s(a \otimes_k a') = \pi(s(a)s(a') - s(aa')) \tag{73}$$

$$= \pi \circ s(a)\pi \circ s(a') - \pi \circ s(aa') \tag{74}$$

$$= 1_A(a)1_A(a') - 1_A(aa') \quad (75)$$

$$= aa' - aa' = 0. \quad (76)$$

Therefore $\mathfrak{B}_s : A^{\otimes 2} \to M$.

Moreover $\mathfrak{B}_s$ is k-linear, since:

$$\mathfrak{B}_s(a + cb \otimes_k a + c'b') = s(a + cb)s(a + c'b') - s((a + cb)(a' + c'b')) \quad (77)$$

$$= (s(a) + cs(b))(s(a) + c's(b')) - s(aa' + cba' + c'ab' + cbc'b') \quad (78)$$

$$= s(a)s(a') + s(cb)s(a') + s(c'a)s(b') + s(cb)s(c'b') - s(aa' + cba' + c'ab' + cbc'b') \quad (79)$$

$$= s(a)s(a') + cs(b)s(a') + c's(a)s(b') + cc's(b)s(b') - s(aa') - cs(ba') - c's(ab') - cc's(bb') \quad (80)$$

$$= s(a)s(a') - s(aa') + cs(b)s(a') - cs(ba') + c's(a)s(b') - c's(ab') + cc's(b)s(b') - cc's(bb') \quad (81)$$

$$= (s(a)s(a') - s(aa')) + c(s(b)s(a') - s(ba')) + c'(s(a)s(b') - s(ab')) + cc'(s(b)s(b') - s(bb')) \quad (82)$$

$$= \mathfrak{B}_s(a \otimes_k a') + c\mathfrak{B}_s(b \otimes_k a') + c'\mathfrak{B}_s(a \otimes_k b') + cc'\mathfrak{B}_s(b \otimes_k b'). \quad (83)$$

Therefore $\mathfrak{B}_s : A^{\otimes 2} \to M \in Hom_k(A^{\otimes 2}, M) = CH^2(A^{\otimes 2}, M)$.

2. $s(A) \oplus M$'s multiplicative structure must be of the form:

$$(\forall a, a' \in A)(\forall m, m' \in M)(a, m)(a', m') = (aa', am' + ma' + \mathfrak{B}_s(a, a')). \quad (84)$$

Since $s(A) \oplus M$'s product structure is entirely determined by the map $\mathfrak{B}_s$ which is entirely determined by the choice of π's section $s : A \to M$ then the choice of multiplicative structure on B may be emphasised to depend on s via the notation $A \rtimes_{\mathfrak{B}_s} M$.

3. It was assumed that all k-algebras were to be associative. It will now be verified that (84) defines an associative multiplicative structure on $A \rtimes_{\mathfrak{B}_s} M$, that is it must

be verified when it defines a k-algebra; in fact this condition will be that $\mathfrak{B}_s$ is a 2-cocycle.

For $\mathfrak{B}_s$ induce an associate product on B the following must hold for $(a,m),(a',m'),(a'',m'') \in A \rtimes_{\mathfrak{B}_s} M$:

$$(a,m)((a',m')(a'',m'')) = (a(a'a''),a(a'm''+m'a''+\mathfrak{B}_s(a',a''))+m(a'a'')+\mathfrak{B}_s(a,a'a'')\text{and} \tag{85}$$

$$((a,m)(a',m'))(a'',m'') = ((aa')a'',(aa')m''+(am'+ma'+\mathfrak{B}_s(a,a'))a''+\mathfrak{B}_s(aa',a'') \tag{86}$$

Therefore $A \rtimes_{\mathfrak{B}_s} M$ is associative *if and only if* (85) equalities with (86) *if and only if*:

$$a\mathfrak{B}_s(a',a'') + \mathfrak{B}_s(a,a'a'') = \mathfrak{B}_s(a,a')a'' + \mathfrak{B}_s(aa',a''). \tag{87}$$

That is A is associative *if and only if*

$$0 = a\mathfrak{B}_s(a',a'') - \mathfrak{B}_s(a,a'a'') + \mathfrak{B}_s(aa',a'') - \mathfrak{B}_s(a,a')a'' = b^2\mathfrak{B}_s(a,a'a''). \tag{88}$$

Therefore (88) implies that $A \rtimes_{\mathfrak{B}_s} M$ is an associative k-algebra *if and only if* $A \rtimes_{\mathfrak{B}_s} M \in Ker(b^2)$, where b^2 denotes the 2^{nd} coboundary map of the Hochschild cocomplex.

Conversely, for every (A, A)-bimodule M the following is by definition a k-split exact sequence:

$$\mathfrak{E} : 0 \longrightarrow M \longrightarrow M \oplus A \longrightarrow A \longrightarrow 0 \quad . \tag{89}$$

Moreover if $\mathfrak{B}$ is a 2-cocycle, a verifications similar to (85)-(88), shows that:

$$(\forall a,a' \in A)(\forall m,m' \in M)(a,m)(a',m') = (aa',am'+ma'+\mathfrak{B}(a,a')) \tag{90}$$

describes a well-defined (associative) product structure on $M \oplus A$, making it into a k-algebra. Finally, since M was assumed to be an (A, A)-bimodule then lemma 1 implies (89) is square-zero; whence $\mathfrak{E}$ is a (square-zero) Hochschild extension. □

Example 4. ***Trivial Extension***

If M is an A^e-module then the 0 *map* $0 : A \otimes_k A \to M$ *defines a square-zero extension of A by M:*

$$0 \longrightarrow M \hookrightarrow A \rtimes_0 M \overset{\pi}{\twoheadrightarrow} A \longrightarrow 0 \tag{91}$$

The k-Hochschild extension (91) *is called the* ***Trivial Extension*** *of A by M.*

Proof. By proposition 6 $A \rtimes_0 M$ is a k-algebra with multiplication given by:

$$(\forall a, a' \in A)(\forall m, m' \in M)\ (a, m)(a', m') = (aa', am' + ma'). \tag{92}$$

□

Remark 2. *Example* 4 *may seem a priori non-interesting, however it is of essential importance in the proof of theorem* 2*. In part because it demonstrates that a square-zero extension of A by M must always exist.*

Definition 14. ***$\mathfrak{B}$-Crossed Product***

If A is a k-algebra, M is an (A, A)-bimodule and $\mathfrak{B} : A \otimes_k A \to M$ is a 2*-cocycle then the k-algebra with underlying k-module structure $A \oplus M$ and with multiplicative structure:*

$$(\forall a, a' \in A)(\forall m, m' \in M)(a, m)(a', m') = (aa', am' + ma' + \mathfrak{B}(a, a')) \tag{93}$$

is called the ***$\mathfrak{B}$-Crossed Product of A by M*** *and is denoted by $A \rtimes_{\mathfrak{B}} M$. If $\mathfrak{B}$ arises from a section $s : A \to M$ of π splitting the short-exact sequence of k-modules:*

$$0 \to M \to A \oplus M \overset{\pi}{\to} A \to 0. \tag{94}$$

then $\mathfrak{B}_s$ *will denote the* 2*-cocycle* $\mathfrak{B}_s(a \otimes_k a') := s(a)s(a') - s(aa')$*, in which case* $A \rtimes_{\mathfrak{B}_s} M$ *will denote the* $\mathfrak{B}_s$*-crossed product of* A *by* M*.*

Proposition 7. *Maintaining the notation of proposition* 6*, if* s *and* s' *are sections of* π *and* $\mathfrak{B}_s$ *and* $\mathfrak{B}_{s'}$ *are their associative* 2*-cocycles then* $\mathfrak{B}_s - \mathfrak{B}_{s'}$ *is a* 2*-coboundary.*

Therefore any k-Hochschild extension $\mathfrak{E}_{\mathfrak{B}_s}$ *determines a unique cohomology class independently of the chosen section* s *splitting* π*.*

Proof.

$$(\forall a, a' \in A)\mathfrak{B}_s(a \otimes_k a') - \mathfrak{B}_{s'}(a \otimes_k a')$$

$$= s(a)s(a') - s(aa') - s'(a)s'(a') + s(aa')$$

$$= s(a)s(a') - s(a)s'(a') + s(a)s'(a') - s(aa') - s'(a)s'(a') + s(aa')$$

$$= s(a)(s(a') - s'(a')) + (s(a) - s'(a))s'(a') + (s(aa') - s(aa')).$$

$$= b^1(s - s')(a \otimes_k a'). \tag{95}$$

□

Maintaining the notation of proposition 7, two Hochschild extensions

$$\mathfrak{E}_{\mathfrak{B}} : \ 0 \longrightarrow M \hookrightarrow B \overset{\pi}{\twoheadrightarrow} A \longrightarrow 0 \text{ and}$$

$$\mathfrak{E}_{\mathfrak{B}'} : \ 0 \longrightarrow M \hookrightarrow B' \overset{\pi'}{\twoheadrightarrow} A \longrightarrow 0 \tag{96}$$

are said to be equivalent if and only if there is a k-algebra isomorphism: $\phi : B \to B'$ making the following diagram of A^e-modules commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \hookrightarrow & B & \overset{\pi}{\twoheadrightarrow} & A & \longrightarrow & 0 \\ \Big\| & & \Big\| 1_M & & \Big\downarrow \phi & & 1_A \Big\| & & \Big\| \\ 0 & \longrightarrow & M & \hookrightarrow & B' & \overset{\pi'}{\twoheadrightarrow} & A & \longrightarrow & 0 \end{array} \tag{97}$$

Definition 15. ***Hochschild Classes***

*The equivalence classes of extensions of A by the (A, A)-bimodule M under the Hochschild equivalence relation are called M, A-**Hochschild classes**.*

Lemma 2. *Maintain the notation of* (96)*. Two k-Hochschild extensions $\mathfrak{E}_{\mathfrak{B}}$ and $\mathfrak{E}_{\mathfrak{B}'}$ of A by M are equivalent if and only if $\mathfrak{B} - \mathfrak{B}'$ is a* 2*-coboundary.*

Proof.

— If $\mathfrak{B} - \mathfrak{B}'$ is a 2-coboundary then there exists a k-module homomorphism $\zeta : A \mapsto M$ satisfying $b^1(\zeta) = \mathfrak{B} - \mathfrak{B}'$ (where b^1 is the Hochschild cocomplex's first coboundary map). Choose some sections s of π and s' of π' (by proposition 7 this choice does not affect $\mathfrak{B} - \mathfrak{B}'$'s cohomology class) and define a k-module homomorphism $\Phi : A \rtimes_{\mathfrak{B}_s} M \to A \rtimes_{\mathfrak{B}_{s'}} M$ as:

$$(\forall a \in A)(\forall m \in M)\ \Phi(a, m) := (a, m + \zeta(a)). \tag{98}$$

Φ has a two-sided inverse, the k-module homomorphism taking an element $(a, m) \in A \rtimes_{\mathfrak{B}_{s'}} M$ to the element $(a, m - \zeta(a))$ in $A \rtimes_{\mathfrak{B}_s} M$.

Moreover Ψ is a k-algebra homomorphism since:

$$(\forall a, a' \in A)(\forall m, m' \in M)\Phi((a,m)(a',m'))$$
$$= (aa', a \cdot m' + m \cdot a' + \mathfrak{B}_s(a,a') + \zeta(aa'))$$
$$= (aa', a \cdot m' + m \cdot a' + a \cdot \zeta(a') + \zeta(a) \cdot a' + \mathfrak{B}_{s'}(a,a'))$$
$$= (a, m + \zeta(a))(a', m' + \zeta(a')) = \Phi(a,m)\Phi(a',m').$$

Furthermore since:

$$(\forall a \in A)(\forall m \in M)\ \pi' \circ \Phi(a,m)$$
$$= \pi'(a, m + \zeta(a))$$
$$= a = \pi(a,m)$$

Φ describes an equivalence of the k-Hochschild extensions $\mathfrak{E}_{\mathfrak{B}}$ and $\mathfrak{E}_{\mathfrak{B}'}$.

— Conversely, if $\mathfrak{E}_{\mathfrak{B}}$ and $\mathfrak{E}_{\mathfrak{B}'}$ are isomorphic k-Hochschild extensions then an analogous computation to (95) shows $\mathfrak{B} - \mathfrak{B}'$ is a 2-coboundary [HI].

□

Theorem 1. ***The Hochschild Class Correspondence Theorem*** *(Hochschild $\sim$ 1944)*

If A is a k-algebra and M is an (A, A)-bimodule then $HH^2(A, M)$ is in $1-1$ correspondence with the set of M,A-Hochschild Classes.

Proof. By lemma 2 two extensions $\mathfrak{E}_{\mathfrak{B}}$ and $\mathfrak{E}_{\mathfrak{B}'}$ of A by M are non-isomorphic if and only if $[\mathfrak{B}]$ and $[\mathfrak{B}']$ are distinct cohomology classes in $H^2(Hom_{A^e}(CB_\star(A), M), Hom_{A^e}(b'_n, M))$.

□

2.2 THE (A, A)-BIMODULES: $\Omega^n(A/k)$

If A is a k-algebra then its multiplication map $\mu_A : A \otimes_k A \to A$ is an (A, A)-bimodule homomorphism, therefore μ_A is an A^e-module homomorphism hence its kernel is an A^e-module. This A^e-module is denoted $\Omega^1(A/k)$ and has the following description:

Proposition 8. *If A is a k-algebra then the A^e-module $\Omega^1(A/k)$ is generated as an A^e-module by the tensors in $A \otimes_k A$ of the form $1 \otimes_k a - a \otimes_k 1$, where $a \in A$.*

Moreover there is k-linear map $d : A \to \Omega^1(A/k)$ defined as $d(a) \mapsto 1 \otimes_k a - a \otimes_k 1$ satisfing the following properties :

1. $d(aa') = ad(a') + d(a)a'$
2. $d(a + a') = d(a) + d(a')$
3. $d(k) = 0$

Therefore d is a derivation of A into $\Omega^1(A/k)$.

Proof. If $a_0, .., a_n, b_0, .., b_n \in A$ and $\sum_{i=0}^{n} a_i \otimes_k b_i \in \Omega^1(A/k)$ then:

$$0 = \mu_A(\sum_{i=0}^{n} a_i \otimes_k b_i) = \sum_{i=0}^{n} a_i b_i. \tag{99}$$

Therefore:

$$0 = 0 - 0 = \sum_{i=0}^{n} a_i b_i - \sum_{i=0}^{n} a_i b_i = \sum_{i=0}^{n} a_i b_i - a_i b_i \tag{100}$$

$$= \sum_{i=0}^{n} a_i (1 b_i - b_i 1) \tag{101}$$

$$= \sum_{i=0}^{n} a_i (\mu_A(1 \otimes_k b_i) - \mu_A(b_i \otimes_k 1)) \tag{102}$$

$$= \sum_{i=0}^{n} a_i \mu_A(1 \otimes_k b_i - b_i \otimes_k 1) = \sum_{i=0}^{n} a_i d(b_i). \tag{103}$$

Thus $\Omega^1(A/k)$ is generated as an A^e-module by elements of the form $1 \otimes_k a - a \otimes_k 1$ where $a \in A$. Moreover the association $a \mapsto 1 \otimes_k a - a \otimes_k 1$ describes a map $d : A \to \Omega^1(A/k)$. The k-linearity as well as the properties of d may be deduced as follows:

1.

$$d(aa') = 1 \otimes_k aa' - aa' \otimes_k 1 = \tag{104}$$

$$= 1 \otimes_k aa' - a \otimes_k a' + a \otimes_k a' - aa' \otimes_k 1 \tag{105}$$

$$= (1 \otimes_k aa' - a \otimes_k a') + (a \otimes_k a' - aa' \otimes_k 1) \tag{106}$$

$$= (1 \otimes_k a - a \otimes_k 1)a' + a(1 \otimes_k a' - a' \otimes_k 1) \tag{107}$$

$$= d(a)a' + ad(a') \tag{108}$$

2.

$$d(a+b) = 1 \otimes_k (a+b) - (a+b) \otimes_k 1 = 1 \otimes_k a + 1 \otimes_k b - a \otimes_k 1 - b \otimes_k 1 \tag{109}$$

$$= 1 \otimes_k a - a \otimes_k 1 + 1 \otimes_k b - b \otimes_k 1 = d(a) + d(b) \tag{110}$$

3.

$$d(k) = 1 \otimes_k k - k \otimes_k 1 = k \otimes_k 1 - k \otimes_k 1 = 0 \tag{111}$$

In particular the map d in proposition 8 is a k-derivation of A into $\Omega^1(A/k)$. □

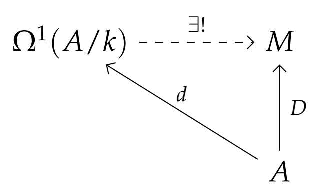

Figure 1 – The universal property of $\Omega^1(A/k)$

Moreover the subsequent result says that for every k-derivation D of A into an (A, A)-bimodule M there exists a unique (A, A)-bimodule map $f : \Omega^1(A/k) \to M$ such that $f \circ d = D$ [7] . Implying:

Proposition 9. ***Universal property of*** $\Omega^1(A/k)$ [8]

If A is a k-algebra and M is an (A, A)-bimodule then there is an isomorphism of A-modules:

$$Hom_{{}_A Mod_A}(\Omega^1(A/k), M) \to Der_k(A, M) \tag{112}$$

Proof. Let $D : A \to M$ be a k-derivation then define the (A, A)-bimodule homomorphism $f : \Omega^1(A/k) \to M$ on $\sum_{i=0}^{n} a_i \otimes_k b_i \in \Omega^1(A/k)$ as:

$$f(\sum_{i=0}^{n} a_i \otimes_k b_i) := \sum_{i=0}^{n} a_i D(b_i). \tag{113}$$

Therefore for any $a \in A$:

$$f(d(a)) = f(1 \otimes_k a - a \otimes_k 1) = -D(1)a + D(a)1 = 0 + D(a) = D(a). \tag{114}$$

Since $d(A)$ generates the (A, A)-bimodule $\Omega^1(A/k)$, the fact that f is an (A, A)-bimodule homomorphism may be verified on the images of d as follows: suppose $a, b, c, e \in A$ then:

$$f(d(a) + bd(c)e) = f(1 \otimes_k a - a \otimes_k 1 + b \otimes_k ce - bc \otimes_k e) \tag{115}$$

$$= -(D(1)a - D(a)1 + D(b)ce - D(bc)e) \tag{116}$$

$$= -(-D(a) + D(b)ce - bD(c)e - D(b)ce) \tag{117}$$

$$= D(a) + bD(c)e \tag{118}$$

7. The universal property of $\Omega^1(A/k)$ is analogous to the universal property of the A-module of Kähler differentials in the case where A was a commutative k-algebra.

8. In other words the functor $Der_k(A, -) : {}_A Mod_A \to {}_A Mod$ is corepresentable by the (A, A)-bimodule $\Omega^1(A/k)$.

Since $(\forall \sum_{i=0}^{n} a_i \otimes_k b_i \in \Omega^1(A/k))\ \sum_{i=0}^{n} a_i b_i = \mu_A(\sum_{i=0}^{n} a_i \otimes_k b_i) = 0$ then:

$$0 = D(0) = D(\sum_{i=0}^{n} a_i b_i) = \sum_{i=0}^{n} D(a_i) b_i + \sum_{i=0}^{n} a_i D(b_i). \tag{119}$$

Therefore (119) implies:

$$\sum_{i=0}^{n} D(a_i) b_i = -\sum_{i=0}^{n} a_i D(b_i). \tag{120}$$

Together (120) and (118) imply:

$$f(d(a) + bd(c)e) = -D(a) - bD(c)e = (-D(a) - 0) + b(-0 - D(c))e \tag{121}$$

$$= (-D(a)1 - 1D(a)) + b(-D(c)1 - cD(1))e \tag{122}$$

$$= f(a) + bf(c)e \tag{123}$$

Therefore f is indeed an (A, A)-bimodule homomorphism. $\square$

Definition 16. $\Omega^n(A/k)$

Let A be a k-algebra and $n \in \mathbb{N}$, then define:

$$\Omega^n(A/k) := Ker(b'_{n-1}) \tag{124}$$

where b'_{n-1} is the $(n-1)^{th}$ differential in the augmented bar resolution of A.

Example 5.

1. $Ker(b'_{-1} : A \to 0) = A = \Omega^0(A/k)$
2. $Ker(b'_0 : A^{\otimes 2} \to A) = \Omega^1(A/k)$

Proof.

1. $Ker(b'_{-1}) = A$.
2. $b'_0 = \mu_A$.

□

2.3 SOME RELATIVE HOMOLOGICAL ALGEBRA

One last ingredient is needed to formulate the first two related results alluded to on page 6 of this master's thesis. This ingredient is a short discussion on the relative homological algebraic framework first introduced in 1967 by Jonathan Mock Beck in his doctoral thesis entitled: *"Triples*[9] *, Algebras and Cohomology"* [TC].

2.3.0.4 $\mathscr{E}_A^k$-*Projective Modules*

Definition 17. ***Projective module***

If A is a k-algebra and P is an A-module, then P is said to be ***projective*** *if and only if for every short exact sequence of A-modules:*

$$0 \longrightarrow M \xrightarrow{\eta} N \xrightarrow{\epsilon} N' \longrightarrow 0 \tag{125}$$

the sequence of k-modules:

$$0 \longrightarrow Hom_A(P,M) \xrightarrow{\eta^\star} Hom_A(P,N) \xrightarrow{\epsilon^\star} Hom_A(P,N') \longrightarrow 0 \tag{126}$$

is exact.

If only certain A-epimorphisms are considered when verifying the universal property of a projective A-module, then there would exist more A-modules which *behave like* projective A-modules. Moreover the acknowledged A-epimorphisms could be fewer thus only the epimorphisms exhibiting some special property could be considered, for example:

Definition 18. $\mathscr{E}_A^k$***-Epimorphism***

9. The word *triple* has fallen out of practice and now is usually referred to as a *monad*.

*For any k-algebra A, an epimorphism ϵ in ${}_A$Mod is an $\mathscr{E}^k_A$-**epimorphism** if and only if ϵ's underlying morphism of k-modules is a k-split epimorphism in ${}_k$Mod.*

The class of these epimorphisms is denoted $\mathscr{E}^k_A$.

Remark 3. *Straightaway from this definition it follows that the class of all epimorphisms in ${}_A$Mod always contains $\mathscr{E}^k_A$ as a subclass (though the containment is not necessarily proper).*

Definition 19. ***$\mathscr{E}^k_A$-Exact sequence***

An exact sequence of A-modules:

$$.. \xrightarrow{\phi_{i-1}} M_i \xrightarrow{\phi_i} M_{i+1} \xrightarrow{\phi_{i+1}} M_{i+2} \xrightarrow{\phi_{i+2}} .. \tag{127}$$

*is said to be $\mathscr{E}^k_A$-**exact** if and only if:*

for every integer i the there exists a morphism of k-modules $\psi_i : M_{i+1} \to M_i$ such that:

$$\phi_i = \phi_i \circ \psi_i \circ \phi_i. \text{ [10]} \tag{128}$$

*In particular as short exact sequence of A-modules which is $\mathscr{E}^k_A$-exact is called an $\mathscr{E}^k_A$-**short exact sequence**.*

Example 6. *The augmented bar complex $\hat{CB_\star}(A)$ of a k-algebra A is $\mathscr{E}^k_{A^e}$-exact.*

Proof. For every $n \in \mathbb{N}$ let $s_n : CB_n(A) \to CB_{n+1}(A)$ be as in (27). s_n is k-linear since:

$$\text{Let } a_0 \otimes_k ... \otimes_k a_{n+1}, a'_0 \otimes_k ... \otimes_k a'_{n+1} \in CB_n(A),\ c \in k \tag{129}$$

$$s_n(a_0 \otimes_k ... \otimes_k a_{n+1} + ca'_0 \otimes_k ... \otimes_k a'_{n+1}) \tag{130}$$

$$= 1 \otimes_k a_0 \otimes_k ... \otimes_k a_{n+1} + c1 \otimes_k a'_0 \otimes_k ... \otimes_k a'_{n+1}) \tag{131}$$

$$= s_n(a_0 \otimes_k ... \otimes_k a_{n+1}) + cs_n(a'_0 \otimes_k ... \otimes_k a'_{n+1}). \tag{132}$$

10. Property (128) is called $\mathscr{E}^k_{A^e}$-admissibility [SA] (alternatively it is called $\mathscr{E}^k_{A^e}$-allowable [MH]).

The s_n show that each b'_n satisfies property (128) since:

$$b'_n \circ s_{n-1} \circ b'_n \tag{133}$$

$$= b'_n \circ (1 + b'_{n+1} \circ s_n) \tag{134}$$

$$= b'_n = b'_n \circ b'_{n+1} \circ s_n \tag{135}$$

Since $b'_\star$ is a coundary map $b_n \circ b_{n+1} = 0$; hence (135) equates to:

$$= b'_n + 0 = b'_n. \tag{136}$$

□

Definition 20. ***$\mathscr{E}^k_A$-Projective module*** [11]

If A is a k-algebra and P is an A-module, then P is said to be ***$\mathscr{E}^k_A$-projective*** *if and only if for every $\mathscr{E}^k_A$-short exact sequence:*

$$0 \longrightarrow M \xrightarrow{\eta} N \xrightarrow{\epsilon} N' \longrightarrow 0 \tag{137}$$

the sequence of k-modules:

$$0 \longrightarrow Hom_A(P, M) \xrightarrow{\eta^\star} Hom_A(P, N) \xrightarrow{\epsilon^\star} Hom_A(P, N') \longrightarrow 0 \tag{138}$$

is exact.

11. This definition is equivalent to requiring that P verify the universal property of projective modules *only* on $\mathscr{E}^k_A$-epimorphisms [MH].

2.3.0.5 *An example: $A^{\otimes n+2}$ is $\mathscr{E}^{k}_{A^e}$-projective for all $n \in \mathbb{N}$.*

Out of convenience it will be proven in a more general form once and for all:

Lemma 3. [12] *If A is a k-algebra and $T :_k Mod \to_A Mod$ is a (contravariant) additive functor then T takes k-split exact sequences to A-split exact sequences.*

Proof. Suppose:

$$0 \longrightarrow M \xrightarrow{\eta} N \xrightarrow{\epsilon} N' \longrightarrow 0 \tag{139}$$

is a split-exact sequence in ${}_kMod^{op}$. Moreover, since (139) is split exact then by definition there are morphisms $s_1 : N \to M$ and $s_2 : N' \to N$ in ${}_kMod$ satisfying $s_1 \circ \eta = 1_M$ and $s_2 \circ \epsilon = 1_{N'}$.

1. $$T(s_1) \circ T(\eta) = T(s_1 \circ \eta) = T(1_M) = 1_{T(M)} \tag{140}$$

   Therefore $T(\eta)$ is split-monic in ${}_AMod$.

2. $$T(\epsilon) \circ T(s_2) = T(\epsilon \circ s_2) = T(1_{N'}) = 1_{T(N')} \tag{141}$$

   Therefore $T(\epsilon)$ is split-epic in ${}_AMod$.

3. If there exists some $x \in T(N)$ such that $\epsilon(x) = 0$ then:

   $$s_2 \circ \epsilon(x) = s_2 \circ 0(x) = 0(x) = 0. \tag{142}$$

   Therefore $1_N(x) = (\eta \circ s_1)(x) + (s_2 \circ \epsilon)(x) = \eta(s_1(x))$; whence $x \in Im(\eta)$. Therefore $Ker(\epsilon) \subseteq Im(\eta)$.

4. $$0 = T(0) = T(\epsilon \circ \eta) = T(\epsilon) \circ T(\eta) \tag{143}$$

12. The dual category of an abelian category is abelian by the *duality principle* [MC] (though ${}_kMod$ need not be a category of Modules).

Therefore $Im(\eta) \subseteq Ker(\epsilon)$.

Therefore:

$$0 \longrightarrow T(M) \xrightarrow{\eta} T(N) \xrightarrow{\epsilon} T(N') \longrightarrow 0 \tag{144}$$

is a split-exact sequence of A-modules.

The contravariant case follows mutatis mutandis (since ${}_k Mod^{op}$ is also an abelian category). □

Proposition 10. *$A^{\otimes n+2}$ is $\mathscr{E}^k_{A^e}$-projective for all $n \in \mathbb{N}$.*

Proof. Suppose (145) is an $\mathscr{E}^k_A$-exact sequence:

$$0 \longrightarrow M \xrightarrow{\eta} N \xrightarrow{\epsilon} N' \longrightarrow 0 \tag{145}$$

Then viewing (145) as a split-exact sequence of k-modules, lemma 3 implies that the additive functors $Hom_k(A^{\otimes n}, -)$ take (145) to an exact sequences of A^e-modules which is A^e-split:

$$0 \longrightarrow Hom_k(A^{\otimes n}, N') \xrightarrow{\eta^\star} Hom_k(A^{\otimes n}, N) \xrightarrow{\epsilon^\star} Hom_k(A^{\otimes n}, M) \longrightarrow 0 \quad . \tag{146}$$

(146) implies the top row of the following diragram of A^e-modules is exact. Furthermore the A^e-module isomorphisms in (43) imply that $Hom_k(A^{\otimes n}, X) \cong Hom_{A^e}(A^{\otimes n+2}, X)$, giving the commutativity of the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Hom_k(A^{\otimes n}, N') & \hookrightarrow & Hom_k(A^{\otimes n}, N) & \twoheadrightarrow & Hom_k(A^{\otimes n}, M) & \longrightarrow & 0 \\ \| & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \| \\ 0 & \longrightarrow & Hom_{A^e}(A^{\otimes n+2}, N') & \hookrightarrow & Hom_{A^e}(A^{\otimes n+2}, N) & \twoheadrightarrow & Hom_{A^e}(A^{\otimes n+2}, M) & \longrightarrow & 0 \end{array} \quad . \tag{147}$$

Whence the bottom row must also be exact [IH]. Therefore $Hom_{A^e}(A^{\otimes n+2}, -)$ takes takes split exact sequences in ${}_{A^e}Mod$ to exact sequences in ${}_kMod$, hence $A^{\otimes n+2}$ is $\mathscr{E}^k_{A^e}$-projective. □

$\mathscr{E}^k_A$-projective A-modules have analogous properties to projective A-modules. For example they admit the following characterization.

Proposition 11. *For any A-module P the following are equivalent:*

$\mathscr{E}^k_A$-SHORT EXACT SEQUENCE PRESERVATION PROPERTY *P is $\mathscr{E}^k_A$-projective.*

$\mathscr{E}^k_A$-LIFTING PROPERTY *For every $\mathscr{E}^k_A$-epimorphism $f : N \to M$ if there exists an A-module morphism $g : P \to M$ then there exists an A-module map $\tilde{f} : P \to N$ such that $f \circ \tilde{f} = g$.*

$\mathscr{E}^k_A$-SPLITTING PROPERTY *Every short $\mathscr{E}^k_A$-exact sequence of the form:*

$$\mathfrak{E}_\pi : 0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0 \tag{148}$$

is A-split-exact.

$\mathscr{E}^k_A$-FREE DIRECT SUMMAND PROPERTY [13] *There exists a k-module F, an A-module Q and an isomorphism of A-modules $\phi : P \oplus Q \xrightarrow{\cong} A \otimes_k F$.*

Proof. See [MH] pages 261 for the equivalence of 1,2 and 3 and page 277 for the equivalence of 1 and 4. □

13. If F is a free k-module, some authors call $A \otimes_k F$ an $\mathscr{E}^k_A$-free module. In fact this gives an alternative proof that $A^e \otimes_k A^{\otimes n} \cong A^{\otimes n+2}$ is $\mathscr{E}^k_{A^e}$-free for every $n \in \mathbb{N}$.

2.3.0.6 $\mathscr{E}_A^k$-homological algebra

Proposition 12. ***Enough $\mathscr{E}_A^k$-projectives***

If A is a k-algebra and M is an A-module then there exists an $\mathscr{E}_A^k$-epimorphism $\epsilon : P \to M$ where P is an $\mathscr{E}_A^k$-projective.

Proof. By proposition 11 $A \otimes_k M$ is $\mathscr{E}_A^k$-projective. Moreover the A-map $\zeta : A \otimes_k M \to M$ described on elementary tensors as $(\forall a \otimes_k m \in A \otimes_k M)\zeta(a \otimes_k m) := a \cdot m$ is epic and is k-split by the section $m \mapsto 1 \otimes_k m$. □

Definition 21. ***$\mathscr{E}_A^k$-projective resolution***

*If M is an A^e-module then a resolution $P_\star$ of M is called an $\mathscr{E}_A^k$**-projective resolution** of M if and only if each P_i is an $\mathscr{E}_A^k$-projective module and $P_\star$ is an $\mathscr{E}_A^k$-exact sequence.*

Example 7. *The augmented bar complex $C\hat{B_\star}(A)$ of A is an $\mathscr{E}_{A^e}^k$-projective resolution of A.*

Proof. In example 2 $C\hat{B_\star}(A)$ was seen to be an acyclic resolution of A. In proposition 10 it was seen that for each $n \in \mathbb{N}$: $C\hat{B_\star}(A)$ was a $\mathscr{E}_{A^e}^k$-projective A^e-module. Finally example 6 implies $C\hat{B_\star}(A)$ is $\mathscr{E}_{A^e}^k$-exact.

Therefore $C\hat{B_\star}(A)$ is an $\mathscr{E}_{A^e}^k$-projective resolution of A. □

Remark 4. *A nearly completely analogous argument to example 7 shows that for any (A, A)-bimodule M, $M \otimes_A C\hat{B_\star}(A)$ is an $\mathscr{E}_{A^e}^k$-projective resolution of M [HI].*

2.3.1 *Relative Homological Algebra*

Nearly all the usual homological algebraic machinery transfers over seamlessly to the relativised framework by making the necessary tweaks [SA] (in fact most arguments are identical with $\mathscr{E}_A^k$ in place of the usual class of *all the epimorphisms of the category $_A Mod$*).

Definition 22. ***$\mathscr{E}^k_A$-relative Tor***

If N is a right A-module, M is an A-module and $P_\star$ is an $\mathscr{E}^k_A$-projective resolution of N then the k-modules $H_\star(P_\star \otimes_A M)$ are called the $\mathscr{E}^k_A$-relative Tor k-modules of N with coefficients in the A-module M and are denoted by $Tor^n_{\mathscr{E}^k_A}(N, M)$.

Remark 5. *The $\mathscr{E}^k_A$-relative Tor functors may differ from the usual (or "absolute") Tor functors. For example consider all the $\mathbb{Z}$-algebra $\mathbb{Z}$, any $\mathbb{Z}$-modules N and M are $\mathscr{E}^{\mathbb{Z}}_{\mathbb{Z}}$-projective. In particular, this is true for the $\mathbb{Z}$-modules $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$. Therefore $Tor^n_{\mathscr{E}^{\mathbb{Z}}_{\mathbb{Z}}}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ vanish for every positive n, however $Tor^n_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ does not. For example, $Tor^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ [IH].* [14]

Similarly there are $\mathscr{E}^k_A$-relative Ext functors:

Definition 23. ***$\mathscr{E}^k_A$-relative Ext***

If N is and M are A-modules and $P_\star$ is an $\mathscr{E}^k_A$-projective resolution of N then the k-modules $H^\star(Hom_A(P_\star, M))$ are called the $\mathscr{E}^k_A$-relative Ext k-modules of N with coefficients in the A-module M and are denoted by $Ext^n_{\mathscr{E}^k_A}(N, M)$.

Remark 6. *The same modules as in remark 5 together with an analogous computation show that a $\mathscr{E}^k_A$-relative Ext functor may differ from an (absolute) Ext functor. Likewise when k is a field they equate [HI].*

Both the definitions of $\mathscr{E}^k_A$-relative Ext and $\mathscr{E}^k_A$-relative Tor are independent of the choice of $\mathscr{E}^k_A$-projective resolution:

Theorem 2. ***$\mathscr{E}^k_A$-Comparison theorem***

If $P_\star$ and $P'_\star$ are $\mathscr{E}^k_A$-projective resolutions of an A-module N then for any A-module M there are natural isomorphisms:

$$H^\star(Hom_{\mathscr{E}^k_A}(P_\star, N)) \xrightarrow{\cong} H^\star(Hom_{\mathscr{E}^k_A}(P'_\star, N)) \tag{149}$$

14. Constrastingly, the two bifunctors $Tor_A(-,-)$ and $Tor_{\mathscr{E}^k_A}(-,-)$ may be identical in some cases (for example when the basering is a field) [HI].

and if $P_\star$ and $P'_\star$ are $\mathscr{E}^k_A$-projective resolutions of a right A-module N then:

$$H_\star(P_\star \otimes_A N) \xrightarrow{\cong} H_\star(P'_\star \otimes_A N) \tag{150}$$

Proof. Nearly identical to the usual comparison theorem, see [MH]. □

For any A-module M $Ext^\star_{\mathscr{E}^k_A}(M,-)$ may behave analogously to the $Ext_A(M,-)$, for example:

Proposition 13. *If X is an A-module and $0 \to N' \to N \to N'' \to 0$ is an $\mathscr{E}^k_A$-short exact sequence then there exists a long exact sequences of k-modules:*

$$... \to Ext^{n+1}_{\mathscr{E}^k_A}(X,N'') \xrightarrow{\partial^{n+1}} Ext^n_{\mathscr{E}^k_A}(X,N') \longrightarrow Ext^n_{\mathscr{E}^k_A}(X,N) \longrightarrow Ext^n_{\mathscr{E}^k_A}(X,N'') \xrightarrow{\partial^n} Ext^{n-1}_{\mathscr{E}^k_A}(X,N') \to ...$$

and

$$... \to Ext^{n+1}_{\mathscr{E}^k_A}(N',X) \xrightarrow{\partial^{n+1'}} Ext^n_{\mathscr{E}^k_A}(N'',X) \longrightarrow Ext^n_{\mathscr{E}^k_A}(N,X) \longrightarrow Ext^n_{\mathscr{E}^k_A}(N',X) \xrightarrow{\partial^{n'}} Ext^{n-1}_{\mathscr{E}^k_A}(N'',X) \to ...$$

Proof. See [RH] page 253. □

Instead of providing a proof of proposition 13, which is analogous to the classical case of Ext_A, it will now instead be shown that proposition 13 need not hold for short exact sequences (which aren't $\mathscr{E}^k_A$-exact). That is $Ext^\star_{\mathscr{E}^k_A}(X,-)$ (resp. $Ext^\star_{\mathscr{E}^k_A}(-,X)$) need not take a short exact sequence to a long exact sequence in general. An issue here is that there exist short exact sequences which *do not* extend to a short exact sequence of $\mathscr{E}^k_A$-projective resolutions (that is a short exact sequences of complexes, such that each complex is an $\mathscr{E}^k_A$-projective resolution).

Example 8. *$\mathbb{Z}/2\mathbb{Z}$ is an $\mathscr{E}_{\mathbb{Z}}^{\mathbb{Z}}$-projective module and*

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0 \tag{151}$$

is a short exact sequence of $\mathbb{Z}$-modules which is not $\mathscr{E}_{\mathbb{Z}}^{\mathbb{Z}}$-short-exact.

Furthermore the exact sequence of $\mathbb{Z}$-modules:

$$0 \to 0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0 \tag{152}$$

is an $\mathscr{E}_{\mathbb{Z}}^{\mathbb{Z}}$-short exact sequence.

Proof. Since $\mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$, proposition 11 implies $\mathbb{Z}/2\mathbb{Z}$ is an $\mathscr{E}_{\mathbb{Z}}^{\mathbb{Z}}$-projective module.

Moreover (151) cannot be $\mathbb{Z}$-split or else $\mathbb{Z}/2\mathbb{Z}$ would be a torsion $\mathbb{Z}$-submodule of the torsion free $\mathbb{Z}$-module $\mathbb{Z}$. □

The $Ext_{\mathbb{Z}}$ and $\mathscr{E}_{\mathbb{Z}}^{\mathbb{Z}}$-relative Ext may differ:

Example 9. *$Ext^1_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and $Ext^1_{\mathscr{E}_{\mathbb{Z}}^{\mathbb{Z}}}(\mathbb{Z},\mathbb{Z}/2\mathbb{Z}) \cong 0$*

Proof. Since (152) is a $\mathscr{E}_{\mathbb{Z}}^{\mathbb{Z}}$-projective resolution of the $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$, there are natural isomorphisms of $\mathbb{Z}$-modules:

$$Ext^1_{\mathscr{E}_{\mathbb{Z}}^{\mathbb{Z}}}(\mathbb{Z},\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}. \tag{153}$$

In contrast, since (152) is a $\mathscr{E}_{\mathbb{Z}}^{\mathbb{Z}}$-projective resolution of the $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$ then theorem 2 implies:

$$Ext^1_{\mathscr{E}_{\mathbb{Z}}^{\mathbb{Z}}}(\mathbb{Z},\mathbb{Z}/2\mathbb{Z}) \cong 0/0 \cong 0. \tag{154}$$

□

Proposition 14. ***Dimension Shifting***

If

$$... \overset{d_{n+1}}{\to} P_n \xrightarrow{d_n} ... \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow 0 \tag{155}$$

is a deleted $\mathscr{E}_A^k$-projective resolution of an A-module M then for every A-module N and for every positive integer n there are isomorphisms natural in N:

$$Ext^1_{\mathscr{E}_A^k}(Ker(d_n), N) \cong Ext^{n+1}_{\mathscr{E}_A^k}(A, N) \tag{156}$$

Proof. By definition the truncated sequence is exact:

$$... \overset{d_{n+j}}{\to} P_{n+j} \xrightarrow{d_{n+j-1}} ... \xrightarrow{d_{n+1}} P_{n+1} \xrightarrow{\eta} Ker(d_n) \longrightarrow 0 \quad , \tag{157}$$

where η is the canonical map satisfying $d_n = ker(d_n) \circ \eta$ (arising from the universal property of $ker(d_n)$). Moreover since (155) is $\mathscr{E}_A^k$-exact, d_n is k-split; whence η must be k-split. Moreover for every $j \geq n+1$, d_j was by assumption k-split therefore (157) is $\mathscr{E}_A^k$-exact and since for every natural number $m > n$ P_m is by hypothesis $\mathscr{E}_A^k$-projective then (157) is an augmented $\mathscr{E}_A^k$-projective resolution of the A-module $Ker(d_n)$.

For every natural number m, relabel:

$$Q_m := P_{m+n} \text{ and } p_m := d_{n+m}. \tag{158}$$

By theorem 2:

$$(\forall N \in_A Mod)(\forall m \in \mathbb{N}) \; Ext^m_{\mathscr{E}_A^k}(Ker(d_n), N) \cong H^m(Hom_A(Q_\star, N)) \tag{159}$$

$$= Ker(Hom_A(p_n, N))/Im(Hom_A(p_{n+1}, N)) \tag{160}$$

$$= Ker(Hom_A(d_{n+m}, N))/Im(Hom_A(d_{n+m+1}, N)) \tag{161}$$

$$= H^{m+n}(Hom_A(P_\star, N)) \tag{162}$$

$$\cong Ext^m_{\mathscr{E}^k_A}(A, N). \tag{163}$$

□

Analogous to the fact that for any A-module P, P is projective if and only if $Ext^1_A(P, N) \cong 0$ for every A-module N there is the following result:

Proposition 15. *P is an $\mathscr{E}^k_A$-projective module if and only if for every A-module N:*

$$Ext^1_{\mathscr{E}^k_A}(P, N) \cong 0 \tag{164}$$

Proof.

— Suppose for every A-module $Ext^1_{\mathscr{E}^k_A}(P, N) \cong 0$ and let

$$0 \to N' \to N \to N'' \to 0 \tag{165}$$

be an $\mathscr{E}^k_A$-short exact sequence of A-modules. Proposition 13 implies there is an exact sequence:

$$Ext^1_{\mathscr{E}^k_A}(A, N'') \xrightarrow{\partial^1} Hom^n_A(P, N') \longrightarrow Hom^n_A(P, N) \longrightarrow Hom^n_A(P, N'') \longrightarrow 0$$

Since it was assumed that $Ext^1_{\mathscr{E}^k_A}(P, N) \cong 0$ then:

$$0 \longrightarrow Hom^n_A(P, N') \longrightarrow Hom^n_A(P, N) \longrightarrow Hom^n_A(P, N'') \longrightarrow 0$$

is exact whence $Hom_A(P, -)$ takes $\mathscr{E}^k_A$-short exact sequences to short exact sequences, therefore P is $\mathscr{E}^k_A$-projective.

— Conversely, since P is an $\mathscr{E}_A^k$-projective module:

$$.. \to 0 \to ... \to 0 \to P \overset{1_P}{\to} P \to 0 \tag{166}$$

is an $\mathscr{E}_A^k$-projective resolution of P of length 0. We denote its corresponding deleted $\mathscr{E}_A^k$-projective resolution by $\mathfrak{P}_\star$. Whence by theorem 2:

$$(\forall X \in_A Mod)\ Ext^1_{\mathscr{E}_A^k}(P, X) \cong H^1(Hom_A(\mathfrak{P}_\star, X)) \cong 0. \tag{167}$$

□

2.3.2 *The Hochschild Cohomology as the $Ext_{\mathscr{E}_{A^e}^k}(A, -)$ functors*

Since $CB_\star(A)$ is an $\mathscr{E}_{A^e}^k$-projective resolution of A then theorem 2 and the definition of the $Ext^\star_{\mathscr{E}_{A^e}^k}(A, -)$ functors imply that:

Proposition 16. *For every A^e module N there are k-module isomorphisms, natural in N:*

$$HH^\star(A, N) \overset{\cong}{\to} Ext^\star_{\mathscr{E}_{A^e}^k}(A, N) \tag{168}$$

Taking short $\mathscr{E}_{A^e}^k$-exact sequences to isomorphic long exact sequences.

Definition 24. ***Hochschild Homology*** [15] [16]

The Hochschild homology $HH_\star(A, N)$ of a k-algebra A with coefficient in the (A, A)-bimodule N is defined as:

$$HH_\star(A, N) := H_\star(P_\star \otimes_A N) \tag{169}$$

15. If A is a commutative k-algebra of essentially-finite type and k is Noetherian then $HH_\star(A, A) \cong \Omega^n_{A|k}$, where $\Omega^n_{A|k}$ are the Kähler n-forms [HI], therefore the Hochschild homology provides yet another noncommutative analogue of $\Omega^n(A|k)$.

16. There is a duality relationship between the Hochschild cohomology and the Hochschild Homology modules of a $\mathbb{C}$-algebra explored in [RG]. In the case where A is the coordinate ring of a smooth affine algebraic $\mathbb{C}$-variety this relationship becomes even clearer [PH].

where $P_\star$ is an $\mathscr{E}^k_{A^e}$-projective resolution of A.

2.3.3 *Two Cohomological Dimensions*

The Hochschild cohomological dimension is the numerical invariant of prime focus in this master's thesis. All the results presented herein revolve around it.

Definition 25. ***Hochschild cohomological dimension***

*The **Hochschild cohomological dimension** of a k-algebra A is defined as:*

$$HCdim(A/k) := \sup_{M \in {}_{A^e}Mod} (sup\{n \in \mathbb{N}^\# | HH^n(A, M) \not\cong 0\}). \tag{170}$$

Where $\mathbb{N}^\#$ is the ordered set of extended natural numbers.

The Hochschild cohomological dimension may be related to the following cohomological dimension and to $\Omega^n(A/k)$ as will be shown in theorem 3 below.

Definition 26. $\mathscr{E}^k_A$***-projective dimension***

*If n is an natural number and M is an A-module then M is said to be of $\mathscr{E}^k_A$**-projective dimension** at most n if and only if there exists a deleted $\mathscr{E}^k_A$-projective resolution of M of length n.*

If no such $\mathscr{E}^k_A$-projective resolution of M exists then M is said to be of $\mathscr{E}^k_A$-projective dimension ∞.

The $\mathscr{E}^k_A$-projective dimension of M is denoted $pd_{\mathscr{E}^k_A}(M)$.

The following is a translation of a classical homological algebraic result into the setting of $\mathscr{E}^k_{A^e}$-projective dimension, $\Omega^n(A/k)$ and Hochschild cohomology:

Theorem 3. *For every natural number n, the following are equivalent:*

1. $HCdim(A/k) \leq n$
2. *A is of* $\mathscr{E}^k_{A^e}$*-projective dimension at most n*
3. $\Omega^n(A/k)$ *is an* $\mathscr{E}^k_{A^e}$*-projective module.*
4. $HH^{n+1}(A, M)$ *vanishes for every* (A, A)*-bimodule M.*
5. $Ext^{n+1}_{\mathscr{E}^k_{A^e}}(A, M)$ *vanishes for every* A^e*-module M.*

Proof.

1 ⇒ 4 By definition of the Hochschild cohomological dimension.

4 ⇔ 5 By proposition 16.

3 ⇒ 2 Since $\Omega^n(A/k)$ is $\mathscr{E}^k_{A^e}$-projective:

$$0 \to \Omega^n(A/k) \to CB_{n-1}(A) \overset{b'_{n-1}}{\to} \ldots \overset{b'_0}{\to} A \to 0 \tag{171}$$

is a $\mathscr{E}^k_{A^e}$-projective resolution of A of length n. Therefore $pd_{\mathscr{E}^k_{A^e}}(A) \leq n$.

3 ⇔ 4 By proposition 14 there are isomorphism natural in M:

$$(\forall M \in_{A^e} Mod) HH^{1+n}(A, M) \cong Ext^{1+n}_{\mathscr{E}^k_{A^e}}(A, M) \tag{172}$$

$$\cong Ext^1_{\mathscr{E}^k_{A^e}}(\Omega^n(A/k), M). \tag{173}$$

Therefore for every A^e-module M:

$$Ext^1_{\mathscr{E}^k_{A^e}}(\Omega^n(A/k), M) \cong 0 \text{ if and only if } HH^{1+n}(A, M) \cong 0. \tag{174}$$

By proposition 15 $\Omega^n(A/k)$ is $\mathscr{E}^k_A$-projective if and only if

$$Ext^1_{\mathscr{E}^k_{A^e}}(\Omega^n(A/k), M) \cong 0. \tag{175}$$

2 ⇒ 1 If A admits an $\mathscr{E}^k_{A^e}$-projective resolution $P_\star$ of length n then theorem 2 implies there are natural isomorphisms of A^e-modules:

$$(\forall j \in \mathbb{N})(\forall M \in_{A^e} Mod) Ext^\star_{\mathscr{E}^k_{A^e}}(A, M) \cong H^\star(Hom_{A^e}(P_\star, M)). \tag{176}$$

Since $P_\star$ is of length n all the maps $p_j : P_{j+1} \to P_j$ are the zero maps therefore so are the maps $p_j^\star : Hom_{A^e}(P_j) \to Hom_{A^e}(P_{j+1})$. Whence (176) entails that for all $j > n+1$ $Ext^\star_{\mathscr{E}^k_{A^e}}(A, M)$ vanishes. By proposition 16 this is equivalent to $HH^j(A, M)$ vanishing for all $j > n+1$ for all $M \in_{A^e} Mod$. Hence A is of Hochschild cohomological dimension at most n.

□

Note 2. *If A is a k-algebra then a minor modification of the above argument (using an $\mathscr{E}^k_A$-projective resolution of an A-module N in place of the bar resolution of the A^e-module A), it can be verified that for any extended natural number n and any A-module N, N is of $\mathscr{E}^k_A$-projective dimension at most n if and only if $Ext^m_{\mathscr{E}^k_A}(N, M) \cong 0$ for all $M \in_A Mod$ and for all $m \geq n$.*

2.4 ANALYSING PROPERTIES OF k-ALGEBRAS VIA THEIR HOCHSCHILD COHOMOLOGICAL DIMENSION

The use of the Hochschild cohomology is that it may be used to characterise quasi-free k-algebras (to be defined below in definition 28). Originally theorem 3 was shown over a field by Cuntz and Quillen for $n = 0, 1$; however here we extend it further to any commutative base ring k and to any n.

2.4.1 $HCdim(A/k) = 0$ *and Inner Derivations*

Corollary 1 generalises a result of Cuntz and Quillen's beyond the case where k is a field:

Corollary 1. [17] *The following are equivalent:*

1. *$HCdim(A/k) = 0$*
2. *A is a projective $\mathcal{E}^k_{A^e}$-module.*
3. *All derivations of A into an A^e-module M are inner.*

Proof. By theorem 3: 1 and 2 are equivalent with $HH^1(A, M) \cong 0$ for every A^e-module M; lemma 5 then rephrases this as saying $Inn_k(A, M) = Der_k(A, M)$ for every A^e-module M. □

17. Over a field k-algebras satisfying any of these properties were called separable by Cuntz and Quillen in [AE].

Example 10. [18] *$\mathbb{Z}$ is a $\mathscr{E}_{\mathbb{Z}}^{\mathbb{Z}}$-projective $\mathbb{Z}$-algebra.*

Proof. $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z}$ therefore $\mathbb{Z}$ is a direct summand of $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}$. Whence the $\mathbb{Z}$-algebra $\mathbb{Z}$ is $\mathscr{E}_{\mathbb{Z}}^{\mathbb{Z}}$-projective by proposition 11. □

Example 11. *All the $\mathbb{Z}$-derivations of $\mathbb{Z}[x_i]_{i\in\mathbb{N}}$ into a $\mathbb{Z}[x_i]_{i\in\mathbb{N}}^e$-module are inner.*

Proof. $\mathbb{Z}[x_i]_{i\in\mathbb{N}}^e \cong \mathbb{Z}[x_i]_{i\in\mathbb{N}\times\mathbb{N}} \cong \mathbb{Z}[x_i]_{i\in\mathbb{N}}$ therefore $\mathbb{Z}[x_i]_{i\in\mathbb{N}}$ is $\mathbb{Z}[x_i]_{i\in\mathbb{N}}^e$-free; whence corollary 1 applies. □

18. The only examples of k-algebras A which satisfy $HCdim(A/k) = 0$ appearing in the literature are k-algebra over a field k those which are Morita equivalent to A. This example is over a ring which isn't a field but it is still (trivially) Morita equivalent to the base ring $\mathbb{Z}$.

2.4.2 $HCdim(A/k) \leq 1$ and Square-Zero Extensions

In the case where k is a field the following corollary yields a result of Cuntz and Quillen's [AE].

Definition 27. ***Lifting of a square-zero k-Hochschild extension***

Let M be an (A, A)-bimodule and:

$$\mathfrak{E}: 0 \longrightarrow M \hookrightarrow B \overset{\pi}{\twoheadrightarrow} A \longrightarrow 0 \tag{177}$$

*be a k-Hochschild extension. Then (177) is said to **lift** if there is a section s of π which is a k-algebra homomorphism.*

Example 12. *Let A be a k-algebra and M be an (A, A)-bimodule.*

The trivial k-Hochschild extension of A by M lifts.

Proof.

— The zero k-map $0 : A \to A \oplus M$ always exists since ${}_{A^e}Mod$.

— The zero map: $A \to A \rtimes_0 M$ is a noncommutative k-algebra homomorphism since:

$$(\forall c \in k)(\forall a, a', a'' \in A)0(aa' + ka'') = 0 = 0(a)0(a') + c0(a''). \tag{178}$$

□

Lemma 4. *Let A be a k-algebra, M be an (A, A)-bimodule and*

$$0 \to M \to B \overset{\pi}{\to} A \to 0 \tag{179}$$

be a k-Hochschild extension of A by M.

Then (179) lifts if and only if (179) is equivalent to the trivial extension.

In particular there is always precisely one M, A-Hochschild class of k-Hochschild extensions that contains a k-Hochschild extension that lifts.

Proof. (179) lifts if and only if there exists a section $s : A \to B$ of π satisfying:

$$(\forall a, a' \in A)\ s(aa') = s(a)s(a') \tag{180}$$

if and only if:

$$(\forall a, a' \in A)\ s(aa') - s(a)s(a') = 0 \tag{181}$$

if and only if:

$$(\forall a, a' \in A)\ \mathfrak{B}_s(a \otimes_k a') = 0. \tag{182}$$

Since the M, A-Hochschild class of (179) is independent of the choice of section of π then (179) lifts if and only if there exists a section s of π such that $\mathfrak{B}_s = 0$; that is (179) lifts if and only if (179) is equivalent to the trivial Hochschild extension of A by M.

Furthermore since the trivial k-Hochschild extension always exists there is always precisely one M, A-Hochschild class of k-Hochschild extensions equivalent to a k-Hochschild extension that lifts. □

Corollary 2. [19]

For a k-algebra A, the following are equivalent:

1. *A is $HCdim(A/k) \leq 1$.*
2. *$\Omega^1(A/k)$ is a $\mathscr{E}^k_{A^e}$-projective A^e-module.*
3. *All k-Hochschild extensions of A by an (A, A)-bimodule lift.*

Proof. Theorem 3 implies 1 and 2 are equivalent to $HH^2(A, M) \cong 0$ for all A^e-modules M. Lemma 4 implies all extensions of A into an (A, A)-bimodule M lift if and only if there is only one M, A-Hochschild class for all (A, A)-bimodules M. Since $HH^2(A, M)$

19. Cuntz and Quillen prove many other results related to quasi-free k-algebras in their article [AE].

is naturally in bijection with the set of M, A-Hochschild classes, all extensions of A into an (A, A)-bimodule M lift if and only if $HH^2(A, M)$ has only one element for all (A, A)-bimodules M if and only if $HH^2(A, M) \cong 0$ for all (A, A)-bimodules M. □

Definition 28. ***Quasi-free*** k***-algebra*** [20]

Any k-algebra satisfying any of the equivalent conditions in corollary 2 *is called a* ***quasi-free*** k***-algebra****.*

2.4.2.1 *An Example*

Definition 29. ***Tensor Algebra on*** M ***over*** B

If B is a k-algebra and M is a (B, B)*-bimodule then the* ***Tensor Algebra on*** M ***over*** B*, denoted* $T_B(M)$ *is the B-algebra defined as:*

$$T_B(M) := B \oplus \bigoplus_{n \in \mathbb{Z}^+} \bigotimes_B^n M \tag{183}$$

with multiplication defined on elementary tensors as:

$$(e_1 \otimes ... \otimes_k e_j) \times (\tilde{e_1} \otimes ... \otimes_k \tilde{e_k}) \mapsto e_1 \otimes ... \otimes_k e_j \otimes \tilde{e_1} \otimes ... \otimes_k \tilde{e_k}. \tag{184}$$

A direct verification shows:

Proposition 17. *The tensor algebra is a (unital associative) B-algebra.*

Proof. See the third chapter of [BA]. □

20. First introduced by Cuntz and Quillen in [AE], due to their lifting property the quasi-free k-algebras are considered a noncommutative analogue to smooth k-algebras; that is k-algebras for which $\Omega_{A|k}$ is a projective A-module.

Proposition 18. ***Universal Property of the tensor algebra***

Let A be a k-algebra, M be an (A,A)-bimodule and define the (A,A)-bimodule homomorphism $f : M \to T_A(M)$ as:

$$(\forall m \in M) f(m) = (0, m, 0, ..., 0, ...). \tag{185}$$

For every homomorphism of k-algebras $h : A \to B$ (giving B the structure of an (A,A)-bimodule) and for every (A,A)-bimodule homomorphism $g : M \to B$ there is exists a unique A-algebra homomorphism $\phi : T_A(M) \to B$ whose underlying A-module homomorphism satisfies $\phi \circ f = g$.

Proof. Let B be a k-algebra whose A-algebra structure is given by the k-algebra homomorphism $h : A \to B$ and let $g : M \to A$ be an (A,A)-bimodule homomorphism. We construct the k-algebra homomorphism ϕ extending h whose underlying A-module homomorphism satisfies $\phi \circ f = g$.

For every positive integer n, the map:

$$g'_n : \bigotimes_A^n M \to A \tag{186}$$

$$\text{defined as: } (\forall m_1, .., m_n \in M)\ g'_n(m_1 \times ... \times m_n) \mapsto g(m_1)....g(m_n) \tag{187}$$

is n-fold A-linear (on the right and on the left). By the universal property of the n-fold tensor product there exists a unique A-linear (on the right and on the left) map:

$$\phi_n : \bigotimes_A^n M \to A \tag{188}$$

$$\text{satisfying: } (\forall m_1, .., m_n \in M)\ \phi_n(m_1 \otimes_A ... \otimes_A m_n) \mapsto g(m_1)....g(m_n). \tag{189}$$

Relabel the k-algebra homomorphism h as ϕ_0. Define the k-module homomorphism:

$$\phi := \bigoplus_{n \in \mathbb{N}} \phi_n : T_A(M) \to A. \tag{190}$$

In fact ϕ is a k-algebra homomorphism since for every $(m_1 \otimes_A ... \otimes_A m_k)$, $(m_1 \otimes_A ... \otimes_A m_j) \in T_k(M)$:

$$\phi((m_1 \otimes_A ... \otimes_A m_k)(m_1 \otimes_A ... \otimes_A m_j)) \tag{191}$$

$$= \phi(m_1) \otimes_A ... \otimes_A \phi(m_k)\phi(m_1) \otimes_A ... \otimes_A \phi(m_j) \tag{192}$$

$$= \phi((m_1 \otimes_A ... \otimes_A m_k))\phi((m_1 \otimes_A ... \otimes_A m_j)). \tag{193}$$

Finally, by construction:

$$(\forall m \in M)\ \phi \circ f(m) = \phi_1(m) = g(m). \tag{194}$$

□

Lemma 5. [21]

If A is a quasi-free k-algebra and P is an $\mathscr{E}^k_{A^e}$-projective (A, A)-bimodule then $T_A(P)$ is a quasi-free A-algebra.

Proof. Let

$$0 \to M \to B \overset{\pi}{\to} T_A(P) \to 0 \tag{195}$$

be a k-Hochschild extension of $T_A(P)$ by M. We use the universal property of $T_A(P)$ to show that there must exist a lift l of (195).

Let $p : T_A(P) \to A$ be the projection k-algebra homomorphism of $T_A(P)$ onto A. p is k-split since the k-module inclusion $i : A \to T_A(P)$ is a section of p; therefore p is an $\mathscr{E}^k_{A^e}$-epimorphism and

$$0 \to Ker(p \circ \pi) \to B \to A \to 0 \tag{196}$$

is a k-Hochschild extension of A by the (A, A)-bimodule $Ker(p \circ \pi)$. Since A is a quasi-free k-algebra there exists a k-algebra homomorphism $l_1 : A \to B$ lifting $p \circ \pi$. Hence B

21. Cuntz and Quillen proved lemma 5 in the case where k was a field.

inherits the structure of an (A, A)-bimodule and π may be viewed as an (A, A)-bimodule homomorphism. Moreover l_1 induces an A-algebra structure on B.

Let $f : P \to T_A(P)$ be the (A, A)-bimodule homomorphism satisfying the universal property of the tensor algebra on the (A, A)-bimodule P. Since $\pi : B \to A$ is an $\mathscr{E}^k_{A^e}$-epimorphism and since P is an $\mathscr{E}^k_{A^e}$-projective (A, A)-bimodule, proposition 11 implies that that there exists an (A, A)-bimodule homomorphism $l_2 : P \to B$ satisfying $\pi \circ l_2 = f$.

Since $l_2 : P \to B$ is an (A, A)-bimodule homomorphism to a A-algebra the universal property of the tensor algebra $T_A(P)$ on the (A, A)-bimodule P (proposition 18) implies there is an A-algebra homomorphism $l : T_A(P) \to B$ whose underlying function satisfies: $l \circ f = l_2$.

Therefore $l \circ \pi \circ l_2 = l_2$; whence $l \circ \pi = 1_{T_A(P)}$; that is l is a A-algebra homomorphism which is a section of π, that is l lifts π. □

Example 13. *Let $n \in \mathbb{N}$. The $\mathbb{Z}$-algebra $T_{\mathbb{Z}}(\bigoplus_{i=0}^{n} \mathbb{Z})$*[22] *is quasi-free.*

Proof. Since all free $\mathbb{Z}$-modules are projective $\mathbb{Z}$-modules and all projective $\mathbb{Z}$-modules are $\mathscr{E}^{\mathbb{Z}}_{\mathbb{Z}}$-projective modules, the free $\mathbb{Z}$-module $\bigoplus_{i=0}^{n} \mathbb{Z}$ is $\mathscr{E}^{\mathbb{Z}}_{\mathbb{Z}}$-projective. Whence lemma 5 implies $T_{\mathbb{Z}}(\bigoplus_{i=0}^{n} \mathbb{Z})$ is a quasi-free $\mathbb{Z}$-algebra. □

Example 14. *If A is a quasi-free k-algebra then $T_A(\Omega^1(A/k))$ is a quasi-free A-algebra.*

Proof. By corollary 2 if A is quasi-free $\Omega^1(A/k)$ must be an $\mathscr{E}^k_{A^e}$-projective (A, A)-bimodule; whence lemma 5 applies. □

22. The $\mathbb{Z}$-algebra $T_{\mathbb{Z}}(\bigoplus_{i=0}^{n} \mathbb{Z})$ is called a **free associative $\mathbb{Z}$-algebra on $n+1$ letters**.

2.5 CUNTZ-QUILLEN N-FORMS

In their paper [AE] Cuntz and Quillen define noncommutative n-forms in a different manner than in this master's thesis. This portion of this master's thesis now closes with a short side-note describing the similarities between these two notions. Explicitly it is shown that $\Omega^n(A/k)$ is $\mathscr{E}^k_{A^e}$-projective if and only if the A^e-module of Cuntz-Quillen n-forms is $\mathscr{E}^k_{A^e}$-projective. Theorem 3 is then reformulated in terms of the Cuntz-Quillen n-forms.

Denote by $\bar{A}$ the k-module A/k. The A^e-modules $\Omega^n_k(A)$ have the following homological description (reminiscent of $\Omega^n(A/k)$).

Proposition 19. ***Normalized Bar Resolution***

If A is a k-algebra then there is an $\mathscr{E}^k_{A^e}$-projective resolution of A denoted by $\bar{CB}_\star(A)$ called the ***normalized bar Resolution of*** *A defined as:*

$$\bar{CB}_n(A) := A \otimes_k \bar{A}^{\otimes n} \otimes_k A \tag{197}$$

Whose boundary operators are defined as:

$$\bar{b}'_n(a_0 \otimes ... \otimes a_{n+1}) := \sum_{i=0,..,n} (-1)^i a_0 \otimes ... \otimes \bar{a}_i \bar{a_{i+1}} \otimes ... \otimes a_{n+1} \tag{198}$$

(By convention: b'_0 is the augmentation map $A \otimes_k A \to A$ and b'_{-1} is the zero map from A to 0).

Proof. The proof is analogous to example 7 and can be found on page 281 of [MH]. □

Definition 30. ***Cuntz-Quillen n-Forms***

For any natural number n and any k-algebra A the module of n-Cuntz-Quillen forms on A is defined as:

$$\Omega^n_k(A) := Ker(\bar{b}'_{n-1} : \bar{CB}_n \to \bar{CB}_{n-1}) \tag{199}$$

By "coincidence" there are the following examples:

Example 15.

1. $\Omega^1(A/k) = \Omega^1_k(A)$
2. $\Omega^0(A/k) = \Omega^0_k(A)$

Proof. By definition $b'_0 = 0 = \bar{b}'_0$ and $b'_1 = \mu_A = \bar{b}'_1$. Therefore proposition 30 together with the definition of $\Omega^1(A/k)$ and $\Omega^0(A/k)$ imply the conclusion. □

The "coincidence" of example 15 in fact runs deeper:

Proposition 20. *If A is a k algebra and n is a natural number then the following are equivalent:*

— $\Omega^n(A/k)$ *is* $\mathscr{E}^k_{A^e}$*-projective.*

— $\Omega^n_k(A)$ *is* $\mathscr{E}^k_{A^e}$*-projective.*

Proof. Example 7 implied that $CB_\star(A)$ is an $\mathscr{E}^k_{A^e}$-projective resolution of A; likewise proposition 19 implies that $\bar{CB}_\star(A)$ is also an $\mathscr{E}^k_{A^e}$ projective resolution of A.

Whence theorem 2 entails that for every A^e-module M there are natural isomorphisms:

$$(\forall n \in \mathbb{N}) H^n(Hom_{A^e}(CB_\star(A), M)) \overset{\cong}{\to} Ext^n_{\mathscr{E}^k_{A^e}}(A, M) \overset{\cong}{\to} H^n(Hom_{A^e}(\bar{CB}_\star(A), M)). \quad (200)$$

By definition $\Omega^n(A/k)$ is the n^{th} syzygy [23] of $CB_\star(A)$, likewise proposition 30 implies $\Omega^n_k(A)$ is the n^{th} syzygy of $\bar{CB}_\star(A)$ therefore for every A^e-module M there are natural isomorphisms:

$$(\forall i, n \in \mathbb{N}) \text{ with } i > 0\text{: } Ext^i_{\mathscr{E}^k_{A^e}}(\Omega^n(A/k), M) \overset{\cong}{\to} Ext^{i+n}_{\mathscr{E}^k_{A^e}}(A, M) \overset{\cong}{\to} Ext^i_{\mathscr{E}^k_{A^e}}(\Omega^n_k(A), M) \text{ [IH]}. \quad (201)$$

Therefore (201) implies: $\Omega^n(A/k)$ is $\mathscr{E}^k_{A^e}$-projective [MH] if and only if
for every positive integer i: $Ext^i_{\mathscr{E}^k_{A^e}}(\Omega^n(A/k), M)$ vanishes for all A^e-modules M
if and only if $Ext^i_{\mathscr{E}^k_{A^e}}(\Omega^n_k(A), M)$ vanishes for every A^e-module M
if and only if $\Omega^n_k(A)$ is $\mathscr{E}^k_{A^e}$-projective [MH]. □

23. The n^{th} **syzygy** of a chain complex $< C_\star, \partial_\star >$ is the kernel of n^{th} boundary map ∂_n.

2.5.1 *Reformulating Theorem* 3

Theorem 3 may now be expressed in terms of the Cuntz-Quillen n-forms.

Theorem 4. *For every natural number n, the following are equivalent:*

1. $HCdim(A/k) \leq n$
2. *A is of $\mathscr{E}^k_{A^e}$-projective dimension at most n*
3. *$\Omega^n(A/k)$ is an $\mathscr{E}^k_{A^e}$-projective module.*
4. *$\Omega^n_k(A)$ is an $\mathscr{E}^k_{A^e}$-projective module.*
5. *$HH^{n+1}(A, M)$ vanishes for every (A, A)-bimodule M.*
6. *$Ext^{n+1}_{\mathscr{E}^k_{A^e}}(A, M)$ vanishes for every A^e-module M.*

Proof.

The equivalence of 1, 2, 3 , 5 and 6 follow from theorem 3. The equivalence of 3 and 4 are entailed by proposition 20. □

3

A LOWER BOUND FOR THE HOCHSCHILD COHOMOLOGICAL DIMENSION

3.1 A FEW HOMOLOGICAL DIMENSIONS

Assumption 1. *Unless otherwise specified, for the remainder of this text any k-algebra will always be commutative.*

In commutative setting we now provide a method of obtaining examples of k-algebras which are not quasi-free. More generally the purpose of this section is to provide a lower-bound for the Hochschild Cohomological dimension of certain commutative k-algebras.

The argument revolves around bounding the Hochschild dimension of a regular commutative k-algebra (to be defined below in definition 39) below via a series of intermediary numerical invariants associated to the algebra A.

3.1.1 *Regular Sequences And Flat Dimension*

Definition 31. ***Regular element***

*Let A be a commutative ring and M be an A-module. A non-zero element x in a commutative ring A is said to be M-**regular** (or an M-regular element), if and only if the A-module map $\lambda_x : M \to M$ defined on elements m of M as $m \mapsto x \cdot m$ is an injection and not a surjection.*

*If M is the A-module A then x is simply said to be **regular** (or a regular element) on A.*

Example 16. *If k is a commutative integral domain then the element x in $k[x]$ is regular on $k[x]$.*

Proof. k is an integral domain then $k[x]$ is an integral domain [AA]. Thus multiplication by any element on the left *(x in particular)* is injective. Moreover x is by definition not a unit in $k[x]$. □

Definition 32. *M**-Regular sequence***

*Let A be a commutative ring and M be an A-module. A sequence of elements $x_1, .., x_n$ in A is called an **M-regular sequence** if x_1 is M-regular on M and for each $i \in \{2, .., n\}$ x_i is regular in $M/(x_1, .., x_{i-1})M$.*

*If there is an ideal I in A such that $\{x_1, .., x_n\} \subseteq I$ then the regular sequence $x_1, .., x_n$ is said to be a M-**regular sequence in** I.*

*Moreover if $M = A$ then $x_1, .., x_n$ is called a **regular sequence**.*

Example 17. *If k is a commutative integral domain, then $x_1, .., x_n$ is a regular sequence in $k[x_1, .., x_n]$.*

Proof. For $0 < i < n$ set $k_i := k[x_1, .., x_{i-n}] \cong k[x_1, ...x_n]/(x_n, ., x_i)$ and set $k_n := k$. Then x_i is a regular sequence in $k_i[x_i]$ by example 16 and the result follows by iteration of example 16. □

3.1.1.1 *Flat Dimension And Regular Sequences*

The first bound between the Krull dimension and the Hochschild Cohomological dimension is a ring theoretical dimension, the flat dimension.

Definition 33. ***A-Flat Dimension***

If A is a commutative ring then the A-flat dimension $fd_A(M)$ of an A-module M is the extended natural number n, defined as the shortest length of a resolution of M by A-flat A-modules. If no such finite n exists n is taken to be ∞.

Example 18. *If A is a commutative ring and M is a flat A-module then $fd_A(M) = 0$.*

Proof. $0 \to M \overset{1_M}{\to} M \to 0$ is an A-flat resolution of M of length 0. □

Lemma 6. *If A is a commutative ring then for any A-module M the following are equivalent:*

1. *The A-flat dimension of M is at most n.*
2. *For every left A-module N, $Tor_A^{n+1}(M, N)$ is the trivial A-module.*

Proof. Similar to the proof of theorem 3, see page 461 of [IH] for details. □

3.1.1.2 *The Koszul Complex*

Definition 34. ***Exterior Power of a Module***

*If A is a commutative ring, n is a positive integer, M is an A-module and σ is a permutation in the permutation group S^n with signature $sgn(\sigma)$ then the n^{th}**-exterior power of** M **over** k is defined as the A-module:*

$$\bigwedge_A^n(M) := M^{\otimes n} / \{a_1 \otimes_A ... \otimes_A a_n - sgn(\sigma) a_{\sigma(1)} \otimes_A ... \otimes_A a_{\sigma(n)} | a_1, ..., a_n \in A, \sigma \in S^n\} M. \tag{202}$$

An element of the equivalence class of $a_1, ..., a_n$ in $\bigwedge_A^n(M)$ is denoted by $a_1 \wedge ... \wedge a_n$.

Lemma 7. *If A is a commutative ring and d is a positive integer then for every positive integer n there is an isomorphism of A-modules:*

$$\Xi_n : \bigwedge_A^n (A^d) \to A^{\binom{d}{n}}. \tag{203}$$

Where Ξ_n maps the set $\{e_{i_1} \wedge ... \wedge e_{i_n} | 1 \leq i_1 < ... < i_n \leq d\}$ in $\bigwedge_A^n (A^d)$ to a basis of $A^{\binom{d}{n}}$.

Proof. See [BA] page 517. □

A regular sequence of a ring is related to its flat dimension as follows:

Lemma 8. ***Koszul Complex***

If A is a commutative ring, $x_1, .., x_d$ is a regular sequence in A then and $\pi : A \to A/(x_1, .., x_d)$ is the canonical projection of A onto $A/(x_1, .., x_d)$ then there is a A-free resolution of $A/(x_1, .., x_d)$ of length d described as:

$$... \to \bigwedge_A^{n+1} (A^d) \xrightarrow{d_{n+1}} \bigwedge_A^{n} (A^d) \xrightarrow{d_n} ... \xrightarrow{d_3} \bigwedge_A^{2} (A^d) \xrightarrow{d_2} \bigwedge_A^{1} (A^d) \xrightarrow{d_1} A \xrightarrow{\pi} A/(x_1, .., x_d) \to 0$$

where for every $(n \in \mathbb{N})$ d_n is defined on a basis element $e_{i_1} \wedge ... \wedge e_{i_n}$ in $\bigwedge_A^{n+1} (A^d)$ as:

$$d_n(e_{i_1} \wedge \ldots \wedge e_{i_n}) = \sum_{j=1}^{n} (-1)^{i+1} x_{i_j} \cdot e_{i_1} \wedge \ldots \wedge \widehat{e_{i_j}} \wedge \ldots \wedge e_{i_n}. \tag{204}$$

(where $\hat{e_{i_j}}$ denotes the omission of the term e_{i_j} in the expression $e_{i_1} \wedge ... \wedge e_{i_n}$). This resolution is denoted by $K_\star(A; x_1, .., x_n)$.

Proof. The A-freeness of $K_\star(A; x_1, .., x_d)$ follows from lemma 7. Moreover, $K_\star(A; x_1, .., x_n)$'s exactness is verified on page 152 of [HA]. Finally, for $n > d > 0$ since $\binom{n}{d} = 0$, the isomorphisms Ξ_n of lemma 7 implies $\bigwedge_A^n (A^d) \cong 0$; whence $K_\star(A; x_1, .., x_d)$ is of length d. □

Proposition 21. *If n is a positive integer and if there exists a regular sequence $x_1,..,x_n$ in A of length n then:*

$$n = fd_A(A/(x_1,..,x_n)). \tag{205}$$

Proof. Denote $A/(x_1,..,x_n)$ by $\tilde{A}$.

— Since $K_\star(A;x_1,..,x_n)$ is a free deleted resolution of $\tilde{A}$ of length n and free A-modules are flat A-modules [IH] $K_\star(A;x_1,..,x_n)$ is a flat resolution of $\tilde{A}$ of length n. Therefore lemma 6 implies:

$$fd_A(\tilde{A}) \leq n. \tag{206}$$

— Since $K_\star(A;x_1,..,x_n)$ is an A-flat resolution of $\tilde{A}$ then there are natural isomorphisms:

$$Tor^n_A(\tilde{A},\tilde{A}) \cong H_n(K_\star(A;x_1,..,x_n) \otimes_A \tilde{A}, d_\star \otimes_A 1_{\tilde{A}}) \text{ [IH]}. \tag{207}$$

However $(x_1,..,x_n)$ is an ideal in A, therefore for all y in A and for every $i \in \{1,..,n\}$ yx_i is in $(x_1,..,x_n)$, whence $y\bar{x}_i = \bar{0}$. Therefore:

$$d_n \otimes_A 1_{\tilde{A}}((x_{p_1} \wedge ... \wedge x_{p_i}) \otimes_A \bar{y}) \tag{208}$$

$$= \sum_{j=1}^{n} (-1)^{i+1}((x_{p_1} \wedge ... \wedge \hat{x_{p_j}} \wedge ... \wedge x_{p_i}) \otimes_A y\bar{x}_{p_j}) \tag{209}$$

$$= \sum_{j=1}^{n} (x_{p_1} \wedge ... \wedge \hat{x_{p_j}} \wedge ... \wedge x_{p_i}) \otimes_A \bar{0} = 0. \tag{210}$$

Hence:

$$Tor^n_A(\tilde{A},\tilde{A}) = Ker(d_n \otimes_A 1_{\tilde{A}})/Im(d_{n+1}) \tag{211}$$

$$= (\bigwedge_A^n (A^d) \otimes_A \tilde{A})/0 \overset{\Xi_n}{\to} (A^{\binom{m}{m}} \otimes_A \tilde{A})/0 \cong (A \otimes_A \tilde{A})/0 \tag{212}$$

$$= \tilde{A}. \tag{213}$$

Therefore by lemma 6:

$$fd_A(\tilde{A}) \geq n. \tag{214}$$

Hence:

$$fd_A(\tilde{A}) = n. \tag{215}$$

□

Example 19. *The $\mathbb{Z}[x]$-flat dimension of $\mathbb{Z}$ as a $\mathbb{Z}[x]$-module is precisely* 1.

Proof. By example 16 x is a regular sequence on $\mathbb{Z}$, moreover $\mathbb{Z}[x]/(x) \cong \mathbb{Z}$. Therefore proposition 21 therefore implies:

$$fd_{\mathbb{Z}[x]}(\mathbb{Z}) = 1. \tag{216}$$

□

Example 20. *The $\mathbb{Z}[x_1, .., x_n]$-module $\mathbb{Z}$'s $\mathbb{Z}[x_1, .., x_n]$-flat dimension is precisely n.*

Proof. $x_1, .., x_n$ is a regular sequence in $\mathbb{Z}[x_1, .., x_n]$ by example 17, whence proposition 21 and $\mathbb{Z}[x_1, .., x_n]/(x_1, .., x_n) \cong \mathbb{Z}$ therefore implies:

$$fd_{\mathbb{Z}[x_1,..,x_n]}(\mathbb{Z}) = n. \tag{217}$$

□

Corollary 3. *If A is a local ring with maximal ideal $\mathfrak{m}$ and $x_1, .., x_n$ is a regular sequence in $\mathfrak{m}$ then:*

$$fd_A(A/(x_1, .., x_n)) = n. \tag{218}$$

Proof. Proposition 21 with the assumption that A is local. □

Example 21. *The $\mathbb{Z}[x_1,..,x_n]$-module $\mathbb{Z}$'s $\mathbb{Z}[x_1,..,x_n]$-flat dimension is precisely n.*

Proof. A direct consequence of example 19 and lemma 3. □

As in example 21, regular sequences provide a direct and precise way of the computing flat dimension of a ring.

One more ingredient related to the flat dimension will soon be needed.

Proposition 22. *If A is a commutative ring and $\mathfrak{m}$ is a maximal ideal of A then for any A-module M $fd_{A_\mathfrak{m}}(M_\mathfrak{m})$ is a lower-bound for $fd_A(M)$.*

Proof. CASE 1: $fd_A(M)$ IS FINITE

1. Let d be the A-flat dimension of M. By definition, there is a deleted A-flat resolution $\mathbf{F}_\star$ of M of length d. Since localization is exact [AA], $A_\mathfrak{m} \otimes_A F_\star$ is an exact sequence augmentable to $A_\mathfrak{m} \otimes_A M \cong M_\mathfrak{m}$.
2. Again since localization is exact, $A_\mathfrak{m}$ is a flat A-module. Since the tensor product of flat modules is again flat [IH] each $A_\mathfrak{m} \otimes_A F_i$ in $A_\mathfrak{m} \otimes_A \mathbf{F}_\star$ is flat as an $A_\mathfrak{m}$-module.
3. Therefore, $A_\mathfrak{m} \otimes_A F_i$ is an $A_\mathfrak{m}$-flat resolution of $M_\mathfrak{m}$ of length d. Whence, by definition the A-flat dimension of $M_\mathfrak{m}$ can therefore be at most d.

CASE 2: $fd_A(M)$ IS INFINITE

By definition of $fd_{A_\mathfrak{m}}(M_\mathfrak{m})$:

$$fd_{A_\mathfrak{m}}(M_\mathfrak{m}) \leq \infty = fd_A(M). \tag{219}$$

□

Example 22. *For any prime integer p the $\mathbb{Z}[x_1,..,x_n]$-module $\mathbb{Z}_{(p)}$'s $\mathbb{Z}[x_1,..,x_n]_{(x_1,..,x_n,p)}$ -flat dimension is at most n.*

Proof. For any prime integer p the ideal $(x_1,..,x_n,p)$ is a maximal ideal in $\mathbb{Z}[x_1,..,x_n]$ [AA]. Therefore proposition 22 and example 21 imply $\mathbb{Z}_{(p)}$'s $\mathbb{Z}[x_1,..,x_n]_{(x_1,..,x_n,p)}$-flat dimension is at most $fd_{\mathbb{Z}[x_1,..,x_n]}(\mathbb{Z}) = n$. □

3.1.2 *Projective Dimension*

Definition 35. ***A-Projective Dimension***

If A is a commutative ring and M is an A-module then the A-projective dimension $pd_A(M)$ of M is the extended natural number n, defined as the shortest length of a deleted A-projective resolution of M. If no such finite n exists n is taken to be ∞.

Lemma 9.

If A is a commutative ring and M is an A-module then $fd_A(M) \leq pd_A(M)$.

Proof. Since all A-projective A-modules are A-flat then any A-projective resolution is an A-flat resolution. □

Lemma 10.

If A is a commutative ring then for any A-module M the following are equivalent:

- *The A-projective dimension of M is at most n.*
- *For every A-module N, the A-module $Ext^A_{n+1}(M,N)$ is trivial.*
- *For every A-module N and every integer $m \geq n+1$: $Ext^A_m(M,N) \cong 0$.*

Proof. Nearly identical to the proof of theorem 3, see page 456 of [IH] for details. □

3.1.2.1 *Cohen-Macaulay At A Maximal Ideal*

Definition 36. ***Cohen-Macauley at a maximal ideal*** [1]

*A commutative ring A is said to be **Cohen-Macaulay at** a maximal ideal $\mathfrak{m}$ if and only if either:*

— *$Krull(A_\mathfrak{m})$ is finite and there is an $A_\mathfrak{m}$-regular sequence $x_1, ..., x_d$ in $A_\mathfrak{m}$ of maximal length $d = Krull(A_\mathfrak{m})$ such that $\{x_1, .., x_d\} \subseteq \mathfrak{m}$.*

— *$Krull(A_\mathfrak{m})$ is infinite and for every positive integer d there is an $A_\mathfrak{m}$-regular sequence $x_1, .., x_d$ in $\mathfrak{m}$ on A of length d.*

Example 23. *$\mathbb{Z}[x_1, .., x_n]$ is Cohen Macaulay at the maximal ideal $(x_1, .., x_n, p)$.*

Proof. For legibility the ideal $(x_1, .., x_n, p)$ will be denoted I.

I is a maximal ideal in $\mathbb{Z}[x_1, .., x_n]$ [AA]. The ring $\mathbb{Z}[x_1, .., x_n]_I$ is of Krull dimension $Krull(\mathbb{Z}[x_1, .., x_n]) = n + Krull(\mathbb{Z}) = n + 1$. Since $\mathbb{Z}$ is an integral domain then p is a regular sequence on $\mathbb{Z} \cong \mathbb{Z}[x_1, .., x_n]/(x_1, .., x_n)$. Since $x_1, .., x_n$ was a regular sequence in $\mathbb{Z}[x_1, .., x_n]$ *(by example* 16*)*, $p, x_1, .., x_n$ must be a regular sequence on $\mathbb{Z}[x_1, .., x_n]$ [SP]. Moreover $\frac{p}{1}, \frac{x_1}{1}, .., \frac{x_1}{1}$ is a regular sequence in $\mathbb{Z}[x_1, .., x_n]_I$ [SP]. Therefore there is a regular sequence in $\mathbb{Z}[x_1, .., x_n]_I$ of length equal to $\mathbb{Z}[x_1, .., x_n]_I$'s Krull dimension, whence that sequence must be maximal [SP]. Finally since $p, x_1, .., x_n$ is contained in the maximal ideal I (in fact it generates it [SP]) the localized sequence $\frac{p}{1}, \frac{x_1}{1}, .., \frac{x_1}{1}$ is contained in the maximal ideal I in $\mathbb{Z}[x_1, .., x_n]_I$ [SP]. Thus $\mathbb{Z}[x_1, .., x_n]$ is Cohen-Macaulay at I. □

Note 3. *In particular if A is a commutative Cohen-Macaulay ring at the maximal ideal $\mathfrak{m}$ such that $A_\mathfrak{m}$ is of finite Krull dimension and $x_1, .., x_n$ is a maximal regular sequence in $A_\mathfrak{m}$ then the A-module $A_\mathfrak{m}/(x_1, .., x_n)$ will play an important role in the rest of this argument.*

1. Usually it is also required that a Cohen-Macaulay ring also be Noetherian.

Proposition 23. *If A is a commutative ring which is Cohen Macaulay at the maximal ideal $\mathfrak{m}$ and $Krull(A_\mathfrak{m})$ is finite then:*

$$Krull(A_\mathfrak{m}) = fd_{A_\mathfrak{m}}(A_\mathfrak{m}/(x_1,..,x_n)) \leq pd_A(A_\mathfrak{m}/(x_1,..,x_n)) \tag{220}$$

Proof. Since A is Cohen-Macaulay at the maximal ideal $\mathfrak{m}$, there is a regular sequence $x_1,..,x_n$ in $\mathfrak{m}$ of length $n = Krull(A_\mathfrak{m})$. Denote $A_\mathfrak{m}/(x_1,..,x_n)$ by $\xi_\mathfrak{m}$. By corollary 3:

$$Krull(A_\mathfrak{m}) = fd_{A_\mathfrak{m}}(\xi_\mathfrak{m}). \tag{221}$$

Proposition 22 applied to (221) entails:

$$Krull(A_\mathfrak{m}) = fd_{A_\mathfrak{m}}(\xi_\mathfrak{m}) \leq fd_A(\xi_\mathfrak{m}) \tag{222}$$

Lastly lemma 9 bounds (222) above as follows:

$$Krull(A_\mathfrak{m}) = fd_{A_\mathfrak{m}}(\xi_\mathfrak{m}) \leq fd_A(\xi_\mathfrak{m}) \leq pd_A(\xi_\mathfrak{m}). \tag{223}$$

□

3.1.3 *Global Dimension*

Definition 37. ***Global Dimension***

*The **global dimension** $D(A)$ of a ring A, is defined as the supremum of all the A-projective dimensions of its A-modules. That is:*

$$D(A) := \sup_{M \in {}_A Mod} pd_A(M). \tag{224}$$

Two classical results on Global dimension are now presented. They do not play a direct role in this paper but are presented only to showacase a more familiar interpretation of the global dimension of a ring.

Theorem 5. ***Auslander–Buchsbaum-Serre Theorem***

If k is a commutative Noetherian local ring then:

$$D(k) = Krull(k) \text{ if and only if } k \text{ is regular} \tag{225}$$

Proof. See [IH]. □

Proposition 24. *If k is a commutative Noetherian ring then $D(k)$ equals to the supremum of $D(k_{\mathfrak{m}})$ taken over every maximal ideal $\mathfrak{m}$ of k.*

Proof. See [IH]. □

Example 24. *The global dimension of $\mathbb{Z}$ is equal to 1.*

Proof. Since $\mathbb{Z}$ is a PID [AA] every maximal ideal in $\mathbb{Z}$ is of the form (p) for some prime integer p [AA]. Since the localization of a commutative Noetherian ring is again Noetherian [CA] each $\mathbb{Z}_{(p)}$ is a Noetherian ring. Since (p) is a maximal ideal in $\mathbb{Z}_{(p)}$ then $1 \leq Krull(\mathbb{Z}_{(p)}) \leq Krull(\mathbb{Z}) = 1$. Whence theorem 5 implies $D(\mathbb{Z}) = 1$; therefore proposition 24 entails:

$$Krull(\mathbb{Z}) = 1 = D(\mathbb{Z}). \tag{226}$$

□

3.1.4 *Relative Dimension Theory*

The homological dimension theory presented thus far has been purely ring theoretic, entirely overlooking the k's role in any k-algebra A.

The $\mathscr{E}_A^k$-projective dimension and the A-projective dimension of a k-algebra A may be related as follows:

Theorem 6. ***Hochschild*** *(∼1958)*

If k is of finite global dimension, A is a k-algebra which is flat as a k-module and M is an A-module then:

$$pd_A(M) - D(k) \leq pd_{\mathscr{E}_A^k}(M) \tag{227}$$

The proof of theorem 6 relies on the following lemma:

Lemma 11. *If A is a k-algebra which is flat as a k-module then:*

$$(\forall M \in_k Mod)\ pd_A(A \otimes_k M) \leq pd_k(M) \tag{228}$$

Proof. If M is of k-projective dimension ∞ then the result is immediate. Suppose M is of projective dimension $n < \infty$ and $P_\star$ is a k-projective resolution of length n. Since A is k-flat $A \otimes_k P_\star$ is exact and for each $i \in \mathbb{N}$ $A \otimes_k P_i$ is A-projective since if P_i is projective there exists a k-module Q and a set I such that $Q \oplus P \cong \bigoplus_{i \in I} k$ therefore:

$$A \otimes_k Q \oplus A \otimes_k P \cong A \otimes_k (Q \oplus P) \cong A \otimes_k \bigoplus_{i \in I} k \cong \bigoplus_{i \in I} A, \tag{229}$$

thus $A \otimes_k P_i$ is the direct summand of a free A-module; whence it is A-projective. Therefore $A \otimes_k P_\star$ A-projectively resolves $A \otimes_k M$; whence

$$pd_A(A \otimes_k M) \leq n = pd_k(M). \tag{230}$$

□

Lemma 12. *If A is a k-algebra then for any k-module M there is an $\mathscr{E}_A^k$-exact sequence:*

$$0 \longrightarrow Ker(a) \longrightarrow A \otimes_k M \xrightarrow{\alpha} M \longrightarrow 0 \tag{231}$$

Where α be the map defined on elementary tensors $(a \otimes_k m)$ in $A \otimes_k M$ as $a \otimes_k m \mapsto a \cdot m$.

Proof. α is k-split by the map $\beta : M \to A \otimes_k M$ defined on elements $m \in M$ as $m \mapsto 1 \otimes_k m$. Indeed if $m \in M$ then:

$$\alpha \circ \beta(m) = \alpha(1 \otimes_k m) = 1 \cdot m = m. \tag{232}$$

□

Lemma 13. *If M and N are A-modules then:*

$$pd_A(M) \leq pd_A(M \oplus N). \tag{233}$$

Proof.

$$(\forall n \in \mathbb{N})(\forall X \in_A Mod)\ Ext_A^n(M, X) \oplus Ext_A^n(N, X) \cong Ext_A^n(M \oplus N, X). \tag{234}$$

Therefore $Ext_A^n(M \oplus N, X)$ vanishes only if both $Ext_A^n(M, X)$ and $Ext_A^n(N, X)$ vanish. Lemma 10 then implies: $pd_A(M) \leq pd(M \oplus N)$. □

PROOF OF THEOREM 6

Proof.

CASE 1: $pd_{\mathscr{E}_A^k}(M) = \infty$

By definition $pd_A(M) \leq \infty$ therefore trivially if $pd_{\mathscr{E}^k_A}(M) = \infty$ then:

$$pd_A(M) \leq pd_{\mathscr{E}^k_A}(M) + D(k). \tag{235}$$

Since k's global dimension is finite hence (235) implies:

$$pd_A(M) - D(k) \leq \infty = pd_{\mathscr{E}^k_A}(M). \tag{236}$$

CASE 2: $pd_{\mathscr{E}^k_A}(M) < \infty$

Let $d := pd_{\mathscr{E}^k_A}(M) + D(k)$. The proof will proceed by induction on d.

BASE: $d = 0$

Suppose $pd_{\mathscr{E}^k_A}(M) = 0$.

By theorem 3 M is $\mathscr{E}^k_A$-projective. Lemma 12 implies there is an $\mathscr{E}^k_A$-exact sequence:

$$0 \longrightarrow Ker(\alpha) \longrightarrow A \otimes_k M \xrightarrow{\alpha} M \longrightarrow 0 \quad . \tag{237}$$

Proposition 11 implies that (237) is A-split therefore M is a direct summand of the A-module $A \otimes_k M$. Hence lemma 13 implies:

$$pd_A(M) \leq pd_A(M \otimes_k A). \tag{238}$$

Lemma 11 together with (238) imply:

$$pd_A(M) \leq pd_A(M \otimes_k A) \leq pd_k(M). \tag{239}$$

Definition 38 and (239) together with the assumption that $pd_{\mathscr{E}^k_A}(M) = 0$ imply:

$$pd_A(M) \leq pd_k(M) \leq D(k) = D(k) + 0 = D(k) + pd_{\mathscr{E}^k_A}(M). \tag{240}$$

Since k's global dimension is finite then (240) implies:

$$pd_A(M) - D(k) \leq pd_{\mathscr{E}_A^k}(M). \tag{241}$$

INDUCTIVE STEP: $d > 0$

Suppose the result holds for all A-modules K such that $pd_{\mathscr{E}_A^k}(K) + D(k) = d$ for some integer $d > 0$. Again appealing to lemma 12, there is an $\mathscr{E}_A^k$-exact sequence:

$$0 \longrightarrow Ker(\alpha) \longrightarrow A \otimes_k M \xrightarrow{\alpha} M \longrightarrow 0 \quad . \tag{242}$$

Proposition 11 implies $A \otimes_k M$ is $\mathscr{E}_A^k$-projective; whence (242) implies:

$$pd_{\mathscr{E}_A^k}(Ker(\alpha)) + 1 = pd_{\mathscr{E}_A^k}(M). \tag{243}$$

Since $Ker(\alpha)$ is an A-module of strictly smaller $\mathscr{E}_A^k$-projective dimension than M the induction hypothesis applies to $Ker(\alpha)$ whence:

$$pd_A(Ker(\alpha)) + 1 \leq pd_{\mathscr{E}_A^k}(Ker(\alpha)) + 1 + D(k) \leq pd_{\mathscr{E}_A^k}(M) + D(k). \tag{244}$$

The proof will be completed by demonstrating that: $pd_A(M) \leq pd_A(Ker(\alpha)) + 1$. For any $N \in_A Mod$ $Ext_A^{\star}(-, N)$ applied to (242) gives way to the long exact sequence in homology, particularly the following of its segments are exact:

$$Ext_A^{n-1}(A \otimes_k M, N) \to Ext_A^{n-1}(Ker(a), N) \overset{\partial^n}{\to} Ext_A^n(M, N) \to Ext_A^n(A \otimes_k M, N) \tag{245}$$

Since $A \otimes_k M$ is $\mathscr{E}_A^k$-projective $pd_{\mathscr{E}_A^k}(A \otimes_k M) = 0$, therefore by the base case of the induction hypothesis $pd_A(A \otimes_k M) \leq pd_{\mathscr{E}_A^k} + D(k) = D(k)$;thus for every positive integer $n \geq D(k)$ (in particular d is at least n):

$$(\forall N \in_A Mod)\ Ext_A^{n-1}(A \otimes_k M, N) \cong 0 \cong Ext_A^n(A \otimes_k M, N); \tag{246}$$

whence ∂^n must be an isomorphism. Therefore lemma 10 implies $pd_A(M)$ is at most equal to $pd_A(Ker(\alpha)) + 1$.

Therefore:

$$pd_A(M) \leq pd_A(Ker(\alpha)) + 1 \leq pd_{\mathscr{E}_A^k}(Ker(\alpha)) + 1 + D(k) \leq pd_{\mathscr{E}_A^k}(M) + D(k). \tag{247}$$

Finally since k is of finite global dimension then (247) implies:

$$pd_A(M) - D(k) \leq pd_{\mathscr{E}_A^k}(M); \tag{248}$$

thus concluding the induction.

□

3.1.5 $\mathscr{E}^k$-Global Dimension

The final numerical invariant used herein will now be presented before the last central result of this masters' thesis is presented.

Definition 38. ***$\mathscr{E}^k$-Global dimension***

*The $\mathscr{E}^k$-**global Dimension** $D_{\mathscr{E}^k}(A)$ of a k-algebra A is defined as the supremum of all the $\mathscr{E}^k_A$-projective dimensions of its A-modules. That is:*

$$D_{\mathscr{E}^k}(A) := \sup_{M \in {}_A Mod} pd_{\mathscr{E}^k_A}(M). \tag{249}$$

Remark 7. *The classical global dimension ignores the influence of k on a k-algebra A; however the relative theory takes it into account.*

Example 25. $D_{\mathscr{E}^{\mathbb{Z}_{(p)}}}(\mathbb{Z}_{(p)}[x_1, .., x_n]) = n$

Proof. See theorem 2 in [RG] with $R := \mathbb{Z}_{(p)}$. □

3.2 A LOWER BOUND ON THE HOCHSCHILD COHOMOLOGICAL DIMENSION

This original result is the second central result of this master's thesis and it is now presented. One of its central purposes is to generalize the claim made by Cuntz and Quillen at the beginning of [AE] stating that commutative k-algebras over a field are not quasi-free is they are of Krull dimension above 1.

Note 4. *Let A be a k-algebra, $i : k \to A$ the morphism defining the k-algebra A and $\mathfrak{m}$ a maximal ideal in A. For legibility the $\mathscr{E}^{k_{i^{-1}[\mathfrak{m}]}}_{A_\mathfrak{m}}$-projective dimension of an $A_\mathfrak{m}$-module N will be abbreviated by $pd_{\mathscr{E}_{\mathfrak{m},k}}(N)$ (instead of writing $pd_{\mathscr{E}^{k_{i^{-1}[\mathfrak{m}]}}_{A_\mathfrak{m}}}(N)$).*

Lemma 14. *If A is a commutative k-algebra and $\mathfrak{m}$ is a non-zero maximal ideal in A then for every A-module M:*

$$pd_{\mathscr{E}_{\mathfrak{m},k}}(M_{\mathfrak{m}}) \leq pd_{\mathscr{E}_A^k}(M), \tag{250}$$

where $i : k \to A$ is the inclusion of k into A.

Proof. Since $\mathfrak{m}$ is a prime ideal in A, $i^{-1}[\mathfrak{m}]$ is a maximal ideal in $k_{i^{-1}[\mathfrak{m}]}$, whence the localized ring $k_{i^{-1}[\mathfrak{m}]}$ is a well-defined sub-ring of $A_{\mathfrak{m}}$. Let

$$... \overset{d_{n+1}}{\to} P_n \xrightarrow{d_n} ... \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0 \tag{251}$$

be an $\mathscr{E}_A^k$-projective resolution of an A-module M. The exactness of localization [CA] implies:

$$... \overset{d_{n+1}}{\to} P_n \otimes_A A_{\mathfrak{m}} \xrightarrow{d_n \otimes_A A_{\mathfrak{m}}} ... \xrightarrow{d_2 \otimes_A A_{\mathfrak{m}}} P_1 \otimes_A A_{\mathfrak{m}} \xrightarrow{d_1 \otimes_A A_{\mathfrak{m}}} P_0 \otimes_A A_{\mathfrak{m}} \xrightarrow{d_0 \otimes_A A_{\mathfrak{m}}} M \otimes_A A_{\mathfrak{m}} \to 0 \tag{252}$$

is exact. It will now be verified that (252) is a $\mathscr{E}_{\mathfrak{m},k}$-projective resolution of the $A_{\mathfrak{m}}$-module $M_{\mathfrak{m}}$.

THE $d_n \otimes_A A_{\mathfrak{m}}$ ARE $k_{i^{-1}[\mathfrak{m}]}$-SPLIT

Since (251) was k-split then for every $i \in \mathbb{N}$ there existed a k-module homomorphism $s_i : P_{n-1} \to P_n$ (where for convenience write $P_{-1} := M$) satisfying $d_i = d_i \circ s_i \circ d_i$. Since $A_{\mathfrak{m}}$ is a $k_{i^{-1}[\mathfrak{m}]}$-algebra $A_{\mathfrak{m}}$ may be viewed as a $k_{i^{-1}[\mathfrak{m}]}$-module therefore the maps: $s_i \otimes_A 1_{A_{\mathfrak{m}}}$ are $k_{i^{-1}[\mathfrak{m}]}$-module homomorphisms; moreover they must satisfy:

$$d_i \otimes_A 1_{A_{\mathfrak{m}}} = d_i \otimes_A 1_{A_{\mathfrak{m}}} \circ s_i \otimes_A 1_{A_{\mathfrak{m}}} \circ d_i \otimes_A 1_{A_{\mathfrak{m}}}. \tag{253}$$

Therefore (252) is $k_{i^{-1}[\mathfrak{m}]}$-split-exact.

THE $P_i \otimes_A A_{\mathfrak{m}}$ ARE $\mathscr{E}_{\mathfrak{m},k}$-PROJECTIVE

For each $i \in \mathbb{N}$ if P_i is $\mathscr{E}_A^k$-projective therefore proposition 11 implies there exists some A-module Q and some k-module X satisfying:

$$P_i \oplus Q \cong A \otimes_k X. \tag{254}$$

Therefore:

$$\begin{aligned}(P_i \otimes_A A_{\mathfrak{m}}) \oplus (Q \otimes_A A_{\mathfrak{m}}) &\cong (P_i \otimes_A Q) \otimes_A A_{\mathfrak{m}} \cong (A \otimes_k X) \otimes_A A_{\mathfrak{m}} \\ &\cong (A \otimes_k X) \otimes_A (A_{\mathfrak{m}} \otimes_{k_{i^{-1}[\mathfrak{m}]}} k_{i^{-1}[\mathfrak{m}]})\end{aligned} \tag{255}$$

Since A,k and $k_{i^{-1}[\mathfrak{m}]}$ are commutative rings the tensor products $- \otimes_A -$, $- \otimes_k -$ and $- \otimes_{k_{i^{-1}[\mathfrak{m}]}} -$ are symmetric [IH], hence (255) implies:

$$\begin{aligned}(P_i \otimes_A A_{\mathfrak{m}}) \oplus (Q \otimes_A A_{\mathfrak{m}}) &\cong (A \otimes_k X) \otimes_A (A_{\mathfrak{m}} \otimes_{k_{i^{-1}[\mathfrak{m}]}} k_{i^{-1}[\mathfrak{m}]}) \\ &\cong (A_{\mathfrak{m}} \otimes_A A) \otimes_{k_{i^{-1}[\mathfrak{m}]}} (k_{i^{-1}[\mathfrak{m}]} \otimes_k X)\end{aligned} \tag{256}$$

Since A is a subring of $A_{\mathfrak{m}}$ then (256) implies:

$$(P_i \otimes_A A_{\mathfrak{m}}) \oplus (Q \otimes_A A_{\mathfrak{m}}) \cong A_{\mathfrak{m}} \otimes_{k_{i^{-1}[\mathfrak{m}]}} (k_{i^{-1}[\mathfrak{m}]} \otimes_k X). \tag{257}$$

$(k_{i^{-1}[\mathfrak{m}]} \otimes_k X)$ may be viewed as a $k_{i^{-1}[\mathfrak{m}]}$-module with action $\hat{\cdot}$ defined as:

$$(\forall c \in k)(\forall (c' \otimes_k x) \in k_{i^{-1}[\mathfrak{m}]} \otimes_k X)\ c \hat{\cdot} (c' \otimes_k x) := c \cdot c' \otimes x. \tag{258}$$

Since $(k_{i^{-1}[\mathfrak{m}]} \otimes_k X)$ is a $k_{i^{-1}[\mathfrak{m}]}$-module then for each $i \in \mathbb{N}$ $(P_i \otimes_A A_{\mathfrak{m}})$ is a direct summand of an $A_{\mathfrak{m}}$-module of the form $A_{\mathfrak{m}} \otimes_{k_{i^{-1}[\mathfrak{m}]}} X'$ where X' is a $k_{i^{-1}[\mathfrak{m}]}$-module, thus proposition 11 implies that $P_i \otimes_A A_{\mathfrak{m}}$ is $A_{\mathfrak{m}}$-projective.

Hence (252) is an $\mathscr{E}_{\mathfrak{m},k}$-projective resolution of $M \otimes_A A_{\mathfrak{m}} \cong M_{\mathfrak{m}}$; whence:

$$pd_{\mathscr{E}_{\mathfrak{m},k}}(M_{\mathfrak{m}}) \leq pd_{\mathscr{E}^k_A}(M). \tag{259}$$

□

All the homological dimensions discussed to date are related as follows:

Proposition 25. *If A is a commutative k-algebra and $\mathfrak{m}$ be a non-zero maximal ideal in A such that $A_{\mathfrak{m}}$ is flat as a $k_{i^{-1}[\mathfrak{m}]}$-module and $D(k_{i^{-1}[\mathfrak{m}]})$ is finite then there is a string of inequalities:*

$$fd_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) - D(k_{i^{-1}[\mathfrak{m}]}) \leq pd_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) - D(k_{i^{-1}[\mathfrak{m}]}) \leq pd_{\mathscr{E}_{\mathfrak{m},k}}(M_{\mathfrak{m}}) \leq pd_{\mathscr{E}^k_A}(M) \leq D_{\mathscr{E}^k}(A) \tag{260}$$

Proof.

1. By definition: $pd_{\mathscr{E}^k_A}(M) \leq D_{\mathscr{E}^k}(A)$.
2. By lemma 14: $pd_{\mathscr{E}_{\mathfrak{m},k}}(M_{\mathfrak{m}}) \leq pd_{\mathscr{E}^k_A}(M)$
3. Since $A_{\mathfrak{m}}$ is flat as a $k_{i^{-1}[\mathfrak{m}]}$-module and $D(k_{i^{-1}[\mathfrak{m}]})$ is finite theorem 6 entails: $pd_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) - D(k_{i^{-1}[\mathfrak{m}]}) \leq pd_{\mathscr{E}_{\mathfrak{m},k}}(M_{\mathfrak{m}})$
4. Lemma 9 implies:

$$fd_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) \leq pd_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}). \tag{261}$$

Since the global dimension of $k_{i^{-1}[\mathfrak{m}]}$ was assumed to be finite (261) implies:

$$fd_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) - D(k_{i^{-1}[\mathfrak{m}]}) \leq pd_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) - D(k_{i^{-1}[\mathfrak{m}]}). \tag{262}$$

□

Lemma 15. *If A is a commutative k-algebra and M and N be A-modules, then there are natural isomorphisms:*

$$Ext^n_{\mathscr{E}^k_A}(M,N) \cong HH^n(A, Hom_k(M,N)) \cong Ext^n_{\mathscr{E}^k_{A^e}}(A, Hom_k(M,N)). \tag{263}$$

Proof.

- For any (A,A)-bimodule X, $X \otimes_A M$ is an (A,A)-bimodule [IH][Cor. 2.53].
- Moreover there are natural isomorphisms:

$$Hom_{{}_AMod}(X \otimes_A M, N) \xrightarrow{\cong} Hom_{{}_AMod_A}(X, Hom_{{}_kMod}(M,N)) \text{ [IH][Thrm. 2.75].} \tag{264}$$

  In particular (264) implies that for every n in $\mathbb{N}$ there is an isomorphism which is natural in the first input:

$$Hom_{{}_AMod}(A^{\otimes n} \otimes_A M, N) \xrightarrow{\psi_n} Hom_{{}_AMod_A}(A^{\otimes n}, Hom_{{}_kMod}(M,N)). \tag{265}$$

  Whence if $b'_{n+1} : A^{\otimes n+3} \to A^{\otimes n+2}$ is the n^{th} map in the Bar complex (recall example 7) and for legibility denote $Hom_{{}_AMod_A}(b'_n, Hom_k(M,N))$ by β_n. The naturality of the maps ψ_n imply the following diagram of k-modules commutes:

$$\begin{array}{ccc} Hom_{{}_AMod}(A^{\otimes n+2} \otimes_A M, N) & \overset{\psi_n}{=\!=\!=\!=} & Hom_{{}_AMod_A}(A^{\otimes n+2}, Hom_{{}_kMod}(M,N)) \\ {\scriptstyle \psi^{-1}_{n+1} \circ \beta_n \circ \psi_n} \downarrow & & \downarrow {\scriptstyle \beta_n} \\ Hom_{{}_AMod}(A^{\otimes n+3} \otimes_A M, N) & \overset{\psi_{n+1}}{=\!=\!=\!=} & Hom_{{}_AMod_A}(A^{\otimes n+3}, Hom_{{}_kMod}(M,N)) \end{array} . \tag{266}$$

- Therefore for every n in $\mathbb{N}$:

$$(\psi^{-1}_{n+2} \circ \beta_{n+1} \circ \psi_{n+1}) \circ (\psi^{-1}_{n+1} \circ \beta_n \circ \psi_n)$$

$$= \beta_{n+1} \circ \beta_n = 0. \tag{267}$$

Whence $< Hom_{_A Mod}(A^{\otimes \star+2} \otimes_A M, N), (\psi_{\star+1}^{-1} \circ \beta_\star \circ \psi_\star) >$ is a chain complex. Moreover the commutativity of (266) implies:

$$(\forall n \in \mathbb{N})\ H^n(Hom_{_A Mod}(A^{\otimes \star+2} \otimes_A M, N)) = Ker(\psi_{\star+1}^{-1} \circ \beta_\star \circ \psi_\star) / Im(\psi_{n+2}^{-1} \circ \beta_{n+1} \circ \psi_{n+1})$$

$$\cong Ker(\beta_n) / Im(\beta_{n+1}) = H^n(Hom_{_A Mod_A}(A^{\otimes \star+2}, Hom_{_k Mod}(M, N))).$$

$$= HH^n(A, Hom_k(M, N)) \tag{268}$$

Furthermore proposition 16 implies there are natural isomorphisms:

$$HH^n(A, Hom_k(M, N)) \cong Ext^n_{\mathscr{E}^k_{A^e}}(A, Hom_k(M, N)); \tag{269}$$

Whence for all n in $\mathbb{N}$ there are natural isomorphisms:

$$H^n(Hom_{_A Mod}(A^{\otimes \star+2} \otimes_A M, N)) \cong HH^n(A, Hom_k(M, N)) \cong Ext^n_{\mathscr{E}^k_{A^e}}(A, Hom_k(M, N)). \tag{270}$$

— Finally if M is an A-module then $< Hom_{_A Mod}(A^{\otimes \star+2} \otimes_A M, N), (\psi_{\star+1}^{-1} \circ \beta_\star \circ \psi_\star) >$ calculates the $\mathscr{E}^k_A$-relative Ext groups of M with coefficients in N; therefore there are natural isomorphisms:

$$H^n(Hom_{_A Mod}(A^{\otimes \star+2} \otimes_A M, N)) \cong Ext^n_{\mathscr{E}^k_A}(M, N) \text{ [HI][pg. 289]}. \tag{271}$$

— Putting it all together, for every n in $\mathbb{N}$ there are natural isomorphisms:

$$Ext^n_{\mathscr{E}^k_{A^e}}(A, Hom_k(M, N)) \cong HH^n(A, Hom_k(M, N)) \cong Ext^n_{\mathscr{E}^k_{A^e}}(A, Hom_k(M, N)). \tag{272}$$

□

Theorem 7.

Let A be a commutative k-algebra and $\mathfrak{m}$ be a non-zero maximal ideal in A such that $A_{\mathfrak{m}}$ is flat as a $k_{i^{-1}[\mathfrak{m}]}$-module and $D(k_{i^{-1}[\mathfrak{m}]})$ is finite.

1. *For every A-module M there is a string of inequalities:*

$$fd_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) - D(k_{i^{-1}[\mathfrak{m}]}) \leq pd_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) - D(k_{i^{-1}[\mathfrak{m}]})$$

$$\leq pd_{\mathscr{E}_{\mathfrak{m},k}}(M_{\mathfrak{m}}) \leq pd_{\mathscr{E}_A^k}(M) \leq D_{\mathscr{E}^k}(A) \leq HCdim(A|k) \tag{273}$$

2. *If A is Cohen-Macaulay at some maximal ideal $\mathfrak{m}$*

   Then $Krull(A_{\mathfrak{m}}) - D(k_{i^{-1}[\mathfrak{m}]}) \leq HCdim(A|k)$.

   In this scenario: if $A_{\mathfrak{m}}$ is of Krull dimension at least $2 + D(k_{i^{-1}[\mathfrak{m}]})$ then A is not Quasi-free.

Proof.

1. For any A-modules M and N lemma 15 implied:

$$Ext^{\star}_{\mathscr{E}_A^k}(N, M) \cong HH^{\star}(A, Hom_k(N, M)). \tag{274}$$

   Therefore taking supremums over all the A-modules M, N, of the integers n for which (274) is non-trivial implies:

$$D_{\mathscr{E}^k}(A) = \sup_{M,N \in {}_A Mod} (sup(\{n \in \mathbb{N}^{\#} | Ext^n(M, N) \neq 0\})) \tag{275}$$

$$= \sup_{M,N \in {}_A Mod} (sup(\{n \in \mathbb{N}^{\#} | HH^n(A, Hom_k(N, M)) \neq 0\})). \tag{276}$$

   $Hom_k(N, M)$ is only a particular case of an A^e-module; therefore taking supremums over *all* A-modules bounds (276) above as follows:

$$D_{\mathscr{E}^k}(A) = \sup_{M,N \in {}_A Mod} (sup(\{n \in \mathbb{N}^{\#} | HH^{\star}(A, Hom_k(N, M)) \neq 0\})) \tag{277}$$

$$\leq \sup_{\tilde{M} \in {}_{A^e} Mod} (sup(\{n \in \mathbb{N}^{\#} | HH^n(A, \tilde{M}) \neq 0\})). \tag{278}$$

The right hand side of (278) is precisely the definition of the Hochschild cohomological dimension. Therefore

$$D_{\mathscr{E}^k}(A) \leq HCdim(A|k) \tag{279}$$

Proposition 25 applied to (279) then draws out the conclusion.

2. CASE 1: $Krull(A_{\mathfrak{m}})$ IS FINITE

Since A is Cohen-Macaulay at $\mathfrak{m}$ there is an $A_{\mathfrak{m}}$-regular sequence $x_1, .., x_d$ in $\mathfrak{m}$ of length $d := Krull(A_{\mathfrak{m}})$ in $A_{\mathfrak{m}}$. Therefore proposition 21 implies:

$$Krull(A_{\mathfrak{m}}) = fd_{A_{\mathfrak{m}}}(A_{\mathfrak{m}}/(x_1, .., x_n)). \tag{280}$$

Part 1 of theorem 7 applied to (280) implies:

$$Krull(A_{\mathfrak{m}}) - D(k_{i^{-1}[\mathfrak{m}]}) = fd_{A_{\mathfrak{m}}}(A_{\mathfrak{m}}) - D(k_{i^{-1}[\mathfrak{m}]}) \leq HCdim(A|k). \tag{281}$$

Moreover the *characterization of quasi-freeness* given in corollary 2 implies that A cannot be quasi-free if:

$$2 + D(k_{i^{-1}[\mathfrak{m}]}) \leq Krull(A_{\mathfrak{m}}). \tag{282}$$

CASE 2: $Krull(A_{\mathfrak{m}})$ IS INFINITE

For every positive integer d there exists an $A_{\mathfrak{m}}$-regular sequence $x_1^d, .., x_d^d$ in $\mathfrak{m}$ of length d. Therefore proposition 21 implies:

$$(\forall d \in \mathbb{Z}^+) \; d = fd_{A_{\mathfrak{m}}}(A_{\mathfrak{m}}/(x_1^d, .., x_d^d)). \tag{283}$$

Therefore part one of theorem 7 implies:

$$(\forall d \in \mathbb{Z}^{+})\ d - D(k_{i^{-1}[\mathfrak{m}]}) = fd_{A_{\mathfrak{m}}}(A_{\mathfrak{m}}/(x_1^d,..,x_d^d)) - D(k_{i^{-1}[\mathfrak{m}]}) \leq HCdim(A|k). \quad (284)$$

Since $D(k)$ is finite:

$$\infty - D(k_{i^{-1}[\mathfrak{m}]}) = \infty \leq HCdim(A|k). \quad (285)$$

Since $Krull(A_{\mathfrak{m}})$ is infinite (285) implies:

$$Krull(A_{\mathfrak{m}}) - D(k_{i^{-1}[\mathfrak{m}]}) = \infty = HCdim(A|k). \quad (286)$$

In this case corollary 2 implies that A is not quasi-free.

□

4

CONCLUSION: NEGATIVE EXAMPLES

Example 26. ***Arithmetic Polynomial-Algebras***

The $\mathbb{Z}$-algebra $\mathbb{Z}[x_1, .., x_n]$ fails to be quasi-free for values of $n > 1$.

Proof. In example 23 it was observed that $\mathbb{Z}[x_1, ...x_n]$ is Cohen-Macaulay at the maximal ideal $(x_1, ...x_n, p)$ and is of Krull dimension $n + 1 = Krull(\mathbb{Z}[x_1, ...x_n])$. In example 24 it was observed that $D(\mathbb{Z}) = 1$; whence by 2 of theorem 7: $\mathbb{Z}[x_1, .., x_n]$ fails to be Quasi-free if $2 \leq Krull(\mathbb{Z}[x_1, ...x_n]) - D(\mathbb{Z}) = (n + 1) - 1 = n$. □

4.0.0.1 *Cuntz's and Quillen's Formulation over a field*

Cuntz's and Quillen's classical claim [AE] may be recovered as a special case of theorem 7.

Definition 39. ***Regular $\mathbb{C}$-algebra***

*A commutative $\mathbb{C}$-algebra A is called **regular** if and only if for each maximal ideal $\mathfrak{m}$ in A: $Krull(A_{\mathfrak{m}})$ is finite and there is a regular sequence $x_1, .., x_d$ in $\mathfrak{m}$ of length $d = Krull(A) = Krull(A_{\mathfrak{m}})$ such that the set $\{x_1, .., x_d\}$ generates the maximal ideal $\mathfrak{m}$.*

By definition:

Proposition 26. *If A is a commutative regular $\mathbb{C}$-algebra then A is Cohen-Macaulay at all of its maximal ideals.*

Corollary 4. *If A is a regular commutative $\mathbb{C}$-algebra then A is not quasi-free if its Krull dimension exceeds* 1.

Proof. The condition for 2 in theorem 7 will be verified to hold.

1. Since $\mathbb{C}$ is a field then all $\mathbb{C}$-modules are free [IH], therefore every $\mathbb{C}$-module is projective M [IH]. By definition of the $\mathbb{C}$-projective dimension of a k-module M:

$$(\forall M \in_{\mathbb{C}} Mod) pd_{\mathbb{C}}(M) = 0. \tag{287}$$

   Whence $D(\mathbb{C}) = 0$.

2. Since $\mathbb{C}$ is a field, the $\mathbb{C}$-module A is free [IH] therefore it is $\mathbb{C}$-projective [IH] and so it is $\mathbb{C}$-flat [IH].

3. Since every $\mathbb{C}$-algebra has a maximal ideal [AA] and A was assumed to be regular as a $\mathbb{C}$-algebra. Then there exists some maximal ideal $\mathfrak{m}$ in A for which A is Cohen-Macaulay at $\mathfrak{m}$.

Fix a maximal ideal $\mathfrak{m}$ in A, since $Krull(A)$ was assumed to equal $Krull(A_{\mathfrak{m}})$ then 1,2 and 3 verify that if: $Krull(A) = Krull(A_{\mathfrak{m}}) > 1$ then theorem 7's 2 is applicable; whence A fails to be quasi-free. □

4.0.0.2 *An application to affine algebraic $\mathbb{C}$-Varieties*

Algebraic $\mathbb{C}$-varieties By viewing polynomials in $\mathbb{C}[x_1,..,x_n]$ as functions on $\mathbb{C}^n$, to $\mathbb{C}^n$ there may be associated a topological space whose underlying pointset is itself and who's topology is generated by the sets $D(f) := \{z \in \mathbb{C}^n | f(z) \neq 0\}$ where $f \in \mathbb{C}[x_1,..,x_n]$ (these are called **principal open sets**). This topological space is called **affine n-space** and is denoted by $\mathbb{A}^n_{\mathbb{C}}$.

An **affine $\mathbb{C}$-algebra** A is a $\mathbb{C}$-algebra which contains no nilpotent elements and can be written as the quotient $\mathbb{C}[x_1,..,x_n]/I$ of a polynomial $\mathbb{C}$-algebra $\mathbb{C}[x_1,..,x_n]$ in n variables by one of its ideals I (where n is some natural number) [LA]. To any such $\mathbb{C}$-algebra A

there may be attributed a topological space $V(A)$ called the **affine algebraic $\mathbb{C}$-variety** associated to A. $V(A)$'s pointset is defined as $\{z \in \mathbb{C}^n : (\forall f \in A) f(z) = 0\}$ and $V(A)$'s topology is defined as the topology induced by the inclusion function $\{z \in \mathbb{C}^n : (\forall f \in A) f(z) = 0\} \subset \mathbb{C}^n$.

If U is an open subset of $V(A)$ then the collection of all $\mathbb{C}$-valued functions f on U, such that for each point x of U there exists an open neighbourhood U_x of x contained in U and $g, h \in A$ satisfying: for all $v \in U_x$ $g(v) \neq 0$ and $f(v) = \frac{h(v)}{g(v)}$, is denoted by $\mathscr{O}_{V(A)}(U)$. $\mathscr{O}_{V(A)}(U)$ may be given the structure of a $\mathbb{C}$-algebra. The elements of $\mathscr{O}_{V(A)}(U)$ are called **regular functions on** U [AA]. Moreover $\mathscr{O}_{V(A)}(V(A)) \cong A$ (as $\mathbb{C}$-algebras) and $V(\mathscr{O}_{V(A)}(V(A))) = V(A)$ [AA].

If A and B are affine $\mathbb{C}$-algebras then a continuous function $\phi : V(A) \to V(B)$ is said to be a **morphism (of affine $\mathbb{C}$-varieties)** if and only if for every open subset U of $V(B)$ and for every regular function f on U $f \circ \phi|_{\phi^{-1}[U]}$ is a regular function on $\phi^{-1}[U]$ [1]. For example, if $V(A)$ and $V(B)$ are affine $\mathbb{C}$-varieties such that $V(A)$ is a topological subspace of $V(B)$, then the inclusion map $i : V(A) \to V(B)$ is a morphism [LA] and $V(A)$ is called an affine **sub-variety** of $V(B)$.

Definition 40. ***Linear algebraic $\mathbb{C}$-group***

A linear algebraic $\mathbb{C}$-group is a triple $G_A := < V(A), m, i >$ of an affine algebraic variety $V(A)$ which is a group, and two morphisms of affine $\mathbb{C}$-varieties $m : V(A) \times V(A) \to V(A)$ and $i : V(A) \to V(A)$ satisfying:

- *$(\forall g, g' \in V(A)$ $m(g, g') = gg'$, where gg' is the multiplication of the elements g and g' of the group $V(A)$.*
- *$(\forall g \in V(A)$ $i(g) = g^{-1}$, where g^{-1} is the multiplicative inverse of the element g of the group $V(A)$.*

*A is called the **coordinate ring of** G_A.*

Proposition 27. *If A is the coordinate ring of a linear algebraic $\mathbb{C}$-group is a regular $\mathbb{C}$-algebra.*

1. $\phi^{-1}[U]$ is open by the continuity of ϕ.

Proof. See [SP]. □

4.0.0.3 $GL_2(\mathbb{C})$

The set of all 2×2 matrices with coefficients in $\mathbb{C}$ may be viewed as the affine-algebraic $\mathbb{C}$-variety $V(\mathbb{C}[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}])$ [LA]. Since the sub-collections of all these matrices consisting of the invertible 2×2 matrices (that is the collection of all 2×2 matrices with entries in $\mathbb{C}$ whose determinant does not vanish) forms a group [AA], the sub-variety of $V(\mathbb{C}[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}])$ consisting of all invertible 2×2 can then viewed as a linear algebraic $\mathbb{C}$-group. Explicitly:

Proposition 28. *The determinant $det : \mathbb{A}^4_{\mathbb{C}} \to \mathbb{C}$ is a polynomial function, in particular (det) is an ideal in $\mathbb{C}[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}]$.*

Proof. See [LA]. □

The collection consisting of all the invertible 2×2 matrices with entries in $\mathbb{C}$ may be viewed as the sub-variety of $V(\mathbb{C}[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}])$ on which the determinant function does not vanish, that is:

Definition 41. ***General Linear $\mathbb{C}$-group***

*The **General Linear $\mathbb{C}$-group** denoted $GL_2(\mathbb{C})$, is defined as the triple:*
$< V(\mathbb{C}[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}]_{(det)}), m_{GL_2(\mathbb{C})}, i_{GL_2(\mathbb{C})} >$ *where $m_{GL_2(\mathbb{C})}$ takes a pair of matrices $X, Y \in V(\mathbb{C}[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}]_{(det)})$ to their matrix multiplication XY and $i_{GL_2(\mathbb{C})}$ takes a matrix $Y \in V(\mathbb{C}[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}]_{(det)})$ to its inverse matrix Y^{-1}.*

Note 5. *It is confirmed in [LA] that det, $m_{GL_2(\mathbb{C})}$ and $i_{GL_2(\mathbb{C})}$ are indeed morphisms of affine $\mathbb{C}$-varieties.*

Example 27. *The $\mathbb{C}$-algebra $\mathbb{C}[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}]_{(det)}$ is regular.*

Proof. Since $\mathbb{C}[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}]_{(det)}$ is the coordinate ring of a linear algebraic $\mathbb{C}$-group proposition 27 implies it is a regular $\mathbb{C}$-algebra. □

4.0.0.4 $T_2(\mathbb{C})$

Definition 42. ***Upper-Triangular Subgroup of*** $GL_2(\mathbb{C})$

A ***upper-triangular subgroup of*** $GL_2(\mathbb{C})$ *denoted* $T_2(\mathbb{C})$ *is the subgroup of* $GL_2(\mathbb{C})$ *isomorphic to the group of all (invertible) upper triangular matrices with coefficients in* $\mathbb{C}$.

Lemma 16. $T_2(\mathbb{C})$ *is an affine algebraic* $\mathbb{C}$*-group and its coordinate ring is isomorphic to the* $\mathbb{C}$*-algebra*
$\mathbb{C}[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}]_{(det)}/(x_{2,1})$.

Proof. See [BG]. □

Example 28. *The* $\mathbb{C}$*-algebra*$\mathbb{C}[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}]_{(det)}/(x_{1,2})$ *is regular* $\mathbb{C}$*-algebra.*

Proof. Since $T_2(\mathbb{C})$ is an affine algebraic $\mathbb{C}$-group proposition 27 implies
$\mathbb{C}[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}]_{(det)}/(x_{1,2})$ is a regular $\mathbb{C}$-algebra. □

4.0.0.5 *An application to affine algebraic varieties*

Let be an affine algebraic $\mathbb{C}$-variety $V(A)$. For any point x in $V(A)$ the ideal generated by the collection of regular functions on $V(A)$ vanishing at the point x is denoted by $\mathscr{I}(x)$; in fact $\mathscr{I}(x)$ is a maximal ideal in A [SP]. Moreover for any affine-algebraic variety $V(A)$ there exists a point x such that $A_{\mathscr{I}(x)}$ is regular. Since every regular local $\mathbb{C}$-algebra is Cohen Macaulay at its maximal ideal, then A is Cohen-Macaulay at $\mathscr{I}(x)$. Since $\mathbb{C}$ is a field it is a regular local ring of krull dimension 0 theorem 5 implies $D(k) = Krull(k) = 0$, moreover $A_{\mathscr{I}(x)}$ is a $\mathbb{C}$-vector space whence it is a $\mathbb{C}$-free and so is a $\mathbb{C}$-flat module. Therefore theorem 7 applies if $Krull(A) \geq 2$. In summary:

Corollary 5. [2]

If $V(A)$ *is an affine* $\mathbb{C}$*-variety and A's Krull dimension is greater than* 1 *then the* $\mathbb{C}$*-algebra A is not quasi-free.*

2. Corollary 5 implies that any affine algebraic $\mathbb{C}$-variety which is not a disjoint union of curves or points has a coordinate ring which fails to be quasi-free over $\mathbb{C}$.

Example 29. *The* $\mathbb{C}$*-algebra* $\mathbb{C}[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}]_{(det)}$ *is not quasi-free.*

Proof. $\mathbb{C}[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}]_{(det)}$ is of Krull dimension $4 > 1$ [LA] therefore theorem 7 applies. □

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