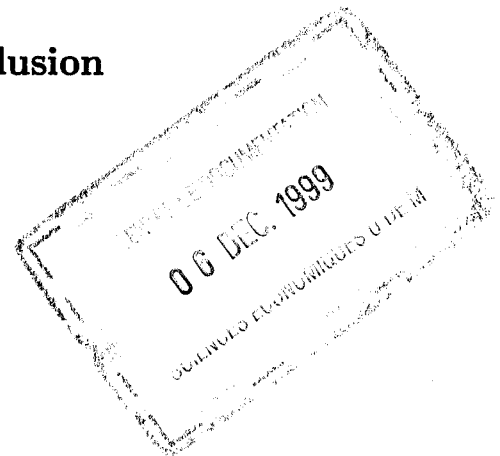


Université de Montréal

Three Essays on Collusion

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Département des sciences économiques

Faculté des arts et des sciences

Thèse présentée à la Faculté des études supérieures

en vue de l'obtention du grade de

Philosophiæ Doctor (Ph.D.)

en sciences économiques

octobre, 1999

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Université de Montréal
Faculté des études supérieures

Cette thèse intitulée:

Three Essays on Collusion

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Thèse acceptée le : ...30 novembre 1999.....

Liste des articles et auteurs

Collusion in a Model of Repeated Auctions

Paul Johnson and Jacques Robert

On Cartel Stability

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Search, Matching and Moral Hazard

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Summary

This thesis defines collusion broadly as play in a repeated game which differs from play in a one shot game. The analysis of collusion is an important part of many branches of economics. In industrial organization, for example, if collusion were not present then we could restrict investigation to the study of oligopolistic and competitive markets. Another important part of modern economic theory is the analysis of situations where there exists some kind of privileged information. Game theorists would say that in such a scenario there exists incomplete information. The predominant theme of the three essays composing this thesis is the study of repeated games under incomplete information.

Typically, repeated game analyses have assumed the presence of complete information. However, many examples of repeated interaction must be treated in an incomplete information scenario. Auctions, the subject of the first essay, are the most natural example of this. The goal of this study is to understand how bidders collude in auctions. The main innovation is an explicit treatment of the repeated nature of the game to endogenize the threats necessary to support non competitive behavior. This analysis yields several testable implications about the behavior of colluding agents in auctions which are not apparent from the few models which have studied collusion in auctions from a static point of view.

Nearly every model constructed to study collusion makes predictions which are at odds with accepted stylized facts. The most prominent of these paradoxes is the stability which theoretical models predict yet empirical and allegorical evidence rejects. This is the subject of the second essay. Incomplete information takes the

form of how players evaluate future payoffs. This is an important detail because future payoffs are the only tool a cartel can use to enforce the play of collusive equilibria. The developed model imposes that patience is private information and heterogeneous and develops predictions which contrast strongly with accepted models of collusion. The predictions of the model are supported with discussion of some empirical evidence on cartels.

The third essay develops a model which can have sociological as well as economic applications, in that it studies how rational agents form surplus creating partnerships in a repeated, incomplete information environment. Previous work has assumed an exogenous production technology which partners use to create this surplus. Furthermore an agent's type, which affects surplus creation, has always been assumed to be observable. This essay studies matching with an endogenous production technology, in the sense that the surplus is a function of the level of collusion which can be supported. Collusion can be supported to varying degree based upon the type (patience, or discount factor) of each agent in the partnership. A special attention is turned to contrasting the implications of the model in the presence of complete as opposed to incomplete information.

Introduction

The predominant theme of the three essays composing this thesis is the study of collusion under incomplete information. Collusion is defined as equilibrium strategies to the repeated game which differ from the repetition of the equilibrium strategies to the stage game. The analysis of collusion is an important part of many branches of economics. In industrial organization, for example, if collusion were not present then we could restrict investigation to the study of oligopolistic and competitive markets. Incomplete information is a way which game theorists study situations where all information is not common knowledge. Its applications are ubiquitous in modern game theoretic economics with applications as diverse as the study of oligopoly and the cold war.

The first essay studies collusion in a set of markets which were the subject of the first systematic investigation into the economics of incomplete information: Vickrey's (1961) seminal study of auctions. With few exceptions, the auction literature has maintained the assumption that agents participate in a one shot game: an auction is held and the participants part company never to meet again. This is the point of departure for this work as a repeated environment is explicitly developed. The repeated nature of such an auction immediately suggests the possibility of bidder collusion. Recently there has been some progress made into understanding the behavior of rational agents colluding in auctions, but this work has abstracted away from a repeated game analysis. For internal consistency this is reason enough to study collusion in an explicitly repeated environment. Studying collusion in explicitly repeated auctions is also of interest from a more

pragmatic point of view. Firstly, the predictions generated by a static model do not always carry over to a dynamic model (in a way to be made clear below). Secondly, a dynamic model yields richer results than its static counterpart. And thirdly, an entire class of results on auctioneer behavior emerges which does not arise in a static model.

One of the earliest examinations of collusion in auctions was done by Comanor and Schankerman (1976) who studied a paradox in auction behavior: that of identical bids submitted by bidders (under usual hypotheses identical bids should be observed with zero probability). McAfee and McMillan (1992) were able to resolve this paradox by showing that if bidders were unable to effectuate sidepayments amongst themselves, perhaps because of a "paper trail" which makes costly detection very likely, then the optimal response of a cartel would be for every bidder to bid the reserve price and let the auctioneer act as a randomization device. Of course if the auctioneer were to award the good in a predetermined way, say to the bidder who's name was first alphabetically, then a little bit more coordination by the cartel is required. The key insight here is that a cartel must renounce efficiency in order to overcome incentive compatibility problems brought about by the existence of privately held information. Though they only examined a static model, McAfee and McMillan assumed the existence of some sort of trigger-strategy which makes adherence to the cartel more profitable than defection.

"Collusion in a Model of Repeated Auctions" studies collusion in auctions from an explicitly repeated game point of view. Thus instead of assuming the existence of some kind of trigger strategy, trigger strategies are explicitly constructed. An immediate outcome is that one can easily give an example of a series of auctions

where McAfee and McMillan's (1992) strategy of bidding the reserve price is not supportable. Thus, added to the familiar incentive compatibility constraints is a new dynamic "trigger" constraint (or in the terminology used by Abreu, Pearce, and Stacchetti (1986), "admissibility"). Roughly, the intuition is that a bidder can value an object very highly today—so much so that she is prepared to face any credible punishment in return for possessing the item today. The inefficiency of the McAfee and McMillan "flat" bidding function is the driving force behind this behavior. The obvious question is that if for certain series of auctions bidding the reserve price is not an admissible scheme, then what is? and what is the optimal scheme? Furthermore, what can be said about cartel behavior across time: can incomplete information cause some kind of instability in the cartel? Does the role of the auctioneer differ from that of a static model where his main weapon in combatting collusion is the reserve price? This essay is an attempt to answer the above questions.

The essay uses techniques first developed by Abreu, Pearce, and Stacchetti (1986) to endogenize punishments. These punishments differ from the usual repeated game analyses in that an auction is a game of incomplete information. The main results are that a cartel permits a *finite* number of bidding levels, and that collusion is found to be stable across time. Perhaps the most interesting insights are the number of predictions which arise due to the explicitly dynamic nature of the model. For example auctioneer profits can be affected by the frequency at which auctions are held.

Unfortunately for cartels, collusion is rarely, if ever, long lasting. This last

observation is troubling for economists, since most repeated game models of collusion predict perfect cartel stability over time. For a literature which has been in existence since the time of Friedman (1971), and has been the subject of so much research effort this is problematic. The second essay, "On Cartel Stability" offers a reconciliation of theory with the accepted stylized fact that cartels are generally unstable.

One of the most commonly made assumptions in the literature on collusion is that one player is just as patient as another player—players share a common discount factor. Harrington (1989) is an interesting exception to this tendency, in that he permits colluding firms to have different discount rates. Despite this heterogeneity, these differing discount rates remain perfectly known. In this essay a model where discount factors are private information is developed. Not only are discount factors private information, but they are subject to change as the game progresses.

Suppose that discount factors (patience) are unobservable and change as players play a repeated game. A maximizing cartel is faced with a tradeoff: on one hand, the more profitable the collusion is the more difficult it is to support, thus a small drop in a discount factor causes costly defection; on the other hand, the less profitable collusion is the less difficult it is to support, thus small changes in discount rates are unlikely to destabilize the cartel resulting in mutually profitable cooperation. If punishments were simple as in Abreu (1988), a razor edge result would be obtained, as any agent with a discount factor below a certain level would defect to the maximum extent permitted by the structure of the game, whereas agents with discount factors above a certain level would respect cartel rules. This

essay aims to characterize the efforts of an optimizing cartel (a concept which is carefully defined) when confronted with such a scenario.

Agents draw iid discount factors in every period. I argue that something akin to defection and punishment arises on the equilibrium path (the revelation principle is invoked throughout, so defection where agents misreport their discount factors does not occur). The most striking result of this section is the structure which optimal collusive schemes must take: unlike Abreu's (1988) simple strategy profiles and unlike Abreu, Pearce, and Stacchetti's (1986) and Abreu, Pearce, and Stacchetti's (1990) bang-bang results, a "punishment" must be made to fit the "crime". In other words, future payoffs depend non trivially on current action, so minor "infractions" are followed by play which is more profitable than the play following a major "infraction" of cartel rules. The essay concludes with a brief discussion of some empirical evidence which appears to be in line with the predictions of the model. This entails a discussion of Barbezat's (1989) study of the International Steel Cartel which existed in Europe between the first and second world war. His discussion implies that member nations of the cartel were subject to different internal pressures leading to non respect of quotas. In response, punishments were structured accordingly. Bertrand's (1999) study of firms' implicit contracts with workers is also discussed.

The final essay of the thesis examines a model which can have economic as well as sociological implications. The essay develops an extension of Becker's (1973) marriage model with the search frictions used by Smith (1997). These models study how rational agents choose partners in order to gain access to a production

technology requiring the inputs of two agents. Marriage was the motivating example of Becker (1973) but such a model can be applied to many other situations. This literature aims to characterize the conditions under which agents of a certain type only accept partners of similar type, i.e. when the matching is assortative. "Search, Matching and Moral Hazard" examines Becker's model in two different informational scenarios when search is costly and the production technology is endogenous to the model.

To see how the production technology is endogenized, consider the meaning and implications of type in a two sided matching model. Oftentimes type is not terribly relevant. For instance if a firm hires a contractor to accomplish a specific task, then all details about the service: terms of delivery and payment may be perfectly measurable and enforceable in a contract. Thus the type of the parties is superfluous since the transaction takes place, in essence, on a spot market. In longer term relationships, type may be more relevant. For instance, if the same firm hires a permanent employee, then type becomes more relevant. It is probably important to the firm to have employees who are willing work hard at their specified tasks. Similarly it is probably important to a worker to work in an agreeable work environment and receive adequate compensation. The firm and the worker may have information on a potential match (through references or reputation), and this may be of help in the matching choice. Alternatively, the firm and worker may have unreliable or insufficient information on which to make a decision (a new firm without a reputation, or a worker with no prior experience or who was self employed). In any case the profitability of a match is a function of the willingness (or ability) to work overtime on a project, or the awarding of

performance through pay increases among many other things.

The previous discussion argues that type need not necessarily be job specific productivity, education or other factors which *directly* enter into the production function of the marriage model studied by Becker. Alternatively, type could be a commitment to do household chores, visit in-laws or not hold grudges after arguments. This suggests that type enters into the joint production function in a special way, as one could always renege on a promise to do dishes or take out the garbage. This possibility of non fulfilment of a promise (implicit or explicit) hints that this type of scenario can be modeled as a repeated game where cooperative (collusive) outcomes may occur. This is the approach which this essay takes.

There are two main innovations in this essay. First is the aforementioned endogenization of the production technology. The technology employed by Becker (1973) or Smith (1997) does not allow for the possibility of agents' "defecting". The other innovation is the treatment of a matching model under incomplete information. The incomplete information is akin to the incomplete information in the second essay, in that it is the patience of the agents (thus their ability to collude) which can vary. Additionally, evolutionary game theory is used to justify the choice of an equilibrium criterium which selects a single equilibrium. This is necessary since even a concept as strong as sequential equilibrium does not permit the modeler to study under what conditions assortative matching obtains. Attention is focused on how the ability of agents to defect from agreements affects matching, as well as contrasting the equilibrium arising under the complete and incomplete informational scenarios.

The main insight of the essay is that the feasibility of assortative matching is

closely linked to the ease with which different levels of collusion can be supported. Contrast this with the findings of Becker (1973) and Smith (1997). Becker finds that assortative matching obtains under almost any condition, whereas Smith requires a log supermodularity condition on the production technology. For example, assume that there are two types of agents. The patient type can support the play of a highly profitable form of collusion. The impatient type can only support the play of a lesser profitable form of collusion. If the patient player finds it "difficult" to support the very profitable form of collusion, then assortative matching may be eschewed in favor of an equilibrium where players are indifferent between matching with a patient or impatient partner. The imposition of incomplete information adds another layer of complexity to the problem, and one of the main goals of the essay is to contrast assortative matching under incomplete and complete information scenarios.

Résumé

Le thème principal de cette thèse, composée de trois essais, est l'étude de la collusion en information incomplète. La collusion se définit comme des stratégies d'équilibre dans le cadre d'un jeu répété qui sont différentes de la répétition des stratégies d'équilibres du jeu d'une seule étape. L'analyse de la collusion est importante pour beaucoup de branches de l'économie. En organisation industrielle par exemple, s'il n'existait pas de possibilité de collusion, on pourrait restreindre la recherche à l'étude des marchés oligopolistiques et concurrentiels. L'information incomplète représente l'environnement utilisé par les théoriciens des jeux pour étudier des modèles dans lesquels l'information n'est pas de connaissance commune. Ses applications sont omniprésentes dans la science économique moderne de la théorie des jeux parce qu'elle est devenue un élément standard d'un grand nombre de modèles, avec des applications diverses telles que l'analyse des oligopoles ou de la guerre froide.

Dans le premier essai, on étudie la collusion dans un ensemble de marchés qui a été l'objet des premières recherches systématiques dans l'économie de l'information incomplète : les ventes aux enchères de l'étude "originale" de Vickrey (1961). Sauf dans un nombre limité d'exceptions, la littérature sur les enchères a maintenu l'hypothèse que les agents participent à un jeu d'une seule période : une vente aux enchères a lieu et les participants se séparent ensuite pour ne plus jamais se rencontrer. C'est le point de départ de ce travail dans lequel un environnement répété est développé explicitement. La répétition d'une telle vente aux enchères suggère immédiatement la possibilité de collusion entre les enchérisseurs

(les acheteurs). Il y a eu récemment quelques progrès dans la compréhension du comportement des agents rationnels coopérant dans les ventes aux enchères, mais ces travaux ne se sont pas attachés à l'analyse en jeux répétés. Pour des raisons de cohérence interne, il est donc raisonnable d'étudier la collusion dans un environnement explicitement répété.

L'étude de la collusion dans des ventes aux enchères explicitement répétées a également de l'intérêt d'un point de vue plus pragmatique. D'abord, les prédictions générées par un modèle statique ne se transposent pas toujours à un modèle dynamique (d'une façon qu'on explicitera plus bas). Ensuite, un modèle dynamique apporte des résultats plus riches que son équivalent statique. Enfin, il émerge des modèles dynamiques une classe entière de résultats sur le comportement des vendeurs, qui ne peuvent être obtenus dans les modèles statiques.

L'une des premières recherches sur la collusion dans les ventes aux enchères a été effectuée par Comanor et Schankerman (1976) qui ont étudié un paradoxe dans le comportement d'enchère : les mises identiques soumises par les acheteurs (sous les hypothèses habituelles, des mises identiques devraient être observées avec probabilité zéro). McAfee et McMillan ont pu résoudre ce paradoxe en montrant que si les acheteurs étaient incapables d'effectuer des paiements de transferts entre eux, par exemple à cause de la possibilité de retracer tout accord écrit, qui rendrait la détection très probable, alors la réponse optimale d'un cartel serait que chaque acheteur mise le prix de réserve et laisse le vendeur agir de façon à rendre le résultat aléatoire. Bien sûr, si le vendeur accorde le bien selon un processus prédéterminé, par exemple à l'acheteur dont le nom arrive en premier dans l'ordre alphabétique, le cartel a besoin d'un peu plus de coordination. L'intuition clé ici est que le cartel

doit renoncer à l'efficacité de façon à surmonter les problèmes incitatifs introduits par l'existence d'information privée. Bien qu'ils se soient contentés d'examiner un modèle statique, McAfee et McMillan ont supposé l'existence d'une forme de stratégie de la gachette qui rend l'adhésion au cartel plus profitable que la défection.

"Collusion in a Model of Repeated Auctions" (Collusion dans un modèle d'enchères répétées) étudie la collusion dans les ventes aux enchères du point de vue d'un jeu explicitement répété. Ainsi, au lieu de supposer l'existence d'une forme de stratégie de la gachette, ces stratégies sont explicitement construites. Un résultat immédiat est que l'on peut facilement donner un exemple d'une série d'enchères dans lesquelles la stratégie de McAfee et McMillan de miser le prix de réserve n'est pas soutenable. Par conséquent, en plus des habituelles contraintes de compatibilité avec les incitations, on a une contrainte de la gachette dynamique nouvelle (ou, selon la terminologie d'Abreu, Pearce et Stacchetti (1986), une contrainte d'"admissibilité"). En gros, l'intuition est qu'un acheteur peut donner beaucoup de valeur à un objet aujourd'hui, si bien qu'il est prêt à faire face à toute punition crédible du moment qu'il peut le posséder tout de suite. L'inefficacité de la fonction de mise "plate" de McAfee et McMillan est la force qui sous-tend ce comportement. La question évidente est que si, pour certaines suites de mises, le prix de réserve n'est pas un schéma admissible, alors qu'est-ce qui l'est? Et quel est le schéma optimal? De plus, que peut on dire du comportement du cartel dans le temps? L'information incomplète peut-elle créer une sorte d'instabilité dans le cartel? Le rôle du vendeur est-il différent de son rôle dans un modèle statique où sa meilleure arme pour combattre la collusion réside dans le prix de réserve? Cet

essai propose des réponses à ces questions.

Cet essai utilise des techniques d'abord développées par Abreu, Pearce et Stacchetti pour endogénéiser les punitions. Ces punitions diffèrent de celles qu'on trouve habituellement dans les jeux répétés en cela qu'une enchère est un jeu à information incomplète. Les résultats principaux sont qu'un cartel autorise un nombre *fini* de niveaux de mises et que la collusion est stable dans le temps. Le phénomène le plus intéressant est sans doute le nombre de prédictions qui émergent grâce à la nature explicitement dynamique du modèle. Par exemple, les profits du vendeur peuvent être affectés par la fréquence avec laquelle les ventes aux enchères sont tenues.

Malheureusement pour les cartels, la collusion dure rarement longtemps. Cette observation est troublante pour les économistes parce que la plupart des modèles de collusion en jeu répété prédisent une parfaite stabilité du cartel dans le temps. Pour une littérature qui existe depuis Friedman (1971) et qui a fait l'objet de tant d'efforts de recherche, c'est problématique. Le second essai, intitulé "On Cartel Stability" (Sur la stabilité des cartels), réconcilie la théorie avec le fait stylisé que les cartels sont généralement instables.

L'une des hypothèses les plus communes dans la littérature sur la collusion est que les joueurs ont le même degré de patience—qu'ils ont le même taux d'escompte. Harrington (1989) est une exception intéressante à cette tendance; il permet en effet aux firmes en collusion d'avoir des taux d'escompte différents. Malgré cette hétérogénéité, les différents taux d'escompte restent parfaitement connus. Dans cet essai, on impose une hétérogénéité et une inobservabilité sur les

taux d'escomptes des agents.

Supposons que les facteurs d'escompte (les degrés de patience) sont inobservables et varient à mesure que les joueurs jouent le jeu répété. Un cartel maximisateur fait face à un arbitrage : d'un côté, plus la collusion est profitable et plus elle est difficile à soutenir; ainsi, une petite chute du taux d'escompte entraîne des défections coûteuses. De l'autre côté, moins la collusion est profitable, moins il est difficile de la soutenir, ainsi de petites variations dans les taux d'escomptes sont peu susceptibles de déstabiliser un cartel résultant d'une collusion mutuellement bénéfique. Si les punitions étaient simples, comme dans Abreu (1988), on pourrait obtenir comme résultat que les agents qui ont un facteur d'escompte en dessous d'un certain niveau font défaut autant qu'il est permis par la structure du jeu alors que les agents qui ont un facteur d'escompte supérieur à un certain niveau respectent les règles du cartel. Cet essai vise à caractériser les efforts d'un cartel optimisateur (un concept qui est défini avec précaution) qui est confronté à un tel scénario.

Supposons que les facteurs d'escompte des agents sont tirés à chaque période d'une distribution iid. Je soutiens qu'un phénomène relié à la défection et aux punitions émerge sur le sentier d'équilibre (on invoque le principe de révélation tout au long du travail si bien que la forme de défection par laquelle les agents mentent sur leurs véritables facteurs d'escompte n'a jamais lieu). Le résultat le plus édifiant de cette section est la structure que doivent prendre les schémas optimaux de collusion : au contraire des profils simples de stratégie de Abreu (1988) et des résultats "bang-bang" de Abreu, Pearce et Stacchetti (1986 et 1990), chaque "punition" doit être à la mesure du "crime". En d'autres termes, les

gains futurs dépendent de façon non triviale de l'action courante si bien que des "infractions" mineures sont suivies d'un jeu qui est plus profitable que le jeu suivant les "infractions" majeures aux règles du cartel. L'essai est conclu par une brève discussion des évidences empiriques qui sont dans la lignée des prédictions du modèle. Ceci entraîne une discussion de l'étude de Barbezat (1989) sur le cartel international de l'acier qui existait en Europe entre la première et la seconde guerre mondiale. Son analyse indique que les nations membres du cartel étaient sujettes à des pressions internes menant au non respect des quotas. En réponse, les punitions étaient structurées de façon appropriée.

Le dernier essai de la thèse étudie un modèle qui peut avoir des implications sociologiques aussi bien qu'économiques. L'essai développe une extension du modèle de mariage de Becker (1973) avec les coûts de recherche utilisés par Smith (1997). Ces modèles étudient comment les agents rationnels choisissent leurs partenaires de façon à avoir accès à une technologie de production requérant les inputs des deux agents. Le mariage était l'exemple utilisé par Becker mais un tel modèle peut être appliqué à beaucoup d'autres situations. Cette littérature vise à caractériser les conditions sous lesquelles des agents d'un certain type acceptent seulement des partenaires d'un type similaire, c'est-à-dire quand l'appariement est assorti. "Search, Matching and Moral Hazard" (Recherche, Appariement et Aléas Moral) étudie le modèle de Becker dans deux scénarios informationnels différents quand la recherche est coûteuse et la technologie de production endogène au modèle.

Pour voir comment la technologie de production est endogénéisée, considérons la signification et les implications du type dans deux modèles d'appariement par-

allèles. Souvent, le type n'est pas vraiment important. Par exemple, si une firme engage un contracteur pour accomplir une tâche spécifique, les détails du service (termes de la livraison et paiement) peuvent être parfaitement mesurables et exécutables dans un contrat. Alors, le type des parties est superflu puisque la transaction a lieu, essentiellement, sur un marché courant. Dans des relations de plus long terme, le type peut être plus important. Par exemple, si la même firme engage un employé permanent, le type devient plus déterminant. Il est probablement important pour la firme d'avoir des employés prêts à travailler fort sur leurs tâches spécifiques. De la même façon, il est probablement important pour un travailleur d'avoir un environnement de travail agréable et de percevoir une compensation adéquate. La firme et le travailleur peuvent avoir de l'information sur un appariement potentiel (à travers des références ou la réputation) et ceci peut aider au choix d'appariement. D'un autre côté, la firme et le travailleur peuvent avoir, pour prendre une décision, de l'information non fiable ou insuffisante (une nouvelle firme sans réputation ou un travailleur sans expérience préalable ou qui était son propre employé). Dans tous les cas, la profitabilité de l'appariement est une fonction, entre autres choses, de la volonté (ou de la capacité) à travailler en temps supplémentaire sur le projet ou de la rémunération de la performance par des augmentations de salaire.

La discussion précédente fait valoir que le type ne représente pas nécessairement la productivité spécifique à l'emploi, l'éducation ou d'autres facteurs qui entrent *directement* dans la fonction de production du modèle de mariage étudié par Becker. Le type peut être un engagement à faire des corvées ménagères, à visiter la belle-famille ou à ne pas garder de rancune après une dispute. Ceci suggère

que le type entre dans la fonction de production jointe d'une façon particulière car l'on pourrait toujours renier une promesse de faire la vaisselle ou de sortir les poubelles. Cette possibilité de ne pas remplir ses engagements (implicites ou explicites) suggère que ce type de scénario peut être modélisé comme un jeu répété où peuvent apparaître des revenus de la coopération (de la collusion). C'est l'approche que prend cet essai.

Il y a deux grandes innovations dans cet essai. D'abord, il y a l'endogénéisation mentionnée plus haut de la technologie de production. La technologie employée par Becker ou Smith ne permet pas la "défection" des agents. L'autre innovation est le traitement d'un modèle d'appariement en information incomplète. L'information incomplète est ici reliée à l'information incomplète qu'on a dans le second essai en cela que c'est la patience des agents (donc leur inclination à la collusion) qui peut varier. De plus, on utilise la théorie des jeux évolutionniste pour justifier le choix d'un critère d'équilibre qui sélectionne un seul équilibre. Ceci est nécessaire parce que même un concept aussi puissant que l'équilibre séquentiel ne permet pas au modélisateur d'étudier sous quelles conditions on obtient un appariement assorti. L'attention se concentre sur la façon dont la capacité des agents à violer leurs engagements affecte l'appariement aussi bien que sur la comparaison des équilibres émergeant de scénarios en information complète et incomplète.

L'apport principal de cet essai est que la réalisabilité de l'appariement assorti est étroitement liée à la facilité avec laquelle différents niveaux de collusion peuvent être supportés. Comparons ceci avec les résultats de Becker et de Smith. Becker trouve que l'appariement assorti peut être obtenu sous presque n'importe quelles conditions alors que Smith requiert une condition de log supermodularité

sur la technologie de production. Par exemple, supposons qu'il y a deux types d'agents. Le type patient peut supporter le jeu d'une forme hautement profitable de collusion. Le type impatient peut seulement supporter le jeu d'une forme de collusion moins profitable. Si le joueur patient trouve "difficile" de supporter la forme de collusion très profitable, le matching assorti peut être contourné en faveur d'un équilibre où les joueurs sont indifférents entre s'apparier avec un partenaire patient ou impatient. L'imposition de l'information incomplète ajoute un degré de complexité au problème et l'un des buts importants de l'essai est de comparer l'appariement assorti sous différents scénarios en information complète et incomplète.

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Acknowledgments

I am indebted to a great many people and organizations who have helped me during my studies at the Université de Montréal.

Firstly I would like to thank the Centre de Recherche sur les Transports (CRT) and the Centre de Développement en Economie (CRDE) for providing top notch facilities which I used for four years. The CRT supported me through my first two years through a series of scholarships. The CRDE supported me through travel assistance to numerous conferences and most importantly through inviting guest speakers to the department which greatly benefitted me. I was also supported through the Département des Sciences Economiques at the Université de Montréal for my first three years. I would also like to thank my advisor, Jacques Robert, for supporting me through my last two years through grants from the Social Sciences and Humanities Research Council of Canada (SSHRC).

I benefitted from my interaction with my fellow students through a series of student seminars organized by Lars Vilhuber and later by Stefan Ambec. Without Karine Gobert, who graciously translated the Résumé, I think I would have required another year to get this section readable.

The single largest source of help throughout my writing this thesis has come from my advisor Jacques Robert. The first essay of this thesis stems from joint work between us. His contribution did not end there as he provided invaluable input and criticism for all aspects of the thesis. In addition to his intellectual contribution he provided me with financing so that I never had to worry about from where funds for conferences or living would come.

Lastly I would like to thank my family. My parents have always been willing to support me throughout any endeavor which I chose to undertake. I consider myself very lucky. During the writing of this thesis my family grew to include my fiancée Isabelle and her family Gilles, Nicole, Philippe, Marylise and Emilie. Isabelle was essential for my sanity throughout the past three years, and our numerous trips to Joliette to visit her family let me forget about economics for at least a little while.

Collusion in a model of Repeated Auctions

Paul Johnson and Jacques Robert

Abstract

A model of first price sealed bid auctions is developed where bidders meet repeatedly while independently drawing private valuations in each period. Attention is focused on symmetric collusive bidding equilibria when sidepayments are not allowed. Via an approach introduced by Abreu, Pearce and Stacchetti endogenous punishments are characterized and used in the construction of optimal collusive bidding strategies. This analysis differs from usual repeated game treatments due to the presence of incomplete information. Optimal collusive bidding strategies are generally inefficient and have a bang-bang nature which implies that defection is never observed. Auctioneer responses are also studied, which in this explicitly dynamic setting give rise to insights not apparent in a static formulation.

Acknowledgments: The authors benefitted from the helpful comments of Yves Sprumont as well as participants at the 1998 Canadian Economic Theory Conference and the 1998 North American Summer Meetings of the Econometric Society. Of course errors remain our own. Support was provided by Canada's SSHRC, the Département de Sciences Economiques at the Université de Montréal and by the Centre de Recherche sur les Transports at the Université de Montréal.

1.1 Introduction

It is well recognized that auctions account for a significant share of current economic activity. From a pragmatic point of view, this is possibly why auctions have been the focus of so much research effort for nearly forty years. With few exceptions, the literature has maintained the assumption that agents participate in a one shot game: an auction is held and the participants part company never to meet again. This is the point of departure for this work, and is justified by the fact that in most auctions there is a core of long run bidders¹. The repeated nature of such an auction immediately suggests the possibility of bidder collusion. Recently there has been some progress made into understanding the behavior of rational agents colluding in auctions, but this work has abstracted away from a repeated game analysis. For internal consistency this is reason enough to study collusion in an explicitly repeated environment. Studying collusion in auctions in a repeated setting is pertinent for several additional reasons. Firstly, the predictions generated by a static model do not always carry over to a dynamic model (in a way to be made clear below). Secondly, a dynamic model yields richer results than its static counterpart. And thirdly, an entire class of results on auctioneer behavior emerges which does not arise in a static model.

One of the earliest examinations of collusion in auctions was done by Comanor and Schankerman (1976) who examined rotating bid schemes (the most famous case-study is perhaps the "phases-of-the-moon scheme" detailed by Smith (1961)). They also studied a paradox in auction behavior: that of identical bids submitted by bidders². McAfee and McMillan (1992) were able to resolve this paradox

by showing that if bidders were unable to effectuate sidepayments amongst themselves, perhaps because of a "paper trail" which makes costly detection very likely, then the optimal response of a cartel would be for every bidder to bid the reserve price and let the auctioneer act as a randomization device. Of course if the auctioneer were to award the good in a predetermined way, say to the bidder whose name was first alphabetically, then a little bit more coordination by the ring is required. The key insight here is that a ring must renounce efficiency in order to overcome incentive compatibility problems brought about by the existence of privately held information. Though they only examined a static model, McAfee and McMillan assumed the existence of some sort of trigger-strategy which makes adherence to the cartel more profitable than defection.

When the restriction on sidepayments is lifted, the analysis changes rather drastically. As Graham and Marshall (1987) show, second price and English auctions are susceptible to a mechanism called a pre-auction knockout which not only succeeds in winning the object at the reserve price (when the cartel is all inclusive), but always awards the object to a buyer who values it the most. Efficiency is retained. In their paper, McAfee and McMillan (1992) also study "strong cartels". These cartels can effectuate transfers amongst themselves and can exclude non serious bidders. They also propose a mechanism which is efficient and wins the item at the reserve price. Again, these models abstract away from repeated play which must be used to justify obedience to such mechanisms.

Friedman (1971) was the first to formalize the "folk" theorem showing that repeated partnerships enable players to coordinate to equilibria which Pareto dominate any single stage Nash equilibrium. Usual analysis has abstracted away from

any uncertainty, but if a model of auctions is to be studied one must keep in mind that uncertainty is the *raison d'être* for auctions. In fact it is this incomplete information in an auction which changes the analysis of repeated play, since players (in a sense) play a different game in each period. The perfect information assumption has been relaxed by Green and Porter (1984) as well as by Abreu, Pearce, and Stacchetti (1986) and Abreu, Pearce, and Stacchetti (1990), mainly through the study of Cournot oligopoly where each period's price gives an imperfect signal about (private information) firm-specific quantities. The uncertainty in auctions is somewhat different from the uncertainty described above. In Cournot competition an "auction-like" uncertainty would be more akin to firms holding private information about stochastic costs, as opposed to the demand side uncertainty studied by Green and Porter and Abreu, Pearce and Stacchetti.³ Nevertheless, the approach introduced by Abreu, Pearce and Stacchetti proves to be particularly useful in studying repeated auctions.

An immediate outcome of studying explicitly repeated auctions is that one can easily give an example of a series of auctions where McAfee and McMillan's strategy of bidding the reserve price is not supportable. Thus, added to the familiar incentive compatibility constraints is a new dynamic "trigger" constraint (or in the terminology used by Abreu, Pearce and Stacchetti, "admissibility"). Roughly, the intuition is that a bidder can value an object very highly today—so much so that he is prepared to face any credible punishment in return for possessing the item today. The inefficiency of the McAfee and McMillan "flat" bidding function is the driving force behind this behavior. Section 1.2 gives a simple example of this phenomenon. The obvious question is that if for certain

series of auctions bidding the reserve price is not an admissible scheme, then what is? and what is the optimal scheme? Furthermore, what can be said about cartel behavior across time: can incomplete information cause some kind of instability in the cartel? Does the role of the auctioneer differ from that of a static model where his main weapon in combatting collusion is the reserve price? This paper, therefore, is an attempt to answer the above questions.

Section 1.3 presents the model to be studied through a series of assumptions. Section 1.4 shows that the method of Abreu, Pearce and Stacchetti can be used in auction games. Some of the major points of the paper are stated in section 1.5 where optimal collusive bidder behavior is studied. The role of the auctioneer is presented in section 1.6 where several implications arise only due to the repeated nature of the scenario. Section 1.7 briefly concludes. Many of the proofs can be found in the appendix.

1.2 An Example

To motivate the following sections, this section presents a simple example. Consider two bidders who meet repeatedly in a first price sealed bid auction, discounting future earnings with a common discount rate $\delta \in (0, 1)$, drawing independent valuations, denoted v , each period from a uniform distribution over $[0, 1]$. Following McAfee and McMillan, suppose they decide to use a form of tacit collusion to increase their expected profits where each bids zero (the reserve price) in each period. The auctioneer is assumed to randomize equally in the case of a draw. Suppose that this collusion specifies that each player use Nash strategies forever if

a winning bid above zero is ever observed. In order for any bidder type to prefer colluding it is necessary and sufficient that the highest valuation type prefer to adhere to cartel rules:

$$\frac{1}{2} + \frac{\delta}{1-\delta} \frac{1}{4} \geq 1 + \frac{\delta}{1-\delta} \int_0^1 v(1-v)dv.$$

The sufficiency follows from the fact that the incentive to cheat for a player with valuation one is greater than for any other valuation. The left hand side represents the gains to obeying the collusive rule for an agent with valuation one, and the right hand side represents the gains to defecting from the collusive rule for an agent with valuation one. However, for any $\delta < 6/7$ this collusive rule will eventually provoke defection (the admissibility constraint is not satisfied). The inefficiency of the collusive rule drives a high valuation agent to prefer cheating. The obvious question becomes what is the form of collusion which should be used to guarantee the participants the highest discounted expected payoffs? Can some kind of defection be permitted? How should defectors be punished?

1.3 Assumptions and Notation

A set $N = \{1, 2, \dots, n\}$ of ex-ante identical potential buyers compete in an infinitely repeated auction game with discounting. Detailed assumptions can be found below.

The Stage Game

(Assumption 1). Each potential buyer is ex-ante symmetric, drawing an independent private value for the object in question from a common continuous distribution function $F(\cdot)$ with strictly positive continuously differentiable density, $f(\cdot)$, defined on a compact support $[0, a]$. For reasons to be made clear later assume that the hazard function, $H(v) := \frac{1-F(v)}{f(v)}$, is strictly decreasing in v .⁴ Furthermore F and f are common knowledge.

(Assumption 2). Each potential buyer is risk neutral.

(Assumption 3). The auctioneer allocates objects via a first price auction, randomizing equally among the winners in the case of a draw. The auctioneer publicly announces the amount of the winning bid, but does not announce any other information (including the identity of the winner).

(Assumption 4). Sidepayments are not permitted between potential buyers.

(Assumption 5). The seller's reserve price is normalized to zero.

(Assumption 6). Unicity of equilibrium to the stage game is assumed.⁵

The Repeated Game

Let $\beta(t) : [0, a] \times [0, c] \times \dots \times [0, c] \rightarrow [0, c]$ be a bidding function at time t . This bidding function sends current valuation as well as all previous winning bids, $(b^w(t) \in [0, c])$ into the bidding space $[0, c]$. Let $\beta = \times_{t \in \mathbb{N}} \beta(t)$ be the strategy set of available bidding functions.

(Assumption 7). Players discount future stage profits with a common discount rate $\delta \in (0, 1)$.

(Assumption 8). Valuations are drawn independently and identically across time.

(Assumption 9). The cartel restricts itself to symmetric and undominated (in the sense of Pareto) collusive schemes.

(Assumption 10). The equilibrium concept used is that of Rubinstein's (1979) (subgame) perfect equilibrium.

Assumption 7 above permits the restriction that each potential buyer submit bids in a compact interval $[0, c]$. This can be done by taking c to be defined as the highest possible payoff one can receive in the entire game, i.e. $c = \frac{a}{1-\delta}$. Assumption 9 can be done away with by imposing the Nash bargaining axioms. This would not be a departure from the mainstream of the literature since modelers usually choose such a focal point when confronted with a continuum of possible payoff vectors. However, providing a criterium for the selection of equilibria in repeated games is outside the scope of this paper. Our objective is simply to study the best symmetric collusive strategy. Remark that under the informational structure imposed by Assumption 3 a cartel must, and can, be all encompassing. This is because a cartel cannot distinguish cheating by a member or non-member, and it can credibly menace any deviation with a punishment. Finally, note that due to the assumptions of ex-ante symmetry and continuous distribution function, a theorem from Milgrom and Weber (1982) can be used to show the existence of a symmetric bidding equilibrium where the bidding functions are increasing in valuation.

1.4 Static Representation of Repeated Auctions

With these assumptions and notations in mind, we now proceed to analyze the repeated game as a single stage game. Abreu, Pearce, and Stacchetti (1986) were the first to use a generalization of the techniques of dynamic programming to analyze noncooperative games. This permits a very convenient representation of repeated games as much more tractable single stage games. The intuition is the following. The single stage representation of a repeated game “factorizes” the gains accruing to each player into two parts. The first part is the gain from the current stage game and the second part is expected, future gains which can be a function of any observable action taken in the current period. So this representation needs two elements: a payoff function for the stage game and a function describing future gains. Proposition 1 states that an equilibrium to the single stage game which satisfies a “self generation” criterium is an equilibrium to the repeated game. Proposition 2 states that any perfect equilibrium to the repeated game can be represented as a Nash equilibrium to the single stage game. Thus the single stage representation is equivalent to the dynamic game. Before stating these propositions some definitions are presented.

Definition 1 Let W be a bounded Borel subset of \mathbb{R} . Let $\beta : [0, a] \rightarrow [0, c]$ and $U : [0, c] \rightarrow W$ be Borel measurable functions and call (β, U) a collusive mechanism. (β, U) is called admissible with respect to W iff for all $b \notin \beta([0, a])$:

$$\pi(v, \beta(v)) + \delta \mathbb{E}_{b^w} (U(b^w) \mid \beta(v)) \geq \pi(v, b) + \delta \mathbb{E}_{b^w} (U(b^w) \mid b)$$

$\pi(v, b)$ is the expected payoff of an agent with valuation v who bids b given that the other agents follow strategy β . $U(b^w)$ represents the expected gains of all agents, given that the winning bid is b^w . The operator $\mathbb{E}_{b^w}(\cdot | b)$ is the conditional expectation of future gains, from the perspective of an agent having bid b where all other players follow strategy β . Note that this conditional expectation is well defined due to the assumptions on (Borel) measurability of β and U , as well as the boundedness of W .

Furthermore define $u(v; \beta, U) := \pi(v, \beta(v)) + \delta \mathbb{E}_{b^w}(U(b^w) | \beta(v))$. Note that an ex-ante value can be attached to $u(v; \beta, U)$, denoted as $\mathbb{E}_v u(v; \beta, U)$, by taking the expectation over the valuation. Finally a set valued function $B(W)$ is defined.

Definition 2 For any (Borel) $W \subset \mathbb{R}$ define:

$$B(W) := \{\mathbb{E}_v u(v; \beta, U) \mid (\beta, U) \text{ is admissible wrt } W\}$$

Definition 3 A bounded (Borel) W such that $W \subset B(W)$ is called self generating.

Propositions 1 and 2 are now corollaries to the propositions on self-generation and factorization in Abreu, Pearce and Stacchetti, since we are assured of the existence of the conditional expectation. The reader is referred to Abreu, Pearce, and Stacchetti (1986) for the proofs. In the following take V to be the set of perfect payoffs.

Proposition 1 Take any bounded (Borel) $W \subset \mathbb{R}$ which is self generating. Then $B(W) \subset V$.

Proposition 2 $V = B(V)$.

1.5 Optimal Collusive Rules

The goal of this section is to characterize optimal collusive schemes with the help of the Abreu, Pearce and Stacchetti static game. Results are presented in two subsections. Subsection 1.5 presents propositions which are used in Subsection 1.5 to present the problem of a maximizing cartel succinctly. Many of the proofs appear in the appendix

Punishments and Rewards

In explicitly repeated auctions there are two types of constraints which need to be satisfied. One corresponds to the usual self selection constraint: if β is the collusive bidding rule, then type v prefers bidding $\beta(v)$ to bidding $\beta(v')$. The other constraint will be referred to as admissibility. This constraint states that if β is the bidding rule, type v prefers bidding in $\beta([0, a])$, as opposed to bidding outside $\beta([0, a])$. These two constraints are qualitatively different because in the latter case, a defection can be detected.

Typically the incentive compatibility constraints force the mechanisms to be structured in such a way that types self select. In the problem at hand, the mechanism must indeed be so structured since valuations are not observable. Consider a collusive mechanism $\{\beta(\cdot), U(\cdot)\}$ where β is the bidding function mapping types into bids and where $U(b^w)$ is the future expected payoffs when the winning bid is b^w . Let $Q(v; \beta)$ denote the probability that a player bidding $\beta(v)$ wins the auction

given that β is the bidding function used by everyone else. Since the auctioneer awards the object to the highest bidder, and in the case of draws randomizes, $Q(v; \beta)$ can be written explicitly. Obviously if v is in an interval where β suggests submitting an increasing bid, then $Q(v; \beta) = F(v)^{n-1}$. Otherwise if β is constant within the interval $[v_i, v_j]$ then for all $v \in [v_i, v_j]$ we have:

$$Q(v; \beta) = \frac{F(v_j)^n - F(v_i)^n}{n(F(v_j) - F(v_i))} \equiv Q(v_i, v_j).$$

Incentive compatibility can thus be written:

$$\begin{aligned} u(v; \beta, U) &:= [v - \beta(v)]Q(v; \beta) + \delta \mathbf{E}_{b^w}(U(b^w) \mid \beta(v)) \\ &\geq [v - \beta(v')]Q(v'; \beta) + \delta \mathbf{E}_{b^w}(U(b^w) \mid \beta(v')) \quad \forall v' \in [0, a]. \end{aligned}$$

Where

$$\mathbf{E}_{b^w}(U(b^w) \mid \beta(v)) = \left[\int_v^a U(\beta(s)) dF(s)^{n-1} + U(\beta(v)) F(v)^{n-1} \right].$$

Using the logic of Myerson (1981) we know that incentive compatibility is equivalent to demanding that

$$\frac{d}{dv} u(v; \beta, U) = Q(v; \beta)$$

$Q(v; \beta)$ is nondecreasing in v .

We can thus obtain a more convenient characterization of incentive compatibility which lets us explicitly solve for the bidding function in terms of type conditional

probabilities of winning:

$$\beta(v) = v - \left[\frac{\int_0^v Q(s; \beta) ds + \delta \int_0^v [U(\beta(s)) - U(\beta(v))] dF(s)^{n-1}}{Q(v; \beta)} \right] \quad (1.1)$$

It is important to note that the bidding function will generally be composed of (positively) sloped and flat regions. From McAfee and McMillan (1992) we know that a perfectly flat bidding scheme is a “folk” type result which speaks to the case when δ is arbitrarily close to unity.

If we are to study collusion in auctions from an explicitly repeated point of view, as was suggested in the Introduction and in Section 1.2, another constraint must be imposed: admissibility. The admissibility constraints will be relevant whenever β contains flat regions. The following lemma furnishes some preliminary results. Its proof can be found in the appendix.

Lemma 1 Let \underline{U} denote the lowest expected profit credibly attainable. An incentive compatible mechanism $\{\beta(\cdot), U(\cdot)\}$ is admissible if and only if for all v^* at the highest edge of a flat bidding range we have:

$$\begin{aligned} [v^* - \beta(v^*)]Q(v^*; \beta) + \delta U(\beta(v^*))F(v^*)^{n-1} + \delta \int_{v^*}^a U(\beta(s))dF(s)^{n-1} \geq \\ [v^* - \beta(v^*)]F(v^*)^{n-1} + \delta \underline{U}F(v^*)^{n-1} + \delta \int_{v^*}^a U(\beta(s))dF(s)^{n-1}. \end{aligned}$$

The above is similar to the example introduced in Section 1.2, in that we compare gains to deviation with gains to compliance. The proof consists primarily of showing the existence of discontinuities in the bidding function, showing that the imposition of the harshest possible punishment is always beneficial. Finally we

prove that if all v^* at the highest edge of a flat bidding range respect admissibility then all types do.

Using the incentive compatibility constraint (1.1) and the above lemma, we can rewrite admissibility as:

$$\delta[U(\beta(v^*)) - \underline{U}] \geq \left[\int_0^{v^*} Q(s; \beta) ds + \delta \int_0^{v^*} [U(\beta(s)) - U(\beta(v^*))] dF(s)^{n-1} \right] \frac{F(v^*)^{n-1} - Q(v^*; \beta)}{F(v^*)^{n-1} Q(v^*; \beta)}. \quad (1.2)$$

Note that when the bidding function is strictly increasing admissibility is automatically satisfied.

Define $\bar{\mathcal{M}}$ as an incentive compatible and admissible mechanism which yields the highest level of expected utility \bar{U} . Define $\underline{\mathcal{M}}$ as an incentive compatible and admissible mechanism which yields the lowest level of expected utility \underline{U} . From section 5 of Abreu, Pearce, and Stacchetti (1986) the set of perfect payoffs is compact, so we are assured that $\bar{\mathcal{M}}$ and $\underline{\mathcal{M}}$ are well defined. The remainder of this section aims to characterize $\bar{\mathcal{M}}$ and $\underline{\mathcal{M}}$.

In order to satisfy admissibility, credible punishments must be available. Proposition 3 shows that the harshest possible punishment gives the same payoffs as the single stage Nash equilibrium. Its proof can be found in the appendix.

Proposition 3 *No perfect strategy can give less than the payoff associated with the stage game Nash equilibrium repeated ad infinitum.*

This proposition is an important step in endogenizing punishments as it charac-

terizes the gains to optimal (in the sense of Abreu (1986)) punishments. The proof relies heavily on the fact that $H(v)$ is decreasing in v along with the fact that the only information released by the auctioneer is the winning bid (Assumption 3).

Inequality (1.2) leads us to an important result on collusion in repeated auctions.

Proposition 4 *An optimal collusive mechanism, $\bar{\mathcal{M}} = \{\beta(\cdot), U(\cdot)\}$, must exhibit the bang-bang property:*

$$U(b) = \bar{U} \quad \text{for almost every } b \in \beta([0, a]).$$

Proof of Proposition 4 Let $\{Q(\cdot; \beta), U(\cdot)\}$ be an optimal collusive mechanism, generating \bar{U} , for which the bang-bang property does not hold. Therefore, there exists an interval, $[v_1, v_2]$, such that $U(\beta(v)) < \bar{U}$ for all $v \in [v_1, v_2]$. In this case we show that there exists an alternative incentive compatible, admissible mechanism which generates higher rents for all types. This mechanism consists of raising $U(\beta(v))$ by some small amount ΔU whenever $v \in [v_1, v_2]$. Denote this new continuation function by U^* . All flat bidding regions above v_2 are cut into two regions otherwise, the probability of winning is held constant for all other types.

From equation (1.1), if we increase continuation payoffs in $[v_1, v_2]$ and wish to preserve type conditional probabilities of winning an auction ($Q(\cdot, \beta)$), it is necessary to change bids in order to retain incentive compatibility. Bids of types lower than v_1 are unaffected by such a change. Studying inequality (1.2) immediately

shows that types in $[v_1, v_2]$ will satisfy admissibility. The variation in the rents to $v < v_2$ are given by:

$$\Delta u(v_2) = [F(v_2)^{n-1} - F(v_1)^{n-1}] \Delta U$$

which is strictly positive for all $\Delta U > 0$.

Consider the intervals $[\underline{v}_i, \bar{v}_i]$ where $\underline{v}_i \geq v_2$ and for which the bidding function is flat. Find recursively a \hat{v}_i where the following holds:

$$(\hat{v}_i - \underline{v}_i)[Q(\underline{v}_i, \bar{v}_i) - Q(\underline{v}_i, \hat{v}_i)] = \Delta u(\underline{v}_i) = \Delta u(\bar{v}_i).$$

Create a new incentive compatible bidding function (using the new continuation function U^*), β^* different from β only in that $Q(\cdot; \beta^*)$ exhibits discontinuities at all \hat{v}_i but constant on all the $[\underline{v}_i, \hat{v}_i]$ and $[\hat{v}_i, \bar{v}_i]$.

We now have $\Delta u(v) = (v - \hat{v}_i)[Q(\hat{v}_i, \bar{v}_i) - Q(\underline{v}_i, \bar{v}_i)] \geq 0$ for all $v \in [\hat{v}_i, \bar{v}_i]$. Additionally for all $v \in [\underline{v}_i, \hat{v}_i]$ we have $\Delta u(v) = [v - \underline{v}_i][Q(\underline{v}_i, \hat{v}_i) - Q(\underline{v}_i, \bar{v}_i)] + \Delta u(\underline{v}_i) \geq 0$. Hence the variation in rents for all types is non negative by construction. It remains to verify that the new mechanism is admissible for all $v > v_2$ if the original mechanism was.

First consider checking admissibility in $[\hat{v}_i, \bar{v}_i]$. With the original mechanism we have

$$\bar{v}_i - \beta(\bar{v}_i) = (\bar{v}_i - \hat{v}_i) + \frac{u(\hat{v}_i; \beta, U)}{Q(\underline{v}_i, \bar{v}_i)}.$$

Under the new mechanism we have that:

$$\bar{v}_i - \beta^*(\bar{v}_i) = (\bar{v}_i - \hat{v}_i) + \frac{u(\hat{v}_i; \beta^*, U^*)}{Q(\hat{v}_i, \bar{v}_i)}.$$

Since $u(\hat{v}_i; \beta^*, U^*) = U(\hat{v}_i; \beta, U)$, and $Q(\underline{v}_i, \bar{v}_i) < Q(\bar{v}_i, \hat{v}_i)$ we have that:

$$\bar{v}_i - \beta^*(\bar{v}_i) < \bar{v}_i - \beta(\bar{v}_i)$$

So the admissibility constraint, which can be written:

$$\delta[\bar{U} - \underline{U}]F(\bar{v}_i)^{n-1} \geq (\bar{v}_i - \beta^*(\bar{v}_i))[F(\bar{v}_i)^{n-1} - Q(\bar{v}_i; \beta^*, U^*)],$$

is sure to be satisfied, since the original mechanism was assumed to be admissible and $Q(\bar{v}_i; \beta^*, U^*) > Q(\bar{v}_i, \beta, U)$ and $(\bar{v}_i - \beta^*(\bar{v}_i)) < (\bar{v}_i - \beta(\bar{v}_i))$.

Finally, check admissibility in $[\underline{v}_i, \hat{v}_i]$. Notice that as ΔU approaches zero, \hat{v}_i must approach \underline{v}_i . It follows that for a ΔU sufficiently close to zero, the admissibility constraint will be respected for \hat{v}_i . \square

Consider the McAfee and McMillan flat bidding scheme. In this instance, the incentive compatibility constraint is trivial since there is only one "advised" bid (the reservation price). However their static formulation abstracted away from admissibility as was alluded to in the example of section 1.2. Proposition 4 states that an optimal collusive scheme should be structured so that any bid in $\beta([0, a])$ be treated as respecting cartel rules, and any bid not in $\beta([0, a])$ should be treated as defection⁶.

It follows directly from Proposition 4 that the bidding function takes on the familiar form:

$$\beta(v) = v - \frac{\int_0^v Q(s; \beta) ds}{Q(v; \beta)}. \quad (1.3)$$

It is also easy to verify that any flat section in the bidding function must be preceded and followed by discontinuities. Suppose that a flat section somewhere in the bidding function starts at valuation $v^* > 0$ and ends at valuation $v^{**} < a$. Since the probability of a bidder with valuation v^* winning is discretely greater than the probability of a bidder with valuation $v^* - \varepsilon$ for any $\varepsilon > 0$, the denominator of the above equation increases discretely at v^* . Therefore $\beta(v^*)$ must increase discretely as well. The argument that a flat section must be followed by a discontinuity is precisely the same.

Collusive Schemes as Optimization Problems

Define the operator Ψ in the following manner:

$$\Psi(K) = \sup_{\beta} \int_0^a H(v) [Q(v; \beta) - F(v)^{n-1}] dF(v) \quad (1.4)$$

subject to:

$$F(v)^{n-1} K \geq \frac{\int_0^v Q(s; \beta) ds}{Q(v; \beta)} [F(v)^{n-1} - Q(v; \beta)] \quad \forall v \in [0, a]. \quad (1.5)$$

The constraint is the admissibility constraint where the bid has been substituted out using equation (1.3) and where the variable K can be seen to take the place of $\delta[\bar{U} - \underline{U}]$. $\Psi(K)$ in equation (1.4) represents the difference in the per period rents between collusion using bidding function β and playing the Nash stage game equilibrium strategy. Note that if

$$\frac{\delta}{1-\delta}\Psi(K) \geq K$$

then the collusive scheme giving payoffs of $\Psi(K)$ every period is admissible. This amounts to comparing K with the future payoffs generated by K .

The following lemma regroups several preliminary results. Its proof is relegated to the appendix.

Lemma 2 i) $\Psi(\cdot)$ is non decreasing in its argument.

ii) $\Psi(0) = 0$.

iii) For any $K \geq a[1 - (1/n)]$, $\Psi(K) = \int_0^a H(v)[\frac{1}{n} - F(v)^{n-1}]dF(v)$.

iv) There is a finite θ such that $\Psi(K) \leq \theta K$ for all K .

The above lemma contains some immediately interpretable results. The fact that $\Psi(\cdot)$ is non decreasing in its argument, means that gains to collusion are non decreasing in δ . And in fact, when players are perfectly myopic ($\delta = 0$) then collusion gains and the gains to the Nash stage game equilibrium strategy are equal. This of course follows from the fact that $\Psi(0) = 0$. The interpretation of part *iii* is straightforward to interpret and is best seen as a result of McAfee and McMillan (1992) who only consider the incentive compatibility constraint. The

interpretation of condition *iv* will be postponed until the statement of proposition 5.

It is possible to prove that $\Psi(K)$ subject to constraint (1.5) is continuous by appealing to the Theorem of the Maximum, however this does not imply that collusive profits are continuous in δ . Consider Figures 1.1 and 1.2 which plot $\Psi(K)$ against K . The maximal admissible and incentive compatible collusive payoffs are given by the intersection of $\Psi(K)$ and the line with slope $\frac{1-\delta}{\delta}$. This follows because whenever $\Psi(K) \geq \frac{1-\delta}{\delta}K$ we know that the future expected collusive rents are large enough to satisfy admissibility required to enforce this collusion. As in Figure 1.1, if $\Psi(K)$ is concave in K , then collusive gains are indeed continuous in K . However if $\Psi(K)$ is not globally concave, as is the case in Figure 1.2 then a small change in δ could generate a large change in the intersection of $\Psi(K)$ with $\frac{1-\delta}{\delta}$. The relationship between the concavity of $\Psi(K)$ and the parameters of the problem is a complex one and we have no reason to believe that $\Psi(K)$ is globally concave for all distributions satisfying Assumption 1. Nevertheless we can state the following proposition.

Proposition 5 *i) For all δ there exists an optimal collusive scheme.*

ii) There exists a $\hat{\delta} > 0$ such that if $\delta < \hat{\delta}$ no collusion is possible.

Proof of Proposition 5 *i)* Let $\Psi^* = \sup_K \{ \sup_{\beta} \int_0^a H(v)[Q(v; \beta) - F(v)^{n-1}] dF(v) \}$ subject to constraint (1.5). Let K^* and β^* be the arguments which obtain the sup. Assume that Ψ^* is not admissible, i.e. $\frac{\delta}{1-\delta}\Psi(K^*) < K^*$. But there exists some sequence $K_n \rightarrow K^*$ such that $\frac{\delta}{1-\delta}\Psi(K_n) \geq K_n$ for all n . If the sequence $\{K_n\}$ is

non monotone or monotone decreasing, then we have a contradiction since $\Psi(K)$ is non decreasing in K . So assume that $K_n \uparrow K^*$. Since $K_n \uparrow K^*$, there exists a sequence $\varepsilon_n \downarrow 0$ such that $K_n + \varepsilon_n = K^*$. Making a substitution we obtain $\frac{\delta}{1-\delta}\Psi(K^*) < K_n + \varepsilon_n$. But for n sufficiently large, we have $\frac{\delta}{1-\delta}\Psi(K^*) < K_n$ which is a contradiction since $K^* \geq K_n$ and Ψ is nondecreasing.

ii) From Lemma 2 we know that there exists a finite number θ such that $\Psi(K) \leq \theta K$ for all K . For a collusive scheme to be admissible and incentive compatible we require that $\frac{\delta}{1-\delta}\Psi(K) \geq K$. Putting $\frac{1}{\theta} = \frac{\hat{\delta}}{1-\hat{\delta}}$ assures us that for all $\delta < \hat{\delta}$, this cannot be the case.

The next proposition narrows down the class of optimal collusive bidding functions.

Proposition 6 *If $\bar{U} > \underline{U}$ then an optimal collusive bidding function can contain no continuous increase in the bidding function.*

The proof of proposition 6 can be found in the appendix. The implications of this proposition are strong as it implies that an optimally colluding cartel will use a bidding function which has a finite range. For example all bidders with type in $[v_0, v_1]$ bid b_0 , all bidders with type in $[v_1, v_2]$ bid b_1 etc. Athey, Bagwell, and Sanchirico (1998) find a similar result for collusion in a Cournot oligopoly with incomplete information about cost.

1.6 Auctioneer Behavior

The tractability of the model used up to now has come at the expense of many simplifying assumptions. This current section argues that despite the simplicity of the model, analysis of collusion in an explicitly repeated environment can lead to interesting and robust insights as to how an auctioneer can best structure a series of auctions to resist bidder collusion. This section has three subsections, each presenting a different way that auctioneers could design auctions to lessen their losses due to collusion. The following lemma is a preliminary result used in each of the following subsections and its proof can be found in the appendix.

Lemma 3 If an optimal collusive bidding function exhibits one or more discontinuities, then there exists at least one type indifferent between adhering to and defecting from prescribed behavior.

Reserve Prices and Bidding Ceilings

The use of reservation prices to combat collusion has been studied by Graham and Marshall (1987) as well as McAfee and McMillan (1992). This analysis is strengthened by the observation that auctioneers in fact do use reserve prices to increase their profits in the face of collusion.⁷ In McAfee and McMillan (1992), since the cartel colludes so that all types are awarded the good with equal probability, the objective of the auctioneer is to solve the following:

$$\max_r : (r - v_0)[1 - F(r)^n],$$

where r denotes the reserve price and v_0 the seller's valuation.

In the more general case studied in this paper, the task of finding an optimal reserve price is a more daunting one. This is principally due to the fact that an optimal collusive rule depends on the reservation price in a non trivial way (as was the case studied by McAfee and McMillan (1992)). The objective of the auctioneer is:

$$\max_r : \int_r^a \beta(v, r) dF(v)^n$$

where the bidding function not only depends on the valuation but also on the reserve price. It is an easy exercise to extend the analysis of subsection 1.5 to accommodate a non zero reserve price. This is necessary to do in order to derive an optimal collusive bidding function for a given reserve price. Assuming differentiability and concavity of the problem, we are looking for r^* such that:

$$-\beta(r^*, r^*) n f(r^*) F(r^*)^{n-1} + \int_{r^*}^a \frac{\partial \beta}{\partial r}(v, r^*) dF(v)^n = 0.$$

In McAfee and McMillan (1992), $\frac{\partial \beta}{\partial r}(v, r)$ is equal to one. In a more traditional non collusive setting (for example Myerson (1981)) $\frac{\partial \beta}{\partial r}(v, r)$ is equal to $\frac{F(r)^{n-1}}{F(v)^{n-1}}$. In a general collusive setting, it is not possible to assign a value to $\frac{\partial \beta}{\partial r}(v, r)$. This is due to the ambiguous effect of a reserve price on the admissibility constraint: a change in the reserve price affects the admissibility constraint through \underline{U} ; additionally a change in the reserve price modifies a collusive bidding function by changing type conditional probabilities of winning the auction. Hence, the impact of the reserve

price on the ability of the cartel to collude is ambiguous, a fortiori so is its impact on the equilibrium bidding schedule.

A tool to increase auctioneer profits which is only apparent from an explicitly repeated context is a bidding ceiling. Consider the example presented in section 1.2 but with a bidding ceiling of $1/4$. Single stage expected Nash profits now increase and become $31/162$. Therefore in order for there to be no type of bidder who does not want to defect from the strategy of bidding zero it is necessary and sufficient that:

$$\frac{1}{2} + \frac{\delta}{1-\delta} \frac{1}{4} \geq 1 + \frac{\delta}{1-\delta} \frac{31}{162}$$

Where the left hand side represents the gains to obeying the collusive rule for an agent with valuation one, and the right hand side represents the gains to defecting from the collusive rule for an agent with valuation one. This equation is only satisfied for δ larger than $6/7$. That bidding ceilings can positively effect auctioneer profits is perhaps surprising, but the intuition is quite straightforward if one thinks in terms of a repeated context: lowering the bidding ceiling makes future punishments less severe, and a less severe punishment can support a less profitable form of collusion. The following proposition gives some general conditions as to when lowering a bidding ceiling can be useful.

Proposition 7 *Suppose that the current price ceiling, \bar{b} , is strictly greater than the highest bid prescribed by the optimal collusive bidding function. Then auctioneer profits can be strictly increased by a lowering of the bid ceiling.*

Proof of Proposition 7 Assume, without loss of generality, that the bidding ceiling is less than or equal to the highest single stage Nash bid. Now remark that since \bar{b} is strictly greater than the highest prescribed bid, \bar{b} can be lowered by $\epsilon > 0$ while still preserving this inequality. Furthermore since lowering \bar{b} by any amount increases the single-stage Nash expected payoffs, any indifferent types will now strictly prefer defecting from the cartel to obedience. Therefore, the cartel must remedy this problem by decreasing the length of certain “flat spots” on the bidding function or adding more discontinuities. Either response results in lower profits to the cartel and higher profit to the auctioneer. If the optimal bidding function contains no discontinuities, then there may not be an indifferent agent (lemma 3). However, by choosing the bidding ceiling such that the Nash single stage payoff and the collusive (bidding zero) single stage payoff differ by less than $a \frac{n-1}{n} \frac{1-\delta}{\delta}$, then at least one type strictly prefers defection. This new bidding ceiling obviously leads to a higher average winning bid. \square

Stated differently, the above says that a necessary condition for an auctioneer to maximize profits is to have the bidding ceiling equal to the highest prescribed bid of an optimal collusive bidding function. This is a strong result since it implies that auctioneers always profit from the imposition of a ceiling. Unfortunately, while bidding ceilings and reserve prices may be useful in increasing auctioneer payoffs the following proposition shows that they are not sufficient for maximizing auctioneer profits. The intuition is similar to that of proposition 7 in that one can outlaw other portions of the range of bids in order to weaken future punishments.

Proposition 8 *Controlling the bidding ceiling and reserve price is not sufficient for an auctioneer to maximize profit.*

Proof of Proposition 8 Suppose that the auctioneer has maximized his profit with respect to the bidding ceiling and floor.⁸ From lemma 3 and proposition 7 there exists at least one type indifferent between defecting from and adhering to cartel rules. Denote the prescribed bids of this bidding function $\{b_1, \dots, b_k\}$, such that $b_i > b_{i-1}$ for all i . Proposition 7 implies that $\bar{b} = b_k$. Now outlaw bidding in an interval just below b_k : $(b_k - \varepsilon, b_k)$. Single stage Nash payoffs must increase when any interior portion of the bidding range is outlawed. Since there exists a type indifferent between adhering and defecting under the old system, this type now strictly prefers defecting. Therefore, in order for cartel profits not to decrease, the cartel must come up with a more profitable collusive scheme. By assuming that the cartel had chosen an optimal collusive bidding function, the only way for the cartel to improve payoffs is to permit bidding at $b_k - \varepsilon$. The key is recognizing that since ε can be taken to be arbitrarily small the admissibility constraint will not be binding between bidding $b_k - \varepsilon$ and b_k . Consider three possibilities: 1) replace b_k by $b_k - \varepsilon$, 2) replace b_{k-1} by $b_k - \varepsilon$, 3) add a new bidding level at $b_k - \varepsilon$. In each case if a cartel were able to increase its profitability this would imply that it had originally chosen a non optimal collusive bidding function—a contradiction. \square

The surprising results on bidding ceilings is quite intuitive, yet to our knowledge bidding ceilings do not exist.

Tiebreaking Rules

The inefficiency of the McAfee and McMillan collusive scheme has been retained to a certain extent; thus one would expect to observe identical bids. Due to the presence of these identical bids the choice of tiebreaking rule used by the auctioneer becomes important through its effect on the admissibility constraint. Consider the example in section 1.2. Assuming a randomization on the part of the auctioneer in case of ties (ex post randomization) we see that the McAfee and McMillan scheme is enforceable only when $\delta \geq 6/7$. Now consider what happens if the auctioneer, prior to bids being submitted, publicly designates a winner in case of equality in the highest bid (ex ante randomization). Assume the designation is made by a flip of a coin. In this case the necessary and sufficient condition for obedience to this scheme is the following:

$$0 + \frac{\delta}{1-\delta} \frac{1}{4} \geq 1 + \frac{\delta}{1-\delta} \int_0^1 v(1-v)dv.$$

The difference here is the first term on the left hand side. This term reflects that a player with a valuation of one has been designated as the loser in case of a tie. The gains to defecting from cartel rules have increased as is reflected by solving for δ in the above inequality to reveal $\delta \geq 12/13$. The above argument is formalized in the following proposition.

Proposition 9 *Bidder profits under ex post randomization are always at least as large as under ex ante randomization.*

Proof of Proposition 9 Consider the optimal collusive scheme $\{v_1^*, v_2^*, \dots\}$, and

suppose $v \in [v_k^*, v_{k+1}^*]$. The admissibility constraint under ex post randomization is:

$$vQ(v_{k+1}^*, v_k^*) + \frac{\delta}{1-\delta} \sum_i Q(v_{i+1}^*, v_i^*) \int_{v_i^*}^{v_{i+1}^*} H(v) dF(v) \geq vF(v_{k+1}^*)^{n-1} + \frac{\delta}{1-\delta} \int_0^a H(v) F(v)^{n-1} dF(v).$$

Now consider the admissibility constraint under ex ante randomization:

$$vF(v_k^*)^{n-1} + \frac{\delta}{1-\delta} \sum_i Q(v_{i+1}^*, v_i^*) \int_{v_i^*}^{v_{i+1}^*} H(v) dF(v) \geq vF(v_{k+1}^*)^{n-1} + \frac{\delta}{1-\delta} \int_0^a H(v) F(v)^{n-1} dF(v).$$

Comparing $Q(v_{k+1}^*, v_k^*)$ with $F(v_k^*)^{n-1}$ implies that one can use lemma 3 to arrive at the following conclusions. If there exists one or more discontinuities in the bidding function then ex ante randomization strictly dominates ex post randomization. If there exists no discontinuity then ex ante randomization weakly dominates ex post randomization. \square

The above argument implies that a cartel would always prefer to face ex post randomization. The existence of rotating bidding schemes may be interpreted to be the cartel's reaction to an "unaccommodating" auctioneer. That the effect on auctioneer profits of such a simple change in the randomization rule is unambiguous, it is surprising that one does observe auctioneers who use ex post randomization.

Bundling

Oftentimes services can be procured through auctions occurring with a regular frequency. One might think of a garbage collection contract being awarded by a municipality through auction every year. For the present purposes, the important part of this scenario is that the auction occurs every year. Once again consider the example of section 1.2, but now assume that the auctioneer sells 2 items at every other period—the auctioneer bundles items. In this case a necessary and sufficient condition for obedience to the McAfee and McMillan scheme is the following inequality:

$$2\frac{1}{2} + \frac{\delta^2}{1 - \delta^2} \frac{1}{4} \geq 2 + \frac{\delta^2}{1 - \delta^2} \int_0^1 v(1 - v)dv.$$

Solving for δ one obtains $\delta \geq \sqrt{12/13}$. One observes two effects here. The first is the reduction in the effective discount rate from δ to δ^2 . The second is the increase in the gains to cheating. Again we formalize the above intuition into the following proposition.

Proposition 10 *Bidder profits are always decreasing in the number of items bundled together.*

Proof of Proposition 10 For simplicity consider bundling two items together, and compare the corresponding admissibility constraint with the admissibility constraint when there is no bundling. Again consider the collusive scheme $\{v_1^*, v_2^*, \dots\}$, and suppose $v \in [v_k^*, v_{k+1}^*]$. One can write down the admissibility constraints and compare $2v[F(v_{k+1}^*)^{n-1} - Q(v_{k+1}^*, v_k^*)]$ and $v[F(v_{k+1}^*)^{n-1} - Q(v_{k+1}^*, v_k^*)]$. The effect

of the decrease in the discount factor is straightforward. Using lemma 3 one can reason along the same lines as in the previous proposition. If the bidding function has at least one discontinuity then domination by bundling is strict. If the bidding function contains no discontinuity then the domination is weak. \square

It is important to stress that this last argument has ignored important considerations such as any costs a municipality may incur in making a long term commitment to a single service supplier.

1.7 Conclusion

Collusion without sidepayments in explicitly repeated auctions has been studied. It is the repeated nature of the model which differentiates it from earlier work. Techniques first developed by Abreu, Pearce and Stacchetti were used to endogenize punishments. These punishments differ from the usual trigger strategies in that an auction is a game of incomplete information. Collusion is characterized by its stability and inefficiency. Collusion is effectuated by submitting identical bids, or using a rotating bidding scheme—predictions which are commonly used cartel tactics. Studying an explicitly repeated game leads to several insights about optimal auctioneer behavior not apparent when studying a static game. The main finding of the paper, that cartels are obliged to use a randomization rule to overcome incentive compatibility problems, is supported by the observation that identical bids and rotating bid schemes are observed.

1.8 Appendix

Proof of Lemma 1 Suppose β to be constant on $[v_i, v_j]$. First remark that $Q(v; \beta)$ is discontinuous at v_j , and that in particular $Q(v_j; \beta) < F(v_j)^{n-1}$. Suppose that the condition in the statement of the lemma is violated and that β is continuous at v_j . Then since $U(\beta(v_j + \varepsilon)) \geq \underline{U}$ for all $\varepsilon > 0$, there exists a small ε such that bidders of type v_j prefer to bid $\beta(v_j + \varepsilon)$, violating the incentive compatibility constraints. Now suppose that β is discontinuous at v_j . From equation (1.1) one can verify that β is also discontinuous at v_j . Therefore there exists a $b \notin \beta([0, a])$ which is arbitrarily close to $\beta([v_i, v_j])$ yet which discretely increases the probability of winning. A (credible) punishment must be available to dissuade such a deviation. This punishment can be made as severe as possible without harming cartel gains, in so far as it is only used off the equilibrium path, i.e. $b^w \notin \beta([0, a])$.

To prove the only if part, notice that admissibility implies that all types respect the above inequality, and in particular v_j respects the above inequality.

To prove the if part, define $\beta^+ = \lim_{v \downarrow v_j} \beta(v)$. Similarly define $\beta^- = \lim_{v \uparrow v_j} \beta(v)$. Since β is discontinuous at v_j we have that $\beta^+ > \beta^-$. Any bid in $(\beta^-, \beta^+]$ will win the auction with probability $F(v_j)^{n-1}$. The inequality in the statement of the lemma implies that v_j prefers following his equilibrium strategy rather than deviating. Fix $v < v_j$. By monotonicity of preferences, v also prefers bidding β^- than bidding in $(\beta^-, \beta^+]$. A similar argument shows that all $v > v_j$ prefer bidding β^+ to bidding in $(\beta^-, \beta^+]$. \square

Proof of Proposition 3 Notice first that Assumption 3 implies that punishments cannot be bidder specific—all players must receive the same expected payoff from punishment.

Gains to a player with valuation v from the collusive mechanism (β, U) are:

$$\int_0^v Q(s; \beta) ds + \delta \int_0^a U(\beta(v)) dF(v)^{n-1},$$

which can be used to calculate ex-ante rents:

$$\int_0^a H(v) Q(v; \beta) dF(v) + \delta \int_0^a U(\beta(v)) dF(v)^{n-1}. \quad (1.6)$$

Consider infinite, or unrelenting, punishments. Let (β_1, U_1) be an incentive compatible collusive mechanism which minimizes (1.6):

$$\min_{\beta_1, U_1} : \int_0^a H(v) Q(v; \beta_1) dF(v) + \delta \int_0^a U_1(\beta_1(v)) dF(v)^{n-1}$$

Since payoffs are constrained to be in V , the perfect equilibrium set, and $V = B(V)$, there exists another collusive couple, (β_2, U_2) giving the same payoffs as $\int_0^a [\mathbb{E}_b(U_1(b) | \beta_1(v))] dF(v)$. Working iteratively in this manner one obtains that

the payoff to this punishment is:

$$\sum_{i=1}^{\infty} \delta^{i-1} \int_0^a H(v) Q(v; \beta_i) dF(v).$$

Incentive compatibility requires $Q(\cdot; \beta)$ to be non decreasing. From Lemma 4 in the appendix the above expression is minimized when we rule out constant bidding regions, i.e. when the bidding rules are constrained to induce efficiency so that $Q(v; \beta) = F(v)^{n-1}$. The single stage Nash strategy accomplishes this task while satisfying incentive compatibility. Note that when bidding rules are constrained to induce efficiency payoffs are necessarily equal to the Nash equilibrium stage game payoffs.

The next step is to show that participants always bid with probability one. Suppose a mechanism which randomly selects agents not to bid. The identity of the winning bidder is unobservable, so future gains can only be a function of the winning bid. Therefore for a certain agent with a given type to be indifferent between bidding and not bidding, all agents with lower valuation must strictly prefer not bidding and all agents with higher valuation must strictly prefer bidding. Consider a mechanism that in the first round outlaws bidding below $r > 0$, and let μ_0 be the future expected gains if nobody bids. Admissibility of the mechanism implies:

$$\delta \mu_0 F(r)^{n-1} \geq r F(r)^{n-1} + \delta \underline{U} F(r)^{n-1}.$$

And since $F(r)^{n-1} > (1 - F(s))F(s)^{n-1}$ for all s in $[0, r]$ we have that:

$$rF(r)^{n-1} + \delta \underline{U}F(r)^{n-1} > \int_0^r (1 - F(s))F(s)^{n-1} ds + \delta \underline{U}F(r)^{n-1}.$$

This previous expression represents using the Nash bidding strategy in the interval $[0, r]$. Therefore we have contradicted the supposition that outlawing bidding in a certain region dominated using the Nash bidding strategy. \square

Proof of Lemma 2 i) Fix $K^* > K$. Obviously any $Q(\cdot; \beta)$ satisfying constraint (1.5) for K will satisfy (1.5) for K^* .

ii) When $K = 0$ the left hand side of (1.5) is zero, so in order for the constraint to be respected the right hand side must be less than or equal to zero. By proposition 3, the only bidding function satisfying this requirement is the single stage Nash equilibrium bidding function.

iii) Incentive compatibility implies that due to informational asymmetries collusive gains can be no higher than those from using a scheme awarding the good to all types with equal probability (McAfee and McMillan (1992) Theorem 1). When such a scheme is used, a necessary and sufficient condition for all types to respect admissibility is for type a to respect admissibility. For type a to respect admissibility implies $K \geq a[1 - (1/n)]$.

iv) Suppose Q is the function derived from the bidding function which solves (1.4) subject to (1.5). Let Q advise bidding a constant amount on $[v_0, v_1], [v_2, v_3], \dots [v_{m-1}, v_m]$

$\Psi(K)$ can be written:

$$\begin{aligned} \Psi(K) &= \sum_{i=1}^m \int_{v_{2(i-1)}}^{v_{2i-1}} [1 - F(v)][Q(v_{i-1}, v) - F(v)^{n-1}] dv = \\ &= \sum_{i=1}^m \int_{v_{2(i-1)}}^{v_{2i-1}} [F(v)^{n-1} - Q(v_{i-1}, v)] \left(\int_{v_{2(i-1)}}^v [1 - F(x)] dx \frac{f(v)}{F(v) - F(v_{i-1})} - [1 - F(v)] \right) dv. \end{aligned} \quad (1.7)$$

The above equality is obtained from the fact that

$$\frac{\partial Q(v_{i-1}, v)}{\partial v} = [F(v)^{n-1} - Q(v, v_{i-1})] \frac{f(v)}{F(v) - F(v_{i-1})}.$$

From inequality (1.5) we have that

$$\frac{Q(v; \beta)}{\int_0^v Q(s; \beta) ds} F(v)^{n-1} K \geq F(v)^{n-1} - Q(v_{2(i-1)}, v) \quad \forall v \in [v_{2(i-1)}, v_{2i-1}].$$

Since $Q(v_{2(i-1)}, v) \leq F(v)^{n-1}$ and $Q(s, \beta) \geq F(s)^{n-1}/n$, we have:

$$\gamma K \geq nK \frac{F(v)^{n-1} F(v)^{n-1}}{\int_0^v F(s)^{n-1} ds} \geq [F(v)^{n-1} - Q(v_{2(i-1)}, v)],$$

where $\gamma = \max_v : \frac{nF(v)^{2n-2}}{\int_0^v F(s)^{n-1} ds}$. Note that γ is finite. Substituting this into equation (1.7) we obtain:

$$\Psi(K) \leq \gamma K \sum_{i=1}^m \int_{v_{2(i-1)}}^{v_{2i-1}} \left(\int_{v_{2(i-1)}}^s \left[\frac{1 - F(x)}{f(x)} - \frac{1 - F(s)}{f(s)} \right] \frac{dF(x)}{F(s) - F(v_{2(i-1)})} \right) dF(s).$$

Since the term in brackets is decreasing in $v_{2(i-1)}$, we have

$$\Psi(K) \leq \gamma K \int_0^a \int_0^s \left(\frac{1-F(x)}{f(x)} - \frac{1-F(s)}{f(s)} \right) \frac{dF(x)}{F(s)} dF(s).$$

This proves the existence of a finite number, θ such that $\Psi(K) \leq \theta K$. \square

The proof of proposition 6 makes reference to the following lemma which we state and prove before the proof of proposition 6.

Lemma 4 For any $0 < v'_0 < v_1 < a$ it is the case that:

$$\int_{v'_0}^{v_1} H(v) \frac{F(v_1)^n - F(v'_0)^n}{n(F(v_1) - F(v'_0))} dF(v) \geq \int_{v'_0}^{v_1} H(v) F(v)^{n-1} dF(v).$$

Proof of Lemma 4 After an integration by parts the above inequality can be written:

$$\int_{v'_0}^{v_1} H'(v) \frac{F(v_1) - F(v)}{(F(v_1) - F(v'_0))} dv \geq \int_{v'_0}^{v_1} H'(v) \frac{F(v_1)^n - F(v)^n}{(F(v_1)^n - F(v'_0)^n)} dv.$$

Since $H'(v) < 0$, it suffices to show:

$$\frac{F(v_1) - F(v)}{(F(v_1) - F(v'_0))} \leq \frac{F(v_1)^n - F(v)^n}{(F(v_1)^n - F(v'_0)^n)} \quad \forall v \in (v'_0, v_1).$$

Since it is the case that $F(v'_0) < F(v) < F(v_1)$, $F(v)$ can be written:

$$F(v) = \lambda F(v'_0) + (1 - \lambda)F(v_1) \quad \text{for some } \lambda \in (0, 1).$$

Therefore after substitution and a rearrangement it is sufficient to show that:

$$\begin{aligned} [\lambda F(v'_0) + (1 - \lambda)F(v_1)]^n [F(v_1) - F(v'_0)] \\ \leq [\lambda F(v'_0)^n + (1 - \lambda)F(v_1)^n] [F(v_1) - F(v'_0)] \end{aligned}$$

Which is always satisfied since $F(v)^n$ is a convex function of $F(v)$. \square

Proof of Proposition 4 Suppose an optimal collusive bidding function which is continuously increasing for some interval $[v_0, v_1] \subset [0, a]$. To show a contradiction, modify the bidding rule so that it awards the object to types in $[v_0, v_1]$ with equal probability and respects incentive compatibility. There are two cases which will be considered separately: 1) Type v_1 is weakly prefers the old scheme. 2) Type v_1 strictly prefers the new scheme. In each treatment incentive compatibility is verified while using the original value \bar{U} , then it is shown that the new bidding function generates a higher \bar{U} .

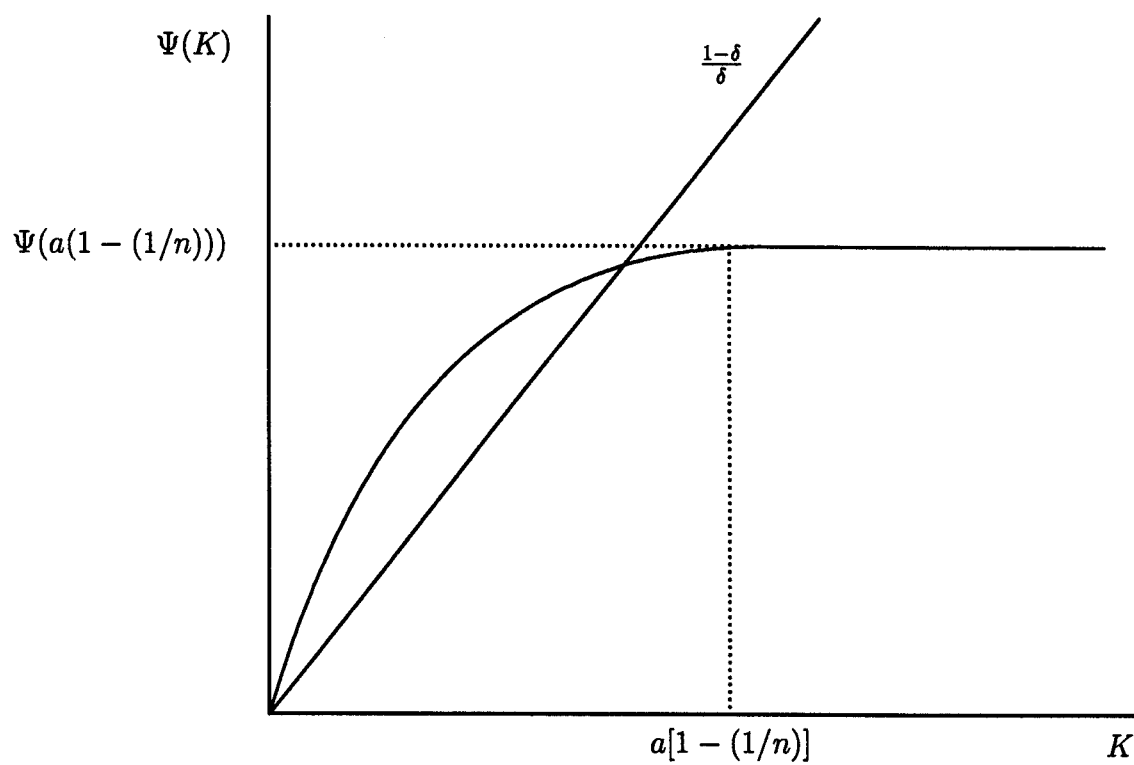
Case 1 Suppose that after invoking b'_0 , type v_1 is just as well off as under the old bidding function. Therefore bidding levels b_1 and b_2 can be maintained and all incentive compatibility constraints are satisfied. Now apply lemma 4 to see that the expected value to collusion has been increased. Suppose that type v_1 strictly

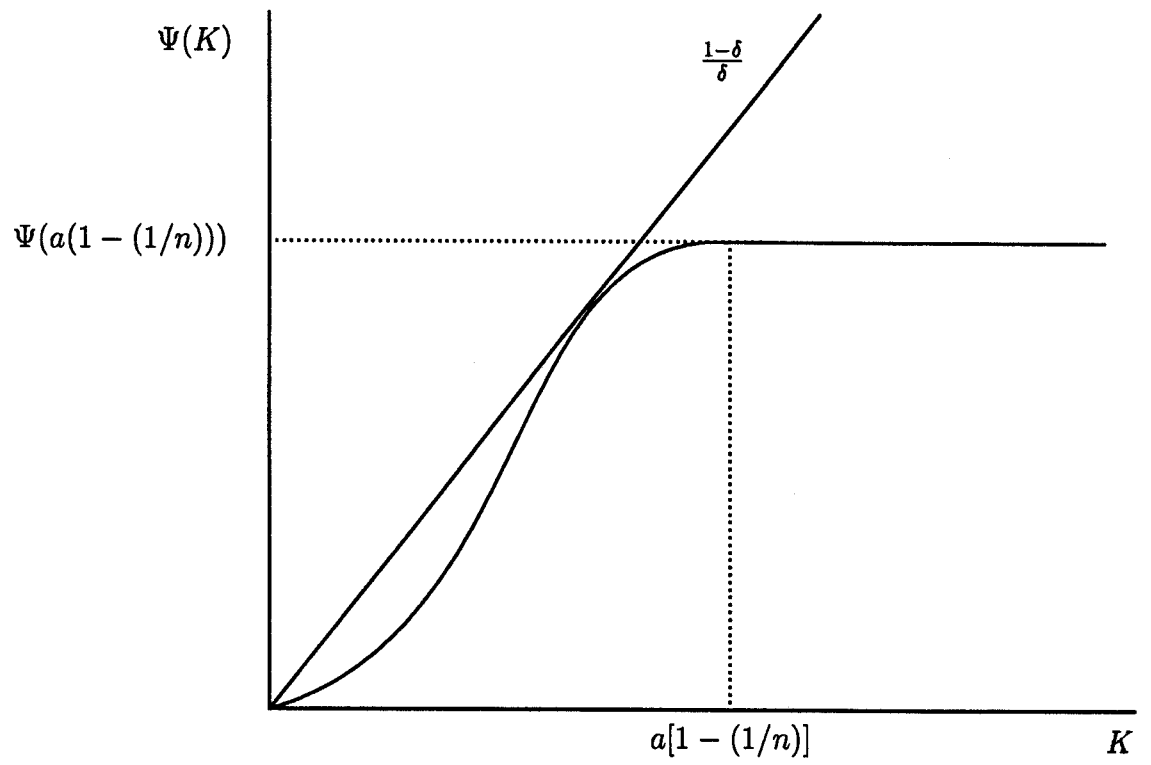
prefers the old scheme. Then if b_1 and b_2 are held constant, the type indifferent between bidding b'_0 and b_1 will be located to the left of v_1 . Now raise b_1 and b_2 such that the indifferent type between bidding b'_0 and b_1 remains v_1 and such that the indifferent type between bidding b_1 and b_2 remains v_2 . Notice that this change in the bidding function has not changed the probabilities that any type greater than v_1 wins the auction and all incentive compatibility and admissibility constraints are respected. Now apply lemma 4.

Case 2 Suppose that type v_1 strictly prefers the new scheme. Then the type indifferent between bidding b'_0 and b_1 moves to the right of v_1 . Call this new indifferent type v'_1 . Incite v'_1 to move back to v_1 by decreasing the continuation payoff if the winning bid is b'_0 . Such a change produces a more profitable bidding function by lemma 4, but may not produce higher rents. Invoke Proposition 4 to be assured of the existence of a collusive mechanism with a bidding function which is never continuously increasing and having the bang-bang property. This bang-bang mechanism uses a bidding function which is even more profitable. Furthermore, such a bang-bang mechanism generates $\bar{U} = \int_0^a H(v)Q(v; \beta^*)dF(v)$, where β^* is the newest bidding function. So since the bang-bang mechanism produces rents which are generated by a bidding function which dominates the original scheme, the proposition is proved. \square

Proof of Lemma 3 Suppose there does not exist an indifferent type under an optimal collusive bidding rule. Call this rule β and denote the single stage profits

it generates by $\int_0^a H(v)Q(v; \beta)dF(v)$. Due to the lack of an indifferent type, there exists $\eta > 0$ such that β' is incentive compatible if it satisfies $\int_0^a |\beta(v) - \beta'(v)|dv < \eta$. Now we can choose a (finite) sequence of bidding functions $(\beta^k)_{k=1}^K$ such that for all $k \in \{1, 2, \dots, K-1\}$: 1) $\int_0^a |\beta^k(v) - \beta^{k+1}(v)|dv < \eta$, 2) $\beta^K(v) = 0$ for all v , 3) $\beta^1 = \beta$, and 4) $\int_0^a H(v)(Q(v; \beta^{k+1}) - Q(v; \beta^k))dF(v) > 0$. We are assured of 4) because $H(v)$ was assumed strictly decreasing, and we are assured that $K > 1$ because there is at least one discontinuity in the bidding function. Obviously this sequence contains at least two bidding functions which are incentive compatible and at least one which gives higher single stage payoffs than β , contradicting the hypothesis on the optimality of β . \square

Figure 1.1: Graph of $\Psi(K)$

Figure 1.2: Graph of $\Psi(K)$

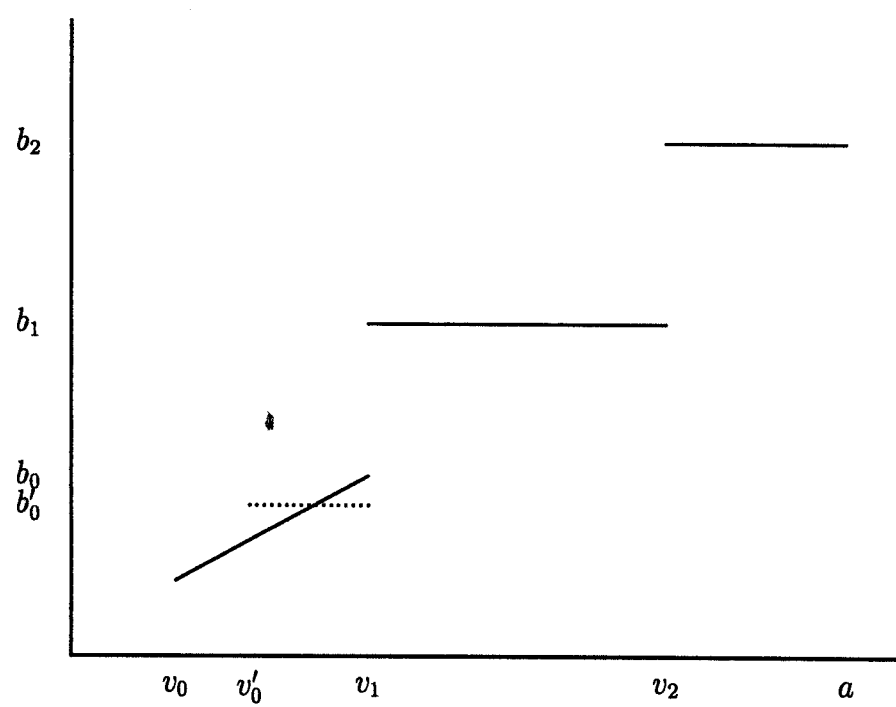


Figure 1.3: A bidding function

Notes

¹In fact it is difficult to give an example of a purely one shot auction.

²Under usual hypotheses identical bids should be observed with zero probability.

³In a recent paper Athey, Bagwell, and Sanchirico (1998) have explored this issue.

⁴This is the case for most common distributions. See Bagnoli and Bergstrom (1989) for more details.

⁵Quoting from Wilson (1992): "...for symmetric first-price auctions there is some presumption that the symmetric equilibrium is the unique equilibrium."

⁶Note that contrary to Abreu, Pearce, and Stacchetti (1986) the bang-bang property is defined so that it invokes the harshest punishment as well. This is not a necessary condition for Proposition 4, but a sufficient one. This is because a less than maximal punishment may suffice to ensure obedience to a bidding rule specifying bidding zero regardless of valuation. To see that a maximal punishment is sufficient, notice that deviation will be observed with probability zero thus will not affect payoffs. Furthermore, a harsher punishment can always support a scheme supportable by a weaker punishment. Note that if the admissibility constraint is binding, then the harshest possible punishments must be used as well.

⁷See the discussion in section II. of Graham and Marshall (1987).

⁸It can be shown that profits move continuously in the bidding ceiling and reserve price. Since we can consider a compact region over which to select the reserve price, we are assured of the existence of a maximum.

On Cartel Stability

Paul Johnson

Abstract

This paper studies collusion when discount rates differ among cartel members, with the goal of establishing a model where cartel instability arises. The revelation principle is invoked throughout, thus it is not literally true that defection in the sense of misreporting type arises, though it is argued that equilibrium strategies can be interpreted to suggest "undercutting" behavior. The model has players drawing iid discount factors in every period. The main implication is that the bang-bang results of Abreu, Pearce and Stacchetti, as well as the simple punishment results of Abreu no longer hold. In other words, players choose actions based on type and each action invokes a unique stream of future payoffs. The model's predictions are also compared with the some empirical evidence on collusion.

Acknowledgments: This research was supported by the SSHRC. I would like to thank Jacques Robert for extremely useful criticism and comments. I would also like to thank seminar participants at the summer 1999 meetings of the Société Canadienne d'Economie and the Canadian Economics Association. Of course, any remaining errors must remain mine alone..

2.1 Introduction

Collusion and cartels, as economists define the words, take many forms. A cartel of countries can collude to prevent over exploitation of fish stocks, or a cartel of firms can collude to raise prices. Despite the different forms that collusion can take, all cartels share certain properties. First, collusion is an attempt at a Pareto improvement (at least for the members of the cartel). If countries sign no treaty to protect fish stocks, then the end result is that each country catches fewer fish than if a treaty had been signed and respected. Second, cartel members always have some motivation not to respect a collusive agreement. Firms competing à la Bertrand and having agreed to fix price above marginal cost, could increase short term profits by undercutting other cartel members. Unfortunately for cartels, this latter effect invariably seems to dominate the former. Collusion is almost never stable: colluding bidders chisel one another (see Smith (1961)), member countries of international cartels do not respect quotas (see Barbezat (1989)), firms do not respect price guidelines (see Grossman (1996)) and firms do not respect agreements with workers (see Bertrand (1999)).

This last observation is troubling for economists, since most repeated game models of collusion predict perfect cartel stability over time. For a literature that has been in existence for over 25 years and the subject of so much research effort, this is problematic. While some may be quick to point out the widely known work of Porter (1983) as well as Green and Porter (1984)¹, theirs is not a wholly satisfying resolution to this paradox for two reasons. First, their model relies entirely on the imperfect nature of information at every stage of the game.

However, there is much evidence that cartels remain unstable in games of perfect monitoring. Secondly, defection or "undercutting" never arise in their model. In fact the reversionary periods should not even be considered to be punishments since it is common knowledge that each player has produced the cartel suggested quantity. The paradox still being unsolved, this paper offers a reconciliation of theory with the accepted stylized fact that cartels are generally unstable.

One of the most commonly made assumptions in the literature on collusion is that one player is just as patient as another player—players share a common discount rate. Two interesting examples are Harrington (1989) and Lehrer and Pauzner (1999). Harrington (1989) permits colluding firms to have different discount rates. Lehrer and Pauzner (1999) study how the set of feasible payoffs change when discount rates are heterogeneous. Despite the heterogeneity in both these papers, these differing discount rates remain perfectly known. This is the point of departure of this paper—not only is a possible heterogeneity imposed, but this heterogeneity is assumed to be private information. This adds a more complicated structure to the game, but it would seem that this is more realistic than the simplifying symmetry and perfect information assumptions.

This introduction has hinted that with the introduction of private information, a model emerges which replicates an important stylized fact of collusion. The revelation principle is used extensively, and as such it is not true that the end result has agents "defecting" by misreporting their types (discount rates.) Rather, agents can be thought of as truthfully reporting their levels of patience and are advised to take actions which could be interpreted as a "defection" or a "punishment." Perhaps the word "undercutting" is best suited in this sense, since

it carries none of the connotation of disagreement which "cheating" or "defection" do. This undercutting result flows directly from optimizing behavior: an agent undercuts if and only if undercutting is better for her than not undercutting. There is some evidence that cartels recognize that undercutting may be inevitable due to the presence of private information. The International Steel Cartel detailed in Barbezat (1989) is a good example of this type of behavior. This case study will be discussed in more detail in section 2.5.

The model developed in this paper assumes agents draw iid discount factors in every period. This assumption need not be interpreted literally as section 2.6 argues that instead of supposing that preferences change in every period, we can equivalently assume that expectations about future market conditions could change. In this model, I argue, something akin to defection and punishment arises on the equilibrium path (of course the caveat about the revelation principle must be kept in mind). The most striking result of this assumption is the structure which optimal collusive schemes must take: unlike Friedman's (1971) unrelenting punishments, Abreu's (1988) simple strategy profiles and unlike Abreu, Pearce, and Stacchetti's (1986) and Abreu, Pearce, and Stacchetti's (1990) bang-bang results, "punishments" must be made to fit the "crime". In other words future payoffs depend non trivially on current action. There is some evidence that cartel punishments are neither "bang-bang" nor unrelenting as in Friedman (1971).

Section 2.2 formally introduces the game to be studied through several assumptions and some notation. Section 2.3 adapts the method of Abreu, Pearce, and Stacchetti (1986) for the present purposes. Section 2.4 presents two examples. Section 2.5 addresses the problem of cartel behavior when discount rates

are private information, and may change with time. Section 2.6 considers some applications.

2.2 Assumptions and Notation

This paper studies a class of games satisfying certain properties giving a flavor of the prisoner's dilemma: cooperative play results in payoffs which Pareto dominate noncooperative play. Roughly, these properties state that the more profitable is a cartel, then the greater are gains to defection and the greater are losses if others defect. Many games of interest to analyze in a repeated context satisfy these properties. The most natural example is collusion in Cournot (quantity) competition. Broadly speaking, the game to be studied is symmetric and is one of perfect information. Furthermore it is repeated infinitely often. The distinguishing feature of this model is that discount rates (preferences) are ex post asymmetric and private information. An important restriction is that the model imposes symmetry on the strategies along the lines of Abreu, Pearce, and Stacchetti (1986). This restriction requires that after any history, all players play the same strategy. This, admittedly, entails a loss of generality since, in general, optimal symmetric collusion is not optimal (asymmetric) collusion. Nevertheless, symmetry in strategy provides significant simplification and does not change the fact that changing discount factors can produce cartel instability.

Denote the stage game by $\Gamma = (S_1, \dots, S_n, \pi_1, \dots, \pi_n)$. Where $N = \{1, \dots, n\}$ is the set of players and S_i is the pure strategy space of player i . Play occurs simultaneously in the stage game. The risk neutral preferences of player i are

measured by the bounded payoff function $\pi_i : S \rightarrow \mathbb{R}^2$. Symmetry is imposed by the following assumptions:

$$A1. S_i = S_j \quad \forall i, j \in N.$$

$$A2. \pi_i(s_1, \dots, s_i, \dots, s_j, \dots, s_n) = \pi_j(s_1, \dots, s_j, \dots, s_i, \dots, s_n) \quad \forall i, j \in N.$$

With the assumption of symmetry, attention can be restricted to arbitrary player $i \in N$. The following two technical assumptions will be needed:

$$A3. S_i \text{ is a compact interval of the real numbers.}$$

$$A4. \pi_i \text{ is once continuously differentiable.}$$

Note that due to A3, for any $s_i, s'_i \in S_i$ such that $s_i > s'_i$, there exists an $s_i^* \in S_i$ such that $s_i > s_i^* > s'_i$. This important fact will be referred to as the 'continuity property' of the game.

It will be convenient to relabel elements of the action space in the following manner:

$$s_i \geq s'_i \Leftrightarrow \pi_j(s_i, s_i, \dots, s_i) \geq \pi_j(s'_i, s'_i, \dots, s'_i).$$

We now assume that profits are strictly decreasing in one's own actions and strictly increasing in the actions of another player:

$$A5.$$

$$\begin{aligned} \frac{\partial \pi_i}{\partial s_i}(s) &< 0 \quad \forall s, \\ \frac{\partial \pi_i}{\partial s_j}(s) &> 0 \quad \forall s, i \neq j. \end{aligned}$$

It follows that there is a unique best response for player i to any s_{-i} ; call this best

response s_{min} . We can conclude that if $s_i \geq s'_i$ then there is a stronger temptation to defect from playing s_i than from playing s'_i :

$$s_i \geq s'_i \Leftrightarrow [\pi_i(s_{min}, s_{-i}) - \pi_i(s_i, s_{-i})] \geq [\pi_i(s_{min}, s_{-i}) - \pi_i(s'_i, s_{-i})] \quad \forall s_{-i}$$

Note that, in essence, a complete and transitive order is being assumed to exist on the strategy space. A5 gives us strict dominance of an action (s_{min}) which ensures existence and unicity of a Nash equilibrium to Γ . In order to avoid cases where the stage game achieves a Pareto optimum the following is assumed:

A6. For any interior point s of S we have:

$$\frac{\partial \pi_i}{\partial s_1}(s) + \dots + \frac{\partial \pi_i}{\partial s_i}(s) + \dots + \frac{\partial \pi_i}{\partial s_n}(s) > 0.$$

It follows that $s_{max} := \{s_i \mid \pi_j(s_i, s_i, \dots, s_i) \geq \pi_j(s'_i, s'_i, \dots, s'_i) \quad \forall s'_i \in S_i\}$ is the unique most profitable collusion which satisfies anonymity. The following assumption permits a considerable simplification of the analysis since it, along with A3 and A4, implies that the Nash stage game equilibrium strategy is an optimal punishment in the sense of Abreu (1988).

A7. $\pi(s_{min})$ equals the opportunity cost of playing the game. Normalize both to zero.

The repeated game is the infinite replication of the stage game. Gains will be discounted by player i with discount factor $\delta_i \in [a, b] \subset [0, 1)$. Discount factors are drawn according to:

A8. At the beginning of every period each player draws a discount fac-

tor with the atomless probability measure f defined on the measurable space $([a, b], \mathcal{B}([a, b]))$. Where $\mathcal{B}([a, b])$ denotes the the Borel subsets of $[a, b]$.

This is a highly restrictive assumption, nevertheless it can be seen to be necessary to obtain tractable results. Suppose, momentarily, that discount factors are drawn according to a Markov process. Expectations about others' discount factors is an important part of any strategy but since it may not be the case that actions and discount factors are one to one in the strategy, the recursive nature of the problem is lost. In other words, one may only be able to infer a (non singleton) set of discount factors from any given action. This being the case, conditional probabilities depend on the entire history of actions in a non trivial way. Though perhaps surprising at first sight, A9 can be interpreted in ways which do not assume that preferences change. Such a discussion will be delayed until section 2.6.

Rubinstein's (1979) concept of (subgame) perfect equilibrium will be used as the equilibrium concept to the repeated game. Since future gains are discounted, this implies that a strategy is an equilibrium if and only if there exists no profitable one shot deviation (Abreu (1988), Proposition 1). Furthermore, we will only study symmetric strategy profiles (see Abreu, Pearce, and Stacchetti (1986)). This does imply a loss of generality, yet it is argued in section 2.4 that the main conclusions of the model are robust to a weakening of this restriction.

2.3 Static Representation

The method of Abreu, Pearce, and Stacchetti (1986) is a convenient tool to analyze (subgame) perfect symmetric strategies in repeated games. The restriction to symmetric strategies is important since after any history the strategy of player i is required to be the same as the strategy for player j for all $i, j \in N$. Their method entails “factorizing” the gains accruing to each player into two parts so the game can be analyzed as a static game. The first part is the gain from the current stage game and the second part is expected, future gains which can be a function of any observable action taken in the current period. So this representation needs two elements: a payoff function for the stage game and a function describing future gains. In Abreu, Pearce, and Stacchetti (1986) Proposition 1 states that an equilibrium to the new static game which satisfies a “self generation” criterium is an equilibrium to the repeated game. Proposition 2 states that any perfect equilibrium to the repeated game can be represented as a Nash equilibrium to the new static game. Thus the static representation is equivalent to the dynamic game. The following closely mimics the presentation in Abreu, Pearce and Stacchetti to modify their static representation to meet our needs.

Define Σ to be the set of Borel measurable functions from $[a, b]$ into S_i : $\Sigma := \{\sigma \mid \sigma : [a, b] \rightarrow S_i\}$. Define $\Omega(W)$ to be the set of Borel measurable functions from S into a bounded set $W \subset \mathbb{R}$: $\Omega(W) := \{u \mid u : S \rightarrow W\}$. Due to the restriction made on symmetric strategies $(\sigma_i, u_i) = (\sigma_j, u_j)$ for all i and j . When there is no danger of confusion I shall write δ to denote $\prod_{i \in N} \delta_i$ and $\sigma(\delta)$ to denote

$\times \sigma(\delta_i)$. For any pair $(\sigma, u) \in \Sigma \times \Omega(W)$ write:

$$\mathbb{E}_\delta(\sigma; u) := \mathbb{E}_\delta[\pi_i(\sigma(\delta)) + \delta_i u(\sigma(\delta))]$$

$$\mathbb{E}_{\delta_{-i}}(\sigma; u)(\delta_i) := \mathbb{E}_{\delta_{-i}}[\pi_i(\sigma(\delta_i, \delta_{-i})) + \delta_i u(\sigma(\delta_i, \delta_{-i}))]$$

Note that $\pi_i(\sigma(\delta))$ and $u(\sigma(\delta))$ denote gains to the stage game and future expected gains respectively for an arbitrary player and for a given vector of discount rates δ . Note that these expectations are well defined because compositions of Borel measurable functions are Borel measurable, as well as the fact that π and W are bounded.

Definition 1 For any $W \subset \mathbb{R}$, $(\sigma, u) \in \Sigma \times \Omega(W)$ is called admissible with respect to W iff the following two conditions are satisfied:

$$u(s) \in W \quad \text{a.e. } s \in S$$

$$\mathbb{E}_{\delta_i}(\sigma; u)(\delta_i) \geq \mathbb{E}_{\delta_{-i}}(\gamma, \sigma_{-i}; u)(\delta_i) \quad \forall \gamma \in \Sigma, \text{ a.e. } \delta_i \in [a, b]$$

Definition 2 For $W \subset \mathbb{R}$, $B(W) := \{\mathbb{E}_\delta(\sigma; u) \mid (\sigma, u) \text{ admissible wrt } W\}$.

Thus for every $w \in B(W)$ there exists at least one pair (σ, u) admissible with respect to W with value w . Choose one such pair, $\sigma : B(W) \rightarrow S$ and $u : B(W) \rightarrow \Omega(W)$. Thus, if we define $W \subset \mathbb{R}$ to be self generating if $W \subset B(W)$, propositions 1 and 2 from (Abreu, Pearce, and Stacchetti 1986) follow. If we define V to be the set of perfect equilibrium expected payoffs, then the construction in APS assures us that there exists a strategy with value $w \in V$ which does not

depend on (privately observable) past discount factors.

Since W is the range of u , we are demanding that for (almost) any draw of discount factors, $\delta = \prod_{i \in N} \delta_i$, we have that $u(\sigma(\delta)) \leq \mathbb{E}_\delta(\sigma; u)$. It is important to note that the left hand side of the previous inequality does not merely hold in expectation but for any conceivable vector of discount factors. If this were only true in expectation, we would not be checking the entire range of u . Thus there might be realizations of δ such that $u(\sigma(\delta)) > \mathbb{E}_\delta(\sigma; u)$, which would violate self-generation, and thus the static game would not be equivalent to the repeated game.

2.4 Examples

This section presents three examples. The first goal is to illustrate how the methods of section 2.3 can be applied to the problem at hand. Secondly, these examples show that changing discount factors can engender undercutting along the equilibrium path.

A simple two player symmetric game will be used. This example is consistent with A5 in that essentially it is a prisoner's dilemma with a continuum of actions. Action spaces are $S_1 = S_2 = [0, 1]$, and payoffs are given by:

$$\pi_i(s_i, s_j) = \begin{cases} s_j + (s_i - s_j)^2 & s_i \leq s_j \\ s_j - (s_i - s_j)^2 & s_i \geq s_j. \end{cases}$$

Let $\delta = 1/2$ with probability $1/2$ and $\delta = 0$ with probability $1/2$. Let $\sigma(1/2) =$

s and $\sigma(0) = 0$. We can calculate per period gains: $\pi(\sigma(1/2)) = (s - s^2)/2$ and $\pi(\sigma(0)) = (s + s^2)/2$. Let x be the continuation value if both types play $\sigma(1/2)$, and let y be the continuation value if one type plays $\sigma(1/2)$ and the other plays zero. If both players play zero, let the continuation value equal zero. Since we are restricting study to symmetric strategies, payoffs must be the same for both players irrespective of previous asymmetric actions. The problem then can be written:

$$\bar{U} = \max_{s,x,y} : \quad \frac{s}{2} + \frac{1}{8}x + \frac{1}{8}y$$

Subject to the following constraints:

$$\frac{s - s^2}{2} + \frac{x}{4} + \frac{y}{4} \geq \frac{s + s^2}{2} + \frac{y}{4},$$

$$0 \leq x, y \leq \bar{U}.$$

The first constraint is the incentive compatibility constraint for type 1/2, and the second constraint is self-generation. These constraints yield:

$$x \in [4s^2, \frac{4}{7}s + \frac{y}{7}]$$

$$y \in [7x - 4s, \frac{4}{7}s + \frac{x}{7}].$$

Thus the program is solved for $s \cong 1/6$, $x \cong 1/9$ and $y \cong 1/9$. It may be surprising that $x = y$, since one might think that the incentive constraints for type 1/2 may not be satisfied. However, if type 1/2 were to play action 0, then zero future gains

would occur with probability $1/2$. If one player were to play s then zero future gains would never occur.

Now I justify why $\sigma(0) = 0$ and why the continuation value when both players play zero is zero. Clearly, in any equilibrium, type 0 must play $s = 0$, since she places no weight on future gains and $s = 0$ is a dominant strategy to the stage game. Letting the continuation value when both players play zero be positive is not optimal since it does not change \bar{U} and makes the incentive compatibility constraint for type $1/2$ more difficult to respect.

The second example shows that more than two values can be taken by the continuation function. Suppose $\delta = 1/2$ with probability 0.576 and $\delta = 0$ with complementary probability. Again let s be the action taken by type $1/2$ (type zero will play action zero), and let x be the continuation value if both players play s and let y be the continuation value if one player plays s and the other plays zero (0 will be obtained if both types play zero). The problem can be written similarly to the first example and optimal values can be calculated to be $s \cong 0.179246$, $x \cong 0.139005$ and $y \cong 0.104008$. Note that the continuation value is clearly not bang-bang, nor is the "punishment" scheme simple.

Consider, briefly, the possibility that asymmetric strategies are used. If $\delta = 1/2$ with probability p and zero with probability $1 - p$, then we can write down the optimization problem as follows:

$$\bar{U} = \max_{s,x,y,z} : ps + \frac{p^2}{2}w + \frac{p(1-p)}{2}x + 0 \cdot y + 0 \cdot z$$

w is continuation gains to a player who has played s so long as her partner has

played s . x is continuation gains to a player who has played s so long as her partner has played zero. y is continuation gains to a player who has played zero so long as her partner has played s . z is continuation gains to a player who has played zero so long as her partner has played zero. The constraints can be written:

$$ps - (1 - p)s^2 + \frac{p}{2}w + \frac{1 - p}{2}x \geq p(s + s^2) + \frac{p}{2}y + \frac{1 - p}{2}z,$$

$$0 \leq w, x, y, z \leq \bar{U}.$$

Ideally, we would like to set $w = x = \bar{U}$ and $y = z = 0$. However, it is necessary to check if there exists an asymmetric equilibrium strategy which gives one player \bar{U} and the other player zero. If there does not exist such an asymmetric equilibrium, then it is necessary to have $x < w$. In any case, allowing asymmetric strategies does not change the implication that defection occurs on the equilibrium path.

2.5 Results

This section aims to show that the results of Abreu, Pearce, and Stacchetti (1986) on the optimality of “bang-bang” equilibria as well as the results of Abreu (1988) on the optimality of “simple” punishments do not carry over to a context in which defection arises. Loosely, it is argued that optimal collusion, in general, must be “more type revealing” than bang-bang collusion.

First we write down the optimization problem faced by the cartel at time 1:

$$\bar{U} = \max_{\sigma(\cdot), u(\cdot)} : \mathbb{E}_{\sigma}(\sigma; u)$$

Subject to, for almost every δ_i , almost every vector $\delta := (\delta_1, \dots, \delta_n)$, and all $\gamma \in \Sigma$:

$$\mathbb{E}_{\delta_{-i}}(\sigma; u)(\delta_i) \geq \mathbb{E}_{\delta_{-i}}(\gamma, \sigma_{-i}; u)(\delta_i)$$

$$0 \leq u(\sigma(\delta)) \leq \bar{U}$$

The first constraint is admissibility and the second is self generation. It is important to stress that for the second inequality $u(\sigma(\delta))$ represents future gains if a vector, δ , of discount rates were drawn. Note that $u(\sigma(\delta))$ does not represent an expectation.

Let (σ^1, u^1) solve the above problem. Based on the discount factors drawn at time 1, the cartel will propose another admissible and self-generating couple, (σ^2, u^2) whose ex-ante value is less than or equal to the ex-ante value of (σ^1, u^1) and greater than zero: $0 \leq \mathbb{E}_{\delta}(\sigma^2; u^2) \leq \mathbb{E}_{\delta}(\sigma^1; u^1)$. This follows since u^1 promises an element of a self generating set for almost every draw of discount factors (i.e. u^1 promises an element of the interval $[0, \bar{U}]$).

In the following, it will be convenient to write $\pi(\sigma(\delta_i)) = \mathbb{E}_{\delta_{-i}}[\pi_i(\sigma(\delta_i, \delta_{-i}))]$; similarly $u(\sigma(\delta_i)) = \mathbb{E}_{\delta_{-i}}[u(\sigma(\delta_i, \delta_{-i}))]$. The first result comes directly from the admissibility constraint.

Proposition 1 $\pi(\sigma(\cdot))$ is decreasing in δ , and $u(\sigma(\cdot))$ is increasing in δ .

Proof of Proposition 1 Fix $\delta_i > \delta'_i$. From admissibility we have:

$$\delta[u(\sigma(\delta_i)) - u(\sigma(\delta'_i))] \geq \pi(\sigma(\delta'_i)) - \pi(\sigma(\delta_i)) \geq \delta'[u(\sigma(\delta_i)) - u(\sigma(\delta'_i))]. \quad (2.1)$$

Therefore, since $\delta_i > \delta'_i$, we have that $u(\sigma(\delta_i)) \geq u(\sigma(\delta'_i))$. Additionally, we have $\pi(\sigma(\delta'_i)) \geq \pi(\sigma(\delta_i))$. \square

Using these monotonicity properties, one can obtain a characterization of collusive rules when discount rates change. First notice that both the expressions:

$$\dot{\pi}(\sigma(\delta_i)) := \lim_{h \rightarrow 0} (1/h)[\pi(\sigma(\delta_i + h)) - \pi(\sigma(\delta_i))]$$

$$\dot{u}(\sigma(\delta_i)) := \lim_{h \rightarrow 0} (1/h)[u(\sigma(\delta_i + h)) - u(\sigma(\delta_i))]$$

exist almost everywhere since each is bounded and monotone in δ . Using this fact we can state the following proposition whose proof is an elementary application of inequalities (2.1):

Proposition 2 *For any admissible couple (σ, u) , the following relation holds for almost every δ_i :*

$$-\dot{\pi}(\sigma(\delta_i)) = \delta_i \dot{u}(\sigma(\delta_i)).$$

Using this result we can modify the program of an optimal symmetric cartel in the first period and write it a bit more explicitly:

$$\bar{U} = \max_{\sigma(\cdot), u(\cdot)} : \int_a^b [\pi(\sigma(\delta)) + \delta u(\sigma(\delta))] f(d\delta)$$

Subject to, for almost every δ_i , and almost every vector $\delta := (\delta_1, \dots, \delta_n)$:

$$-\dot{\pi}(\sigma(\delta_i)) = \delta \dot{u}(\sigma(\delta_i))$$

$$\mathbb{E}_{\delta_{-i}}(\sigma; u)(a) \geq \pi^*(\sigma)$$

$$0 \leq u(\sigma(\delta)) \leq \bar{U}$$

The first constraint is the admissibility constraint. Define the following: $\pi^*(\sigma) := \max_{s_i} \int_{\delta_{-i}} \pi_i(s_i, \sigma(\delta_{-i})) f(d\delta_{-i})$. Thus, the second constraint states that playing a one period best response to σ (which is s_{min}) is not profitable. Due to the monotonicity of σ and u , we only need check this for the least patient type. The third constraint is self generation.

In general, optimal collusion will require the division of the space of discount factors into three distinct intervals. Any type in the first interval, $[0, \delta_1]$, will gain $\pi^*(\sigma)$ in the current period and will invoke continuation gains of zero. Any type in the second interval, $[\delta_1, \delta_2]$, will have current gains strictly and continuously decreasing in discount factor, and continuation gains continuously increasing in discount factor. Any type in the third interval, $[\delta_2, 1]$, will have constant current gains and continuation gains equal to the ex ante expected value of optimal collusion. It is important to notice that any particular example of a game satisfying the assumptions of section 2.2, could have $a > \delta_1$, or $a > \delta_2$. The former case would give that nobody plays $\pi^*(\sigma)$ and the collusion never completely falls apart. The latter case would result in a collusion where every type is advised to play the same action and the collusion is stable. It can be seen that such a collusion would entail the playing of $s^* = \{s \mid \pi(s) + a \frac{\pi(s)}{1 - \mathbb{E}\delta} = \pi(s_{min})\}$, where $\mathbb{E}\delta = \int_a^b \delta f(d\delta)$.

The following justifies the existence of the first region $[0, \delta_1]$.

Proposition 3 *Let $a = 0$. If (σ, u) is an optimal admissible couple, then $\sigma(0) = s_{min}$ and $u(\sigma(0)) = 0$.*

Proof of Proposition 3 It should be clear that any type zero agent must play s_{min} since s_{min} is a strictly dominant action to the stage game and no weight is placed on future gains. It remains to show that $u(\sigma(0)) = 0$. Suppose that $u(\sigma(0)) > 0$. On one hand, this would not change the actions of the type zero agent and not increase \bar{U} since any future gains to a type zero agent would be weighted by zero. On the other hand this would cause all types sufficiently close to $\delta = 0$ to decrease their actions. A6 assures us that such an eventuality decreases the profitability of collusion, which contradicts the fact that (σ, u) was assumed optimal. \square

The property discussed above has a corollary which demonstrates the existence of some δ_1 , such that every type in the interval $[0, \delta_1]$ plays s_{min} and receives zero future payoffs.

Proposition 4 *Let $a = 0$ and suppose that (σ, u) is an optimal admissible couple. Then, there exists a type $\delta_1 \geq 0$ which plays s_{min} and receives zero future payoffs.*

Proof of Proposition 4 From proposition 3 we have $\pi(\sigma(0)) = \pi^*(\sigma)$ and $u(\sigma(0)) = 0$. By incentive compatibility we have $\pi^*(\sigma) \geq \pi(\sigma(\delta)) + 0u(\sigma(\delta))$ for all δ . Recall that $\pi(\sigma(\delta)) = -\delta u(\sigma(\delta))$. If the previous inequality is strict then there will exist $\delta_1 > 0$ such that for all $\delta < \delta_1$ incentive compatibility requires that $\pi(\sigma(\delta)) = \pi^*(\sigma)$ and $u(\sigma(\delta)) = 0$. \square

Now consider types in the upper range of the support of δ . It is claimed that in this range, different types may take the same action. We would like every type to play as high an action as possible, which requires appropriate recompensation through increased future gains. However, this is not always possible as the following example demonstrates.

Consider the game presented in section 2.4, where δ can equal $1/2$ or $1/4$ with equal probability. Note that the most profitable symmetric collusion a player with type $1/4$ can support is the playing of 0.4 because $\mathbb{E}\delta = 3/8$. This follows because $s(1 + \frac{1/4}{1-3/8}) = s + s^2$ is solved for 0.4 . By a similar argument note that a player with type $1/2$ can support the playing of 0.8 . Letting s_1 and s_2 be the actions taken by $1/2$ and $1/4$ respectively, and letting x be the continuation gains if both players play s_1 , y be the continuation gains if one player plays s_1 and the other plays s_2 , and z be the continuation gains if both players play s_2 , one can pose the problem in the following way:

$$\bar{U} = \max_{s_1, s_2, x, y, z} : \frac{1}{2}[s_1 + s_2] + \frac{1}{8}(x + y) + \frac{1}{16}(y + z).$$

Subject to:

$$-(s_1 - s_2)^2 + \frac{1}{4}(x - z) \geq 0,$$

$$(s_1 - s_2)^2 + \frac{1}{8}(z - x) \geq 0,$$

$$-2s_1s_2 + \frac{1}{4}(y + z) \geq 0,$$

$$0 \leq x, y, z \leq \bar{U}.$$

The constraints are the incentive compatibility constraints for type 1/2 and type 1/4, the constraint stating that 1/4 prefers playing s_2 to playing 0, and self generation. This program can be solved to show that the optimal symmetric collusion advises both types to play action 0.4 and promises future payoffs of 0.64. Note that despite the fact that discount factors are subject to change, collusion is perfectly stable and a bang-bang continuation function, or simple punishment, is sufficient to support optimal collusion.

This example shows that bunching can occur “at the top”. The intuition is the following. In order that 1/2 be motivated to increase s_1 , she must be appropriately recompensed through an increase in x and y (proposition 2). But when x and y increase, it is necessary that \bar{U} , the value of collusion, increase by at least as much in order to respect self generation. But \bar{U} is endogenous to the problem and as such, the increase in \bar{U} brought about by the increase in s_1 may not be large enough to ensure admissibility. Thus the tension between the admissibility and self generation constraints may provoke bunching near the most patient type.

The two previous cases have explored the extremes of the possible support of the discount factor. Now we study the case where all possible discount factors do not fall into one of these two categories. Let these intermediate discount factors be in the interval $[\delta_1, \delta_2]$. It follows that for types in $[0, \delta_1]$ or in $[\delta_2, 1)$, collusion is not type revealing. The following proposition shows that for types in $[\delta_1, \delta_2]$, collusion is perfectly type revealing.

Proposition 5 *Optimal collusion must reveal almost any type not in $[0, \delta_1] \cup [\delta_2, 1)$.*

Proof of Proposition 5 Suppose (σ, u) admissible with respect to a bounded self generating Borel set W . Furthermore assume to the contrary that σ is not fully type revealing ($\sigma^{-1}(s_i) := \{\delta_i \mid \sigma(\delta_i) = s_i\}$ is not a singleton) on an interval $[\underline{\delta}, \bar{\delta}]$ such that $\underline{\delta} > a$ and $\bar{\delta} < b$. There must be a discontinuity in both $\pi(\sigma(\cdot))$ and $u(\sigma(\cdot))$ at $\bar{\delta}$. Due to the fact that f is assumed atomless and π is assumed continuous such a discontinuity in $\pi(\sigma(\cdot))$ must be due to a discontinuity in $\sigma(\cdot)$.

Pick $\hat{\delta} \in (\underline{\delta}, \bar{\delta})$ and define a new strategy, σ^* , to be identical to σ everywhere except on the interval $[\hat{\delta}, \bar{\delta}]$. On this interval define σ^* by:

$$\sigma^*(\delta) = \sigma^*(\delta') \quad \forall \delta, \delta' \in [\hat{\delta}, \bar{\delta}], \quad (2.2)$$

$$\sigma^*(\delta) > \sigma(\delta) \quad \forall \delta \in [\hat{\delta}, \bar{\delta}], \quad (2.3)$$

$$\sigma(\bar{\delta} + \varepsilon) > \sigma^*(\delta) \quad \forall \delta \in [\hat{\delta}, \bar{\delta}], \quad \forall \varepsilon > 0. \quad (2.4)$$

Remark that we can write (2.4) due to the continuity property as well as due to the fact that the admissibility of (σ, u) implies that $\sigma(\bar{\delta} + \varepsilon) > \sigma(\bar{\delta})$ for all $\varepsilon > 0$. We can write (2.3) if $\sigma(\underline{\delta}) \neq s_{max}$. This is guaranteed since $b < 1$.

Now it is necessary to find a continuation function, u^* , such that (σ^*, u^*) is admissible with respect to a self generating set. Define u^* to be equal to u everywhere except when the discount factor lies in the interval $[\hat{\delta}, \bar{\delta}]$. In this interval define u^* by:

$$\pi(\sigma^*(\hat{\delta})) + \hat{\delta}u^*(\sigma^*(\hat{\delta})) = \pi(\sigma(\hat{\delta})) + \hat{\delta}u(\sigma(\hat{\delta})). \quad (2.5)$$

This is sufficient to show that $[\underline{\delta}, \hat{\delta}]$ and $[\hat{\delta}, \bar{\delta}]$ self select due to the fact that future

payoffs are increasing in the discount rate. Now it remains to show that $\bar{\delta} + \varepsilon$ prefers reporting $\bar{\delta} + \varepsilon$ as opposed to $\bar{\delta}$ for all $\varepsilon > 0$. From the admissibility of (σ, u) we have:

$$\pi(\sigma(\bar{\delta} + \varepsilon)) + (\bar{\delta} + \varepsilon)u(\sigma(\bar{\delta} + \varepsilon)) > \pi(\sigma(\underline{\delta})) + (\bar{\delta} + \varepsilon)u(\sigma(\underline{\delta})). \quad (2.6)$$

Showing the desired self selection means verifying the following:

$$\pi(\sigma(\bar{\delta} + \varepsilon)) + (\bar{\delta} + \varepsilon)u(\sigma(\bar{\delta} + \varepsilon)) \geq \pi(\sigma^*(\bar{\delta})) + (\bar{\delta} + \varepsilon)u^*(\sigma^*(\bar{\delta})) \quad (2.7)$$

Due to the continuity property, $\sigma^*(\bar{\delta})$ can be chosen so that $|\pi(\sigma^*(\bar{\delta})) - \pi(\sigma(\bar{\delta}))| < \eta_1$ for all $\eta_1 > 0$. Thus, $\sigma^*(\bar{\delta})$ can be chosen such that for any $\eta_2 > 0$ we have $|u^*(\sigma^*(\bar{\delta})) - u(\sigma(\bar{\delta}))| < \eta_2$ by equation (2.5). Using (2.5) and (2.6) it is sufficient to show that for any fixed $\eta_3 > 0$:

$$|\pi(\sigma(\underline{\delta})) + (\bar{\delta} + \varepsilon)u(\sigma(\underline{\delta})) - \pi(\sigma^*(\bar{\delta})) - (\bar{\delta} + \varepsilon)u^*(\sigma^*(\bar{\delta}))| < \eta_3.$$

This can be seen to be verified since:

$$\begin{aligned} &|\pi(\sigma(\underline{\delta})) + (\bar{\delta} + \varepsilon)u(\sigma(\underline{\delta})) - \pi(\sigma^*(\bar{\delta})) - (\bar{\delta} + \varepsilon)u^*(\sigma^*(\bar{\delta}))| \leq \\ &|\pi(\sigma(\underline{\delta})) - \pi(\sigma^*(\bar{\delta}))| + (\bar{\delta} + \varepsilon)|u(\sigma(\underline{\delta})) - u^*(\sigma^*(\bar{\delta}))| < \\ &\eta_1 + (\bar{\delta} + \varepsilon)\eta_2. \end{aligned}$$

Since $[\hat{\delta}, \bar{\delta}]$ play a higher action this must result in more profitable collusion due to A6. Since $\bar{\delta} < b$ then self-generation follows trivially since the convex hull

of the range of u equals the convex hull of the range of u^* . \square

It is worthwhile pointing out why the above argument does not work for $[0, \delta_1]$ or $[\delta_2, 1)$. For $\delta \in [\delta_2, 1)$ in order to respect admissibility it is necessary to increase the range of the continuation function. As was argued earlier, this may not respect self generation. For $\delta \in [0, \delta_1]$ there need not be a discontinuity in $\sigma(\cdot)$ at δ_1 . This follows since types below δ_1 are constrained to play s_{min} (see proposition 4).

The structure of optimal collusion which is detailed above is important for the following reason. If collusion is to be (partially) type revealing then two different types may take different actions. And by incentive compatibility, this implies that the less patient is an agent then the higher will be her current payoff, and the lower will be her future payoffs. In other words, impatient agents will “undercut” today and be “punished” in the future (due to the restriction on symmetric strategies, all players must be punished equally). This is the behavior that I argue can be interpreted as cartel instability.

For $\delta \in [0, \delta_1) \cup (\delta_2, 1)$, continuity of $\pi(\sigma(\delta))$ and $u(\sigma(\delta))$ follows trivially (they are constant). The following proposition shows that continuity of σ and u holds in general for all $\delta \in [0, 1)$.

Proposition 6 *If (σ, u) is an optimal admissible collusive strategy then, both σ and u are continuous in δ .*

Proof of Proposition 6 Suppose that σ is an optimal collusive action strategy which has a point of discontinuity at δ . Let the limit from the left of $\sigma(\delta)$ be $\sigma^-(\delta)$ and let the limit from the right of $\sigma(\delta)$ be $\sigma^+(\delta)$. From admissibility we know that $\sigma^-(\delta) < \sigma^+(\delta)$. Admissibility also implies that δ is just indifferent between playing

$\sigma^-(\delta)$ and $\sigma^+(\delta)$. Now advise a new action s^* and a new continuation function defined in the following manner. Choose s^* such that $\pi(\sigma^+(\delta)) - \pi(s^*) = \Delta < \pi(\sigma^+(\delta)) - \pi(\sigma^-(\delta))$. Increase the expected continuation gains to a player having played s^* by Δ/δ (the convex hull of the range of u will remain unchanged). Now type δ is indifferent between choosing any of the three actions: $\sigma^-(\delta)$, s^* , $\sigma^+(\delta)$. We can advise similar new actions for all types in a neighborhood less than δ . Such a change will alter the distribution of actions, and as such may cause a type greater than δ to wish to play s^* . If this is the case then we know that δ strictly prefers playing s^* to playing $\sigma^-(\delta)$ or $\sigma^+(\delta)$, thus we can lower continuation gains until δ is again indifferent between the three actions. Such a decrease in continuation gains will dissuade any type greater than δ from playing s^* . Thus we have increased the actions of certain types, which, by A6, ensures higher profitability of collusion contradicting the supposition of optimal discontinuous collusion. \square

As a corollary to the above proposition we can use proposition 2 and integrate by parts to obtain the following characterization for optimal symmetric collusion:

$$\pi(\sigma(\delta_i)) + \delta_i u(\sigma(\delta_i)) = \pi(\sigma(a)) + a u(\sigma(a)) + \int_a^{\delta_i} u(\sigma(s)) ds \quad \forall \delta_i \in [a, b]$$

It is worthwhile pointing out that the above propositions can be taken together to enable us to say what an optimal collusive mechanism is, and what it is not. Specifically, the above narrows the choice of optimal collusive strategies to one of four possibilities: $\sigma(\cdot)$ is constant for all $\delta \in [a, b]$; $\sigma(\cdot)$ is constant for all $\delta \in [a, \delta_1]$ and increasing for all $\delta \in [\delta_1, b]$; $\sigma(\delta)$ is increasing for all $\delta \in [a, \delta_2]$ and constant for all $\delta \in [\delta_2, b]$; or $\sigma(\cdot)$ is constant for all $\delta \in [a, \delta_1]$, increasing for all $\delta \in [\delta_1, \delta_2]$

and constant for all $\delta \in [\delta_2, b]$. This rules out cases where, for example, $\sigma(\cdot)$ is increasing then constant then increasing.

The above studied optimal collusion. But the instability of such collusion prevents a cartel from always offering optimal gains. Consider the measure zero case where in the first round all players draw the lowest possible discount factor. Unless it is the case that $a > \delta_2$, the ensuing payoffs must be very low and, thus, cannot be optimal. So the mechanism at time t , (σ^t, u^t) , along with the draw of discount factors at time t , chooses the mechanism at time $t + 1$. At period one, since the cartel is not constrained by past commitments, it may use the optimal mechanism. After types are reported, with positive probability, it will be the case that the cartel promises future gains which are not Pareto optimal. Thus, future mechanisms may be "less" type revealing, discontinuous and thus not optimal.

Now consider how to replicate any payoff v in the interior of V , the set of perfect payoffs. v is equal to a weighted sum of current and future payoffs. Future payoffs are weighted by the expectation of the discount factor, $\mathbb{E}(\delta) = \int_a^b \delta f(d\delta)$. Consider any current and future payoffs which yield v . It is easy to see that a decrease in current payoffs by $\mathbb{E}(\delta)$ units along with an increase in future payoffs by one unit yields v . We can always increase and decrease current and future payoffs so long as they are not extremal points of V . This implies that there are an infinite number of ways to mete out non-extremal continuation payoffs.

There exists some evidence giving support to the kind of cartel behavior described in this section. Barbezat (1989) studies the International Steel Cartel which comprised Germany, France, Belgium and Luxembourg starting in 1926.

He reports "For an over quota production between 7.5 and 28 percent they paid one dollar; between 28 and 30.8 percent two dollars; and beyond 30.8 percent ... four dollars." (footnote 9.) This is precisely the type of punishment implied by the preceding proposition. The cartel seemed to recognize that member countries may have been subject to internal pressures which significantly changed incentives. The degree of these internal pressures may, of course, have varied across countries and across time (Germany's war reparations, for example). The above proposition says that the cartel may be able to do better by letting types sort themselves. Note also that these punishments were neither unrelenting as in Friedman (1971), nor were they optimal in the sense of Abreu (1988).

2.6 Interpretation of Changing Discount Rates

The previous sections have assumed discount rates to be the only heterogeneity between agents. This was imposed for several reasons. First, it is fairly straightforward to incorporate differing discount rates into a fairly tractable model. Second, the commonly made assumption of symmetry and perfect information can be argued to be unrealistic. Third, and most important, patience provides a linkage between future rewards and current actions. Despite the plausibility of heterogeneous patience, no rigorous explanation for this heterogeneity has been advanced. Since the overwhelming majority of the literature has focused on the symmetric case, it would be useful to be able to provide a more rigorous explanation for this departure. Harrington (1989) argues that either imperfections in the capital market or differing degrees of agency problems among firms could account for such

a heterogeneity. These are plausible explanations and one could imagine these factors changing over time. However they are explanations which rely on factors entirely outside the scope of his model and, as such, cannot be assessed. The goal of this subsection, therefore, is to provide not only a plausible but a (mostly) endogenous explanation of a heterogeneity which plays the same role as patience.

Any privately observed variable which provides a linkage between future rewards and current actions could be used to derive results similar to those of the previous sections. As a simple example, consider a model where patience is symmetric and common knowledge, but where payoffs are subject to permanent multiplicative shocks. Furthermore, suppose agents have different (private) expectations about the size of these shocks. In this case the relevant payoff to agent i would be $\pi_i(s) + \delta\rho_i u_i(s, \delta\rho_i)$. Where ρ_i is a parameter reflecting i 's expectations about future profitability in an industry. The parameter ρ_i has now taken the place of δ_i in the previous analysis. As another example suppose that in every period each agent must pay k_i dollars worth of interest on loans to creditors. If agent i is unable to reimburse k_i in any period, then he must declare bankruptcy and future play stops for him. If current profits are greater than or equal to k_i , then agent i has no incentive to cheat from a well constructed collusive scheme assuming stability of discount rates and expectations on future payoffs. Now assume that the amount agent i must reimburse in any period changes in an unverifiable way. Obviously, a player might not have a choice but to defect from specified rules. This one time defection could have devastating effects on the participants since not only do firms suffer from being cheated, but a possibly painful punishment strategy must be played.

Recent work by Bertrand (1999) has empirically examined relationships between firms and workers. MacLeod and Malcomson (1989) show that repetition and sufficiently large discount factors permits firms and workers to achieve Pareto improving equilibria which are unobtainable in a static game. Bertrand (1999) finds that firms facing competition from foreign industries are more likely to renege on agreements with their workers: "Competitive pressures, by lowering expected future rents and shortening the corporate horizon, reduce" the cost associated with renegeing on an implicit contract (page 9).

In these two examples the discount rate heterogeneity has merely been replaced by heterogeneity in another variable which links future rewards to current actions. However, this would seem to be a more economically interesting perspective because it demonstrates how expectations about observable parameters change current behavior. Contrast this with the approach of using unobservable preferences.

It must be stressed that the variable changing expectations about future payoffs must be private information, since if this were not the case then a cartel would be able to restructure the collusive scheme. The above analyses lets us make predictions on when price wars should occur. If it can be supposed that expectations play a major role in an industry, then it can be posited that price wars should be more frequent when expectations are more variant. Thus it is necessary that expectations systematically change in variance along the business cycle. Similarly, if it can be supposed that indebtedness is a major factor in an industry, then price wars should be more frequent when indebtedness is high.

2.7 Conclusion

This paper has presented a model where colluding optimizing agents exhibit undercutting behavior which was interpreted as cartel instability. This instability is brought about as a direct result of allowing discount factors, or patience, to vary across time and across agents. It was argued that the bang-bang results of Abreu, Pearce, and Stacchetti (1986) as well as the simple punishment results of Abreu (1988) do not carry over to such a scenario. Instead punishments, in optimal collusion must be made to "fit the crime", and as such actions reveal type. The interpretation of changing discount factors need not rely on changes in preferences, since it was argued that heterogeneous private expectations about future market profitability, or credit crunches may result in similar results. The model makes predictions which evidence from Barbezat (1989) and Bertrand (1999) appear to back up.

Notes

¹These papers studied Cournot collusion when price was an imperfect signal of firm specific quantities.

²An unsubscripted symbol will refer to the cartesian product of the symbol in question over N , and the subscript $-i$ will refer to the cartesian product of the variable in question over $N \setminus \{i\}$.

Search, Matching and Moral Hazard

Paul Johnson

Abstract

This paper re-examines Becker's (1973) marriage model with search frictions along the lines of Smith (1997). There are two main innovations. First, we interpret surplus creation to be the result of Pareto improving collusion. Thus, parties in a relationship can choose to exercise a moral hazard variable or "defect." Collusion can be of differing profitability, so type is patience (discount factor). Since players are free to search for new partners in every period, usual enforcement mechanisms (trigger strategies) cannot be used. Instead, players are interpreted to burn a sufficient quantity of money upon meeting a new partner so that defection from collusion is rendered unprofitable. The second innovation is the examination of the search equilibrium under the assumption that type is unobservable. An interesting implication of type unobservability is the appearance of "defection" equilibria.

Acknowledgments: This research was supported by the SSHRC. I would like to thank Jacques Robert for much insightful criticism and discussion. This paper also benefitted from the comments of seminar participants at the microeconomic theory seminar at Penn State University. Errors must remain mine alone.

3.1 Introduction

Becker (1973) studied a matching model where type is idiosyncratic, production is exogenous to the model and search is frictionless. Smith (1997) and Shimer and Smith (1997) studied Becker's matching model from the perspective where type is still idiosyncratic, and production is still exogenous but where there exist search frictions. The goal of this literature is to study under what conditions partners, in equilibrium, are of similar type – the matching is *assortative*. This literature has studied the formation of long run relationships, yet has not permitted the set of equilibrium payoffs to increase due to repeated play. The current paper offers a production process susceptible to Pareto improving collusion while retaining a search friction.

Consider the meaning and implications of type in a two sided matching model. Oftentimes type is not terribly relevant. For instance if a firm hires a contractor to accomplish a specific task, then all details about the service: terms of delivery and payment may be perfectly measurable and enforceable in a contract. Thus the type of the parties is superfluous since the transaction takes place, in essence, on a spot market. In longer term relationships, type may be more relevant. For instance, if the same firm hires a permanent employee¹, then the relationship may be subject to more moral hazard on both sides due to the incompleteness of a contract. It is probably important to the firm to have employees who are willing work hard at their specified tasks². Similarly, it is probably important² to a worker to work in an agreeable work environment and receive adequate compensation. The firm and the worker may have information on a potential match (through references

or reputation), and this may be of help in the matching choice. Alternatively, the firm and worker may have unreliable or insufficient information on which to make a decision (a new firm without a reputation, or a worker with no prior experience or who was self employed). In any case, the profitability of a match is a function of the willingness (or ability) to work overtime on a project, or the awarding of performance through pay increases among many other things. The firm-worker example seems to be pertinent to the assortative matching literature since many economists, including Kremer and Maskin (1996)³ note and rationalize a recent trend of higher intra-firm wage correlation.

The previous discussion argues that type need not necessarily be job specific productivity, education or other factors which *directly* enter into the production function of Becker's (1973) marriage model. Instead, type is the ability of parties to overcome the presence of opportunities to take advantage of one another. This paper interprets a relationship subject to moral hazard as a relationship susceptible to Pareto improving collusion. For instance, a marriage affords both parties opportunities to renege on promises (implicit or explicit) to visit in-laws or share household chores. However, the repeated nature of a relationship between spouses may permit the creation of a much larger surplus than between casual acquaintances. The interpretation of marriage as Pareto improving collusion has been made by Lundberg and Pollak (1993) who analyze bargaining within marriages. They use a noncooperative equilibrium as a threat point, whereas others have used divorce as a threat point.

On a more abstract level, this paper proposes a search model where type is heterogeneous and search is costly. Moreover, the game for which partners are

searched has a flavor of the prisoner's dilemma in that more profitable outcomes occur if people can somehow collude. This motivates the interpretation of type as patience, or discount factor. Smith (1997) and Shimer and Smith (1997) explore a search model where type is heterogeneous in a non transferable and transferable utility paradigm respectively. This paper studies the non transferable utility paradigm. Their work demonstrates that the existence of a search equilibrium is far from trivial to show with a continuum of types. In a game with a production technology reflecting the ability of agents to collude (cooperate), the regularity conditions of these two papers are not met⁴. It is for this reason that a considerable simplification is made in assuming that there exist only two types of agents.

The very fact that agents search implies that they are free to choose with whom to play the game. Thus the traditional repeated game analysis which implicitly assumes that players have no other outside opportunities cannot be applied here. Thus, there needs to be a way which collusive outcomes can be supported when players are not obliged to remain with a single partner and receive a punishment. The main obstacle to overcome is the ability of players to defect while changing partners in every period. Such a strategy can be rendered unprofitable by giving players the ability to (verifiably) burn a quantity of money before playing with a new partner so that changing partners becomes prohibitively expensive. This is the main thrust of Carmichael and MacLeod (1997) who use the giving of intrinsically valueless gifts in lieu of burning money. Their paper leaves no room for the exchange of money since the net effect of such would be nil. However, it is shown that under certain circumstances having the ability to transfer money

is a Pareto improvement in this context. Their's is a paper using evolutionary game theory to refine the equilibrium concept, thus we can think of this as a "No Parasites" (hereafter NP) requirement. It is important to note that this constraint is particular to collusion games—in Smith (1997), Kremer and Maskin (1996) or Becker (1973) no such defection is possible.

Ghosh and Ray (1996) study the ability of agents to collude in a search model where discount factors and histories are unobservable. They assume a fixed proportion of players having a discount factor strictly greater than zero and the remainder have a discount factor of zero, so (non) assortative matching is not an issue. In their paper, newly matched agents initially cooperate to a certain extent after which type is revealed and full cooperation ensues. Their equilibrium is particularly interesting since it is robust to bilateral deviation, i.e. no matched pair can deviate by forgoing the unprofitable "courtship" at the beginning of a relationship. In general, this result does not carry over to the current paper since it is money burning in *future* relationships which makes collusion supportable. However, the interested reader should pay special attention to subsection 3.4 to see how Ghosh and Ray's (1996) notion of bilateral rationality carry over in this context.

If matching is to be assortative then there is a positive probability that any potential partner will be rejected – a waiting time, or exogenous search friction. This is one reason why matching may fail to be assortative. The game studied in this paper affords another explanation as to why matching may be assortative. Since all profitable relationships are subject to moral hazard, the NP constraint must be respected in every match. But since the profitability of a match is positively

related to the extent of moral hazard present, a more profitable match makes the NP constraint more difficult to satisfy. Allegorical experience might lead one to believe that the winning of a partner's trust and confidence is a more difficult obstacle than the *physical* meeting of a prospective partner. Furthermore, more profound relationships usually require a courtship which is more taxing than do more ephemeral relationships.

It will be assumed that any player's history is private information. This is to avoid the possibility of a trigger strategy of the type "refuse to play with anyone who has ever cheated" could support cooperative play. I argue that it may be unwise to permit the use of such a strong tool in this case that history were observable. For example, long term relationships may be kept mostly implicit to increase flexibility between a worker and firm or between a husband and wife. Due to the implicit nature of the contract it may be a very difficult task to assess blame. For example, divorce proceedings sometimes degenerate due to the parties' differing interpretation of the implicit marriage contract; or a worker may choose to quit a job because she did not receive a raise, however from the firm's perspective the raise may not have been merited.

The equilibrium concept used in the search model is important as it quite drastically changes certain conclusions. Obviously a stronger concept than Nash equilibrium will be needed, since the strategy "reject every potential partner" is a Nash equilibrium. This naturally leads to examining perfect bayesian equilibrium (or a variant like Kreps and Wilson's (1982) sequential equilibrium ⁵). However, I show that even a strong refinement like perfect bayesian equilibrium leads to unpalatable results; essentially the model loses all ability to predict when assor-

tative matching arises. The approach I take is one based on payoff dominance. A key component to which is the necessity to have unanimous consent to form a partnership.

The paper studies two informational scenarios: one where type is observable and the other where type is unobservable. This is a departure from the matching literature which has only treated the case where type is observable. Section 3.3 assumes that type is observable. Intuitively, this is equivalent to all information on a prospective partner being available. Section 3.4 assumes that type is unobservable. Intuitively, this is equivalent to partial or no information (imperfect or incomplete information) available on a prospective partner. The remainder of the paper includes section 3.2 which presents the model to be used and section 3.5 which briefly concludes.

3.2 Model

The model used throughout this paper aims to be as simple as possible. The game at the heart of this model is inspired by the well known prisoner's dilemma with the addition of allowing for different "levels" of collusion.

THE PLAYERS: There exists a continuum of players of finite mass. All players have preferences which are representable by a utility function of the form:

$$\sum_{t=1}^{\infty} \delta^{t-1} u_t$$

where δ is a discount factor and u_t is the quantity “utils” consumed in period t . Players are risk neutral so utils will be interpreted as monetary sums. Opportunity costs are zero, so an unmatched player receives a surplus of zero. A proportion, $p > 0$, of players have discount rate δ_h , and a proportion, $1 - p$, of players have a discount rate δ_l . Note that p will be, in part, chosen by the type of equilibrium. The proportion of type δ_h players is common knowledge. Further restrictions will be placed on δ_h and δ_l below.

THE GAME: The game to be played will be denoted Γ . Γ is a two person symmetric prisoner’s dilemma game with three pure strategies available to each player. $\Pi : S \times S \rightarrow \mathbb{R}$ denotes the payoff function sending the set of strategies (S) into the reals. Like Smith (1997) this paper assumes that utility is non transferable. There exists a unique single stage Nash equilibrium in pure strategies. Denote the action corresponding to this pure strategy by s^* and normalize $\Pi(s^*, s^*) = 0$ to be the same as the opportunity cost of playing the game. The two remaining pure strategies will be denoted by s_h and s_l . s^* is a uniform best response to both s_h and s_l . The following shorthand will be useful. $\pi(s^*)$ will denote payoffs when both players play s^* . $\pi(s_h)$ will denote payoffs when both players play s_h . $\pi(s_l)$ is defined similarly. $\pi^*(s_h)$ will denote the payoff to playing s^* when one’s opponent plays s_h . $\pi^*(s_l)$ is defined similarly. The notation $\pi(s_h, s^*)$ will denote the payoffs to playing s_h when one’s partner has played s^* .

THE REPEATED GAME The repeated game consists of the infinite repetition of the stage game. Payoffs and discount rates are such that:

$$\frac{\pi(s_h)}{1 - \delta_h} \geq \pi^*(s_h) > \frac{\pi(s_h)}{1 - \delta_l} > \frac{\pi(s_l)}{1 - \delta_l} \geq \pi^*(s_l). \quad (3.1)$$

Intuitively, Γ is merely a prisoner's dilemma, where action s_h is an equilibrium to the infinitely repeated game when the discount factor is δ_h , but not when the discount factor is δ_l . Action s_l is an equilibrium to the repeated game when the discount factor is either δ_h or δ_l . Note that since the opportunity cost of playing the game is equal to the payoffs to the single stage Nash equilibria, reliance on an unrelenting Nash equilibrium punishment is optimal in the sense of Abreu (1986). It is easy to give an example of such a game. If $\delta_h = 2/3$ and $\delta_l = 1/3$ then such a game might resemble the following:

	s_h	s_l	s^*
s_h	10, 10	-3, 12	-5, 15
s_l	12, -3	5, 5	-1, 6
s^*	15, -5	6, -1	0, 0

The real interest of the model is a search structure which is imposed upon the players playing Γ . Play begins where every player is unmatched. The extensive form of the searching game is as follows.

1. Searching players are matched at random with a player from the unmatched population. Histories are private information.

2. If so desired, players may verifiably burn a quantity of money. Money may also be transferred to one's partner.
3. Players observe, or not, their partner's type and choose to play Γ , or not, for one period with their partner. Unanimity is required for players to play Γ .
4. If Γ is played, actions are observed and payoffs received.
5. Players decide whether to accept playing next period with their current partner, or search for a new partner whose type is unknown.

Note that in this scenario search has no direct costs and the game may be played with a new partner in every period. The distinction between type being observable is an important one, and will be studied in the remainder of the paper.

POPULATION DEMOGRAPHICS: An important component of any matching model is population demographics. For instance, if a matching equilibrium is assortative and no new players enter into the population then after the first period of matching $p = 1/2$. If a matching equilibrium is non assortative and no new players enter the population, then after the first period all players will have been matched. This latter case drastically changes the implications of the following sections since players' histories are now observable, thus burning money is no longer necessary. To avoid this difficulty, I assume that in every period a mass, μ , of new agents are born and enter the population unmatched of which a fixed proportion, β , are of type δ_h .

It should be obvious that if the equilibrium were non assortative, then $p = \beta$.

It remains to study the case where matching is assortative. We therefore have the following proposition:

Proposition 1 *If the equilibrium is assortative, then we have the following:*

$$\begin{aligned} \beta = \frac{1}{2} &\Rightarrow p = \frac{1}{2} \\ \beta \neq \frac{1}{2} &\Rightarrow p = \frac{\beta - \sqrt{\beta - \beta^2}}{2\beta - 1} \end{aligned}$$

Proof of Proposition 1 If $\beta = \frac{1}{2}$ it is obvious that $p = \frac{1}{2}$. We are looking for a state in which the total mass of agents does not change, nor does the proportion, p , of type δ_h agents. If the total mass is constant then we must have $\mu = s(p^2 + (1-p)^2)$, where s is the total mass of all agents in the economy. If the proportion of type δ_h agents is constant, then we must have:

$$p = \frac{\frac{s}{2}(1-p^2 - (1-p)^2) + \beta\mu}{s(1-p^2 - (1-p)^2) + \mu}. \quad (3.2)$$

Substituting out s in equation (3.2) yields $\beta = \frac{p^2}{p^2 + (1-p)^2}$. Using the quadratic formula to solve for p yields:

$$p = \frac{\beta \pm \sqrt{\beta - \beta^2}}{2\beta - 1}.$$

If $\beta > \frac{1}{2}$ then $p \in (\frac{1}{2}, \beta)$. Since $\beta < \frac{\beta}{2\beta-1}$ when $\beta > \frac{1}{2}$, we can see that the relevant root for this case is $\frac{\beta - \sqrt{\beta - \beta^2}}{2\beta - 1}$. When $\beta < \frac{1}{2}$, we have that the denominator is negative, therefore the numerator must be negative also. This implies that the

relevant root must be: $\frac{\beta - \sqrt{\beta - \beta^2}}{2\beta - 1}$. \square

3.3 Full Information

In this section the simplest informational structure is imposed: type (discount factor) is observable. Understanding the NP constraint is critical to understanding the following results. This constraint requires that no player can profitably unilaterally deviate from a search equilibrium by cheating on a new partner in every period. Primarily, this constraint is satisfied through the verifiable burning of a quantity of money. It will be shown that transferring money also will be useful under certain conditions. Note that money burning is only used to satisfy the NP constraints and not as a signal since type is observable; this is a major difference between this and the following section. Recall that a quantity T of money can be verifiably burned only before the type of one's partner is known.

There are three equilibria of interest to this game. The first is an assortative equilibrium where a type matches only with a similar type. Non assortative equilibria have all types automatically matching with any other type. Non assortative equilibria come in two flavors. δ_l is always obliged to play s_l with any type, however δ_h could play either s_l or s_h if matched with another type δ_h . These equilibria are examined in the subsections below.

Strategies are composed of three elements. The first element will be a type dependent quantity of money to burn. The second element will be an action (element of $\{s_h, s_l, s^*\}$) dependent on type as well as the quantity of money burned by both agents. The third element will be a decision (dependent on the quantity

of money burned and the history of actions played) to remain with one's current partner or search for a new one.⁶ Note that this third element of a strategy enables matches to be ended if a partner defects. An equilibrium will be a strategy which satisfies the No Parasites (NP) constraint, as well as the Individual Rationality (IR) constraint for both types of agent. The IR constraint is standard. The NP constraint will be more formally presented in each of the following subsections which treat the assortative and non assortative cases separately.

As alluded to in the introduction, the equilibrium concept is important. For instance, consider the strategy "burn nothing and play s^* ". Such a strategy is clearly a Nash equilibrium. More interestingly, such a strategy is also a perfect bayesian equilibrium. This motivates the use of the payoff dominance concept to refine equilibria. The following subsections show that this equilibrium concept gives the model much more predictive power than it otherwise would have.

Assortative

If it is the case that the search equilibrium is assortative (in the sense that type δ_l is only accepted by type δ_l), there may be two quantities of money to be burned. The case where only one quantity of money is burned is discussed later. Let T_h^A be the quantity of money burned by type δ_h and let T_l^A be the quantity of money burned by type δ_l in an assortative equilibrium. T_h^A and T_l^A must satisfy the

following No Parasites (NP) constraints:

$$\frac{-T_h^A + \frac{p\pi(s_h)}{1-\delta_h}}{1 - \delta_h(1-p)} \geq \frac{-T_h^A + p\pi^*(s_h)}{1 - \delta_h}$$

$$\frac{-T_l^A + \frac{(1-p)\pi(s_l)}{1-\delta_l}}{1 - \delta_l p} \geq \frac{-T_l^A + (1-p)\pi^*(s_l)}{1 - \delta_l}.$$

Or, more conveniently:

$$T_h^A \geq \frac{1}{\delta_h} [\pi^*(s_h)(1 - \delta_h(1-p)) - \pi(s_h)] \quad (3.3)$$

$$T_l^A \geq \frac{1}{\delta_l} [\pi^*(s_l)(1 - \delta_l p) - \pi(s_l)]. \quad (3.4)$$

An equilibrium concept even as strong as perfect bayesian equilibrium does not delete strategies of the type where $T_h^A + \varepsilon$ and $T_l^A + \varepsilon$ ($\varepsilon > 0$) must be burned in order that agents play together. This is the first instance when the payoff dominance criterium is employed. Clearly, burning T_h^A or T_l^A satisfying the NP constraints with strict inequality is dominated by the burning of T_h^A or T_l^A satisfying the NP constraints with equality. Hereafter T_h^A and T_l^A will be understood to denote the lowest values respecting (3.3) and (3.4) respectively.

It may well be the case that T_h^A or T_l^A be negative. If this is the case then the physical friction of finding a match dominates the friction due to the ability of players to defect. This is undesirable since players would have to "burn" a negative quantity of money. It is obviously not acceptable to have agents literally burning negative quantities of money, but the use of transfers can serve the same purpose. The following proposition gives the condition under which transferring money can make the NP constraints be satisfied with equality.

Proposition 2 *If $pT_h^A + (1-p)T_l^A \geq 0$, then the NP constraints can be satisfied with equality through the use of transfers.*

Proof of Proposition 2 If both T_h^A and T_l^A are greater than zero, then the inequality in the statement of the proposition is valid, and transfers need not be used. Suppose that $T_h^A < 0$ and $T_l^A > 0$, and instead of burning money, δ_l transfers money to a prospective partner while δ_h neither burns nor exchanges money. Denote the transfer by t and the amount of money burned by b . In order that δ_l 's NP constraint be satisfied we require that $pt \geq T_l^A$ (if the inequality is strict, then simply set $b > 0$ to satisfy the NP constraint with equality). In order that δ_h 's NP constraint be satisfied with equality we require that $(1-p)t \geq -T_h^A$, or $(1-p)T_l^A + pT_h^A \geq 0$. Supposing that $T_h^A > 0$ and $T_l^A < 0$ leads to exactly the same condition. \square

Note that when $T_h^A < 0$ and $T_l^A < 0$, then the NP constraints cannot be satisfied with equality. Furthermore, when p is close to zero or unity, transfers can always be used to satisfy the NP constraints with equality. Thus, we obtain:

$$T_h^A = \max \left\{ \frac{1}{\delta_h} [\pi^*(s_{\delta_h})(1 - \delta_h(1-p)) - \pi(s_{\delta_h})], 0 \right\} \quad (3.5)$$

$$T_l^A = \max \left\{ \frac{1}{\delta_l} [\pi^*(s_{\delta_l})(1 - \delta_l p) - \pi(s_{\delta_l})], 0 \right\}. \quad (3.6)$$

The amount of money to be burned cannot be too high so as to make playing the game unprofitable. We thus have the following individual rationality (IR)

constraints:

$$T_h^A \leq \frac{p\pi(s_h)}{1 - \delta_h}$$

$$T_l^A \leq \frac{(1 - p)\pi(s_l)}{1 - \delta_l}.$$

It is straightforward to check that (3.1) guarantees that the IR and NP constraints are internally consistent.

Non assortative

A non assortative search equilibrium is one where both types accept playing with both types. It will always be the case that δ_l , when paired with δ_h or δ_l , will play s_l . However, if δ_h were paired with δ_h either s_h or s_l could be played. This is an important distinction. I will call the type of equilibrium where δ_h plays s_l with δ_h and δ_l “non assortative”; “hybrid” will refer to the equilibrium where δ_h plays s_h with δ_h and s_l with δ_l . Since type is assumed to be observable, no signal need be sent. In a non assortative search equilibrium money must still be burned to satisfy the NP constraints. Note that since type is observable, we may permit each type to burn a different amount of money, despite the fact that each type may be equally as “productive.” In a hybrid equilibrium it is useful to recall that money is required to be burned before one’s opponent’s type is observed, therefore δ_h must burn a single quantity of money regardless of her partner.

Define T_h^{NA} and T_l^{NA} to be the amount of money burned in a non assortative

equilibrium by δ_h and δ_l respectively. The NP constraint for δ_l is written:

$$T_l^{NA} \geq \frac{1}{\delta_l} [\pi^*(s_{\delta_l}) - \pi(s_{\delta_l})]. \quad (3.7)$$

Since δ_h plays s_l with δ_h and δ_l then her NP constraint is:

$$T_h^{NA} \geq \frac{1}{\delta_h} [\pi^*(s_l) - \pi(s_{\delta_l})] \quad (3.8)$$

Now consider a hybrid equilibrium. These two equilibria are identical from the perspective of δ_l , so we will have $T_l^{NA} = T_l^H$. Since δ_h plays s_h when paired with δ_h then δ_h 's NP constraint will differ from T_h^{NA} above. Call the new amount of money to be burned T_h^h . T_h^h must be such that δ_h prefers not defecting once matched with type δ_h :

$$\frac{\pi(s_h)}{1 - \delta_h} \geq \pi^*(s_h) - \delta_h T_h^h + \frac{p\pi(s_h) + (1 - p)\pi(s_l)}{1 - \delta_h},$$

which yields:

$$T_h^h \geq \frac{1}{\delta_h} \left[(\pi^*(s_h) - \pi(s_h)) - \frac{\delta_h(1 - p)}{1 - \delta_h} (\pi(s_h) - \pi(s_l)) \right]. \quad (3.9)$$

Similarly T_h^h must be such that δ_h prefers not defecting once matched with type δ_l :

$$\frac{\pi(s_l)}{1 - \delta_h} \geq \pi^*(s_l) - \delta_h T_h^h + \frac{p\pi(s_h) + (1 - p)\pi(s_l)}{1 - \delta_h},$$

which yields:

$$T_h^h \geq \frac{1}{\delta_h} \left[(\pi^*(s_l) - \pi(s_l)) + \frac{\delta_h p}{1 - \delta_h} (\pi(s_h) - \pi(s_l)) \right]. \quad (3.10)$$

Since the right hand side of inequality (3.9) may be less than or greater than the right hand side of inequality (3.10), T_h^h must satisfy both constraints. The NP constraints for δ_l are identical in a hybrid equilibrium to the NP constraints in a non assortative equilibrium.

Note that in non assortative and hybrid equilibrium it will never be useful to use transfers to burn negative quantities of money. The intuition is that since a partner is found at the beginning of play, the friction due to the ability of players to defect will always dominate the friction due to the time to look for a new partner.

Payoff dominance requires that T_l^{NA} , T_h^{NA} and T_h^h satisfy the NP constraints with equality. Perfect bayesian equilibrium does not select the most profitable equilibrium between non assortative and hybrid equilibria for δ_h . Define T_h^{NA} and T_h^h as above. We have that $T_h^h > T_h^{NA}$. Due to this, there exist beliefs entailing that the only rational strategy based on these beliefs is to burn T_h^{NA} and play s_l with both δ_l and δ_h . Thus burning T_h^{NA} and playing s_l is a perfect bayesian equilibrium, even if burning T_h^h and playing s_h with δ_h brings higher expected payoffs.

Comparison

In a frictionless world where δ_h and δ_l have the ability to commit to playing s_h and s_l respectively, assortative matching is the first best. This section uses the payoff dominance criterium to predict whether matching will be assortative or not when there are search frictions and moral hazard.

If we ignore the NP constraints, type δ_l always prefers a non assortative equilibrium since in either equilibrium a match is equally as profitable and in a non assortative equilibrium search time is minimized. The following proposition illustrates how the NP constraints change this conclusion. Additionally, the proposition illustrates the importance of treating different types differently even if they are equally as productive.

Proposition 3 *If $T_l^A > 0$ or $pT_h^A + (1 - p)T_l^A \geq 0$, then type δ_l is indifferent between hybrid, non assortative and assortative equilibria.*

Proof of Proposition 3 It is clear that δ_l is indifferent between a hybrid and non assortative equilibrium. Therefore, we show that δ_l is indifferent between a non assortative and assortative equilibrium. The money that δ_l burns in expectation in an assortative equilibrium minus the money that δ_l burns in expectation in a non assortative equilibrium is equal to:

$$\frac{\pi(s_l)p}{1 - \delta_l p}$$

The expected payoff to δ_l in a non assortative equilibrium minus the expected

payoff in an assortative equilibrium is equal to:

$$\frac{\pi(s_l)p}{1 - \delta_l p}. \quad \square$$

Although the above proof is entirely algebraic, the proposition is intuitive. When the NP constraint is just binding, agents are just indifferent between cheating and playing the equilibrium strategy (cooperating). Suppose δ_l has found a match, so that in a non assortative or assortative equilibrium her gains to obedience are the same. Therefore gains to defection are equal in the two cases:

$$\pi^*(s_l) + \delta_l \left[\frac{-T_l^A + (1-p)\pi(s_l)}{1 - \delta_l p} \right] = \pi^*(s_l) + \delta_l \left[-T_l^{NA} + \frac{\pi(s_l)}{1 - \delta_l} \right].$$

Thus, the value of being unmatched is equal in the two cases.

We can also conclude that if δ_l 's NP constraint cannot be satisfied with equality, then non assortative matching is strictly preferred to assortative matching. This, along with proposition 3, assures us that δ_l would always be willing to accept δ_h as a partner. Therefore, since a match requires consent of both partners, the type of equilibrium will be dependent on the preferences of δ_h . The following lemmas study δ_h 's preferences as to different types of equilibrium as a function of the parameters of the game. Together, these lemmas lead to the main result of the section.

Lemma 1 Suppose $T_h^A > 0$ or $pT_h^A + (1-p)T_h^A \geq 0$. Then whenever $\pi^*(s_h) - \pi^*(s_l) < \frac{\pi(s_h) - \pi(s_l)}{1 - \delta_h}$, δ_h is indifferent between a hybrid and non assortative equilibrium. Similarly, whenever $\pi^*(s_h) - \pi^*(s_l) > \frac{\pi(s_h) - \pi(s_l)}{1 - \delta_h}$, δ_h is indifferent between a

hybrid and assortative equilibrium.

Proof of Lemma 1 First notice that T_h^h is determined by (3.10) rather than (3.9) iff

$$\pi^*(s_h) - \pi^*(s_l) \leq \frac{\pi(s_h) - \pi(s_l)}{1 - \delta_h}.$$

(3.10) will define T_h^h iff

$$\pi^*(s_l) - \pi(s_l) + \frac{\delta_h p}{1 - \delta_h} [\pi(s_h) - \pi(s_l)] \geq \pi^*(s_h) - \pi(s_h) - \frac{\delta_h(1-p)}{1 - \delta_h} [\pi(s_h) - \pi(s_l)].$$

Simplification of the above term yields the promised result.

Now suppose that $\pi^*(s_h) - \pi^*(s_l) < \frac{\pi(s_h) - \pi(s_l)}{1 - \delta_h}$. We know that T_h^h is determined by (3.10). δ_h is indifferent between hybrid and assortative equilibria iff:

$$-T_h^h + \frac{p\pi(s_h)}{1 - \delta_h} + \frac{(1-p)\pi(s_l)}{1 - \delta_h} = -T_h^{NA} + \frac{\pi(s_l)}{1 - \delta_h}.$$

Substitution for T_h^{NA} and T_h^h and simplification shows this to always be the case.

Now suppose that $\pi^*(s_h) - \pi^*(s_l) > \frac{\pi(s_h) - \pi(s_l)}{1 - \delta_h}$. Assortative equilibria will be will give the same gains as a hybrid equilibria iff:

$$\frac{-T_h^A + \frac{p\pi(s_h)}{1 - \delta_h}}{1 - \delta_h(1-p)} = -T_h^h + \frac{p\pi(s_h) + (1-p)\pi(s_l)}{1 - \delta_h}.$$

By the previous lemma, we know that T_h^h is defined through (3.9). After substituting in for T_h^A and T_h^h we see that δ_h is always indifferent between hybrid and

assortative equilibria. \square

The following lemma studies the relationship between assortative and non assortative matching.

Lemma 2 Suppose $T_h^A > 0$ or $pT_h^A + (1-p)T_h^A \geq$. Then whenever $\pi^*(s_h) - \pi^*(s_l) < \frac{\pi(s_h) - \pi(s_l)}{1 - \delta_h}$, δ_h prefers assortative to non assortative matching. Similarly, whenever $\pi^*(s_h) - \pi^*(s_l) > \frac{\pi(s_h) - \pi(s_l)}{1 - \delta_h}$, δ_h prefers non assortative to assortative matching.

Proof of Lemma 2 So long as δ_h 's NP constraints bind, rents to δ_h from assortative matching can be readily calculated to be:

$$\frac{\pi(s_h) - (1 - \delta_h)\pi^*(s_h)}{\delta_h(1 - \delta_h)}.$$

Rents to δ_h from non assortative matching can be calculated to be:

$$\frac{\pi(s_l) - (1 - \delta_h)\pi^*(s_l)}{\delta_h(1 - \delta_h)}.$$

Comparison of these two expressions yields the promised result. \square

Thus using the above lemmas, when $\pi^*(s_h) - \pi^*(s_l) < \frac{\pi(s_h) - \pi(s_l)}{1 - \delta_h}$, δ_h orders the types of equilibrium in the following manner:

assortative \succ non assortative \sim hybrid.

When $\pi^*(s_h) - \pi^*(s_l) > \frac{\pi(s_h) - \pi(s_l)}{1 - \delta_h}$, δ_h orders the types of equilibrium in the

following manner:

non assortative \succ assortative \sim hybrid.

Note that hybrid equilibria do not occur.

Obviously $\pi^*(s_h) - \pi^*(s_l) \leq \frac{\pi(s_h) - \pi(s_l)}{1 - \delta_h}$ is the key property of the game which determines if matching is assortative or non assortative. This property links the difference in the gains to defection and the difference in the gains to collusion in an interpretable way. Intuitively, if additional collusion yields only a small increase in the profitability of the relationship yet a large increase in the gains to exercising the moral hazard variable, then non assortative matching will ensue. Becker's (1973) marriage model predicts that in a nontransferable utility scenario assortative matching will arise for any (monotone increasing) production technology. Smith (1997) shows that when players are impatient and search is lengthy, search is assortative when the production technology is log-supermodular. Therefore, for the case when production is subject to collusion players choose the equilibrium based on the "ease" with which different levels of collusion can be supported.

The above has given conditions under which equilibrium is assortative or non assortative so long as $T_h^A > 0$ or $pT_h^A + (1 - p)T_l^A \geq 0$. Since whenever $\pi^*(s_h) - \pi^*(s_l) > \frac{\pi(s_h) - \pi(s_l)}{1 - \delta_h}$, equilibrium is non assortative, it remains to study the equilibrium when $\pi^*(s_h) - \pi^*(s_l) \leq \frac{\pi(s_h) - \pi(s_l)}{1 - \delta_h}$ and δ_h 's assortative NP constraint does not bind.

If the NP constraints were just binding, then, for δ_h , the rents from an assortative equilibrium minus the rents from a non assortative equilibrium equal

$\frac{\pi(s_h) - \pi(s_l)}{1 - \delta_h} - (\pi^*(s_h) - \pi^*(s_l))$. First suppose $T_h^A < 0$ and $T_l^A \geq 0$ and $pT_h^A + (1 - p)T_l^A < 0$. In this case δ_h burns no money but receives a transfer from δ_l . The discounted expected value of the loss due to the NP constraint not binding can be calculated to be: $\frac{T_h^A + p(1-p)T_l^A}{1 - \delta_h(1-p)}$. Thus δ_h prefers an assortative equilibrium whenever:

$$\frac{\pi(s_h) - \pi(s_l)}{1 - \delta_h} - (\pi^*(s_h) - \pi^*(s_l)) > -\frac{T_h^A + (1-p)pT_l^A}{1 - \delta_h(1-p)}.$$

Now suppose that $T_h^A < 0$ and $T_l^A < 0$. In this case δ_h burns no money and receives no transfer from δ_l . The discounted expected value of the loss due to the NP constraint not binding can be calculated to be: $\frac{T_h^A}{1 - \delta_h(1-p)}$. Thus δ_h prefers an assortative equilibrium whenever:

$$\frac{\pi(s_h) - \pi(s_l)}{1 - \delta_h} - (\pi^*(s_h) - \pi^*(s_l)) > -\frac{T_h^A}{1 - \delta_h(1-p)}.$$

We can finally state the main proposition of this section.

Proposition 4 *Matching will be assortative whenever:*

$$\frac{\pi(s_h) - \pi(s_l)}{1 - \delta_h} - (\pi^*(s_h) - \pi^*(s_l)) > \max \left\{ 0, \min \left\{ \frac{-T_h^A}{1 - \delta_h(1-p)}, \frac{-[T_h^A + (1-p)pT_l^A]}{1 - \delta_h(1-p)} \right\} \right\}.$$

Similarly, matching will be non assortative whenever:

$$\frac{\pi(s_h) - \pi(s_l)}{1 - \delta_h} - (\pi^*(s_h) - \pi^*(s_l)) < \max \left\{ 0, \min \left\{ \frac{-T_h^A}{1 - \delta_h(1 - p)}, \frac{-[T_h^A + (1 - p)pT_l]}{1 - \delta_h(1 - p)} \right\} \right\}.$$

Cycling

Until this section, we have ignored the effects of population demographics on equilibrium. Since the choice of equilibrium (to a certain extent) chooses the proportion of patient players, such effects should be studied. Of course, such effects may be trivial in that the combination of β , μ and the type of equilibrium may cause one type of equilibrium to always be played. The following example shows that this need not always be the case.

Recall that if the equilibrium is non assortative, then $p = \beta$. If the equilibrium is assortative and $p \neq 1/2$ then $p = \frac{\beta - \sqrt{\beta - \beta^2}}{2\beta - 1}$. Now suppose the following:

1. $\beta > \frac{1}{2}$,
2. $[\pi^*(s_h)(1 - \delta_h(1 - \beta)) - \pi(s_h)] > 0$,
3. $\pi^*(s_h) \leq \frac{\pi(s_h) - \pi(s_l)}{1 - \delta_h}$,
4. for $p = \frac{\beta - \sqrt{\beta - \beta^2}}{2\beta - 1}$, we have $T_h^A < 0$, and $pT_h^A + (1 - p)T_l^A < 0$,
5. for $p = \frac{\beta - \sqrt{\beta - \beta^2}}{2\beta - 1}$, we have $\pi^*(s_l) < \frac{\pi(s_l) - \frac{\delta_h p}{1 - \delta_h(1 - p)}\pi(s_h)}{1 - \delta_h}$.

1. assures us that if the equilibrium is assortative, then $p < \beta$. 2. assures us that if $p = \beta$ then $T_h^A > 0$. 3. assures us that if $T_h^A > 0$, then the equilibrium

is assortative (proposition 4). 4. assures us that for a certain value of p δ_h 's NP constraint cannot bind. 5. assures us that for this value of p , type δ_h prefers non assortative matching.

5. requires some justification. The value to δ_h of non assortative matching can be calculated to be $\frac{\pi(s_l) - (1 - \delta_h)\pi^*(s_l)}{\delta_h(1 - \delta_h)}$. The value to δ_h of assortative matching when $T_h = 0$ is: $\frac{p\pi(s_h)}{(1 - \delta_h)(1 - \delta_h(1 - p))}$. 5. is simply the combination of these two expressions.

Consider the environment in period one. Since $p = \beta$, we are assured that $T_h^A > 0$ (by 2.). By 3. the equilibrium is assortative. In period two, p changes such that $p = \frac{\beta - \sqrt{\beta - \beta^2}}{2\beta - 1}$, so the non negativity constraint on T_h^A binds by 4, thus $T_h^A = 0$. Now we use 5. to see that non assortative matching is preferred to assortative matching. Of course in period 3, $p = \beta$ and the cycle repeats itself. So in general, in every odd numbered period there is assortative matching and in every even numbered period there is non assortative matching.

However, if such an environment were to exist then the amount of money to be burned (T_h and T_l , presumably) is miscalculated. In this case there would be different quantities of money to burn in odd and even periods. Let the superscript o denote an assortative (odd) period, and the superscript e denote a non assortative (even) period. Thus, we can use the NP constraints, the non negativity constraints, and the payoff dominance criterium to write the following explicitly:

$$\begin{aligned}
T_h^e &= \max \left\{ 0, \frac{1}{\delta_h} \left[\pi^*(s_h) - \frac{\pi(s_h)}{1 - \delta_h} + \frac{\delta_h \pi(s_l)}{1 - \delta_h} \right] \right\}, \\
T_h^o &= \max \left\{ 0, \frac{1}{\delta_h} \left[-\pi^*(s_l) - \delta_h(1 - \beta)\pi^*(s_h) + \frac{\delta_h}{1 - \delta_h} \pi(s_h) - \frac{\pi(s_l)}{1 - \delta_h} \right] \right\}, \\
T_l^e &= \max \left\{ 0, \frac{1}{\delta_l} [\pi^*(s_l) - \pi(s_l)] \right\}, \\
T_l^o &= \max \left\{ 0, \frac{1}{\delta_l} \left[\pi^*(s_l) - \frac{1 - \delta_l(1 - \beta)}{1 - \delta_l} \pi(s_l) \right] \right\}.
\end{aligned}$$

The NP constraints assure us that no agent has an incentive to defect in any given period. However, it may be the case that an agent be prepared to “sit out” a period in anticipation of the next. The algebra involved in checking such an eventuality is cumbersome, nevertheless the following lemmas provide an alternative method of analyzing this case.

Lemma 3 If $T_l^o > 0$, then δ_l obtains identical payoffs in odd and even rounds.

Proof of Lemma 3 The lemma can be proved in a fashion similar to the earlier indifference propositions. We know that T_l^e is always greater than zero, so if $T_l^o > 0$, δ_l is indifferent between defecting and cooperating in the two cases. Since cooperating gains are equal in the two cases, gains to defection must be equal in the two cases. Thus, gains to being unmatched are equal in the two cases. \square

The previous lemma implies that if $T_l^o > 0$, then δ_l will always prefer participating as opposed to “sitting out” (since future gains are discounted). Additionally, if $T_l^o = 0$, then δ_l strictly prefers non assortative (even) periods to assortative (odd) periods, thus, δ_l may prefer to sit out during assortative periods.

The following lemma studies how δ_h acts in such a scenario.

Lemma 4 δ_h must eventually find it preferable to participate in the economy.

Proof of Lemma 4 Suppose not. Therefore in every period only δ_l participates. (δ_l must participate since any such period would be equivalent to non assortative matching.) Thus the mass of type δ_h agents approaches infinity, and as such p approaches unity and T_h approaches $\frac{1}{\delta_h}[\pi^*(s_h) - \pi(s_h)]$ which is a contradiction since by assumption 2, δ_h would gain strictly positive payoffs by participating. \square

Thus, δ_h may prefer to sit out for several periods, but must eventually re-enter the economy. In this case, cycling has a period of greater than two.

3.4 Private Information

In this section type is assumed unobservable. The NP constraints must be retained as agents are still able to change partners at will. The presence of private information does, however, require the introduction of incentive compatibility constraints: type δ_h must prefer sending signal (burning) T_h rather than T_l , and type δ_l must prefer sending signal (burning) T_l to sending signal T_h . These constraints will be referred to by IC_h and IC_l respectively. Thus an equilibrium must satisfy IC_h and IC_l in addition to the NP and IR constraints. In this context money burning not only serves to satisfy a NP constraint, but also serves as a signal along the lines of Camerer (1988). Note that IC_l requires that δ_l prefer burning T_l to burning T_h and playing s^* .

There are four equilibria of interest which arise in this informational setting:

assortative, non assortative, hybrid and “defection”. The notions of assortative, non assortative and hybrid equilibria are akin to those developed in the previous section. Defection equilibria have both types burning the same amount of money, but playing different actions. This type of equilibrium is only possible when type is unobservable. Each of the different types of equilibria will be discussed in a series of subsections.

Assortative Equilibria

Assortative equilibria under incomplete information are similar to assortative equilibria under full information. The constraints for each type can be written in the following manner. For type δ_h :

$$\frac{-T_h^A + p \frac{\pi(s_h)}{1-\delta_h}}{1 - \delta_h(1-p)} \geq \begin{cases} 0 \\ \frac{-T_h^A + p\pi^*(s_h)}{1-\delta_h} \\ \frac{-T_l^A + (1-p) \frac{\pi(s_l)}{1-\delta_h}}{1-p\delta_h} \end{cases}$$

For type δ_l :

$$\frac{-T_l^A + (1-p) \frac{\pi(s_l)}{1-\delta_l}}{1 - \delta_l p} \geq \begin{cases} 0 \\ \frac{-T_l^A + (1-p)\pi^*(s_l)}{1-\delta_l} \\ \frac{-T_h^A + p\pi^*(s_h)}{1-\delta_l} \end{cases}$$

These are the familiar IR, NP and new IC constraints, respectively of each type.

The IR and NP constraints are identical to the IR and NP constraints of section

3.3. The NP constraints must respect the constraint that no negative quantities of money be burned.

The only justification needed is to explain why the constraint where δ_h burns T_l^A and then defects, is not included. Similarly the constraint where δ_l burns T_h^A and then plays (faithfully) s_h is not included either. To see why, consider δ_h burning T_l^A . We have that there is a unique $\delta \in (0, 1)$ such that:

$$\frac{-T_l^A + (1-p)\frac{\pi(s_l)}{1-\delta}}{1-\delta p} = \frac{-T_l^A + (1-p)\pi^*(s_l)}{1-\delta}.$$

For any discount factors above δ the left hand side is strictly greater than the right hand side. Since we know that the left hand side is weakly greater than the right hand side for δ_l , we are assured that the inequality is strict for δ_h . A similar argument shows why δ_l will never faithfully play s_h .

Let us now reconsider Ghosh and Ray's (1996) notion of bilateral rationality. While the equilibrium concept of this paper rules out unilateral deviations, newly matched agents may have an incentive to reach an agreement wherein no money is burned. Such a bilateral deviation might be profitable since it is only expectations about *future* market conditions which deter deviation. Ghosh and Ray's (1996) work show that type unobservability is essential to achieve an equilibrium which satisfies bilateral rationality. It is also essential that a bilateral proposal offering to forego the burning of money be less profitable than burning money. This is true if there is a sufficiently large proportion of agents ready to defect and cause sufficiently large losses once such a proposal is accepted. In an assortative equilibrium where type is unobservable, δ_h has no incentive to bilaterally renegotiate

money burning whenever $T_h \leq -(1-p)\pi(s_h, s^*)$.

Non assortative

Non assortative equilibria need not satisfy incentive compatibility constraints, nevertheless the analysis is changed under private information since only one quantity of money can be burned. This follows since if there were two distinct quantities of money to burn, it would always be optimal to burn the lesser quantity of money. Thus there will only be one NP constraint:

$$T_h^{NA} \geq \frac{\pi^*(s_{\delta_l}) - \pi(s_{\delta_l})}{\delta_l} \quad (3.11)$$

which will be satisfied with equality if the payoff dominance criterium is used. Additionally, there will only be one IR constraint: $-T_h^{NA} + \pi(s_l)/(1 - \delta_l) \geq 0$. This is always satisfied so long as (3.11) holds with equality due to (3.1).

Hybrid Equilibria

A hybrid equilibrium when type is unobservable is similar to a hybrid equilibrium when type is observable: δ_h signals her type by burning a greater amount of money than δ_l , but if she is unlucky enough to have drawn a partner who has not signalled that she is of type δ_h , then she does not search the next period in the hope of finding a patient partner. Of course a hybrid equilibrium will involve δ_l burning money as in a non assortative equilibrium, thus we may write T_l^{NA} and T_l^H interchangeably.

Let T_l^{NA} be the signal sent by δ_l and let T_h^h be the signal sent by δ_h . Unob-

servability of type requires two additional constraints.

IC_l requires δ_l to prefer burning T_l^{NA} to burning T_h^h :

$$T_h^h - (1 - \delta_l)T_l^{NA} \geq p \left[\pi^*(s_l) - \frac{\pi(s_l)}{1 - \delta_l} \right] + \frac{\delta_l}{1 - \delta_l} \pi(s_l).$$

Note that the above excludes the possibility that δ_l defects upon meeting another δ_l ; implicitly the constraint:

$$\frac{\pi(s_l)}{1 - \delta_l} \geq \pi^*(s_l) + \delta_l \left[-T_l^{NA} + \frac{\pi(s_l)}{1 - \delta_l} \right]$$

is assumed satisfied. But this is exactly the NP constraint which defines T_l^{NA} .

IC_h requires that δ_h prefer burning T_h^h to burning T_l^{NA} :

$$T_h^h - T_l^{NA} \leq \frac{p[\pi(s_h) - \pi(s_l)]}{1 - \delta_h}$$

Defection Equilibria

A defection equilibrium has both δ_h and δ_l burning an identical quantity of money, T^D , and upon meeting a new partner δ_h and δ_l play different actions. There are three types of defection equilibria which are of interest:

Defection 1: δ_h only plays s_h with a permanent partner.

Defection 2: δ_h only plays s_l with a permanent partner.

Defection 3: δ_h plays both s_h and s_l with a permanent partner.

These definitions do not speak to the behavior of δ_l nor to the initial actions taken by prospective partners. For example, a defection 1 equilibrium could have

δ_h playing s_h or s_l upon meeting a new partner, whereas δ_l could play s^* . Furthermore, δ_l , after meeting somebody who has played s^* , could choose to play s_l permanently. Initial actions will influence the quantity of money to be burned by each type. However, notice that in a type 2 defection equilibrium δ_l must play s^* at every round. For instance, in a defection 1 equilibrium where δ_h always plays s_h and δ_l always plays s^* , the value which δ_h assigns to being unmatched decreases since there is a probability $(1 - p)$ that she will receive payoffs of $\pi(s_h, s^*)$. Recognizing this, δ_h will need to burn less money upon meeting a new partner to respect the NP constraints. The next proposition uses this insight to arrive at an important result:

Proposition 5 *If the NP constraint binds in a defection 1 equilibrium, then δ_h is indifferent between this equilibrium and an assortative equilibrium when type is observed.*

If the NP constraint binds in a defection 2 equilibrium, then δ_h is indifferent between this equilibrium and a non assortative equilibrium when type is observed.

If the NP constraint binds in a defection 3 equilibrium, then δ_h is indifferent between this equilibrium and a hybrid equilibrium when type is observed.

Proof of Proposition 5 For each equilibrium, if the NP constraint binds, then δ_h is just indifferent between staying with a long term partner and defecting. Thus the values to being unattached in a defection 1, 2 or 3 equilibrium must be just equal to the value of being unattached in an assortative, non assortative or hybrid equilibrium respectively. \square

A corollary to the above proposition is that in a defection equilibrium the amount

of money burned is less than the amount of money burned in a full information equilibrium. Also note that defection equilibria make δ_h strictly worse off when NP does not bind.

It is easy to give a simple, albeit somewhat trivial, example of a defection equilibrium where δ_h plays s_h and δ_l plays s^* . Suppose the following:

1. p is arbitrarily close to unity,
2.
$$\frac{\pi(s_h) - (1 - \delta_h)\pi^*(s_h)}{\delta_h} \geq \frac{\pi(s_l) - (1 - \delta_l)\pi^*(s_l)}{\delta_l},$$
3.
$$\pi^*(s_h) - \pi^*(s_l) < \frac{\pi(s_h) - \pi(s_l)}{1 - \delta_h}.$$

From 1, both NP constraints will be binding in an assortative equilibrium (using transfers). δ_h always prefers playing s_h to s_l so long as 3. holds. The incentive compatibility constraint for δ_l is not typical, as it says that δ_l prefers to masquerade as δ_h rather than playing an assortative equilibrium. This is given by condition 2. Since the mass of δ_l types has been assumed to be arbitrarily small, δ_h would tolerate an infestation rather than raise T_h or play s_l . A similar example can be given where δ_h plays s_l . When type is observable a defection equilibrium could not arise as δ_h would simply refuse to play with δ_l .

In general, there are three stages in the construction of a defection equilibrium. First the NP constraint for δ_h is used to define the quantity of money which is burned in equilibrium. This amount will be denoted T^d . Second, the incentive compatibility constraint must be satisfied for type δ_l . This constraint states that δ_l prefers burning T^d to burning another amount which may result in another type of equilibrium. Third, δ_h must prefer a defection equilibrium to raising T^d

by a quantity which would render δ_l 's masquerade unprofitable. Similarly, δ_h must prefer a defection equilibrium to a non assortative or hybrid equilibrium. Note that if the perfect bayesian equilibrium concept is to be used, a defection equilibrium could exist for a larger set of economies.

Implications of Type Unobservability

As was mentioned in the introduction of this section, money burning in this informational context not only serves to fulfill an NP constraint but also as a way of communicating information. This is an important distinction with section 3.3 where type was observable. This subsection has for goal the study of the implications of the introduction of the incentive compatibility constraints on equilibrium.

The first result verifies that proposition 3 holds when type is unobservable.

Proposition 6 *When type is unobservable, so long as δ_l 's NP constraint is binding, then she is indifferent between hybrid, non assortative and assortative equilibria and prefers defection equilibria.*

Proof of Proposition 6 We need verify that the introduction of the IC constraints does not change proposition 3. The only way in which this would be untrue would be if T_l^A , T_l^{NA} or T_l^H were changed. None can be decreased since an NP constraint would be violated. Suppose T_l^A or T_l^H increases (due to a violation of IC_h). Then δ_l would strictly prefer a non assortative equilibrium. But δ_l could refuse to increase T_l^A or T_l^H and a non assortative equilibrium would ensue, contradicting the payoff dominance criterium.

To see that δ_l prefers a defection equilibrium, simply recall the IC_l constraint

in the definition of a defection equilibrium. This constraint requires that δ_l prefer a defection equilibrium to an assortative equilibrium. \square

Note that if δ_l 's NP constraint is not binding then δ_l strictly prefers a non assortative or hybrid equilibrium to an assortative equilibrium. Thus, we conclude that, like when type was observable, δ_h selects the type of equilibrium.

An important corollary to the above proposition is that the introduction of the incentive compatibility constraints can never force T_l to be increased. Suppose that IC_h was violated in an assortative equilibrium at the values of T_h^A and T_l^A defined by the NP constraints. The only way to satisfy IC_h is to raise T_l^A (lowering T_h^A is not possible, since that would violate an NP constraint). Thus if T_l^A were to be increased, then δ_l would be strictly worse off than in a non assortative equilibrium (from the above proposition). But by refusing to increase T_l^A , δ_l can guarantee that a non assortative equilibrium ensue. Thus δ_l would never increase T_l^A . The same argument holds for T_l^H .

In the previous section we were able to order δ_h 's preferences over equilibria in an intuitive way so long as the NP constraint was binding (i.e. so long as the physical friction of search did not dominate the moral hazard friction of search). The assumption that type is unobservable now affords another reason for the NP constraint not to bind – the IC constraints. Thus like in the previous section the analysis is complicated when the NP constraints do not bind, nevertheless certain results can be obtained.

Proposition 7 examines the impact of incentive compatibility so long as the NP constraints for the defection equilibria binds. Proposition 8 examines the

choice between an assortative and defection 1 equilibrium, provided that IC_i is violated and that the NP constraint does not bind for the defection 1 equilibrium. Proposition 9 examines the choice between a non assortative and defection 2 equilibrium, provided that the NP constraint does not bind for the defection 2 equilibrium.

Proposition 7 *If δ_h 's NP constraint bind in a defection 1 equilibrium, then δ_h can always guarantee herself the same payoff as an assortative type observable equilibrium.*

Similarly, if δ_h 's NP constraint binds in a defection 2 equilibrium, then δ_h can always guarantee herself the same payoff as a non assortative type observable equilibrium.

Proof of Proposition 7 This follows directly from proposition 5. \square

Stated differently, when the NP constraints bind for the defection equilibria, incentive compatibility does not affect δ_h 's payoffs. Note that although δ_h may guarantee herself the same expected payoffs as in an assortative full information equilibrium, the equilibrium is quite different. Additionally, when the NP constraints bind a defection 1 or 2 equilibrium is always preferred to a defection 3 equilibrium.

Proposition 8 *Suppose IC_i is violated in an assortative equilibrium and the NP constraint for a defection 1 equilibrium does not bind. Then a defection 1 equilib-*

rium is preferred to an assortative equilibrium iff:

$$T_h^A + \pi(s_h, s^*) \geq - \left[\frac{\pi(s_h) - \pi^*(s_h)(1 - \delta_h)}{\delta_h} - \frac{\pi(s_l) - \pi^*(s_l)(1 - \delta_l)}{\delta_l} \right].$$

Proof of Proposition 8 If IC_l is violated in an assortative equilibrium then,

T_h^A must be increased by

$$\alpha := \frac{\pi(s_h) - \pi^*(s_h)(1 - \delta_h)}{\delta_h} - \frac{\pi(s_l) - \pi^*(s_l)(1 - \delta_l)}{\delta_l}.$$

Thus, gains to an assortative equilibrium will be equal to gains to a full information assortative equilibrium minus $\frac{\alpha}{1 - \delta_h(1 - p)}$.

If the NP constraint for a defection 1 equilibrium does not bind, then $T^d = T_h^A + \pi(s_h, s^*)$ is less than zero. (Note that here we have assumed that δ_l plays s^* , it might be the case that δ_l plays s_l and remains with a type δ_l agent. If this is the case, the necessary change should be obvious.) Thus gains to a defection 1 equilibrium will be equal to the gains to a full information assortative equilibrium plus $\frac{T_h^A + \pi(s_h, s^*)}{1 - \delta_h(1 - p)}$

Comparing the losses in the two cases leads to the promised result. \square

Proposition 9 Suppose NP does not bind for a defection 2 equilibrium. Then a defection 2 equilibrium will be preferred to a non assortative equilibrium iff:

$$\frac{1}{\delta_l} [\pi^*(s_l) - \pi(s_l)] \geq -(1 - p)\pi(s_l, s^*)$$

Proof of Proposition 9 Gains to a non assortative equilibrium when type is

unobservable can be calculated to be:

$$\frac{\pi(s_l) - (1 - \delta_l)\pi^*(s_l)}{\delta_h(1 - \delta_h)} - (T_l^{NA} - T_h^{NA}).$$

Assuming that the NP constraint does not bind for a defection 2 equilibrium, gains are equal to:

$$\frac{\pi(s_l) - (1 - \delta_l)\pi^*(s_l)}{\delta_h(1 - \delta_h)} + (T_h^{NA} + (1 - p)\pi(s_l, s^*)).$$

Combining the two expressions, cancelling out T_h^{NA} , and posing $T_l^{NA} = \frac{1}{\delta_l}[\pi^*(s_l) - \pi(s_l)]$ leads to the promised result. \square

The most striking implication of considering the model when type is unobservable is the appearance of defection equilibria. Typical analyses of collusion have shared the implication that defection should never be observed since a cartel could always structure collusion in such a way as to dissuade non respect of an agreement. The current analysis differs in that cartels (which in this case are formed of only two agents) form endogenously. This results in a trade off: to increase the profitability of a match with a patient type it is necessary to decrease the profitability of meeting an impatient type. Defection equilibria are a key ingredient to the following proposition.

Proposition 10 *δ_h receives higher payoffs when type is observable, whereas δ_l receives higher payoffs when type is unobservable so long as her NP constraints are binding.*

Proof of Proposition 10 The first part of the proof shows that δ_h is worse off when types are unobservable. Suppose, to the contrary, that δ_h is strictly better off when type is unobservable. Recall that δ_h chose the type of equilibrium when type was observable. Therefore, if the resulting equilibrium is assortative, non assortative or hybrid, then the only way for δ_h to be better off is for the amount of money she burns to decrease. This is not possible since a violation of an NP constraint would ensue. The remaining possibility is that a defection equilibrium be played. But proposition 5 assured us that δ_h was always at least as well off in an assortative or non assortative full information equilibrium as in a defection equilibrium.

Next, we examine the welfare of type δ_l . Since we have assumed that δ_l 's NP constraints are binding, δ_l is always indifferent between assortative, non assortative and hybrid equilibria when type is observable. When type is unobservable a violation of IC_h will not affect the amount of money that δ_l burns. If IC_l is violated in a hybrid or assortative equilibrium, then δ_l would refuse to raise T_l , since a non assortative equilibrium would obtain, leaving him just as well off. The IC_l constraint in defection equilibria ensures us that δ_l prefers a defection equilibrium to an assortative equilibrium. Thus, since defection equilibria only occur when type is unobservable, we can conclude that if a defection equilibrium obtains, then δ_l is better off. \square

Perhaps an interesting result can be obtained by noticing that the scenario where type is unobservable may Pareto dominate the scenario where type is observable. We know that δ_h is indifferent between a defection and assortative

equilibrium. δ_l prefers a defection equilibrium to a non assortative equilibrium iff:

$$\pi^*(s_h) \left[p - \frac{1 - \delta_h(1 - p)}{\delta_l} \right] + \pi^*(s_l) \frac{1 - \delta_l}{\delta_l} + \frac{\pi(s_h)}{\delta_l} + \pi(s_l) \frac{1 - \delta_l(1 - p)}{\delta_l(1 - \delta_l p)} - (1 - p)\pi(s_h, s^*) \geq 0.$$

The intuition as to why an incomplete informational scenario may dominate a complete information scenario is that the action set of δ_h when type is unobservable is contained in the action set of δ_h when type is observable. Namely, when type is unobservable, δ_h may not refuse to play with δ_l before both have played.

3.5 Conclusion

This paper re-examined Becker's (1973) marriage model with search frictions. Two major changes to the model were studied. First the production technology was endogenized to reflect the possibility for Pareto improving collusion. The partnership is faced with the obstacle of agents defecting in every period and matching with a new partner. This obstacle is overcome by making meeting new partners prohibitively expensive. Second was the examination of the model under incomplete information: unobservability of type. An equilibrium concept stronger than perfect bayesian equilibrium is necessary to select a single equilibrium. This paper uses a payoff dominance concept. The goal of the paper was to study when only like types match together (assortative matching) and when types match with different types (non assortative matching). The key condition was argued to be the "ease" with which the different degrees of collusion were supported. The ad-

dition of incomplete information permits the existence of defection equilibria and diminishes the payoffs of patient types while increasing the payoffs of impatient types.

Notes

¹The firm may want to avoid the transaction costs of hiring a contractor for each new task. However, a rationale as to why firms choose to outsource or not is beyond the scope of this paper

²This is along the lines of the efficiency wage literature.

³ Kremer and Maskin (1996) study a matching model, where workers match with other workers.

⁴See Shimer and Smith (1997) footnote 5.

⁵The game to be studied is not a finite game since the amount of money to be burned is an element of the real numbers. This poses a technical obstacle since the notions of perfect bayesian and sequential equilibrium are defined for finite games.

⁶If a player wishes to reject her partner before playing, without loss of generality, assume that both players play s^* , and thus receive zero payoffs.

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