Université de Montréal

Trois essais sur les conséquences de l'absence d'engagement en économique

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Sommaire

Cette thèse est constituée de trois essais concernant la théorie des contrats. À prime abord, les sujets de ces trois essais peuvent paraître fort distincts mais ils sont toutefois liés par le thème récurrent des conséquences économiques d'une capacité limitée d'engagement dans une relation d'affaires bilatérale.

Le premier essai présente un algorithme de programmation dynamique que j'ai développé afin de résoudre le modèle exposé dans Gauthier, Poitevin et González (1997). Le second essai explore la nature des contrats bilatéraux optimaux en information incomplète lorsque l'information est révélée de façon séquentielle à la partie informée. L'analyse y est menée sous différentes hypothèses quant à la capacité d'engagement des parties au contrat. Le troisième essai examine la teneur des comportements stratégiques des parties contractuelles lorsque la partie disposant du plus faible pouvoir de négociation peut procéder de manière privée à un investissement qui accroît le surplus social.

Si l'on écarte le délicat problème du partage du surplus social, la nature des allocations économiques optimales dans une relation bilatérale, réalisables par l'emploi de contrats complets, en information complète et avec pleine capacité d'engagement des parties au contrat, est assez bien comprise dans la profession. À l'instar du système des prix, ces allocations réalisent l'égalité des taux marginaux de substitution des deux parties entre toutes les paires imaginables de biens économiques. Cependant, les contrats obtenus sous ces hypothèses ressemblent généralement fort peu à ceux que l'on peut observer empiriquement. En relâchant les hypothèses précédentes, la théorie moderne des contrats parvient à expliquer la nature des contrats comme résultant d'une adaptation optimale à un environnement contractuel imparfait.

Parmi les hypothèses les plus susceptibles de ne pas être vérifiées dans la réalité, celle d'information parfaite des agents ainsi que celle d'une capacité d'engagement illimitée, sur tout horizon temporel et en toute circonstance, semblent les moins plausibles. La première de ces hypothèses est relâchée dans les deux derniers essais alors que la seconde l'est tout au long de la thèse.

En information incomplète ou imparfaite, on peut démontrer que les objectifs d'assurance du revenu et d'efficacité dans la production sont généralement incompatibles, même si des contrats avec plein engagement sont disponibles. Afin de bien isoler les effets des déficiences quant à la capacité d'engagement des parties à un contrat en information incomplète ou imparfaite, je pose l'hypothèse, dans les deux derniers essais, que les agents n'ont aucune préférence pour le risque. Cette hypothèse n'est pas appropriée dans le premier essai; d'abord, parce que l'aversion stricte pour le risque y constitue la source des gains à l'échange et ensuite, parce que l'analyse y est menée en information complète.

De manière générale, il ressort de mes travaux qu'une capacité limitée d'engagement dans les contrats restreint plus ou moins sévèrement l'ensemble des gains à l'échange réalisables. Il existe des moyens pour parvenir à créer un surplus économique – éventuellement sous-optimal – même si les agents économiques sont opportunistes et ne sont pas tenus de respecter leurs engagements passés. En ce sens, un environnement contractuel parfait n'est pas absolument requis pour qu'une économie puisse fonctionner à un niveau supérieur à l'autarcie. En horizon fini, les contrats robustes au problème de capacité limitée d'engagement, tels que prescrits en théorie, ont souvent une structure plus simple que les contrats de premier rang. Ils ressemblent davantage aux contrats réels et, en ce sens, il est plausible de penser que ce type de problème constitue un trait caractéristique de la nature des relations d'affaires dans l'économie. En horizon infini, lorsque les agents exploitent le potentiel de gain économique réalisable par des échanges différés, le problème d'engagement limité restreint la gamme des échanges possibles. Malgré un problème d'engagement, deux agents pourront s'auto-assurer mutuellement mais de manière imparfaite. Encore une fois, les contrats prescrits par cette théorie de second rang ressemblent davantage aux contrats observés dans la réalité.

Résumé

Je présente ici un résumé de chacun des trois essais inclus dans la thèse. Le premier article étant rédigé en français, j'ai choisi d'en donner également un résumé en anglais.

Solution numérique au modèle d'échange intertemporel avec contraintes auto-exécutoires

Les résultats présentés dans cet article ont été obtenus en collaborant à la recherche de Gauthier, Poitevin et González (1997) (GPG). Dans cet article, nous analysons une relation de partage de risque de long terme entre deux agents averses au risque évoluant dans un environnement stochastique, dont la capacité d'engagement est réduite au point qu'ils doivent recourir pour échanger à des contrats auto-exécutoires. Un contrat auto-exécutoire est requis lorsque les parties à ce contrat ne peuvent s'engager dans le futur à en respecter les termes. Ce type de contrat est conçu de manière telle qu'en respecter les termes en toute circonstance est une stratégie faiblement dominante pour chaque partie.

Dans l'article de GPG, l'espace des contrats est augmenté en autorisant l'emploi ex ante d'un paiement collatéral permettant d'accroître la capacité d'engagement des deux parties. Ce paiement est effectué avant que ne se réalise l'incertitude. Les paiements ex ante et ex post optimaux sont alors caractérisés. L'apport, en termes de bien-être, de la possibilité d'effectuer un

paiement ex ante est enfin calculée numériquement afin de pouvoir comparer l'évolution dynamique de la consommation avec et sans ce paiement.

Dans l'article présenté ici, je présente un algorithme itératif que j'ai développé afin de calculer les contrats auto-exécutoires présentés dans GPG. Les contrats auto-exécutoires optimaux peuvent être représentés comme la solution d'une équation fonctionnelle du type de l'équation de Bellman. Leur résolution diffère toutefois de celles des équations fonctionnelles traditionnelles, telles qu'on en retrouve, par exemple, dans la littérature macro-économique parce que la fonction de valeur inconnue apparaît à la fois dans le maximand de même que dans la correspondance de transition qui définit l'ensemble des instruments du problème de maximum de l'équation fonctionnelle. De fait, les algorithmes standards pour résoudre les équations fonctionnelles ne convergent pas dans ce cas.

Mon algorithme résout cette difficulté en créant deux opérateurs de contraction, appliqués alternativement à chaque itération: le premier transpose la fonction de valeur et la correspondance de transition en une nouvelle fonction de valeur alors que le second les transpose en une nouvelle correspondance de transition. La convergence de cet algorithme est démontrée dans l'article.

J'ai transposé cet algorithme en langage *Matlab* en représentant les fonctions sur un ordinateur à l'aide de splines cubiques Hermitiens. Dans l'article, je présente de manière heuristique les résultats fondamentaux du modèle en commentant des graphes des simulations obtenues grâce à cet algorithme. Les programmes que j'ai écrits pour appliquer cet algorithme sont présentés en annexe de la thèse.

Solution numérique au modèle d'échange intertemporel avec contraintes auto-exécutoires

This paper is a companion paper to Gauthier, Poitevin et González (1997) (GPG). In that paper, we analyze a long-term risk-sharing contract between two risk-averse agents that evolve in a stochastic environment. Because of *their limited commitment capability, the agents must rely on self-enforcing contracts to govern their relationship. Self-enforcing contracts are designed in such a way as to make sure that following the contract terms is a weakly dominated strategy, for each agent, in every possible circumstance.*

In GPG, the contracting space is enlarged to allow for the use of a collateral ex ante payment to improve the commitment capabilities of both agents. That payment is to be made before the state of nature is realized. The optimal ex ante and ex post payments are characterized. In that paper, the value of introducing the ex ante payment is assessed numerically in order to compare the dynamics of consumption with and without this collateral transfer.

In this paper, I present an iterative algorithm that I have developed to compute the self-enforcing contracts presented in GPG. Optimal self-selecting contracts are represented as the solution of a Bellman-like functional equation. Their resolution differs from that of standard functionals equations one can find, for instance, in macroeconomics, in that the unknown value function enters both in the maximand as well as in the correspondence of transition that defines the instrument set of the program of the functional. As such, standard recursive algorithms used to solve functionals fail to converge to the solution in this case.

My algorithm addresses that difficulty by creating two mapping operators that are applied alternately at each iteration: one that maps the value function and the correspondence of transition into a new value function and one that maps them into a new correspondence of transition. The convergence of the algorithm is established in the paper.

The algorithm was implemented in Matlab *using Hermitian cubic splines to represent the functions on a computer. The basic results are presented heuristically using the graphs of simulations obtained thanks to the algorithm. The listings of the programs written to implement the algorithm are presented in an appendix to this thesis.*

Sequential Screening

Dans cet article, je caractérise les contrats optimaux en présence de sélection adverse séquentielle. Une situation de sélection adverse séquentielle peut se produire lorsque certains acteurs économiques ont plusieurs caractéristiques aléatoires inobservées qui se réalisent de manière séquentielle. Un exemple de sélection adverse séquentielle consiste en un monopoleur qui souhaite discriminer initialement ses clients sur la base de leur demande espérée et, ex post, sur la base de leur demande réelle.

Le modèle présenté est du type principal-agent avec communication, adapté aux jeux à deux étapes. À la première période, le principal peut offrir un contrat complet à un agent. L'agent appartient à une population comportant deux types ex ante. Le type ex ante détermine la probabilité de chaque agent d'avoir un type particulier ex post. Le type ex post détermine la fonction de coût de l'agent pour effectuer une tâche particulière (production d'un bien, réalisation d'un achat, etc.) qui importe au principal.

En information complète, le principal offrirait des contrats différents aux deux types d'agents ex ante; chaque contrat comportant des dispositions particulières concernant les types ex post. Le problème consiste à déterminer la nature des contrats optimaux en information incomplète quand le principal ignore tant le type ex ante que le type ex post de l'agent. Par ailleurs, chaque agent ignore initialement son type ex post qui ne lui sera révélé qu'au début de la seconde période.

Le modèle est résolu en appliquant le principe de révélation pour les jeux séquentiels. L'analyse est menée sous différentes formes d'engagement possibles dans les contrats, soit plein engagement sur les deux périodes, absence d'engagement à long terme et engagement avec renégociation. Aux contraintes habituelles de compatibilité des incitations courantes, les contrats offerts à la première période doivent tenir compte des contraintes d'incitation globales qui déterminent le comportement de l'agent quant à l'annonce honnête de son type ex ante.

Afin de pouvoir composer facilement avec les multiples contraintes qui restreignent l'espace des contrats réalisables, j'ai choisi de procéder par programmation dynamique. Je calcule d'abord l'ensemble des vecteurs d'utilité réalisables ex post, sous l'hypothèse que les incitations à la révélation sont satisfaites ex post. Je montre ensuite qu'on peut associer un contrat optimal ex post à chacun de ces vecteurs de sorte que ceux-ci peuvent être retenus pour résumer un contrat. Je calcule ensuite les préférences indirectes du principal à l'égard de ces vecteurs et je montre que celles-ci sont convexes. Ceci me permet d'analyser le problème global depuis la première période.

Je montre comment l'arbitrage traditionnel entre l'extraction d'une rente informationnelle et l'efficacité dans la production, dans les modèles à une seule étape, est transposé ici en un arbitrage entre rente informationnelle et efficacité dans le choix ex ante du type de contrat. Le contrat de plein engagement sépare en effet les types ex ante en offrant un contrat efficace (qui sépare efficacement les types ex post) à l'agent efficace ex ante (contre une rente ex ante) et un contrat inefficace au type inefficace, même si ce dernier peut fort bien se révéler efficace ex post. Ce contrat optimal est toutefois fort sensible aux possibilités de renégociation. En l'absence d'engagement, le contrat optimal ne sépare pas les types ex ante.

Specific Investment, Commitment and Observability

Dans cet essai, je cherche à évaluer dans quelle mesure le potentiel d'asymétrie d'information peut inciter un agent à effectuer un investissement spécifique à une relation bilatérale lorsque la partie adverse est dans l'incapacité à s'engager à en compenser les coûts. Je montre que le contrat optimal issu de cette situation de hold-up a une structure simple en équilibre: il revient à laisser à l'agent investisseur le choix d'un point sur sa courbe de coût moyen de long terme et de lui payer ses coûts totaux à ce point.

Le problème fondamental auquel doit faire face l'investisseur tient au fait que, s'il investit pour abaisser ses coûts variables ex post, il n'a aucune garantie que l'autre partie conviendra ex post d'assumer une juste part des coûts fixes d'investissement. Toutefois, en investissant de manière privée, l'agent investisseur peut obtenir un avantage informationnel ex post quant à sa structure de coût. En exploitant cet avantage, il lui est possible d'extraire ex post une rente informationnelle qui, à l'équilibre, le compensera justement pour ces coûts d'investissement.

Contrairement à la plupart des modèles de type principal-agent, conceptualisés comme des jeux bayesiens en information incomplète, je mène l'analyse en situation d'information complète mais imparfaite. Au début du jeu, il n'existe pas d'ambiguïté quant à la structure de coût initiale de l'agent investisseur. Ce n'est qu'une fois que celui-ci a investi de manière privée que l'asymétrie d'information ex post apparaît. La décision d'investissement est évidemment endogène. En ce sens, ce modèle peut être conceptualisé comme une généralisation des modèles traditionnels de principal-agent où l'agent choisit son type de manière optimale dans une phase initiale du jeu. Cette forme structurelle a comme avantage que la forme réduite du modèle ne dépend pas d'une abstraite distribution de types mais du coût des facteurs d'investissement sur les marchés. Dans le même ordre d'idée, les rentes informationnelles y sont interprétées comme des paiements de facteurs.

Afin de solutionner le modèle, j'applique le concept courant d'équilibre en stratégies behaviorales, tel que présenté dans Myerson (1991). En information imparfaite, ce concept requiert que les anticipations de chaque agent, à l'égard de la stratégie, non observée, jouée par l'autre agent, soient compatibles, à l'équilibre, avec les stratégies effectivement adoptées. Il résulte de l'équilibre que l'agent investisseur est conduit à randomiser sa décision d'investissement en optant pour une stratégie mixte. À l'instar de Fudenberg et Tirole (1990), je montre que cet équilibre peut être purifié au sens d'Harsanyi. C'est en randomisant sa stratégie d'investissement que l'agent parvient à créer un « bruit » endogène ex post, sur sa structure de coûts variables, suffisamment important pour lui donner un avantage informationnel.

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À mon père.

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Patrick González,
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Chapitre 1

Solution numérique au modèle d'échange intertemporel avec contraintes auto-exécutoires

1.1 Introduction

Dans cet article, je présente un algorithme permettant de résoudre numériquement le calcul des contrats dits *auto-exécutoires*. Un contrat est *auto-exécutoire* lorsqu'il est conçu de manière telle qu'en respecter les termes est une stratégie faiblement dominante pour toute partie à ce contrat.

De manière générale, des coûts de transaction doivent être encourus pour assurer que chaque partie à un contrat en respecte les termes, même si cela va à l'encontre de ses intérêts immédiats (*enforcement costs*). Ces coûts peuvent être prohibitifs au point de rendre certains types d'échange impraticables. Toutefois, lorsque les parties à un contrat évoluent dans une relation de long terme d'échange répété, il demeure possible d'économiser une partie de ces coûts en usant des gains à l'échange futurs comme d'une carotte (ou d'un bâton) pour inciter les parties à la coopération. En information complète, les parties à l'échange sont enjointes par le contrat à ne pas manquer à leurs engagements immédiats si elles souhaitent préserver les dispositions futures à la coopération de leurs partenaires. Cet arrangement n'est évidemment possible que si les parties accordent suffisamment d'importance aux mérites d'une coopération future.

Thomas et Worrall (1988), Marcet et Marimon (1994), Kocherlakota (1996) et plus récemment Gauthier, Poitevin et González (1997) ont analysé les propriétés des contrats auto-exécutoires de partage de risque. Le modèle de base est le suivant : deux agents averses au risque sont chacun dotés d'un processus de revenu individuel aléatoire i.i.d. Ces processus sont toutefois négativement corrélés entre eux si bien qu'en échangeant, les deux agents peuvent réduire la variance de leur consommation individuelle. Ceci requiert aujourd'hui que le plus riche des deux partage avec l'autre. Cet arrangement est profitable aux deux parties dans la mesure où les rôles pourront éventuellement être inversés.

Pour que le plus riche aujourd'hui accepte de partager son bien, il doit être raisonnablement convaincu que son voisin voudra bien à son tour partager le sien demain. Réduite à une seule période, cette situation correspond en effet à un jeu à somme nulle. S'il est possible de signer un contrat avec plein engagement, tout arrangement de ce type, qui bénéficie aux deux parties, est possible. Dans le cas contraire, le plus riche aujourd'hui n'acceptera de partager que dans la mesure où il estime que ce qu'il perd aujourd'hui est compensé par ce qu'il pourrait recevoir demain et que, en tout état de cause, son vis-à-vis devrait connaître demain des incitations au partage similaires aux siennes. Un contrat auto-exécutoire gère ces incitations en faisant en sorte que l'ampleur des sacrifices exigé d'un agent riche aujourd'hui soit à la mesure des bénéfices qu'il peut en retirer demain.

L'algorithme présenté ici a été développé au cours des travaux présentés dans Gauthier, Poitevin et González (1997) (GPG). Dans cet essai, les auteurs montrent comment l'emploi optimal de biens collatéraux peut augmenter le potentiel de gains à l'échange réalisables à l'aide de contrats auto-exécutoires. Par ailleurs, ils illustrent les propriétés des allocations qui résultent de l'emploi de contrats auto-exécutoires. Seul un nombre limité de propriétés de ces allocations peuvent être établies analytiquement. En effet, les contrats auto-exécutoires optimaux sont définis par une équation fonctionnelle, issue d'un programme dynamique, qui ne peut être complètement résolue par simple analyse. De manière générale, on doit recourir à l'analyse numérique pour calculer ces contrats. C'est à cette fin que j'ai développé mon algorithme.

Il existe plusieurs algorithmes pour résoudre les programmes dynamiques tels qu'on les rencontre, le plus souvent, dans la littérature macro-économique.[1] Ainsi, pour identifier la fonction de valeur inconnue f satisfaisant l'équation fonctionnelle de Bellman suivante,

$$f(V,s) = \max_{V' \in \Gamma(V,s)} u(V',s) + \beta \sum_{s' \in S} f(V',s')\pi(s,s')$$

où l'argument V est le vecteur de variables d'état du système à la période courante ; V' est le vecteur de variables d'état à la période future qui représente également un ensemble d'instruments courants permettant d'affecter le programme ; u est la fonction de rendement courant (par exemple, une fonction d'utilité) ; $s \in S$ est un aléa stochastique ; $\pi(s,s')$ exprime la probabilité de transition, d'une période à l'autre, d'un état s à s' et Γ est la correspondance de transition. $\Gamma(V,s)$ détermine l'évolution de V' en contraignant l'ensemble des valeurs que ce vecteur peut prendre. La correspondance de transition résume, par exemple, les contraintes de ressources dans le modèle néoclassique standard.

Un algorithme récursif traditionnel pour résoudre ce type de problème consiste à calculer la limite d'une suite $\{f^m\}$ telle que

$$f^{m+1}(V,s) = \max_{V' \in \Gamma(V,s)} u(V',s) + \beta \sum_{s' \in S} f^m(V',s')\pi(s,s')$$

1. Cf. Lucas, Stokey et Prescott (1989) de même que Christiano (1994).

avec une fonction initiale f^1 donnée. Il suffit alors d'établir la convergence de cette suite, dans un espace de fonctions approprié et pour une fonction initiale f^1 connue, vers la solution désirée f. Pour fonctionner, ce type d' algorithme nécessite que la correspondance de transition Γ soit connue, autrement le problème de maximum n'est généralement pas défini. Or, il se trouve que dans le calcul des contrats auto-exécutoires, la correspondance de transition Γ est elle-même une fonction composite de la fonction de valeur inconnue f et est donc, a priori, inconnue. L'algorithme que je présente ici contourne cette difficulté majeure. En deux mots, mon algorithme définit une suite de correspondances de transition Γ^m qui converge vers la vraie correspondance à mesure que la suite de fonctions de valeur $\{f^m\}$ en fait autant.

Les résultats traditionnels sur la convergence des algorithmes récursifs, tels qu'ils peuvent être retrouvés chez Lucas, Stokey et Prescott (1989), ne s'appliquent pas directement ici. Par exemple, les conditions de Blackwell pour établir la propriété de contraction de l'opérateur maximum ne sont généralement pas satisfaites ou sont, à tout le moins, impossibles à vérifier. Il demeure néanmoins possible d'établir la convergence d'une suite particulière de fonctions vers la solution. L'approche moderne de la programmation dynamique[2] distingue deux méthodes générales pour résoudre les problèmes à horizon infini, soit la méthode de *contraction* qui consiste à obtenir f en appliquant le maximum (l'opérateur de contraction) de manière itérative et la méthode d'*approximations successives* qui consiste à obtenir f comme la limite d'une suite $\{f^m\}$ où chaque f^m est la solution du problème sur un horizon de taille m fini. Mon algorithme s'inspire de ces deux approches.

Dans le problème étudié ici, la fonction f représente la frontière de Pareto des gains à l'échange réalisables dans un environnement contractuel particulier. La variable d'état V, quant à elle, est associée au niveau d'utilité intertemporelle attribué par le contrat à l'un des deux agents (l'autre jouissant de $f(V)$). Dans un environnement stationnaire, f représente également

2. Cf. Sniedovich (1992).

les gains à l'échange futurs. La frontière f entre dans la correspondance de transition parce que tout degré de coopération courante – qui fonde ces gains à l'échange et qui implique qu'un agent riche aujourd'hui renonce à une partie de son revenu courant – ne peut être assuré que dans la mesure où cet agent anticipe d'être compensé par une redistribution qui lui sera favorable des gains à l'échange futurs, lesquels sont donnés par f. Cette restriction est caractérisée par des contraintes dites auto-exécutoires. Ce type de modèle ne se prête pas à une formulation sur un horizon fini : les gains à l'échange étant assurément nuls en période terminale, ils ne peuvent inciter les parties à adopter un comportement coopératif à la pénultième période ; ainsi, aucun gain à l'échange ne peut donc être généré en cette période et, en répétant le raisonnement de manière récursive, en aucune période. Pour pouvoir générer un comportement coopératif aujourd'hui, il faut pouvoir mettre en gage des gains à l'échange futur. La teneur de ces gains, tels que représentés par f, nous est, *a priori*, inconnue mais l'on sait qu'ils ne peuvent surpasser ceux réalisables par une allocation de premier rang. J'exploite cette idée de la manière suivante : au lieu de supposer que le contrat implicite liant les agents doive satisfaire des contraintes auto-exécutoires sur tout horizon, je restreint cet horizon à m périodes à partir de la période courante t. Cette formulation correspond à un contrat où sont appliquées des pénalités exogènes extrêmement élevées à partir de la période $t+m$, lesquelles garantissent alors qu'une solution de premier rang peut être obtenue après $t+m$. J'obtiens ainsi une solution coopérative pour chaque horizon de contraintes m et une suite de frontières de Pareto associées $\{f^m\}$. Il suffit alors de démontrer que cette suite converge vers la solution recherchée.

Mon algorithme exploite la représentation du problème sous la forme d'une équation fonctionnelle en f. Avec cette approche, on procède au calcul d'un contrat (ou d'une allocation) optimal en deux étapes. D'abord, on calcule f par itération ; puis, on en infère le contrat associé, lequel peut être représenté comme une fonction composite de f. À chaque itération m, le calcul d'un point particulier sur la frontière f^m est réalisé en maximisant le bien-être d'un des agents sous la contrainte que le bien-être de l'autre agent atteigne un niveau déterminé.

Il est possible de montrer que les allocations optimales satisfont un système d'équations de différence stochastiques non linéaires. Une autre approche consiste à résoudre directement ce système. Marcet et Marimon (1994) ont retenu cette approche. Celle-ci les conduit à maximiser une pondération linéaire du bien-être des deux agents et à retenir le facteur de cette pondération comme variable d'état. Cette approche n'est correcte que si la frontière de Pareto est concave.

Mon approche comporte deux avantages sur celle de Marcet et Marimon. D'une part, elle ne requiert pas nécessairement que la frontière de Pareto soit concave ; bien que je recours à cette hypothèse suffisante pour établir mes résultats (chez Marcet et Marimon, cette hypothèse est nécessaire). D'autre part, elle est très robuste : une application mécanique de mon algorithme apporte généralement une solution. En revanche, résoudre un système d'équations différentielles non linéaires demeure une entreprise dont l'issue est souvent incertaine.

Le reste de cet article est divisé de la manière suivante. Dans la section 1.2, je présente une version du modèle étudié dans GPG. La présentation et l'analyse du modèle qui y sont faites demeurent sommaires ; les lecteurs intéressés sont priés de se référer à GPG. Les contrats auto-exécutoires sont définis à la section 1.3. Leur formulation récursive est exposée à la section 1.4 où j'établis la convergence de mon algorithme. Dans la section 1.5, j'explique comment j'ai programmé mon algorithme sur un ordinateur, à l'aide des fonctions splines. La conclusion suit. Un substrat des résultats de GPG est présenté à l'appendice 1.A. L'analyse y est informelle et repose principalement sur l'examen de graphiques obtenus grâce à mon algorithme.

1.2 Le modèle

Considérez une relation bilatérale entre deux agents, notés u et v, connaissant un horizon de vie infini. Ces agents évoluent en information complète dans une économie stochastique stationnaire. À chaque période t, l'économie est dans un état de la nature s_t résultant d'une variable aléatoire[3] $\tilde{s}_t$ qui suit une loi fixe sur $S = \{1, 2, ..., \overline{s}\}$. Les variables aléatoires $\tilde{s}_t$ sont indépendamment et identiquement distribuées dans le temps avec $\text{Prob}(\tilde{s}_t = s) = \pi^s$.

L'état s détermine le revenu individuel, non capitalisable, de chaque agent. Ainsi, à chaque s correspond un vecteur de dotations $\overline{y}^s = [y_u^s, y_v^s]$. Je note $y^s = y_u^s + y_v^s$ le revenu agrégé. Ces dotations doivent être entendues comme représentant les fluctuations marginales du revenu – les agent peuvent disposer accessoirement d'une autre source de revenu constant non modélisée ici.[4]

Les agents ont des préférences intertemporelles conventionnelles données à la période t par

$$\mathrm{E}\left\{\sum_{n=0}^{\infty} \beta^n u(\tilde{c}_{t+n})\right\}$$

pour l'agent u où $u(\cdot)$ est une fonction strictement concave, β est un facteur d'escompte strictement inférieur à 1 et $\tilde{c}_{t+n}$ représente la consommation, potentiellement aléatoire, à la période $t+n$. L'utilité de l'agent v est notée de manière similaire avec une fonction v, laquelle peut être simplement concave. En absence d'échange, puisqu'ils ne peuvent épargner, les agents consomment simplement leurs dotations individuelles soit, $c_t = y_u^{s_t}$ pour l'agent u et $y^{s_t} - c_t$ pour l'agent v. Cette situation représente la solution autarcique. On suppose toutefois que la décomposition $\overline{y}^s$ du revenu agrégé y^s est telle que les processus individuels de revenu sont négativement corrélés entre eux. Il s'en suit que

3. Dans cet essai, la notation $\tilde{\cdot}$ est employée pour désigner des variables aléatoires.

4. Cette distinction est importante pour justifier plus loin la possibilité des agents de procéder à des transferts ex ante.

les agents peuvent accroître leur utilité en échangeant de manière intertemporelle. Un agent dont la réalisation du revenu est particulièrement élevée peut en céder une partie à son vis-à-vis dont le revenu courant est plutôt faible. Ceci permet aux agents de lisser leur consommation, tant à travers le temps qu'à travers les états de la nature, en en réduisant la variance.

Afin de pouvoir décrire les allocations possibles résultant de tels échanges, j'introduit immédiatement les notations suivantes. Une histoire h_t est une suite d'états de la nature réalisés jusqu'à la période t :

$$h_t = \{s_1, s_2, \ldots, s_{t-1}, s_t\}.$$

Pour $n \geq 0$, je note $h_{t+n} \succ h_t$ pour signifier que h_{t+n} est une continuation possible à h_t. Si $s_t \in h_t$, alors $\{h_{t-1}, s_t\} \equiv h_t$. L'ensemble des continuations possibles à h_t est noté

$$H(h_t) = \{h_{t+n} | n \geq 0, h_{t+n} \succ h_t\}.$$

Une allocation établie après h_t – ou un contrat – est une fonction

$$\begin{aligned} \delta_{h_t} : H(h_t) &\rightarrow X \\ h_{t+n} &\mapsto \delta_{h_t}(h_{t+n}) = x_{t+n} \end{aligned} \tag{1.1}$$

qui détermine notamment la consommation de l'agent u en $t+n$, pour chaque état s possible en $t+n$, lorsque h_{t+n-1} a été réalisée.[5] Avec cette notation générale, X représente l'ensemble des instruments contractuels disponibles. Ainsi, le modèle de GPG autorise l'emploi ex ante, avant que l'état de la nature ne se réalise, d'un paiement B_t de l'agent v à l'agent u. Dans ce cas, $x_t \in X$ inclut non seulement le vecteur $c_t \in R^S$ de consommations pour

5. Je suppose implicitement que le revenu agrégé est consommé par les deux agents. C'est évident dans une allocation de premier rang. Kocherlakota (1996) a montré que c'est aussi le cas pour les allocations de second rang en absence d'engagement.

l'agent u à la période t mais aussi le paiement B_t ;

$$x_t = [B_t, c_t^1, c_t^2, \dots, c_t^{\bar{s}}].$$

Lorsque deux contrats δ_{h_t} et δ_{h_q} ont des prescriptions identiques dans le futur, *i.e.* que, pour toute histoire $h \in H(h_t) \cap H(h_q)$ on a $\delta_{h_t}(h) \equiv \delta_{h_q}(h)$, je note cette relation $\delta_{h_t} \sim \delta_{h_q}$.

Soit,

$$\begin{aligned} S_u(x_t, s_t) &= u(c^{s_t}) - u(y_u^{s_t}) \\ S_v(x_t, s_t) &= v(y^s - c^{s_t}) - v(y_v^{s_t}) \end{aligned}$$

les surplus courants d'utilité sous le contrat pour chaque agent, par rapport à l'utilité qu'il peuvent atteindre en autarcie. Le surplus d'utilité espéré de l'agent u, après h_{t+n}, sous un contrat δ_{h_t}, se note dès lors

$$U(\delta_{h_t}, h_{t+n}) = \mathrm{E}\left\{\sum_{i=1}^{\infty} \beta^{i-1} S_u(\delta_{h_t}(\tilde{h}_{t+n+i}), \tilde{s}_{t+n+i}) | h_{t+n}\right\}.$$

De manière similaire, le surplus d'utilité de l'agent v est notée $V(\delta_{h_t}, h_{t+n})$. Avec cette notation, la décomposition suivante est immédiate,

$$U(\delta_{h_t}, h_t) = \mathrm{E}\{S_u(\delta_{h_t}(\tilde{h}_{t+1}), \tilde{s}_{t+1}) + \beta U(\delta_{h_t}, \tilde{h}_{t+1}) | h_t\}$$

ou encore

$$U(\delta_{h_t}, h_t) = \mathrm{E}\{S_u(\delta_{h_t}(h_{t+1}), \tilde{s}_{t+1}) + \beta U(\delta_{\tilde{h}_{t+1}}, \tilde{h}_{t+1}) | h_t\} \tag{1.2}$$

où $\delta_{h_{t+1}} \sim \delta_{h_t}$, pour toute réalisation $h_{t+1} \in H(h_t)$ de $\tilde{h}_{t+1}$.

Soit $\overline{\Gamma}^0$, l'ensemble des contrats qui respectent les contraintes de ressources de l'économie. Une allocation optimale de premier rang après h_t satisfait

$$f^0(V) = \max_{\delta_{h_t} \in \overline{\Gamma}^0} \mathrm{E}\{U(\delta_{h_t}, h_t)\} \text{ s. c. } \mathrm{E}\{V(\delta_{h_t}, h_t)\} \geq V, \tag{1.3}$$

où V est un niveau d'utilité réalisable pour l'agent v. L'ensemble de ces allocations optimales, indexées par V, forme la frontière de Pareto de premier rang f^0.

Ces allocations sont réalisables à l'aide de contrat avec plein engagement. On peut montrer que les allocations de premier rang égalisent les taux marginaux de substitution de la consommation des deux agents, dans le temps et à travers les états s, et qu'elles sont stationnaires et ne dépendent pas de l'histoire, *i.e.* que $\delta_h(\{h_{t-1}, s_t\}) = \delta_h(\{h_{q-1}, s_q\})$ si $s_t = s_q$.

Le problème consiste à déterminer la nature des allocations réalisables lorsque les contrats avec plein engagement ne sont pas disponibles. C'est notamment le cas lorsqu'aucun tiers ne peut appliquer une sanction si une des parties manque à ses engagements. La solution retenue par GPG consiste à construire des contrats auto-exécutoires, lesquels sont décrits dans la section suivante.

1.3 Contrats auto-exécutoires

Implicitement, un contrat avec plein engagement est respecté par les parties parce qu'il prévoit des sanctions suffisamment importantes pour rendre toute déviation non attrayante. Si ces sanctions sont absentes, un agent peut trouver profitable de dévier des termes du contrat. Il est facile de prouver que, en information complète, plus les sanctions sont importantes, moins les agents seront tentés de dévier des termes du contrat et plus l'ensemble des allocations réalisables sera grand. Il en résulte qu'en choisissant un cadre contractuel, les agents ont intérêt à opter pour celui qui offre les sanctions les plus importantes en cas de défaut. En l'absence de telles sanctions, il est possible de montrer que la pire allocation qu'ait rationnellement à subir un agent consiste à devoir consommer en autarcie dans le futur. Dans le contexte présent, cette « sanction » peut être appliquée par n'importe quelle partie en refusant tout simplement d'échanger. Elle forme la base des contrats auto-exécutoires : les agents s'entendent pour cesser de coopérer (et consommer dès lors en autarcie) si l'un d'entre eux brise le contrat.

Afin d'accroître l'ensemble des allocations réalisables par contrats auto-exécutoires, GPG utilisent le paiement ex ante B_t pour modifier optimalement ex ante, la distribution des revenus ex post. Par voie de conséquence, les surplus courants d'utilité ex post, par rapport à l'autarcie, en sont modifiés. Afin de pouvoir considérer cette possibilité, je note ces surplus

$$\hat{S}_u(x_t, s_t) = u(c^{s_t}) - u(y_u^{s_t} + B_t)$$
$$\hat{S}_v(x_t, s_t) = v(y^s - c^{s_t}) - v(y_v^{s_t} - B_t).$$

Un contrat δ_{h_t} est auto-exécutoire pour m périodes s'il satisfait les contraintes auto-exécutoires ex ante et ex post suivantes :

$$U(\delta_{h_t}, h_{t+k}) \geq 0 \quad \forall k, 0 \leq k \leq m, \forall h_{t+k} \in H(h_t) \tag{1.4}$$

$$\hat{S}_u(\delta_{h_t}(h_{t+k}), s_{t+k}) + \beta U(\delta_{h_t}, h_{t+k}) \geq 0 \quad \forall k, 1 \leq k \leq m, \forall h_{t+k} \in H(h_t) \tag{1.5}$$

$$V(\delta_{h_t}, h_{t+k}) \geq 0 \quad \forall k, 0 \leq k \leq m, \forall h_{t+k} \in H(h_t) \tag{1.4'}$$

$$\hat{S}_v(\delta_{h_t}(h_{t+k}), s_{t+k}) + \beta V(\delta_{h_t}, h_{t+k}) \geq 0 \quad \forall k, 1 \leq k \leq m, \forall h_{t+k} \in H(h_t). \tag{1.5'}$$

(1.4) et (1.4′) sont les contraintes auto-exécutoires ex ante, en début de période pour les agents u et v. (1.5) et (1.5′) sont les contraintes ex post, une fois que le paiement ex ante a été versé et que l'état de la nature a été réalisé. Lorsqu'elles sont satisfaites, ces contraintes indiquent que chaque agent anticipe qu'en respectant le contrat, il obtiendra un gain d'utilité net par rapport à l'autarcie – soit ce qu'il obtiendrait en brisant le contrat. Il n'est pas dans l'intérêt d'aucun agent de briser aujourd'hui un contrat qui satisfait ces contraintes car il renoncerait du coup à un surplus d'utilité positif.

L'ensemble des contrats auto-exécutoires après h_t pour m périodes est noté $\overline{\Gamma}^m$. Comme l'environnement stochastique est i.i.d. et que les contraintes sont prospectives (*forward looking*), cet ensemble ne dépend pas de h_t. Il est aussi immédiat qu'un contrat auto-exécutoire pour $m+1$ périodes aujourd'hui l'est nécessairement pour m périodes demain de sorte que $\overline{\Gamma}^{m+1} \subseteq \overline{\Gamma}^m$. Cette relation d'inclusion entraîne que $\overline{\Gamma}^\mu$ peut être noté $\cap_{m=1}^{\mu}\overline{\Gamma}^m$ et $\overline{\Gamma} \equiv \cap_{m=1}^{\infty}\overline{\Gamma}^m$ correspond alors à l'ensemble des contrats dont l'implémentation peut être faite sans supposer de capacité d'engagement par les agents, sur aucun horizon. Je note $\delta_0(h) = x_0, \forall h$ le contrat trivial correspondant à l'autarcie, tel que $S_u(x_0, s) = S_v(x_0, s) = \hat{S}_u(x_0, s) = \hat{S}_v(x_0, s) = 0, \forall s \in S$. Il satisfait immédiatement les contraintes (1.4)-(1.5'). Ce contrat est donc auto-exécutoire et fait en sorte que $\overline{\Gamma}^m \neq \emptyset, \forall m$.

La proposition suivante établit une propriété de fermeture des contrats auto-exécutoires ; soit qu'on peut toujours remplacer n'importe quelle branche d'un contrat auto-exécutoire par un autre contrat auto-exécutoire qui le domine au sens de Pareto sans que le contrat cesse d'être auto-exécutoire.

Proposition 1.1. *Soit deux histoires successives $h_{t+1} \succ h_t$ et deux contrats δ_{h_t} et $\delta_{h_{t+1}}$ appartenant respectivement à $\overline{\Gamma}^{m+1}$ et $\overline{\Gamma}^m$, tels que $\delta_{h_t} \sim \delta_{h_{t+1}}$. S'il existe un $\delta'_{h_t+1} \in \overline{\Gamma}^m$ tel que $U(\delta'_{h_{t+1}}, h_{t+1}) \geq U(\delta_{h_{t+1}}, h_{t+1})$ et $V(\delta'_{h_{t+1}}, h_{t+1}) \geq V(\delta_{h_{t+1}}, h_{t+1})$, alors il existe un contrat $\delta'_{h_t} \in \overline{\Gamma}^{m+1}$, $\delta'_{h_t} \sim \delta'_{h_{t+1}}$, qui domine δ_{h_t} au sens de Pareto.*

Preuve. Tout contrat futur auto-exécutoire est trivialement auto-exécutoire dans le futur. S'il domine, au sens de Pareto, une branche d'un contrat auto-exécutoire courant, alors le contrat obtenu par substitution de cette branche est aussi supérieur au sens de Pareto. Cette substitution relâche donc les contraintes auto-exécutoires courantes en accroissant le surplus futur de chaque agent : le contrat construit par substitution est donc, lui aussi, auto-exécutoire. □

Notez que les fonctions $\hat{S}_u$ et $\hat{S}_v$, qui apparaissent dans les contraintes (1.5) et (1.5′), ne sont pas nécessairement quasi-concaves dans leurs arguments $(c^{s_{t+k}}, B_{t+k})$ (inclus dans $x_{t+k} = \delta_{h_t}(h_{t+k})$). L'ensemble des contrats auto-exécutoires pour m périodes, $\overline{\Gamma}^m$, n'est donc pas assurément convexe. Il est toutefois possible d'en garantir la convexité en l'étendant aux contrats formés en randomisant sur $\overline{\Gamma}^m$.[6] Introduire explicitement cette randomization alourdirait considérablement l'exposé qui suit. Je suppose donc, dans le reste de cet article, que $\overline{\Gamma}^m$ est convexe, fusse au prix de supposer que $\hat{S}_u$ et $\hat{S}_v$ sont quasi-concaves ou d'étendre cet ensemble aux contrats randomisés.

1.4 Représentation récursive des contrats auto-exécutoires optimaux

Dans GPG, les auteurs cherchent à caractériser les contrats auto-exécutoires optimaux en résolvant le problème suivant :

$$f(V) = \max_{\delta_{h_0} \in \overline{\Gamma}} U(\delta_{h_0}, h_0) \text{ s.c. } V(\delta_{h_0}, h_0) \geq \min\{V, \overline{V}\}, \qquad (1.6)$$

$$\text{où } V \geq 0 \text{ et } \overline{V} = \max_{\delta_{h_0} \in \overline{\Gamma}} V(\delta_{h_0}, h_0) \text{ s.c. } U(\delta_{h_0}, h_0) \geq 0. \qquad (1.7)$$

Il s'agit, de fait, d'identifier les points de la frontière de Pareto de second rang f réalisables à l'aide de contrats auto-exécutoires sur tout horizon. Ici, $V(\delta_{h_0}, h_0) \geq \min\{V, \overline{V}\}$ est la contrainte parétienne traditionnelle. Il est no-

6. Cf. Pearce and Stacchetti (1993) pour une discussion de cette technique. J'ai pris cette précaution en programmant mon algorithme mais elle s'est avérée, à l'usage, non nécessaire : le contrat optimal étant caractérisé par une randomisation dégénérée.

tamment démontré dans GPG que cette frontière satisfait le système d'équations fonctionnelles suivant :

$$\overline{V} = \max_{\delta\in\Delta} \mathrm{E}\{S_v(x,\tilde{s}) + \beta\tilde{V}\} \tag{1.8}$$

$$\text{s. c. } \mathrm{E}\{S_u(x,\tilde{s}) + \beta f(\tilde{V})\} \geq 0; \tag{1.9}$$

$$f(V) = \max_{\delta\in\Delta} \mathrm{E}\{S_u(x,\tilde{s}) + \beta f(\tilde{V})\} \tag{1.10}$$

$$\text{s. c. } \mathrm{E}\{S_v(x,\tilde{s}) + \beta\tilde{V}\} \geq \min\{V,\overline{V}\}; \tag{1.11}$$

où Δ est défini par

$$\Delta = \{\delta = [x, V^1, V^2, \ldots V^{\bar{s}}] | x \in X, \forall s \in S : V^s \in [0,\overline{V}],$$
$$\hat{S}_u(x,s) + \beta f(V^s) \geq 0, \hat{S}_v(x,s) + \beta V^s \geq 0\} \tag{1.12}$$

et $\tilde{V}$ est une variable aléatoire discrète de support $\{V^1, \ldots, V^{\bar{s}}\}$ et de distribution donnée par les π^s.

Mon objectif est d'obtenir un algorithme permettant de résoudre ce système d'équations fonctionnelles et d'en établir la convergence vers le couple $(\overline{V}, f)$ défini par (1.6) et (1.7). Le système (1.8)-(1.12) connaît toujours une solution triviale $(\overline{V}, f) = (0,0)$ qui peut être atteinte grâce au contrat trivial δ_0. Il s'agit de s'assurer que lorsque $\overline{V}$, tel que donné par (1.6), est strictement positif, mon algorithme converge vers cette solution non triviale.

Dans les pages qui suivent, j'établis les points suivants :

1. Il existe une suite $\{(\overline{V}^m, f^m)\}$ connue et qui converge simplement vers une solution au système (1.8)-(1.12). Cette suite accepte une forme récursive.

2. Il existe une famille de contrats auto-exécutoires permettant d'obtenir les surplus associés à toute solution au système (1.8)-(1.12).

3. La suite $\{(\overline{V}^m, f^m)\}$ converge vers $(\overline{V}, f)$, soit la solution aux équations (1.6) et (1.7).

4. La suite $\{f^m\}$ converge uniformément sur $[0, \overline{V}]$.

5. La suite $\{\delta^m\}$ de contrats associés à la suite $\{(\overline{V}^m, f^m)\}$ converge vers le contrat limite $[0, \overline{V}]$.

Considérez d'abord la suite de $\{(\overline{V}^m, f^m)\}_{m=0,1,\ldots}$ définie ainsi :

$$\overline{V}^m = \max_{\delta_{h_0} \in \Gamma^m} V(\delta_{h_0}, h_0) \text{ s.c. } U(\delta_{h_0}, h_0) \geq 0 \tag{1.13}$$

$$\text{et } f^m(V) = \max_{\delta_{h_0} \in \Gamma^m} U(\delta_{h_0}, h_0) \text{ s.c. } V(\delta_{h_0}, h_0) \geq \min\{V, \overline{V}^m\}. \tag{1.14}$$

Chaque f^m est une fonction et les $\overline{V}^m$ sont des scalaires. Les fonctions f et f^m diffèrent en ce sens que la frontière f^m inclut des points accessibles qu'en recourrant à des contrats qui ne sont auto-exécutoires que pour les m premières périodes. Pour $m = 0$, $(\overline{V}^0, f^0)$ n'est autre que la solution de premier rang donnée à l'équation (1.3) pour un surplus de réserve positif $V \geq 0$ de l'agent v. Les contraintes (1.4)-(1.5′) garantissent que chaque couple $(\overline{V}^m, f^m)$ existe et que la suite est bien définie : sachant que $\overline{\Gamma}^m$ n'est jamais vide, il existe au moins un contrat pour lequel (1.4′) est satisfaite de sorte que $\overline{V}^m \geq 0$; (1.14) étant dual à (1.13) pour $V \geq \overline{V}^m$, une solution pour (1.14) existe en ces points. V restreignant l'espace des solutions, il s'en suit alors que celles-ci existent aussi pour $0 \leq V \leq \overline{V}^m$.

La frontière f^m donne un maximum bien défini sur $V \geq 0$ et est strictement concave sur $[0, \overline{V}^m]$ car la contrainte $V(\delta_{h_0}, h_0) \geq V$ est nécessairement serrante sur cet intervalle – autrement on pourrait réduire la consommation de l'agent v à la période suivante dans au moins un état s et accroître strictement $U(\delta_{h_0}, h_0)$. Il s'en suit que, à mesure que V décroît sur cet intervalle, la valeur de f^m s'accroît strictement de sorte que l'allocation de la consommation pour deux points distincts, V_1 et V_2 appartenant à $[0, \overline{V}^m]$, ne peut être identique. Ainsi, toute combinaison convexe de ces deux allocations réalisera une valeur de f^m plus élevée que la combinaison convexe de $f^m(V_1)$ et $f^m(V_2)$. La proposition suivante établit que cette suite possède une structure récursive.

Proposition 1.2. *La suite $\{\overline{V}^m, f^m\}$ obéit aux équations récursives suivantes :*

$$\overline{V}^{m+1} = \max_{\delta \in \Delta^m} \mathrm{E}\{S_v(x, \tilde{s}) + \beta \tilde{V}\} \tag{1.15}$$

$$s.\ c.\ \mathrm{E}\{S_u(x, \tilde{s}) + \beta f^m(\tilde{V})\} \geq 0; \tag{1.16}$$

$$f^{m+1}(V) = \max_{\delta \in \Delta^m} \mathrm{E}\{S_u(x, \tilde{s}) + \beta f^m(\tilde{V})\} \tag{1.17}$$

$$s.\ c.\ \mathrm{E}\{S_v(x, \tilde{s}) + \beta \tilde{V}\} \geq \min\{V, \overline{V}^{m+1}\}; \tag{1.18}$$

où Δ^m est défini, pour $m \geq 0$, par

$$\Delta^m = \{\delta = [x, V^1, V^2, \dots V^{\overline{s}}] | x \in X, \forall s \in S : V^s \in [0, \overline{V}^m], \\ \hat{S}_u(x, s) + \beta f^m(V^s) \geq 0, \hat{S}_v(x, s) + \beta V^s \geq 0\} \tag{1.19}$$

où $\tilde{V}$ est une variable aléatoire discrète de support $\{V^1, \dots, V^{\overline{s}}\}$ et de distribution donnée par les π^s.

Preuve. La preuve consiste essentiellement à montrer que, à l'instar de la proposition 1.1, toutes les suites d'un contrat auto-exécutoire pour $m+1$ périodes sont optimales parmi les contrats auto-exécutoires pour m périodes.

L'équation (1.2) implique que tout contrat peut être décomposé en un vecteur de K instruments courants x et un ensemble de $\overline{s}$ contrats futurs associés $\{\delta_{\{h,s\}}\}_S$. Notez $\tilde{\delta}_{h_1}$ un contrat aléatoire, de distribution donnée par les π_s, sur cet ensemble. Étant donné que si un contrat est auto-exécutoire pour $m+1$ périodes, toutes ses suites $\delta_{\{h,s\}}$ le sont pour les m dernières périodes, on a

$$\overline{V}^{m+1} = \max_{\substack{x \in X \\ \delta_{\{h,s\}} \in \overline{\Gamma}^m \\ \forall s \in S}} \mathrm{E}\{S_v(x, \tilde{s}) + \beta V(\tilde{\delta}_{h_1}, \tilde{h}_1)\}$$

$$\text{s. c. } \mathrm{E}\{S_u(x, \tilde{s}) + \beta U(\tilde{\delta}_{h_1}, \tilde{h}_1)\} \geq 0.$$

$$\text{où} \quad \mathrm{E}\{V(\tilde{\delta}_{h_1}, \tilde{h}_1)\} = \sum_S \pi_s V(\delta_{\{h,s\}}, \{h, s\}).$$

Je réfère plus simplement à une histoire $\{h, s\}$, pour un état s quelconque, par h_1. Les $\overline{s}$ contrats δ_{h_1} sont auto-exécutoires (pour m périodes) ; il s'en suit que tous les surplus $V(\delta_{h_1}, h_1)$ et $U(\delta_{h_1}, h_1)$ sont positifs. On a donc que $V(\delta_{h_1}, h_1) \leq \overline{V}^m$ puisque $\overline{V}^m$ est un maximum et domine $V(\delta_{h_1}, h_1) \leq \overline{V}^m$ sur $\overline{\Gamma}^m$ avec $U(\delta_{h_1}, h_1) \geq 0$. On substitue ainsi chaque $V(\delta_{h_1}, h_1)$, où $h_1 = \{h, s\}$, par $V^s \in [0, \overline{V}^m]$.

Soit un contrat optimal δ_{h_0} incluant des δ_{h_1} auxquels on a associé des V^s. On doit maintenant établir que $U(\delta_{h_1}, h_1) = f^m(V^s)$. Par définition de $f^m(V^s)$, on doit avoir $U(\delta_{h_1}, h_1) \leq f^m(V^s)$. Supposez qu'il y ait inégalité ; alors il existe un autre contrat $\delta'_{h_1} \in \overline{\Gamma}^m$ subséquent à s qui domine strictement δ_{h_1}. Par la proposition 1.3, on peut remplacer toute branche d'un contrat auto-exécutoire par un contrat auto-exécutoire dominant. δ'_{h_1} peut donc remplacer δ_{h_1} dans δ_h. Le contrat ainsi modifié domine donc strictement le contrat δ_{h_0} qui ne peut alors être optimal. Cette écriture incorpore les contraintes auto-exécutoires futures. Il suffit donc de vérifier le respect des contraintes auto-exécutoires courantes présentes dans la définition de Δ^m.

Le même argument peut s'appliquer à l'équation (1.14), pour tout V donné. C'est-à-dire qu'on peut résumer un contrat auto-exécutoire optimal pour $m+1$ périodes par le choix d'un vecteur d'instruments courants x et d'un vecteur de niveaux d'utilité $U^s \in [0, f^m(0)]$ sur la frontière de Pareto f^m de l'équation (1.17), tels que les contraintes auto-exécutoires courantes soient satisfaites. Le contrat étant, par définition, auto-exécutoire en s pour l'agent v, on peut toujours effectuer le changement de variable $V^s = \min_{V \geq 0} [f^m]^{-1}(U^s)$ et obtenir un $V^s \geq 0$. La restriction $V^s \leq \overline{V}^m$ est sans conséquence puisque si $V > \overline{V}^m$, $f^m(V) = f^m(\overline{V}^m)$ et le maximum n'est pas affecté. □

La proposition suivante établit, en quelque sorte, l'inverse du résultat précédent, soit qu'on peut toujours reconstruire un contrat à partir de sa forme récursive.

Proposition 1.3. *Considérez une solution $(\overline{V}, f)$ au système* (1.8)-(1.12) *; alors pour tout $V \in [0, \overline{V}^0]$, il existe un contrat $\delta \in \overline{\Gamma}$ tel que $f(V) = U(\delta, h_0)$ et* $\min\{V, \overline{V}\} = V(\delta, h_0)$.

Preuve. Soit $x(V) = [B, c^1(V), \ldots, c^s(V)]$ et $[V^1(V), \ldots, v^s(V)]$ les instruments optimaux de (1.10) en V. Il suffit de reconstruire δ de la manière suivante :

$$\begin{aligned}
&\delta(h_0) = x(V);\\
&\delta(h_1) = x(V^{s_1}(V)) \quad \forall h_1 \in H(h_0);\\
&\delta(h_2) = x(V^{s_2}(V^{s_1}(V))) \quad \forall h_2 \in H(h_0);\\
&\qquad \ldots\\
&\delta(h_n) = x(V^{s_n}(V^{s_{n-1}}(\cdots V^{s_2}(V^{s_1}(V))\cdots))) \quad \forall h_n \in H(h_0);\\
&\qquad \ldots \text{ etc.}
\end{aligned}$$

Il est immédiat que ce contrat appartient à $\overline{\Gamma}$ et satisfait les égalités énoncées dans la proposition. □

Supposez maintenant qu'on établisse la convergence de la suite $\{\overline{V}^m, f^m\}$ vers un couple $(\overline{V}^\infty, f^\infty)$. Par la proposition précédente, cette limite serait une solution au système (1.8)-(1.12). Dans la prochaine proposition, je m'emploie à démontrer la convergence simple de la suite $\{\overline{V}^m, f^m\}$.

Proposition 1.4. *La suite $\{\overline{V}^m, f^m\}$ converge simplement.*

Preuve. $\overline{V}^{m+1}$ étant la solution d'un programme plus contraint que celui de $\overline{V}^m$, il s'en suit immédiatement que $0 \leq \overline{V}^{m+1} \leq \overline{V}^m$. Toute suite monotone décroissante minorée converge, il s'en suit que $\{\overline{V}^m\}$ converge.

Pour tout $V \geq 0$, on a

$$\begin{aligned} \min\{V, \overline{V}^m\} &= V \quad \text{si } V < \overline{V}^m, \\ \text{ou } \min\{V, \overline{V}^m\} &= \overline{V}^m \quad \text{si } V \geq \overline{V}^m. \end{aligned}$$

Considérez la suite $\{f^m(V)\}$ pour tout $V > \overline{V}$. Puisque la suite monotone $\{\overline{V}^m\}$ converge, il existe un entier $M_V \geq 0$ tel que $\overline{V}^m - \overline{V} < V - \overline{V}$, pour tout entier m tel que $m \geq M_V$. On a donc $V > \overline{V}^m$ et $\min\{V, \overline{V}^m\} = \overline{V}^m$, $\forall m \geq M_V$. La suite $\{f^m(V)\}$ se confond donc avec la suite $\{f^m(\overline{V}^m)\}$, laquelle est duale à la suite $\{\overline{V}^m\}$ qui converge (il s'agit d'une suite de zéros). Si, en revanche, $V \leq \overline{V}$, alors on a nécessairement $0 \leq f^{m+1}(V) \leq f^m(V)$, puisque le problème f^{m+1} inclut plus de contraintes que le problème f^m. Cette suite étant minorée et monotone décroissante, elle converge.[7] On en conclut que la suite $\{f^m\}$ converge simplement. □

Soit $(\overline{V}^\infty, f^\infty)$ la limite de la suite $\{\overline{V}^m, f^m\}$. Il s'agit maintenant d'établir que cette solution au système (1.8)-(1.12) est bien celle donnée par (1.6) et (1.7).

Proposition 1.5. *La limite* $(\overline{V}^\infty, f^\infty)$ *correspond au couple* $(\overline{V}, f)$ *donné par* (1.6) *et* (1.7).

Proof. Sans perte de généralité, nous pouvons étendre les domaines de f et des f^m à $[0, \overline{V}^0]$. De plus, il suffit de prouver que $\{f^m\}$ converge vers f pour tout V pour établir que $\{\overline{V}^m\}$ converge vers $\overline{V}$.

1. Supposez que $f^\infty(V) > f(V)$ pour un $V \in [0, \overline{V}^0]$. La proposition 1.3 nous dit alors qu'il existe un contrat $\delta^\infty \in \overline{\Gamma}$ qui procure $f^\infty(V) > f(V)$

7. Les preuves sur la convergence de suites monotones minorées sont classiques. Cf., par exemple, Michel (1989).

à l'agent u et V à l'agent v ce qui est impossible puisque $f(V)$ est censé représenter la frontière de Pareto. On en conclut que $f^{\infty}(V) \leq f(V)$, $\forall V$ et $\overline{V}^{\infty} \leq \overline{V}$.

2. Supposez que $f^{\infty}(V) < f(V)$ pour un certain V. Puisque $f^0(V) \geq f(V), \forall V$, il existe un nombre $m \geq 0$ tel que $f^m(V) \geq f(V), \forall V$ et $f(V) > f^{m+1}(V)$ pour certaines valeurs $\hat{V}$ de V. Substituez alors $(\overline{V}^m, f^m)$ par $(\overline{V}, f)$ dans le système (1.15)-(1.19). Puisque $(\overline{V}, f)$ solutionne (1.8)-(1.12), pour toute valeur de V, on obtient $(\overline{V}, f(V))$ dans les membres de gauche. Mais puisque $(\overline{V}, f)$ donnent des valeurs inférieures à $(\overline{V}^m, f^m)$, le programme de droite est davantage contraint de sorte que $\overline{V} \leq \overline{V}^{m+1}$ et $f(V) \leq f^{m+1}(V), \forall V$; ce qui contredit l'existence des $\hat{V}$.

□

La convergence uniforme de $\{f^m\}$ peut être facilement établie sur $[0, \overline{V}]$ parce que tous les f^m sont strictement concaves sur cet ensemble.

Proposition 1.6. $\{f^m(V)\}$ *converge uniformément sur le compact* $[0, \overline{V}]$.

Preuve. La suite $\{f^m\}$ converge simplement vers f (proposition 1.4) sur le compact $[0, \overline{V}^0]$. De plus, chaque f^m est strictement concave sur $[0, \overline{V}] \supseteq [0, \overline{V}^m]$. Si l'on restreint le domaine à $[0, \overline{V}]$, on obtient une suite de fonctions strictement concaves. Le théorème 10.8 de Rockafellar (1970) commande alors que $\{f^m\}$ converge uniformément sur $[0, \overline{V}]$. □

Considérez d'abord l'argmax de (1.17). S'il existe plusieurs valeurs de B qui solutionnent (1.17), je retiens toujours la plus petite. Cette restriction, sans conséquence à l'égard de la maximisation, me permet de définir la suite $\{\delta^m\}$ de contrats solutions à (1.17) ; soit une suite $\{\delta^m\}$ bien définie de vecteurs de fonctions de V. La dernière partie de cette section cherche à établir la convergence uniforme de cette suite de contrats optimaux associés à la suite $(\overline{V}^m, f^m)$ vers le contrat δ solutionnant (1.10).

Proposition 1.7. *Supposez que les Γ^m soient strictement convexes. Alors, la suite $\{\delta^m\}$ converge sur $[0, \overline{V}]$ vers δ.*

Preuve. Considérez la suite de correspondances $\{\Gamma^m\}$ où Γ^m, définie sur $[0, \overline{V}^0]$, détermine l'ensemble des vecteurs δ appartenant à Δ^m qui satisfont (1.18). Le contrat permettant de générer $\overline{V}^m$ appartient à Γ^m qui est donc non vide. Sous l'hypothèse que l'ensemble des instruments est strictement convexe, Γ^m donne des ensembles strictement convexes et compacts puisqu'elle est décrite par des inégalités faibles et des fonctions continues. En usant des résultats présentés dans Lucas, Stokey et Prescott (1989), on peut établir que Γ^m est continue sur $[0, \overline{V}^0]$. On a, comme précédemment $\Gamma^{m+1}(V) \subseteq \Gamma^m(V)$ et $\cap_{m=1}^{\infty}\Gamma^m(V)$ est simplement noté $\Gamma(V)$.

Soit A^0 le graphe de Γ^0. Le maximand de (1.17) est défini sur A^0 et est noté $\mathcal{U}^m$. Ainsi, pour $d = [x, V^1, V^2, \ldots, V^s]$,

$$\mathcal{U}^m(d) = \mathrm{E}\{u(x, \tilde{s}) + \beta f^m(\tilde{V})\}.$$

Soit $\delta(V)$ la solution du problème limite en $(\overline{V}, f)$ et $\{\hat{\delta}^m\}$ une suite de fonctions

$$\hat{\delta}^m(V) = \underset{d \in \Gamma(V)}{\operatorname{argmax}}\ \mathcal{U}^m(d).$$

Cette fonction existe parce que $\mathcal{U}^m$ est strictement concave et que $\Gamma(V)$ est convexe et non vide.

Puisque[8] $\delta(V)$ maximise $\mathcal{U}$ sur $\Gamma(V)$ et $\hat{\delta}^m(V)$ maximise $\mathcal{U}^m$ sur $\Gamma(V)$,

8. Les lignes qui suivent, pour démontrer la convergence de $\hat{\delta}^m$ vers δ, sont adaptées de la preuve du théorème 3.8 présenté dans Lucas, Stokey, and Prescott (1989) à la page 64.

on a, pour $V \in [0, \overline{V}]$,

$$0 \leq \mathcal{U}(\delta(V)) - \mathcal{U}(\hat{\delta}^m(V))$$
$$0 \leq \mathcal{U}^m(\hat{\delta}^m(V)) - \mathcal{U}^m(\delta(V)).$$

L'addition de ces inégalités résulte en

$$0 \leq \mathcal{U}(\delta(V)) - \mathcal{U}(\hat{\delta}^m(V)) \\ \leq [\mathcal{U}(\delta(V)) - \mathcal{U}^m(\delta(V))] + [\mathcal{U}^m(\hat{\delta}^m(V)) - \mathcal{U}(\hat{\delta}^m(V))]. \quad (1.20)$$

Pour chacun des deux termes entre crochets à droite on a

$$||\mathcal{U} - \mathcal{U}^m|| \equiv \sup_{V' \in [0,\overline{V}]} |\mathcal{U}(d(V')) - \mathcal{U}^m(d(V'))| \\ \geq |\mathcal{U}(d(V)) - \mathcal{U}^m(d(V))| \quad \forall V \in [0, \overline{V}].$$

On a de suite

$$0 \leq \mathcal{U}(\delta(V)) - \mathcal{U}(\hat{\delta}^m(V)) \leq 2||\mathcal{U}^m - \mathcal{U}|| \quad \forall V \in [0, \overline{V}].$$

Puisque $\{f^m\}$ converge uniformément vers f, $\{\mathcal{U}^m\}$ en fait autant. Il s'en suit que pour tout $\gamma > 0$, il existe un entier $M_\gamma \geq 1$ tel que

$$0 \leq \mathcal{U}(\delta(V)) - \mathcal{U}(\hat{\delta}^m(V)) \leq 2||\mathcal{U}^m - \mathcal{U}|| < \gamma, \quad \forall V \in [0, \overline{V}], \forall m \geq M_\gamma. \quad (1.21)$$

étant donné que $[0, \overline{V}]$ est compact, il suffit, pour montrer que $\{\hat{\delta}^m\}$ converge uniformément vers δ, d'établir que

$$\forall \epsilon > 0, \exists M \text{ tel que } ||\delta(V) - \hat{\delta}^m(V)|| < \epsilon \quad \forall V \in [0, \overline{V}], \forall m \geq M. \quad (1.22)$$

étant donnée les propriétés énoncées de Γ, on peut appliquer le lemme 3.7

présenté dans Lucas, Stokey, and Prescott (1989) qui démontre qu'on peut établir (1.22) en prouvant que pour tout $\gamma > 0$, il existe un entier N, indépendant de V, tel que

$$|\mathcal{U}(\delta(V)) - \mathcal{U}(\hat{\delta}^m(V))| < \gamma \quad \forall m \geq N. \tag{1.23}$$

L'équation (1.21) indique alors que tout entier $N \geq M_\gamma$ satisfait (1.23).

Ayant établi la convergence de $\{\hat{\delta}^m\}$ vers δ, je cherche maintenant à faire de même pour $\{\delta^m\}$. Seule la convergence simple est démontrée,[9] pour une valeur arbitraire de $V \in [0, \overline{V}]$ donnée. Afin d'alléger la notation dans ce qui suit, $\Gamma(V)$ est simplement noté Γ et $\delta(V)$ est noté δ.

Soit $\text{Int}(\Gamma)$ et $\partial\Gamma$ l'intérieur et la frontière de Γ (pour un V donné) et $A(d, d') = \{d_\alpha | d_\alpha = \alpha d + (1-\alpha)d', \alpha \in [0,1])\}$ le segment dont d et d' sont les extrémités. Je définis la suite $\{e^m\}$ telle que

$$e^m = \sup_{\substack{d,d' \in \Gamma^m \setminus \text{Int}(\Gamma) \\ A(d,d') \cap \Gamma \subseteq \partial\Gamma \cup \emptyset}} ||d - d'||.$$

Tout couple (d, d') tel que $d = d' \in \partial\Gamma$ satisfait les contraintes et fait en sorte que e^m existe pour tout m. La valeur e^{m+1} étant maximisée sur un ensemble plus restreint que celui de e^m, il s'en suit que $0 \leq e^{m+1} \leq e^m$ et la suite converge. Puisque les d et d' doivent ultimement être choisis sur la frontière $\partial\Gamma = \Gamma \setminus \text{Int}(\Gamma)$ et puisque cette frontière est strictement concave (Γ étant strictement convexe), les couples satisfaisant $A(d, d') \cap \Gamma \subseteq \partial\Gamma \cup \emptyset$ doivent ultimement être identiques : $d = d'$. On en conclut que e^m converge vers zéro.

Considérez maintenant la relation entre $\hat{\delta}^m$ et δ^m pour toute valeur $V \in [0, \overline{V}]$ donnée : ces contrats appartiennent respectivement à $\partial\Gamma$ et $\partial\Gamma^m$ puisque la contrainte parétienne est toujours serrante ; ainsi $\hat{\delta}^m(V), \delta^m(V) \in \Gamma^m \setminus$

9. La démonstration de la convergence uniforme de δ^m fera l'objet de futures recherches.

$\text{Int}(\Gamma)$. Je démontre maintenant que la relation

$$A(\hat{\delta}^m, \delta^m) \cap \Gamma \subseteq \partial\Gamma \cup \emptyset \tag{1.24}$$

est toujours vérifiée. Puisque $\mathcal{U}^m$ est strictement concave et que les Γ^m sont convexes, $\mathcal{U}^m$ atteint son maximum en un point unique sur Γ^m et Γ (possiblement le même). Si $\mathcal{U}^m(\delta^m) = \mathcal{U}^m(\hat{\delta}^m)$, alors $\delta^m = \hat{\delta}^m \in \partial\Gamma$ et (1.24) est vérifiée. Si $\mathcal{U}^m(\delta^m) > \mathcal{U}^m(\hat{\delta}^m)$ et (1.24) n'est pas vérifiée, alors il existe un $\delta^m_\alpha \in A(\hat{\delta}^m, \delta^m) \cap \Gamma$ tel que $\delta^m_\alpha \notin \partial\Gamma$. Puisque $\mathcal{U}^m$ est strictement concave, $\mathcal{U}^m(\delta^m_\alpha) \geq \mathcal{U}^m(\hat{\delta}^m)$ avec égalité si et seulement si $\delta^m_\alpha = \hat{\delta}^m$. Puisque $\mathcal{U}^m$ est maximisée en $\hat{\delta}^m$ sur Γ alors que δ^m_α est réalisable on doit donc avoir $\delta^m_\alpha = \hat{\delta}^m$ mais $\hat{\delta}^m \in \partial\Gamma$ donc δ^m_α n'existe pas.

Par construction, on a alors

$$0 \leq ||\hat{\delta}^m - \delta^m|| \leq e^m.$$

Puisque $e^m \to 0$, δ^m converge simplement vers la même limite que $\hat{\delta}^m$, soit δ. □

1.4.1 Autres fonctions initiales

Afin d'obtenir f, nous avons calculé la limite de la suite $\{f^m\}$, définie à partir de la fonction initiale de premier rang f^0, mais il est clair que nous aurions pu également calculer la limite d'une autre suite convergeant vers la même solution f. Souvent, de telles suites existent. Ainsi, considérez deux types de mécanismes contractuels en relation de dominance. Si f_A et f_B sont les frontières de Pareto correspondantes à ces mécanismes, on aura $f_B \geq f_A$. Les démonstrations précédentes peuvent être reprises en substituant f^0 par f_B. Étant donné que la distance $\rho(f^0, f_A) \geq \rho(f_B, f_A)$, il est plausible qu'on obtienne plus rapidement convergence en calculant f_A à partir de f_B

plutôt qu'à partir de f^0.[10] C'est exactement l'approche que j'ai retenue pour comparer les mérites relatif du paiement ex ante dans le modèle de GPG. J'y calcule d'abord la frontière de Pareto f_B du modèle avec paiement ex ante, à partir de f^0. Puis, je calcule la frontière f_A du modèle où ce paiement est contraint à zéro, en prenant f_B comme fonction initiale. Les gains en temps de calcul se sont avérés considérables.

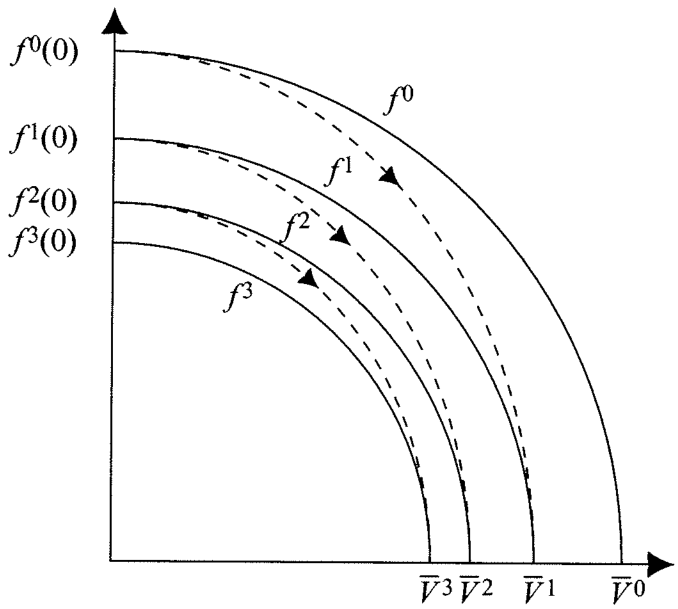

FIG. 1.1: *Convergence de la suite* $\{\overline{V}^m, f^m\}$.

La mécanique de convergence de la suite $\{\overline{V}^m, f^m\}$, telle que formalisée dans les programmes (1.17) et (1.15), est illustrée intuitivement à la figure 1.1 à partir de la frontière de premier rang f^0. Le point $\overline{V}^0$ est déterminé sur l'abcisse au zéro de f^0. On calcule ensuite $\overline{V}^1$ à partir de $\overline{V}^0$ et f^0 ; puis on calcule f^1 définie sur $[0, \overline{V}^1]$; puis $\overline{V}^2$ à partir de $\overline{V}^1$ et f^1 ; puis

10. Cette approche séquentielle n'est indiquée que si l'on désire à la fois calculer f_A et f_B. Rien ne laisse penser, en effet, qu'il soit plus rapide de calculer f_A en deux étapes plutôt que de procéder directement à partir de f^0 ; de fait, l'inégalité du triangle $\rho(f^0, f_A) \leq \rho(f^0, f_B) + \rho(f_B, f_A)$ suggère plutôt le contraire.

f^2 définie sur $[0, \overline{V}^2]$ et ainsi de suite jusqu'à convergence. Dans la section suivante, j'illustre comment j'ai effectué ces calculs, en utilisant notamment les fonctions splines.

1.5 Calcul numérique de la solution

Le couple $(\overline{V}, f)$ est calculé en appliquant de manière itérative l'opérateur $T : R \times \mathcal{C} \to R \times \mathcal{C}$, où $\mathcal{C}$ est un espace de fonctions auquel appartient f, donné par (1.17) et (1.15) ; *i.e.*

$$(\overline{V}, f) = \lim_{m \to \infty} T^m(\overline{V}^0, f^0).$$

Une solution $(\hat{V}, \hat{f}) = (\overline{V}^{m+1}, f^{m+1})$ est obtenue lorsque

$$\lambda |\hat{V} - \overline{V}^m| + \sup_{V \in [0, \overline{V}^{m+1}]} |\hat{f}(V) - f^m(V)| < \epsilon,$$

où λ et ϵ sont des réels positifs quelconques (ϵ est devrait être « petit »). Pour ce faire, il me faut calculer $(\overline{V}^0, f^0)$, soit la restriction au quadrant positif de la frontière de Pareto de premier rang pour laquelle aucune autre contrainte auto-exécutoire n'est imposée. La technique suivante s'est avérée tout à fait satisfaisante.

On peut représenter les contrats de premier rang par des vecteurs δ de dimension $\overline{s} + K$ comme précedemment. La solution de premier rang étant stationnaire (cf. GPG), f^0 satisfait,

$$\begin{aligned}
(1 - \beta) f^0(V) &= \mathrm{E}\{S_u(x(V), \tilde{s})\} \\
(1 - \beta) V &= \mathrm{E}\{S_v(x(V), \tilde{s})\} \\
\mathrm{E}\{S_u(x(\overline{V}^0), \tilde{s})\} &= 0 \\
\mathrm{E}\{S_v(x(0), \tilde{s})\} &= 0,
\end{aligned}$$

où, $\forall s \in S$, on a l'égalité des taux marginaux de substitution,

$$\frac{S'_u(x^s(V), s)}{S'_v(x^s(V), s)} = -\frac{\partial f^0}{\partial V}(V).$$

En posant $V = \overline{V}^0$, on obtient le système suivant de $S + 2$ équations avec $S + 2$ inconnues, soit les $x(\overline{V}^0)$, $\overline{V}^0$ et le gradient $\partial f^0 / \partial V(\overline{V}^0)$.

$$\begin{aligned} (1-\beta)\overline{V}^0 &= \mathrm{E}\{S_v(x(\overline{V}^0), \tilde{s})\} \\ \mathrm{E}\{S_u(x(\overline{V}^0), \tilde{s})\} &= 0 \\ \frac{S'_u(x^s(\overline{V}^0), s)}{S'_v(x^s(\overline{V}^0), s)} &= -\frac{\partial f^0}{\partial V}(\overline{V}^0) \quad \forall s \in S. \end{aligned}$$

Après résolution numérique, on obtient ainsi $\overline{V}^0$.

Je définis ensuite $k = \tan^{-1}(-\partial f^0 / \partial V(V))$. La fonction $-\partial f^0 / \partial V$ étant positive, on a $k \in [0, \pi]$. Pour chaque $s \in S$, je cherche alors la fonction $z^s : [0, \pi] \to R$ qui satisfait

$$\frac{S'_u(z^s(k), s)}{S'_v(z^s(k), s)} = \tan(k). \tag{1.25}$$

Le domaine de k étant borné, cette équation peut aisément être résolue par simple collocation.[11] z étant une fonction continue et différentiable, sous l'hypothèse usuelle que u et v soient deux fois différentiables, j'ai choisi de la représenter par un spline cubique Hermitien à L sections.[12] Une telle fonction comporte $2(L + 1)$ paramètres à identifier, correspondant aux images et aux dérivées de la fonction aux $L + 1$ points (*breaks*) délimitant les sections.

11. Cf. Judd (1992). Dans bien des cas, il existe une solution analytique à cette équation. Par exemple, si $u(c^s(k)) = -\exp(-\sigma_u c^s(k))/\sigma_u$ et $v(c^s(k)) = -\exp(-\sigma_v(y^s - c^s(k)))/\sigma_v$, on obtient immédiatement

$$c^s(k) = \frac{\sigma_v y^s - \log k}{\sigma_u + \sigma_v}.$$

12. Cf. de Boor (1992).

En différentiant (1.25), j'obtient

$$\frac{S'_u(z^s(k),s)}{S'_v(z^s(k),s)}\left(\frac{S''_u(z^s(k),s)}{S'_u(z^s(k),s)}-\frac{S''_v(z^s(k),s)}{S'_v(z^s(k),s)}\right)\frac{\partial z}{\partial k}(k)=1. \tag{1.26}$$

En choisissant $L+1$ points sur $[0,\pi]$, j'obtiens $2(L+1)$ équations. Il me suffit alors de résoudre ce système en $\{z^s(k^l), \partial z^s/\partial k(k^l)\}_{l=0\ldots L}$ pour obtenir une approximation très satisfaisante des $z^s(k)$. Comme valeurs de départ, j'ai retenu les coefficients correspondants à l'approximation linéaire des z^s obtenue en résolvant (1.25) en $k=\pi/2$.

Une fois les $z^s(k)$ calculés, j'obtiens directement

$$F(k) = \mathrm{E}\{u(z^s(k))\}/(1-\beta), \tag{1.27}$$

$$V(k) = \mathrm{E}\{v(z^s(k))\}/(1-\beta). \tag{1.28}$$

Je détermine alors k_0 et k_1 tels que $V(k_0)=0$ et $V(k_1)=\overline{V}^0$. Par un simple changement de variable, j'obtiens enfin

$$f^0(V) = F(k_0+(k_1-k_0)V/\overline{V}^0).$$

Une fois que j'ai obtenu f^0 et $\overline{V}^0$, il me suffit d'appliquer la procédure décrite en début de section. La section suivante aborde certains points de détails concernant ce calcul.

1.6 Détails du calcul de f

1.6.1 Calcul aux bords

Le calcul de $f^m(V)$ peut être difficile lorsque V est proche de $\overline{V}^m$. L'essentiel de la maximisation consiste alors à trouver un ensemble d'instruments qui satisfassent les contraintes. Hors de l'ensemble des instruments admissibles, les algorithmes de maximisation que j'ai employés minimisent la violation des contraintes, tentant ainsi d'approcher l'ensemble des points admissibles. Malheureusement, cette procédure a pour effet de retenir des instruments non optimaux en $V = \overline{V}^m$. En calculant d'abord $f^m(V')$ pour un V' intérieur à $[0, \overline{V}^m]$, où les contraintes sont moins susceptibles d'être serrantes, et en utilisant en retour les instruments optimaux en $V' < V$ comme point de départ pour calculer $f^m(V)$, on minimise ce problème puisque tout contrat δ^m auto-exécutoire en V' satisfait au moins $\delta^m \in \Gamma^m$ en V.

Une autre façon de minimiser cette distorsion consiste à exploiter la dualité du problème. En tout temps et pour tout V, on peut substituer le problème (1.17) par son problème dual en $U = f^{m+1}(V)$.

$$[f^{m+1}]^{-1}(U) = \max_{\delta \in \Delta^m} \mathrm{E}\{S_v(x, \tilde{s}) + \beta \tilde{V}\}$$
$$\text{s. c. } \mathrm{E}\{S_u(x, \tilde{s}) + \beta f^m(\tilde{V})\} \geq \min\{U, [f^{m+1}]^{-1}(0)\}.$$

De fait, j'ai appliqué systématiquement cette méthode en programmant l'algorithme : à chaque itération, la « moitié » gauche des points (déterminée par la droite à quarante-cinq degrés) a été calculée en résolvant le problème primal et l'autre moitié en calculant le problème dual.

1.6.2 Interpolation de f^m

Lorsqu'on peut établir la différentiabilité[13] des f^m, la technique d'interpolation évoquée précédemment s'est avérée très efficace pour obtenir une approximation de f^m à chaque itération. Si l'on détermine une suite de grilles $\{\Theta^m\}$ sur $[0, \overline{V}^m]$ et que l'on calcule $f^m(V), \forall V \in \Theta^m$, les conditions de Khun-Tucker permettent non seulement d'identifier l'image de f^m en Θ^m mais également sa dérivée en ces points. En effet, par la condition d'enve-

13. La formulation du problème en forme ex ante permet, dans certaines circonstances, de réduire le nombre de variables d'états nécessaires à la représentation du système et c'est pour cela qu'elle fut ici adoptée. Il convient toutefois de noter que dans ce contexte, la frontière de Pareto peut ne pas être différentiable sur certains points intérieurs de son domaine. Un tel exemple est présenté à l'appendice 1.B.

loppe, $\partial f^{m+1}/\partial V(V) = -\lambda$ où λ est le multiplicateur associé à la première contrainte de (1.17).[14]

Une fois $\hat{V}$ et $\hat{f}$ calculés, il est facile d'en dériver la famille de contrats optimaux $\delta(V) = [x(V), V^1(V), V^2(V), \ldots, V^{\bar{s}}(V)]$ associée. Ces fonctions ont été estimées par simple interpolation linéaire à partir des solutions en Θ^m calculées à la dernière itération. Cette approche est conséquente avec le fait que les fonctions $\{x(V), \tilde{V}(V)\}$ sont généralement simplement $\mathcal{C}^0$ et peuvent inclure des points de non différentiabilité (Cf. l'appendice 1.B).

1.6.3 Zone de premier rang

De manière générale, la principale caractéristique de la solution de premier rang est qu'elle garde constante, sur tout horizon et en toute circonstance, la dérivée de la frontière de Pareto. Dans le cas i.i.d., cette caractéristique implique que V demeure constant. Considérez l'ensemble Θ_0 des points $V \geq 0$ qui satisfont les contraintes (1.5) et (1.4) avec f^0. Ces points, s'il en existe, sont tels que $f(V) = f^0(V)$. Le temps de calcul nécessaire pour obtenir f est donc considérablement réduit si nous restreignons notre évaluation aux $V \in [0, \overline{V^m}] \setminus \Theta_0$.

1.7 Conclusion

Dans cet article, j'ai exposé un algorithme permettant de résoudre une famille de modèles d'allocations parétiennes de second rang soumises à des contraintes auto-exécutoires. La convergence uniforme de la frontière de Pareto et des contrats optimaux a été établie.

La structure des contrats auto-exécutoires optimaux ayant une forme récursive, ceux-ci peuvent être approximés par itération. Cependant, du fait que

14. Lorsque l'application inverse $[f^m]^{-1}(U)$ est calculée, il suffit de prendre l'inverse du multiplicateur correspondant.

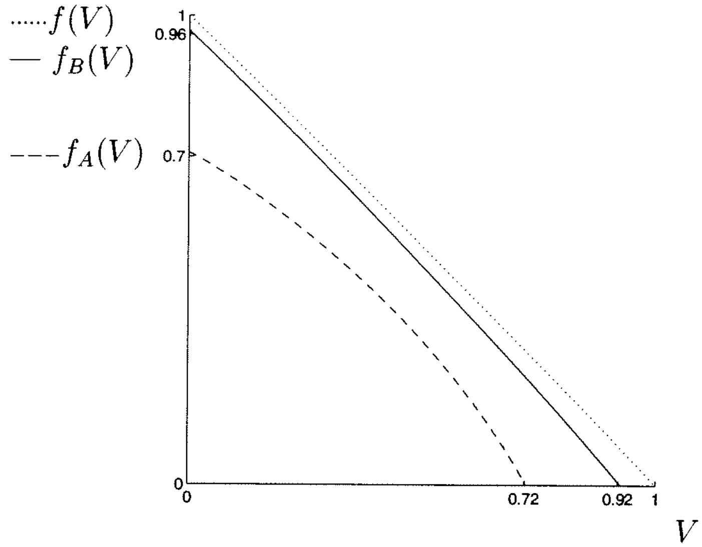

FIG. 1.2: *Frontières de Pareto pour M1.*

la frontière de Pareto inconnue apparaisse dans l'ensemble des contraintes définissant la correspondance de transition, il convient de faire converger tant la frontière que la correspondance de transition pour parvenir à la solution.

J'ai programmé l'algorithme en construisant à chaque itération une approximation de la frontière de Pareto par un spline cubique Hermitien. Ce type d'approximation est particulièrement bien adaptée à ce type de problème parce qu'elle utilise l'information sur le gradient de la frontière, laquelle est un résidu naturel des routines de maximisation.

L'étude des contrats auto-exécutoires est loin d'être complétée. Le modèle de GPG (de même que celui de Thomas and Worrall (1988)) n'a jusqu'ici été considéré qu'un dans un cadre i.i.d. La généralisation au cas markovien reste à faire. Par ailleurs, il serait intéressant d'adapter l'algorithme présenté ici aux modèles incluant, en sus de V, d'autres variables d'état comme le capital.

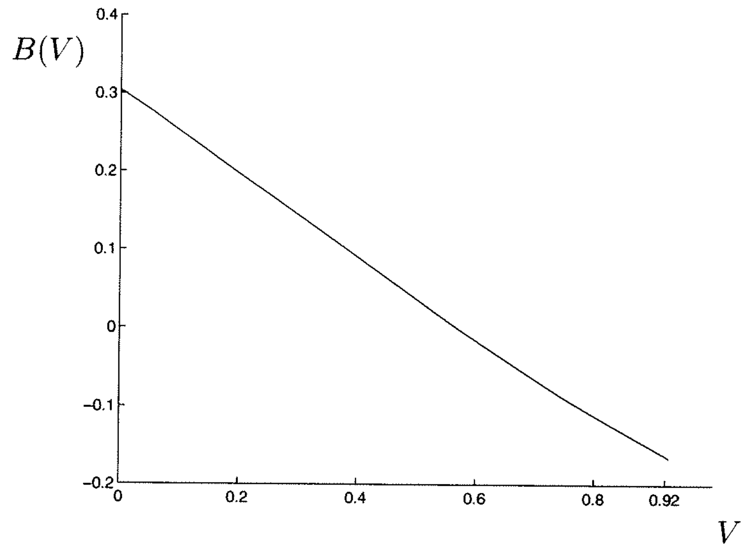

FIG. 1.3: *Paiement ex ante B pour M1.*

1.A Résultats des simulations

Dans cette section, je présente les résultats de deux simulations[15] du modèle de GPG. L'intérêt premier de ces simulations est de comparer les performance relatives des contrats avec plein engagement, sans engagement et sans engagement avec paiement ex ante. Les frontières de Pareto associées à ces deux dernier cas sont notées respectivement f_A et f_B. Les résultats sont discutés directement à partir des graphiques obtenus après calcul. Les deux modèles M1 et M2 dont je présente ici la solution sont les suivants :

M1 L'agent u a des préférences *CARA* $u(c) = -\exp(-\sigma c)/\sigma$ avec $\sigma = 4/3$. Cette valeur fait en sorte que l'aversion relative au risque lorsque l'agent u consomme sa dotation moyenne est la même dans les modèles M1 et M2. L'agent v a des préférences linéaires : dans ce cas, seules les dotations de u importent ; elles sont de $y_u^s = 1 + (s-1)/10$, pour

15. J'ai effectué des simulations pour plus d'une centaine de configurations différentes du modèle, sans problème particulier.

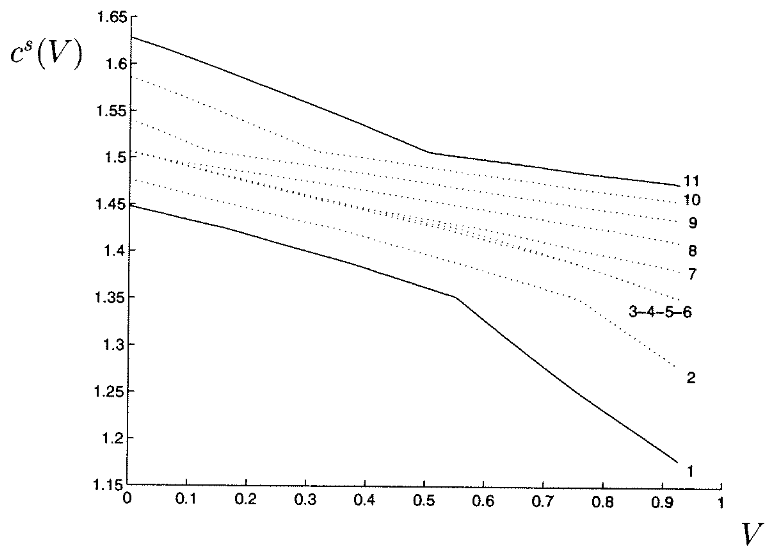

FIG. 1.4: *M1 avec B. Consommation.*

$s = 1, 2, \ldots, 11$. Chaque état est équiprobable avec $\pi^s = 1/11$. Le facteur d'escompte β commun est 0.85.

M2 Le modèle est symétrique : u et v ont des préférences identiques *CRRA* $u(c) = (1 - c^{1-\gamma})/(1-\gamma)$ avec facteur d'aversion relative pour le risque $\gamma = 2$. Les dotations de u sont identiques à celles du modèle précédent. Celles de v sont parfaitement négativement corrélées, *i.e.* que $y_v^s = 2 - (s-1)/10$. Il s'en suit que les dotations agrégées sont constantes avec $y_u^s + y_v^s = 3$. L'optimum de premier rang (pour une division égalitaire du surplus) commande évidemment un partage égal de cette dotation agrégée. β est fixé à 0.75.

Considérez la figure 1.2 (figure 1.11 pour le modèle 2). Les V sont en abscisse. Les fonctions d'utilité ont été normalisées de sorte que $f^0(0) = \overline{V}^0 = 1$ pour la frontière de premier rang (en pointillé). Elle domine naturellement les deux autres : $f^0 > f_B > f_A$. Si l'on accroissait le facteur d'escompte β, les deux courbes f_B et f_A (respectivement en traits plein et pointillé) se rapprocheraient éventuellement de la frontière de premier rang (il s'agit là

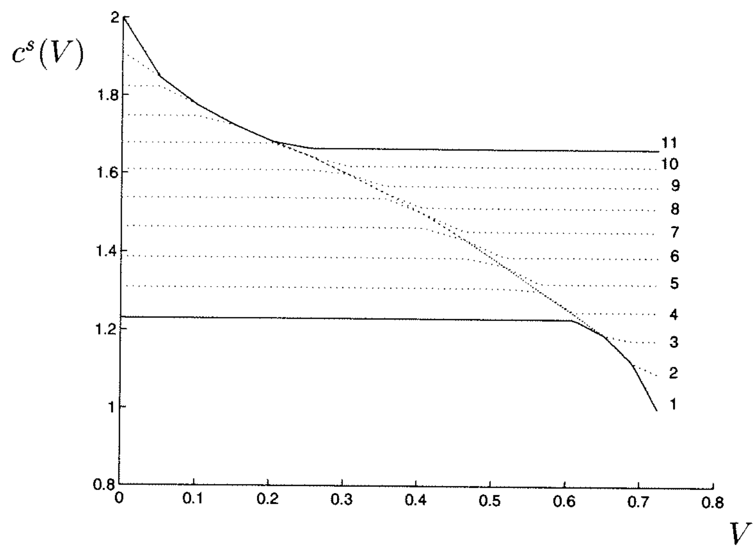

FIG. 1.5: *M1 sans B. Consommation.*

d'une interprétation relative puisque, de fait, la frontière de premier rang est également affectée par un changement de β). C'est là une conséquence directe du *folk theorem.* Pour un β assez élevé, des portions de ces frontières se confonderaient avec la frontière de premier rang. On peut montrer que ces portions définissent de fait l'ensemble ergodique de la variable V.

Notez comment l'emploi d'un paiement ex ante accroît l'espace des utilités par rapport à la frontière de second rang sans paiement f_A. Le paiement ex ante permet de générer plus de 90% du surplus de premier rang alors que la restriction aux paiements ex post n'en autorise, au plus, que 70%. Les zéros des frontières sur l'abscisse correspondent aux $\overline{V}$ de chaque frontière.

La dépendance du paiement ex ante par rapport à l'état de la relation est représentée à la figure 1.3 (figure 1.12 pour M2). $B(V)$ apparaît comme une fonction pratiquement linéaire en V. Il en résulte que B est extrêmement corrélé avec V comme en témoigne le graphique 1.7.

La figure 1.4 (figure 1.13 pour M2) représente la consommation de l'agent u en fonction de l'état s (courbe) et de V (abscisse) lorsqu'un paiement ex ante est utilisé de manière optimale. Les fonctions sont définies sur $[0, \overline{V}]$,

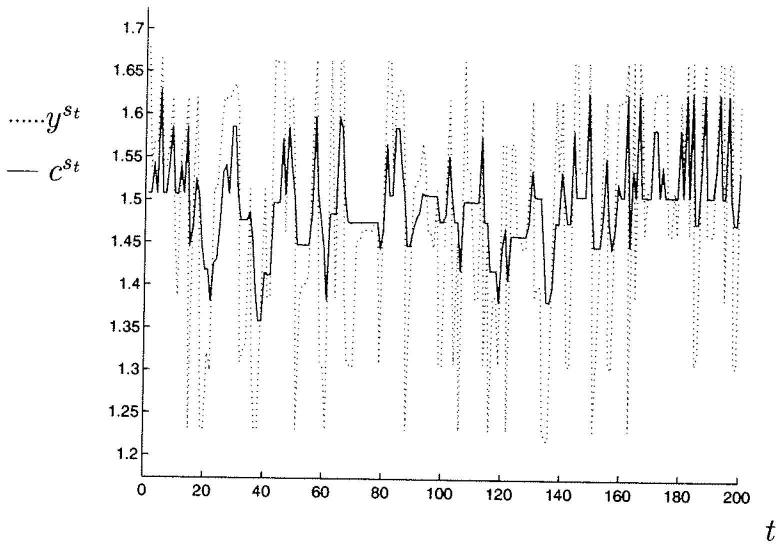

FIG. 1.6: *M1. Évolution de la consommation.*

$\overline{V} \leq 1$. Si elles étaient représentées dans le graphique, les courbes de consommation en autarcie apparaîtraient comme des droites horizontales (indépendantes de V) de niveau correspondant aux dotations y_u^s. Dans le modèle 1, où l'agent v a des préférences linéaires, la consommation de premier rang (non représentée) serait pour sa part résumée par un ensemble de courbes c^s *confondues*, décroissantes en V : ceci correspond à la solution de pleine assurance (la consommation ne dépend pas de s). Le modèle de second rang offre une solution mitoyenne : les courbes ne sont pas confondues mais elles sont tout de même moins dispersées qu'en autarcie. Notez qu'une assurance partielle dans les états $s \in \{3, 4, 5, 6\}$ est offerte.

En l'absence de paiement ex ante, à la figure 1.5 (figure 1.14 pour M2), la consommation a une structure très particulière qui a été caractérisée par Thomas et Worrall (1988). Chaque fonction $c^s(V)$ est incluse dans un intervalle fixe $[\underline{c}^s, \overline{c}^s]$. Ainsi[16], $c^6(V) \in [1,4\ ,\ 1,6]$. De plus, le surplus V demeure constant si l'état est stationnaire, *i.e.* $s_{t+1} = s_t$ (cf. la figure 1.10) et, avec

16. Les nombres rapportés sont des approximations visuelles obtenues à partir du graphique.

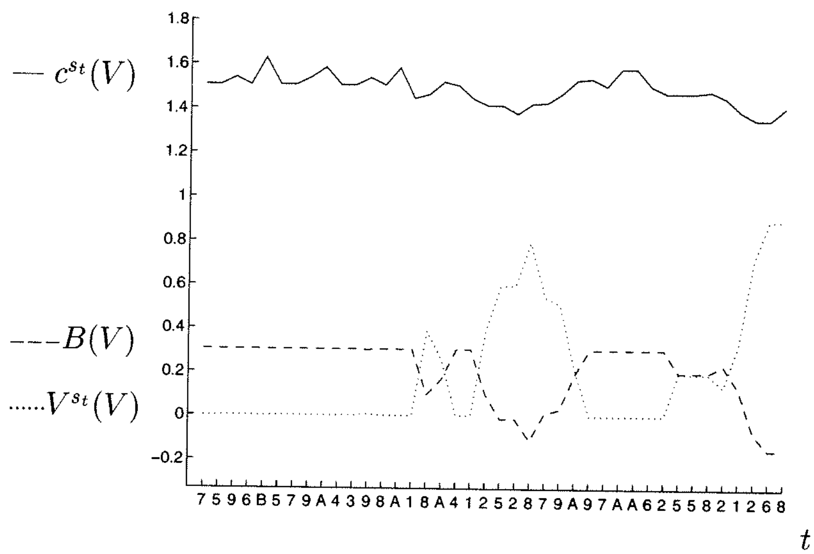

FIG. 1.7: *M1. 40 premières périodes de la simulation avec B.*

lui, le niveau de la consommation.

La figure 1.6 montre l'évolution de la consommation pour une suite aléatoire de 200 chocs i.i.d. Toutes les simulations sont faites en posant $V_0 = 0$. La consommation dans le modèle avec paiement ex ante est en trait plein et celle sans paiement ex ante est en trait pointillé. L'agent v ayant des préférences linéaires, la consommation de premier rang est ici constante et égale à l'espérance des dotations de u, soit $c = 1,5$. Notez que les deux consommations évoluent autour de cette moyenne mais que celle avec paiement ex ante est moins variable. La structure markovienne des contrats est apparente, particulièrement avec B, dans la persistance des processus. Notez comment l'emploi du paiement ex ante B permet de limiter les fluctuations.

Les figures 1.7 et 1.8 donnent le détail de ces simulations pour les 40 premières périodes. Les périodes sont en abscisse et la valeur de l'état (de 1 à 11) est écrite en hexadécimal où A = 10, B = 11.[17] La ligne en trait plein est la consommation ; la ligne en pointillée est la variable d'état V —

17. Une erreur lors de la programmation des graphique a fait en sorte que ces chiffres et lettres sont décalés d'une case vers la gauche.

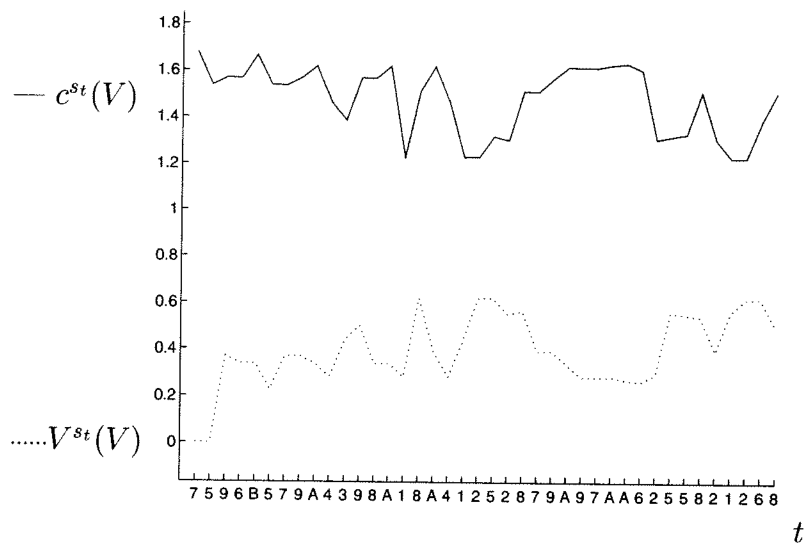

FIG. 1.8: *M1. 40 premières périodes de la simulation sans B.*

notez qu'elle est toujours positive. $B(V)$ est en tirets. Cette variable étant presque linéaire en V (cf. la figure 1.3), elle est extrêmement corrélée avec V. De fait, B sert de tampon et fait en sorte que la consommation peut être gardée relativement constante malgré la présence de contraintes auto-exécutoires. Il s'en suit que c est peu corrélée avec V, soit l'histoire de la relation. Comparez avec la figure 1.8 générée à l'aide de la même suite aléatoire de chocs mais en restreignant l'emploi de B. La ligne en trait plein y représente la consommation ; la ligne pointillée est la variable d'état V, toujours positive. En l'absence de paiement ex ante, la consommation est fortement corrélée avec V.

La simulation pour M2 est représentée dans les figures 1.15, 1.16 et 1.17. Remarquez comment la division initiale du surplus ($V_0 = 0$) n'a plus d'importance après quelques périodes, *i.e.* que la variable d'état V entre rapidement dans l'ensemble ergodique du système. Dans la figure 1.16, la ligne en trait plein est la consommation ; la ligne en pointillée est la variable d'état V et $B(V)$ est en tirets. Notez comment la séquence de chocs relativement mauvaise en milieu d'échantillon entraîne un accroissement progressif de V sans

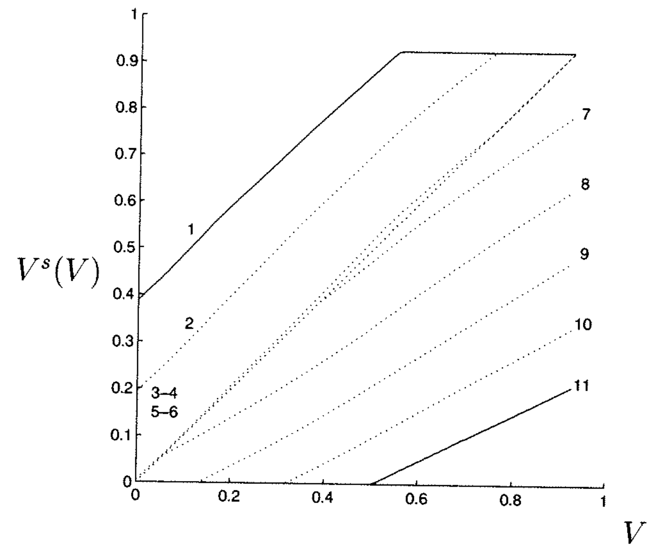

FIG. 1.9: *M1 avec B. Dynamique de V.*

que cela n'affecte trop le lissage de la consommation. La même séquence entraîne beaucoup plus de variations dans la consommation si les agents ne peuvent recourir au paiement ex ante.

Dans les figures 1.9 et 1.10, chaque courbe représente une des fonctions $V^s(V)$ du contrat optimal, soit l'espérance d'utilité future accordée par le contrat à l'agent v si le contrat lui a promis V par le passé et que l'état s se produise. Dans le modèle de premier rang, avec des chocs i.i.d., V demeure constant, *i.e.* que les courbes V^s correspondantes seraient toutes confondues avec la droite à 45 degrés. Le contrat de second rang essaie de limiter les fluctuations de V : c'est ici possible dans les états $s \in \{3, 4, 5, 6\}$.

Les courbes de la figure 1.18 représentent les $f(V^s(V))$. La courbe implicite définie par les intersections n'est autre que la frontière de Pareto de second rang sans B de la figure 1.11. En effet, supposez que deux courbes V^s et $V^{s'}$ se confondent avec la droite à 45 degrés sur un intervalle $\mathcal{V}$. Il s'en suit que pour $V \in \mathcal{V}$, $V^s(V) = V^{s'}(V) = V$; et donc que $f(V^s(V)) = f(V^{s'}(V)) = f(V)$.

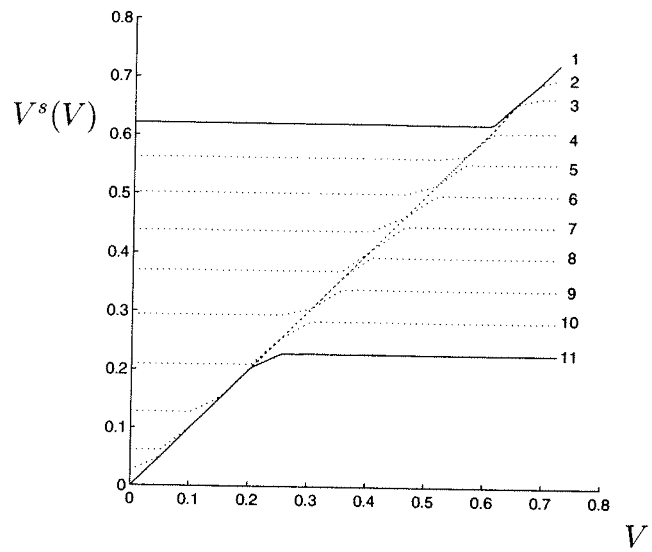

FIG. 1.10: *M1 sans B. Dynamique de V.*

Les figures 1.19 et 1.20 ont été conçues pour accompagner les résultats de la simulation présentée dans GPG. La figure 1.19 illustre les intervalles (indicés par s) dans lesquels évolue la consommation. La gradation de la variabilité de la consommation est apparente entre l'autarcie (en points noirs), et la solution de premier rang (FC), en passant par les modèles de Thomas et Worrall ($B = 0$) et celui de GPG (B) qui permet un paiement ex ante. Dans la figure 1.20, le résultat d'une simulation de la consommation, sous différents régimes contractuels et pour une même série de dotations stochastiques, est superposé sur celui de la consommation autarcique (en gris).

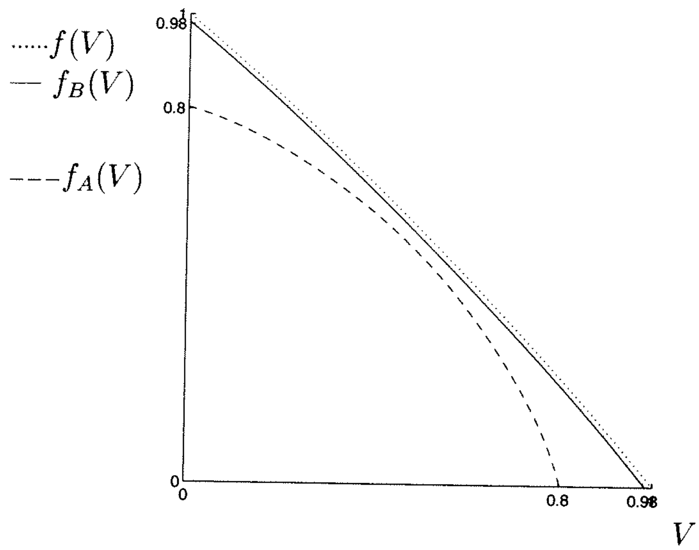

FIG. 1.11: *Frontières de Pareto pour M2.*

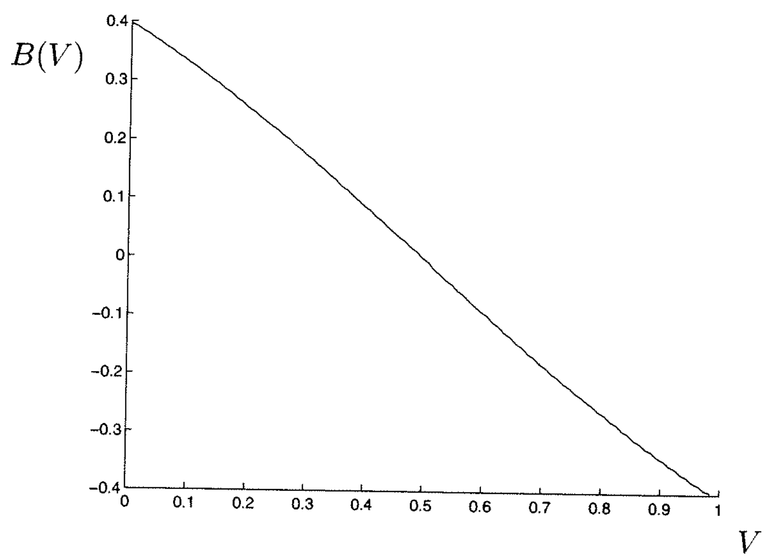

FIG. 1.12: *M2 avec B. Paiement ex ante.*

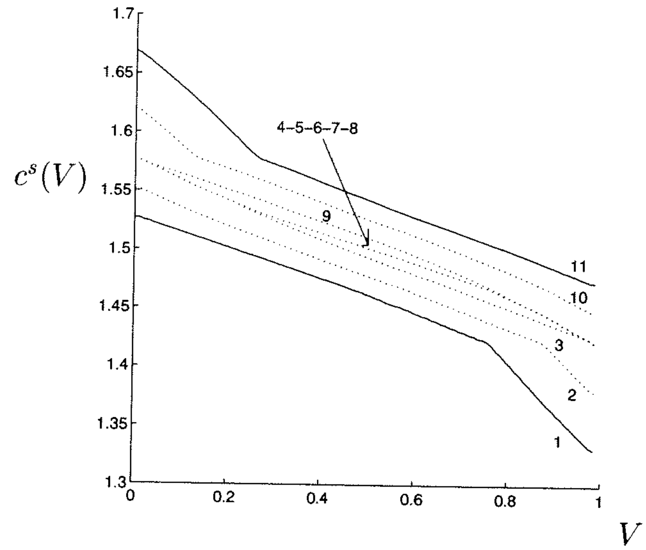

FIG. 1.13: *M2 avec B. Consommation.*

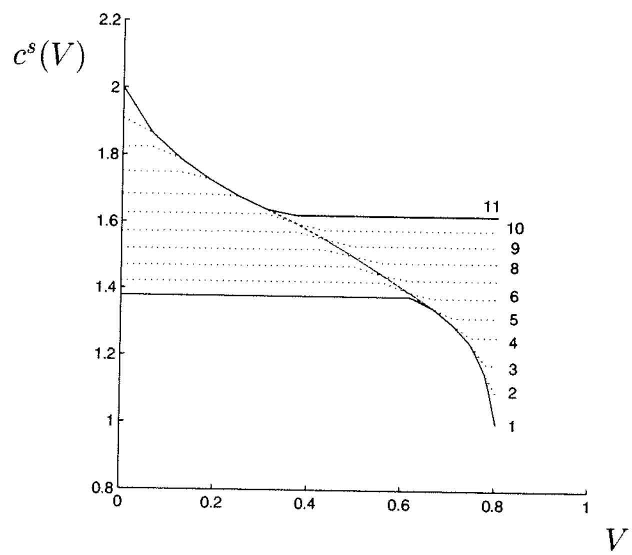

FIG. 1.14: *M2 sans B. Consommation.*

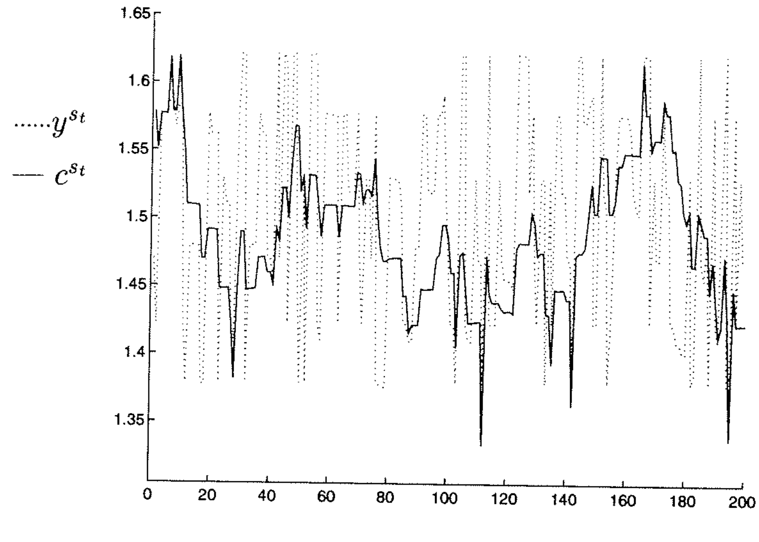

FIG. 1.15: *M2. Évolution de la consommation.*

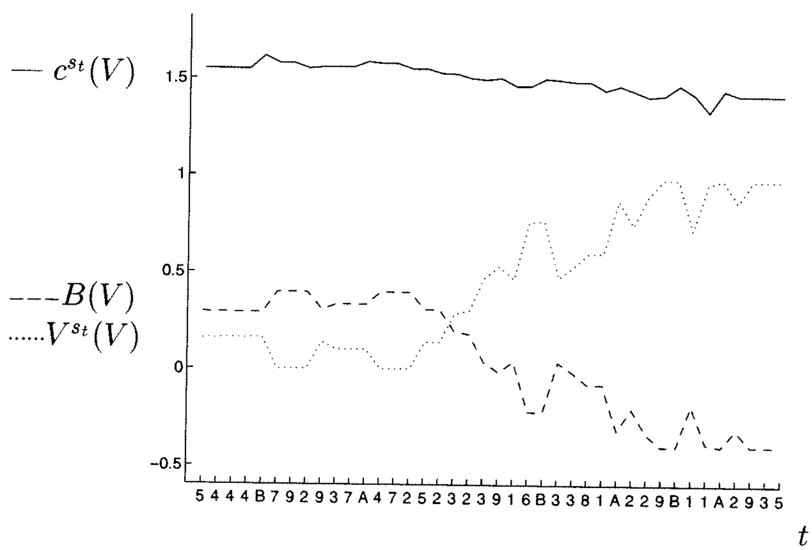

FIG. 1.16: *M2. 40 dernières périodes de la simulation avec B.*

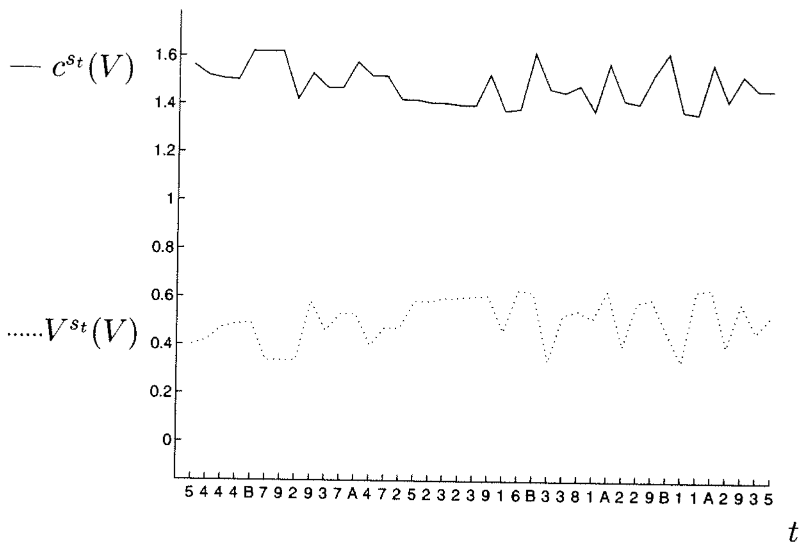

FIG. 1.17: *M2. 40 dernières périodes de la simulation sans B.*

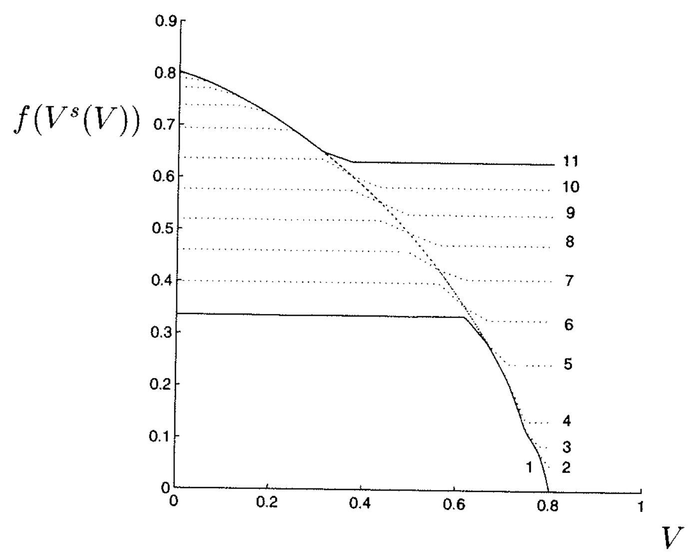

FIG. 1.18: *M2 avec B.*

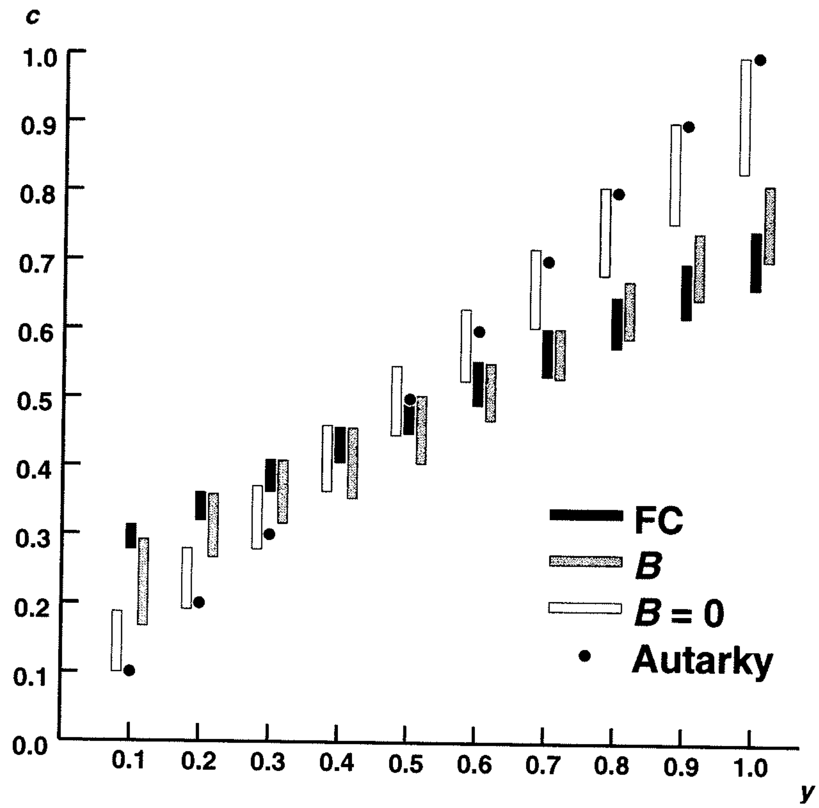

FIG. 1.19: *Intervalles de consommation.*

1.B Différentiabilité de la fonction de valeur

Dans le cadre de ma recherche avec GPG, il m'est vite apparu que la différentiabilité de la frontière de Pareto ne pouvait être prise pour acquise. Dans cette note, je traite succinctement de ce problème et je présente un exemple simple où la frontière possède un coude à l'intérieur de son domaine.

La différentiabilité de la frontière de Pareto peut poser problème avec l'approche ex ante retenue dans GPG. Thomas et Worrall (1988) (TW) posent le problème de Pareto avec contraintes auto-exécutoires de manière ex post, une fois que l'état s initial est révélé. GPG posent le problème ex ante, avant que l'état initial ne soit connu. Les contrats qui résultent de ces deux approches sont identiques mais les frontières de Pareto désignent des objets différents

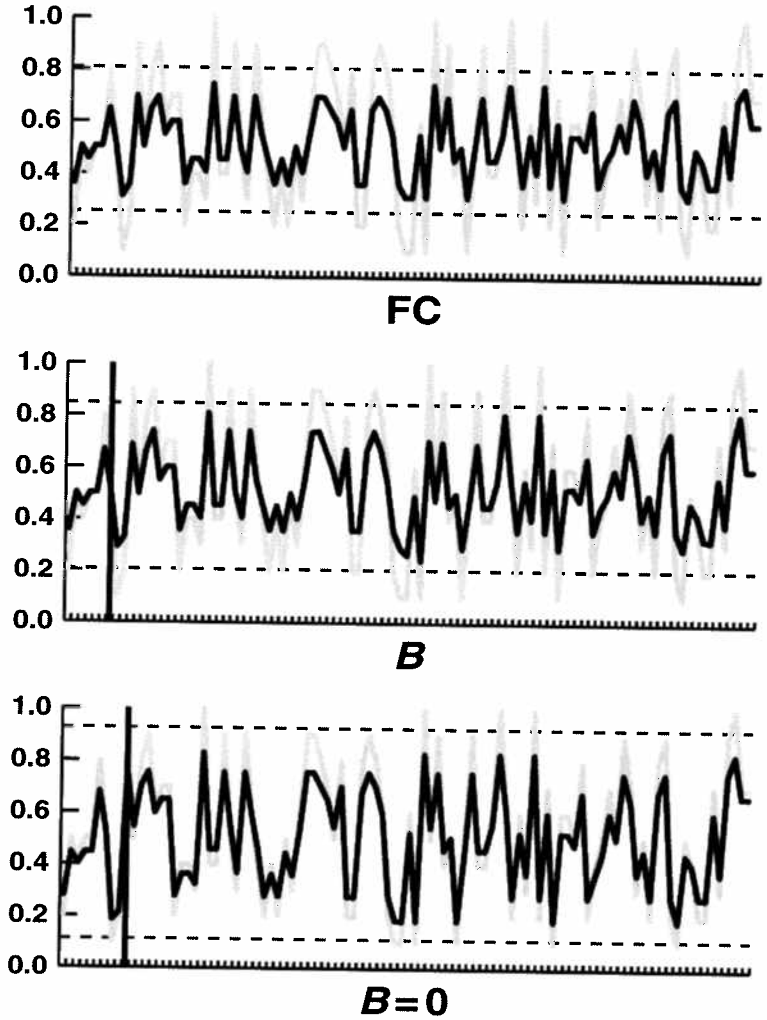

FIG. 1.20: *Simulation.*

dans les deux modèles : on peut concevoir la frontière de GPG comme l'espérance des $\overline{s}$ frontières de TW. Dans leur traitement du problème, TW ont suggéré d'établir la différentiabilité de f de la manière suivante :

1. prendre un contrat optimal auto-exécutoire δ ;

2. en dériver un contrat non optimal associé δ_γ, identique à δ sauf dans l'éventualité d'une histoire h_{t-1}^s pour lequel le transfert ex post courant est légèrement modifié d'une quantité γ ;

3. construire une pseudo-frontière à l'aide de δ_γ, fonction de γ, laquelle co ıncide avec la vraie frontière de Pareto en $\gamma = 0$;

4. recourir au lemme de Benveniste-Scheinkman[18] pour établir la différentiabilité de la vraie frontière en faisant varier γ dans un voisinage ouvert de zéro.

De manière générale, cette preuve requiert que le contrat δ_γ soit lui aussi auto-exécutoire. Pour que cela soit possible, il faut que, sous δ, et après h_{t-1}^s, l'agent u soit non contraint. Dans le modèle de TW, ceci est toujours possible sur le segment ouvert[19] $(0, \overline{U}_s)$ et la preuve de différentiabilité (à droite) est donc établie sur cet ouvert. Il est clair que la fonction ne peut pas être différentiable aux extrémités du domaine puisque celui-ci est un compact.

Le lemme de Benveniste-Scheinkman nécessite de construire une fonction différentiable, inférieure à la fonction maximum, dans un voisinage du point où l'on cherche à établir la différentiabilité. Dans leur preuve, TW n'ont considéré qu'un voisinage à droite : telle que présentée dans leur article, leur exposition n'établit que la différentiabilité à droite sur $(0, \overline{U}_s)$. Il est possible de raffiner leur argument pour étendre ce voisinage : parallèlement, on peut prouver la différentiabilité à droite de la fonction inverse, laquelle correspond à l'inverse de la différentielle à gauche de la fonction. Pour ce faire, il faut

18. Benveviste et Scheinkman (1979).
19. La notation $\overline{U}_s$ retenue ici est celle de TW.

donc que l'autre agent soit également non contraint. Cette opération étant menée *en un point où les contraintes auto-exécutoires sont non serrantes*, les deux dérivées correspondront et l'existence de la dérivée sera confirmée. Mais cette exigence est déjà vérifiée car si $U \in (0, \overline{U}_s)$, alors $V \in (f_s(0), f_s(\overline{U}_s))$.

Dans la formulation ex ante du modèle de GPG, il s'agit également d'établir la différentiabilité pour des points intérieurs à $[0, \overline{V}]$. Cependant, si le facteur d'escompte est suffisamment faible, la condition requérant que les deux agents soient non contraints peut ne pas être remplie pour certain points appartenant à l'intérieur de $(0, \overline{V})$. Dans ce cas, il est alors impossible de faire varier le paiement ex post d'une quantité γ dans un voisinage ouvert autour de zéro. Il s'en suit que, bien qu'il soit possible d'établir la différentiabilité à gauche et à droite en ces points, ces dérivées ne concordent pas et la fonction affiche de fait un coude à cet endroit.

La différence entre les frontières de TW et GPG tient au fait que tous les points de non différentiabilité sont reportées aux extrémités des $\overline{s}$ frontières de Pareto chez TW alors que la frontière de GPG constitue une sorte d'agrégat sophistiqué de ces S fonctions – ceci a pour conséquence de reporter certains points de non différentiabilité à l'intérieur du domaine $[0, \overline{V}]$ comme l'illustre l'exemple suivant.

Considérez la solution dynamique pour la consommation déterminée par TW et GPG :

$$c_{t+1}^s = \begin{cases} \underline{c}^s & \text{si } c_t < \underline{c}^s \\ c_t & \text{si } \underline{c}^s \leq c_t < \overline{c}^s \\ \overline{c}^s & \text{si } c_t \geq \overline{c}^s \end{cases}$$

c'est-à-dire que la consommation évolue à l'intérieur d'intervalles fixes, en demeurant le plus possible stationnaire. Considérez le cas où l'agent v a des préférences linéaires, avec deux états de la nature $y_1 < y_2$ équiprobables, et un facteur d'escompte β suffisamment faible de sorte que la solution pour c

soit caractérisée par des intervalles *disjoints.* À long terme, la consommation évolue entre c_1 et c_2 avec probabilité 1/2. Ici, $c_1 = \overline{c}^1$ et $c_2 = \underline{c}^2$. Si $c_{t-1} = c_1$ ou $c_{t-1} = c_2$, l'utilité attendue de ce contrat δ pour l'agent u est donc

$$\begin{aligned} U &= (1/2)\sum_{i=0}^{\infty} \beta^i (u(c_1) + u(c_2) - \overline{u}) \\ &= \frac{1}{2(1-\beta)}(u(c_1) + u(c_2) - \overline{u}), \end{aligned}$$

où $\overline{u} = u(y_1) + u(y_2)$. De même pour v

$$\begin{aligned} V &= (1/2)\sum_{i=0}^{\infty} \beta^i (\overline{y} - c_1 - c_2) \\ &= \frac{1}{2(1-\beta)}(\overline{y} - c_1 - c_2), \end{aligned}$$

où $\overline{y} = y_1 + y_2$. Sous ce régime, l'agent v est désormais toujours contraint dans l'état 1 alors que l'agent u l'est toujours dans l'état 2.

Je cherche à établir les dérivées à gauche et à droite de la fonction de valeur en (V, U). Le point (V, U) est un point intérieur de la frontière de Pareto puisque les intervalles ne sont pas dégénérés. Pour calculer la dérivée à gauche, on cherche à identifier les contrats optimaux auto-exécutoires qui procurent légèrement moins d'utilité à l'agent v et davantage à l'agent u.[20] Puisque l'agent v butte sur sa contrainte dans l'état 1, il n'est pas possible d'accroître la consommation de u dans cet état. Conséquemment, un contrat réalisable et préférable δ_γ^g pour l'agent u augmentera la consommation dans l'état 2 de γ. L'agent u obtient donc $c_2 + \gamma$ dans l'état 2 et c_1 dans l'état 1. Si l'état 2 se réalise, les possibilités de consommation sous ce contrat δ_γ^g demeurent les mêmes ; sinon, le contrat offrira dès lors un profil de transferts

20. De fait, il y a abus de langage ici puisqu'il s'agit d'un seul et même contrat. Il s'agit, en fait, d'établir les c_{t-1} correspondant à des points (U_t, V_t) légèrement à gauche ou à droite de (U, V).

identiques à δ. δ^g_γ procure donc une utilité de

$$\begin{aligned} U^g_\gamma &= (1/2)(u(c_1) + u(c_2 + \gamma) - \overline{u}) + (1/2)\beta(U + U^g_\gamma) \\ &= \frac{u(c_1) + u(c_2 + \gamma) - \overline{u} + \beta U}{2 - \beta}. \end{aligned}$$

Après substitution, on obtient $U^g_\gamma - U = \dfrac{u(c_2 + \gamma) - u(c_2)}{2 - \beta}$, et de même pour v, $V^g_\gamma - V = \dfrac{-\gamma}{2 - \beta}$. Par la règle de l'Hospital, on obtient la dérivée à gauche de la frontière de Pareto

$$f'_g(V) = \lim_{\gamma \to 0} \frac{U^g_\gamma - U}{V^g_\gamma - V} = \lim_{\gamma \to 0} \frac{u(c_2 + \gamma) - u(c_2)}{-\gamma} = -u'(c_2).$$

Pour établir la dérivée à droite, il faut chercher les contrats auto-exécutoires optimaux qui procurent plus d'utilité à v et moins à u. Par un raisonnement similaire, un tel contrat diminuera la consommation de u dans l'état 1 d'une quantité γ (puisqu'il n'est pas contraint dans cet état) et gardera constante sa consommation dans l'état 2. On calcule ainsi

$$f'_d(V) = \lim_{\gamma \to 0} \frac{U^d_\gamma - U}{V^d_\gamma - V} = \lim_{\gamma \to 0} \frac{u(c_1 - \gamma) - u(c_1)}{\gamma} = -u'(c_1).$$

Le résultat est immédiat : si les intervalles sont disjoints on a

$$f'_g(V) = -u'(c_2) > -u'(c_1) = f'_d(V).$$

La dérivée en V (qui constitue ici, incidemment, l'ensemble ergodique) n'existe tout simplement pas et la frontière de Pareto affiche un coude.

Chapitre 2

Sequential Screening

2.1 Introduction

In this paper, I address the normative question of designing optimal screening complete contracts when agents have *sequential types*, that is, types that unfold dynamically as a sequence. The analysis is made under the assumption that, at any point in time, the agent behavior is resumed by a single dimensional characteristic.

The economic analysis of screening contracts with sequential types is a useful generalization of the standard model where all private information is resumed by a single static characteristic. Assuming sequential types allows for situations where the private information has a more complex nature. The celebrated trade-off between efficiency in production space and informational rent extraction, in the single static type set up, is transposed, for sequential types, in a trade-off between efficiency in *contract* space and informational rent extraction.

With sequential types, screening is achieved using the direct mechanism proposed by Myerson (1986), for which he demonstrates that a form of revelation principle holds. This mechanism requires that players announce privately their type sequence as it unfolds and that they follow a proper action after each announcement. In the initial stages of the mechanism, these actions will take the form of a choice of a binding contract to be followed afterward. It is only in the final stage of the mechanism that a particular choice of a physical action (consumption, production, etc.) takes place. Transfers are usually conditioned on the whole history of type announcements.

I limit myself to two-stage sequences: this is enough to analyze issues like ex post vs. ex ante pricing or the effect of lack of commitment on screening contracts. A more serious restriction of this paper is that most of the results are derived for situations where the players only have two possible types at each stage of the game.

Sequential types screening contracts have been explored before by Baron and Besanko (1984) in a principal-agent framework. The hypotheses I make in my model are closely related to that of their Theorem 3. The main difference between both papers lies in the scope of the analysis. In particular, Baron and Besanko recognize but bypass the issue of global incentive compatibility (that is, the fact that no player should be tempted to misreport its type ex ante) while I handle it in a systematic fashion. All the results presented in Theorem 3 of their paper are based on the heroic assumption that global incentive compatibility constraints do not bind at the optimum and they give an example of a type distribution for which this is the case. In most of the examples I study here, some of these incentive constraints are binding at the optimum.

An assumption that explains these differences is that consumption and production take place in each period in Baron and Besanko's model while all such activities are relegated at the end of the second period in my model. As a result, in their model, there are much stronger incentives for the principal to induce type revelation in the first period since that has, loosely speaking, a "first-order" effect on first-period production. This, in turn, relaxes the "second-order" incidence of global incentive compatibility constraints.

The economic interpretation of sequential screening contracts is very different in both papers. Baron and Besanko emphasize the trade-off, when sequential types are correlated, between inducing information revelation ex post or ex ante. I emphasize that at each period, there is a trade-off between efficiency in the contract space and rent extraction.

While their model can be seen as a useful multiperiod generalization of the one-period model, it is unsatisfactory with respect to the analysis of the combination of incentives one can expect when information is more complex by nature. This is the case, for instance, of a monopolist who wishes to discriminate consumers ex ante according to their expected demand and ex post according to their actual demand (I study this particular case in section 2.8).

The rest of this paper is divided as follows. The model is presented in the next section. In section 2.5, the analysis is completed under the assumption that the agent has two possible ex post types. Section 2.6 describes the optimal contract under various commitment assumptions. In sections 2.8 and 2.9, I apply my results to two economic problems encountered in the literature. The conclusion follows.

2.2 The model

I consider a principal-agent relationship in a two-period economy with incomplete information. Ex ante, in period 1, the agent has a private type $m \in M = \{\underline{m}, \overline{m}\}$. Ex post, in period 2, the agent is in a random state $\theta \in \Theta$ where Θ is an ordered set of N elements, independent of m. At times, I will refer to the agent's individual state as his *ex post type*. The type m of the agent characterizes his distribution probability $f(m)$ over Θ, represented as a $N \times 1$ vector, and his ex ante reservation utility (best external opportunity) $\overline{u}(m)$.[1] The state θ affects the ex post cost function $c(q, \theta)$ of the agent

[1] In section 2.6, I address the issue of defining ex post reservation utility. It is subsumed here by the certainty equivalent of the expected ex post opportunities.

to produce some good q.[2] c is assumed strictly convex in $q \in Q$ where Q is closed and bounded below by $q_0 = \min Q$. The lower the θ, the lower the cost:

$$-c_\theta(q, \theta) \leq 0 \tag{2.1}$$

where the equality stands only when $q = q_0$, that is $c_\theta(q_0, \theta) = 0$, for all θ. Furthermore, the l.h.s. of (2.1), that is the marginal saving of having a better type, is assumed a decreasing strictly concave function of q:

$$-c_{q\theta} < 0 \quad \text{and} \quad -c_{qq\theta} < 0. \tag{2.2}$$

The principal does not observe neither the ex ante type m nor the ex post state θ and the actual costs borne by the agent. Nevertheless, he has a strictly positive Bayesian prior p about the probability that the agent is of ex ante type $\underline{m}$. Both p and all the $f(m)$ are common knowledge to both players. I assume that the principal has all the bargaining power over the duration of the relationship. Various commitment capabilities, that of full commitment, no commitment and commitment with renegotiation are explored in the analysis.

The agent can communicate freely with the principal at all times. Since new information about θ is revealed to the agent at the end of period 1, I model this economy as a two-stage game with communication (Myerson 1986). More specifically, the course of the game is as follows. At the beginning of period one, the principal offers a contract (to be defined later) to the agent. The agent then accepts or refuses the contract. If he refuses, then the game ends and both players get their reservation utility (normalized to zero for the principal). If he accepts, the game moves to period 2 once the uncertainty about θ has been resolved. There, depending on the commitment assumption, the contract may be renegotiated. After that, production and

[2] As usual, q could be an "action" that affects the utility of the principal.

exchange take place according to the provisions of the final contract. The economic problem is to compute the optimal contract that will be offered to the agent by the principal under these various commitment assumptions.

Both the agent and the principal are assumed to be risk-neutral with respect to income. This assumption can be rationalized on the basis of preferences alone or because the gains of trade from the relationship are but a small fraction of their total income. If we normalize the price of the good q to 1, the principal's payoff is $q - t$, where t is a transfer from the principal to the agent, while the agent gets $u = t - c(q, \theta)$. I decompose the transfer t into actual cost and utility u for the agent so that the principal's ex post payoff becomes $q - c(q, \theta) - u$.

A set of behavioral strategies for the agent is a binary rule, to accept or to refuse the initial contract and a choice of messages to be sent to the principal about his type and the state θ that will be realized. These communication possibilities are resumed by invoking the revelation principle for multistage games under which the agent truthfully announces his type, at each stage, and is then asked to follow the prescriptions of a (possibly randomized) history-dependent scheme. I also assume that the agent accepts any contract that is individually rational. A strategy for the principal is an offer of a complete contract δ at the beginning of period one. Ultimately, a contract should specify a production level and a monetary transfer.

Under these hypotheses, the agent accepts any contract that leaves him with at least his reservation utility and truthfully reveals his type m and the state θ he is in, if the contract δ satisfies the usual incentive compatibility constraints. The principal's payoff out of a contract δ, evaluated at the initial contracting stage, is thus $\mathrm{E}(q_\delta - c(q_\delta, \theta) - u_\delta | \delta)$ while the agent expects $\mathrm{E}(u_\delta | \delta, m)$.

2.3 The equilibrium

Our solution concept is to look for a Perfect Bayesian Equilibrium (PBE) where the principal offers the contract δ that maximizes his payoff, under the relevant participation and incentive constraints. In the appendix, I show that we can restrict our attention to non randomized schemes. In that class, a contract is simply a pair of functions q_δ and t_δ that map the type space $M \times \Theta$ into the production and transfers spaces. Since the final transfer and the agent's utility are diffeomorphic, I define a contract as a pair of function q_δ and u_δ. For notational simplicity, I will omit the subscript δ whenever there is no possible confusion. A contract δ is thus a pair of functions q and u that map the type space into production and utility for the agent. These functions will be represented by pairs of $N \times 1$ vectors, one for each m, that is,

$$\delta = \{(q(m), u(m))\}_{m \in M}$$

where $q(m)$ and $u(m)$ are vectors of dimension N with indices that match that of the θ.

Let Γ be the set of contracts that comply with global incentive compatibility. Under full commitment, the optimal contract solves[3]

$$\max_{\delta \in \Gamma} \mathrm{E}(q(m) - c(q(m), \theta) - u(m)) \tag{2.3}$$
$$\text{subject to} \quad \mathrm{E}(u(m)|m) \geq \overline{u}(m) \quad \forall m \in M.$$

In this paper, we will take the less direct approach of dynamic programming.[4]

[3] I use the following convention: $\mathrm{E}(z)$ where z is a vector means $\mathrm{E}(\tilde{z})$ where $\tilde{z}$ is a random variable whose support are the elements of z.

[4] This approach was first devised by Townsend (1982).

More specifically, I rewrite (2.3) as

$$\max_{u(\underline{m}),u(\overline{m})} \mathrm{E}(s(u(m),m) - u(m)), \tag{2.4}$$

$$\text{subject to} \quad u(m) \in U \quad \forall m \in M,$$

$$\mathrm{E}(u(m) - u(m')|m) \geq 0 \quad \forall (m, m') \in M^2, \tag{2.5}$$

$$\mathrm{E}(u(m)|m) \geq \overline{u}(m) \quad \forall m \in M, \tag{2.6}$$

where U is the set of vectors of utility (v.o.u.) feasible under ex post incentive compatibility and

$$s(u(m), m) = \max_{q(m) \in Q(u(m))} \mathrm{E}(q(m) - c(q(m), \theta)|u(m)). \tag{2.7}$$

There, $Q(u(m))$ is the set of q that can be achieved under ex post incentive compatibility once $u(m)$ has been promised to the agent (see the next section). Once we have characterized $U(m)$, $Q(u(m))$ and $s(x, m)$, we can carry on the analysis in $R^{2\times N}$ where the optimal $u = (u(\underline{m}), u(\overline{m}))$ lies.

2.4 The Contract Space

A v.o.u. $u(m)$ is *feasible ex post under self-selection* (f.e.u.s.s.) for type m if there exists a $q(m) \in R^N$ such that the ex post incentive compatibility constraints are satisfied. Letting the "(m)" notation aside, these constraints can be written as

$$u_i \geq u_j + c(q_j, \theta_j) - c(q_j, \theta_i) \quad \forall (i, j) \in I^2. \tag{2.8}$$

Here the indices i and j in $I = \{1 \ldots N\}$ refer to the elements of the $N \times 1$ vectors $u(m)$ and $q(m)$. From then on, I assume that θ_i increases with i. $Q(u(m))$ is then the set of $q(m)$ that satisfy (2.8). A v.o.u. $u(m)$ is f.e.u.s.s. if $Q(u(m))$ is not empty.

The constraints (2.8) have the usual interpretation: they state that an agent of ex post type θ_i prefers to truthfully announce his type, in which case he produces q_i and ends up with transfer $t_i = u_i + c(q_i, \theta_i)$, than to announce θ_j for which transfer t_j was devised, given that he would then have to support the cost differential $c(q_j, \theta_j) - c(q_j, \theta_i)$ to produce q_j.

By assumption (2.2), each of the constraints (2.8) that define $Q(u(m))$ are monotonous in each q_i so that the whole set of constraints is quasi-concave in q; this makes $Q(u(m))$ a convex set. Since the maximand of program (2.7) is strictly concave, its associated argmax is a well defined function of $u(m)$ into the space of production vectors. Since that relation will hold at the optimum of (2.4), there is no loss of generality in considering only contracts for which $q(m)$ solves (2.7). Since these contracts are completely defined by u, I will talk of u as a *contract* and, more specifically, of $u(m)$ as a contract intended at type m.

I note U the set of contracts f.e.u.s.s. It is never empty: consider, for instance, $u(m)$ such that $u_i = \overline{u} - c(\overline{q}, \theta_i)$, where $\overline{q}$ and $\overline{u}$ are positive numbers that satisfy $\overline{u} - c(\overline{q}, \theta_N) > 0$; it is straightforward then that u satisfies (2.8) for $\overline{q}$.

Multiplying (2.8) by $(\theta_j - \theta_i)$ and using (2.1), I get the following obvious property of contracts f.e.u.s.s.

$$(u_i - u_j)(\theta_i - \theta_j) \leq 0 \quad \forall (i,j) \in I^2, i \neq j; \tag{2.9}$$

with equality when $q = q_0$. Hence, all contracts f.e.u.s.s. must guarantee that utility decreases with the ex post type.

The set of self-selecting constraints (2.8) can be rearranged by pairs:

$$c(q_i, \theta_j) - c(q_i, \theta_i) \geq u_i - u_j \geq c(q_j, \theta_j) - c(q_j, \theta_i) \quad \forall i \in I, \forall j \in I_i, \tag{2.10}$$

where $I_i = I \setminus \{1 \ldots i\}$. Using (2.1) and (2.9), it is obvious from (2.10) that q_i decreases with θ_i (with i).

Rearranging (2.10), a necessary condition for u to be f.e.u.s.s. is thus

$$u_i - u_j \geq u_j - u_k \quad \forall i \in I \setminus \bar{i}, \forall j \in I_i, \forall k \in I_j. \tag{2.11}$$

I say that a contract u is *internally consistent* if it satisfies these constraints. I then show the following.

Proposition 2.1. *A vector of utilities u is f.e.u.s.s. if and only if it decreases with θ and it is internally consistent. The set U of contracts f.e.u.s.s. is a convex cone.*

Proof. All missing proofs are in the appendix. □

Let q^* be the vector of efficient production values that, for all state i, equate marginal cost $c_q(q_i^*, \theta_i)$ to marginal (unitary) benefit for the principal. Let $U^* \subseteq U$ bet the set of contracts f.e.u.s.s. that allow efficient production q^*.

Proposition 2.2. *U^* is closed, convex and of non empty interior.*

Proof. Simply build the set of contracts associated to q^* such that

$$u_i^* = b_i + \overline{u} + q_i^* - c(q_i^*, \theta_i), \tag{2.12}$$

where $\overline{u}$ is an arbitrary constant and the b_i's belong to an open ball $B(\epsilon)$ of radius ϵ around $\tilde{u} \in U$, that is $\sum_I b_i^2 < \epsilon^2$. Take $\epsilon = 0$ so that $u^* = \tilde{u}$ and suppose now that $\tilde{u} \notin U^*$. Then, at least for some i, (2.8) must be violated for some j. This would yields

$$q_i^* - c(q_i^*, \theta_i) < q_j^* - c(q_j^*, \theta_i), \tag{2.13}$$

which is a sheer impossibility since q^* maximizes social surplus so that q_i^* should creates at least as much surplus than q_j^* for type θ_i. Hence, $\tilde{u}$ belongs to U^*. Suppose now that there is no ball $B(\epsilon)$ around $\tilde{u}$ so that all points in $B(\epsilon)$ belong to U^*. Then one of the incentive compatibility constraints has to be binding at $\tilde{u}$ otherwise increasing slightly ϵ from zero would be possible. But this implies that (2.13) must hold with equality for some (i, j) which is also impossible. Hence, such a ball $B(\epsilon)$ exists and that makes U^* of non-empty interior. Closeness and convexity comes from the fact that U^* is described by the weak inequalities (2.8) evaluated at q^*.

Finally, if $u^* \in U^*$, then it accepts decomposition (2.12) where

$$b_i + \overline{u} = u_i^* + c(q_i^*, \theta_i) - q_i^*.$$

Hence, equation (2.12) does completely characterizes all contracts in U^*.

Note that if a contract u allows ex post efficiency, then shifting u by a constant does not affect that property; that is, if $u \in U^*$, then $u + \overline{u}e \in U^*$ where $\overline{u}$ is an arbitrary constant and e is a vector of ones. □

A contract for agent m is noted again $u(m)$. I note $v(u(m), m)$ the value to the principal of a self-selecting contract $u(m)$ offered to some ex ante type m. This value is the expected difference between the maximal ex post expected surplus $s(u(m), m)$ he might gain from $u(m)$ and the expected utility compensation $\mathrm{E}(u(m)|m)$; hence,

$$v(u(m), m) = s(u(m), m) - \mathrm{E}(u(m)|m). \tag{2.14}$$

Working ex ante with function v is useful since it gives us the principal's ex post valuation of any ex ante promise of a v.o.u. $u(m)$. Arbitrage between informational rent extraction and efficiency will be made ex ante in the space of contracts. Put differently, given any level of utility $E(u(m)|m)$ the principal may wish to confer to some agent m, the principal will have the choice to offer a set of more or less efficient ex post contracts $u = \{u(\underline{m}), u(\overline{m})\}$ as the next proposition shows.

Proposition 2.3. *For any expected utility level* $\mathrm{E}(u(m)|m)$, *there exist a* $u^* \in U^*$ *such that* $\mathrm{E}(u^*|m) = \mathrm{E}(u(m)|m)$. *Likewise, for any expected utility level* $\overline{v} = v(u(m), m)$, *we can find a contract* $u^* \in U^*$ *that yields* $\overline{v}$ *to the principal.*

Given the decomposition (2.14) of the principal's expected utility from any contract $u(m)$, I get the following insight on the form of the principal's indifference curves over U.

Proposition 2.4. *Let* $\overline{v} = v(u'(m), m)$ *for some contract f.e.u.s.s.* $u'(m) \in U$ *and let* $\Gamma_{\overline{v}} = \{u(m) \in U | v(u(m), m) = \overline{v}\}$ *be the set of contracts f.e.u.s.s. for which the principal is (ex post) indifferent. By proposition 2.3, we know that* $U^* \cap \Gamma_{\overline{v}}$ *is not empty. Let* u^* *be a point in that intersection. Then, for all* $u(m) \in \Gamma_{\overline{v}}$, $\mathrm{E}(u(m)|m) \leq \mathrm{E}(u^*|m)$ *and the equality stands only if* $u(m) \in U^*$.

Corrolary 2.1. *If* $u^*(m) \in \mathrm{argmax}_{u(m)}\, v(u(m), m)$, *subject to* $\mathrm{E}(u(m)|m) \geq \overline{u}(m)$, *then* $u^*(m) \in U^*$.

Proposition 2.4 simply states that the indifference curve $\Gamma_{\overline{v}}$ associated to some utility level $\overline{v}$ is everywhere below the hyperplane $\mathrm{E}(u(m)|m)$ that passes through some u^* that yields $\overline{v}$ and is common with that hyperplane everywhere on $\Gamma_{\overline{v}}(m) \cap U^*$. Corollary 2.1 states that if a contract maximizes the principal's payoff under only an ex ante participation constraint of the agent, then it is efficient. Such an indifference curve is represented in figure 2.1, when $N = 2$. It bends inward, outside U^*, because these contracts are distorted and cannot achieve the optimal level of social surplus. For any given level of utility $\overline{v}$ for the principal, that distortion must be compensated by a reduction in the expected payment to the agent, hence in his expected consumption level $\mathrm{E}(u(m)|m)$. That distortion is minimized to zero for efficient contracts in U^*. Hence, all distorted contracts must lie below the hyperplane $\mathrm{E}(u(m)|m)$ that goes through some $u^* \in U^*$ on that indifference curve.

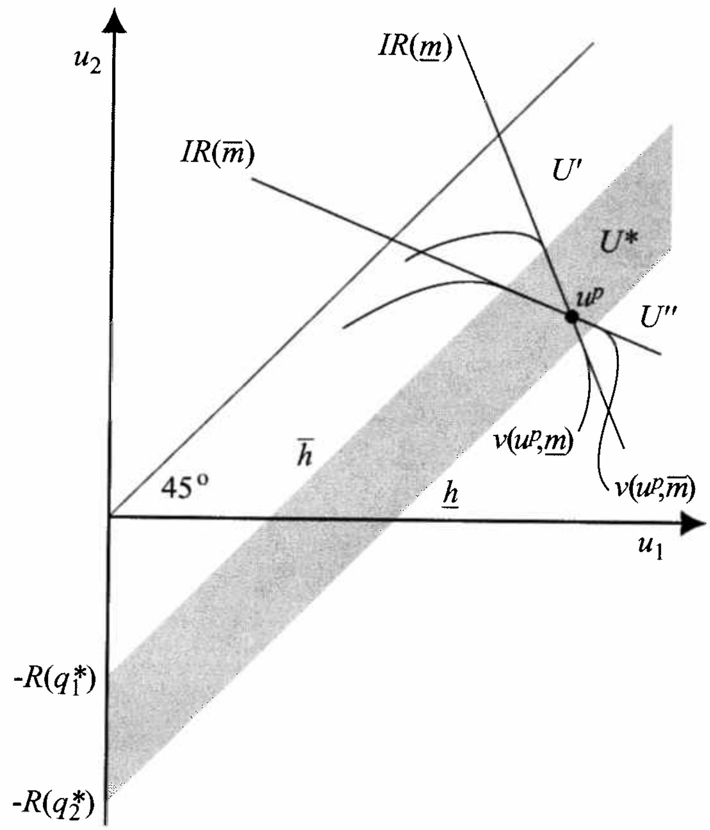

Figure 2.1: Efficient contract.

2.5 Ex Ante Contracting Under Full Commitment

I now considerably restrict the scope of the analysis by assuming that θ can take only two values, that is $N = 2$ and $\Theta = \{\underline{\theta}, \overline{\theta}\}$. That simplification is made to avoid having to establish a set of sufficient conditions under which function v is quasi-concave on a relevant portion of its domain.[5] Let $f(m) = \text{Prob}(\underline{\theta}|m)$ such that $f(\underline{m}) > f(\overline{m})$. Hence, $\underline{m}$ is the "good" ex ante type since he is most likely to have a "good" ex post type $\underline{\theta}$.

I let aside temporarily the notation for the ex ante type so that $u(m)$ is simply noted u. When $N = 2$, internal consistency is satisfied if u_i decreases with i (where $i = 1$ refers to ex post type $\underline{\theta}$); hence U amounts to all u such that $u_1 \geq u_2$; that is, the cone under the 45 ° line in the non-negative orthant of R^2. Consider now U^*; that set is given by all $u \in U$ that satisfy (2.8) at q^*, that is,

$$c(q_1^*, \theta_2) - c(q_1^*, \theta_1) \geq u_1 - u_2 \geq c(q_2^*, \theta_2) - c(q_2^*, \theta_1). \tag{2.15}$$

Hence, U^* is the set delimited by the hyperplanes $\overline{h}$ and $\underline{h}$

$$\overline{h} : u_2 = u_1 - R(q_1^*)$$
$$\underline{h} : u_2 = u_1 - R(q_2^*),$$

where

$$R(q) = c(q, \theta_2) - c(q, \theta_1) \tag{2.16}$$

is the ex post rent function. U is thus composed of three disjoint sets: U',

[5] Remember that the analysis of adverse selection models is usually done under some assumption about the probability distribution of the types (for instance, the monotone likelihood ratio assumption in the continuous case); such kind of assumption is needed here for higher dimensional setting. This will be the object of future research.

between the 45° line and $\overline{h}$, U^* and U'', outside U^* on the $\underline{h}$'s side (see figure 2.1). Contracts in U' are thoses where the ex post type $\overline{\theta}$ is asked to produce ineffficently; thoses in U'' require that the good ex post type $\underline{\theta}$ produces inefficiently.

For any given $u \in U$, let $q_1(u)$ and $q_2(u)$ be the solution to program (2.7). These values must satisfies the self-selecting constraints (2.10) that define $Q(u)$, that is,

$$c(q_1,\theta_2) - c(q_1,\theta_1) \geq u_1 - u_2 \geq c(q_2,\theta_2) - c(q_2,\theta_1). \qquad (2.17)$$

This leads us to the following proposition.

Proposition 2.5. *When $N = 2$, v is quasi-concave over $U' \cup U^*$ and strictly quasi-concave over U' for all distributions f.*

Corrolary 2.2. *Suppose $u(m) \notin U''$ and let $u^* \in U^*$ such that $\mathrm{E}(u(m)|m) = u^*$. Let $u_\lambda(m)$ be any convex combination of $u(m)$ and u^*,*

$$u_\lambda(m) = \lambda u(m) + (1-\lambda)u^* \quad \lambda \in (0,1).$$

Then $v(u_\lambda(m), m) \geq v(u(m), m)$ with equality only if $u(m) \in U^*$.

Corrolary 2.2 implies that the valuation of contracts is well behaved in U': if we keep the expected transfers constant, then contracts become more and more valuable to the principal as they get closer to U^*. The principal has strictly concave indifference curves over U'. This makes him locally behave ex ante like a risk averse[6] agent although such behavior was ruled out on the basis of preferences alone. This is because the principal prefers to pay the agent any given expected transfer with a non-degenerated lottery (based on θ) than with a sure amount of money; for lotteries, contingent on the agent's performance, allow provisions for ex post efficient incentives. Put differently,

[6]Because the agent is risk-neutral, utility and monetary transfers are interchangeable in this set-up.

the principal is willing to pay a positive premium to get rid of a lottery and thus behaves like a risk averse agent.[7]

Once a value $v(u(m)|m)$ has been assigned to any contract offered (and chosen) by an agent of ex ante type m, computing the best self-selecting ex ante contract amounts to solve the following program:

$$u \in \underset{\substack{u=\{u(\underline{m}),u(\overline{m})\},\\ u\in U\times U}}{\operatorname{argmax}} \; \mathrm{E}(v(u(m),m)) \tag{2.18}$$

subject to

$$IC: \quad \mathrm{E}(u(m)-u(m')|m) \geq 0 \quad \forall (m,m') \in M^2, \tag{2.19}$$

$$IR: \quad \mathrm{E}(u(m)|m) \geq \overline{u}(m) \quad \forall m \in M. \tag{2.20}$$

where the expectation in the maximand is based on the principal's prior p about the ex ante type population and $\overline{u}(m)$ is the reservation utility level of agent m.

Ex ante informational asymmetries do not necessarily command the use of inefficient contracts. When the ex ante efficient agent has a high reservation utility, first best may be achieved despite his capacity of mimicking the bad type. This is illustrated in figure 2.1. There, the ex ante participation constraint of the efficient agent $IR(\underline{m})$ is so high that it crosses that of the inefficient agent, $IR(\overline{m})$, within U^*. Hence, the principal can optimally offer a single ex ante pooling contract u^p at that point, and achieve efficiency. That contract leaves no expected informational rent to any agent. The link between pooling contracts and U^* is exposed in the following proposition.

Proposition 2.6. *If the optimal contract $(u(\underline{m}), u(\overline{m}))$ is pooling ex ante types, that is, $u(\underline{m}) = u(\overline{m}) = u^p$, then it is efficient.*

Proof. Suppose that the optimal contract is a pooling contract and that $u^p \notin U^*$. Consider offering with u^p an efficient contract intended at type $\underline{m}$.

[7]See Caillaud, Dionne, and Julien (1996) for a similar result.

Built that contract by adding payment $t(\underline{m})$ to u^p such that

$$t_i(\underline{m}) = \mathrm{E}(u^p|\underline{m}) - u_i^p + q_i^* - c(q_i^*, \theta_i) - s^*(\underline{m}),$$

where

$$s^*(m) = \mathrm{E}(q^* - c(q^*, \theta)|m). \tag{2.21}$$

Such additional transfers worth zero for agent $\underline{m}$ while they have an expected value of

$$s^*(\overline{m}) - s^*(\underline{m}) + \mathrm{E}(u^p|\underline{m}) - \mathrm{E}(u^p|\overline{m}) \tag{2.22}$$

for agent $\overline{m}$. If (2.22) was non-positive, then the contract $u^p + t(\underline{m})$ could be offered to agent $\underline{m}$ without affecting any incentive compatibility constraint. The contract $(u^p + t(\underline{m}), u^p)$ would obviously be prefered by the principal and that would lead to a contradiction since (u^p, u^p) was assumed optimal. Hence, it must be that (2.22) is strictly positive. Yet, we could construct a similar contract intended at type $\overline{m}$ so that (2.22) should be strictly negative. Since both conditions cannot be met by u^p at the same time, it must be that $u^p \in U^*$. □

It should be noted that the pooling contract associated to u^p is pooling types ex ante but not ex post. Since $u^p \in U^*$, then ex post types are asked to produce at their own efficient level, hence the contract is separating types ex post. In fact, what proposition 2.6 tells us is that if it is optimal to pool ex ante types, then it must be that we are separating efficiently ex post types.

As long as the reservation utility of the ex ante good type $\underline{m}$ is not too high with respect to that of type $\overline{m}$, he will be offered an efficient contract as the following proposition shows.

Proposition 2.7. *When $N = 2$ and the individual rationality constraints of both agents do not cross in U'', then agent $\underline{m}$ is offered an efficient contract.*

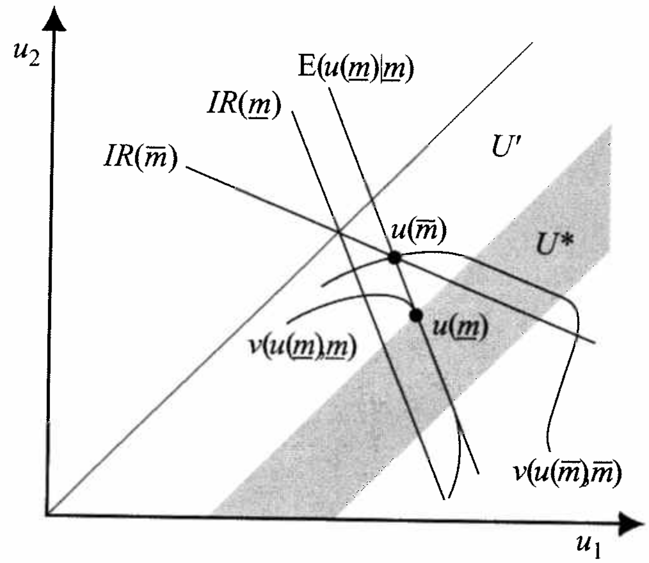

Figure 2.2: Sequential Screening Contracts.

Furthermore, $IR(\overline{m})$ is binding at an optimum.

Proof. The proof is trivial when the IR lines cross in U^*. Besides, if the optimal contract was pooling ex ante type, then agent $\underline{m}$ would be offered an efficient contract by proposition 2.6. Hence, I want to show that the proposition holds even if the optimal contract separates ex ante types. Consider first that, for each agent, either his incentive compatibility or individual rationality constraint is binding, or both, at the optimum. Otherwise, reducing his expected payment by the minimum strictly positive slack value of these two constraints would have no effect on the agent performance, would not create any bad incentive and, yet, would improve the principal's payoff so that the original contract would not be optimal after all. Now, unless the contract is pooling ex ante types, the incentive compatibility constraints for types $\underline{m}$ and $\overline{m}$ cannot be binding at the same time since the 2×1 vector $u(\underline{m}) - u(\overline{m})$ can only be orthogonal to a single vector $[f(m), 1 - f(m)]$. Furthermore,the incentive compatibility constraint of agent $\overline{m}$ cannot be binding at an optimum. To see this, suppose that $IC(\overline{m})$ is binding so that $IC(\underline{m})$ is free.

This implies that $u(\overline{m})$ and $u(\underline{m})$ belong to the same hyperplane that goes through some point $u^* \in U^*$ that yields an identical payoff to agent $\overline{m}$. Since $IC(\underline{m})$ is free, $u(\underline{m})$ is lying between $u(\overline{m})$ and u^*. Corrolary 2.2 then tells us that $v(u(\underline{m}), \overline{m}) > v(u(\overline{m}), \overline{m})$. Hence, the principal would be better off if $\overline{m}$ would choose contract $u(\underline{m})$. Since the original contract is dominated by a pooling contract that offers $u(\underline{m})$, it was not optimal in the first place. Since $IC(\overline{m})$ is free, it must be that $IR(\overline{m})$ is binding. Now, suppose that $u(\underline{m})$ is not efficient. Since $IC(\overline{m})$ is free at the optimum, proposition 2.3 allows us to assign to $\underline{m}$ a contract $u^* \in U^*$ that brings him the same expected transfer as $u(\underline{m})$ and strictly improves the principal's payoff. Hence, $u(\underline{m})$ must be efficient. □

The following corrolary extend the scope of proposition 2.6 when $N = 2$.

Corrolary 2.3. *Assume that the individual rationality constraints of both agents do not cross in U''. Then if the optimal contract $(u(\underline{m}), u(\overline{m}))$ is in U^*, there exists a pooling contract $u^p \in U^*$ that is optimal as well.*

When $\overline{u}(\underline{m})$ is sufficiently low, so that $IR(\underline{m})$ is not binding at an optimum, the optimal ex ante contract realizes the marginal trade off between rent extraction and efficiency so familiar in the one period setting except that it takes place here in the contract space. Such a contract is represented in figure 2.2. If $IR(\underline{m})$ is not binding, then $IC(\underline{m})$ must be binding; hence, given that the efficient agent is offered an efficient contract, the program becomes

$$\max_{u(\underline{m}), u(\overline{m})} pv(u(\underline{m}), \underline{m}) + (1-p)v(u(\overline{m}), \overline{m})$$

$$\text{s.t. } \mathrm{E}(u(\underline{m}) - u(\overline{m}) | \underline{m}) = 0$$

$$\mathrm{E}(u(\overline{m}) | \overline{m}) = 0.$$

After substituting for the constraints and taking the constant term $p(s^*(\underline{m}) -$

$\overline{u}(\underline{m})$) out of the maximand, the principal is maximizing

$$\max_{u(\overline{m})} -pr(u(\overline{m})) + (1-p)s(u(\overline{m}), \overline{m}). \tag{2.23}$$

where $r(u(\overline{m})) = \mathrm{E}(u(\overline{m})|\underline{m}) - \overline{u}(\underline{m})$ is the ex ante expected rent left to agent $\underline{m}$. Increasing the ex post efficiency $s(u(\overline{m}), \overline{m})$ of the bad ex ante type contract $u(\overline{m})$ increases the rent $\mathrm{E}(u(\overline{m})|\underline{m}) - \overline{u}(\underline{m})$ that must be left to the good type. If the reservation utility of the efficient agent is low, that process is costly and the inefficient agent is offered a very inefficient contract, that is a contract, close to the 45 ° line, that involves little risk, hence, little ex post type separation.[8] Program (2.23) trades off these rents in the first period because the the information revealed can then be used to devise an optimal self-selecting scheme in the second period (at least for type $\underline{m}$). Under such contract, an agent who is inefficient ex ante (type $\overline{m}$) but efficient ex post (type $\underline{\theta}$) will be offered to produce inefficiently. The traditional trade off between informational rent extraction and efficiency applies here to the extent that efficiency is measured with respect to contracts, not production.[9]

[8]I do not provide a full characterization of what is going on when the IR curves cross in U''. This case will generally result in the agent $\underline{m}$ being offered and efficient contract while the ex ante efficient type $\overline{m}$ will be offered an inefficient contract that involves, ex post, overproduction to a point where marginal cost is higher than marginal utility.

[9]The analysis is made under the implicit assumption that the principal never wish to shut down the ex ante inefficient agent. That is ensured if the proportion of ex ante type $\overline{m}$ is not too low, that is,

$$p \leq \frac{v(u(\overline{m}), \overline{m})}{v(u(\overline{m}), \overline{m}) + r(u(\overline{m}))},$$

where $u(\overline{m})$ is the optimal separating contract for type $\overline{m}$. Otherwise, (in the case where the reservation utilities are the same for both ex ante types) the principal will maximize his payoff by offering the efficient agent $\underline{m}$ an efficient contract in U^* that sets him on his participation constraint.

2.6 Contracting Under No Commitment

The analysis so far was performed under the assumption that the principal and the agent could commit themselves to a long-term contract. In some cases, the players cannot credibly commit themselves ex ante to abide by any contract. Even when such commitment is possible, ex ante contracting might leave the players ex post in a position where Pareto improving amendments of the original contract are available. These are the cases of non commitment or default and of commitment with renegotiation. Both cases will be treated by imposing additional constraints on the set of feasible ex ante contracts. Contracts robust to the possibility of default must satisfy ex post individual rationality (or participation constraints) while those robust to renegotiation must be *renegotiation-proof.* I examine the impact of these constraints in turn.[10]

The non commitment hypothesis does not trivialize ex ante contracting. On the agent's side, it means that we must deal with as much participation constraints as there are ex post states θ since the ultimate decision to abide by the contract will be taken ex post. The expected individual rationality constraints (2.20) are replaced by

$$IR: \quad u(m) \geq \overrightarrow{u}(m) \quad \forall m \in M.$$

where $\overrightarrow{u}(m)$ is the $N \times 1$ vector of best payoffs an agent of ex ante type m can gain by leaving the relationship in period 2. Geometrically, the effect of such constraints is to replace each half-space $E(u(m)|m) \geq \overline{u}(m)$ by a perpendicular convex cone whose vertices are given by $\overrightarrow{u}(m)$. The literature

[10]The possibility of default or renegotiation introduces new nodes in the course of the game. Care must be taken in handling the players beliefs at these nodes while constructing a PBE. But PBE are not very restrictive with respect to the admissible ex post beliefs off equilibrium paths: to pin down an equilibrium, I assume that if the agent initially refuses the contract or if he proposes an unexpected renegotiation at the interim stage, then the principal is therefore convinced that the agent is of the most efficient type ex ante and in the most favorable state θ.

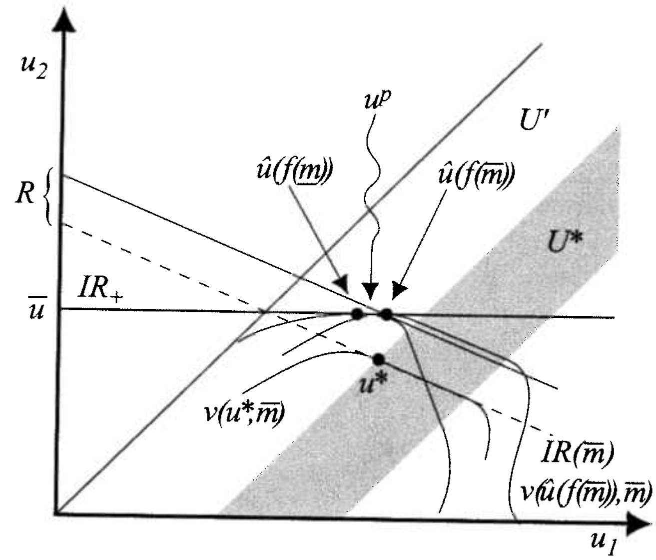

Figure 2.3: The Classical Principal-Agent Problem.

on how to deal with these constraints is still in its infancy and is beyond the scope of this paper[11] so I will restrict myself to the specific familiar case where $\overrightarrow{u}(m)$ is a vector composed of a single constant $\overline{u}(m)$. In that case, the cone points on the 45 ° line. When $N = 2$, imposing ex post participation constraints amounts to rotate back to the horizontal the ex ante participation constraint of the agent. This will restrict the set of contracts available to the principal. Obviously, it must be assumed that the principal bears the same commitment incapacity or that the agent has serious liquidity constraints for that restriction to be effective. Otherwise, the principal can simply post a bond, whose payments are contingent on the agent's ex post behavior, that the agent will buy ex ante.

In figure 2.3, for instance, I represent the classical principal-agent problem when there is a single ex ante type m and $N = 2$. When a full commitment contract is available, then full efficiency can be achieved with a contract u^*, anywhere along the intersection between IR and U^* where the utility

[11]See Julien (1994).

of the principal is maximized at $v(u^*, m)$. But if the contract has to be individually rational ex post or, equivalently, if we assume that the initial contract is not binding, then, in effect, we are restricting the contract space to contracts above the horizontal line IR_+. In this case, the principal's utility is maximized at $\hat{u}(f(m))$ yielding $v(\hat{u}(f(m)), m)) < v(u^*, m)$. This contract gives an expected informational rent of R to the agent and is not efficient since it lies outside U^*. Contract $\hat{u}(f(m))$ is easily identified by solving the first-order condition $v_1(\hat{u}(f(m)), m) = 0$, that is,[12]

$$(1 - f(m))(1 - c_q(\hat{q}, \overline{\theta}))\hat{q}' - f(m) = 0.$$

Since $\hat{q}$ is the inverse function of the ex post rent function (2.16), its derivative is the inverse of the derivative of the rent function, hence,

$$(1 - f(m))(1 - c_q(\hat{q}, \overline{\theta}) - f(m)(c_q(\hat{q}, \overline{\theta}) - c_q(\hat{q}, \underline{\theta})) = 0.$$

Hence, contract $\hat{u}(f(m))$ realizes the familiar trade off between marginal expected production inefficiency $(1 - f(m))(1 - c_q(\hat{q}, \overline{\theta})$ and the marginal expected informational rent to be left to ex post efficient type $\underline{\theta}$, $f(m)(c_q(\hat{q}, \overline{\theta}) - c_q(\hat{q}, \underline{\theta}))$. As $f(m)$ increases – say, from $f(\overline{m})$ to $f(\underline{m})$) – the contract is pushed away from U^* along IR_+:

$$\frac{\partial \hat{u}_1}{\partial f(m)}(f(m)) = \frac{1}{(1 - f(m))v_{11}} < 0. \tag{2.24}$$

with $\hat{u}_2(f(m)) = \overline{u}$. If there are many ex ante types and commitment is a problem, the principal will not try to separate ex ante types as the following proposition shows.

Proposition 2.8. *Suppose that* $\overline{u}(\underline{m}) = \overline{u}(\overline{m})$, $N = 2$ *and that both players can't credibly commit themselves in the long run. Then the optimal contract* $(u(\underline{m}), u(\overline{m}))$ *is a pooling contract* (u^p, u^p) *where* u^p *is inefficient and lies*

[12] See the proof of proposition 2.5.

somewhere between $\hat{u}(f(\underline{m}))$ and $\hat{u}(f(\overline{m}))$.

Full commitment contracts achieve higher levels of efficiency by using ex post penalties as an instrument to induce ex ante type separation, while keeping expected transfers low. These penalties are no longer avalaible under no commitment so that separating ex ante types is no longer an interesting option.

The difference between contracts $\hat{u}(f(\underline{m}))$ and $\hat{u}(f(\overline{m}))$ lies in the amount of distorsion that is put on the production plan of the ex post type $\overline{\theta}$ (the distance from the 45 $^\circ$ line). As usual, that distorsion is imposed in order to separate the ex post types while minimizing the expected rent to be left to the ex post efficient type $\underline{\theta}$. The distorsion is greater for the ex ante type $\underline{m}$ because he is more likely to be ex post of type $\underline{theta}$. If both ex ante type are present, at least one ex ante type m will obviously be offered a contract u^p on IR_+. Strict ex ante separation could then be achieved by offering a contract $u(m')$ to the other ex ante type that lies above IR_+ and his indifference curve going through u^p. All these contracts yield less utility to the principal than u^p so that (u^p, u^p) is to be expected as an ex ante pooling contract. Hence, ex ante separation does not occur not because it is unfeasible but because it does not maximize the principal's payoff.

The contract u^p does separate ex post types θ. It is efficient with respect to weighting the need for ex post efficiency in production and ex post rent minimization under ex post revelation constraints. The weight used to locate u^p is the unconditionnal probability of facing an ex post efficient type

$$f(\tilde{m}) = pf(\underline{m}) + (1-p)f(\overline{m})$$

which lies between $f(\overline{m})$ and $f(\underline{m})$. In effect, the absence of commitment yields a similar outcome to the one we would get by postponing the contracting date once the uncertainty about θ had resolved. Then, both ex ante types $\underline{m}$ and $\overline{m}$ would be behaviorally equivalent, regardless of their ex post type, and the principal would offer a single menu based on his unconditional prior $f(\tilde{m})$.

2.7 Renegotiation-proof Contracts

In this section, I explore the issue of renegotiation in my framework. I assume that any offer of renegotiation is to be made by the principal and that it only needs to provide the agent as much utility as he can expect under the status quo (the ex ante contract) to be accepted. Since the principal has no private information, offers of renegotiation won't carry any signalling feature. Renegotiation may be considered at different points in time in my model:

interim-m: just after the contract has been signed ex ante but prior to the agent announcing his type m;

ex post-m: just after the agent has announced his type m but before the ex post type θ is realized;

interim-θ: just after θ is realized but before the agent has announced his ex post type;

ex post-θ: just after the agent has announced his ex post type but prior to production taking place.

The first and the last forms of renegotiation are of lesser interest here since they don't affect the nature of the final contract once all the proper incentives, induced by the common knowledge possibility of renegotiation, have been incorporated in the analysis[13].

For a contract to be robust to the possibility of renegotiation, it should always prescribe allocations that are on the Pareto frontier at any node where

[13] In particular, the possibility of ex post-θ renegotiation is eliminated if we assume that the announcement of the ex post type is implicitely made by the very irreversible act of producing. Cf. Beaudry and Poitevin (1995).

renegotiation is assumed possible. Consider first the possibility of ex post-m renegotiation. The contract pairs I derive in section 2.5, when the IR curves cross in U', are not renegotiation-proof when they prescribe an inefficient contract for type $\overline{m}$, that is $u(\overline{m}) \notin U^*$. A contract that is robust to ex post-m renegotiation will thus have to be in U^*. By corrolary 2.3, for any such contract, there is a pooling contract u^p in U^* that does just as well. Hence, any optimal ex post-m renegotiation-proof contract $(u(\underline{m}), u(\overline{m}))$ will solve

$$\mathrm{E}(v(u(m), m)) = \max_{\substack{u^p \in U^* \\ \mathrm{E}(u^p|m) \geq \overline{u}(m), \\ \forall m \in M}} \mathrm{E}(v(u^p, m)).$$

It is straightforward to see that any such contract, conditionally on belonging to U^* and satisfying the participation constraints, will in fact minimize the expected transfers to the agent,

$$\max_{\substack{u^p \in U^* \\ \mathrm{E}(u^p|m) \geq \overline{u}(m), \\ \forall m \in M}} \mathrm{E}(v(u^p, m)) = \mathrm{E}(s^*(m)) - \min_{\substack{u^p \in U^* \\ \mathrm{E}(u^p|m) \geq \overline{u}(m), \\ \forall m \in M}} \mathrm{E}(u^p).$$

The effect of renegotiation constraints is to make separation more costly or, put differently, rent extraction more difficult to the principal. Hence, unless his participation constraint is so high that pooling in U^* would have had occured anyway (when the IR constraints of both agents crosses in U^*), the ex ante efficient agent will gather more surplus when ex post-m renegotiation is possible.

If a contract is robust to ex post-m renegotiation, it is naturally robust to interim-θ renegotiation. To the extent that the possibility of renegotiation is costly ex ante to the principal (it constrains the set of feasible contracts), he might wish to devote resources to improve his commitement capabilities and relax the ex post-m renegotiation-proof contraints. Besides, it is plausible that such commitment decreases over time. This raises the possibility that the principal can commit himself not to renegotiate the contract in the short

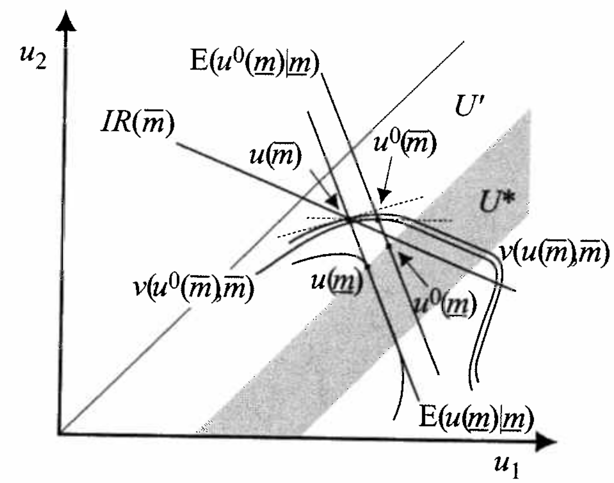

Figure 2.4: Interim-θ Renegotiation.

term (ex post-m type of renegotiation) but not in the long term (interim-θ type of renegotiation).

I define a interim-θ renegotiation-proof contract as a pair of contracts $(u(\underline{m}), u(\overline{m}))$ such that $v(u(m), m) \geq v(u(m)+t)$, for all non-negative $N \times 1$ vectors t. The idea is that it should not be possible to improve the principal utility, once the types m have been separated, simply by giving more utility to the agent with respect to what he can expect under the original contract he chose. Equivalently, one can states that $u(m)$ is interim-θ renegotiation-proof for ex ante type m if

$$0 \in \operatorname*{argmax}_{t \geq 0} v(u(m) + t, m).$$

Hence, assuming differentiability of v at $u(m)$, $v_{u_i}(u(m)) \leq 0$, for all $i \in I$, is a necessary condition for $u(m)$ to be renegotiation-proof. When $N = 2$, this implies that the indifference curve of the principal at $u(m)$ should not be upward sloping. Obviously, all contracts U^* are interim-θ renegotiation-proof so that only contracts designed for type $\overline{m}$ may be subject to that kind of renegotiation. Hence, to find the best interim-θ renegotiation-proof contract, I add the constraint

$$\frac{v_1}{v_2}(u(\overline{m}), \overline{m}) \geq 0$$

to program (2.18). Since[14] $v_2 = -(1+v_1)$, for the optimal separating contract, the interim-θ renegotiation-proof constraint will bind. That is, we will need to set $v_1(\overline{m}, \overline{m}) = 0$ like we did for the no commitment case.

The effect of ex-post-θ renegotiation is illustrated in figure 2.4. There, I assume that the IR curve of the efficient agent is sufficiently low so that it is not binding. The optimal self-selecting pair of contracts is to offer an inefficient contract $u(\overline{m})$ to type $\overline{m}$ so that the marginal production distortion cost of that contract equals the marginal informational rent that has to be left to the efficient type, for any contract u^* in the intersection of the indifference curve that goes through $u(\overline{m})$ and U^*. But $u(\overline{m})$ is not interim-θ renegotiation-proof because the indifference curve of level $v(u(\overline{m}), \overline{m})$ that goes through it is upward-sloping at $u(\overline{m})$. Once the agent has committed himself to $u(\overline{m})$, the principal would then try to renegotiate at $u^0(\overline{m})$. That renegotiation would be anticipated by $\underline{m}$ and he will pretend to be of type $\overline{m}$ because $\mathrm{E}(u^0(\overline{m}) - u(\underline{m})|\underline{m}) > 0$. Contract $(u^0(\underline{m}), u^0(\overline{m}))$, on the other hand, is renegotiation-proof and it specifies, for the ex ante inefficient type $\overline{m}$, a similar contract to the non commitment case.

The interim-θ renegotiation constraint is binding for agent $\overline{m}$ because the ex ante separation of types $\underline{m}$ and $\overline{m}$ distorts the interim efficiency of the ex post contract offered to him. This does not occur with standard models of renegotiation since the global incentive compatibility constraints, with respect to the revelation of the ex ante type, are not present.

[14]See the proof of proposition 2.5 in the appendix.

The possibility of ex post-m renegotiation implies that contracts should lie in U^* while that of interim-θ renegotiation only implies that contracts should lie on decreasing portions of the principal's indifference curves. Since all contracts in U^* satisfy that last property anyway, the effect of ex post-m renegotiation on the principal expected profit is more stringent than that of interim-θ renegotiation.

The possibility of interim-θ renegotiation is less harmful to the principal than that of ex post-m renegotiation because it does not preclude separation with respect either to the ex ante type or the ex post type. Ex post-m renegotiation precludes such separation for ex ante types. Unlike the case of ex post-m renegotiation, interim-θ renegotiation still allows the principal to devise distorsions in production plans to better extract rent from the ex post efficient agent. In both types of renegotiation, the ex ante type m is announced to the principal, on the equilibrium path, prior the opportunity to renegotiate arises. That announce has a dramatic effect when the opportunity to choose an ex ante efficient contract in U^* is still available but it only leads to a revision of the ex post principal's prior with respect to the ex post type in the case of interim-θ renegotiation.[15]

In the following two sections, I apply some of the results of this paper to problems that I have encountered in the literature.

2.8 Optional Calling Plans and Tapered Tariffs

Clay, Dibley, and Srinagesh (1992) and, more recently, Miravete (1996) have studied the merits of Optimal Calling Plans (OCP) and Tapered Tariffs (TT) as optimal nonlinear pricing schemes for a monopolist in the telecommunication industry. Quoting from Clay, Dibley and Srinagesh' (hereafter, CDS)

[15]Renegotiating a contract in U^* under interim-θ renegotiation is possible but would not maximize the ex post payoff of the player that proposes the renegotiation (the principal).

paper:

> Optional calling plans are usually sets of two-part tariffs, each consisting of an entry fee and a user charge, from which a consumer chooses and determines the effective price at the start of a billing period. Tapers are declining block tariffs, for which the effective price is determined at the end of the billing period.

I will use the CDS' model to illustrate how optimal screening mechanisms should be designed but the analysis applies to Miravete's paper as well. These authors associate OCP to ex ante pricing and TT to ex post pricing and they intend to explain why both might be observed in reality. Their analysis is of some interest; first, because they use a model where the agents have an ex ante type and a random demand ex post; second, because it is essentially flawed so that it is a perfect ground to demonstrate intuitively the superiority of the contracts presented in this paper. Both CDS and Miravete fail to recognize the importance of designing contracts that achieve screening whenever that is possible. Miravete, for instance, explicitly invoke Myerson's (1979) revelation principle to design his mechanism. But this is inappropriate since that principle was not made for multistage games. As a result, his direct mechanism is dominated by other equilibria where the agent and the principal send themselves messages that allow them to improve their utility.

I resume now the CDS model. There is a single profit-maximizing monopolist that faces a population of consumers composed of two ex ante types. Type A consumers have a demand of $q_A(p)$ while type B have a demand of $q_H(p)$ in state H and a lower demand of $q_L(p)$ in state L. Type A consumers are assumed to have a higher demand than the expected demand of type B consumers. The monopolist does not observe neither the type nor the state of any consumer. The monopolist has the choice to contract (using take-it-or-leave-it offers) under full commitment before each consumer learns their ex post type (that is, for type B consumers) or after, once the population is composed of types A, H and L. A tariff scheme is made of an entry fee e and a price p at which the consumer may buy the good. Using ex ante contracts, they assume that the monopolist would offer a pair of two-part tariffs, one for type A and one for type B, that is, a choice of an OCP, while ex post contracting TT is to be associated with a finer tariff scheme that allows for two prices that decrease with quantity bought.

These authors claim that if $q_L(p)$ is sufficiently low, then ex post pricing (TT) will dominate ex ante pricing (OCP). Their logic is that if $q_L(p)$ is low, then the monopolist will be forced, under OCP, to lower down the entry fee for type B and that will tighten the incentive compatibility constraint of type A. Under TT, a low value of $q_L(p)$ can be accommodated by setting prices so as to leave type L consumer out of the market. I will argue that this analysis is flawed (at the very least, from a normative point of view) because the ex ante contracting scheme they use is suboptimal as they fail to recognize the dynamic nature of programming incentive schemes with sequential adverse selection.

Formally, the optimal ex post contracting scheme TT is found by solving

$$
\begin{aligned}
P_{TT}: \quad & \max_{\substack{e_A, e_H, e_L \\ p_A, p_H, p_L}} \ \mathrm{E}(e + (p - c)q(p)) \\
\text{subject to} \quad & IC: \quad u(i,i) \geq u(i,j) \quad \forall i,j \in \{A, H, L\} \\
& IR: \quad u(i,i) \geq \overline{u}(i) \quad \forall i \in \{A, H, L\}.
\end{aligned}
$$

Here the principal maximizes its expected surplus from the entry fees and his profit margin over his constant marginal cost c. The incentive compatibility constraints IC are stated using consumer surplus to represent preferences induced from the Marshallian demand functions; $u(i,j)$ is thus the consumer surplus (net of the entry fee) an agent of type i may realized by taking the two-tariff plan designed for consumer j. $\overline{u}(i)$ is the consumer surplus consumer i may realize by switching to an other company and that forms the basis of the individual rationality constraints IR.

CDS compare the value of program P_{TT} to the following ex-ante program,

$$\begin{aligned} P_{OCP}: \quad & \max_{\substack{e_A, e_B \\ p_A, p_B}} \mathrm{E}(e + (p-c)q(p)) \\ \text{subject to} \quad IC: \quad & \mathrm{E}(u(i,i)|i) \geq \mathrm{E}(u(i,j)|i) \quad \forall i,j \in \{A,B\} \\ IR: \quad & \mathrm{E}(u(i,i)|i) \geq \mathrm{E}(\overline{u}(i)|i) \quad \forall i \in \{A,B\}. \end{aligned}$$

Here, the principal offers two OCP plans: one for type A and one for type B. Incentive compatibility and individual rationality constraints only have to hold in expectation because the contract is signed prior the consumer of type B learns the state he is in. Their point is that when q_L is sufficiently low, then program P_{TT} may dominate program P_{OCP}. They provide an example with linear demand

$$q_A(p) = 9.5 - p \quad q_H(p) = 10 - p \quad q_L(p) = 5 - p$$

and probabilities

$$\mathrm{Prob}(i = A) = 1/2 \quad \mathrm{Prob}(i = H) = \mathrm{Prob}(i = L) = 1/4.$$

There, the $u(i,j)$ function of type i is given by $\chi(z_i \leq p_j)\,(p_j - z_i)^2/2 - e_j$, where z_i is the intercept of his linear demand function on the price axis and $\chi(cnd)$ is the indicator function that takes the value one when the condition cnd is true. In that case, program P_{TT} yields a profit of 37.53 while the P_{OCP} program yields only[16] 34 with the following instruments:

$$e_A = 20 \quad e_B = 7 \quad p_A = 2 \quad \text{and} \quad p_B = 4.$$

[16] I have corrected that figure from the value 32 the authors computed under the strange hidden assumption that consumers of type L would consume a ***negative*** amount of the good under their reservation plan.

The flaw in the analysis is that the contracts computed under program P_{OCP} are by no means optimal as they unnecessarily constrain the set of instruments available to the principal. Consider the following class of contracts: instead of having a single contract for type B consumers, let them pay ex ante an entry fee of e_L that allows them to buy at price p_L and authorize the consumer that has declared himself of type B to pay ex post an additional fee of e_H to get a price reduction from p_L to p_H. That part of the contract is intended at type H once they have learned their type. I make sure that these contracts are incentive compatible and satisfy all the necessary participation constraints. The profit-maximizing contract in that class solves

$$P^* : \quad \max_{\substack{e_A, e_B, e_H \\ p_A, p_B, p_H}} \mathrm{E}(e + (p - c)q(p))$$

subject to

$$\begin{aligned} IC: \quad & u(A,A) \geq \max_{i \in \{B,H\}} u(A,i) \\ & \mathrm{E}(u(\tilde{B},\tilde{B})|B) \geq \mathrm{E}(u(\tilde{B},A)|B) \\ & u(i,i) \geq u(i,j) \quad \forall i,j \in \{H,L\} \\ IR: \quad & \mathrm{E}(u(i,i)|i) \geq \mathrm{E}(\overline{u}(i)|i) \quad \forall i \in \{A,B\}, \end{aligned}$$

where $\tilde{B}$ is the random ex post B type. The principal can ensure himself a profit of 40.25 with the solution contract

$$e_A = 24.25 \quad e_L = 6.25 \quad e_H = 19.5 \quad p_A = 2 \quad p_L = 5 \quad p_H = 2,$$

so that this contract dominate both the OCP and the TT plan.

Hence, ex post contracting does not dominate ex ante contracting for the latter, when judiciously designed, can duplicate all of the outcomes of the former. The optimal contract I design is very intuitive. Suppose for instance that consumers of type A plan to use their phone for a commercial purpose while that of type B are residential customers. For home-based workers, the phone company has little options but to use a non linear tariff scheme to separate these commercial customers whose use of the system, albeit lesser than a classical corporation, is expected to be heavier than that of residential customers. Nevertheless, the company would like to discriminate, among its residential customers, those whose mother is sick and are more likely to make a lot of phone calls. I assume that residential customers whose mother is sick are even heavier user of the system than the commercial ones. According to the optimal contracting scheme, the company will first ask its customers to declare themselves either as commercial or residential. Self-declared residential customers pay as little as 6.25 to get into the low consumption plan but they get to pay an additional fee of 19.5 if they later realized that their mother is sick. That plan is unattractive for type A consumers because it implies a sure entry fee of $25.75 > 24.25$. The point of the optimal revelation mechanism is that type B consumers won't be allowed to get the commercial plan: ex post pricing will be conditional on the announcement history.

The idea that ex post pricing or ex post contracting could strictly dominate ex ante pricing under full commitment and with complete contracts is flawed. Heuristically, consider simply that complete ex ante contracts can replicate the outcome of any ex post contract. Under the profit-maximizing firm hypothesis, one must assume that if firms fail to implement the optimal sequential OCP scheme that I have described, then it is because that scheme is unavailable for some reason. I have shown in section 2.7 that, if we consider the possibility of interim-θ renegotiation, an optimal ex ante menu of contracts could include, at the same time, a no commitment contract that satisfy ex post individual rationality and a full commitment contract that satisfy only an expected participation constraint. Hence, a sensible way of rationalizing the choice of both OCP and TT schemes by profit-maximizing firms could be that these firms have limited commitment capability. [17]

2.9 Contracting Under No Commitment and Ex Ante Screening

My interest in sequential screening contracts evolved while trying to resolve the following paradox. In González (1997b),[18] I show that the hold-up problem of investment can be mitigated if the investing party can make his move privately. In my model, there is a seller that can make an investment e at an opportunity cost $\psi(e)$ to reduce his variable costs $c(q,e)$ of producing a quantity q of some good; and a buyer that offers a take-it-or-leave-it contract to the seller afterward. Investing is assumed socially efficient but has no value outside the relationship. Under full information, the buyer has no incentive to repay the fix cost of investment when offering a contract and will simply offer to pay the variable costs. Expecting such a contract, the seller does not invest. But when the seller can make his investment privately, then he can induce the buyer to propose a self-selecting contract by playing a mixed strategy with respect to investment. That will allow some investment to take place because optimal self-selecting contracts yield rents to efficient types. These rents will be used to repay the fixed costs; in equilibrium, rents match exactly the fixed costs.

Laffont and Tirole (1993) have proposed a very similar problem but the investment e determines the likelihood of having a low marginal cost. Like in my model, the buyer will offer an optimal self-selecting contract to discriminate among ex-post marginal costs and will thus give high rents to efficient type. The fixed cost is also financed by rents although the relation there is only statistical. But, in their model, the agent plays a *pure* strategy with respect to investment at the equilibrium solution. Hence, the issue of the ob-

[17] I intend to do more research on that subject.

[18] Chapter 3 of this thesis.

servability is inconsequential since pure equilibrium strategies can be inferred by any player.

Both are very similar models of imperfect information (there is no ex ante type). Why then such different conclusions with the issue of observability? The answer lies in the implications of no commitment, or ex post rationality constraints, on sequential screening contracts. At the contracting stage, Laffont and Tirole impose that the contract should satisfy individual rationality constraints for all possible realizations of marginal cost. This means that the contract is signed *after* the seller learns his variable costs. In my model, there is no such uncertainty but the variable costs of the agent could be interpreted as his expected variable costs. The point is that I don't impose such individual rationality constraints with respect to the variable costs. My model is thus one where contracting take place *before* the agent learns precisely his variable costs.

These two different timing structures can be analyzed in the set-up developed in this paper. Investment e is the ex ante type while the realization of marginal cost is the ex post type. My model is one where full commitment ex ante contracts are available while Laffont and Tirole' approach is one where ex post individual rationality constraints are imposed. But we saw in proposition 2.8 that with ex post individual rationality constraints, the pooling solution will prevail. Hence, if the agent expects a pooling contract, there is no need nor wish for him to randomize his investment decision in equilibrium. On the contrary, under full commitment ex ante contracting, proposition 2.7 shows that the optimal self-selecting contract will generally separate ex ante types, that is, investment levels. Payoffs, conditionally on investment, are higher for a seller that has invested; that can only be sustained if investment is not observed and the seller plays a mixed strategy.

2.10 Conclusion

In this paper, I have sketched a theory of sequential screening contracts in a principal-agent framework where the agent learns his two-dimensional type sequentially. Contrary to Baron and Besanko (1984), I have handled the global incentive compatibility constraints at the initial contracting stage. These constraints turn out to be binding at the optimum. I have shown how the trade-off between efficiency and rent extraction is transposed in the contract space with sequential types. The form of the optimal contract depends crucially on the nature of the commitment assumption. Under full commitment, the optimal contract will often imply ex ante as well as ex post separation of types. Ex ante pooling will occur only if ex post efficiency in production can be ensured for all ex ante types. By contrast, under no commitment, the optimal contract will pool types ex ante and will be inefficient ex post. Two forms of commitment with renegotiation were analyzed. If the contract can be renegotiated once the ex ante type has been announced but prior to the agent learning his ex post type, then the renegotiation-proof contract will be an efficient contract that pools type ex ante. If the contract can only be renegotiated once the agent has learned his ex post type, ex ante type separation might still be optimal but the renegotiation constraints will be binding and that will reduce the set of feasible contracts. The ex ante inefficient agent will be offered a contract similar to the one of the non commitment case while the efficient agent will be offered an efficient contract.

I have applied my results to the problem of comparing ex ante *vs* ex post pricing in the telecommunication industry. Unlike some recent efforts in the pricing literature, the contracts I get are derived under the revelation principle for multistage games and are thus robust to any form of communication. I have also applied my analysis of the effect of commitment on sequential screening contracts to settle a seemingly paradoxical feature of two similar models of investment.

2.A Appendix

2.A.1 Non optimality of randomized schemes

In section 2.3, I state that we can restrict our attention to non randomized contract schemes. According to the revelation principle for multistage games, the general contractual process should be subject to the outcome of two lotteries, one for each stage, whose distributions depend on the history of the announcements made by the agent and the outcomes of the lotteries. Let L_1 and L_2 be the supports of these lotteries and λ_1 and λ_2 their outcome. A contract is then a set of four functions; two that map history and announcements into the spaces ΔL_1 and ΔL_2 of distributions onto L_1 and L_2, while the last two map into R. Formally, δ is represented by

$$\begin{aligned}
\delta_1 &: M \to \Delta L_1 \\
\delta_2 &: M \times L_1 \times \Theta \to \Delta L_2 \\
q_\delta &: M \times L_1 \times \Theta \times L_2 \to R \\
t_\delta &: M \times L_1 \times \Theta \times L_2 \to R
\end{aligned}$$

where $\delta_1(m)$ and $\delta_2(m, \lambda_1, \theta)$ are the lotteries.

Yet, I claim that the optimal contract won't be randomized. To see this, suppose δ is an optimal randomized self-selecting contract and let F be the joint distribution of m and θ ($dF(\theta, m) = p_m f_m(\theta)$). δ must then satisfy the following equations:

$$\begin{aligned}
\delta \in \operatorname*{argmax}_{\delta \in \Gamma} \int_{\Theta \times M} \int_L \int_L &(q_\delta(m, \lambda_1, \theta, \lambda_2) - c(q_\delta(m, \lambda_1, \theta, \lambda_2), \theta) \\
&- u_\delta(m, \lambda_1, \theta, \lambda_2)) d\delta_2(\lambda_2; m, \lambda_1, \theta) d\delta_1(\lambda_1; m) dF(\theta, m) \quad (2.25)
\end{aligned}$$

subject to

$$IC: \quad m \in \operatorname*{argmax}_{m' \in M} U(\delta, m, m') \quad \forall m \in M, \tag{2.26}$$

where

$$U(\delta, m, m') = \int_\Theta \int_L \int_L u_\delta(m', \lambda_1, \theta, \lambda_2) d\delta_2(\lambda_2; m', \lambda_1, \theta) d\delta_1(\lambda_1; m') f_m(\theta),$$

and

$$IR: \quad \max_{m' \in M} U(\delta, m, m') \geq \overline{u}(m) \quad \forall m \in M, \tag{2.27}$$

$$\begin{aligned} IC_2: \quad \theta \in \operatorname*{argmax}_{\theta' \in \Theta} \int_L (u_\delta(m, \lambda_1, \theta', \lambda_2) + c(q_\delta(m, \lambda_1, \theta', \lambda_2), \theta') \\ - c(q_\delta(m, \lambda_1, \theta', \lambda_2), \theta)) d\delta_2(\lambda_2; m, \lambda_1, \theta') \geq 0 \\ \forall m \in M, \quad \forall \lambda_1 \in L, \quad \forall \theta \in \Theta. \end{aligned} \tag{2.28}$$

Equation (2.25) states that δ is optimal, (2.26) and (2.28) are the ex ante and ex post incentive compatibility constraints and (2.27) is the participation constraint of the agent. Consider the non randomized contract $\overline{\delta}$

$$\begin{aligned} q_{\overline{\delta}}(m, \theta) &= \int_L \int_L q_\delta(m, \lambda_1, \theta, \lambda_2) d\delta_2(\lambda_2; m, \lambda_1, \theta) d\delta_1(\lambda_1; m) \\ u_{\overline{\delta}}(m, \theta) &= \int_L \int_L u_\delta(m, \lambda_1, \theta, \lambda_2) d\delta_2(\lambda_2; m, \lambda_1, \theta) d\delta_1(\lambda_1; m) \end{aligned}$$

that specifies the expected instruments of δ conditional on the agent's type. Contract $\overline{\delta}$ is strictly preferred, by the principal, to δ because of the strictly convex costs. By simple substitution, it is then straightforward to see that (2.26) and (2.27) are unaffected by switching to $\overline{\delta}$. Finally, if θ maximizes

(2.28) for all possible values of λ_1, it does so for any convex combination over L so that, once I integrate the maximand times $d\delta_1(\lambda_1; m)$ over L, θ still maximizes it. Simple substitution then shows that $\overline{\delta}$ satisfies the same requirements as δ for (2.28). Since any randomized contract is strictly dominated by a non randomized scheme, there is no loss of generality to restrict our attention to the latter.

Proof of proposition 2.1.
I have already shown the necessary part. To see that these two conditions are sufficient, I only need to find a q that complies with incentive compatibility. Consider q such that its elements q_i's solve

$$c(q_i, \theta_N) - c(q_i, \theta_i) = u_i - u_N \quad \forall i \in I.$$

Such a q unambiguously exists and is unique because of assumption (2.1). The incentive compatibility constraints (2.10) become

$$u_i - u_N \geq u_i - u_j \geq u_j - u_N \quad \forall i \in I \setminus \bar{\imath}, \forall j \in I_i.$$

The l.h.s. is trivially true because utility decreases with i while the r.h.s. is true because u satisfies internal consistency.

Hence U is completely defined by (2.9) and by the internal consistency constraints (2.11). These are linear weak inequalities so that U is closed and convex. Clearly, if u and u' satisfy these conditions, so is their sum $u + u'$ and αu, the product of u by a non negative scalar α. This makes U a convex cone. □

Proof of proposition 2.3.
Taking the conditional expectation on (2.12) yields

$$\mathrm{E}(u^*|m) = \mathrm{E}(b|m) + \overline{u} + s^*(m)$$

where $s^*(m)$ is defined like in equation (2.21) of proposition 2.6. It suffices then to choose b and $\overline{u}$ so that

$$E(b|m) + \overline{u} = E(u(m)|m) - s^*(m).$$

Likewise, any utility level of the principal $v(u(m), m)$ can be reached through the use of an efficient contract by setting

$$E(b|m) + \overline{u} = -v(u(m), m).$$

□

Proof of proposition 2.4.
Along the indifference curve $\Gamma_{\overline{v}}$, we have

$$v(u(m), m) = v(u^*, m)$$
$$s(u(m), m) - E(u(m)|m) = s(u^*, m) - E(u^*|m).$$

Hence,

$$E(u^*|m) - E(u(m)|m) = s(u^*, m) - s(u(m), m) \geq 0.$$

Suppose the equality stands and $u(m) \notin U^*$; then, by proposition 2.3, there is a $u^*(m) \in U^*$ such that $E(u(m)|m) = E(u^*(m)|m)$. Since $u^*(m)$ is efficient, it must be strictly preferred by the principal to $u(m)$, hence $v(u^*(m), m) > v(u(m), m) = v(u^*, m)$ but that implies $s(u^*(m), m) > s(u^*, m)$ which is impossible.

To prove the corrolary, let $\overline{v} = v(u^*(m), m)$ and suppose again that $u^*(m) \notin U^*$. Take $u^* \in \Gamma_{\overline{v}}$. By proposition 2.3, we must have $E(u^*(m)|m) < E(u^*|m)$ but that is impossible since $v(u^*(m), m) = v(u^*, m)$ and $s(u^*, m)$ is maximal. □

Proof of proposition 2.5.
Let the "(m)" notation aside. By proposition 2.4, we know that the principal

has linear indifference curves over U^* so that v is trivially quasi-concave on that portion of its domain. Let $q_1(u)$ and $q_2(u)$ be the solution of program (2.7). I want to prove that

$$v(u) = f(q_1(u) - c(q_1(u), \theta_1) - u_1) + (1-f)(q_2(u) - c(q_2(u), \theta_2) - u_2),$$

is strictly quasi-concave over U'.

By assumption (2.2), the rent function $c(q, \overline{\theta}) - c(q, \underline{\theta})$ is monotonously strictly increasing. Hence, it has a well defined inverse $\hat{q}$ which I apply on constraints (2.17) to get:

$$q_1(u) \geq \hat{q}(u_1 - u_2) \geq q_2(u).$$

Depending on the value of $\hat{q} = \hat{q}(u_1 - u_2)$, the solution of (2.7) can be of three types:

$$(q_1(u), q_2(u)) = \begin{cases} (\hat{q}, q_2^*) & \text{if } \hat{q} > q_1^* \quad (u \in U'') \\ (q_1^*, q_2^*) & \text{if } q_1^* \geq \hat{q} > q_2^* \quad (u \in U^*) \\ (q_1^*, \hat{q}) & \text{if } q_2^* \geq \hat{q} \quad (u \in U') \end{cases} \tag{2.29}$$

where the (q_1^*, q_2^*) stands for the unconstrained solution that maximizes social surplus.

Since $\hat{q}$ is a function of the difference $u_1 - u_2$ alone, I can write

$$\frac{\partial \hat{q}}{\partial u_1} = -\frac{\partial \hat{q}}{\partial u_2} = \hat{q}' > 0.$$

One can check that the second derivative,

$$\hat{q}'' = -(\hat{q}')^3 (c_{qq}(\hat{q}, \theta_2) - c_{qq}(\hat{q}, \theta_1)) < 0,$$

is negative using assumption (2.2) so that $\hat{q}$ is a strictly concave function. Strict quasi-concavity of v can be checked by verifying if its bordered determinant is positive,

$$|B| = \begin{vmatrix} 0 & v_1 & v_2 \\ v_1 & v_{11} & v_{12} \\ v_2 & v_{21} & v_{22} \end{vmatrix} > 0.$$

When $u \in U'$, we have

$$\begin{aligned} v_1 &= (1-f)(1-c_q(\hat{q},\theta_2))\hat{q}' - f \\ &= -v_2 - 1 \\ v_{11} &= -(1-f)c_{qq}(\hat{q},\theta_2)(\hat{q}')^2 - (v_1+f)\hat{q}'' \\ &= -v_{12} = -v_{21} = v_{22}. \end{aligned}$$

and $|B| = -v_{11}$. That value is positive since $v_1 + f = (1-f)(1-c_q(\hat{q},\theta_2))\hat{q}' > 0$ for $\hat{q} < q_2^*$, when $u \in U'$.

The proof of the corrolary is trivial if $u(m) \in U^*$. Suppose that $u(m) \in U'$ and, without loss of generality, let $u^* \in \overline{h}$ so that $u_\lambda(m) \in U'$. Since $u(m)$ and u^* imply the same expected payment, it must be that $v(u(m), m) < v(u^*, m)$ and since v is strictly quasi-concave on U', this implies that $v(u_\lambda(m), m) > v(u(m), m)$. □

Proof of corrolary 2.3.
By proposition 2.7, we know that $IR(\overline{m})$ is binding at the optimum and that $IC(\underline{m})$ and $IR(\underline{m})$ cannot be free at the same time. Obviously, $u(\underline{m})$ must lie on or below the hyperplane delimited by $IR(\overline{m})$ otherwise, choosing $u(\overline{m})$ on $IR(\overline{m})$ would be a dominated strategy for $\overline{m}$. Suppose that $IC(\underline{m})$ is binding, then the pooling contract such that $u^p = u(\overline{m})$ is optimal as well. If it is $IR(\underline{m})$ that binds, consider the pooling contract u^p where both $IR(\underline{m})$ and $IR(\overline{m})$ bind. That contract must be in U^* otherwise $IC(\underline{m})$ could not be satisfied in the first place. □

Proof of proposition 2.8.

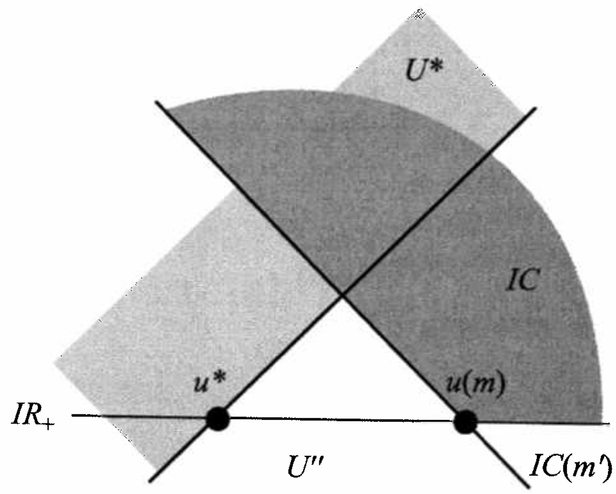

Figure 2.5: Proof of proposition 2.8.

Since both agents $\underline{m}$ and $\overline{m}$ have the same ex post reservation utility, both contracts $u(\underline{m})$ and $u(\overline{m})$ must lie on or above the same horizontal hyperplane IR_+. Obviously, at least one of these two contracts, say $u(m)$, must lie on IR_+ otherwise the transfers of both contracts could be reduced by a same amount, at the principal benefit.

Suppose now that, for some $m \in M$, $u(m)$ lies in U'' at the optimum; then the contract designed for $m' \neq m$ would be sought by maximizing $v(u(m'), m')$ subject to the incentive compatibility constraints and IR_+: $u(m')$ would be set in U^* but the resulting separating contract would be dominated by a pooling contract u^* in U^* (see figure 2.5). Now, any incentive compatible separating contract where $u(m)$ is not in U'' is weakly dominated by a pooling contract that offers $u(m)$ so that we can restrict our search for an optimum to pooling contracts that lies on IR_+. In that class, the optimal pooling contract (u^p, u^p) solves,

$$v(u^p, \tilde{m}) = \max_{u^p} \mathrm{E}(v(u^p, m)) \quad \text{s.t.} \quad (u^p, u^p) \in IR_+.$$

where the distribution associated to $\tilde{m}$ is $f(\tilde{m}) = pf(\underline{m}) + (1-p)f(\overline{m})$. Since

u_1^p is monotonic with respect to f from (2.24), it must be that u^p lies between contracts $\hat{u}(f(\underline{m}))$ and $\hat{u}(f(\overline{m}))$ on $I\!R_+$. □

Chapitre 3

Specific Investment, Commitment and Observability

3.1 Introduction

The problem of contracting under asymmetric information has been the object of considerable attention by economists for more than twenty-five years. In particular, adverse selection and moral hazard have been identified as prime sources of economic inefficiency. Adverse selection refers to situations where an economic agent has private information relevant to efficiency that he may wish to hide. Moral hazard refers to situation where efficiency requires that an economic agent take a particular action that can't be perfectly monitored. In a bilateral relationship where asymmetric information is one-sided, that is, there is a completely informed party (the agent) and an uninformed party (the principal), adverse selection refers to ex ante asymmetric information, prior to contracting taking place, while moral hazard can be interpreted as ex post asymmetric information.

When the knowledge of the agent characteristics and of his prior actions is relevant to devise ex post efficiency, both adverse selection and moral hazard can be seen as the same phenomena of asymmetric information under different contracting timing. Yet, while the nature of asymmetric information under moral hazard is conceived as the outcome of some decision process from the agent, it is usually assumed exogenous under adverse selection. Models of adverse selection are built using Bayesian game theory and rely on the notion of "type" to describe the informational advantage of the agent. As a consequence, any advantage the agent may get from his particular type is named an informational "rent" since it is not produced from anything else. That formulation is unsatisfactory in most cases studied in the literature for the type too often refers to characteristics that are obviously reproducible and marketable. Modeling asymmetric information as an adverse selection problem is but a partial analysis that tells us what happens once the types are fixed but fails to tell us where the types come from.

Traditional models of moral hazard have flaws of their own. In particular, they neglect the fact that, although the agent's moves cannot be perfectly monitored, this information is theoretically available from the agent himself. In fact, any contract whose ex post efficiency is hampered by a moral hazard problem should be corrected ex post, once the moral hazard problem has become an adverse selection problem.

With quasi-linear payoffs functions and when complete contracts are available, both problems can be solved using transfers. Achieving efficiency under adverse selection was resolved by the works of Groves (1973), d'Aspremont and Gérard-Varet (1979) and Rogerson (1992). Likewise, moral hazard can be eliminated[1] if arbitrarily high rewards (penalties) can be imposed in the event of a satisfactory (unsatisfactory) performance (Holmström 1979). Hence, if both problems subsist (as they undoubtedly do in many real-life situations), then it must be because such transfers are economically costly,

[1]By that, I mean that it is possible to induce the agent to take any desired action using transfers. Nevertheless, to the extent that the outcome of this action may depend on a random component and if the agent is risk averse, it will generally be impossible to achieve first-best efficiency with respect to insuring the agent's income. This does not occur in my model because I assume that the agent is risk neutral.

impossible to enforce or because contracts are incomplete.

In this paper, I built a model that addresses all the previous points under the assumption that contracts are complete but impossible to enforce in the long run. There is a principal that wishes that his agent invests in a specific costly technology in order to reduce the variable cost of producing some good. Unfortunately, the principal is limited to short-term contracts and cannot sign any binding contract prior the investment stage. Under perfect information, the agent would never invest because he would justifiably fear that the principal would refuse to pay for the investment cost at the contracting stage. But if the agent invests privately and if that piece of information is valuable to the principal, then it will be shown that the principal will be willing to yield informational rents to the agent, through a screening mechanism, and that these rents, in turn, will indirectly finance investment. In equilibrium, there will be a direct link between the terms of the ex post screening contract that will be offered by the principal and the cost of investment. Informational rents, so pervasive in the adverse selection literature, will be interpreted as a factor payment of investment.

My model can be associated to a classical principal-agent model *à la* Guesnerie and Laffont (1984) to which I add an initial period where the agent has the opportunity of choosing his "type", at a price (the cost of investment). In equilibrium, the agent randomizes on his investment support thus inducing a common-knowledge "type" distribution that is the basis of the subsequent play.

That problem can be cast as the classical hold-up problem. Consider a firm that must sink an investment to produce some specific good. The profitability of such venture depends on whether the firm expects or not to recover its sunk costs from future sales. Once the investment is sunk, the firm is exposed to the risk that its client reduces its demand or the price it is willing to pay. Even if the good is highly valuable, it is still advantageous for the client never to pay a price in excess of the lowest price at which the firm is ready to produce, that is, variable marginal cost. When the good and the investment is specific to a client and has little intrinsic value outside the relationship, the firm may be forced to lower its price to variable marginal cost. There is then less incentive for the firm to invest in the first place and the social benefits of investment may be lost.

This fundamental issue is known as the hold-up problem (or Williamson's problem) in the literature. There are three sets of circumstances when this problem may be totally solved; that of vertical integration, pure market or bargaining power and commitment. If the client and the firm vertically integrate, the issue of who shall absorb the investment costs becomes economically irrelevant. If the firm expects to have all the bargaining power in the future, it needs not to worry about prices. Finally, if the firm's clients or possibly some institutions like banks can commit themselves by signing binding contracts, then the firm can sell today its future production at a sure price, maybe contingent on the state of nature, that internalizes investment costs.

But vertical integration is often an unrealistic option. It creates problems of its own by substituting internal management of resources, which can be subject to costly moral hazard effects, for market transactions. And potential competition together with antitrust laws suggest that most firms cannot rely on pure market power to finance their investment. Hence, the general analysis of investment usually implicitly assumes that binding contracts are available.

Binding contracts overcome this problem by internalizing the costs of investment in the bargaining process. As it was shown by Rogerson (1992), efficient investment can be achieved under various assumptions about the information structure when parties can commit themselves ex ante, that is prior to the investment stage, using non-renegotiable binding contracts. Yet, in many cases, such contracts are unavailable. For example, with international business transactions, for instance, it may prove difficult or even impossible for a local firm to efficiently sue a foreign firm for a breach of contract.

Firms doing business with the government may reasonably be doubtful that the return they expect from some specific long-term project, will effectively be paid fully in the future, in all circumstances, because of the government's sensitivity to public opinion and its ability to change the law (we refer the reader to the regulation literature). In other cases, the client may not even be identified at the investment stage. Consider, for example, a firm developing a new product. Even when binding contracts are effective, enforcing them usually involves the judiciary system and that can have a very costly and unpredictable outcome. For instance, if the "specific" good involved is a common good of a "specific" quality that can be observed by the firm and its client but not by the courts. The firm would still be able to get a reduced price from the market but would lose the specific value added in quality.

In this paper, I relax these assumptions to emphasize that the privacy of the investment decision provides the firm with a sufficient strategic advantage for some investment to take place. Being "tough" with a supplier is an option for the client when he has a good knowledge of the firm's cost structure. Without that knowledge, the client runs the risk of making an unacceptable offer to the supplier that can lead to delays in production. An uninformed client then has a weaker bargaining position that can increase the supplier incentives to invest prior to bargaining.

This idea can be traced back to Tirole's (1986) analysis of the hold-up problem under asymmetric information. Tirole shows that Williamson's conjecture about suboptimal investment when parties have limited commitment capabilities holds quite generally (Williamson, 1983, 1985). He points out that the party who makes the ex ante investment does so in a suboptimal fashion because the ex post bargaining procedure used to redistribute the ex post gains from trade does not fully compensate the investing party for its ex ante costs. In particular, Tirole argues that this phenomenon may be sensitive to whether the investment decision is observable or not because the information structure usually affects the outcome of any given bargaining process. He illustrates in simple examples how mixed-strategy equilibria, with respect to the investment decision, can emerge in that context. Unobserved mixed strategies allow the supplier to "hide" its investment behind a veil of endogenously created noise, thereby reducing the asymmetrically uninformed client's bargaining power.

I expand on this idea in the following directions. First, I consider a much richer class of contracts: in Tirole's analysis, parties can only devise contracts with respect to the price of some fixed-size project whose (ex post) cost is a function of investment alone. This leaves little room for mechanism design as the trade-off between rent extraction and efficiency in production can only be carried out at the expense of a strictly positive probability of disagreement. Here, I allow for contracts that modulate quantities to be produce so that production and exchange always take place[2].

Secondly, I show that the optimal contract has a deceptively simple structure: it amounts to let the agent choose ex post any point on his long run average cost ($LRAC$) curve to the left of the socially optimal level, that is, some quantity of the good to be produced and an optimal investment level (capacity) matching that quantity, and to offer him a cost-plus contract that matches that level; that is to pay him the $LRAC$ unit price plus his (minimized) investment expenditures. Such a scheme serves two purposes: first, it guarantees, ex ante, that the agent will be paid fully for both his fixed and variable costs; second, it ensures that, ex post, the agent is cost-efficient with respect to its investment level. Nevertheless, to the extent that the agent does not internalize the principal preferences, production is generally socially inefficient with respect to the first best.

The rest of the paper is divided as follows. Section 3.2 presents the model with discrete investment. Section 3.3 solves the continuous investment case. The importance of the privacy of the investment decision is discussed in the

[2]An alternative formulation would specify a fixed-size project, variable costs with respect to an "effort" variable and contract modulation on realized (observable) costs. See Laffont and Tirole (1993).

conclusion.

3.2 The model

3.2.1 Environment

Consider a two-period relationship between two risk-neutral players. There is a firm (hereafter, the agent) that produces an output q which is only or more valuable to a single client (the principal). The principal values each unit of q at one.

To reduce production costs, the agent has the opportunity to invest in a specific technology. The cost of producing the output q is then $c(q, e)$ where $e \in E$ is the prior investment made by the agent in the specific technology and E is the relevant close set of possible investment levels. The cost of that specific investment e is $\psi(e)$. Hence $\psi(e)$ represents investment fixed costs and $c(q, e)$, variable costs. Producing nothing, $q = 0$, engages no variable costs, that is, $c(0, e) = 0$, $\forall e \in E$.

$c(\cdot, e)$ and $\psi(\cdot)$ are assumed to be strictly increasing convex functions, $c(\cdot, e)$ being strictly convex. To make sure that production is desirable, whatever the level of investment, the following hypothesis is assumed throughout:

$$c_q(0, e) < 1 \quad \forall e \in E, \tag{3.1}$$

where subscripts denote partial derivatives. Investment is assumed to reduce costs, $c_e < 0$, at a decreasing rate $c_{ee} > 0$. We will also need a natural sorting condition

$$c_{qe} < 0 \tag{3.2}$$

which says that investment decreases marginal cost at any given level of production (that is, investment increases capacity), and also two technical conditions

$$c_{qqe} < 0 \tag{3.3}$$

$$c_{qqq} \geq 0. \tag{3.4}$$

The first one ensures that second-order conditions for the principal's program are always satisfied while the second one allows us to disregard stochastic contracting schemes.

Investment e is costly reversible because of the specific nature of the assets involved. We simplify the analysis by assuming full irreversibility so that investment has a scrap value of zero and cannot be used to any other production. Hence $\psi(e)$ should be viewed as a sunk cost once e is realized. Nevertheless, the agent has the opportunity not to invest at all by choosing $\underline{e} = \min E$ which yields, by assumption,[3] $\psi(\underline{e}) = 0$.

The principal does not observe neither investment nor costs in our set-up but he is aware of the production frontier of the agent, that is, he knows ψ and c. I also assume that he cannot sign binding contracts prior to investment and that it is common knowledge that he holds all the bargaining power ex ante and at the interim stage. The problem is then to give the agent incentives to invest. More precisely, I want to show to what extent the unobservability of investment does provide such incentives.

3.2.2 The game

The course of events is as follows: in the first period, the principal and the agent "meet" and the agent observes the principal's preferences. The agent then privately invests in some specific technology. Once the investment has been sunk, the players enter the interim phase. The principal then makes a

[3]Remember that we are dealing with "opportunity cost". Hence, setting $e = \underline{e}$ may be interpreted as opting for a general purpose technology which is less "costly" in the sense that it can be used to produce other profitable goods. Setting $\psi(\underline{e}) = 0$ is then just a matter of normalization.

take-it-or-leave-it offer to the agent with respect to production. That offer takes the form of a binding contract that specifies a non-negative transfer t to be paid to the agent for a delivery of a non-negative amount q of the good.[4] The agent has then the choice either to accept or to refuse that contract. If he accepts the contract, then production and exchange take place in the second period. There is no other round of negotiation: if the agent refuses the first offer, then the relationship ends, the principal gets nothing and the agent gets stuck with an idle investment e.

3.2.3 Strategies and payoffs

A pure strategy for the principal is simply an offer of a binding contract δ prior to the production stage but after the investment has been sunk. By the Revelation principle, and assuming that δ satisfies incentive compatibility, such a contract δ can be thought as a pair of functions (q, t) that map the agent "type" e into a production plan $q(e)$ and a transfer $t(e)$ to be paid on delivery. Without loss of generality, I restrict the space of contracts to contracts that can, at least, break even with respect to variable costs, for some level of investment $e \in E$. Let Γ be the set of such possible contracts.

A pure strategy for the agent must specify an investment level e at the investment stage; a binary decision, to accept or not the contract, for every possible contract in Γ; and, conditionally on the acceptance of the contract, a level of production q. Obviously, the principal would never be tempted to offer a strictly positive transfer for a null amount of q; likewise, the agent will never accept to support strictly positive variable costs for a null transfer. Hence, we can assume that the provision $q = 0 \Rightarrow t = 0$ is part of any contract in Γ so that the strategies "refusing the contract" and "accepting the contract and producing nothing" yield the same payoffs for both players. A pure strategy for the agent can thus be reduced to an investment decision

[4] As usual, such a contract could be implemented as a non-linear pricing scheme that specifies an entry fee and a linear demand that the agent can freely satisfy.

e and a production decision q for every possible contract in Γ.

A behavioral strategy for the principal is a probability distribution $h \in H$ over Γ. A behavioral strategy for the agent is a probability distribution $f \in F$ over E and a map $g \in G$ from Γ into the set of probability distributions over the provisions of any contract that maximizes the agent ex post payoff. F, G and H simply stand for the sets of probability distributions over which the players must choose.

The principal's payoff at the end of the game is $q - t$, where t is specified according to the q chosen by the agent and the contract δ he himself offered. The agent's payoff is $t - c(q, e) - \psi(e)$.

3.2.4 The equilibrium

Contrary to a large body of the literature on principal-agent models, this is a game of *imperfect* information, not of *incomplete* information. At the beginning of the game, before the agent makes his investment, players have a perfect knowledge of each other characteristics. It is only after the agent invest that an informational asymmetry appears since the agent then knows his variable costs structure while the principal can only guess about the level of investment that was made.

Even if both players are symmetrically and completely informed at the beginning of the game, beliefs are important in this set-up. When choosing his investment level e, the agent does not know for sure what contract will be offered ex post. Likewise, the principal does not know e when he makes his offer. Furthermore, he does not know for sure if the agent will accept any contract he might propose. Both players must rely on their beliefs to make their move. Let $\tilde{f}$ and $\tilde{g}$ be the beliefs of the principal, that is, $\tilde{f}$ is a probability distribution over E and $\tilde{g}$ is defined like g. Likewise, let $\tilde{h}$ be the beliefs of the agent over Γ.

An equilibrium in behavioral strategy for this game requires that the players' beliefs are compatible with Bayes's law and that, conditional on these beliefs, their moves are optimal. Formally, given $\tilde{f}$, $\tilde{g}$ and $\tilde{h}$, we have an equilibrium if

$$\begin{aligned} h &\in \text{argmax}_H \; \mathrm{E}_{\tilde{f}\circ\tilde{g}}(q-t) \\ (f,g) &\in \text{argmax}_{F\times G} \; \mathrm{E}_{\tilde{h}\circ\tilde{g}}(t-c(q,e)-\psi(e)), \end{aligned}$$

where the subscripts next to the expectation refer to the probability distribution used for the expected value.

That definition is not problematic for the principal for he can only be in a single information state in this game. The issue is less trivial for the agent for he may be offered various contracts and can thus end up in various information states at the interim stage. This leaves the door open for uninteresting equilibria where the agent refuses contracts that satisfy individual rationality to equality. Consider the following family of equilibria: the agent expects a contract that does not allow the payment of sunk costs of investment and hence optimally chooses not to invest; the principal expects that the agent will refuse the contract anyway and so proposes such a contract that, if accepted, nevertheless extracts all the possible surplus from an inefficient agent; the agent then is indifferent ex post between producing or not for zero profits (the contract satisfies rational individuality) and does not accept the contract. To exclude that possibility, I assume that the agent accepts any contract that satisfies ex post individual rationality at the interim stage.[5]

Besides, randomized schemes, that is, contracts that bind players to the outcome of a lottery over standard contracts, will be excluded from the analysis. Laffont and Tirole (1993) showed that these contracts are dominated if assumption (3.4) holds.

[5]Fudenberg and Tirole (1991) make a similar hypothesis when discussing mechanism design (pages. 244-245).

3.2.5 Solution of the equilibrium

In what follows, I will assume that $E = \{\underline{e}, \overline{e}\}$ and I will show that there exists a single equilibrium in behavioral strategy where the agent accepts any contract that complies with individual rationality. An investment strategy for the agent will thus be a real number $f \in [0, 1]$, such that $\Pr(e = \overline{e}) = f$. The following assumption ensures that investment beyond $\underline{e}$ is desirable,

$$\hat{q}(\underline{e}) - c(\hat{q}(\underline{e}), \underline{e}) < \hat{q}(\overline{e}) - c(\hat{q}(\overline{e}), \overline{e}) - \psi(\overline{e}) \tag{3.5}$$

where, for $e \in E$, $\hat{q}(e)$ solves $c_q(\hat{q}(e), e) = 1$. Throughout this paper, I will use the caret notation to indicate efficient level values; hence $\hat{q}(e)$ is the efficient level of production that equates marginal benefit to marginal cost given investment level e and $\hat{q}(\hat{e})$ is the efficient level of production given efficient (first-best) level of investment $\hat{e}$ that maximizes social surplus.

By the Revelation principle, we can restrict ourselves, without any loss of generality, to contracts that are self-selecting in the sense that it is always optimal for the agent to reveal truthfully his investment level e and to be asked to produce $q(e)$ for a compensation $t(e)$ afterward.[6] Besides, meaningful contracts must satisfy rationality constraints in the sense that the agent gets at least as much utility in following the contract prescriptions, whatever the level of investment e he actually made, than in leaving the relationship for his reservation utility.

To grasp the problem of limited commitment capabilities, we first characterize the solution under full commitment prior the investment stage. In that context, the principal can easily achieve an efficient allocation that gives him all the surplus. That allocation is the triplet $\{\hat{e}, \hat{q}, \hat{t}\}$ that maximizes the

[6]The Revelation principle is usually stated in a Bayesian framework but its logic applies here as well.

principal's surplus given the ex ante rationality constraint for the agent:

$$\{\hat{e}, \hat{q}, \hat{t}\} = \underset{e,q,t}{\operatorname{argmax}}\, q - t \tag{3.6}$$

$$\text{subject to} \quad t - c(q,e) - \psi(e) \geq 0. \tag{3.7}$$

Under our assumptions, $\hat{e} = \overline{e}$. That allocation can be implemented by a whole family of contracts: each of these contracts requires efficient production $\hat{q}(\overline{e})$ and pays the minimal total costs to achieve that level if $\overline{e}$ is announced and imposes a non-negative penalty p otherwise. Such contracts satisfy the ex ante participation constraints (3.7) with equality and implementing any other allocation is clearly a weakly dominated strategy for the agent. Hence, the observability of investment is not an issue under full commitment, regardless of imperfect information.

When the principal cannot commit himself to stand by any arrangement made prior to the investment decision, the problem of underinvestment will naturally arise. Under commitment, the agent is willing to invest because the principal commits himself to pay him sufficient transfers so that (3.7) holds. If there is no commitment and the investment decision is observed, then the agent cannot expect to be left with any ex post gain. In the second period, both players will recognize that $\psi(e)$ is a sunk cost because of the specific nature of the investment. Hence, the principal knows that any contract that covers only variable costs will be accepted, so that q and t will only have to satisfy

$$t - c(q,e) \geq 0.$$

This implies an expected loss of $\psi(e)$ for the agent which he minimizes ex ante by inefficiently setting $e = \underline{e}$.[7]

[7] This result is robust (although at a smaller scale) even if the bargaining power is more evenly shared between both agents. The point is that the agent that invests only gets a fraction of the returns through the bargaining process.

Things are different if the investment level is not observed by the principal. Some investment may take place. The general idea is the following: the agent plays a mixed strategy for his investment decision. Ex post, the principal offers a contract that solves the adverse selection problem created by the random selection of e. Optimal self-selecting contracts offer the efficient (low-cost) agent informational rents ex post to induce him to reveal his "type". As long as these rents match investment costs, the agent will be indifferent ex ante between investing or not and will thus accept to randomize.

The strategy we use to solve the problem follows that of Fudenberg and Tirole (1990). First, I assume that the agent makes his investment decision according to some mixed (behavioral) strategy $f \in [0,1]$. Then I compute the associated optimal self-selecting contract that would be offered by the principal at the interim stage. Such a contract will be called *rational* with respect to f since it is a best response for the principal if indeed the agent played his investment decision according to f. Finally, once the behavior of the principal has been described, I compute f given the anticipated optimal reaction by the principal. This amounts to identify an equilibrium in behavioral strategy for that game.

I will say that a contract is *rational* with respect to f if it maximizes the expected utility of the principal given the incentive and rationality constraints at the interim stage. That is,

$$\delta \in \underset{q,t}{\operatorname{argmax}}(1-f)(q(\underline{e}) - t(\underline{e})) + f(q(\overline{e}) - t(\overline{e})) \tag{3.8}$$

among all feasible contracts that satisfy, $\forall e \in E$,

$$\begin{aligned} \mathrm{IIC}(e): &\quad t(e) - c(q(e), e) \geq t(\tilde{e}) - c(q(\tilde{e}), e) \quad \forall \tilde{e} \in E, \\ \mathrm{IR}(e): &\quad t(e) - c(q(e), e) \geq 0. \end{aligned} \tag{3.9}$$

IIC(e) are the Interim Incentive Compatibility constraints that ensure that the agent won't gain by lying about his type. IR(e) are the Interim

Rationality constraints: they take into account the agent's opportunity of not producing (even if investment was sunk) and thus minimizing his sales and variable costs to zero.

A rational contract is thus a contract that solves a standard principal-agent model with adverse-selection like in Guesnerie and Laffont (1984). The two following propositions proceed from their classical analysis.

Proposition 3.1. *If a contract δ is rational, then*

monotonicity: *q, t and the agent's ex post profits are rising with the investment level;*

active constraints: *either condition A or B holds:*

A *$IR(\bar{e})$ is binding in program (3.8);*

B *only $IIC(\bar{e})$ and $IR(\underline{e})$ are binding in program (3.8).*

Proof. See the appendix. □

Proposition 3.1 establishes that there can be two types of rational contracts: a type A-rational contract that leaves no ex post rent to the agent who has invested (at the expenses of shutting down the inefficient agent) and a type B-rational that exploits both agents' capacities of production, leaving an ex post rent to the efficient agent in the process. Since investment is costly, it is easy to see that the agent will never invest if he expects a A-rational contract to be offered. Such a contract would bring the principal a utility of zero if the agent had not invested (he would never accept it), hence he will never offer it. It follows that A-rational contracts cannot hold in equilibrium.

I thus restrict the analysis on rational B-rational contracts. I use the Kuhn-Tucker conditions associated with program (3.8) to completely characterize a rational contract.

Proposition 3.2. *A contract δ is B-rational for distribution f if and only if it satisfies the monotonicity requirements of proposition 3.1, conditions B, and the following equations:*

$$q(\overline{e}) = \hat{q}(\overline{e}) \tag{3.10}$$

$$(1-f)(1-c_q(q(\underline{e}),\underline{e})) - f r_q(q(\underline{e})) = 0 \tag{3.11}$$

where

$$r(q(\underline{e})) = c(q(\underline{e}),\underline{e}) - c(q(\underline{e}),\overline{e}) \tag{3.12}$$

is the rent left to the efficient type. If f admits a B-rational contract, then it is unique.

Proof. Here I substituted for $t(\underline{e})$ and $t(\overline{e})$ using IR($\underline{e}$) and IIC($\overline{e}$). The proof then amounts to verify that the Kuhn-Tucker conditions for program (3.8) are satisfied. The second order condition

$$\text{SOC:} \quad -(1-f)c_{qq}(q(\underline{e}),\underline{e}) - f r_{qq}(q(\underline{e})) \leq 0 \tag{3.13}$$

is always satisfied by assumption (3.3) and equation (3.12). Uniqueness of the contract comes from the fact that if (3.11) has a solution, then it has a strictly non-zero gradient with respect to f and $q(\underline{e})$; hence there is a bijection linking these variables. See equation (3.14) below. □

Note that equation (3.11) might not admit a solution in $q(\underline{e})$ if f is too high. This would only mean that the probability of facing an efficient agent is so high that it is better for the principal to shut down a potential inefficient agent by setting $q(\underline{e}) = 0$, that is, it is better to offer a A-rational contract.

The determination of a B-rational contract is illustrated in Figure 3.1. In the space $Q \times T$ where the figure is drawn, the indifference curves of the

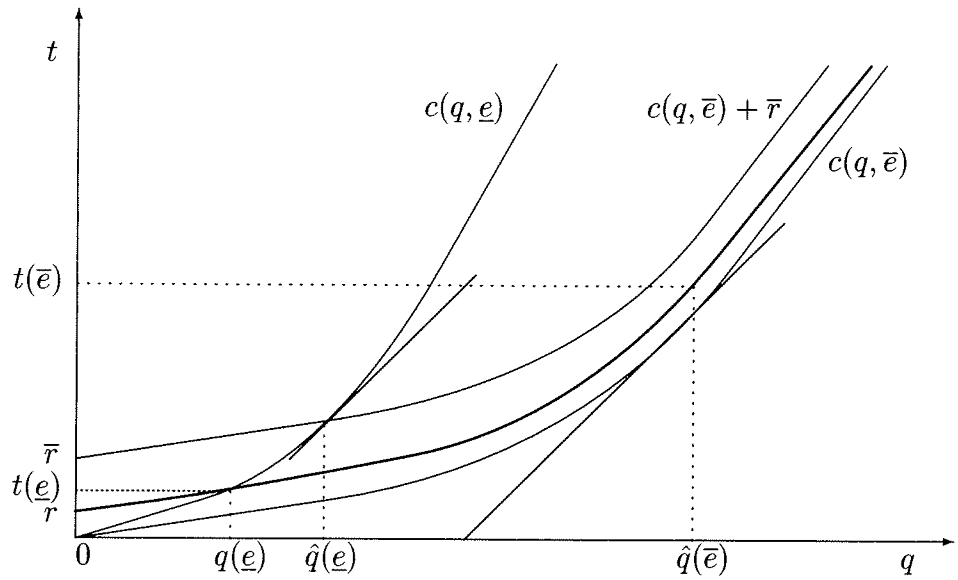

Figure 3.1: Ex-Post Contract.

principal are given by straight lines of slope 1 and are increasing in the south east direction. The profit gathered by an inefficient agent is $t - c(q, \underline{e})$. By proposition 3.1, we know that this profit is zero (IR($\underline{e}$) is binding). Hence, the inefficient agent will be offered any contract along the isoprofit curve $t - c(q, \underline{e}) = 0$. Since IIC($\bar{e}$) is binding, the (ex post) isoprofit curve for the efficient agent will be given[8] by $t - c(q, \bar{e}) - r = 0$. Since the efficient agent must be left with a non-negative rent, the isoprofit curves are the affine (parallel) curves above $c(q, \bar{e})$. Truthful revelation requires that, once the principal has fixed a plan $(q(\underline{e}), t(\underline{e}))$ for the inefficient type, the efficient agent must not be tempted to lie about his type. Hence the contract for the efficient agent

[8]By IIC($\bar{e}$) and IR($\underline{e}$) we have

$$\begin{aligned}\pi(\bar{e}) = t(\bar{e}) - c(q(\bar{e}) &= t(\underline{e}) - c(q(\underline{e}), \bar{e}) \\ &= c(q(\underline{e}), \underline{e}) - c(q(\underline{e}), \bar{e}) \\ &= r(q(\underline{e}))\end{aligned}$$

agent. That rent is thus a decreasing function of f for a B-rational contract:

$$\frac{\partial r(Q(f))}{\partial f} = r_q(Q(f))\frac{\partial Q}{\partial f}(f) < 0$$

where r_q is positive by the sorting condition (3.2) and is bounded within $[0, \overline{r}]$. As I show below, there will be an equilibrium rent r that will finance the opportunity cost $\psi(\overline{e})$ of investment as long as $\psi(\overline{e}) < \overline{r}$.

I now turn to the choice of an optimal investment strategy for the agent. Since no A-rational contracts could be sustained in equilibrium, the agent will expect some B-rational contract δ for some belief f the principal might have about the probability that investment took place. A contract will be said *consistent* for belief f if f is indeed an optimal randomization from the agent's point of view, that is, δ is *consistent* with belief f if it is B-rational for f and is such that

$$f \in \underset{\tilde{f}}{\operatorname{argmax}}\, \tilde{f}(r(q(\underline{e})) - \psi(\overline{e})). \tag{3.16}$$

This yields a "bang-bang" type of behavior. The agent will invest only if he expects that the rents from investment will match investment costs.

An equilibrium in behavioral strategy for this game is simply a couple (f, δ) such that δ is a consistent B-rational contract for distribution f. I now show that this equilibrium is unique.

Proposition 3.3. *There is a single equilibrium. If $\psi(\overline{e}) \geq \overline{r}$, it involves no investment from the agent ($f = 0$) and a pooling contract $\{\hat{q}(\underline{e}), c(\hat{q}(\underline{e}), \underline{e})\}_{e \in E}$. If $\psi(\overline{e}) < \overline{r}$, then the equilibrium (f, δ) is such that $f > 0$ solves*

$$r(q(\underline{e})) = r(Q(f)) = \psi(\overline{e}) \tag{3.17}$$

and δ is the unique B-rational contract associated to f.

must lie above or on the isoprofit thick curve that crosses $(q(\underline{e}), t(\underline{e}))$. This contract will optimally lie on the curve at $\hat{q}(\overline{e})$ where marginal cost equals marginal utility and where transfers are minimized. Among all such candidate solutions, which can be indexed by $q(\underline{e})$, the optimal one will trade off, in equation (3.11), the requirements for efficient production for the inefficient agent, as measured by the expected difference between marginal utility and marginal cost, $1 - c_q(q(\underline{e}), \underline{e})$, and the extra marginal rent $r_q(q(\underline{e}))$ that must be left to the efficient agent each time the production required of an inefficient agent is increased. Equivalently, this optimal quantity $q(\underline{e})$ can be expressed as a function $Q(f)$ of f that solves equation (3.11). Differentiating (3.11) for a B-rational contract, $Q(f)$ can be shown to be a decreasing function of f:

$$\frac{\partial Q}{\partial f}(f) = \frac{1 - c_q(Q(f), \overline{e})}{\text{SOC}} < 0, \tag{3.14}$$

where SOC is the negative second-order condition for program (3.8). The numerator is positive because marginal cost increases with q, reaching 1 only at $q(\overline{e}) \geq Q(f)$ for a B-rational contract. Inverting Q, I get

$$f = \frac{1 - c_q(q(\underline{e}), \underline{e})}{1 - c_q(q(\underline{e}), \overline{e})}. \tag{3.15}$$

Hence, not B-rational contract would stand for $f = 1$. Given (3.14) and condition (3.3), one can show that the ratio on the r.h.s. of (3.15) decreases monotonously as $q(\underline{e})$ is raised, that is, as we slide upward $c(q, \overline{e}) + r$ by raising r and moving along $c(q, \underline{e})$. It reaches zero when $c_q(q(\underline{e})) = 1$, that is when $q(\underline{e}) = \hat{q}(\underline{e})$ and $f = 0$. Hence, if there is no chance that the principal is facing a low cost agent, he will simply ask an efficient production plan from the almost surely high cost agent. That point yields a bound $\overline{r} = r(\hat{q}(\underline{e})) = r(Q(0))$ on the maximum rent that can be left to an efficient

Proof. See the Appendix. □

Thus, when the opportunity costs of investing is not too high, f is strictly positive and average investment exceeds $\underline{e}$ by $f(\overline{e} - \underline{e}) > 0$, where it would have been $\underline{e}$ if it had been common knowledge that the principal could observe investment. Unobservability of investment increases the total surplus because it allows the agent to secure some returns from his investment by endogenously creating random noise that modifies the ex post bargaining process.

Rents match investment costs in equilibrium. This implies that the agent can expect to be paid both his variable and fixed costs when making his investment decision. Inefficiency with respect to the the first best solution arises because $q(\underline{e}) < \hat{q}(\underline{e})$ and $f \neq 1$. Besides, the optimal contract is directly related to investment costs through equation (3.17).

3.2.6 Purification of the equilibrium

The idea of a mixed strategy with respect to investment may sound strange at first. When the time comes to proceed with some investment, businessmen do not usually base their decision on a roll of dice. Nevertheless, business decisions are frequently made at the margin and the mixed strategy formalization can be shown to be nothing more than the result of such decision. To see that, we follow Fudenberg and Tirole (1990) in invoking the Harsanyi's concept of equilibrium *purification.* This involves embedding the model in a somewhat enlarged model where the agent's ex ante preferences are "fuzzy" instead of being common knowledge. This added incomplete information will have negligible effects on the equilibrium contract and expected level of investment but it implies that any agent does indeed follow a pure strategy, investing or not, which is determined, at the margin, by small idiosyncratic differences. As I make that incompleteness disappear, I obtain at the limit the mixed strategy equilibrium of proposition 3.3.

More formally, consider a family of incomplete information models indexed by n where, for each n, the agent's investment costs for $\overline{e}$ are $\psi(\overline{e}) + \epsilon$ where ϵ is a random variable with convex support and with a strictly increasing absolutely continuous distribution function G^n which is common knowledge. Let T^n be the inverse function of G^n. Assume that, as n increases, the family of distributions G^n converges to the point zero. That is, $\lim_{n\to\infty}\{G^n(\epsilon) - G^n(-\epsilon)\} = 1$ for all ϵ. Likewise, this implies that for all bounded sequence $\{f^n\}$ with $f^n \in [0,1]$ for all n, $\lim_{n\to\infty} T^n(f^n) = 0$. The optimal contract to be offered by the principal was shown to be a function of his prior about the likelihood that the agent invested and is obviously independent of the realization of ϵ when contracting take place. Let f^n be the equilibrium prior to model n. I have already shown that optimal contracts could be indexed by $q(\underline{e})$. Hence, the agent will expect a contract $Q(f^n)$ that will yield a rent $r(Q(f^n))$ to an efficient agent. The agent will thus invest in pure strategy if he has low investment costs

$$\epsilon \leq r(Q(f^n)) - \psi(\overline{e})$$

and will not invest otherwise. The unconditional probability that an agent invests is thus, in conformity with the equilibrium prior, given by

$$G^n(r(Q(f^n)) - \psi(\overline{e})) = f^n.$$

Invert the r.h.s to get

$$r(Q(f^n)) - \psi(\overline{e}) = T^n(f^n).$$

Taking the limit on both sides yields

$$\lim_{n\to\infty}\{r(Q(f^n)) - \psi(\overline{e})\} = 0;$$

that is

$$\lim_{n\to\infty} f^n = f$$

where f is the equilibrium mixed strategy of equation (3.17).

3.3 Continuous investment

Up to now, the agent could only choose between two discrete investment levels, $\underline{e}$ and $\overline{e}$. Does allowing for a convex (pure) strategy space would change the result? Although non-convexity usually create a need for randomization in a Nash equilibrium, this is not the source of randomization here. In this section, I assume that the investment set is convex by setting $E = \{e | e \geq \underline{e}\}$. A contract, as before, will be a pair of functions $\{q(e), t(e)\}$ that map the investment set into actual production and transfers.

Define the optimal social surplus, conditional on e, as

$$S(e) = \hat{q}(e) - c(\hat{q}(e), e) - \psi(e).$$

I assume that $S(e)$ is positive in $e = \underline{e}$ and has a single zero $\overline{e}$ strictly to the right of $\underline{e}$. This implies that the set $\text{argmax}_e\, S(e)$ is a bounded singleton and that social surplus eventually becomes strictly negative if e is high enough (see figure 3.2). All that is required to satisfy this condition is that ψ rises sufficiently fast.[9]

Next I argue that I can restrict myself to self-selecting contracts that set $q(e) - t(e) \geq 0$, $\forall e \in E$. To see that, note that there is no loss of generality in considering only self-selecting contracts because of the Revelation principle. Then, among these contracts, any contract that sets $q(e) - t(e) < 0$ for some e is weakly dominated, from the principal's point of view, by the contract,

[9] This is a sufficient condition for my proof to work. It is, by no mean, necessary.

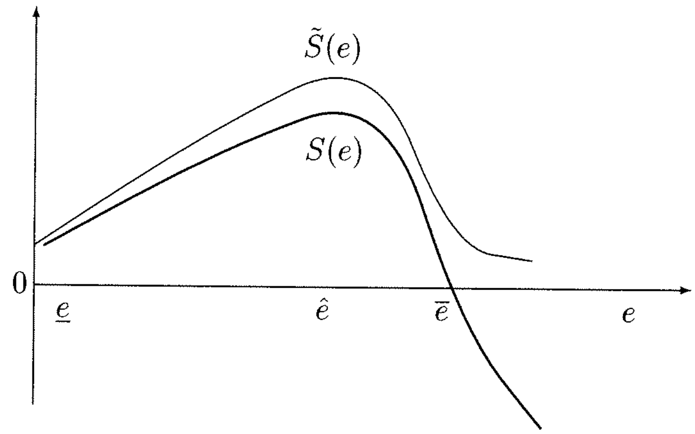

Figure 3.2: Social Surplus.
Social surplus should be positive in $\underline{e}$ *and have a single zero at* $\overline{e}$. *Here* $S(e)$ *would be acceptable but not* $\tilde{S}(e)$ *because it damps too slowly to cross the e-axis.*

otherwise identical, that shuts down these types ($q(e) = 0$). That contract may not be self-selecting but there is a self-selecting contract that duplicates its outcome and yields the same utility for the principal, again because of the Revelation principle. Hence, there is no loss of generality in restricting our attention to self-selecting contracts that set the utility of the principal to a non-negative value for all e.

Since the optimal social surplus is negative if $e > \overline{e}$, so is the realized social surplus, and since the principal's surplus from any optimal contract is bounded at zero for any e, we can say that the agent will never invest more than $\overline{e}$. For then on, I thus restrict the pure strategy space to $E^* = [\underline{e}, \overline{e}]$.

Here, I take a somewhat less direct approach than in the preceding section. First, I restrict the strategy space of the agent to strategies that can be represented as $\mathcal{C}^1$ functions. As it stands, the analysis does not preclude the existence of more sophisticated equilibria.[10] Secondly, I assume that

[10]Fudenberg and Tirole (1990) took a more direct and general approach when computing a Perfect Bayesian Equilibrium with convex strategies sets. In the end, they end up with an equilibrium in $\mathcal{C}^1$ functions.

the randomization that the agent uses is such that we ought to look for an interior maximum when computing the principal's response. I compute the equilibrium and check at the end if this assumption about the agent's strategy is satisfied.

Assumption 3.1. *A strategy for the agent is an absolutely continuous distribution F over E^* with $F(\underline{e}) = 1 - F(\overline{e}) = 0$. F is assumed to be such that $q(e)$ is strictly increasing at the maximum.*

I first show that the agent never invests more than the socially optimal level of investment in equilibrium.

Proposition 3.4. *In equilibrium, the agent will randomize over the domain $\hat{E} = [\underline{e}, \hat{e}]$.*

Proof. See the appendix. □

Then, I can easily compute the principal's optimal reaction for any prior belief about F.

Proposition 3.5. *Assume that the principal expects the agent to randomize with F over $\hat{E}$ where F satisfies Assumption 3.1. He will then offer a menu of contracts $\delta = \{q(e), t(e)\}_{e \in \hat{E}}$ such that for all $e \in \hat{E}$, $q(e)$ solves*

$$f(e)(1 - c_q(q(e), e)) + (1 - F(e))c_{qe}(q(e), e) = 0 \tag{3.18}$$

and transfers are given by

$$t(e) = c(q(e), e) - \int_{\underline{e}}^{e} c_e(q(\tilde{e}), \tilde{e}) d\tilde{e} \tag{3.19}$$

Proof. See the Appendix. □

This menu of contracts bears the same interpretation as in proposition 3.2. In equation (3.18), the principal equalizes the marginal surplus of having type e producing marginally more

$$f(e)(1 - c_q(q(e), e))$$

with the marginal increase in rents $-c_{qe}(q(e), e)$ that have to be left to more efficient types (in proportion $1 - F(e)$). Under assumption 3.1, the solution to (3.18) implies that production plans and rents increase with e so that the second-order conditions for the principal's program are satisfied.

As in the discrete case, an equilibrium solution is sought by looking in the class of contracts $\{q(e), t(e)\}_{e \in \hat{E}}$ that satisfy (3.18). The equilibrium optimal contract should leave the agent indifferent between all levels of investment so that he will be willing to randomize with F. Rents must match the investment cost in equilibrium. Once this is done, it turns out that I am left with an ordinary linear differential equation in F whose unique solution is given in the next proposition.

Proposition 3.6. *In equilibrium, the agent will play a random strategy F over $\hat{E}$ such that*

$$F(e) = \begin{cases} \frac{1}{x(e)} \int_{\underline{e}}^{e} x(\tilde{e}) a(\tilde{e}) d\tilde{e} & \text{if } e < \hat{e} \\ 1 & \text{if } e = \hat{e} \end{cases} \tag{3.20}$$

where

$$a(e) = \frac{\dot{\psi}(e)}{1 - c_q(q(e), e)} \tag{3.21}$$

$$x(e) = \exp\left(\int_{\underline{e}}^{e} a(\tilde{e}) d\tilde{e}\right) \tag{3.22}$$

and $q(e)$ solves

$$0 = \dot{\psi}(e) + c_e(q(e), e) \tag{3.23}$$

Proof. See the Appendix. □

In the proof, (3.23) is used to show that $q(e)$ is strictly increasing in e so that our first-order approach is valid.

The equilibrium contract has a surprisingly simple and very intuitive structure which is described in the following proposition.

Proposition 3.7. *The equilibrium contract δ is a menu of cost-plus contracts; that is, it amounts to let the agent choose any point to the left of $\hat{q}(\hat{e})$ on his long-run average cost (*LRAC*) curve and to pay him the minimum total costs to achieve that level of production.*

Proof. Consider equation (3.23) that defines the optimal production plans, that is, the function $q(e)$. This equation is the first-order condition of program

$$\min_{e \in \hat{E}} \psi(e) + c(q, e)$$

which define points on the $LRAC$ curve. With respect to transfers, using (3.19) and (3.23), I get

$$t(e) = c(q(e), e) + \psi(e). \tag{3.24}$$

That amounts to exactly compensate the agent for both its variable and fixed costs which are minimized for $q(e)$ at e on the $LRAC$ curve. □

By (3.24), we see that if investment is kept private, no hold-up will occur and the firm can invest with confidence. The complex part of the contract lays in the buying policy $q(e)$ that prescribes lowering the principal's demand from a poorly capitalized firm ($e < \hat{e}$). It is given by (3.23) and manages to equate ex ante marginal cost of investment and ex post marginal savings on variable costs. These $q(e)$'s are nothing more than the points to the left of $\hat{q}(\hat{e})$ on the $LRAC$ curve. That is, for any $q(e)$, e is the optimal capacity that minimizes costs to produce $q(e)$. Hence, the optimal self-selecting contract can be implemented ex post as follows: let the agent choose any level $q \in [q(\underline{e}), \hat{q}(\hat{e})]$ and offer him a cost-plus contract that pays only the minimum costs, both fixed and variable, necessary to produce q. That contract trivially satisfies the ex ante and the ex post rationality constraints since it pays both fixed and variable costs. The fact that the contract lies on the $LRAC$ curve implies that it is ex post cost efficient although it is not socially efficient, as it does not lie on the long run marginal cost curve, because the principal would have preferred more investment ex ante and more production ex post once e is sunk.

In figure 3.3, I drew the traditional envelope representation of short-run average costs curves. Under full commitment, the optimal contract would equate marginal utility of the principal to the long-run marginal cost at $\hat{q}(\hat{e})$. That point is both ex ante and ex post efficient. In the absence of commitment and under full observability of the investment decision, the agent would simply refuse to invest more than $\underline{e}$; hence the solution would be ex ante inefficient. Efficient bargaining would yield an ex post efficient production level at $\hat{q}(\underline{e})$ and the loss of welfare associated to the absence of commitment capabilities could be measured by the distance $\hat{q}(\hat{e}) - \hat{q}(\underline{e})$. If the investment decision is kept private, then the equilibrium contract will allow the agent to choose any production level in $[q(\underline{e}), \hat{q}(\hat{e})]$, say $q(e)$, and will pay him the minimum total costs to achieve that level. It is easy to see that this contract is self-selecting; without loss of generality, consider potential deviation from an ex ante point of view: if the agent expects to invest e, then the contract requires that $q(e)$ should lie on the $LRAC$ curve. He will never choose to produce to the left of $q(e)$ because he would then receive lesser payments in fixed costs and he would have higher than minimum variable costs to produce at $q < q(e)$. Likewise, producing $q(e') > q(e)$ would imply higher actual variable costs than the minimum allowed and if he could expect a profit from higher fixed costs reimbursement, $\psi(e') > \psi(e)$, than this would imply

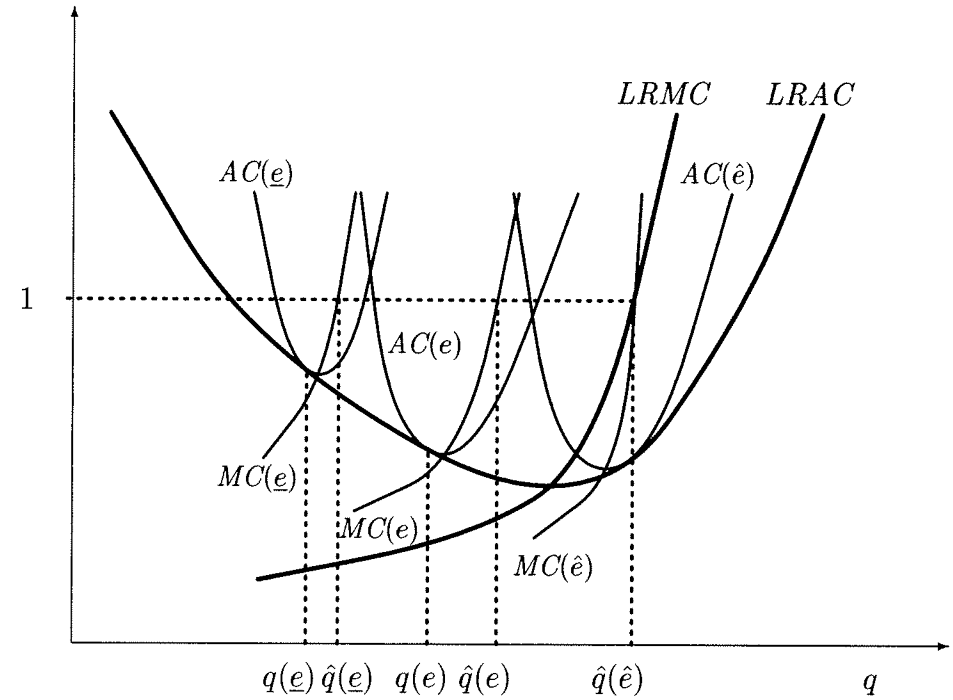

Figure 3.3: Optimal Continuous Equilibrium Contract.
Since the contract pays total costs, it trivially satisfies the rationality constraints. Since it lies on the LRAC *curve, it is self-selecting.*

that e is a better cost-efficient way than e' to produce $q(e')$, a contradiction with $q(e)$ being on the $LRAC$ curve in the first place. Keeping investment private and producing at any $q(e)$ is both ex ante inefficient with respect to $\hat{q}(\hat{e})$ and ex post inefficient with respect to $\hat{q}(e)$ but, whenever $q(e) > \hat{q}(\underline{e})$, it does increase social welfare.

3.3.1 Analytical example

I provide here a simple example that has an analytical solution. Suppose that

$$c(q,e) = \exp(q - \hat{q} - e + \hat{e})$$

and that the opportunity cost of investment is given by

$$\psi(e) = \frac{\exp(e) - \exp(\underline{e})}{\exp(\hat{e})}$$

where $e \geq \underline{e}$. One can check that under this specification, the parameters $\hat{q}$ and $\hat{e}$ turn out to be the efficient levels of production and investment. Hence I expect the agent to randomize over $[\underline{e}, \hat{e}]$. I normalize the q units so that $\hat{q} - 2(\hat{e} - \underline{e}) \geq 0$. In equilibrium, the contractual quantities can be computed from (3.42), hence

$$q(e) = \hat{q} - 2(\hat{e} - e),$$

which implies that q increases with investment, therefore validating the first-order approach. Using (3.21), I have

$$\begin{aligned} a(e) &= \frac{\exp(e)}{\exp(\hat{e}) - \exp(e)} \\ &= \frac{\exp(e - \hat{e})}{1 - \exp(e - \hat{e})} \end{aligned}$$

Integrating and taking the exponential yields

$$x(e) = \exp\left(\int_{\underline{e}}^{e} a(\tilde{e})d\tilde{e}\right) \tag{3.25}$$

$$= \exp(\left(\int_{\underline{e}}^{e} \frac{\exp(\tilde{e}-\hat{e})}{1-\exp(\tilde{e}-\hat{e})}d\tilde{e}\right) \tag{3.26}$$

$$= \exp\left([-\ln(1-\exp(\tilde{e}-\hat{e}))]_{\underline{e}}^{e}\right) \tag{3.27}$$

$$= \frac{1-\exp(\underline{e}-\hat{e})}{1-\exp(e-\hat{e})}. \tag{3.28}$$

Both $a(e)$ and $x(e)$ are well defined functions for $e < \hat{e}$. Using (3.20) I get[11]

$$F(e) = \frac{\exp(e)-1}{\exp(\hat{e})-1}$$

$$f(e) = \frac{\exp(e)}{\exp(\hat{e})-1}.$$

3.4 Discussion

Note that the contract I obtain is robust to ex post mutually beneficiary renegotiation if the agent reveals his investment e by the very act of producing the irreversible quantity $q(e)$.[12] We saw that preserving the privacy of the agent's investment decision gives him incentives to proceed with strictly

[11] Although increasing investment costs are sufficient to obtain an interior solution to the principal's problem, note that, in this particular case, the endogenous hazard rate is monotonic:

$$\frac{\partial(1-F)/f}{\partial e}(e) = -\exp(\hat{e}-e) < 0.$$

Another case where this is so is when the variable cost function is log separable in q and e:

$$c(q,e) = \phi(q)/\gamma(e)$$

where γ is log-concave and the investment cost function ψ is linear.

[12] See Beaudry and Poitevin (1995) for a discussion of that point.

positive (expected) investment. It is straightforward then to show that going from full observability to unobservability increases welfare. Suppose that the principal observes the investment sunk by the firm. As it was argued before, this leads to a minimal level of investment and the firm gets no rent, neither ex ante nor ex post. The principal would then offer the contract $\underline{\delta}$ of proposition 3.1 which was shown to be dominated by rational contracts.[13] Hence, if the principal can commit himself either to observe investment or not, he should do the latter. Thus, by modifying the information structure ex ante so that it affects the ex post bargaining solution, the principal can actually mitigate his commitment problem and improve his utility.[14]

I wish to emphasize that when the investment set is convex, unobservability matters only to the extent that the principal wishes and is able to extract information about costs, related to the investment decision, through some form of mechanism design. To see that, consider the case where the principal is legally constrained to behave as a non-discriminating monopsony in the model of section 3.3. He would then compute a single buying policy $\{q, t\}$ that would maximize his expected utility given any prior F on the likelihood of investment. Expecting to be able to sell only q for t, the agent would simply try to minimize his costs by choosing e in

$$\operatorname*{argmin}_{e} c(q, e) + \psi(e)$$

which is a singleton. Hence, the equilibrium would necessarily be in pure strategies. Likewise, consider the model of Laffont and Tirole (1993). These authors show that the unobservability of the firm's ex post variable costs can induce it to invest if the investment is negatively correlated with ex post variable costs. To solve the ex post information extraction problem, the

[13]See the proof in the Appendix.

[14]See Crémer (1995) for a similar result where the principal optimally chooses a poor monitoring technology to strengthen the agent's incentives. In a recent paper, Barros (1997) applies that result to an oligopolistic market.

client must abandon ex post rents to the firm. Since these rents decrease with variable costs, which are assumed negatively correlated with investment, it is possible for the firm to finance its investment through these rents. In their model, the investment e determines the likelihood $F(\theta|e)$ (strictly convex in e), over a fixed support Θ of having a variable costs parameter (that plays a similar role as e in my model) less or equal to θ. If the contract is signed ex post, that is once θ is drawn, then the information about e is irrelevant for the principal. If the contract is signed ex ante, then the principal would wish to separate investment "types" using the fact that high types e should be more willing than low types to accept contracts that call for high penalties if costs are high since they know that low variable costs are more likely. But their model proscribes such penalties by imposing ex post participation constraints. As it turns out, for any prior G about the likelihood of investment, the optimal contract in that set-up can be shown to separate ex post the θ's but not the investment types. Assume that $\pi(\theta)$ is the expected ex post profit the agent can gather from that contract when θ is drawn; he will thus choose e in

$$\underset{e}{\operatorname{argmax}} \int_{\Theta} \pi(\theta) dF(\theta|e) - \psi(e) \equiv \underset{e}{\operatorname{argmax}}\, \pi(\overline{\theta}) - \int_{\Theta} \dot{\pi}(\theta) F(\theta|e) d\theta - \psi(e)$$

which involves a strictly concave program in e and hence yields also a singleton. There again, observability is irrelevant in equilibrium.[15]

Imposing ex post participation constraints implies that the principal always pools investment "types".[16] The agent then expects a single complex contract, that is, a separating contract with respect to ex post variable costs but not with respect to investment, which usually maps into a single global maximum response strategy in terms of investment. Hence, the resulting equilibrium is in pure strategies and the observability of investment becomes

[15] Mixed strategies equilibria are common when the agent's strategies set is not convex. In that context, unobservability will again increase welfare.

[16] See González (1997a); chapter 2 of this thesis.

innocuous. Here, I do not impose ex post individual rationality constraints: this is sensible given the assumption that commitment is possible at the interim stage.

There is no exogenous uncertainty in our basic model formulation. Adding exogenous noise like in Laffont and Tirole's model would not change qualitatively the results *as long as the model is such that the principal still wishes to discriminate among investment "types".*[17]

There have been other approaches to the issue of observability of the investment decision in the hold up problem. Besanko and Spulber (1992), for instance, use a principal-agent model with signaling features to analyze the investment problem. In that paper, the principal is able to commit himself to a rate of return regulation which gives sufficient incentives for the agent to invest through an Averch-and-Johnson effect. Investment is observed in their model and is used as a signal that reveals, in equilibrium, an efficiency parameter unobserved by the principal. Besanko and Spulber emphasize that the mechanism-design approach (like the one I use in this paper) does not provide any incentives when investment is observed. Yet, as our paper shows, this is not the case when investment is not observed.

Whether we should assume that investment is observable or not depends pretty much on the problem at hand. Assuming that investment is observable and verifiable, as Besanko and Spulber (1992) do, is a sensible way to go in the context of natural monopolies, like public utilities, where specific investment take the form of huge infrastructures (e.g. building a dam). But whenever the specific investment is easily concealable, like R&D, or of a "softer" nature, like the negotiation of access to crucial human resources, then pure unobservability is more realistic.

My results apply generally to Bayesian principal-agent models with adverse selection. In these models, transfers are decomposed in "costs" that

[17]This might involve random cost structures with moving supports that are functions of investment and/or relaxing ex post participation constraints so that the principal will use actuarial risk to discriminate agents with respect to their investment decision.

retribute factors of production and "informational rents" that are left to the agent to make him reveal his type. Here, I challenge that interpretation by hypothesizing that, in most cases, the agent's "type" is the result of an ex-ante maximizing choice where the agent optimally ''invest" by choosing an appropriate "type". Hence, we should not talk of a "rent" but of the retribution of a factor of production, that is, of the investment. This interpretation is important, not only for the sake of economic literacy but because my approach yields contract designs that are robust to the endogenous formation of types. Bayesian principal-agent models with adverse-selection yield, at best, very short-term predictions under the problematic assumptions that the distribution of types is common knowledge and unaffected by the agent behavior. My imperfect information formulation is robust to ex ante investment by the agent and requires only the cost function of investment to be common knowledge.

3.A Proofs

Proof of proposition 3.1.
Consider first the contract $\underline{\delta}$ such that

$$q(\underline{e}) = q(\overline{e}) = \hat{q}(\underline{e})$$
$$t(\underline{e}) = t(\overline{e}) = c(\hat{q}(\underline{e}), \underline{e})$$

such that $\hat{q}(\underline{e})$ solves

$$c_q(\hat{q}(\underline{e}), \underline{e}) = 1$$

Such a contract trivially satisfies IR(e) and IIC(e), $\forall e \in E$, and gives the principal a utility of

$$\hat{q}(\underline{e}) - c(\hat{q}(\underline{e}), \underline{e}) \tag{3.29}$$

Since marginal cost is monotonously increasing, assumption (3.1) yields

$$c_q(q, \underline{e}) < 1 \quad \forall q < \hat{q}(\underline{e})$$

integrating on q yields

$$\int_0^{\hat{q}(\underline{e})} c_q(q, \underline{e}) dq < \int_0^{\hat{q}(\underline{e})} q dq$$
$$c(\hat{q}(\underline{e}), \underline{e}) < \hat{q}(\underline{e})$$

so that (3.29) is strictly positive. Since a solution to program (3.8) can't yield no less, I conclude that a rational contract must yield a strictly positive utility to the principal. Take δ_0 to be the "zero" contract $\{(0,0),(0,0)\}$; that contract yield zero utility to the principal hence it is not rational.

Summing IIC($\underline{e}$) and IIC($\overline{e}$), one gets

$$\big(c(q(\overline{e}), \overline{e}) - c(q(\underline{e}), \overline{e})\big) - \big(c(q(\overline{e}), \underline{e}) - c(q(\underline{e}), \underline{e})\big) \leq 0 \tag{3.30}$$

hence

$$\int_{\underline{e}}^{\overline{e}} \int_{q(\underline{e})}^{q(\overline{e})} c_{qe}(q, e) dq de \leq 0 \tag{3.31}$$

which implies $q(\overline{e}) \geq q(\underline{e})$ by the sorting condition (3.2). Reporting that result in IIC($\overline{e}$), we immediately get

$$t(\overline{e}) - t(\underline{e}) \geq c(q(\overline{e}), \overline{e}) - c(q(\underline{e}), \overline{e}) \geq 0$$

and

$$t(\overline{e}) - c(q(\overline{e}), \overline{e}) \geq t(\underline{e}) - c(q(\underline{e}), \overline{e}) \geq t(\underline{e}) - c(q(\underline{e}), \underline{e})$$

so that transfers and profits increase with e.

Now IR($\overline{e}$) is either binding or not. Suppose that it is binding together with IR($\underline{e}$). Then IIC($\overline{e}$) would be violated

$$t(\overline{e}) - c(q(\overline{e}), \overline{e}) = t(\underline{e}) - c(q(\underline{e}), \underline{e}) < t(\underline{e}) - c(q(\underline{e}), \overline{e})$$

unless $\delta = \delta_0$ but the zero contract is not rational. Likewise, suppose that IR($\overline{e}$) and IIC($\overline{e}$) are binding together; then IR($\underline{e}$) would be violated

$$0 = t(\overline{e}) - c(q(\overline{e}), \overline{e}) = t(\underline{e}) - c(q(\underline{e}), \overline{e}) > t(\underline{e}) - c(q(\underline{e}), \underline{e})$$

unless again δ is the zero contract.

Assume now that IR($\overline{e}$) is not binding and suppose that IR($\underline{e}$) is not binding at the maximum in (3.8). Then the maximand could be increased by lowering all the $t(e)$'s by a same amount thus leading to a contradiction; hence, IR($\underline{e}$) is binding. By the same argument, assume that IIC($\overline{e}$) is not binding; then I could lower $t(\overline{e})$ and the maximand in (3.8) would not yield a maximum. Hence IIC($\overline{e}$) is binding. □

Proof of proposition 3.3.
We have already seen that all rational contracts, whether of type A or B, set $q(\overline{e}) = \hat{q}(\overline{e})$; hence I discriminate contracts through $q(\underline{e})$ which is equal to zero for all A-rational contracts (the inefficient agent is shut down) and to $Q(p)$ for B-rational contracts, that is, when $p \leq r^{-1}(0)$. I extend the definition of Q over all $[0, 1]$ with $Q(p) = 0$ when $p > r^{-1}(0)$.

Define $q_r = r^{-1}(\psi(\overline{e}))$ as the randomization point. Given any expected contract indexed by $q(\underline{e})$, the agent will optimally choose f in order to maximize (3.16). Hence we define the correspondence Γ

$$p \in \Gamma(q(\underline{e})) = \begin{cases} 0 & \text{if} \quad q(\underline{e}) < q_r \\ [0,1] & \text{if} \quad q(\underline{e}) = q_r \\ 1 & \text{if} \quad q(\underline{e}) > q_r \end{cases} \tag{3.32}$$

Combining these relations gives us a hemi-continuous compact-valued correspondence $Q \circ \Gamma$ that maps $[0,1]$ onto itself; hence, it has a fixed point by Kakutani's theorem.

Proof of uniqueness. First note that q_r is unique. Then recall that all equilibria must involve B-rational contracts and that all contracts can be indexed by $q(\underline{e})$. Suppose there is two distinct equilibria (p_1, δ_1) and (p_2, δ_2). Since Q is a function, I must have $p_1 \neq p_2$. Assume that δ_1 and δ_2 are the same, then necessarily $q(\underline{e}) = q_r$. Since $Q^{-1}(q(\underline{e}))$ is not a singleton, it must be that either $q_r = Q(0) = \hat{q}(\underline{e})$ or $q_r = Q(1) = 0$. In either case, $Q^{-1}(q_r) \cap [0,1]$ is a singleton, hence a contradiction. Thus it must be that δ_1 and δ_2 set $q(\underline{e})$ differently. Then $p_1 = 0$ implies $p_2 > 0$ but also

$$q_1 = Q(0) \Rightarrow q_2 < q_1 \leq q_r \Rightarrow p_2 = 0$$

a contradiction. And $p_1 = 1$ implies $p_2 < 1$ but also

$$q_1 = Q(1) \Rightarrow q_2 > q_1 \geq q_r \Rightarrow p_2 = 1$$

also a contradiction. Obviously, the same reasoning applies for p_2 hence I are left with the possibility that p_1 and p_2 belongs to $(0,1)$, that is the principal expects the agent to randomize in both equilibria which is impossible since there exists only a single q_r. □

Proof of proposition 3.4.
Let $\hat{E} \subseteq E^*$ be the actual support over which the agent randomizes in equilibrium. Assume that $\hat{E}$ is compact so that $\underline{e}^* = \min \hat{E}$ and $\overline{e}^* = \max \hat{E}$ are

well defined. Let δ be an optimal incentive compatible contract given that the agent plays a strategy F over $\hat{E}$ and let $r(e)$ be the maximal ex post rent an agent of type e can get. Since agent e can duplicate the variable cost of any agent $e' < e$, r must be non decreasing with e. In particular, that implies that $r(\underline{e}^*) = \min r(e)$. Now $r(\underline{e}^*)$ should be null otherwise δ would be weakly dominated by a contract that sets it to zero. But if $\underline{e}^* > \underline{e}$, then $\underline{e}^*$ implies a costly investment level that won't be matched by a positive rent $(r(\underline{e}^*) = 0)$. Investing $\underline{e}^*$ would imply a sure loss and would not be played by the agent. Hence the equality $\underline{e}^* = \underline{e}$ must stands. Now, let $R(q(e), e)$ be the rent gathered by an agent of type e that produces $q(e)$ from a self-selecting contract.

$$R(q, e) = t - c(q, e) - \psi(e).$$

If $q(\overline{e}^*) < \hat{q}(\overline{e}^*)$, then the marginal benefit to the principal of increasing $q(\overline{e}^*)$ is higher than the marginal variable cost compensation. Setting $q(\overline{e}^*) = \hat{q}(\overline{e}^*)$ is thus optimal for the principal since that can be done without affecting, at the margin, incentive compatibitity (increased production plans are marginally more costly to types $e < \overline{e}^*$). If $\overline{e}^* < \hat{e}$, then the agent could have expected an increase in profit from the contract $(q(\overline{e}^*), t(\overline{e}^*))$ by investing more. The marginal gain of investing more at $\overline{e}^*$ is

$$\frac{\partial[R(q(\overline{e}^*), e) - \psi(e)]}{\partial e}(\overline{e}^*) = -\frac{\partial c_e(\hat{q}(\overline{e}^*), e)}{\partial e}(\overline{e}^*) - \dot{\psi}(\overline{e}^*) \geq 0,$$

and is equal to zero only at $\overline{e}^* = \hat{e}$. Because a strictly positive arbitrage opportunity is incompatible with an equilibrium, it must be that $\overline{e}^* = \hat{e}$. $\square$

Proof of proposition 3.5.

Under the hypothesis about F, the optimal contract solves[18]

$$\max_{\delta} \int_{\underline{e}}^{\hat{e}} (q(e) - r(e) - c(q(e), e)) f(e) de \tag{3.33}$$

$$\begin{aligned} \dot{q}(e) &\geq 0 \\ q(\underline{e}) &\geq 0 \\ \dot{r}(e) &= -c_e(q(e), e) \\ r(\underline{e}) &= 0. \end{aligned} \tag{3.34}$$

The Hamiltonian function is

$$H(e) = (q(e) - r(e) - c(q(e), e)) f(e) - \mu(e) c_e(q(e), e),$$

from which I get the first-order conditions

$$\frac{\partial H(e)}{\partial q(e)} = f(e)(1 - c_q(q(e), e)) - \mu(e) c_{qe}(q(e), e) = 0 \tag{3.35}$$

$$\dot{\mu}(e) = -\frac{\partial H(e)}{\partial r(e)} = f(e). \tag{3.36}$$

The boundary condition at $e = \hat{e}$ is unconstrained hence $\mu(\hat{e}) = 0$. Integrating (3.36) I get

$$-\mu(e) = \mu(\hat{e}) - \mu(e) = \int_{e}^{\hat{e}} \dot{\mu}(\tilde{e}) d\tilde{e} = \int_{e}^{\hat{e}} f(\tilde{e}) d\tilde{e} = 1 - F(e) \tag{3.37}$$

The optimal quantity for type e is thus implicitly given by

$$f(e)(1 - c_q(q(e), e)) + (1 - F(e)) c_{qe}(q(e), e) = 0 \tag{3.38}$$

[18] See Laffont and Tirole (1993).

from which I can compute the necessary rents

$$r(e) = r(e) - r(\underline{e}) = -\int_{\underline{e}}^{e} c_e(q(\tilde{e}), \tilde{e}) d\tilde{e} \tag{3.39}$$

and transfers

$$t(e) = r(e) + c(q(e), e) \tag{3.40}$$

□

Proof of Proposition 3.6.
For e to be played in equilibrium, I need $r(e) = \psi(e)$ over $\hat{E}$, hence

$$\dot{\psi}(e) = \dot{r}(e) \tag{3.41}$$

$$= -c_e(q(e), e) \quad \text{over } \hat{E}. \tag{3.42}$$

That expression gives $q(e)$ as a function of $\dot{\psi}(e)$ and e. Differentiating (3.42) yields

$$\dot{q}(e) = -\frac{\ddot{\psi}(e) + c_{ee}(q(e), e)}{c_{eq}(q(e), e)} > 0$$

which is positive because ψ is convex and investment has a decreasing marginal effect upon costs ($c_{ee} > 0$). Hence, assumption 3.1 holds in equilibrium.

Once $q(e)$ has been characterized, equation (3.18) becomes

$$f(e)(1 - c_q(q(e), e)) - (1 - F(e))\dot{\psi}(e) = 0. \tag{3.43}$$

This is an ordinary linear differential equation of the form

$$\frac{y'}{a(e)} + y = 1 \tag{3.44}$$

where $y = F(e)$, $y' = f(e)$ and

$$a(e) = \frac{\dot{\psi}(e)}{1 - c_q(q(e), e))} \tag{3.45}$$

with initial condition $y(\underline{e}) = F(\underline{e}) = 0$. Note that since $q(\hat{e}) = \hat{q}(\hat{e})$, $1/a(e)$ will vanish only at $e = \hat{e}$. This implies that (3.44) as a unique solution over $[\underline{e}, e']$, for any $e' < \hat{e}$, which is given by (3.20).[19] There are many solutions to (3.43) in f at $\hat{e}$. Since the principal would never compensate variable cost in excess of his marginal utility, the first term of (3.43) is always non-negative so that all these solutions imply $F(\hat{e}) = 1$ and F is thus a distribution function. The derivative of F at $\hat{e}$ is defined by taking the limit at $\hat{e}$ from the left

$$f(\hat{e}) = \lim_{e \uparrow \hat{e}} dF(e).$$

□

[19]That can be checked by direct substitution. See Hochstadt (1963).

Annexe A

Programmes informatiques

Je présente ici les programmes informatiques que j'ai écrits pour appliquer l'algorithme présenté au chapitre 1. Ces programmes ont été écrits en langage *Matlab* et font notamment emploi des routines développées pour l'optimisation non linéaire (Grace, 1992) et la manipulation des splines (de Boor, 1992).

A.1 Initialisation: `init2.m` et `init2a.m`

Ces deux routines initialisent le programme avant l'appel de la routine principale `mainloop.m`. La routine `init2.m` appelle les routines `param1.m` et `param2.m`. La routine `param1.m` inclut les paramètres du modèle (préférences, environnement, type d'interpolation et degré de précision de l'interpolant). La routine `param2.m` spécifie des paramètres contrôlant la routine d'optimisation sous contrainte. La routine `init2.m` appelle ensuite la routine `cmpfb.m` qui procède à l'évaluation de la frontière de premier rang. Les routines `bounds.m` et `boundac.m` sont ensuite appelées afin de calculer l'ensemble ergodique. Cet ensemble correspond à l'intersection des frontières de premier et second rang lorsque celle-ci est non vide et à la frontière de second rang lorsque l'intersection est vide. Lorsque l'intersection est non vide, le calcul de la frontière de second rang est ainsi facilité. La routine `init2.m` calcule précisément les bornes de l'intersection de la frontière de premier rang avec la frontière de second rang lorsque le paiement ex ante B est libre et lorsqu'il est contraint à zéro. Le calcul de la frontière de ces ensemble se fait en cherchant les zéros (au voisinage des valeurs identifiées par `bounds.m`) des contraintes auto-exécutoires.

```
disp('< init2 >')
flag_init = 1;
%
% --- Initialization Program
%
param1
param2
%
% --- Optimization parameters
options(1) = display;
options(2) = terminate_x;
options(3) = terminate_f;
options(4) = terminate_g;
options(9) = gradient_check;
options(13) = 1;
%
% --- States of nature
%
S = length(y(1,:));
e = ones(1,S);
r = 1:S;
%
d = SM(1,:);
%
% first-best function
%
cmp_fb;
MAXU = fnval(f_nc,0);
MAX_NC = [MAXV,MAXU];
%
% -- Note: these computations are put here because they require
%    to be done AFTER u and v have been normalized
Ey = y*d';
uy = w(u,paru,y(1,:));
Euy = uy*d';
vy = w(v,parv,y(2,:));
Evy = vy*d';
```

```
WY  = [uy;vy];
EWY = [Euy;Evy];
%
%
% --- Bounds from the ergodic set
NGRID=100;
grid = linspace(0,MAXV,NGRID);
B = bounds(grid,u,v,paru,parv,beta,y,c_nc,f_nc);
%
B_J = find(B(1,:)<=B(2,:));
A_J = find((B(1,:)<=0)&(B(2,:)>=0));
%
% --- Check for irregularities (non-convexities of the ergodic set)
[last_B_J,LB_J] = last(B_J);
if LB_J
   if last_B_J-B_J(1)~=LB_J-1 ,disp('IRREGULARITIES!');B_J=[];end
end
%
%
% --- Accurate bounds
old_options = options;
options(13)=0;
for free=0:1
   if free
      label = 'B';
   else
      label = 'A';
   end
   eval(['L_J = ',label,'_J;']);
   if L_J
      if L_J(1)-1
         LOW = constr('boundac',grid(L_J(1)),options, ...
                      grid(L_J(1)-1),grid(L_J(1)),[], ...
                      u,v,paru,parv,beta,y,c_nc,f_nc,[free,-1]);
      else
         LOW = 0;
     end
      if NGRID-last(L_J)
         VHIGH = constr('boundac',grid(last(L_J)),options, ...
                   grid(last(L_J)),grid(last(L_J)+1),[], ...
                   u,v,paru,parv,beta,y,c_nc,f_nc,[free,1]);
         HIGH = fnval(f_nc,VHIGH);
      else
         VHIGH = MAXV;
         HIGH = 0;
      end
```

```
   else
      LOW = []; HIGH = []; VHIGH = [];
   end
   eval(['LOW_',label,' = LOW;']);
   eval(['HIGH_',label,' = HIGH;']);
   eval(['VHIGH_',label,' = VHIGH;']);
end
%
options=old_options;
%
if flagB
    LOW  = LOW_B;
    HIGH = HIGH_B;
   VHIGH = VHIGH_B;
else
    LOW  = LOW_A;
    HIGH = HIGH_A;
   VHIGH = VHIGH_A;
end
%
init2a
```

A.1.1 Configuration: param1.m et param2.m,

Divers modèles peuvent être calculés en ne modifiant que ces deux fichiers.

param1.m

```
% --- param1.m
%
% --- Reset Matlab memory
clear all
%
% --- Tell the main routine if we wish to allow
%     for an ex ante payment (flagB=1)
flagB = 0;
%
% --- Preferences
u = 'ucrra';
paru = [1, 2, .01, 1];
v = 'ulin';
parv = [1, 1, .01, 1];
beta = 0.70;
%
```

```
% --- Environment
y(1,:) = [.5,1.5];
y(2,:) = [0 0];
%
%     Number of states
S = length(y(1,:));
%
%     Stochastic Matrix
ones(1,S)/S;
SM = ans(ones(1,S),:);
%
% --- Initial guess for the second best
%     Pareto frontier (first-best --> fnc)
f0 = 'fnc';
%
% --- Interpolation scheme
%     (Cubic Hermitian --> fcubH)
ft = 'fcubH';
%
%     Normalization of the domain of the
%     first best Pareto frontier
Normf = '[-1,1]';
%
%     Number of pieces in the interpolant
L=32;
N = 32;
%
%     Constraints on instruments?
noneg = 0;
```

* * *

param2.m

```
% --- param2.m
% --- Optimisation parameters
display = 0;
terminate_x = 1e-4;
terminate_f = 1e-4;
terminate_g = 1e-7;
gradient_check = 0;
%
% --- Convergence criteria for F
convergence = 1e-3;
```

```
%
% --- Domain (subset of [0,pi/2]) for computation
%     of first-best consumption function.
%
dom_c_fb = [.3,pi/2-.1];
%
% --- REVERSE set the way guesses are computed for the maximization
%     problems. If REVERSE is set, "recursive guessing", that is
%     giving the optimal instruments for the previous maximization
%     problem as a guess for the current one, is used. Otherwise,
%     the program will use the optimal instruments for that problem
%     from the previous iteration. Near the solution, the latter
%     is more efficient but the former is always more robust.
%
REVERSE = 0;
%
% Method for computing first-best solution
METHOD=2;
```

A.1.2 Calcul de la frontière de premier rang

Ce calcul se résume essentiellement à la résolution d'un système d'équations non linéaires et est beaucoup plus aisé que celui des frontières de second rang. J'ai notamment besoin de cette frontière pour déterminer l'ensemble ergodique du système ainsi que pour initier l'algorithme itératif de résolution des frontières de second rang. La routine `cmpfb.m` appelle les routines `fbest.m` , `fbest2.m` et `W2.m` qui procèdent effectivement au calcul. Les routines `fbest.m` et `fbest2.m` correspondent à des méthodes différentes de calcul et l'utilisateur peut spécifier quelle méthode il souhaite employer. Ces deux fonctions évaluent les consommations optimales $c^* : [0, \overline{V}^*]^S \to R^S$. La routine `W2.m` procède ensuite au calcul de f à partir des $c^*_s(V)$. La routine `cmpfb.m` normalise les préférences de sorte que $f(0) = f^{-1}(0) = \overline{V}^* = 1$.

`cmpfb.m`

```
disp('< cmpfb.m >')
%
% --- Computation of the unconstrained value function
```

```
%
% --- Initial normalization
%
guess = zeros(1,S);
[INSTRUMU,options,lambdaU] = constr('calcfnc', ...
  guess,options,[],[],'gcalcfnc',u,v,paru,parv,beta,y,SM,0,1);
MAXU = -options(8);
[INSTRUMV,options,lambdaV] = constr('calcfnc', ...
  guess,options,[],[],'gcalcfnc',u,v,paru,parv,beta,y,SM,0,2);
MAXV = -options(8);
if MAXV<1e-5
   error(['Gains of trade to low : ',num2str(MAXV)])
end
%
%
paru(1) = paru(1)/MAXU;
parv(1) = parv(1)/MAXV;
%
%
%
if METHOD==1
   breaks = linspace(0,1,L+1);
   [f,cv] = fbest(u,v,paru,parv,beta,y,SM,breaks,0,options);
   MAXV=1;
else
   breaks = linspace(dom_c_fb(1),dom_c_fb(2),L+1);
   %
   c = fbest2(u,v,paru,parv,beta,y,breaks,0,options);
   [f,cv] = W2(c,u,v,paru,parv,beta,y,SM);
   %
   % --- Evaluation of the domain of V for which
   %     the first-best frontier crosses the positive quadrant.
   %     We compute the root of f.
   %
   MAXV = pproot(f,-1);
   %
   % --- Normalization of preferences
   %
   paru(1) = paru(1)/fnval(f,0);
   parv(1) = parv(1)/MAXV;
   %
   %
   % --- Second computation to make sure everything is set
   %
   c = fbest2(u,v,paru,parv,beta,y,breaks,0,options);
   [f,cv] = W2(c,u,v,paru,parv,beta,y,SM);
```

```
end
%
%
%
% --- Store first-best function
f_nc = f;
c_nc = cv;
```

* * *

fbest.m et calcfnc

Cette première méthode de calcul maximise l'utilité de l'agent u sous contrainte que l'utilité de l'agent v atteigne un niveau V donné.

```
function [f,c] = fbest(u,v,paru,parv,beta,y,SM,breaks,graph,options)
if nargin<9,options=[];end
if nargin<8, graph=0;end
%
% FBEST: Compute the first-best consumption functions.
%
% Syntax: c = fbest(u,v,paru,parv,beta,y,SM,breaks)
%
% where
%       u, v, paru, parv and beta resume the preferences;
%       y is 2xS matrix of endowments where S is the number
%         of states.
%
% The function returns c, a S values function of t over
% [breaks(1) breaks(l+1)] that gives optimal consumption
% of u given a ratio of tan(t) of the ponderations in
% the Pareto problem. c is returned as a spline.
%
e = ones(1,2);
[dummy,S] = size(y);
%
%
L =length(breaks);
%
lambda = zeros(1,L);
INSTRUM = zeros(S,L);
guess = zeros(1,S);
for point = 1:length(breaks)
   [solution,options,lambda(point)] = constr('calcfnc', ...
```

```
      guess,options,[],[],'gcalcfnc',u,v,paru,parv,beta,y,SM, ...
      breaks(point),1);
    INSTRUM(:,point) = solution';
    MAX(point) = -options(8);
    guess = solution;
end
%
% --- Interpolation
VAL = reshape([MAX;-lambda],1,2*L);
f = spapi(augknt(breaks,4,2),sort([breaks,breaks]),VAL);
f = sp2pp(f);
c = spapi(augknt(breaks,2),breaks,INSTRUM);
c = sp2pp(c);
```

* * *

```
function [f,g] = calcfnc(x,u,v,paru,parv,beta,y,SM,minV,z)
%
% CALCFNC: Computation of the first-best.
%
% SYNTAX: [f,g] = calcfnc(x,u,v,paru,parv,beta,y,SM,minV,z)
C = y + [x;-x];
%
%
WY(1,:) = w(u,paru,y(1,:));
WY(2,:) = w(v,parv,y(2,:));
%
WC(1,:) = w(u,paru,C(1,:));
WC(2,:) = w(v,parv,C(2,:));
%
EW = SM(1,:)*(WC-WY)';
%
prog = 1/(1-beta)*EW; % 2x1
%
f = -prog(z); % 1x1
g = minV-prog(3-z);
```

* * *

```
function [df,dg] = gcalcfnc(x,u,v,paru,parv,beta,y,SM,minV,z)
S = length(y(1,:));
C = y + [x;-x];
dprog(1,:) = SM(1,:).*w(u,paru,C(1,:),1);
dprog(2,:) =-SM(1,:).*w(v,parv,C(2,:),1);
dprog = 1/(1-beta)*(-dprog);
```

```
%
df = dprog(z,:);
dg = dprog(3-z,:)';
```

fbest2.m

Cette seconde méthode de calcul résout le zéro des équations normales caractérisant le premier rang, soit l'égalité des taux marginaux de substitution.

```
function c = fbest2(u,v,paru,parv,beta,y,breaks,graph,options)
if nargin<9,options=[];end
if nargin<8, graph=0;end
%
% FBEST2: Compute the first-best consumption functions with collocation.
%
% Syntax: c = fbest2(u,v,paru,parv,beta,y,SM,breaks)
%
% where
%       u, v, paru, parv and beta resume the preferences;
%       y is 2xS matrix of endowments where S is the number
%         of states.
%
% The function returns c, a S value function of t over
% [breaks(1) breaks(l+1)] that gives optimal consumption
% of u given a ratio of tan(t) of the ponderations in
% the Pareto problem. c is return as a spline.
%
L = length(breaks)-1;
e = ones(1,2);
[dummy,S] = size(y);
%
%
% keyboard
[knots,addl] = augknt(breaks,4);
%
% --- Collocation points
colpnts = ...
sort([(breaks(1)+breaks(2))/2, ...
breaks,(breaks(L)+breaks(L+1))/2]);
lcn = length(colpnts);
%
% --- Initial guess put in c0
ty = [1 1]*y;
```

```
%
%
% k = pi*[3/16,1/4,5/16];
% C0 = solve_tms(tan(k),ty,u,v,paru,parv);
% TY = [ty;ty;ty];
% vC0 = w(v,parv,TY-C0);
% muC0 = w(v,parv,C0,1);
% mvC0 = w(v,parv,TY-C0,1);
% (cos(k)).^2;COSK2 = [ans;ans]';
% tan(k);TANK = [ans;ans]';
% DC0 = -vC0./(COSK2.*(muC0+TANK.*mvC0));
% C00 = C0(1,:)+DC0(1,:).*(colpnts(1)-3*pi/16);
% C11 = C0(3,:)+DC0(3,:).*(colpnts(lcn)-5*pi/16);
% DC00 = DC0(1,:);
% DC11 = DC0(3,:);
% VAL = [C00',C0',C11',DC00',DC0',DC11'];
% VAL = VAL(:,[1 6 2 7 3 8 4 9 5 10]);
% k = [colpnts(1),k,last(colpnts)];
% c0 = sp2pp(spapi(augknt(k,4,2),sort([k,k]),VAL));
%
% C0 = fnval(c0,colpnts);
% DC0 = fnval(c0,colpnts);
% VAL = [C0,DC0];
% [1:lcn;lcn+(1:lcn)];order = (ans(:))';
% VAL = VAL(:,order);
%
% c0 = spapi(augknt(colpnts,4,2),sort([colpnts,colpnts]),VAL);
%
%
%
C0 = solvetms(tan(pi/4),ty,u,v,paru,parv);
k0 = pi/4*ones(1,S);
mvC0 = w(v,parv,ty-C0,1);
coef_init =-2*mvC0./(w(u,paru,C0,2)+w(v,parv,ty-C0,2));
guess_points = (C0-(coef_init.*k0))'*ones(1,lcn) ...
                + coef_init'*colpnts;
%
c0 = spapi(knots,colpnts,guess_points);
%
%
%
[knots,coef0,n,k,d] = spbrk(c0);
%
%
colmat = spcol(knots,k,colpnts);
[dummy,m]= size(colmat);
```

```
%
%
% --- Collocation: the return function is fbcalc.m; the
%                  gradient is provided by fbgcalc.m
[coef,options] = fsolve('fbcalc',coef0',options, ...
                        'fbgcalc',knots,d,colmat, ...
                         m,colpnts,u,v,paru,parv,y);
%
% --- Result put in c
c = spmak(knots,coef');
%
% -- Graph of the results
% ----- domain
t = breaks(1):(breaks(L+1)-breaks(1))/100:breaks(L+1);
%
% ----- autarchic consumption for u in CY
CY = y(1,:)'*ones(size(t));
Y = ([1 1]*y)'*ones(size(t));
% ----- optimal consumption
C = spval(c,t);
%
% ----- graph
if graph
figure
hold on
for i =1:S
   plot(t,CY(i,:),':',t,C(i,:),'-')
end
hold off
%
% -- Error
sin(t);
sin_t = ans(ones(1,S),:);
cos(t);
cos_t = ans(ones(1,S),:);
Err = sin_t.*w(u,paru,C,1)-cos_t.*w(v,parv,Y-C,1);
figure
hold on
for i=1:S
 plot(t,Err(i,:))
end
hold off
end
```

* * *

```
function c = solve_tms(k,y,u,v,paru,parv)
%
% SOLVETMS:
%
% SYNTAX: c = solve_tms(k,y,u,v,paru,parv)
%
guess_0 = y./2;
guess = y - tms(guess_0,k(1),u,v,paru,parv,y) ...
          * inv(gtms(guess_0,k(1),u,v,paru,parv,y));
%
for i=1:length(k)
   c(i,:) = fsolve('tms',guess,[],'gtms',k(i),u,v,paru,parv,y);
   guess = c(i,:);
end
```

* * *

```
function f = fbcalc(coef,knots,d,colmat,m,colpnts,u,v,paru,parv,y)
%
[dummy,S] = size(y);
C = spval(spmak(knots,coef'),colpnts);
%
tan(colpnts);
tan_X = ans(ones(1,S),:);
([1 1]*y)';
Y = ans(:,ones(1,length(colpnts)));
%
z = w(u,paru,C,1)./w(v,parv,Y-C,1).*tan_X;
f = 2*(z.^(1/2) - z.^(-1/2));
%f = tan_X.*w(u,paru,C,1)-w(v,parv,Y-C,1);
```

* * *

```
function df = fbgcalc(coef,knots,d,colmat,m,colpnts,u,v,paru,parv,y)
[dummy,S] = size(y);
r = length(colpnts);
C = spval(spmak(knots,coef'),colpnts);
%
tan(colpnts);
tan_X = ans(ones(1,S),:);
([1 1]*y)';
```

```
Y = ans(:,ones(1,r));
%
mu = w(u,paru,C,1);
mv = w(v,parv,Y-C,1);
%
su = w(u,paru,C,2)./mu;
sv = w(v,parv,Y-C,2)./mv;
%
z = mu./mv.*tan_X;
(z.^(1/2) + z.^(-1/2));
MC = (su+sv).*ans;
%
MM = sparse(r*d,d*m);
for i =1:r
  z = d*(i-1)+(1:d);
 MM(z,:) = kron(spdiags(MC(:,i),0,d,d),colmat(i,:));
end
%
df = full(MM');
```

A.1.3 W2.m

Cette routine reconstruit la frontière de Pareto à partir des politiques de consommation optimales. Plus précisément, la routine prend en input le vecteur de fonctions $c : [0,\overline{V}]^S \to R^S$, les vecteurs de dotation y et de probabilités d et produit la fonctions $f : [0,\overline{V}] \to [0, f^{-1}(0)]$, telle que $f = d'(u(c) - u(y))/(1-\beta)$.

```
function [f,cv] = W2(c,u,v,paru,parv,beta,y,SM)
%
% W2:
%
% SYNTAX: [f,cv] = W2(c,u,v,paru,parv,beta,y,SM)
%
S = length(y(1,:));
d = SM(1,:);
[knots,coefs,n,k,dd] = spbrk(c);
[breaks,coefs,l,k,dd] = ppbrk(sp2pp(c));
L = length(breaks)-1;
colpnts = ...
sort([(breaks(1)+breaks(2))/2, ...
breaks,(breaks(L)+breaks(L+1))/2]);
colpnts = ...
```

```
sort([(breaks(1)+breaks(2))/2, ...
breaks,(breaks(L)+breaks(L+1))/2]);
sx = size(colpnts);
e = ones(sx);
z = zeros(sx);
%
% --- Optimal consumption
 c= sp2pp(c);
 C = fnval(c,colpnts);
DC = fnval(fnder(c),colpnts);
%
% --- Autarchic consumption
y(1,:)';
CUY = ans(:,e);
y(2,:)';
CVY = ans(:,e);
Y = CUY+CVY;
%
 F =  d*(w(u,paru,C)-w(u,paru,CUY))/(1-beta);
DF =  d*(w(u,paru,C,1).*DC)/(1-beta);
 V =  d*(w(v,parv,Y-C)-w(v,parv,CVY))/(1-beta);
DV = -d*(w(v,parv,Y-C,1).*DC)/(1-beta);
DC = DC./DV(ones(1,S),:);
%
[V,J] = sort(V);
%
lV = length(V);
%
knots = augknt(V,4,2);
domain = sort([V,V]);
%
reshape([F(J);DF(J)./DV(J)],1,2*lV);
 f = sp2pp(spapi(knots,domain,ans));
reshape([C(:,J);DC(:,J)],S,2*lV);
cv = sp2pp(spapi(knots,domain,ans));
```

A.1.4 Ensemble ergodique: `bounds.m` et `boundac.m`

Ces deux sous-routines évaluent la valeur des contraintes auto-éxécutoires pour les contrats de premier rang, tels qu'ils ont été calculés précédemment par `cmpfb.m`. La routine `bounds.m` utilise une grille plus ou moins fine (déterminée par la dimension du vecteur d'entrée V) pour vérifier si ces contraintes sont satisfaites par des contrats de premier rang ; à ces contrats, s'il existent, correspondent des points sur la frontière de premier rang qui appartiennent également à la frontière de second rang. Si l'ensemble de ces points semble connexe sur la frontière, `bounds.m` renvoie en sortie des approximations des deux extrémités de cet ensemble. La routine `init2.m` calcule ensuite précisément les zéros correspondant à ces extrémités. La routine `boundac.m` est une version de `bounds.m` employée par `init2.m`.

```
function B = bounds(V,u,v,paru,parv,beta,y,c,f)
%
% BOUNDS: Evaluation of the bounds B of the intersection
%         of first and second-best Pareto frontiers.
%
% SYNTAX: B = bounds(V,u,v,paru,parv,beta,y,c,f)
%
S = length(y(1,:));
e = ones(1,S);
%
% --- Evaluation of consumption for agent u
CU = fnval(c,V);
y(1,:)';YU = ans(:,ones(1,length(V)));
y(2,:)';YV = ans(:,ones(1,length(V)));
Y = YU+YV;
%
% --- Agent v consume the residual
CV = Y-CU;
F = fnval(f,V);
%
BL = zeros(1,length(V));
BH = BL;
%
% --- Determination of constraint violations
%     (we use the inverse utility functions)
BH = w(u,paru,w(u,paru,CU)+beta*F(e,:),-1)-YU;
BL = -w(v,parv,w(v,parv,CV)+beta*V(e,:),-1)+YV;
%
% --- Deal with imaginary solutions
%     For a solution to exist, BH>=BL.
%     For all B with imaginary parts, we make that impossible
%     by setting BH to -Inf and BL to Inf
J = find((imag(BH)~=0)|(imag(BL)~=0));
BH(J) = -Inf*ones(size(J));
BL(J) = Inf*ones(size(J));
```

```
B = [max(BL);min(BH)];
```

* * *

```
function [f,g] = boundac(guess,u,v,paru,parv,beta,y,c_nc,f_nc,flags)
%
% BOUNDAC: Evaluation of the bounds B of the intersection
%          of first and second-best Pareto frontiers.
%
%
%
free = flags(1);
sgn  = flags(2);
b = bounds(guess,u,v,paru,parv,beta,y,c_nc,f_nc);
f = sgn*guess;
if free
  g = [1 -1]*b;
else
  g = [1;-1].*b;
end
```

A.1.5 `init2a.m`

À partir de la frontière de premier rang (donnée par `cmpfb.m`) et de l'information sur la frontière de second rang – donnée par `init2.m` et `bounds.m` – la routine `init2.a` construit la fonction spline qui constituera la première estimation de l'algorithme récursif.

Dans le cas particulier où

- il y a deux états de la nature ;
- l'agent v est neutre au risque ;
- le paiement ex ante est contraint à zéro ;
- le taux d'escompte est suffisamment faible de sorte que la frontière de second rang est disjointe de la frontière de premier rang ;

alors, la résolution du problème de second rang revient à solutionner un simple système d'équations non linéaires. Dans ce cas, une portion du code résout immédiatement le système grâce aux fonctions d'utilité inverses.

```
disp('< init2a >');
% --- Points and first-best zone
%
if LOW|VHIGH
   ppcut(f_nc,[LOW,VHIGH]);
   POINTS_FB = ppbrk(ans);
    F_FB = fnval(f_nc,POINTS_FB);
   DF_FB = fnval(fnder(f_nc),POINTS_FB);
else
   POINTS_FB = [];
    F_FB = [];
   DF_FB = [];
end
%
% --- MIDV and MIDU are roughly in the ``middle''
%     of the Pareto frontier
MIDV = fsolve('mid',1/2,[],[],f_nc);
MIDU = fnval(f,MIDV);
%
[POINTS_LOW,POINTS_HIGH] = make(MAXU,MAXV,LOW,HIGH,N,MIDV,MIDU);
%
NLOW = length(POINTS_LOW);
NVHIGH = length(POINTS_HIGH);
%
%
FLOW = [];
if NLOW
   FLOW = fnval(f_nc,POINTS_LOW);
end
FHIGH = zeros(size(POINTS_HIGH));
FHIGH(1) = 1/2;
%
for i=1:length(POINTS_HIGH)
   FHIGH(i) = fsolve('mid',FHIGH(max(1,i-1)),[],[],f_nc,POINTS_HIGH(i));
end
VALUES = [FLOW,last(POINTS_LOW),FHIGH,last(POINTS_HIGH)];
OLD_VALUES_DIF = zeros(size(VALUES));
%
%
% --- Definition of initial guesses for instruments
%
```

```
if flagB
  INSTRU = [INSTRUMU,INSTRUMU,zeros(1,2*S),[0 0 1]];
  INSTRV = [INSTRUMV,INSTRUMV,MAXV*ones(1,2*S),[0 0 1]];
else
  INSTRU = [INSTRUMV,0*e];
  INSTRV = [INSTRUMV,MAXV*e];
end
%
LAMBDA_LOW = [];
LAMBDA_HIGH = [];
INSTRUMENTS_LOW  = INSTRU(ones(1,NLOW ),:);
INSTRUMENTS_HIGH = INSTRV(ones(1,NVHIGH),:);
%
%
if flagB
   LBA = -Inf*ones(1,2*S); % 4*S+3);
   LBB = [-Inf -Inf 1/2];
   LB = [LBA,zeros(1,2*S),LBB];
   UBA = Inf*ones(1,2*S); %4*S+3);
   UBB = [Inf Inf 1];
   UB = [UBA,1.2*MAXV*ones(1,2*S),UBB];
else
   LBA = -Inf*e; %ones(1,2*S);
   LB = [LBA,zeros(1,S)];
   UBA =  Inf*e;%ones(1,2*S);
   UB = [UBA,1.2*MAXV*e];
end
%
%
HAS_CONVERGED = 0;
ITERATION = 0;
BYPASS = 0;
%
%
% --- Special case solvable as a zero of a system
%     of non linear equations.
if strcmp(v,'ulin') & S == 2 & LOW == [] & HIGH == [] & flagB==0
   disp('Special case: B = 0, S = 2, v = ulin and beta small.');
   X = fsolve('special',[mean(y(1,:))*[1 1],1,1],options, ...
              'specialg', ...
              y(1,:),d,u,v,paru,parv,beta);
   C1 = X(1);
   C2 = X(2);
   V = X(3);
   U = X(4);
   if U & V
```

```
   fac1 = 1/d(1)-beta;
   MAXV = fsolve('solvMAXV',V,options,'solvMAXVg', ...
               u,paru,parv,fac1,C1,U,V);
   fac2 = 1/d(2)-beta;
   MAXU = U + (w(u,paru,C2+fac2*V)-w(u,paru,C2))/fac2;
   %
   DOMV = sort([linspace(0,MAXV),V]);
   DOMVG = DOMV(find(DOMV<=V));
   IMAGG = U + (w(u,paru,C2+fac2*(V-DOMVG)/parv(1)) ...
                -w(u,paru,C2))/fac2;
   DIMAGG = -w(u,paru,C2+fac2*(V-DOMVG)/parv(1),1)/parv(1);
   DOMVD = DOMV(find(DOMV>=V));
   IMAGD = U + (w(u,paru,C1-fac1*(DOMVD-V)/parv(1)) ...
                -w(u,paru,C1))/fac1;
   DIMAGD = -w(u,paru,C1-fac1*(DOMVD-V)/parv(1),1)/parv(1);
   %
   NG = length(DOMVG)-1;
   ND = length(DOMVD);
   INSG = [C1*ones(1,NG)-y(1,1); ...
           C2-y(1,2)+fac2*(V-DOMV(1:NG))/parv(1); ...
           V*ones(1,NG);DOMVG(1:NG)];
   INSD = [C1-fac1*(DOMVD-V)/parv(1)-y(1,1); ...
           C2*ones(1,ND)-y(1,2); ...
           DOMVD;V*ones(1,ND)];
   INS = [INSG,INSD]';
   %
   [breaksG,coefG] = ppbrk(sp2pp(spapi(augknt(DOMVG,4,2), ...
                      sort([DOMVG,DOMVG]), ...
                     reshape([IMAGG;DIMAGG],1,2*length(DOMVG)))));
   [breaksD,coefD] = ppbrk(sp2pp(spapi(augknt(DOMVD,4,2), ...
                      sort([DOMVD,DOMVD]), ...
                     reshape([IMAGD;DIMAGD],1,2*length(DOMVD)))));
   f = ppmak([breaksG(1:(length(breaksG)-1)),breaksD], ...
             [coefG;coefD],1);
else
   MAXV = 0;
   MAXU = 0;
   DOMV = 0;
   INS = [0 0 0 0];
   f = ppmak([0 1],[0 0 0 0],1)
end
%
POINTS_FB = [];
POINTS_LOW = DOMV;
POINTS_HIGH = [];
INSTRUMENTS_LOW = INS;
```

```
   INSTRUMENTS_HIGH = [];
   LAMBDA_LOW = [];
   LAMBDA_HIGH = [];
   %
   LABEL = 'A';
   %
   eval(['f_',LABEL,' = f;']);
   eval(['POINTS_FB_',LABEL,' = POINTS_FB;']);
   eval(['POINTS_LOW_',LABEL,' = POINTS_LOW;']);
   eval(['POINTS_HIGH_',LABEL,' = POINTS_HIGH;']);
   eval(['INSTRUMENTS_LOW_',LABEL,' = INSTRUMENTS_LOW;']);
   eval(['INSTRUMENTS_HIGH_',LABEL,' = INSTRUMENTS_HIGH;']);
   eval(['LAMBDA_LOW_',LABEL,' = LAMBDA_LOW;']);
   eval(['LAMBDA_HIGH_',LABEL,' = LAMBDA_HIGH;']);
   eval(['MAX_',LABEL,'=[MAXV,MAXU];']);
   eval(['MAXV_',LABEL,'= MAXV;']);
   eval(['MAXU_',LABEL,'= MAXU;']);
else
   disp('Initialization completed. Now, call mainloop.')
end
```

mid.m et make.m

Les calculs sont plus rapides et plus précis si nous divisons la tâche en deux. Considérez un point « mitoyen » $(\tilde{v}, f(\tilde{v}))$ de la frontière de Pareto ; nous calculons f (et f') sur $[\tilde{v}, \overline{v}]$ et f^{-1} $(1/f')$ sur $[f(\tilde{v}), f(0)]$. La routine mid.m estime un tel point $(\tilde{v}, f(\tilde{v}))$ en prenant l'intersection de la frontière avec la droite à 45 degrés. La routine make.m construit ensuite les divers domaines d'approximation.

```
function zero = mid(x,f,value)
% MID: ''zero'' equals zero if x is the intersection of
%      y=x and a Pareto frontiers y=f(x).
%
% SYNTAX: zero = mid(x,f,value)
if nargin<3
   zero = fnval(f,x)-x;
else
   zero = fnval(f,x)-value;
end
```

* * *

```
function  [POINTS_LOW,POINTS_HIGH] = make(MAXU,MAXV,LOW,HIGH,N,MIDV,MIDU)
```

```
% MAKE: make the domain for approximation.
%
% SYNTAX: [POINTS_LOW,POINTS_HIGH] = make(MAXU,MAXV,LOW,HIGH,N,MIDV,MIDU)
%
if nargin < 6
  MIDV = MAXV;
  MIDU = MAXU;
end
%
if LOW & HIGH
   NLOW = max(fix(LOW/(LOW+HIGH)*N),2);
   NHIGH = N-NLOW;
   linspace(0,LOW,NLOW+1);
   POINTS_LOW  = ans(1:NLOW);
   linspace(0,HIGH,NHIGH+1);
   POINTS_HIGH = ans(1:NHIGH);
elseif LOW & ~HIGH
   NLOW = N;
   NHIGH = 0;
   linspace(0,LOW,NLOW+1);
   POINTS_LOW = ans(1:NLOW);
   POINTS_HIGH = [];
elseif ~LOW & HIGH
   NLOW = 0;
   NHIGH = N;
   POINTS_LOW = [];
   linspace(0,HIGH,NHIGH+1);
   POINTS_HIGH = ans(1:NHIGH);
elseif (LOW==0) & (HIGH==0)
   error('MAKE: Already first best!');
else
   NLOW = max(fix(N/2),2);
   NHIGH = N-NLOW;
   linspace(0,MIDV,NLOW+1);
   POINTS_LOW = ans(1:NLOW);
   linspace(0,MIDU,NHIGH);
   POINTS_HIGH = ans;
end
%
% --- Sort point in reverse order
POINTS_LOW = fliplr(POINTS_LOW);
POINTS_HIGH = fliplr(POINTS_HIGH);
```

solvMAXV.m

Cette fonction est employée afin de calculer $\overline{V}$, soit le zéro de f.

```
function zero=solvMAXV(MAXV,u,paru,parv,fac,C1,U,V)
% solvMAXV:
%
% SYNTAX: zero=solvMAXV(MAXV,u,paru,parv,fac,C1,U,V)
zero=U+(w(u,paru,C1-fac*(MAXV-V)/parv(1))-w(u,paru,C1))/fac;
```

* * *

```
function zero=solvMAXVg(MAXV,u,paru,parv,fac,C1,U,V)
zero=-w(u,paru,C1-fac*(MAXV-V)/parv(1),1)/parv(1);
```

prgsbf.m

Ces fonctions calculent les valeurs du programme de second rang, de même que des contraintes, en un point donné de l'ensemble des instruments. La routine associée prgsbg.mcalcule le Jacobien du maximand et des contraintes en ce point.

```
function [f,g] = prgsbf(x,bounds,z,flags,u,v,paru,parv,beta,y,SM,f,WY)
%
% PRGSBF: Values of the second-best Pareto frontier
%         and of the self-enforcing constraints.
%
% SYNTAX: [f,g] = prgsbf(x,bounds,z,flags,u,v,paru,parv,beta,y,SM,f,WY)
 minV = bounds(1);
fbarV = bounds(2);
%
flagB = flags(1);
noneg = flags(2);
%
[dummy,S]= size(y);
e = ones(1,S);
r = 1:S;
%
yu = y(1,:);
yv = y(2,:);
%
d = SM(1,:);
EWY = WY*d';
```

```
WUY = WY(1,:);
WVY = WY(2,:);
%
end
%
if flagB
   % --- Input variables
   a1 = x(r);
   a2 = x(S+r);
   V1 = x(2*S+r);
   V2 = x(3*S+r);
   B1 = x(4*S+1)*e;
   B2 = x(4*S+2)*e;
   q  = x(4*S+3);
   %
   % --- Constructed variables
   F1 = fnval(f,V1);
   F2 = fnval(f,V2);
   %
   T1 = a1+B1;
   T2 = a2+B2;
   %
   CU1 = yu + T1;
   CU2 = yu + T2;
   cu1 = yu + B1;
   cu2 = yu + B2;
   %
   CV1 = yv - T1;
   CV2 = yv - T2;
   cv1 = yv - B1;
   cv2 = yv - B2;
   %
   WU1 = w(u,paru,CU1);
   WU2 = w(u,paru,CU2);
   WV1 = w(v,parv,CV1);
   WV2 = w(v,parv,CV2);
   %
   Wu1 = w(u,paru,cu1);
   Wu2 = w(u,paru,cu2);
   Wv1 = w(v,parv,cv1);
   Wv2 = w(v,parv,cv2);
   %
   ASEU1 = WU1+beta*F1;
   ASEU2 = WU2+beta*F2;
   ASEV1 = WV1+beta*V1;
   ASEV2 = WV2+beta*V2;
```

```
    %
    SEU1 = ASEU1-Wu1;
    SEU2 = ASEU2-Wu2;
    SEV1 = ASEV1-Wv1;
    SEV2 = ASEV2-Wv2;
    %
    prog1 = [ASEU1;ASEV1]*d';
    prog2 = [ASEU2;ASEV2]*d';
    prog = EWY-(q*prog1+(1-q)*prog2);
    contr = -[SEU1,SEV1,SEU2,SEV2,F1,F2,fbarV-F1,fbarV-F2,CU1,CV1,CU2,CV2];
    if ~noneg
      contr = contr(1:(8*S));
    end
    %
else
    % --- Input variables
    a  = x(r);
    V  = x(S+r);
    %
    % --- Constructed variables
    F  = fn_val(f,V);
    CU  = yu+a;
    CV  = yv-a;
    %
    WU = w(u,paru,CU);
    WV = w(v,parv,CV);
    %
    ASEU = WU + beta*F;
    ASEV = WV + beta*V;
    %
    SEU = ASEU-WUY;
    SEV = ASEV-WVY;
    %
    prog = -[SEU;SEV]*d';
    contr= -[SEU,SEV,F,fbarV-F];
end
%
%
f = prog(z);
g = [minV+prog(3-z),contr];
```

* * *

```
function [df,dg] = prgsbg(x,bounds,z,flags,u,v,paru,parv,beta,y,SM,f,WY)
%
% PRGSBG: Values of the gradients of the second-best Pareto frontier
```

```
%         and of the self-enforcing constraints.
%
% SYNTAX: [f,g] = prgsbf(x,bounds,z,flags,u,v,paru,parv,beta,y,SM,f,WY)
%
 minV = bounds(1);
fbarV = bounds(2);
%
flagB = flags(1);
noneg = flags(2);
%
[dummy,S]= size(y);
e = ones(1,S);
r = 1:S;
%
yu = y(1,:);
yv = y(2,:);
%
d = SM(1,:);
D = [d;d];
EWY = WY*d';
WUY = WY(1,:);
WVY = WY(2,:);
%
%
if flagB
   % --- Input variables
   a1 = x(r);
   a2 = x(S+r);
   V1 = x(2*S+r);
   V2 = x(3*S+r);
   B1 = x(4*S+1)*e;
   B2 = x(4*S+2)*e;
   q  = x(4*S+3);
   %
   % --- Constructed variables
    F1 = fnval(f,V1);
    F2 = fnval(f,V2);
   dF1 = fnval(fnder(f),V1);
   dF2 = fnval(fnder(f),V2);
   %
   T1 = a1 + B1;
   T2 = a2 + B2;
   %
   CU1 = yu + T1;
   CU2 = yu + T2;
   cu1 = yu + B1;
```

```
cu2 = yu + B2;
%
CV1 = yv - T1;
CV2 = yv - T2;
cv1 = yv - B1;
cv2 = yv - B2;
%
WU1 = w(u,paru,CU1);
WU2 = w(u,paru,CU2);
WV1 = w(v,parv,CV1);
WV2 = w(v,parv,CV2);
%
Wu1 = w(u,paru,cu1);
Wu2 = w(u,paru,cu2);
Wv1 = w(v,parv,cv1);
Wv2 = w(v,parv,cv2);
%
dWU1 =  w(u,paru,CU1,1);
dWU2 =  w(u,paru,CU2,1);
dWV1 = -w(v,parv,CV1,1);
dWV2 = -w(v,parv,CV2,1);
%
dWu1 =  w(u,paru,cu1,1);
dWu2 =  w(u,paru,cu2,1);
dWv1 = -w(v,parv,cv1,1);
dWv2 = -w(v,parv,cv2,1);
%
ASEU1 = WU1+beta*F1;
ASEU2 = WU2+beta*F2;
ASEV1 = WV1+beta*V1;
ASEV2 = WV2+beta*V2;
%
SEU1 = ASEU1-Wu1;
SEU2 = ASEU2-Wu2;
SEV1 = ASEV1-Wv1;
SEV2 = ASEV2-Wv2;
%
prog1 = [ASEU1;ASEV1]*d';
prog2 = [ASEU2;ASEV2]*d';
%
prog = EWY-q*prog1-(1-q)*prog2;
%
dprog1_a1 =    q *D.*[dWU1;dWV1];
dprog2_a2 = (1-q)*D.*[dWU2;dWV2];
%
dprog1_V1 =    q *beta*D.*[dF1;e];
```

```
dprog2_V2 = (1-q)*beta*D.*[dF2;e];
%
dprog1_B1 =    q *[dWU1;dWV1]*d';
dprog2_B2 = (1-q)*[dWU2;dWV2]*d';
%
dprog_q   = prog1-prog2;
%
dprog     = -[dprog1_a1,dprog2_a2,dprog1_V1,dprog2_V2, ...
              dprog1_B1,dprog2_B2,dprog_q];
%
%
n = 12*S;
JACOB = zeros(n,4*S+3);
%
% --- Derivatives with respect to a1
i =     (r-1)*n;
   0+r; JACOB(i+ans) = dWU1;
   S+r; JACOB(i+ans) = dWV1;
 8*S+r; JACOB(i+ans) =  e;
 9*S+r; JACOB(i+ans) = -e;
%
% --- Derivatives with respect to a2
i = (S + (r-1))*n;
 2*S+r; JACOB(i+ans) = dWU2;
 3*S+r; JACOB(i+ans) = dWV2;
10*S+r; JACOB(i+ans) =  e;
11*S+r; JACOB(i+ans) = -e;
%
% --- Derivatives with respect to V1
i = (2*S + (r-1))*n;
   0+r; JACOB(i+ans) = beta*dF1;
   S+r; JACOB(i+ans) = beta*e;
 4*S+r; JACOB(i+ans) =  dF1;
 6*S+r; JACOB(i+ans) = -dF1;
%
% --- Derivatives with respect to V2
i = (3*S + (r-1))*n;
 2*S+r; JACOB(i+ans) = beta*dF2;
 3*S+r; JACOB(i+ans) = beta*e;
 5*S+r; JACOB(i+ans) =  dF2;
 7*S+r; JACOB(i+ans) = -dF2;
%
% --- Derivatives with respect to B1
i = 4*S*n;
   0+r; JACOB(i+ans) =  dWU1-dWu1;
   S+r; JACOB(i+ans) =  dWV1-dWv1;
```

```
     8*S+r; JACOB(i+ans) =  e;
     9*S+r; JACOB(i+ans) = -e;
    %
    % --- Derivatives with respect to B2
    i = (4*S+1)*n;
     2*S+r; JACOB(i+ans) =  dWU2-dWu2;
     3*S+r; JACOB(i+ans) =  dWV2-dWv2;
    10*S+r; JACOB(i+ans) =  e;
    11*S+r; JACOB(i+ans) = -e;
    %
    JACOB = -JACOB';
    if ~noneg
      JACOB = JACOB(:,1:(8*S));
    end
else
    % --- Instruments variables
    a  = x(r);
    V  = x(S+r);
    %
    % --- Constructed variables
     F = fn_val(f,V);
    dF = fn_val(f,V,1);
    %
    CU = yu+a;
    CV = yv-a;
    %
     WU =   w(u,paru,CU);
     WV =   w(v,parv,CV);
    dWU =  w(u,paru,CU,1);
    dWV = -w(v,parv,CV,1);
    %
    %
    dprog_a = D.*[dWU;dWV];
    dprog_V = beta*D.*[dF;e];
    %
    dprog = -[dprog_a,dprog_V];
    %
    %
    n = 4*S;
    JACOB = zeros(n,2*S);
    %
    % --- Derivatives with respect to a
    i =     (r-1)*n;
       0+r; JACOB(i+ans) = dWU;
       S+r; JACOB(i+ans) = dWV;
    %
```

```
   % --- Derivatives with respect to V
   i =  (S+(r-1))*n;
      0+r; JACOB(i+ans) =  beta*dF;
      S+r; JACOB(i+ans) =  beta*e;
    2*S+r; JACOB(i+ans) =  dF;
    3*S+r; JACOB(i+ans) = -dF;
   %
   JACOB = -JACOB';
end
%
%
df =  dprog(z,:);
dg = [dprog(3-z,:)',JACOB];
```

A.2 Routine principale: `mainloop.m`

Cette routine procède à une itération sur la frontière de Pareto jusqu'à ce que le critère de convergence soit atteint.

```
disp('< mainloop.m >');
options(1)=display;
options(9) = gradient_check;
continue = 1;
LAMBDA = [];
INSTR = [];
 FLOW = [];
 FHIGH = [];
DFLOW = [];
DFHIGH = [];
GOEVENIFNOTCONCAVE = 0;
m = flagB+1;
NHIGH = NVHIGH;
while continue == 1
   OLD_VALUES = VALUES;
   %
   ITERATION = ITERATION+1;
   fprintf('\n I: %g ****************************************\n',ITERATION);
   %
   %
   fprintf('\n f(V): [0,%f] --> [0,%f]\n',[MAXV,MAXU]);
   %
   % --- New domain?
   if (LOW==[])&(~HIGH==[])
```

```
   MIDV = fsolve('mid',MAXV/2,[],[],f);
   MIDU = fnval(f,MIDV);
   [POINTS_LOW,POINTS_HIGH] = make(MAXU,MAXV,LOW,HIGH,N,MIDV,MIDU);
end
%
%
counter = 0;
for UP = (2-(NLOW>0)):(1+(NVHIGH>1))
   if UP==1
      PNTS = POINTS_LOW;
      NPNTS = length(PNTS);
      LABEL = 'LOW';
      [SIZE_INS,dummy] = size(INSTRUMENTS_LOW);
      if SIZE_INS == NPNTS
         INSTRUMENTS = INSTRUMENTS_LOW;
      elseif SIZE_INS < NPNTS
         INSTRUMENTS = INSTRUMENTS_LOW;
         INSTRUMENTS = INSTRUMENTS([1:SIZE_INS, ...
                       SIZE_INS*ones(1,NPNTS-SIZE_INS)],:);
      else
         INSTRUMENTS = INSTRUMENTS_LOW(1:NPNTS,:);
      end
   else
      PNTS = POINTS_HIGH;
      NPNTS = length(PNTS);
      LABEL = 'HIGH';
      [SIZE_INS,dummy] = size(INSTRUMENTS_HIGH);
      if SIZE_INS == NPNTS
         INSTRUMENTS = INSTRUMENTS_HIGH;
      elseif SIZE_INS < NPNTS
         INSTRUMENTS = INSTRUMENTS_HIGH;
         INSTRUMENTS = INSTRUMENTS([1:SIZE_INS, ...
                       SIZE_INS*ones(1,NPNTS-SIZE_INS)],:);
      else
         INSTRUMENTS = INSTRUMENTS_HIGH(1:NPNTS,:);
      end
   end
   %
   clear F;clear LAMBDA;
   for k = 1:NPNTS
      %
      % --- If REVERSE is set, we use recursive guessing.
      %     Otherwise, the instruments of the previous
      %     iteration are used as the current guess.
      %
      if REVERSE|k>=NPNTS
```

```
         GUESS = INSTRUMENTS(max(k-1,1),:);
      else
         GUESS = INSTRUMENTS(k,:);
      end
      [INSTRUMENTS(k,:),options,LAMBDA(:,k)] = constr('prg_sb_f',...
       GUESS,options,LB,UB,'prg_sb_g',[PNTS(k),1.2*MAXU], ...
       UP,[flagB,noneg],u,v,paru,parv,beta,y,SM,f,WY);
      F(k) = -options(8);
      counter = counter+1;
      if counter == 5
         fprintf('|');
         counter = 0;
      else
         fprintf('.');
      end
   end
   eval([ 'F',LABEL,' = F;']);
   eval(['DF',LABEL,' = -(LAMBDA(1,:)).^(3-2*UP);']);
   eval(['LAMBDA_',LABEL,' = LAMBDA.^(3-2*UP);']);
   eval(['INSTRUMENTS_',LABEL,' = INSTRUMENTS;']);
fprintf('\n');
end
%
%
% --- Interpolation by complete hermitian cubic spline
%
[POINTS,J] = sort([POINTS_LOW,POINTS_FB,FHIGH]);
[FLOW,F_FB,POINTS_HIGH];     F_ALL = ans(J);
[DFLOW,DF_FB,DFHIGH]; DF_ALL = ans(J);
% [INSTRUMENTS_LOW,INSTRUMENTS_FB,INSTRUMENTS_HIGH];
% INS_ALL = ans(J,:);
%
MAXV = last(POINTS);
MAXU = F_ALL(1);
%
% --- Quick check for a decreasing Pareto Frontier
if any(diff( F_ALL)>0),
   error('Pareto Frontier not decreasing!');
end
%
FDF = reshape([F_ALL;DF_ALL],1,2*length(F_ALL));
VALUES = [FLOW,last(POINTS_LOW),FHIGH,last(POINTS_HIGH)];
%
NEWf= sp2pp(spapi(augknt(POINTS,4,2),sort([POINTS,POINTS]),FDF));
%
% --- Check for concavity and correction for possible kinks
```

```
%     in the Pareto frontier
%
[J,BREAKS] = isconcav(NEWf);
if any(J)
   disp('Pareto Frontier not concave!');
   disp('Correct for possible kinks?');
   pause
   %
   % Cubic kinks
   %
   d2NEWf = fnder(fnder(NEWf));
   D20 = fnval(d2NEWf,BREAKS(J));
   D21 = fnval(d2NEWf,BREAKS(J+1));
   %
   for i = 1:length(J)
      fl = ppcut(pppce(NEWf,J(i)-1),BREAKS(J(i)+[0 1]));
      fr = ppcut(pppce(NEWf,J(i)+1),BREAKS(J(i)+[0 1]));
      [BR,COEF] = ppbrk(fncmb(fl,1,fr,-1));
      RTS = roots(COEF);
      pick = (RTS>0) & (RTS<(BREAKS(J(i)+1)-BREAKS(J(i))));
      XCUB(i) = pick*RTS'+BREAKS(J(i));
      YCUB(i) = fnval(fr,XCUB);
      D20(i) = fnval(fnder(fnder(fl)),BREAKS(J(i)));
      D21(i) = fnval(fnder(fnder(fr)),BREAKS(J(i)+1));
   end
   KNOTS = sort([augknt(BREAKS,4,2),dup3(XCUB,3)]);
   POINTSX = sort([POINTS,POINTS]);
   SUPX = [BREAKS([J,J+1]),XCUB];
   POINTSX = [POINTSX,SUPX];
   SUPY = [D20,D21,YCUB];
   POINTSY = [FDF,SUPY];
   [POINTSX,J] = sort(POINTSX);
   POINTSY = POINTSY(J);
   NEWf= sp2pp(KNOTS,POINTSX,POINTSY);
end
%
f = NEWf;
%
%
%
% --- Check for convergence
%
if HAS_CONVERGED
   fprintf('Maximum change in MAXV: %f\n',MAXV-OLD_MAXV);
   fprintf('Maximum change in MAXU: %f\n',MAXU-OLD_MAXU);
   z = linspace(0,MAXV,10000);
```

```
   z = abs(fnval(f,z)-fnval(OLD_f,z));
   fprintf('Distance MAX between f and old f: %f\n',max(z));
   fprintf('Distance ABS between f and old f: %f\n',sum(z)/10000);
else
   VALUES_DIF = max(abs(VALUES-OLD_VALUES));
   fprintf('\n Max change : %f\n',VALUES_DIF);
end
%
% --- Either there is convergence or the algorithm
%     does not seems to converge fast enough
%
if (VALUES_DIF < convergence)|HAS_CONVERGED
   %
   if ~HAS_CONVERGED
      disp('CONVERGENCE!');
      disp('Doing a last pass with a finer grid...');
      %
      HAS_CONVERGED = 1;
      %
      OLD_MAXV = MAXV;
      OLD_MAXU = MAXU;
      %
      OLD_f = f;
      N = 2*N;
      %
      [NLOW,dummy] = size(INSTRUMENTS_LOW);
      if NLOW
         INSTRUMENTS_LOW  = INSTRUMENTS_LOW(sort([1:NLOW,1:NLOW]),:);
      end
      [NHIGH,dummy] = size(INSTRUMENTS_HIGH);
      if NHIGH
         INSTRUMENTS_HIGH = INSTRUMENTS_HIGH(sort([1:NHIGH,1:NHIGH]),:);
      end
      NLOW = 2*NLOW;
      NVHIGH = 2*NVHIGH;
      NHIGH = 2*NHIGH;
      MIDV = fsolve('mid',1/2,[],[],f);
      MIDU = fnval(f,MIDV);
      [POINTS_LOW,POINTS_HIGH] = make(MAXU,MAXV,LOW,HIGH,N,MIDV,MIDU);
      mainloop;
   else
      break;
      HAS_CONVERGED = 0;
   end
   %
   % --- Save the previous results
```

```
      if flagB
         disp('Call nownob to redo without B.')
         LABEL = 'B';
         N = N/2;
      else
         LABEL = 'A';
      end
      %
      eval(['f_',LABEL,' = f;']);
      eval(['POINTS_FB_',LABEL,' = POINTS_FB;']);
      eval(['POINTS_LOW_',LABEL,' = POINTS_LOW;']);
      eval(['POINTS_HIGH_',LABEL,' = POINTS_HIGH;']);
      eval(['INSTRUMENTS_LOW_',LABEL,' = INSTRUMENTS_LOW;']);
      eval(['INSTRUMENTS_HIGH_',LABEL,' = INSTRUMENTS_HIGH;']);
      eval(['LAMBDA_LOW_',LABEL,' = LAMBDA_LOW;']);
      eval(['LAMBDA_HIGH_',LABEL,' = LAMBDA_HIGH;']);
      eval(['MAX_',LABEL,'=[MAXV,MAXU];']);
      eval(['MAXV_',LABEL,'= MAXV;']);
      eval(['MAXU_',LABEL,'= MAXU;']);
      %
      continue=0;
      %
   elseif abs(OLD_VALUES_DIF-VALUES_DIF)<1e-3*convergence
      disp(' ');
      disp('Algorithm stuck!');
      continue=0;
   end
   OLD_VALUES_DIF = VALUES_DIF;
   %
   if BYPASS,continue=0;end
   %
end
%
% --- mainloop.m set bypass to 1. So, upon restarting mainloop.m,
%     the loop will be done only one pass. To restart full looping,
%     reset manualy bypass to zero.
%
BYPASS = 1;
```

* * *

```
disp('< nownob.m >')
%
% --- Now no B.
%
%
```

```
% --- Reinitialisation
%
flagB = 0;
 LOW  = LOW_A;
 HIGH = HIGH_A;
VHIGH = VHIGH_A;
%
init2a
```

isconcav.m

Cette routine vérifie grossièrement la concavité de la frontière de Pareto, telle que représentée approximativement par un spline.[1]

```
function [J,BREAKS] = isconcav(f)
%
[BREAKS,COEFS,L,K,D] = ppbrk(f);
TEST = COEFS(:,1).*diff(BREAKS').^2/2;
BOUNDS = [COEFS(2:L,3);fnval(fnder(f),last(BREAKS))];
J = find(abs(TEST)>abs(BOUNDS));
```

A.3 Analyse

Une fois la frontière de Pareto calculée, j'estime le contrat optimal en recourant à une simple interpolation linéaire, point à point, des instruments obtenus à la dernière itération. Cette approche génère des formes fonctionnelles $\hat{f}$ continues mais non nécessairement différentiables ($\hat{f} \in \mathcal{C}^0$) ce qui correspond aux propriétés des fonctions f qu'elles sont censées approcher ($f \in \mathcal{C}^0$).

1. Rien en garantit que l'approximation par un spline préserve la concavité. Toutefois, cette contrainte n'est pas apparue serrante dans les simulations auxquelles j'ai procédé. Sigouin (1997) a développé un algorithme permettant de sélectionner une approximation sous la contrainte que celle-ci soit concave.

`analyse.m`

La routine `analyse.m` définit les domaines des fonctions à représenter approximativement soit

variable	*domaine*	*modèle*
`DOM_NC`	$[0, \overline{V}^*]$	non contraint.
`DOM_B`	$[0, \overline{V}]$	contraint avec paiement ex ante.
`DOM_A`	$[0, \overline{V}_{B=0}]$	contraint sans paiement ex ante.

Elle appelle ensuite les routines `analyseB.m` et `analyseA.m`. Ces routines construisent les contrats auto-exécutoires optimaux dans les cas avec et sans paiement ex ante. Les contrats sont représentés à l'aide de splines continus construits à partir des instruments obtenus lors de la dernière itération sur la frontière de Pareto.

```
disp('< analyse.m >')
REP = [];
EXP_LABEL = 'p';
PRINT = 0;
FILE = 0;
%
gray(10);
G = ans(8,:);
%
DOM_NC = linspace(0,MAX_NC(1));
DOM_B = linspace(0,MAX_B(1));
DOM_A = linspace(0,MAX_A(1));
%
yu = y(1,:)';
yv = y(2,:)';
([1 1]*y)';Y = ans(:,ones(1,100));
YU = yu(:,ones(1,100));
YV = yv(:,ones(1,100));
%
analyseB
analyseA
if DO_GRAPH_ANALYSE
   graphanalyse
end
```

analyseB.m

```
disp('< analyseB.m >')
DOM_B = linspace(0,MAXV_B);
% --- All policy functions are approximated by
%     piecewise linear functions.
%
%
% --- For B
LPFB = length(POINTS_FB_B);
if LPFB
   DOM_FB_B = linspace(POINTS_FB_B(1),last(POINTS_FB_B));
   B_FB = bounds(DOM_FB_B,u,v,paru,parv,beta,y,c_nc,f_nc);
   C_FB_B = fnval(c_nc,DOM_FB_B);
   V_FB_B = DOM_FB_B(ones(1,S),:);
   LLAMBDA_FB_B = zeros(4*S+1,100);
else
   B_FB   = [];
   C_FB_B = [];
   V_FB_B = [];
   LLAMBDA_FB_B = [];
end
%
yu = y(1,:)';
NLOW_B = length(POINTS_LOW_B);
if NLOW_B
   B_LOW_B = (INSTRUMENTS_LOW_B(:,4*S+1))';
   C_LOW_B = (INSTRUMENTS_LOW_B(:,r))'+ ...
             B_LOW_B(e,:)+yu(:,ones(1,NLOW_B));
   V_LOW_B = (INSTRUMENTS_LOW_B(:,2*S+r))';
   LLAMBDA_LOW_B = LAMBDA_LOW_B([1:(2*S+1),4*S+1+r,10*S+1+r],:);
else
   B_LOW_B = [];
   C_LOW_B = [];
   V_LOW_B = [];
end
NHIGH_B = length(POINTS_HIGH_B);
if NHIGH_B
   B_HIGH_B = (INSTRUMENTS_HIGH_B(:,4*S+1))';
   C_HIGH_B = (INSTRUMENTS_HIGH_B(:,r))'+ ...
              B_HIGH_B(e,:)+yu(:,ones(1,NHIGH_B));
   V_HIGH_B = (INSTRUMENTS_HIGH_B(:,2*S+r))';
   LLAMBDA_HIGH_B = LAMBDA_HIGH_B([1:(2*S+1),4*S+1+r,10*S+1+r],:);
else
   B_HIGH_B = [];
   C_HIGH_B = [];
```

```
   V_HIGH_B = [];
end
%
% --- Inversion
OLD_POINTS_HIGH_B=POINTS_HIGH_B;
for i=1:NHIGH_B
   POINTS_HIGH_B(i) = fsolve('mid',POINTS_HIGH_B(max(i-1,1)), ...
                      options,[],f_B,POINTS_HIGH_B(i));
end
IPOINTS_HIGH_B=POINTS_HIGH_B;
POINTS_HIGH_B=OLD_POINTS_HIGH_B;
%
POINTS_B = [POINTS_LOW_B,DOM_FB_B,IPOINTS_HIGH_B];
B_B = [[B_LOW_B;B_LOW_B],B_FB,[B_HIGH_B;B_HIGH_B]];
C_B = [C_LOW_B,C_FB_B,C_HIGH_B];
V_B = [V_LOW_B,V_FB_B,V_HIGH_B];
%
[POINTS_B,J] = sort(POINTS_B);
B_B = B_B(:,J);
C_B = C_B(:,J);
V_B = V_B(:,J);
%
b_B = sp2pp(spapi(augknt(POINTS_B,2),POINTS_B,B_B));
c_B = sp2pp(spapi(augknt(POINTS_B,2),POINTS_B,C_B));
v_B = sp2pp(spapi(augknt(POINTS_B,2),POINTS_B,V_B));
%
% --- Evaluation
BB = fnval(b_B,DOM_B);
%
CU_B = fnval(c_B,DOM_B);
CV_B = Y-CU_B;
V_B = fnval(v_B,DOM_B);
F_B = zeros(size(V_B));
DF_B = F_B; F_A = F_B;DF_A=F_B;
for i=1:S
   F_B(i,:) = fnval(f_B,V_B(i,:));
  DF_B(i,:) = fnval(fnder(f_B),V_B(i,:));
end
SEUB1 = w(u,paru,CU_B)-w(u,paru,YU+BB(1*e,:))+beta*F_B;
SEUB2 = w(u,paru,CU_B)-w(u,paru,YU+BB(2*e,:))+beta*F_B;
SEVB1 = w(v,parv,CV_B)-w(v,parv,YV-BB(1*e,:))+beta*V_B;
SEVB2 = w(v,parv,CV_B)-w(v,parv,YV-BB(2*e,:))+beta*V_B;
```

analyseA.m

```
disp('< analyseA.m >')
DOM_A = linspace(0,MAXV_A);
% --- All policy functions are approximated by
%     piecewise linear functions.
%
%
% --- For a
if length(POINTS_FB_A)
   POINTS_FB_A = linspace(POINTS_FB_A(1),last(POINTS_FB_A));
   C_FB_A = fnval(c_nc,POINTS_FB_A);
   V_FB_A = POINTS_FB_A(e,:);
else
   C_FB_A = [];
   V_FB_A = [];
end
%
yu = y(1,:)';
NLOW_A = length(POINTS_LOW_A);
if NLOW_A
   C_LOW_A = (INSTRUMENTS_LOW_A(:,r))'+yu(:,ones(1,NLOW_A));
   V_LOW_A = (INSTRUMENTS_LOW_A(:,S+r))';
else
   C_LOW_A = [];
   V_LOW_A = [];
end
NHIGH_A = length(POINTS_HIGH_A);
if NHIGH_A
   C_HIGH_A = (INSTRUMENTS_HIGH_A(:,r))'+yu(:,ones(1,NHIGH_A));
   V_HIGH_A = (INSTRUMENTS_HIGH_A(:,S+r))';
else
   C_HIGH_A = [];
   V_HIGH_A = [];
end
%
% --- Inversion
OLD_POINTS_HIGH_A=POINTS_HIGH_A;
for i=1:NHIGH_A
   POINTS_HIGH_A(i) = fsolve('mid',POINTS_HIGH_A(max(i-1,1)), ...
                         options,[],f_A,POINTS_HIGH_A(i));
end
IPOINTS_HIGH_A=POINTS_HIGH_A;
POINTS_HIGH_A=OLD_POINTS_HIGH_A;
%
POINTS_A = [POINTS_LOW_A,POINTS_FB_A,IPOINTS_HIGH_A];
```

```
C_A = [C_LOW_A,C_FB_A,C_HIGH_A];
V_A = [V_LOW_A,V_FB_A,V_HIGH_A];
%
[POINTS_A,J] = sort(POINTS_A);
C_A = C_A(:,J);
V_A = V_A(:,J);
%
c_A = sp2pp(spapi(augknt(POINTS_A,2),POINTS_A,C_A));
v_A = sp2pp(spapi(augknt(POINTS_A,2),POINTS_A,V_A));
%
%
% --- Evaluation
%
CU_A = fnval(c_A,DOM_A);
CV_A = Y-CU_A;
%
V_A = fnval(v_A,DOM_A);
%
F_A = zeros(size(V_A));
DF_A=F_A;
for i=1:S
   F_A(i,:) = fnval(f_A,V_A(i,:));
  DF_A(i,:) = fnval(fnder(f_A),V_A(i,:));
end

%
w(u,paru,yu); WUY = ans(:,ones(1,100));
w(v,parv,yv); WVY = ans(:,ones(1,100));
%
%
SEUA = w(u,paru,CU_A)-WUY+beta*F_A;
SEVA = w(v,parv,CV_A)-WVY+beta*V_A;
```

A.4 Fonctions d'utilité

Mon algorithme peut fonctionner avec n'importe quelle fonction d'utilité concave. À cette fin, je l'ai programmé de manière à ce que les routines acceptent en entrée des variables u et v représentant les fonctions d'utilité souhaitées. Je présente ici les diverses fonctions que j'ai utilisées et programmées à cette fin.

A.4.1 w.m

À chaque fois que le programme doit évaluer l'utilité d'un agent pour un objet x, il appelle la routine w.m. Celle-ci accepte en entrée le type des préférences u (*i.e.* le nom de la fonction d'utilité), son vecteur de paramètres paru, l'argument de la fonction x de même qu'un argument entier der qui modifie la fonction u de la manière suivante: si der < 0, der réfère à la fonction inverse de u, en quel cas der $= -1$ réfère à la fonction inverse, der $= -2$ à sa dérivée, der $= -3$ à sa seconde dérivée, etc. Si der $= 0$ (valeur par défaut), u est employée et les valeurs der $= 1, 2, 3\ldots$ correspondent aux dérivées successives de u. Le premier argument (α) de chaque fonction d'utilité est un paramètre d'échelle positif.

```
function y = w(u,paru,x,der)
% w: Evaluation of utility
if nargin<4
   der = 0;
end
%
y = eval([u,'(paru,x,der)']);
```

A.4.2 ucrra.m

Cette sous-routine définit la classe de fonctions d'utilité exhibant une aversion relative au risque σ constante, *i.e.*

$$-\frac{u''(x)}{u'(x)}x = \sigma.$$

La forme fonctionnelle est

$$u(x) = \alpha\frac{x^{1-\sigma}}{1-\sigma} \quad x \geq \underline{x} > 0, \sigma \in R,$$

où α est un paramètre d'échelle. Lorsque $\sigma = 1$, on peut appliquer la règle de l'Hospital pour obtenir $u(x) = \alpha \log(x)$. La fonction est étendue à gauche de $\underline{x}$ par une approximation linéaire ou quadratique si paru[4] est différent de zéro.

```
function y = ucrra(paru,x,der)
% UCRRA: Constant relative risk utility function (CRRA).
%        If posititive, der specifies the derivative. If negative,
%        abs(der)-1 specifies the derivative of the inverse function.
%
%           paru(1) is a scale factor.
%           paru(2) is the coefficient of relative risk aversion.
%
%        To make sure that preferences are defined over all R,
%        the CRRA is substituted by a quadratic function that matches
%        the CRRA in first two derivatives in paru(3) for x<= paru(3).
%        paru(3) must be strictly positive.
%
% SYNTAX: y = ucrra(paru,x,der)
%
% Copyright Patrick Gonzalez 1997.
if nargin<3, der = 0;end
if paru(3)<=0
   disp('UCRRA: correcting for negative paru(3). Set to .0001');
   paru(3) = .0001;
end
der = fix(der);
if paru(4)
   % --- Computation of the parameters of the quadratic extension
   %
   A = -paru(2)*paru(3)^(-paru(2)-1);
   B = (1+paru(2))*paru(3)^(-paru(2));
   if paru(2) == 1
           C = log(paru(3))-3/2;
   else
           C = paru(2)*(1+paru(2))/(2*(1-paru(2)))*paru(3)^(1-paru(2));
   end
else
   % --- Linear extension
   if paru(2) == 1
       B = 1/paru(3);
       A = log(paru(3))-B*paru(3);
   else
       B = paru(3)^(-paru(2));
       A = paru(3)^(1-paru(2))/(1-paru(2))-B*paru(3);
   end
end
%
```

```
%
y = zeros(size(x));
%
if der >= 0
   J = find(x<=paru(3));
   K = find(x>paru(3));
   %
   if paru(4)
      y(J) =  (der<3)*(A/(2^(der==0))*x(J).^(max([0,2-der])) ...
                    + B*x(J).^(max([0,1-der])) +(der==0)*C);
   else
      y(J) = (der==0)*A+(der<=1)*B*x(J).^(der==0);
   end
   %
   if der==0; % --- Special treatment for 0th derivative
      if paru(2)==1
         y(K) = log(x(K));
      else
         y(K) = x(K).^(1-paru(2))/(1-paru(2));
      end
   else
      if der==1
         coef = 1;
      else
         coef = prod((0:(der-2))+paru(2));
      end
      sgn = -(-1)^(der);
      y(K) = sgn*coef*x(K).^(1-paru(2)-der);
   end
   %
   y = paru(1)*y;
   %
else % --- Inverse function
   iparu_3 = ucrra(paru,paru(3));
   J = find(x<=iparu_3);
   K = find(x>iparu_3);
   x = x/paru(1);
   %
   der = abs(der)-1;
   D = B^2-2*A*(C-x(J));
   %
   scale = (1/paru(2))^der;
   %
   if paru(2)==1
      y = exp(x);
   else
```

```
      if der==0
         if paru(4)
            y(J) = -(B-sqrt(D))./A;
         else
            y(J) = (x(J)-A)/B;
         end
         y(K) = ((1-paru(2))*x(K)).^(1/(1-paru(2)));
      else
         if der==1
            coef_J = 1;
            coef_K = 1;
         else
            coef_J = fact2(der-1)/2;
            coef_K = prod(paru(2)*ones(1,der-1)-(0:(der-2)));
         end
         if paru(4)
             y(J) = (-1)^der*coef_J*D.^(-(der-1/2))*A^(der-1);
         else
             y(J) = (der==1)/B;
         end
         y(K) = coef_K*((1-paru(2))*x(K)).^((der*paru(2)-der+1)/(1-paru(2)));
      end
   end
   %
   y = scale*y;
   %
end
```

A.4.3 uexp.m

Classe de fonctions d'utilité exponentielles.

$$u(x) = -\alpha \exp(-\sigma x)/\sigma \quad x \in R, \sigma \in R,$$

où σ est le coefficient (constant) d'aversion absolue au risque:

$$-\frac{u''(x)}{u'(x)} = \sigma.$$

La routine retourne un message d'erreur si $\sigma x > 744$ car $\exp(z) = \infty$ si $z > 744$ sur l'ordinateur que j'ai employé.

```
function y = uexp(paru,x,der)
if nargin<3, der = 0;end
der = fix(der);
% Exponential utility function
%
if paru(2) == 0
   y = ulin(paru,x,der);
else
   if der < 0 % --- Inverse function
      der = abs(der)-1;
      if der == 0
         y = log(paru(1))-log(paru(2))-log(-x);
      else
         y = (-1)^der*fact(der)/y.^der;
      end
      y = y/paru(2);
   else % --- Exponential function
      TEST = paru(2)*x;
      if TEST > 744
         error('UEXP: x too big relative to paru(2)');
      end
      COEF = -paru(1)*(-1)^der*paru(2)^(der-1);
      y = COEF*exp(-TEST);
   end
end
```

A.4.4 ulin.m

Fonction d'utilité linéaire

$$u(x) = \alpha x$$

```
function y = ulin(paru,x,der)
% Linear utility function
%
if nargin<3,der = 0;end
if der==0
   y = paru(1)*x;
elseif der==1
   y = paru(1)*ones(size(x));
```

```
elseif der>1
   y = zeros(size(x));
elseif der<0
   y = ulin(1/paru(1),x,abs(der)-1);
end
```

A.4.5 uquad.m

Fonction d'utilité quadratique.

$$u(x) = \alpha(-x^2/2 + \beta x) \quad x < \overline{x}, \text{ où } \beta = \tilde{x}/\sigma(\tilde{x}) - (\overline{x} - \tilde{x}).$$

La fonction est étendue à droite en appelant la routine `uright.m`.

```
function y = uquad(paru,x,der)
% Quadratic utility function
%
%    x = X-MAX
%    u(x) = -x^2/2 + bx for x<=0
%    if x>0, then a right radical extension is used
%
%    b = MID/sigma - (MAX-MID)
%
%    sigma is the measure of relative risk aversion in MID
%
%    sigma = MID/(MAX-MID+b)
%
%    sigma must satisfy sigma<MID/(MAX-MID)
%    otherwise an error message is sent.
%
%    SYNTAX: uquad(paru,x,der)
%
%            where paru = [CNST , MAX , MID , sigma]
%
if nargin<3, der = 0;end
der = fix(der);
[m,n] = size(x);
x = x(:);
%
MAX = paru(2);
MID = paru(3);
sigma = paru(4);
%
```

```
cnd1 = MAX-MID;
if cnd1<=0,error('UQUAD: MAX-MID must be strictly positive!'),end
cnd2 = MID/(MAX-MID);
if sigma>=cnd2,fprintf('%f < %f ',[sigma,cnd2]);error('UQUAD: sigma too big!'),end
%
b = MID/sigma - (MAX-MID);
%
if der>=0
   JIN  = find(x <= MAX);
   JOUT = find(x >  MAX);
else
   JIN  = find(x <= 0);
   JOUT = find(x >  0);
end
%
if JOUT
   parf = [paru(1),MAX,0,b];
   UOUT = uright(parf,x(JOUT),der);
end
%
if JIN
   if der >=0
      X = x(JIN)-MAX;
      if     der == 0
         UIN = -X.^2/2 + b*X;
      elseif der == 1
         UIN = -X + b;
      elseif der == 2
         UIN = -ones(size(X));
      else der > 2
         UIN = zeros(size(X));
      end
   else % --- Inverse function
      U = x(JIN)/paru(1);
      D = b^2-2*U;
      if     der == -1
         UIN = MAX + b - D.^(1/2);
      else
         UIN = fact2(-der-2)*D.^(1/2+der+1)*paru(1)^(der+1);
      end
   end
end
%
% --- Round up
y = zeros(size(x));
y(JIN)  = UIN;
```

```
if der>=0
   y = paru(1)*y(JIN);
end
%
y(JOUT) = UOUT;
%
y = reshape(y,m,n);
```

A.4.6 ulexp.m

Cette fonction est discutée dans Gauthier, Poitevin et González (1997). La fonction est étendue à droite en appelant la routine `uright.m`.

```
function y = uquad(paru,x,der)
% Quadratic utility function
%
%    x = X-MAX
%    u(x) = -x^2/2 + bx for x<=0
%    if x>0, then a right radical extension is used
%
%    b = MID/sigma - (MAX-MID)
%
%    sigma is the measure of relative risk aversion in MID
%
%    sigma = MID/(MAX-MID+b)
%
%    sigma must satisfy sigma<MID/(MAX-MID)
%    otherwise an error message is sent.
%
%    SYNTAX: uquad(paru,x,der)
%
%            where paru = [CNST , MAX , MID , sigma]
%
if nargin<3, der = 0;end
der = fix(der);
[m,n] = size(x);
x = x(:);
%
MAX = paru(2);
MID = paru(3);
sigma = paru(4);
%
cnd1 = MAX-MID;
if cnd1<=0,error('UQUAD: MAX-MID must be strictly positive!'),end
```

```
cnd2 = MID/(MAX-MID);
if sigma>=cnd2,fprintf('%f < %f ',[sigma,cnd2]);error('UQUAD: sigma too big!'),end
%
b = MID/sigma - (MAX-MID);
%
if der>=0
   JIN  = find(x <= MAX);
   JOUT = find(x >  MAX);
else
   JIN  = find(x <= 0);
   JOUT = find(x >  0);
end
%
if JOUT
   parf = [paru(1),MAX,0,b];
   UOUT = uright(parf,x(JOUT),der);
end
%
if JIN
   if der >=0
      X = x(JIN)-MAX;
      if     der == 0
         UIN = -X.^2/2 + b*X;
      elseif der == 1
         UIN = -X + b;
      elseif der == 2
         UIN = -ones(size(X));
      else der > 2
         UIN = zeros(size(X));
      end
   else % --- Inverse function
      U = x(JIN)/paru(1);
      D = b^2-2*U;
      if     der == -1
         UIN = MAX + b - D.^(1/2);
      else
         UIN = fact2(-der-2)*D.^(1/2+der+1)*paru(1)^(der+1);
      end
   end
end
%
% --- Round up
y = zeros(size(x));
y(JIN)  = UIN;
if der>=0
   y = paru(1)*y(JIN);
```

```
end
%
y(JOUT) = UOUT;
%
y = reshape(y,m,n);
```

A.4.7 uright.m

Fonction d'utilité radicale. Cette fonction est utilisée pour étendre une fonction d'utilité à la droite de son domaine naturel. Si une fonction $\tilde{u}$ est définie pour $x < \tilde{x}$, on définit $\tilde{u}(x)$ pour $x \geq \overline{x}$ où $\overline{x}$ est inférieur mais voisin à $\tilde{x}$ par

$$u(x) = \tilde{u}(\overline{x}) + 2\tilde{u}'(\overline{x})(-1 + \sqrt{x+1}).$$

La fonction est ainsi continuement différentiable en $\overline{x}$.

```
function y = uright(paru,x,der)
% URIGHT: Right radical extension
%
if nargin<3, der = 0;end
der = fix(der);
%
 MIN = paru(2);
UMIN = paru(3);
DMIN = paru(4);
%
%
if DMIN<=0,DMIN,error('URIGHT: DMIN must be strictly positive!',DMIN);end
test = ((der>=0)&(any(x<MIN)))|((der<0)&any(x<UMIN));
if test,error('URIGHT: All points must be to the right of MIN');end
%
%
if der>=0
   X = x-MIN;
   D = X+1;
   if der == 0;
      y = UMIN+2*DMIN*(-1+D.^(1/2));
   else
      y = (-1)^(der-1)*fact2(der-1)*DMIN/2^(der-1)*D.^(1/2-der);
   end
   y = paru(1)*y;
```

```
else
   U = x/paru(1);
   if der==-1
      y = MIN-1+(1+(U-UMIN)/(2*DMIN)).^2;
   elseif der == -2
      y = (1+(U-UMIN)/(2*DMIN))/(DMIN*paru(1));
   elseif der == -3;
      y = ones(size(U))/(2*DMIN^2*(paru(1))^2);
   else
      y = zeros(size(U));
   end
end
```

* * *

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