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Axiomatic Cost Sharing

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SOMMAIRE

Le sujet de cette thèse est le partage de coût. Plus précisément pour un problème de partage de coût donné, j'étudie différentes méthodes de partage de coût selon l'approche axiomatique. Le problème de partage de coût est un problème où un nombre fini d'agents cherchent à partager le coût joint de la production nécessaire à la satisfaction de leur demande. Une méthode de partage de coût est une fonction qui associe à chaque problème les proportions ou parts du coût total qui doivent être allouées à chacun des agents. L'approche axiomatique vise donc à caractériser un ensemble de méthodes de partage de coût en se basant sur des propriétés ou axiomes mathématiques généraux ou normatifs.

La thèse est divisée en trois chapitres, chacun étant lui-même composé d'une ou plusieurs sections.

Le premier chapitre est une revue de la littérature où sont résumés les résultats les plus importants qui ont suivi l'article de Shapley [1953]. Dans ce chapitre, le partage de coût est présenté d'un point de vue général comme faisant partie intégrante d'une économie de production où l'on aborde à la fois les problèmes d'équité, d'efficacité et de compatibilité des incitations de la méthode de partage de coût.

Le deuxième chapitre s'appuie sur le modèle discret introduit par Moulin [1995] à travers trois sections. La première section caractérise l'ensemble des méthodes qui satisfont les axiomes d'Additivité et "Dummy". Le principal résultat de la section est que cet ensemble est généré par toutes les combinaisons convexes de méthodes dites "path generated". C'est un résultat important pour étudier l'effet des autres axiomes sur la caractérisation de la méthode de partage de coût. La deuxième section étudie la version discrète de la méthode d'Aumann-Shapley. Nous donnons une caractérisation

par les axiomes d'Additivité, de Dummy, et de Proportionalité pour les cas où le nombre d'agents est égal à deux ($n = 2$) et la demande d'un des agents est égale à un ($\exists i, \bar{q}_i = 1$). Dans la troisième section, nous proposons un nouvel axiome dit Invariance à la Mesure (Measurement Invariance). Nous démontrons ensuite que l'ensemble des méthodes satisfaisant les axiomes d'Additivité, de Dummy, et d'Invariance à la Mesure est l'ensemble des méthodes "Simple Random Order Values" (SROV) et que la méthode de Shapley-Shubik est l'unique méthode symétrique de l'ensemble des SROV.

Le troisième chapitre repose sur le modèle continu étudié par Friedman et Moulin [1995]. Dans la première section, nous étudions l'impact de l'axiome d'Ordinalité introduit par Sprumont [1998] sur les méthodes additives de partage de coût et nous généralisons le résultat de la deuxième section du Chapitre 2 au cas continu en remplaçant l'axiome d'invariance à la mesure par celui d'Ordinalité. Dans la deuxième section de ce chapitre, nous considérons une méthode "non-additive", c.-à-d., la méthode proportionnelle ajustée au coût marginal dites "Proportionally Adjusted Marginal Pricing" (PAMP). Nous caractérisons la méthode PAMP par les axiomes d'Indépendance Locale, de Proportionalité, d'Invariance à l'échelle, et de Continuité.

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INTRODUCTION GÉNÉRALE

Nous considérons le problème de partage de coût où un nombre fini d'agents $N = \{1, 2, \dots, n\}$ cherchent à partager le coût joint de la production nécessaire à la satisfaction de leur demande. Chaque agent i a une demande q_i et le coût total est donné par une fonction $C(q_1, \dots, q_n)$. Une méthode de partage de coût est une fonction qui associe à chaque problème représenté par $(q; C)$, les proportions ou parts du coût total qui doivent être allouées à chacun des agents.

Le problème de partage de coût est étroitement relié à des modèles importants en économie normative, par exemple, le problème de l'exploitation des ressources communes, la régulation d'un monopoleur naturel et le partage de coût dans un réseau de télécommunication, etc.

Ce qui nous concerne dans cette thèse est l'analyse axiomatique des différentes méthodes de partage de coût. L'approche axiomatique vise donc à caractériser un ensemble de méthodes de partage de coût en se basant sur les propriétés ou axiomes mathématiques généraux ou normatifs. Cette approche axiomatique a été montrée très fructueusement dans la littérature.

Nous suivons les modèles standards du problème de partage de coût. Il existe trois formulations particulières dans la littérature, qui sont le modèle de demande binaire (chaque agent a une demande 0 ou 1), le modèle discret (les demandes sont des nombres entiers) et le modèle continu. Dans le modèle de demande binaire, le coût de satisfaire les demandes d'une coalition d'agents est donné par $c(S)$. Il est identique au modèle standard des jeux coopératifs avec utilité transférable. Suite à l'article important de Shapley [1953], il existe une vaste littérature sur les solutions

des jeux coopératifs et leurs applications dans le partage de coût (voir Young [1985a] pour une revue de cette littérature).

Le modèle discret a d'abord été introduit par Moulin [1995]. La théorie de la valeur de Shapley a été généralisée au cas où les demandes peuvent être des nombres entiers. En plus des axiomes d'Additivité et "Dummy" de Shapley, quelques nouveaux axiomes ont été ajoutés, telle que la Monotonie dans la demande. La Monotonie dans la demande exige que la part de coût d'un agent soit non-décroissante quand sa demande augmente. Moulin a caractérisé l'ensemble des méthodes de partage de coût satisfaisant les axiomes d'Additivité, de Dummy, et de Monotonie.

Le modèle continu a largement été étudié dans la littérature après l'article de Billera et al [1978], qui utilise la théorie des jeux non-atomiques (Aumann et Shapley [1974]) dans un problème de télécommunication pour fixer les taux de chaque appel. La littérature s'est concentrée sur les prix d'Aumann-Shapley (voir Tauman [1988] pour un survol). En 1995, Friedman et Moulin ont caractérisé la méthode de Shapley-Shubik (S-S) à l'aide de l'axiome de Monotonie dans la demande. Ils ont aussi introduit et caractérisé la méthode de Friedman-Moulin Serial (F-M). Ils ont découvert que les méthodes de S-S, de F-M, et de A-S peuvent se baser sur une représentation caractérisée par les axiomes d'Additivité et de Dummy. Précisément, la méthode S-S est caractérisée par la Monotonie dans la demande et l'Invariance à l'échelle, la méthode A-S est caractérisée par l'Invariance à l'échelle et la Proportionalité, et la méthode F-M est caractérisée par la Monotonie dans la demande et la "Serial Property" (voir Friedman et Moulin [1995]). Récemment, Friedman [1998] et Haimanko [1998] ont montré que l'ensemble des méthodes additives satisfaisant l'axiome "Dummy" est généré par toutes les combinaisons convexes des méthodes dites "path generated".

Un cas spécial du modèle continu est le cas d'un bien homogène. La méthode du coût moyen avait été la seule méthode concevable dans ce dernier cas avant que la méthode sérielle ait été proposée par Moulin et Shenker [1992]. Cette dernière méthode a des propriétés normatives et stratégiques remarquables (Moulin et Shenker [1992][1994]). Dans un article récent, Moulin et Shenker [1999] ont introduit l'axiome de Distributivité et caractérisé l'ensemble de méthodes satisfaisant les axiomes d'Additivité, de Distributivité, et de rendements constants. Cet ensemble inclut la méthode du coût moyen, la méthode sérielle, la méthode incrémentale, et beaucoup d'autres.

Le premier chapitre de cette thèse est une revue de la littérature. Il illustre comment la théorie des jeux coopératifs a été appliquée à des problèmes de partage de coût et a motivé l'approche axiomatique récente. Nous discutons essentiellement l'approche axiomatique de partage de coût, mais nous discutons aussi brièvement les questions de l'incitation-compatibilité et de l'efficacité de ces méthodes.

Dans le second chapitre, on considère le modèle discret. Dans la première section, on étudie l'ensemble des méthodes qui satisfont les axiomes d'Additivité et Dummy. On montre que cet ensemble est généré par toutes les combinaisons convexes de méthodes dites "path generated". Un chemin est une fonction croissante de $\{0, 1, \dots, \sum q_i\}$ à $[0, q]$. Étant donné un chemin, une méthode "path generated" assigne la somme des coûts marginaux à chaque agent le long du chemin. Dans le cadre des jeux coopératifs, Weber [1988] a caractérisé l'ensemble des valeurs satisfaisant les axiomes d'Additivité et de Dummy comme étant les valeurs d'ordre aléatoire. Dans le modèle de demande binaire, les valeurs d'ordre aléatoire sont identiques aux combinaisons convexes de méthodes "path generated", donc notre caractérisation généralise le résultat de Weber [1988]. Cette caractérisation est aussi très utile pour analyser les implications des autres axiomes, par exemple, l'axiome de Monotonie dans la

demande. A titre d'application, en utilisant un lemme de représentation de cette section, Sprumont [1998b] a caractérisé les méthodes d'ordre aléatoire par les axiomes d'Additivité, de Dummy, de Cohérence stricte et de Monotonie dans la demande. Une méthode d'ordre aléatoire simple est une méthode qui utilise le même ordre aléatoire pour tout vecteur de demande ayant les mêmes agents actifs. A l'aide du résultat de la section, Moulin [1999] a caractérisé la méthode S-S par les axiomes d'Additivité, de Dummy, et de Lower Bound.

Dans la section 2, nous discutons la méthode discrète de A-S. La méthode discrète de A-S est été introduite par Moulin [1995], mais sans caractérisation. Nous essayons d'examiner cette méthode et d'obtenir une caractérisation par les axiomes d'Additivité, de Dummy, et de Proportionalité. La Proportionalité est une propriété qui exige que la part de coût soit proportionnelle à la demande quand la fonction de coût est homogène. Nous avons construit un exemple pour montrer qu'en général, les axiomes d'Additivité, de Dummy, et de Proportionalité ne sont pas suffisants pour caractériser la méthode de A-S. Par contre, pour le cas de deux agents avec un agent ayant une demande d'une unité, nous avons montré que la méthode de A-S est la seule méthode qui satisfait les trois axiomes ci-dessus. Récemment, en utilisant l'approche de la théorie des jeux, E. Calvo et al [1998] ont donné une caractérisation récursive de la méthode de A-S par les axiomes d'Efficacité et de Contribution équilibrée (Myerson [1977], Hart et Mas-Colell [1989]). Mais nous pensons qu'il est possible de trouver une caractérisation sans avoir recours à la théorie des jeux.

Dans la section 3 du chapitre 2, nous étudions l'impact d'un axiome d'invariance à la mesure sur les méthodes additives. L'axiome d'invariance à la mesure est la version discrète de l'axiome d'invariance à l'échelle. Nous démontrons que l'ensemble de méthodes satisfaisant les axiomes d'Additivité, de Dummy, et d'invariance à la

mesure est composé des méthodes d'ordre aléatoire et que la méthode de S-S est la seule méthode symétrique dans cet ensemble. La méthode discrète de A-S ne satisfait pas cet axiome.

Dans le chapitre 3, nous considérons le modèle continu. Dans la section 1, nous examinons encore les méthodes additives satisfaisant l'axiome de Dummy. Nous donnons une caractérisation du sous-ensemble des méthodes qui satisfont les axiomes d'Additivité, de Dummy, et d'Ordinalité. L'axiome d'Ordinalité, qui a été introduit par Sprumont [1998a], est intuitivement la combinaison des axiomes de Monotonie dans la demande et d'Invariance à l'échelle. Plus précisément, cet axiome exige que les parts de coût soient invariants par rapport à toute transformation croissante de l'unité du bien. Parallèlement au modèle discret, nous montrons que l'ensemble des méthodes qui satisfont les axiomes d'Additivité, de Dummy, et d'Ordinalité est composé de toutes les méthodes d'ordre aléatoire simple, et que la méthode de S-S est la seule méthode symétrique dans cet ensemble.

Dans la section 2 du chapitre 3, nous considérons une méthode "non-additive", la méthode proportionnelle ajustée au coût marginal dite "Proportionally Adjusted Marginal Pricing" (PAMP). Cette méthode peut être aussi considérée comme une contrepartie modifiée de la "separable cost-remaining benefit method" (SCRBM, voir Young [1985b]) dans le modèle continu. Nous caractérisons la méthode PAMP par les axiomes d'Indépendance Locale, de Proportionalité, d'Invariance à l'échelle, et de Continuité.

General Introduction

A finite number of agents $N = \{1, 2, \dots, n\}$ share a production facility. Each agent $i \in N$ demands a quantity of an idiosyncratic output (good or service) q_i . They have to decide how to divide the total cost of production $C(q_1, \dots, q_n)$. The only information available to them are the cost data at various demand levels summarized by the cost function C and the reported demand profile $q = (q_1, \dots, q_n)$. They seek a solution or cost sharing method which provides them with a systematic way to allocate the cost among them for each such problem summarized by a pair $(q; C)$.

The cost sharing problem defined above exists in many important models of normative economics, from the exploitation of common resources, the regulation of a natural monopoly, cost sharing in a computer or telecommunication network, cooperation in production, and so on.

What we are concerned with in this thesis is the axiomatic analysis of various cost sharing methods. That is, we use normative or mathematical structural axioms to characterize and compare various cost sharing methods. This axiomatic approach to the cost sharing problem has proved to be very fruitful.

We follow the standard modeling practice of the cost sharing problem. There are three particular formulations in the literature. They are the binary demand model (each agent's demand is 0 or 1), the discrete model (integer demands), and the continuous model (demands are real numbers). In the binary demand model, the cost of satisfying the demands of a coalition S (a subset of agents) is conveniently denoted as $c(S)$. This is identical to the standard model of cooperative games with transferable utility. Following Shapley's [1953] seminal paper, there has been a vast literature on the solutions of cooperative games and their applications in cost sharing (see Young [1985a] for a review of this literature).

Moulin [1995] proposed the discrete model which allows the demands to vary in integer quantity. He extended the Shapley value theory to the discrete model by introducing the Demand Monotonicity axiom and retaining Shapley's two original axioms, Additivity and Dummy. The Demand Monotonicity requires that one's cost share should not decrease as one's demand increases. Moulin characterized the class of cost sharing methods satisfying Additivity, Dummy and Demand Monotonicity.

For the continuous model, the literature has been inspired by the paper of Billera et al [1978], in which non-atomic game theory (Aumann and Shapley [1975]) was used. Most of the literature focuses on the so-called Aumann-Shapley prices (see Tauman [1988] for the survey). In 1995, Friedman and Moulin provided an alternative axiomatic approach for the continuous model. They first proved a representation result for all cost sharing methods characterized by the axioms of Additivity and Dummy (hereafter called *additive* methods). Then, they showed that the well-known Shapley-Shubik (S-S) method is characterized by Demand Monotonicity and Scale Invariance, the Aumann-Shapley (A-S) method is characterized by Proportionality and Scale Invariance, and the Friedman-Moulin Serial (F-M) method is characterized by Demand Monotonicity and the Serial Property. The Proportionality axiom requires that when the cost function is homogeneous, the method uses average cost as price for each agent. The Scale Invariance axiom requires that cost shares be independent of the measuring scales of the goods. And the Serial Property says that when the cost function is homogeneous, cost shares are calculated by the well-known serial cost sharing method first proposed in the seminal paper of Moulin and Shenker [1992]. More recently, Friedman [1998] and Haimanko [1998] showed that the set of additive methods satisfying the dummy axiom is generated by all the convex combinations of the *path* generated methods.

In a recent paper, Sprumont [1998] proposed an Ordinality axiom which generalizes Scale Invariance to the invariance of cost shares with respect to any monotonic non-linear changes of the measurements of any good. He provided a characterization of the Shapley-Shubik method by Additivity, Dummy, Symmetry, and Ordinality. More importantly, he used Ordinality to explore non-additive methods.

In another direction, Moulin and Shenker [1999] provided an extensive study of the homogeneous good model based on two structural axioms: Additivity and Distributivity. Before, Average Cost Pricing had been the only conceivable method in this model. In 1992, Moulin and Shenker proposed the alternative *serial cost sharing* method, which can be derived by the following two properties: the "Equal Treatment of Equals" and that "cost shares are independent of larger demands". The serial cost sharing method has other remarkable normative and strategic properties (see Moulin and Shenker [1992][1994]). Recently, the above two authors [1999] introduced the Distributivity axiom and investigated the family of methods satisfying Additivity, Distributivity, and Constant Returns. The Distributivity axiom requires that cost sharing methods commute with the *composition* of cost functions. Moulin and Shenker characterized the family of methods satisfying the above three axioms and showed that this family is very rich, including the average cost pricing, serial cost sharing, incremental cost sharing, and more.

The first chapter of this thesis is a review of the literature. It summarizes how cooperative game theory has been used in cost sharing and motivated the recent axiomatic approach. While the most part of this review concentrates on axiomatic cost sharing, the review also includes a brief survey of the literature on the issues of incentive-compatibility and efficiency a cost sharing method generates.

Chapter 2 deals with the discrete model. There are three sections in this chapter.

Section 1 studies the set of additive methods. We show that this set is the set of all convex combinations of the so-called *path generated methods*. A path is a monotonic mapping from $\{0, 1, \dots, \sum q_i\}$ to $[0, q]$. Given a path, a path generated method assigns to each agent the sum of his (or her) marginal costs along this path. In the cooperative game model, Weber [1988] characterized the set of values satisfying Additivity and Dummy as being the *random order values*. The random order values are the convex combinations of the *incremental values*, each of which is associated with a random order of the agents (permutations of $\{1, 2, \dots, n\}$). Note that in the binary demand model or the cooperative game model, random order methods are identical to the convex combinations of the path generated methods since all the paths in $\{0, 1\}^N$ correspond (one-to-one) to the permutations of the agents. But in our discrete model, the random order values in the Weber's definition only correspond to a subset of paths (edges of the demand interval $[0, q]$). Therefore, our characterization generalizes Weber's.

This characterization result of additive methods is very useful in analyzing the implications of other axioms, e.g. Demand Monotonicity. As an application, Sprumont [1998b] used the (discrete) representation lemma to characterize the *simple* random order methods by the axioms of Additivity, Dummy, Strict Coherence (an informal explanation is given below) and Demand Monotonicity. By simple random order method we mean that the method uses the same random order method for all the demand vectors having the same active agents. Another application of this characterization result is Moulin [1999]'s characterization of the S-S method by Additivity, Dummy, and the Lower Bound axiom. See the survey of Moulin [1999] for more applications.

Section 2 (in chapter 2) discusses the discrete Aumann-Shapley method. The

discrete A-S method was first proposed (but not characterized) by Moulin [1995]. It is defined as the Shapley value of the replica game (in which each unit of each good is considered as a player) corresponding to the cost sharing problem. The discrete A-S is the arithmetic average of the path generated methods. It satisfies Additivity, Dummy, and Proportionality. Recall the characterization of the A-S method (Samet and Tauman [1982]). We want to know if we can find a similar characterization for the discrete Aumann-Shapley method. Unfortunately, in the discrete model, we do not have a scale invariance axiom which can play the same role as the Scale Invariance axiom in the continuous model. All the discrete versions of the scale invariance (Measurement Invariance, Moulin's Ordinality, Sprumont's Coherence) we have seen so far are so strong that they force additive methods to be simple random order methods (so the discrete A-S is excluded). On the other hand, we show by an example that the three axioms, Additivity, Dummy and Proportionality are not sufficient to characterize the discrete A-S. In the two-agent case where one agent's demand is fixed at one unit, we show that the discrete A-S is the only method satisfying Additivity, Dummy and Proportionality. In general, we introduce a condition called Constant Cost Sharing Ratios (CCSR) and show that Additivity, Dummy, Proportionality, and CCSR characterize the discrete A-S method. E. Calvo et al [1998] obtained a recursive characterization of the discrete A-S method using the game-theoretic approach (by converting the cost sharing problem into a *multichoice game*). The discrete A-S method is then characterized by *efficiency* and *balanced contributions*, which is similar to Myerson's [1977] and Hart and Mas-Colell's [1989] characterizations of the Shapley Value. However, our characterization does not rely on game theory.

In Section 3 of this chapter 2, we study the impact of a measurement invariance axiom on *additive* methods. The measurement invariance axiom is a discrete version

of the well-known *scale invariance* axiom. We show that the set of methods satisfying Additivity, Dummy, and Measurement Invariance consists of the *simple random order* methods, and that the Shapley-Shubik method is the unique symmetric method in that set. There are other formulations of scale invariance for the discrete model. Moulin [1995] used the Ordinality axiom (it is different from Sprumont's Ordinality [1998a], see below). Informally speaking, the axiom says that if the cost function is flat between two consecutive demands, erasing one unit of this good does not affect cost shares. Recently, Sprumont [1998b] proposed a Coherence axiom which requires that a cost sharing method should not *always* prescribe different cost shares between a given problem and its refined problem *no matter what the refined cost function turns out to be*. A cost function refines another cost function if the first one provides 'finer' cost data than the second. Generally speaking, Coherence is probably the most pertinent scale invariance in the discrete model. See Sprumont [1998b] for details. The Measurement Invariance, on the other hand, is probably the crudest version of the scale invariance.

Chapter 3 deals with the continuous model. In Section 1, we still consider the set of additive methods. But we study a subset of additive methods constrained by an axiom called Ordinality first introduced by Sprumont [1998]. Ordinality requires that cost shares be invariant with respect to any increasing (e.g., non-linear) transformations of the measurement of any good. Thus, it completely dispenses with any conventions to be used to measure the goods. This axiom is compelling in cost sharing problems which involve non-physical goods, e.g., services. Mathematically, Ordinality combines the properties of Scale Invariance and Demand Monotonicity. We show that the set of additive methods satisfying Ordinality consists of all simple random order methods, and as a corollary, the Shapley-Shubik is the only symmetric method in that set.

This result parallels the characterization of simple random order methods by the axioms of Additivity, Dummy, and Measurement Invariance in the discrete model (section 3 in Chapter 2). In spite of the strong restriction on the class of additive methods, Ordinality itself is still a flexible axiom. In fact, it has been combined with other axioms such as Proportionality or the Serial Property, to explore *non-additive* methods (see Sprumont [1998a]).

In Section 2 of Chapter 3, we propose and study a non-additive method called the *Proportionally Adjusted Marginal Pricing* method (PAMP). This method is not derived from Ordinality, but instead, is derived from a new axiom (with other axioms) called Local Independence. By Local Independence we mean that cost shares only depend on the information about costs around the final demand profile q (to be precise, the costs and the first order derivatives of the cost function at q). We are interested in PAMP because of its connection with the well-known Ramsey pricing as well as the *separable cost-remaining benefit* method (SCRB) (Moulin [1989], p139), which is frequently used in applications (see Young [1985c]). We provide a characterization of PAMP by Local Independence, Scale Invariance, Proportionality, and Continuity.

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**Chapter 1:
Overview of the Literature**

1.1. Introduction

1.2. Cooperative Game Theory-An Anecdote

1.3. Axiomatic Cost Sharing

1.4. Economic Efficiency and Demand Revelation

1.5. Summary

1 Introduction

A group of agents share a production facility with variable returns. Each demands a (variable) quantity of output(s). The total (non-separable) cost of these output(s) must be shared “fairly” among these agents. This definition of a cost sharing problem is quite general. The agents in a cost sharing problem can be interpreted either as people who share the joint production, or as goods whose real contributions to the total joint cost must be calculated, or as services a public facility provides, and more. The problem just defined is also non-trivial and often challenging because of the assumptions of variable returns of the production technology and the heterogeneity of the individual demands as well as, perhaps more important, the controversial notion of “equity” or “fairness”.

The formal modeling of the cost sharing problem is usually divided into two kinds of models, namely the models with preferences and the models without preferences but with demands. The models with preferences are used to discuss the efficiency and incentive compatibility issues along with the fairness issue in choosing cost sharing methods. The models without preferences are used to focus on the fairness issue. The advantages of the second kind of models are among others their simplicity and applicability. It is this kind of models that are called *axiomatic cost sharing* in the literature and will be discussed in this thesis.

Our general model of the cost sharing problem, then, can be formally described as follows. A finite set of agents $N = \{1, 2, \dots, n\}$ share a cost function C which is defined on a domain of all the conceivable demand profiles q , which are the vectors in the space R_+^N representing the list of demands. A problem is a pair $(q; C)$. A solution is a vector x in R_+^N assigning agent i the cost share x_i and satisfying the budget balance $\sum x_i = C(q)$. A method (or a rule, interchangeably called) is a mapping associating

with each problem a solution.

The above generally described cost sharing problems are, in fact, widespread in practice. The following three historical examples of cost sharing problems provide a snapshot of how the problems are identified, formulated and solved. They also provide anecdotes from the classical cooperative game theory approach to the more recent axiomatic approach, both of which will be discussed below. These examples have had significant influence in the literature.

The first example is the multipurpose reservoir. A dam on a river is planned to serve several different regional interests, such as flood control, hydro-electric power, navigation, irrigation, and municipal supply. The dam can be built to different heights, depending on which purposes are to be included. The cost function associated with such a problem exhibits decreasing marginal costs per acre-foot of water up to some critical height of the dam. The problem is how to apportion the cost among the different purposes. The problem was modeled as a cooperative game and the Shapley value (Shapley [1953]) was used as a solution. This problem has a rich history (see Ransmeier [1942] and Parker [1943]) and has been of both theoretical and practical importance. On the one hand, the idea of the "core" concept (Gillies [1959]) in cooperative game theory was foreshadowed in the cost-benefit analysis of this project (see Ransmeier [1942], p220, Young [1985], p8). On the other hand, certain cost sharing formulas recommended for this problem are still in use today by water resource agencies.

The second example was provided by Billera, Heath, and Raanan [1978] to set telephone billing rates which would allocate the cost arising in serving the consumers. The well-known Aumann-Shapley prices were used by casting the problem into a nonatomic cooperative game (Aumann and Shapley [1974]). Later, Aumann-Shapley

prices were characterized axiomatically by Billera and Heath [1982] and Mirman and Tauman [1982]. A vast literature on the Aumann-Shapley prices followed (see the survey by Tauman [1988]).

The third example is the Airport landing fees for Birmingham airport (see Littlechild and Thompson [1977]). The cost of a runway is determined by the size of the largest aircraft using it. A cooperative game model was defined by the problem and the well-known Shapley value was applied to that game. It turns out that the cost of serving the smallest types of aircraft is divided equally among the aircraft of all types, then the incremental cost of serving the second smallest type is divided equally among all aircraft except those of the smallest type, and so on. Interestingly, this equal splitting of the incremental cost among relevant agents coincides with the idea of serial cost sharing (see Moulin and Shenker [1992]).

The above examples typically have two characteristics. The first is that costs must be allocated exactly, with no profit or deficit. The second is that there is no objective basis at hand for attributing costs directly to specific agents, or products, or services. The problem is to find criteria and methods for allocating the costs in a just, equitable, fair, and reasonable way. Therefore, equity and fairness¹ are the ultimate concerns of cost sharing (however, see section 1.4 for the issues of economic efficiency and incentive-compatibility in cost sharing).

Despite these practical applications, there has until recently been relatively little theory about how cost sharing should be accomplished. In a classical example of pricing a multi-output monopoly which is constrained by budget balance, economic efficiency of the pricing method is emphasized, rather than the fairness property of the pricing. There had been a large literature centering on "Ramsey pricing" (Ramsey

¹In fact, the meaning of the word *fairness* in cost sharing is context-dependent. We will see that this notion can be best approached from various perspectives using the axiomatic approach.

[1927], Boiteux [1971], Baumol and Bradford [1970]). The main idea of Ramsey pricing is that in industries with declining average costs the percentage mark-up over marginal cost is greater the more inelastic the demand for the good is². Paradoxically, in spite of its theoretic significance, Ramsey pricing has not been well accepted in practice. It has two problems. The first is that it is equity-blind. Another problem with Ramsey pricing is that it is not practical since it relies on information about market demands that are hard to estimate.

A close relative to the cost sharing problem, the rationing problem has a long history dating back to Aristotle. A given amount of divisible good must be divided among agents with different claims (see O'Neill [1982], and Rabinovich [1973] for examples of rationing problems). The amount is short of the total claims. A solution is a vector which specifies how much each agent gets. A rationing method provides each rationing problem a solution. Aristotle proposed that "equals should be treated equally, unequals unequally according to their differences and similarities". According to this principle, the Proportional method (the proportional division with respect to the claims) has been suggested and it has been the dominant method for rationing problems (until other methods have been discovered recently by the axiomatic approach, see the section 1.3).

While Aristotle's principle of equity has far-reaching implications in distributive justice, its limit in cost sharing is immediate. In the case of a homogeneous good and constant returns of scale, Aristotle's principle implies proportional division. In general cases, such as variable returns of scale and heterogeneous outputs, it is not clear how to interpret the second part of the principle.

The recent axiomatic cost sharing literature explores the logical limits of Aris-

²See the interesting explanation by W. Arthur Lewis [1949]: The principle is ... that those who cannot escape must make the largest contribution to indivisible cost, and those to whom the commodity does not matter much may escape.

tote's principle within a number of specifications of our general cost sharing model defined in the beginning. This literature has drawn on classical cooperative game theory. Particularly, it has been inspired by the seminal paper of Shapley [1953]. Shapley [1953] developed a set of simple but persuasive axioms, mainly the Additivity and Dummy axioms, which lead to the selection of a value (Shapley value) for each player (agent) in a cooperative game (see the next section). Briefly, a cooperative game is a function associating with each subset (called coalition) of players a real number. A value is a solution mapping each game to a vector such that the sum of the components of this vector is equal to the number (the worth) associated with the grand coalition (the whole player set). In fact, Shapley showed that there exists a unique symmetric solution, the Shapley value, satisfying the Additivity and Dummy axioms³.

Shapley's Additivity axiom is a mathematical invariance property. It requires that the solution commutes with the addition of two games. In other words, it says that it does not matter whether we compute cost shares on the two separate components of the total costs or on the combined costs. A standard interpretation is accounting decentralization (Billera and Heath [1982], Mirman and Tauman [1982])⁴. Additivity is not an equity axiom and has no ethical meaning, and had once been a source of controversies (Moriarty [1981]). Now this axiom has become standard in the literature. Shapley's second axiom, the Dummy axiom, is an equity axiom which requires that if a player in a game contributes nothing to each coalition, his value should be zero⁵. This dummy axiom has been well accepted.

Shubik [1962] was the first to propose using the Shapley value to cost sharing

³Weber [1988] generalized Shapley's result by dropping the Symmetry axiom and showed that the pair of Additivity and Dummy characterizes the set of random order values. See Chapter II.

⁴A. Roth [1988] argued that Additivity can be considered as a risk neutrality property.

⁵The Dummy axiom rules out the equal division solution, namely $c(N)/n$, where $n = |N|$ is the total number of players.

problems. In the binary demand case in which each agent's demand is either 0 or 1, the problem is equivalent to a cooperative game (see the next section). Then, the Shapley value can be used as a solution. In the variable demand case, a so-called stand alone cost game can be generated for each given cost sharing problem, which associates with each coalition the costs of fully serving the agents in the coalition and not serving agents outside the coalition (see the formal definition in the next section). Applying the Shapley value to the stand alone game a cost sharing problem generates defines the so-called Shapley-Shubik method (see the formula in section 3.2).

Aumann and Shapley [1974] further generalized the Shapley value theory to games with infinite number of agents, the so-called non-atomic games. Briefly, a non-atom game is function v defined on a σ -field \mathcal{B} of all Borel sets of the interval $[0, 1]$. The sets in \mathcal{B} are called coalitions. Aumann and Shapley [1974] extended the Shapley's axioms in n -person game theory (usually called cooperative game theory) to non-atomic games and defined the so-called Aumann-Shapley value. In continuous cost sharing model (see the classification below), a cost sharing problem corresponds to a non-atomic game and the Aumann-Shapley value of the non-atomic games corresponds to the well-known Aumann-Shapley (pricing) method. See Tauman [1988] for details.

However, the above game theoretic approach to cost sharing has limitations. In the variable demand case, in order to use the solution concepts in cooperative game theory, first we have to cast a cost sharing problem into a game (the stand alone game). This casting makes a cost sharing method depend only on the stand alone costs, ignoring the information of the cost function at other demand levels. Therefore, the implication of some other axioms, for instance, the Demand Monotonicity (firstly proposed by Moulin [1995]) on cost sharing methods can not be analyzed. This limitation motivates us to take the general axiomatic approach, which is beyond and

independent of the game-theoretic interpretation.

The idea of the axiomatic approach to cost sharing, as the Shapley value approach in the cooperative game theory, is simply to use axioms to select and characterize various cost sharing methods. Following the tradition initiated by Shapley [1953], we adapt his two original Additivity and Dummy axioms to cost sharing models. The interpretations of these two axioms in cost sharing are the same as in cooperative game theory. However, in addition to these two classical axioms, new axioms can be introduced thanks to the variable demands. The two most prominent examples are the Proportionality axiom and the Demand Monotonicity axiom. The Proportionality axiom requires that if the cost function is homogeneous (one-dimensional, see the next paragraph) then cost shares should be proportional to demands (also called Average Cost Pricing). The Proportionality axiom is relatively classical and within the additive methods ⁶ it essentially characterizes the well-known Aumann-Shapley pricing (Billera et al [1982], Samet and Tauman [1982]). The Demand Monotonicity (Moulin [1995]) requires that each agent's cost share should be a non-decreasing function of his demand. Within the additive methods, this axiom essentially characterizes the Shapley-Shubik method.

In the literature, cost sharing problems are conveniently classified into two families of models, namely the homogeneous (demand) model and the heterogeneous (demand) model. In the former case, the demands enter additively into the cost function, and the cost function is called homogeneous. Symmetrically, when each agent's demand is a personalized good (may or may not be the same with each other) and therefore each agent can be identified by his demand, we say the problem is heterogeneous.

The heterogeneous problems are further classified into three kinds of models,

⁶A method is called additive if it satisfies both Additivity and Dummy axioms. Sometimes we explicitly mention the Dummy axiom when we want to emphasize these two axioms.

namely the binary demand model, the discrete model (Moulin [1995]) in which demands are integers, and the continuous model in which demands are real numbers. Since the binary demand model is equivalent to the (monotonic) cooperative game model, it is the last two models that are to be discussed in this thesis.

Until recently, the literature on axiomatic cost sharing had been mainly focusing on the Aumann-Shapley method (see Tauman [1988]). In 1995, Moulin first generalized the axiomatic theory of the Shapley value to the discrete model. He showed that the Aumann-Shapley method violates Demand Monotonicity and the Shapley-Shubik method is recommended instead. His approach opened an alternative route in axiomatic cost sharing, which turned out to be very rich. Apart from the Shapley-Shubik method, other methods have been discovered, e.g., the serial method and the incremental methods (Moulin [1995]).

In 1995, Friedman and Moulin considered the continuous model and provided a systematic study of the family of additive methods. Similar as Moulin's discussion on the discrete model, they showed that the demand monotonicity again puts the Shapley-Shubik method in the front. In particular, they provided a representation formula for the family of additive methods, which has played an important role in analyzing the impact of other axioms (see the following discussion for the refinements of this representation result).

The axiomatic cost sharing literature has since experienced a fast expansion. Recently, Moulin and Shenker [1999] reconsidered the homogeneous model and found that the two prominent methods, average cost pricing and serial cost sharing, are only two extreme examples among a very rich family of methods which combine the properties of proportionality and priority⁷ in a very complex way⁸. This analysis is

⁷The agents are ordered and the incremental costs are charged according to the given ordering. This is called the incremental method (or value).

⁸The combination of proportionality and priority is determined by the divisions of the demand

very deep and complex and can be regarded as an exploration of the second part of the Aristotle's equity principle in full dimension.

In another direction, namely the heterogeneous models, Friedman and Moulin's [1995] integral representation of additive methods has been refined by Friedman [1998] and Haimanko [1998]. They showed that the set of additive methods is the set of all (infinite) convex combinations of the path generated methods. A path is a monotonic mapping from a finite interval, say $[0, 1]$, to the domain of demand vectors. A path generated method assigns to each agent the integral of his marginal costs along this path. In the discrete model, a parallel result has been established by Wang [1999] (see Chapter II). Using the language of isomorphism⁹ or structure equivalence, now we can summarize the structures of three families of additive methods across three different models, namely the rationing model, the discrete cost sharing model, and the continuous cost sharing model as follows: the set of additive methods in the homogeneous model is linearly isomorphic to the set of rationing methods, and the set of rationing methods is isomorphic to the extreme points¹⁰ of the convex set of additive methods in the heterogeneous models¹¹ (see section 1.3). These isomorphisms allow us to follow the "same" method across three different models (see Moulin [1999], page 4).

As we said in the beginning, our discussion of axiomatic cost sharing considers only space, rather than the traditional convex combinations of methods. See Moulin and Shenker [1999] for the detail.

⁹We say a set A in a vector space is isomorphic to a set B in another vector space if there is a linear one-to-one mapping from A to B .

¹⁰A set is called convex if for any two points in the set the whole line connecting these two points is in the set. A point of a convex set is said to be an extreme point if it cannot be the center of a line between two points in the set. The formal definition of convex set and its extreme points can be found in the first paper of the next chapter or in the book of Rockafellar [1970].

¹¹The discrete model deserves special attention. The corresponding rationing problems should be the discrete counterparts of the traditional rationing problems, namely, the amount to be divided, the individual claims, as well as the shares are all integers. See the recent paper of Moulin [1999], "The Proportional Random Allocation of Indivisible Units," for the detail.

the models with demands instead of preferences. Therefore, the issues of efficiency and the incentive properties of the cost sharing methods can not be discussed. To discuss these issues, the models must include preferences. There are two different approaches in the literature. One is in the mode of decentralization, i.e., given a cost sharing (or surplus sharing, symmetrically) method, what is the equilibrium outcome of the generated cost sharing game? There, the incentive compatibility or strategy-proofness is implicitly incorporated into the equilibrium concept. But this approach puts the efficiency issue of the method in the second place, giving no responsibility to the users to compare and choose alternative cost sharing methods (no means to enforce the efficient methods). Recall the problem of the Tragedy of the Commons (Hardin [1968]). We will not discuss this literature in this thesis. See Moulin and Watts [1994] and Watts [1996] for a discussion of it. Another approach, which is more general, is to regard the problem as a social choice problem, and an efficient, incentive-compatible, and equitable cost sharing method is to be selected. This approach is in the mode of centralization. We will discuss it briefly in section 1.4. But a thorough discussion of it will be beyond the scope of the thesis.

In the following sections we will provide a further detailed review of the literature. Section 1.2 is a brief introduction of cooperative game theory. Section 1.3 summarizes the modern axiomatic approach to cost sharing. Section 1.4 briefly discusses the economic efficiency and demand revelation related to cost sharing. Section 1.5 concludes this overview.

2 Cooperative Game Theory-An Anecdote

In this section, we give a brief introduction of cooperative game theory and show how it is used to solve cost sharing problems. The most relevant part is Shapley's value

theory, which inspired the recent axiomatic cost sharing literature.

Let $N = \{1, \dots, n\}$ be the set of agents (users or outputs or services) of a public facility. Each agent will be served at some targeted level if he is included or not served at all if he is not included. The problem is to determine how much to charge for the services, based on the costs of providing them.

Formally, the problem can be modeled in the following two equivalent ways¹². First, in terms of the standard cost sharing modeling, each agent $i \in N$ is represented by a binary variable $q_i = 0$, or 1. So, a demand vector q is a vector in $\{0, 1\}^N$. The cost of serving the agents in a subset $S \subseteq N$ is denoted by $C(1_S, 0_{-S})$, where 1_S is the restriction of the vector $e = (1, \dots, 1)$ on S , $-S$ denotes $N - S$ which is $N \setminus S$, similarly 0_S is the restriction of the zero vector on subset $-S$. Thus, a cost function is a (non-decreasing) mapping $C : \{0, 1\}^N \rightarrow R$ with $C(0, \dots, 0) = 0$. Finally, a problem is a pair $(q; C)$.

The second, and more natural formulation of this (binary demand) problem, is the following cooperative game model:

For each $S \subseteq N$, denote $c(S)$ the cost of serving agents of S . Usually, call $c(S)$ the stand-alone cost of the coalition S . Most often, $c(S)$ represents the least cost of serving the agents in S by the most efficient means. Naturally, c is monotonic, i.e., $c(S) \leq c(T)$ when $S \subseteq T \subseteq N$. The cost of serving no one is assumed to be zero: $c(\emptyset) = 0$.

Thus, a game is a (here, monotonic) mapping $c : 2^N \rightarrow R$ with $c(\emptyset) = 0$, where $2^N = \{S | S \subseteq N\}$.

By the above definitions, obviously,

$$c(S) = C(1_S, 0_{-S}), S \subseteq N. \quad (1)$$

¹²Interestingly, this is exactly the point that the cooperative game model and the cost sharing model divide.

In other words, a binary demand cost sharing model is equivalent to a cooperative game model.

A solution (here a cost sharing method) is a mapping x defined for all (finite set) N and all c on N such that

$$x(c) = (x_1, \dots, x_n) \in R^N \text{ and } \sum x_i = c(N).$$

where x_i is the charge assigned to agent i .

Many solution concepts of cooperative game theory can be used. The “core” (Gillies [1959]) is the set of cost share vectors : $\{(x_1, \dots, x_n) | \sum x_i = c(N)\}$ such that

$$\sum_S x_i \leq c(S), \text{ for all } S \subseteq N.$$

The interpretation is that if cooperation among the agents is to be voluntary, then the calculus of self-interest dictates that no participant or group of participants be charged more than their stand-alone (opportunity) costs. Otherwise they would have no incentive to agree to the proposed allocation.

An equivalent equity interpretation of this incentive compatibility condition, called no subsidizing, is as follows.

$$\sum_S x_i \geq c(N) - c(N - S), \text{ for all } S \subseteq N.$$

Shapley [1953] used the following three axioms to characterize (and derive) a one-point solution, the well-known Shapley value, for cooperative games:

1) Additivity:

$$x(c_1 + c_2) = x(c_1) + x(c_2), \text{ for any two cost games } c_1, c_2.$$

2) Dummy: If $i \in N$ and $c(S \cup i) - c(S) = 0$ for all $S \subseteq N$, then

$$x_i(c) = 0.$$

3) Symmetry: If $i, j \in N$ and $c(S \cup i) - c(S) = c(S \cup j) - c(S)$ for all $S \subseteq N$ ($i, j \notin S$), then

$$x_i(c) = x_j(c).$$

He showed that there is a unique value x satisfying these three axioms, namely,

$$x_i(c) = \sum_{S \subseteq N: i \in S} \frac{|S - i|! |N - S|!}{|N|!} (c(S) - c(S - i)), \quad i = 1, \dots, n.$$

The Shapley value is an important solution concept in cooperative game theory (see the book A. Roth "The Shapley Value" [1988]). It has been characterized from many different angles (e.g., Young [1985b], Myerson [1977], Hart and Mas-Colell [1989]). It generalizes the concept of marginal value, namely the incremental value for a given ordering of the agents (see below), by taking a convex combination of all these incremental values with equal probability to each possible ordering.

Given an ordering of N , say $\{1, 2, \dots, n\}$, the incremental value with respect to this ordering is defined as

$$x_i^{inc}(c) = c(\{1, 2, \dots, i\}) - c(\{1, 2, \dots, i - 1\}), \text{ for each } i \in N. \quad (2)$$

The Shapley value has the following interpretation in cost sharing. Imagine that the participants in a cost sharing problem are rational agents who view the outcome as being subject to uncertainty. They might reason about their prospects as follows. Everyone is thought of as "signing up", or committing themselves to some random order. At each stage of the sign-up the cost sharing method is myopic: each player must pay the incremental cost of being included at the moment of signing. The assessments will therefore depend on the particular order in which the players join. Instead of actually proceeding in this way, rational agents might simply evaluate their prospects by calculating their expected cost shares from such a scheme. Assume that all orderings are a priori equally likely. Then, the formula is agent i 's "expected" cost

share. That is, it is the average marginal contribution to the total cost each player would make.

Another solution concept in cooperative game theory is the “nucleolus” (Schmeidler [1968]). First, given a cost sharing vector x , we say coalition S is better off than T with respect to x if

$$c(S) - \sum_S x_i > c(T) - \sum_T x_i.$$

The number $e(x, S) = c(S) - \sum_S x_i$ is called the excess of S relative to x .

Then, we find an allocation x that maximizes the minimum excess $e(x, S)$ over all proper subsets $\emptyset \subset S \subset N$, i.e.,

$$\begin{aligned} \max \quad & \epsilon \\ \text{s.t.} \quad & e(x, S) \geq \epsilon \quad \forall S \neq \emptyset, N, \\ & \sum_N x_i = c(N). \end{aligned}$$

If the solution is unique, it is called the “nucleolus” of c . If it is not unique, use the following tie-breaking rule. Order $e(x, S)$ ($\emptyset \subset S \subset N$), from lowest to highest and denote this $2^n - 2$ vector by $e(x)$. The “nucleolus” is the vector x that maximizes $e(x)$ lexicographically, i.e., for which the value of the smallest excess is as large as possible and is attained on as few sets as possible, the next smallest excess is as large as possible, and is attained on as few sets as possible, and so on. The “nucleolus” was axiomatically characterized by Sobolev [1975] on the basis of a consistency axiom.

3 Axiomatic Cost Sharing

Axiomatic cost sharing, as we said in the introduction, draws on Shapley’s axiomatic approach to cooperative game theory. It goes beyond the cooperative game approach to cost sharing. The axiomatic approach allows us to introduce and investigate many

other axioms beyond Shapley's two axioms¹³ on cost sharing methods. More interesting cost sharing methods can be studied and selected based on various combinations of these axioms.

A number of papers using the axiomatic approach in cost sharing had been scattered in the literature. Inspired by the Shapley value theory, Loehman and Whinston [1974] were among the first who explicitly used the axiomatic approach in cost sharing. Later, inspired by the non-atomic game theory (Aumann-Shapley [1974], Billera, Heath and Raanan [1978]), Billera and Heath [1982], Mirman and Tauman [1982], Samet and Tauman [1982] used the axiomatic approach to characterize the Aumann-Shapley method.

The paper of Moulin [1995], who firstly extended the Shapley value theory to the discrete cost sharing model by introducing the remarkable Demand Monotonicity axiom, motivated the recent alternative current of the literature. In the same vein, the paper of Friedman and Moulin [1995] initiated the investigation of the family of additive methods in the continuous model. These two papers promoted an extensive exploration of additive methods both in the discrete model and in the continuous model.

For the sake of convenience as well as by the tradition of the literature, we separate our discussion into two main parts, namely the homogeneous model and the heterogeneous model. The later is further classified into two subclasses: the discrete model and the continuous model. The relation between cost sharing and rationing is discussed in the homogeneous model.

¹³The Additivity and Dummy axioms

3.1 The Homogeneous Good Model and the Rationing Problem

A **homogeneous cost sharing problem** is a triple (N, C, q) where N is a finite set of agents $i = 1, \dots, n$, C is a non-decreasing cost function from R_+ into R_+ such that $C(0) = 0$, and $q = (q_1, \dots, q_n)$ is the demand profile in which agent i demand $q_i \geq 0$.

A **solution** to the (cost sharing) problem (N, C, q) is a vector $x = (x_1, \dots, x_n) \in R_+^n$ specifying a cost share for every agent and such that

$$\sum_{i \in N} x_i = C\left(\sum_{i \in N} q_i\right).$$

A **method** is a mapping x associating to any problem (N, C, q) a solution $x(N, C, q)$ (when N is fixed we often write $x(q; C)$).

In the following special case where the technology of the joint production exhibits a constant returns of scale, i.e., the cost function is a linear function, it is obvious that cost shares should be proportional to individual demands of output. This follows from Aristotle's proportionality principle. Now we have the following first axiom:

Constant Returns (CR, also called Separability):

$$\{C(z) = \lambda z \text{ for all } z \geq 0\} \Rightarrow \{x(N, C, q) = \lambda q\} \text{ for all } N, \lambda \geq 0, C, \text{ and } q.$$

The challenge is when the technology exhibits variable returns of scale, i.e., when the cost function is not linear. The following two structural axioms on the cost sharing methods are well-known. Before we present them, we need to define the following class of cost functions.

Let \mathcal{C} be a generic domain of cost functions. In the papers of Moulin and Shenker [1994], and Moulin [1999], \mathcal{C} consists of all cost functions C that can be written as the difference of two convex functions. Then, it contains all the twice continuously

differentiable cost functions, as well as the piecewise linear ones. In the following main theorem 1, the set \mathcal{C} is such a domain.

In the literature, we often use the following Additivity axiom:

Additivity (ADD):

$$x(q; C_1 + C_2) = x(q; C_1) + x(q; C_2), \text{ for all } C_1, C_2 \in \mathcal{C}, \text{ all } q.$$

This axiom allows to decompose the computation of cost shares whenever the cost function can be additively decomposed. Many important results in the sequel rely on Additivity.

The following Distributivity axiom was recently proposed by Moulin and Shenker [1999].

Distributivity (Dis):

$$x(q; C_1 \circ C_2) = x(x(q; C_2), C_1), \text{ for all } C_1, C_2 \in \mathcal{C}, \text{ and all } q.$$

Denote by $\mathcal{M}(CR, ADD)$ the class of methods satisfying Constant Returns and Additivity. Denote by $\mathcal{H}(CR, ADD, Dis)$ the class of methods satisfying Constant Returns, Additivity, and Distributivity.

In order to reveal the structure of these two families of methods, we relate cost sharing to the well-known rationing problem.

A rationing problem is a triple (N, t, x) where N is a finite set of agents, the non-negative number t represents the amount of resources to be divided, the vector $x = (x_1, \dots, x_n)$ specifies for each agent i a claim x_i , and these numbers are such that

$$x_i \geq 0 \text{ for all } i; \quad 0 \leq t \leq \sum_{i \in N} x_i.$$

A **solution** to the rationing problem is a vector $y = (y_1, \dots, y_n)$, specifying a share y_i for each agent i such that

$$0 \leq y_i \leq x_i \text{ for all } i; \sum_{i \in N} y_i = t.$$

A **rationing method** r associates to each rationing problem (N, t, x) a solution $y = r(N, t, x)$. When N is fixed, we simply write $r(t, x)$. Call a rationing method r *monotonic* if

$$\begin{aligned} 0 \leq r(t, x) \leq x, r_N(t, x) = t \text{ for all } t, 0 \leq t \leq x_N, \\ t \leq t' \Rightarrow r(t, x) \leq r(t', x) \text{ for all } t, t', 0 \leq t, t' \leq x_N. \end{aligned}$$

Note that a monotonic rationing method defines for all $x \in R_+^N$ a monotonic continuous path $t \rightarrow r(t, x)$ from 0 to x . Let \mathcal{R} be the set of all monotonic rationing methods.

The rationing problem has a long history (see Rabinovitch [1973] for examples from the Babylonian Talmud). It has recently received an intensive study from the axiomatic perspective (see O'Neill [1982], Aumann and Maschler [1985], Young [1988][1990], Moulin [1987], Chun [1988], Banker [1981], Balinski and Young [1982], Sprumont [1991]).

Denote by Γ_t the cost function $\Gamma_t(z) = \min\{z, t\}$, $z \geq 0$. The following important result links the rationing methods to the cost sharing methods.

Theorem 1 (Moulin [1999]) *The following two mappings, from \mathcal{R} into $\mathcal{M}(CR, ADD)$ and from $\mathcal{M}(CR, ADD)$ into \mathcal{R} :*

$$r \rightarrow x : x(q; C) = \int_0^{q_N} C'(t) dr(t, q) \text{ for all } C \in \mathcal{C}, q,$$

$$x \rightarrow r : r(t, q) = x(\Gamma_t, q) \text{ for all } t, q,$$

define a linear isomorphism between \mathcal{R} and $\mathcal{M}(CR, ADD)$.



$$C(q; q) = \frac{C(q^i)}{C(q^j)} - \frac{n-i+1}{\sum_{i=1}^{j-1} (n-i+1)C(q^i)}$$

the incremental cost $C(q^i) - C(q^{i-1})$ between agents $\{i, i+1, \dots, n\}$, i.e.,

Then, the serial cost sharing is defined as follows: For $i = 1, \dots, n$ divide equally

$$q_n = q_1 + q_2 + \dots + q_n.$$

$$q_1 = nq_1; q_2 = q_1 + (n-1)q_2; \dots; q_i = q_1 + (n-i+1)q_i; \dots; q_n = q_1 + \sum_{i=1}^{n-1} q_i; \dots;$$

demands, for example, $q_1 \leq q_2 \leq \dots \leq q_n$. Define a sequence $q^i, i = 1, \dots, n$ as follows:

$q_i \geq q_j \Rightarrow q_i' \geq q_j'$, then $x_i(q; C) = x_i(q'; C)$. Formally, ordering agents by increasing

Independence of Higher Demands (for all C, q, q' and i if for all $j, q_j \leq q_j' \Rightarrow q_i = q_i'$ and

Equal Treatment of Equals (equal cost shares for equal demands) and *Independence of Lower Demands* (for all C, q, q' and i if for all $j, q_j \geq q_j' \Rightarrow q_i = q_i'$ and

Serial Cost Sharing is defined (and characterized) by the following two properties:

(see Moulin and Shenker [1994]).

Monotonicity means that $C_1 \leq C_2$ implies $x(q; C_1) \leq x(q; C_2)$ for all C_1, C_2 and all q

or it is characterized by the Constant Returns and Cost Monotonicity. Here, Cost

property and the property of "no advantage of reallocations" (see Moulin [1999]),

example, it is the unique method satisfying the "no charge for null demand"

The average cost pricing has been axiomatically characterized in many ways. For

$$x_i(q; C) = \frac{q_i}{C(q^N)}, i \in N.$$

demands, i.e.,

The **Average Cost Pricing** divides the total cost in proportion to individual

rationing, and so on (see Moulin [1999] for the corresponding rationing methods).

the uniform gains rationing, the incremental cost sharing corresponds to the priority

to the proportional rationing method. And the serial cost sharing corresponds to

In particular, the following well-known average cost pricing method corresponds



Serial cost sharing has also been characterized by Moulin and Shenker [1994].

Moulin and Shenker [1999] investigated the class $\mathcal{H}(CR, ADD, Dis)$. Its structure was shown to be very complex. Indeed, the construction of the class $\mathcal{H}(CR, ADD, Dis)$ relies on the Partition of the Demand Space induced by the Coverings of the Unit Simplex in R_+^N .

Denote by S the unit simplex of R_+^N . A *polytope* of dimension d , $d \leq n$, in S is the convex hull of d affinely independent elements e^1, \dots, e^d of S . An *ordered polytope* is a polytope with an ordering of its vertices $\{e^k\}$, simply denoted as an ordered list $P = \{e^1, \dots, e^d\}$. Denote \mathcal{P} the set of all ordered polytopes.

Two ordered polytopes P^1 and P^2 are said to be *adjacent* if i) their relative interiors are disjoint, ii) the set A of their common vertices is non empty, iii) $P^1 \cap P^2 = \text{conv}(A)$ (the intersection of P^1 and P^2 is the convex hull of the vertices in A), and iv) P^1 and P^2 induce the same ordering of A . In particular, an ordered polytope P and any of its faces are adjacent, provided the face inherits the ordering of P . A face of a polytope is a polytope generated by the subset of the vertices of the original polytope.

Definition 1 : *Ordered Coverings (Moulin and Shenker [1999])*

An ordered covering of the simplex S is a set \mathcal{C}_S (not necessarily finite) of ordered polytopes ($\mathcal{C}_S \subseteq \mathcal{P}$) such that:

- i) their union covers S : $\cup_{\mathcal{C}_S} P = S$,
- ii) if \mathcal{C}_S contains P , it contains all the ordered faces of P ,
- iii) any two ordered polytopes in \mathcal{C}_S are either disjoint or adjacent.

Note that by definition of adjacency, if two elements of \mathcal{C}_S are adjacent, their intersection is a common face, hence it is in \mathcal{C}_S , too. Next, any ordered covering defines a partition of S , namely the family $\{P^0\}$ of the relative interiors of its polytopes.

Correspondingly, the family of cones generated from each polytope in the covering: $\{[P^0]\}$ is a partition of $R_+^N \setminus 0$. See Moulin and Shenker [1999] for details.

Definition 2 *The Family \mathcal{H} of Cost Sharing Methods (Moulin and Shenker [1999])*

To each ordered covering \mathcal{C}_S of S , we associate the following cost sharing method. For any element $P = \{e^1, \dots, e^d\}$ of \mathcal{C}_S and any vector q in $[P]$ we have:

$$\begin{aligned} \{q = \sum_{k=1}^d \lambda_k e^k, \text{ for all } k, \lambda_k \geq 0\} \Rightarrow \\ x(q; C) = \sum_{k=1}^d [C(\lambda_{\{1, \dots, k\}}) - C(\lambda_{\{1, \dots, k-1\}})] \cdot e^k \end{aligned} \quad (3)$$

(where $\lambda_S = \sum_{i \in S} \lambda_i$ with convention $\lambda_\emptyset = 0$).

Remarkably, Moulin and Shenker [1999] showed that the following characterization result holds:

$$\mathcal{H}(CR, ADD, Dis) = \mathcal{H}.$$

To be precise, in order to show that the family \mathcal{H} can be characterized by the combination of Additivity, Constant Returns, and Distributivity, the following two choices must be made: either the domain of cost functions is restricted, or a continuity requirement with respect to variations of the cost functions is added to the three basic axioms.

The subdomain \mathcal{C}^* of \mathcal{C} is defined as :

$$\mathcal{C}^* = \{C \in \mathcal{C} \mid C = C^1 - C^2 \text{ for some convex functions } C^1, C^2 \in \mathcal{C}\}.$$

The domain \mathcal{C}^* contains all twice continuously differentiable functions in \mathcal{C} , as well as all the piecewise linear functions. Moreover it is a dense subset of \mathcal{C} for the topology of pointwise convergence (see Moulin and Shenker [1994], p.184).

Next, a cost sharing method x is *continuous* (w.r.t. the cost function) if for all q in R_+^N , $x(q; C^s)$ converges to $x(q; C)$ whenever the sequence $\{C^s; s = 1, 2, \dots\}$ converges pointwise to C (where C^s and C are in \mathcal{C}).

Theorem 2 (Moulin and Shenker [1999])

a) A cost sharing method x with domain \mathcal{C}^* , satisfies Separability, Distributivity and Additivity if and only if it is (the restriction to \mathcal{C}^* of) a method in \mathcal{H} .

b) A cost sharing method x with domain \mathcal{C} , satisfies Separability, Distributivity, Additivity and Continuity if and only if it is a method in \mathcal{H} .

3.2 The Discrete Heterogeneous Goods Model

The discrete model was first proposed by Moulin [1995]. It generalizes the cooperative (monotonic) game model by allowing each individual demand to vary in non-negative integer quantities. The key ingredient in Moulin's axiomatic generalization of the Shapley value theory is the Demand Monotonicity axiom.

Formally, **the discrete model** is defined by a triple $(N, \bar{q}, \mathcal{C})$ where \bar{q} is a capacity vector in $\{0, 1, 2, \dots\}^N$ and \mathcal{C} is the set of cost functions $C : [0, \bar{q}] \rightarrow R_+$ (non-decreasing and $C(0) = 0$).

A **(discrete) cost sharing problem** is a tuple $(q; C)$ (fixed population) where $q \in [0, \bar{q}]$ and $C \in \mathcal{C}$. The solution concept and cost sharing method are obviously defined.

It is easy to see that for given N , if the capacity vector is $\bar{q} = (1, \dots, 1)$, the cost sharing problem becomes a monotonic cooperative game with transferable utility.

The Additivity and Dummy axioms in Shapley's paper [1953] were translated word by word by Moulin [1995] to the discrete cost sharing model as follows:

Additivity(ADD):

$$x(q; C_1 + C_2) = x(q; C_1) + x(q; C_2), \forall C_1, C_2 \in \mathcal{C}, q \in [0, \bar{q}].$$

Dummy(DUM): For any $i \in N$,

$$\{C(q) - C(q_i - 1, q_{N \setminus i}) = 0 \ \forall q \in [0, \bar{q}] \text{ s.t. } q_i > 0\} \Rightarrow \{x_i(q; C) = 0\}.$$

For convenience, denote $\partial_i C(q) = C(q) - C(q_i - 1, q_{N \setminus i})$.

Consider the family of cost sharing methods satisfying Additivity and Dummy, namely the set of *additive* methods. For simplicity, we fix the demand profile q for a moment. We will see that the result in the sequel generalizes to variable demands.

The basic elements in the set of additive methods are the following so-called path generated methods (see Section 1 in Chapter 2 for details).

First, a path is a monotonic mapping $P : \{0, 1, \dots, q(N)\} \rightarrow [0, q]$ with $P(0) = 0$ and $P(q(N)) = q$, where $q(N) = \sum_N q_i$. Denote \mathcal{P} the set of all paths to q .

A path generated method, generated by a path $P \in \mathcal{P}$, is a method which charges each agent the sum of his marginal costs along the path.

Note that the path generated methods satisfy the Additivity and Dummy axioms (this fact is easy to be checked). In Section 1 of Chapter 2, we will show that any additive method is a convex combination of the path generated methods.

Moulin [1995] considered the impact of the Demand Monotonicity axiom on the set of additive methods. He showed that the set of cost sharing methods satisfying Additivity, Dummy, and Demand Monotonicity is the set of all convex combinations of *fixed* path methods. A fixed path method is a method generated by a fixed path (see Moulin [1999] for detail).

Sprumont [1998] used the “informational coherence axiom” to characterize the

“simple” random order values and the Shapley-Shubik method¹⁴. Sprumont and Wang [1996] also used the “measurement invariance” to characterize the simple random order values. A simple random order value uses the same random order value for all demand profiles having the same set of active agents (positive demands). For a complete summary of the discrete model, see Moulin [1999].

3.3 The Continuous Heterogeneous Goods Model

The continuous model differs from the discrete model only in that the demand vector is continuously variable. Correspondingly, certain regularity about the cost functions is assumed, e.g., the cost function C is usually assumed to be twice continuously differentiable.

The continuous model has been discussed by Billera and Heath [1978] [1982] and Mirman and Tauman [1982]. As we mentioned in the introduction, this literature mainly focuses on the Aumann-Shapley prices¹⁵.

Friedman and Moulin [1995] discussed three methods, namely the Shapley-Shubik

¹⁴The Shapley-Shubik method applies the Shapley value to the stand-alone cost game a cost sharing problem generates, i.e.,

$$x_i(q; C) = \sum_{s=0}^n \frac{s!(n-s-1)!}{n!} \sum_{S: S \subseteq N \setminus i, |S|=s} [C(q_{S \cup i}, 0) - C(q_S, 0)], \quad i = 1, \dots, n. \quad (4)$$

¹⁵The Aumann-Shapley method (Aumann and Shapley [1974], Billera, Heath and Raanan [1978], Samet and Tauman [1982]) is defined by the following formula:

$$x_i(q; C) = \int_0^{q_i} \partial_i C\left(\frac{t}{q_i} q\right) dt, \quad i = 1, \dots, n. \quad (5)$$

method, the Aumann-Shapley method, and the Friedman-Moulin Serial Methods¹⁶, in the family of additive methods based on a representation result (Lemma 1 below). Friedman and Moulin showed that the Shapley-Shubik method is the unique additive method satisfying Demand Monotonicity, Scale Invariance and Symmetry, the Aumann-Shapley method is the unique proportional extension satisfying Scale Invariance, and the Friedman-Moulin Method is the unique serial extension satisfying Demand Monotonicity.

Friedman [1998] further showed that the set of additive methods (satisfying dummy axiom) can be generated by the convex combinations of the path generated methods (the exact counterpart of the discrete model). Haimanko [1998] also proved the path generating result in the context of non-atomic theory.

Recently, Sprumont [1998] proposed the Ordinality axiom and characterized the Shapley-Shubik method by Additivity, Dummy, Ordinality, and Symmetry. He also proposed and characterized a handful of *non-additive* methods.

Now, assume that \mathcal{C} is the set of all twice continuously differentiable cost functions on R_+^N . Let $q = (q_1, \dots, q_n) \in R_+^N$ be a demand profile for the agents $N = \{1, \dots, n\}$. A **cost sharing problem** is a pair $(q; C)$. A **solution** of the problem $(q; C)$ is a vector (x_1, \dots, x_n) such that $\sum_{i \in N} x_i = C(q)$. A **cost sharing method** x is a mapping associating to each problem $(q; C)$ a solution $x(q; C)$. The Additivity and Dummy axioms for the cost sharing methods are similarly defined as in the discrete model. The following representation lemma is due to Friedman and Moulin (1995).

Lemma 1 Fix $q \in R_+^N$. Let x be additive and satisfy the dummy axiom. Then, for

¹⁶The Friedman-Moulin serial method (Friedman and Moulin [1995]) is defined as:

$$x_i(q; C) = \int_0^{q_i} \partial_i C((te) \wedge q) dt, i = 1, \dots, n, \quad (6)$$

where $(x \wedge y) = (\min\{x_1, y_1\}, \dots, \min\{x_n, y_n\})$.

each $i \in N$, there exists a measure μ_i^q such that

$$x_i(q; C) = \int_{[0, q]} \partial_i C(p) d\mu_i^q(p), \text{ for each } C \in \mathcal{C}, \quad (7)$$

where the measure μ_i^q has the following property: its projection on any interval $[p_i, p'_i]$, $0 \leq p_i < p'_i \leq q_i$, is the Lebesgue measure on R .

To introduce the path generated methods, call a mapping $\gamma : R_+ \times R_+^N \rightarrow R_+^N$ a path if i) $\gamma(0; q) = 0$ ii) $\gamma(\infty; q) = q$ iii) $\gamma(t; q)$ is nondecreasing in t . Let $\Gamma(N)$ be the set of all paths. Given a path $\gamma \in \Gamma(N)$, define a cost sharing method as the Riemann-Stieltjes integral:

$$x_i^\gamma(q; C) = \int_0^\infty \partial_i C(\gamma(t; q)) d\gamma_i(t; q), i = 1, \dots, n \quad (8)$$

Friedman and Moulin [1995] have shown that the above construction is a valid cost sharing method.

Friedman [1998] pointed out that the following three well-known methods are path generated:

- 1) Aumann Shapley: $\gamma_i(t; q) = \min[t, 1]q_i$.
- 2) Friedman-Moulin Serial Cost: $\gamma_i(t; q) = \min[t, q_i]$.
- 3) Random order value (incremental stand-alone cost method) with order i_1, \dots, i_n : $\gamma_{i_j}(t; q) = \min[1, \phi_{i_j}(t)]q_{i_j}$; where $\phi_k(t) = 0$ if $t \leq (k-1)$ and 1 if $t \geq k$ and $(t-k+1)$ otherwise.

Note that the Shapley-Shubik method is obtained by averaging the random order values over all orders. The weighted version of Aumann-Shapley (McLean and Sharkey [1996]) is similarly obtained.

Fix N and q . Let $\Gamma(q)$ be the set of all paths to q in $\Gamma(N)$. Let $CP(q, n) = \{x_\gamma | \gamma \in \Gamma(q)\}$, the set of path generated methods. Let $CS(q, n)$ be the set of additive cost sharing methods satisfying the dummy axiom for fixed q and N .

Theorem 3 (Friedman [1998], Haimanko [1998]) *All additive methods satisfying dummy are the convex combinations of path generated methods, i.e., the following statements are equivalent:*

- 1) $x \in CS(q, n)$.
- 2) *There exists a nonnegative measure v on $CP(q, n)$ such that*

$$x = \int_{\gamma \in \Gamma(q)} x_{\gamma} dv(\gamma).$$

This result is the exact counterpart of Theorem 1 in Section 1 of Chapter 2 in the discrete model. It also establishes an isomorphism between the rationing methods and the extreme points of the set of all additive methods satisfying dummy. There are also many applications of this characterization (see Friedman [1998]).

Sprumont [1998] proposed an alternative approach by introducing the “Ordinality axiom”. The ordinality axiom requires that the cost sharing method be invariant under all increasing transformations of the measuring units of the goods. Remarkably, this axiom is flexible enough to allow cost sharing methods satisfying both ordinality and proportionality, ordinality and the serial property, which are impossible within the additive cost sharing methods. It is therefore a key to the exploration of non-additive methods. Interestingly, it can also provide sharper characterization for a subset of the set of additive methods, namely the simple random order methods (see Theorem 1 in Section 1 of Chapter 3 for details).

Ordinality is not only a mathematical axiom but also a compelling requirement when the cost function involves non-physical goods such as services. See Sprumont [1998] for details.

3.4 Brief Comments of Axiomatic Cost Sharing

We have seen that Additivity is the backbone of the axiomatic characterizations of most cost sharing methods studied so far. It played an important role in establishing the isomorphisms across the three different models (Theorem 1 and Theorem 3 in this chapter, and Theorem 1 in Section 1 of Chapter 2). When two or more equity axioms are in conflict under the restriction of additivity, the additivity axiom should be relaxed first. The ordinality axiom had played an important rule in the exploration of non-additive methods. Distributivity can also be a key to study non-additive methods in the homogeneous case.

A typical feature of axiomatic cost sharing is that most axioms are “structural invariance” axioms. Equity axioms are relatively few.

4 Economic Efficiency and Demand Revelation

Now, we briefly look at the following neglected aspects of cost sharing methods: their efficiency and incentive-compatibility properties, in an economic environment at large. We ask what mechanisms (cost sharing methods) generate the correct incentives to achieve an efficient and fair utilization of the common resources. A general result (see below) shows that it is impossible to have an incentive-compatible and efficient *full* cost allocation mechanism (Groves [1977]).

Consider the following situation. A firm consists of n divisions, $i = 1, \dots, n$, that are, for accounting or control purposes, treated as separate units but are also connected through some firm-wide decisions. To be concrete, consider a single firm-wide decision such as the provision of some service that is made available to all divisions. Two questions confront the firm with respect to this decision. First, the firm must decide the quantity of the good to provide. Second, it must decide how to

finance its provision.

A full cost allocation mechanism is one that divides the total cost of providing the total service fully among the divisions. An optimal decision or efficient decision for the firm is that given the reported valuation from each division at every level of provision, the decision maximizes the total net value (minus total cost). We say a full cost allocation mechanism is *incentive compatible* if and only if it is always in every division's interest to send correct information, regardless of the information sent by other divisions.

The main result we are going to discuss is that there is no incentive compatible full cost allocation method if the decision rule specified must also pick optimal decision when supplied with correct information. Thus, either the full cost feature, incentive compatibility, or optimality of decisions must be sacrificed. This result comes from other results in this line. The first of these is due to Groves [1973] that exhibited a family of incentive-compatible cost allocation rules, but which were not full cost allocation rules. The other key result, established by Green and Laffont [1977], showed that any incentive-compatible procedure for making optimal decisions was equivalent to a member of the family of incentive-compatible cost allocation rules defined by Groves.

Formally, let $i = 0, 1, \dots, n$ denote n divisions and a center ($i = 0$) of a firm. Denote z the center's decision, and $C(z)$ the total cost of the decision z , which is common knowledge. Let $\pi_i(z)$ be the net payoff (valuation) of division i , $i = 1, \dots, n$, which is the private information.

Suppose that $C_1(\cdot), \dots, C_n(\cdot)$ is a full cost allocation mechanism, i.e.,

$$\sum_{i=1}^n C_i(\cdot) = C(z), \text{ for all } z. \quad (9)$$

The net revenue of division i is $\pi_i(z) - C_i(\cdot)$.

Let $m_i(\cdot)$ denote the i th division's "reported" net payoff function. If the divisions do send true net payoff functions, i.e., $m_i(\cdot) = \pi_i(\cdot)$ for all i , then center's optimal decision z^* would be the solution to the program:

$$\max_z \sum_{i=1}^n m_i(z) - C(z).$$

Define the center's decision rule $z^*(m_1, \dots, m_n)$ for choosing z given any messages $m_1(\cdot), \dots, m_n(\cdot)$ from the divisions by:

$$z = z^*(m_1, \dots, m_n) \equiv \arg \max \sum_{i=1}^n m_i(z) - C(z).$$

Given the decision rule $z^*(\cdot)$, and the full cost allocation scheme $C_i(\cdot), i = 1, \dots, n$, each division chooses the best report $m_i(\cdot)$ to the center:

$$\max_{m_i} \pi_i(z(\tilde{m}/m_i)) - C_i(z(\tilde{m}/m_i), \tilde{m}/m_i) \quad (10)$$

where $\tilde{m}/m_i \equiv (\tilde{m}_1, \dots, m_i, \dots, \tilde{m}_n)$, and where C_i is allowed to depend directly on all the message $m_j, j = 1, \dots, n$, as well as on the decision $z(m)$.

We say that the full cost allocation scheme (C_1, \dots, C_n) is incentive-compatible if and only if π_i solves the division's optimization problem, i.e., if $\pi_i(\cdot)$ solves (10) for any $m_j(\cdot), j \neq i$.

Theorem 4 (*Green and Laffont [1977], Hurwicz [1975], [1981], Walker [1978]*) *There is no incentive-compatible full cost efficient allocation scheme. More formally, there is no decision-cost allocation scheme $[z(\cdot), (C_1(\cdot), \dots, C_n(\cdot))]$ such that*

$$\sum_{i=1}^n C_i(z(m), m) = C(z(m)) : \text{full cost allocation} \quad (11)$$

$$\pi_i(\cdot) = \arg \max_{m_i} \pi_i(z(m/m_i)) - C_i(z(m/m_i); m/m_i) \quad (12)$$

for all m : incentive-compatibility

$$z(\pi_1, \dots, \pi_n) = \arg \max \sum_{i=1}^n \pi_i(z) - C(z) : \text{efficiency} \quad (13)$$

In particular, the class of mechanisms (i.e., decision rule-cost allocation scheme pair) $(z(\cdot); (C_1(\cdot), \dots, C_n(\cdot)))$ defined by Groves is:

$$z(m) \equiv \arg \max \sum_{i=1}^n m_i(z) - C(z) \quad (14)$$

$$C_i(z(m), m) \equiv - \sum_{j \neq i} m_j(z(m)) + C(z(m)) + A_i(m/m_i) \quad (15)$$

where $A_i(m/m_i)$ is a function of all reports except that of division i : $m/m_i \equiv (m_1, \dots, m_{i-1}, m_{i+1}, \dots, m_n)$. The relevance and interest in this class rest on the following two propositions.

Theorem 5 (Groves [1973]) *Any member of the class defined by (14) and (15) above has the pair of properties that it is incentive-compatible and results in efficient decisions being chosen.*

Theorem 6 (Green and Laffont [1977]) *Any mechanism (decision rule-cost allocation scheme pair) that is both incentive-compatible and result in efficient decisions is equivalent to a member of the class defined by Groves.*

These results imply that if we insist on full cost allocation mechanisms (methods), we must depart from the Groves mechanisms. Almost all the cost sharing methods we have discussed in our axiomatic cost sharing are not Groves.

A less general but more practical situation is when there is no coordination center which collects the information from each individual (division) and picks up the optimal or efficient quantity of good to provide and enforces the full cost allocation mechanism. In this case, any cost allocation mechanism (whether or not budget balanced) is a decentralized device and generates a game in which each player choose his demand strategically. Moulin and Shenker [1996] analyzed and compared two well-known mechanisms, Marginal Contribution (MC) and Shapley Value (SH), for

a network model. In that model, a given set of users share the submodular cost¹⁷ of access to a network. How can one share the costs in an incentive-compatible manner? Moulin and Shenker compared the welfare properties of the above two mechanisms. They showed that MC is the unique individually rational Clarke-Groves mechanism and serves the efficient set of users, but in general runs a budget deficit. SH is budget balanced and coalitional strategy-proof, but does not realize optimal welfare (not efficient). Among all budget balanced and coalitionally strategy-proof mechanisms, SH is characterized by the property that its worst welfare loss is minimal.

Any cost sharing method generates a game in which each agent simultaneously demands an amount of outputs and the cost of total output demanded is then divided by the given cost sharing method. A. Watts [1996] discussed the minimum requirements that must be placed on a cost sharing mechanism in order to generate a unique Nash equilibrium (corresponding to the strategy-proofness of the sharing mechanism), given that each player's preferences are convex and bi-normal (both the inputs and the outputs are normal goods). The results were applied to several popular cost sharing mechanisms. Particularly, the average cost sharing mechanism generates a unique Nash equilibrium if and only if the cost function is increasing, convex and marginal cost is always strictly greater than average costs, the serial cost sharing mechanism generates a unique equilibrium if and only if cost is increasing and strictly convex (see A. Watts [1996]). Moulin [1996] characterized the incremental methods by coalition strategy-proofness. In his model, each user demands a quantity of a personalized indivisible good. He showed that if the second derivatives of the cost function are of constant sign, the sequential stand alone cost sharing method yields a unique strong equilibrium at every profile of convex preferences in the cost sharing game where each

¹⁷A cost function $C : 2^N \rightarrow R_+$ is called submodular if $C(S) + C(T) \geq C(S \cap T) + C(S \cup T)$ for all $S, T \subseteq N$, where N is the set of all users. Equivalently, C is submodular if the marginal cost $C(S \cup i) - C(S)$ of adding user i to the set of users S is non-increasing in S .

user chooses his (her) own demand. The sequential stand alone cost sharing method shares cost incrementally according to a fixed ordering of the users: the first user always pays stand alone cost, the second pays the stand alone cost of the first two users minus that of the first and so on. This equilibrium defines a coalition strategy-proof social choice function. Under increasing marginal cost and supermodular cost¹⁸, coalition strategy-proofness, characterizes a larger family of cost sharing methods: they give out one unit at a time while charging marginal cost, with the users taking turns according to a sequence fixed in advance.

5 Summary

The axiomatic cost sharing literature focuses on the fairness properties of various cost sharing methods. Various cost sharing methods have been characterized by different combinations of equity axioms and “structural invariance” axioms. A typical feature is that most characterization results are centering around the structural invariance axioms which express the commutativity of cost sharing methods with respect to certain variations in cost sharing problems.

Additivity is the most important structural invariance axiom, which encompasses three different models, the homogeneous model, the discrete model, and the continuous model. It is a decomposition property with respect to cost functions. Distributivity is a property with respect to the composition of cost functions. Scale Invariance is with respect to the linear transformation of the measurement units of the goods. Ordinality generalizes the Scale Invariance by completely dispensing with any measurement conventions.

An interesting finding is the isomorphism between the set of rationing methods

¹⁸The converse of the submodularity, see Moulin and Shenker [1996].

and the set of additive methods. This conclusion hinges on Additivity. It says that the set of rationing methods is linearly isomorphic to the set of additive methods in the homogeneous good model and isomorphic to the extreme points of the set of additive methods in the heterogeneous goods model. Thus, various methods in three different models are linked correspondingly.

Most often a single equity axiom or sometimes two at most could pin down a specific method within a certain family of methods characterized by structural axioms. For examples, among *additive* methods, Proportionality and Scale Invariance characterize Aumann-Shapley method, and Ordinality and Symmetry characterize the Shapley-Shubik method.

On the one hand, the structural axioms show us how far we can go. On the other hand, they demonstrate restrictions. The incompatibility between Proportionality and Demand Monotonicity, for example, hinges on Additivity. To incorporate these two equity axioms, we should drop the additivity axiom. This leads us to explore non-additive methods. Indeed, a few meaningful non-additive methods have been proposed based on the recent Ordinality axiom (Sprumont [1998]).

Obviously, cost sharing problems are not in isolation but often in economic environments in which each agent is a rational player and acts strategically (although we ignore this for simplicity). Indeed, each method generates a cost sharing "game" in which agents choose their demands strategically. From a more general viewpoint, the problem can be regarded as a social choice problem in which each agent is endowed with a preference. The overall question will be what cost sharing mechanism (method) generates an incentive-compatible and efficient utilization of the commonly shared production facility. This will lead us to the more complex issues of efficiency and strategy-proofness in mechanism design. This is beyond the scope of this thesis.

In the following chapters, two different models are discussed separately. Chapter 2 discusses the discrete (heterogeneous) model. Chapter 3 deals with the continuous (heterogeneous) model.

Chapter 2 : The Discrete Model

The Additivity and Dummy Axioms in
the Discrete Cost Sharing Model

Abstract

The paper considers the discrete cost sharing model first studied in Moulin's paper [1995]. It shows that the set of *additive* methods satisfying the *dummy* axiom is the set of all convex combinations of the *path generated* methods.

1 Introduction

Consider the following cost sharing problem. A production facility is shared by a finite number of agents $i = 1, \dots, n$. Each agent i demands an integer amount q_i of a personalized good, and the total production cost expressed by $C(q_1, \dots, q_n)$ must be equitably divided among the n agents.

This model was first proposed by Moulin [1995]. He generalized the Shapley value theory to cost sharing by adding the Demand Monotonicity axiom to Shapley's two original axioms, Additivity and Dummy. He used this Demand Monotonicity axiom to characterize the Shapley-Shubik method, among others.

There are two other related models in the literature. The first model (**Model 1**) is a special case of the above discrete model. It assumes that all individual demands are either 0 or 1. This is equivalent to the standard cooperative game model. The second model (**Model 2**) assumes that all demands are real numbers. This corresponds to the well-known Aumann-Shapley pricing model (see the survey by Tauman [1988]). For both models, there has been a large amount of literature.

We reconsider Moulin's discrete model (we call **Model 3**). We focus on two basic axioms, Additivity and Dummy, and reexamine their implications on the cost sharing methods. Recall that for **Model 1**, Weber [1988] showed that the *additivity* (to be precise, he uses a stronger axiom called *linearity*) and *dummy* axioms characterize the class of random order values. For **Model 2**, Friedman and Moulin [1995] provided a representation formula for all cost sharing methods meeting Additivity and Dummy axioms. Recently, Friedman [1998] and Haimanko [1998] provided characterizations in terms of the *path generated* methods for **Model 2**. For **Model 3**, Moulin [1995] characterized the class of cost sharing methods satisfying Additivity, Dummy and Demand Monotonicity.

The main result of this paper is a characterization of the entire class of methods satisfying the Additivity and Dummy axioms for **Model 3**. We show that these cost sharing methods are the convex combinations of the *path generated* methods (Theorem 1).

2 The Model and the Axioms

The model is essentially the same model discussed by Moulin [1995]. The only difference here is that the demand profile is fixed in the discussion. The reader will immediately see that the model with variable demand profile allows for exactly the same result.

Denote $\mathbb{N} = \{0, 1, 2, \dots\}$. Let $n \in \mathbb{N}$, $n > 0$ and $N = \{1, \dots, n\}$ be the set of agents. Let $q \in \mathbb{N}^n$ be a demand profile, which is fixed throughout this paper. Denote $[0, q]$ the interval: $0 \leq t \leq q$ in \mathbb{N}^n . A cost function is a mapping $C : [0, q] \rightarrow R$ such that $C(0) = 0$ and $C(t) \leq C(t')$ if $t \leq t'$. The set of all cost functions is denoted by \mathcal{C} . A (cost sharing) problem is a pair $(q; C)$ where $C \in \mathcal{C}$. Since we fixed the demand profile we simply call $C \in \mathcal{C}$ a problem. A (cost sharing) method x is a mapping from \mathcal{C} into R_+^n satisfying the *budget balance* condition $\sum x_i(C) = C(q)$.

A few more notations will be used. The interval $]0, q]$ means the set $[0, q] \setminus \{0\}$ and $]0, q[= [0, q] \setminus \{q\}$. If $t \in [0, q]$ and $S \subseteq N$ is a coalition, denote t_S the restriction of t to S . We write $t = (t_S, t_{-S})$ instead of $t = (t_S, t_{N-S})$ and i instead of $\{i\}$. Denote $A(t) = \{i \in N | t_i > 0\}$ the set of active agents at t . Let 1^i stand for the vector in \mathbb{N}^n whose j th component is 1 if $j = i$ and 0 otherwise. Given a problem C and $t_i > 0$, we define $\partial_i C(t) = C(t) - C(t - 1^i)$.

The paper only considers the cost sharing methods satisfying the following two axioms: Additivity and Dummy. A method x is *additive* if $x(C_1 + C_2) = x(C_1) + x(C_2)$

for any $C_1, C_2 \in \mathcal{C}$. It satisfies the *dummy* axiom if $x_i(C) = 0$ whenever $\partial_i C(t) = 0$ for every $t \in [0, q]$ such that $t_i > 0$.

3 The Characterization

At first, the Additivity and Dummy axioms imply the following representation result. It corresponds to Friedman and Moulin's [1995] representation lemma for **Model 2** and to Weber's [1988] Theorem 2 for **Model 1**. The proof of this lemma is in the Appendix.

Lemma 1 (Representation Lemma) *A method x is additive and satisfies the dummy axiom if and only if, for each $i \in A(q)$,¹⁹ there exists a unique mapping $\mu_i : [1^i, q] \rightarrow R_+$ such that*

$$x_i(C) = \sum_{t \in [1^i, q]} \mu_i(t) \partial_i C(t) \text{ for each } C \in \mathcal{C},^{20} \quad (16)$$

and

$$\sum_{i \in A(t)} \sum_{s_{-i} \in [t_{-i}, q_{-i}]} \mu_i(t_i, s_{-i}) = 1 \text{ for each } t \in]0, q].^{21} \quad (17)$$

The collection $\mu = (\mu_i)_{i \in A(q)}$ is called the *weight system* associated with the method x .

Based on this representation lemma, we further show that any additive method satisfying the dummy axiom must be a convex combination of the so-called path generated methods defined below. This alternative characterization is much more intuitive than the representation lemma. Before we state the main theorem we need the following important notions, namely a *path* and a *path generated method*.

¹⁹For $i \notin A(q)$, $x_i(C) = 0$. They are implied by the budget balance condition.

²⁰In general, each μ_i ($i \in A(q)$) depends on q . Since q is fixed in this paper we write μ_i instead of μ_i^q for each $i \in A(q)$.

²¹This ensures the budget balance condition.

Definition 1 A path to q is a mapping $P : \{0, 1, \dots, q(N)\} \rightarrow [0, q]$ such that

- $P(0) = 0$,
- For each $k \in \{0, 1, \dots, q(N)\}$, $P(k)$ is identical to $P(k - 1)$ in all coordinates but one, say, the i th, for which

$$P_i(k) = P_i(k - 1) + 1,$$

where $q(N) = \sum_N q_i$.

Denote \mathcal{P} the set of all paths to q .

Definition 2 A path generated method, generated by a path $P \in \mathcal{P}$, is a method which charges each agent the sum of his marginal costs along the path.

More precisely, let P be a path. The P -path generated method is defined as follows: if C is a problem and $q_i > 0$, we compute for every $t_i \in [1, q_i]$ the unique integer $k =: k(t_i)$ for which $P_i(k - 1) = t_i - 1$ and $P_i(k) = t_i$, and then charge agent i

$$x_i^P(C) = \sum_{t_i \in [1, q_i]} \partial_i C(P(k(t_i))).^{22}$$

Note that the path generated methods satisfy the Additivity and Dummy axioms. And their associated weight systems consist of vectors with components of 0 or 1 only.

Denote CP the set of all the path generated methods w.r.t. the path set \mathcal{P} . Denote CSM the set of all the cost sharing methods satisfying the Additivity and Dummy axioms. Then, CP is a subset of CSM . From Lemma 1, we observe that each method in CSM corresponds uniquely to a weight system, which is a vector in the Euclidean space of dimension $\sum_1^n q_i \prod_{j \neq i} (q_j + 1)$. Therefore, the sets CP and CSM can be thought of as subsets in this vector space. Then, it can be shown that

²²This expression should be understood to be zero if $q_i = 0$.

the set CSM is a convex compact set and the set CP is a subset of the set of extreme points²³ of CSM , $ext(CSM)$, i.e., $CP \subseteq ext(CSM)$ (we omit the proof). Therefore, we conclude that $conv(CP) \subseteq CSM$.²⁴ Our main theorem, given in the following, states that the converse inclusion, $CSM \subseteq conv(CP)$, is also true.

Theorem 1 *The cost sharing methods satisfying the Additivity and Dummy axioms are the convex combinations of the path generated methods x^P ($P \in \mathcal{P}$), i.e.,*

$$CSM = conv(CP). \quad (18)$$

The proof of the theorem, i.e., the proof of $CSM \subseteq conv(CP)$, relies on several lemmas.

Lemma 2 *Let x be an additive method satisfying the dummy axiom. Let μ_i ($i \in A(q)$) be the weight system associated with x (by Lemma 1). Then, for every $t \in]0, q[$,*

$$\sum_{i \in A(q) \cap A(t)} \mu_i(t) = \sum_{i \in A(q) \cap \{i | t_i < q_i\}} \mu_i(t + 1^i) \quad (19)$$

and

$$\sum_{i \in A(q)} \mu_i(1^i) = 1, \quad (20)$$

$$\sum_{i \in A(q)} \mu_i(q) = 1. \quad (21)$$

This lemma is in Sprumont [1998b]. It follows directly from the representation lemma by applying it to the following two types of cost functions:

$$C_t^*(s) = 1 \text{ if } s \geq t \text{ and } 0 \text{ otherwise,}$$

$$C_t^{**}(s) = 1 \text{ if } s > t \text{ and } 0 \text{ otherwise.}$$

²³The definition of extreme point is the following: A point $c \in A$ (A is a convex set) is an extreme point of A if whenever $a, b \in A$, and $c = a/2 + b/2$, then $a=b$.

²⁴The notation "conv" refers to the convex hull.

Note that if we interpret the weights $\mu_i(t)$ ($i \in A(q)$) at any intermediate node t as flows, then equation (19) says that the incoming flow at t equals the outgoing flow from t . Equations (20) and (21) say respectively that there is one unit of flow coming out from the “origin” 0, and going into the “sink” q . Therefore, the weight system associated with a cost sharing method can be regarded as a “flow” running through a “network”.

Formally, a network is a pair $G = (V, E)$, in which $V = V_1 \cup V_2 \cup V_3$, $V_1 = \{0\}$ is the origin, $V_2 = \{t | t \in]0, q[\}$ is the set of the intermediate nodes, and $V_3 = \{q\}$ is the sink. To define the set of arcs E , for each $t \in]0, q[$ and each $i \in A(q)$ such that $t_i > 0$, let $e_i(t)$ be the directed link from $t - 1^i$ to t . Let

$$E(t) = \{e_i(t) | i \in A(q)\} \quad (22)$$

and define

$$E = \cup_{t \in]0, q[} E(t).$$

Let f and $c(e)$ ($e \in E$) be non-negative real numbers. Given these numbers, a *feasible flow* on G is a set of numbers $\mu(e)$ ($e \in E$) such that

$$\sum_{i \in A(q) \cap \{i | t_i < q_i\}} \mu(e_i(t + 1^i)) - \sum_{i \in A(q) \cap A(t)} \mu(e_i(t)) = \begin{cases} f & t = 0 \\ 0 & t \in]0, q[\\ -f & t = q \end{cases} \quad (23)$$

with capacity constraints

$$0 \leq \mu(e) \leq c(e), \quad e \in E. \quad (24)$$

Our proof of Theorem 1 also relies on the following two lemmas from the integer programming theory (Garfinkel, R. S. and G. L. Nemhauser [1972]).

Lemma 3 (Garfinkel, R. S. and G. L. Nemhauser [1972], p66-74) *The constraint matrix corresponding to the flow constraints (23) and (24) is totally unimodular²⁵.*

²⁵An integer $m \times n$ matrix A is totally unimodular if every square, nonsingular submatrix B of A has determinant 1 or -1 .

Lemma 4 (*Garfinkel, R. S. and G. L. Nemhauser [1972], p66-74*) *If A is totally unimodular, then the extreme points (if any) of $S(b) = \{x | Ax = b, x \geq 0\}$ are integer vectors for any arbitrary integer vector $b \geq 0$.*

We are now ready to prove that $CSM \subseteq \text{conv}(CP)$. Let $x \in CSM$. We observe that the weight system μ associated with x is a feasible flow for the numbers $c(e) = 1$ ($e \in E$) and $f = 1$. This directly follows from Lemma 2. Indeed, let $\mu(e_i(t)) = \mu_i(t)$ in the flow constraints (23) and (24). From Lemma 2, the constraints (23) are satisfied for $f = 1$ and $c(e) = 1$ ($e \in E$). The constraints (24) are satisfied due to Lemma 1's budget balance condition. In particular, we also observe that the weight systems associated with the *path generated methods* correspond to the "unit flows", namely, those feasible flows μ for which $\mu(e)$ is zero or one for each $e \in E$. Conversely, a *unit flow* μ defines a path to $q : P = \{t \in [0, q] | t = 0 \text{ or there exists an } i \in A(q) \text{ s.t. } \mu(e_i(t)) = 1\}$ (check that P is indeed a path), which generates a method x^P (Definition 2). Now, consider the set of all feasible flows with $c(e) = 1$ ($e \in E$) and $f = 1$ defined by (23) and (24). It is a compact convex set and can be spanned by its extreme points. However, its extreme points, by lemmas 3 and 4, are integer vectors, and in our case, vectors with components of zero or one. So these extreme points are the unit flows. This proves that the weight system μ associated with x is a convex combination of the *unit flows*. Accordingly, the cost sharing method x is a convex combination of the path generated methods. This completes our proof of the theorem. Q.E.D.

Appendix: The Proof of The Representation Lemma

Before the proof, we provide the following two properties implied by the Additivity Axiom. They will be used in the proof.

Lemma 5 (*Independence of Irrelevant Costs, Moulin [1995]*) *If x satisfies Additivity, then $\forall q \in [0, \bar{q}]$, $C_1, C_2 \in \mathcal{C}$,*

$$\{C_1(q') = C_2(q'), \forall q' \in [0, q]\} \implies x(q; C_1) = x(q; C_2).$$

Lemma 6 *If x satisfies Additivity, then for every fixed $q \in [0, \bar{q}]$, operator $x(q; \cdot)$ extends uniquely to a linear operator on R^D , where $R^D = \{C \mid C : [0, q] \rightarrow R, C(0) = 0\}$, $D = \prod_{i=1}^n (q_i + 1) - 1$.*

Proof: By the Independence of Irrelevant Costs, we only need to consider C on $[0, q]$. Denote $\mathcal{C}(q)$ the restriction of \mathcal{C} on $[0, q]$. Then we see that the set of cost functions on $[0, q]$, i.e., the $\mathcal{C}(q)$ can be viewed as a convex cone in R^D . We first show that the additive operator $x(q; \cdot)$ is also a linear operator w.r.t. positive scalars on the cone $\mathcal{C}(q)$.

By Additivity we know that for any positive rational number R

$$x(q; RC) = Rx(q; C).$$

Let r be any positive real number. Let $\{r_m\}$ be an increasing and $\{R_m\}$ a decreasing sequence of rational numbers converging toward r . Then for each $m = 1, 2, \dots$,

$$r_m < r < R_m,$$

and by Additivity and the fact that cost shares are non-negative (called Positivity) we have

$$\begin{aligned} R_m x(q; C) &= x(q; R_m C) \\ &= x(q; R_m C - rC + rC) \\ &= x(q; rC) + x(q; (R_m - r)C) \\ &\geq x(q; rC), \end{aligned}$$

and

$$\begin{aligned}
 x(q; rC) &= x(q; rC - r_m C + r_m C) \\
 &= x(q; r_m C) + x(q; (r - r_m)C) \\
 &\geq x(q; r_m C) \\
 &= r_m x(q; C).
 \end{aligned}$$

Therefore

$$r_m x(q; C) \leq x(q; rC) \leq R_m x(q; C), \quad m = 1, 2, \dots$$

and by taking limit as $m \rightarrow \infty$ we get

$$x(q; rC) = rx(q; C).$$

This implies that $x(q; \cdot)$ is a linear operator on the cone $\mathcal{C}(q)$ w.r.t. positive scalars.

Since $R^D = \mathcal{C}(q) - \mathcal{C}(q)$, we can extend the linear operator $x(q; \cdot)$ (w.r.t. positive scalars) from $\mathcal{C}(q)$ to R^D . From this linearity w.r.t. positive scalars on R^D it follows that $x(q; \cdot)$ is linear on R^D .

The lemma is proved.

Q.E.D.

From these two lemmas, we provide a proof of the *representation lemma* as follows.

The proof is divided into three steps:

Step 1. For each $i = 1, \dots, n$, there exists a unique vector γ_i in R^D such that

$$x_i(q; C) = \gamma_i * C = \sum_{t \in]0, q]} \gamma_i(t) C(t) \quad \text{for each } C \in \mathcal{C}. \quad (25)$$

This follows from the last lemma and the fact that any linear operator in finite dimensional space R^D is uniquely associated with a vector in R^D , with which it can be expressed as (25).

Step 2. Further, these vectors γ_i ($i = 1, \dots, n$) in Step 1 have the following properties.

$$\sum_{t \in [(0, q'_{-i}), (q_i, q'_{-i})]} \gamma_i(t) = 0, \quad (26)$$

where $(0, q'_{-i}) \neq 0$,

$$\sum_{t \in [(q'_i, 0_{-i}), (q'_i, q_{-i})]} \gamma_i(t) = 0, \quad q'_i \neq 0, q_i,$$

$$\sum_{t \in [(q_i, 0_{-i}), q]} \gamma_i(t) = 1,$$

and

$$\sum_{t \in [p, q]} \gamma_i(t) \geq 0 \text{ for each } p \in]0, q]. \quad (27)$$

These follow from repeatedly applying both Dummy axiom to such cost functions as $\delta_{(0, q'_{-i})}(\cdot)(\delta_{(0, q'_{-i})}(p) = 1 \text{ iff } p \geq (0, q'_{-i}))$ and the Positivity of the cost sharing method to cost functions like δ_p .

Step 3. We deduce the representation formulas. To be specific, consider $i = 1$.

By Step 1 and 2, we have a unique vector γ_1 such that

$$x_1(q; C) = \sum_{t \in]0, q]} \gamma_1(t) C(t)$$

and $\gamma_1(t)$, $t \in]0, q]$ satisfy the equations in Step 2.

For $(0, q'_{-1}) \neq 0$, by (26) we have

$$\gamma_1(q_1, q'_{-1}) = - \sum_{t \in [(0, q'_{-1}), (q_1 - 1, q'_{-1})]} \gamma_1(t).$$

Therefore

$$\begin{aligned} x_1(q; C) &= \sum_{t \in]0, q]} \gamma_1(t) C(t) \\ &= \sum_{t \in [(1, 0_{-1}), (q_1, 0_{-1})]} \gamma_1(t) C(t) + \sum_{q'_{-1} \neq 0} \sum_{t \in [(0, q'_{-1}), (q_1, q'_{-1})]} \gamma_1(t) C(t) \end{aligned}$$

$$\begin{aligned}
&= \sum_{t \in [(1,0_{-1}), (q_1,0_{-1})]} \gamma_1(t)C(t) + \sum_{q'_{-1} \neq 0} [\sum_{t \in [(0,q'_{-1}), (q_1-1,q'_{-1})]} \gamma_1(t)C(t) + \\
&\quad \gamma_1(q_1, q'_{-1})C(q_1, q'_{-1})] \\
&= \sum_{t \in [(1,0_{-1}), (q_1,0_{-1})]} \gamma_1(t)C(t) + \sum_{q'_{-1} \neq 0} [\sum_{t \in [(0,q'_{-1}), (q_1-1,q'_{-1})]} \gamma_1(t)C(t) - \\
&\quad \sum_{t \in [(0,q'_{-1}), (q_1-1,q'_{-1})]} \gamma_1(t)C(q_1, q'_{-1})].
\end{aligned}$$

Denote $\Omega_{-1} = [(0, q'_{-1}), (q_1 - 1, q'_{-1})]$, then

$$\begin{aligned}
&\sum_{t \in \Omega_{-1}} \gamma_1(t)C(t) - \sum_{t \in \Omega_{-1}} \gamma_1(t)C(q_1, q'_{-1}) \\
&= - \sum_{t \in \Omega_{-1}} \gamma_1(t)[C(q_1, q'_{-1}) - C(t)] \\
&= - \sum_{t \in \Omega_{-1}} \gamma_1(t) \left[\sum_{t'_1=t_1}^{q_1-1} (C(t'_1 + 1, q'_{-1}) - C(t'_1, q'_{-1})) \right] \\
&= - \sum_{t \in \Omega_{-1}} \sum_{t'_1=t_1}^{q_1-1} \gamma_1(t)(C(t'_1 + 1, q'_{-1}) - C(t'_1, q'_{-1})) \\
&= - \sum_{t_1=0}^{q_1-1} \left(\sum_{t'_1=0}^{t_1} \gamma_1(t'_1, q'_{-1}) \right) [C(t_1 + 1, q'_{-1}) - C(t_1, q'_{-1})] \\
&= - \sum_{t_1=1}^{q_1} \left(\sum_{t'_1=0}^{t_1-1} \gamma_1(t'_1, q'_{-1}) \right) [C(t_1, q'_{-1}) - C(t_1 - 1, q'_{-1})].
\end{aligned}$$

For each $t_1 \in [1, q_1]$, define

$$\mu_1^q(t_1, q'_{-1}) = - \sum_{t'_1=0}^{t_1-1} \gamma_1(t'_1, q'_{-1}) = \sum_{t_1}^{q_1} \gamma_1(t'_1, q'_{-1}).$$

Then by applying the condition (26) and the Positivity to the cost function $\tilde{C} : \tilde{C}(t) = 1$ if $t \geq (t_1, q'_{-1})$ or $t_{-1} > q'_{-1}$ and 0 otherwise, we can show that

$$\mu_1^q(t_1, q'_{-1}) \geq 0, \quad t_1 \in [1, q_1]. \quad (28)$$

So

$$\sum_{q'_{-1} \neq 0} \sum_{t \in [(0,q'_{-1}), (q_1,q'_{-1})]} \gamma_1(t)C(t) = \sum_{q'_{-1} \neq 0} \sum_{t_1=1}^{q_1} \mu_1^q(t_1, q'_{-1}) \partial_1 C(t_1, q'_{-1}). \quad (29)$$

For the part

$$\sum_{t \in [(1, 0_{-1}), (q_1, 0_{-1})]} \gamma_1(t) C(t),$$

it can be rewritten as

$$\sum_{t_1=1}^{q_1} \mu_1^q(t_1, 0_{-1}) \partial_1 C(t_1, 0_{-1}),$$

where

$$\mu_1^q(t_1, 0_{-1}) = \sum_{t'_1=t_1}^{q_1} \gamma_1(t'_1, 0_{-1}) \geq 0, \quad t_1 \in [1, q_1]$$

(again by the same reason as (28)).

Now combining the above two parts, the formula (16) for $i = 1$ is obtained. The very argument is true for any other $i \in N$ and so their formulas are similarly obtained.

The uniqueness of the weight systems μ follows from the uniqueness of the vectors γ_i ($i = 1, \dots, n$).

The budget balance conditions (17) follow immediately by applying (16) to the cost functions: for each $t \in]0, q]$, $\delta_i(p) = 1$ if $p \geq t$, 0 otherwise.

For the second part of the lemma, it suffices to show that the method defined by the formulas (16) satisfies the budget balance condition. This follows from the facts that any cost function can be expressed as a linear combination of the δ_t ($t \in]0, q]$) functions, and the budget balance condition is satisfied for each such function since it is just the condition (17). The Additivity and Dummy properties are obvious from the formulas (16).

Summing up, the lemma is proved.

Q.E.D.

A Characterization of the
Aumann-Shapley Method in the
Discrete Cost Sharing Model

with Yves Sprumont

Abstract

We discuss the discrete Aumann-Shapley Method (A-S) proposed by H. Moulin ("On Additive Methods to Share Joint Costs", Japanese Economic Review 46 [1995], 303-332). We show that Additivity, Dummy and Proportionality are not sufficient to characterize the A-S method. For the two-agent case in which one agent's demand is fixed at one unit, we show that Additivity, Dummy and Proportionality do characterize the A-S. In general case, we introduce a property called *Consistency of Cost Sharing Ratios (CCSR)*. We use CCSR together with Additivity, Dummy and Proportionality to characterize the A-S.

1 Introduction

In this paper, we continue to consider *additive* methods satisfying the dummy axiom in the discrete cost sharing model. But we focus on a special method in this set, namely the discrete Aumann-Shapley method first proposed (but not characterized) by Moulin [1995].

The discrete Aumann-Shapley method is defined as the arithmetic average of all the path generated methods. Therefore, it occupies a central position in the set of additive methods satisfying the dummy axiom. Alternatively, it can be defined as the Shapley value of the “replica game” a cost sharing problem generates. More precisely, it is computed by associating to each demand profile (q_1, \dots, q_n) and the cost function C the stand-alone cost game with $\sum_{i=1}^n q_i$ players, where each player is a particular unit of a particular good. Then the cost share to a particular good is the sum of the cost shares of all the units of this particular good, which are the Shapley values of these units.

The discrete Aumann-Shapley method is closely related to the Aumann-Shapley method in the continuous model. Given the (continuous) cost function C and demand profile q , we cut each individual demand q_i into K identical demands q_i/K and consider each small demand as a separate entity; then we apply the Shapley value to the stand-alone game among $n \cdot K$ players. When K goes to infinity, this leads to the Aumann-Shapley method, which is the integral of the marginal costs on the diagonal.

As the continuous Aumann-Shapley method, the discrete Aumann-Shapley method satisfies the Proportionality property, i.e., it becomes average cost pricing when the cost function is homogeneous. This method relates to a recent paper by Moulin [1999b] on the problem of rationing indivisible units among agents with indivisible claims. There, the proportional random allocation distributes each available unit

sequentially among unsatisfied claims according to the probability (not the priority) calculated by the proportions of their unsatisfied claims among all unsatisfied claims. Note that each rationing method in rationing problem corresponds to a path generated method in cost sharing problem (see Moulin [1999b]).

What we are interested in here is a characterization of the discrete Aumann-Shapley method. Recall its continuous counterpart. In the continuous model, the Aumann-Shapley method is characterized by Additivity, Dummy, Proportionality and Scale Invariance (Billera and Heath [1982], Mirman and Tauman [1982]). In the discrete model, we attempt to provide a parallel characterization for the discrete Aumann-Shapley method.

It turns out that this is a very challenging problem. The difficulty comes from one of the differences between these two models, namely the fact that the standard Scale Invariance axiom has no obvious counterpart in the discrete model. It has been known that all the discrete versions of the scale invariance we have known so far, such as Moulin's Ordinality (Moulin [1995]), Measurement Invariance in Sprumont and Wang [1996] (see the next paper), or Sprumont's Coherence (Sprumont [1998b]) are too strong to allow the discrete A-S. On the other hand, as we will see in the following, the axioms of Additivity, Dummy and Proportionality are not sufficient to characterize the discrete A-S. Therefore, how to characterize the discrete A-S is not obvious.

We show that for the two-agent case in which one agent's demand is fixed at one unit, Additivity, Dummy and Proportionality do characterize the discrete A-S. In the general case, we introduce a property called *Consistency of Cost Sharing Ratios (CCSR)*. We use CCSR together with Additivity, Dummy and Proportionality to characterize the discrete A-S.

We have to point out that our characterization in the general case (Theorem 2) is not satisfactory. The reason is that the CCSR is a property imposed on the parameters of the cost sharing methods rather than the methods themselves. Therefore, it is still an open question to know what axiom can replace this property.

In a quite different context, Calvo and Santos [1998] provided a characterization of the discrete A-S by axioms of efficiency and balanced contributions using *multichoice* games. However, their approach is not in line of ours, but in the spirit of Hart and Mas-Colell's *potential* approach in cooperative game theory.

2 The Discrete Aumann-Shapley Method

First, let us repeat that the model is the same as in the previous paper, except that this time we allow the demand vector to vary. To be precise, now a problem is a pair $(q; C)$. Given a problem $(q; C)$, a solution is a vector $x(q; C) \in \mathbb{R}_+^N$ such that $\sum_1^n x_i(q; C) = C(q)$. Similarly, a cost sharing method x is a mapping associating with each problem a solution. The Additivity now is read as

$$x(q; C_1 + C_2) = x(q; C_1) + x(q; C_2), \quad \forall q \in \mathbb{N}^n, C_1, C_2 \in \mathcal{C},$$

and the Dummy is

If $\partial_i C(p) = 0 \quad \forall p$, then $x_i(q; C) = 0$.

Now, we introduce the discrete Aumann-Shapley method (Moulin [1995]).

Definition 1 *The Aumann-Shapley method x^{AS} recommends the arithmetic average of the cost shares computed by the path generated methods: for each problem $(q; C)$ and $i \in N$,*

$$x_i^{AS}(q; C) = \frac{1}{|Pq|} \sum_{\pi^q \in Pq} x_i^{\pi^q}(q; C).$$

An explicit formula provided by Moulin [1995] using game-theoretic interpretation, is as follows:

$$x_i(q; C) = \frac{1}{\lambda(q)} \sum_{t \in [0, \bar{q}]} \lambda(t) \lambda(t') \left(\frac{t_i}{\sum_j t_j} - \frac{t'_i}{\sum_j t'_j} \right) C(t), \quad i = 1, \dots, n, \quad (30)$$

with the notation $t'_i = q_i - t_i$ and $\lambda(q) = \frac{(\sum_j q_j)!}{q_1! \dots q_n!}$.

Here, we reformulate the formula to conform to the representation lemma (in the previous paper) as follows.

Let

$$\mu_i(q)(t) = \frac{1}{\lambda(q)} \lambda(t_i - 1, t_{-i}) \lambda(t'), \quad (31)$$

and check that the Aumann-Shapley formula can be expressed as

$$x_i(q; t) = \sum_{t \in [0, q]} \mu_i(q)(t) \partial_i C(t). \quad (32)$$

Apart from Additivity and Dummy, the Aumann-Shapley method satisfies the Proportionality axiom which is formally stated below. (Note that this is obvious from its game-theoretic definition (Moulin [1995])).

Definition 2 *A cost function is homogeneous if there exists a function $c : R_+ \rightarrow R_+$ such that*

$$C(q) = c\left(\sum_{i=1}^n q_i\right), \quad \forall q \in [0, \bar{q}].$$

Definition 3 *A cost sharing method has the Proportionality property if it allocates costs in proportion to demands when the cost function is homogeneous.*

Lemma 1 *For any given $q \in [0, \bar{q}]$, let μ be the weight system of the Aumann-Shapley method. Then we have*

$$\sum_{t' \in [0, q]: t'_N = t} \mu_i(q)(t') = \frac{q_i}{q_N}, \quad \text{for each } i = 1, \dots, n \text{ and } t = 1, \dots, q_N. \quad (33)$$

Proof: By the definition of μ in (31), a direct calculation gives the results.

Corollary 1 *The Aumann-Shapley method satisfies the Proportionality.*

Proof: Suppose that $C(q) = c(q_N)$ where $c : R_+ \rightarrow R_+$. Then $\partial_i C(t) = c'(t_N)$, $i = 1, \dots, n$ and therefore for each $i = 1, \dots, n$

$$\begin{aligned} x_i(q; C) &= \sum_{t \in [0, q]} \mu_i(q)(t) \partial_i C(t) \\ &= \sum_{t_N=1}^{q_N} c'(t_N) \sum_{t' \in [0, q]: t'_N = t_N} \mu_i(q)(t') \\ &= \frac{q_i}{q_N} \sum_{t_N=1}^{q_N} c'(t_N) \\ &= \frac{q_i}{q_N} C(q). \end{aligned}$$

This completes the proof.

Q.E.D.

Recall the continuous Aumann-Shapley pricing method. It is generated by the continuous path-the diagonal, as follows,

$$x_i(q, C) = \int_0^{q_i} \partial_i C\left(\frac{t}{q_i} q\right) dt. \quad (34)$$

(See Aumann and Shapley [1974], Billera, Heath and Raanan [1978], Samet and Tauman [1982].)

Both the discrete and the continuous Aumann-Shapley methods satisfy Proportionality and do not satisfy the Demand Monotonicity (each agent's cost share is a non-decreasing function of his demands, see Moulin [1995]).

3 The Characterization

To simplify the analysis, we first consider the two-agent case.

In this case the Aumann-Shapley method can be expressed as:

$$x_1(q_1, q_2; C) = \frac{q_1!q_2!}{(q_1 + q_2)!} \sum_{t \in [0, q_1]} \frac{(t_1 + t_2)! (q_1 + q_2 - t_1 - t_2)!}{t_1!t_2! (q_1 - t_1)!(q_2 - t_2)!} \quad (35)$$

$$\left(\frac{t_1}{t_1 + t_2} - \frac{q_1 - t_1}{q_1 + q_2 - t_1 - t_2} \right) C(t). \quad (36)$$

Theorem 1 *If either $\bar{q}_1 = 1$ or $\bar{q}_2 = 1$, then x satisfies Additivity, Dummy and Proportionality if and only if x is the Aumann-Shapley method.*

Proof: We only need to check the *only if* part. It suffices to show that the cost sharing method which satisfies Additivity, Dummy and Proportionality is unique under the assumption.

Without loss of generality, assume that $q_2 = 1$ and consider $i = 1$. Since x is Additive and has the Dummy property there exists a vector μ_1 such that

$$x_1(q_1, 1; C) = \sum_{t \in [0, (q_1, 1)]} \mu_1(q_1, 1)(t) \partial_1 C(t).$$

Consider the following two types of cost functions:

Type 1: Homogeneous

$$C(p) = \delta_r(p_1 + p_2), \quad r \geq 1$$

where $\delta_r(z) = 1$ if $z \geq r$ and zero otherwise.

And

Type 2:

$$\delta_{(r_1, 0)}, \quad 0 < r_1 \leq q_1$$

where $\delta_q(q') = 1$ if $q' \geq q$ and zero otherwise.

Let $r = 1$ first. Then by Proportionality

$$x_1(q_1, 1; \delta_r) = \mu_1(q_1, 1)(1, 0) = \frac{q_1}{q_1 + 1}.$$

And let $r_1 = 1$, by Dummy

$$x_1(q_1, 1; \delta_{(r_1, 0)}) = \mu_1(q_1, 1)(1, 0) + \mu_1(q_1, 1)(1, 1) = 1.$$

Therefore

$$\mu_1(q_1, 1)(1, 1) = 1 - \mu_1(q_1, 1)(1, 0) = 1 - \frac{q_1}{q_1 + 1}.$$

Repeating these two steps, we can uniquely determine $\mu_1(q_1, 1)(t)$ for all $t \in [0, (q_1, 1)]$.

So x is uniquely determined. The theorem is proved.

Q.E.D.

The following example shows that in general Additivity, Dummy and Proportionality are not enough to characterize the Aumann-Shapley method.

Let $q_1 = 2, q_2 = 2$ be the demands. The Aumann-Shapley method is (for $i=1$)

$$\begin{aligned} x_1((2, 2); C) &= \frac{1}{6}[3C(2, 2) + 2C(2, 1) + C(2, 0) + 2C(1, 0) \\ &\quad - 2C(0, 1) - C(0, 2) - 2C(1, 2)] \end{aligned}$$

But there is another method which also satisfies Additivity, Dummy and Proportionality. It is

$$\begin{aligned} x_1((2, 2); C) &= \frac{1}{4}[2C(2, 2) + C(2, 1) + C(2, 0) + C(1, 0) \\ &\quad - C(0, 1) - C(0, 2) - C(1, 2)] \end{aligned}$$

Therefore we have to introduce other axiom(s) in order to characterize the Aumann-Shapley method.

Let x be a cost sharing method satisfying Additivity and Dummy. Then by the representation lemma there exist vectors $\mu_i, i = 1, \dots, n$ such that (16) and (17) hold.

Definition 4 Let x be an additive method satisfying Dummy. Say that x satisfies Consistency of the Cost Sharing Ratios (CCSR) if the corresponding $\mu_i, i = 1, \dots, n$ have the following property

$\forall q \in [0, \bar{q}]$, $q' \in [0, \bar{q}]$ and $t \in [0, q] \cap [0, q']$,

$$\frac{\mu_i(q)(t)}{\mu_j(q)(t)} = \frac{\mu_i(q')(t)}{\mu_j(q')(t)}. \quad (37)$$

This axiom can be regarded as a consistency property. The coefficients in the representation of the cost sharing formula may depend on the demand vector q but the ratios between any two agents should not change as demands change.

Lemma 2 *If x satisfies Additivity, Dummy, Proportionality and the CCSR, then*
 $\forall q \in [0, \bar{q}]$, $t \in [0, q]$

$$\frac{\mu_i(q)(t)}{\mu_j(q)(t)} = \frac{t_i}{t_j}. \quad (38)$$

Proof: Consider the $n = 2$ case only. The general case is similar. If $q_1 = 1$ or $q_2 = 1$, then we know by Theorem 1 that Additivity, Dummy and Proportionality imply that x must be the Aumann-Shapley method. It is easy to check that for A-S (38) holds.

Now assume that $q_1 = 2$ and $q_2 = 2$ first. Choose $q' = (2, 1)$. By CCSR, $\forall t \in [0, q']$,

$$\frac{\mu_1(q)(t)}{\mu_2(q)(t)} = \frac{\mu_1(q')(t)}{\mu_2(q')(t)}$$

but from the above argument we already know that

$$\frac{\mu_1(q')(t)}{\mu_2(q')(t)} = \frac{t_1}{t_2}.$$

Therefore

$$\frac{\mu_1(q)(t)}{\mu_2(q)(t)} = \frac{t_1}{t_2}, \quad t \in [0, q'].$$

Symmetrically let $q' = (1, 2)$. We have

$$\frac{\mu_1(q)(t)}{\mu_2(q)(t)} = \frac{t_1}{t_2}, \quad t \in [0, q'].$$

So only $t = (2, 2)$ needs to be checked. But by Proportionality

$$\mu_1(q)(2, 2) = \frac{2}{2+2} = \frac{1}{2}$$

and

$$\mu_2(q)(2, 2) = \frac{2}{2+2} = \frac{1}{2}.$$

Hence,

$$\frac{\mu_1(q)(2, 2)}{\mu_2(q)(2, 2)} = \frac{1/2}{1/2} = \frac{2}{2}.$$

Therefore for any $t \in [0, (2, 2)]$,

$$\frac{\mu_1(q)(t)}{\mu_2(q)(t)} = \frac{t_1}{t_2}.$$

Before jumping to the conclusion, consider one more step forward, i.e., assume now that $q = (3, 3)$. Let $q' = (2, 2)$. Then for all $t \in [0, (2, 2)]$, by the CCSR and the above result we have

$$\frac{\mu_1(q)(t)}{\mu_2(q)(t)} = \frac{\mu_1(q')(t)}{\mu_2(q')(t)} = \frac{t_1}{t_2}.$$

Similarly by Proportionality for $t = (3, 3)$

$$\frac{\mu_1(q)(t)}{\mu_2(q)(t)} = 1 = \frac{3}{3}.$$

Now we show for $t = (3, 2)$ and $t = (2, 3)$. Again consider $q = (3, 2)$, $q' = (2, 2)$, $q'' = (3, 1)$. We have

$$\frac{\mu_1(q)(t)}{\mu_2(q)(t)} = \frac{\mu_1(q')(t)}{\mu_2(q')(t)} = \frac{t_1}{t_2}, \quad \forall t \in [0, q'],$$

and

$$\frac{\mu_1(q)(t)}{\mu_2(q)(t)} = \frac{\mu_1(q'')(t)}{\mu_2(q'')(t)} = \frac{t_1}{t_2}, \quad \forall t \in [0, q''].$$

So only $t = (3, 2)$ (or symmetrically $t = (2, 3)$) needs to check. Again by Proportionality (and the CCSR)

$$\mu_1(q)(t) = \frac{3}{5}$$

and

$$\mu_2(q)(t) = \frac{2}{5}.$$

This implies

$$\frac{\mu_1(q)(t)}{\mu_2(q)(t)} = \frac{3/5}{2/5} = \frac{3}{2}.$$

Now it is clear that by induction (we omit the details) we can conclude that for any $q \in [0, \bar{q}]$ and any $t \in [0, q]$ the equation (38) holds. The lemma is proved. Q.E.D.

In general, we have the following theorem.

Theorem 2 *There exists a unique cost sharing method which satisfies Additivity, Dummy, Proportionality and Consistency of the Cost Sharing Ratios. It is the Aumann-Shapley method.*

Proof: Let x be a method satisfying these four axioms. We are going to show that the coefficients in the representation lemma (Lemma 1 in Section 1 of this chapter) are uniquely determined.

We know that the total number of unknowns, $\mu_1(t), \mu_2(t), t \in]0, q]$, is

$$q_1(q_2 + 1) + q_2(q_1 + 1) = 2q_1q_2 + q_1 + q_2.$$

The budget balance (equation (17 in Lemma 1 in Section 1)) implies that we have

$$q_1 + q_2 + q_1q_2$$

independent equations (each corresponds to a different $t \in]0, q]$) for the unknowns.

By lemma 2, in addition, we have

$$q_1q_2$$

more independent equations (for different t they must be linearly independent between each other). Therefore totally we have exactly

$$2q_1q_2 + q_1 + q_2$$

independent equations and henceforth the unknowns are uniquely determined.

That the Aumann-Shapley method satisfies Additivity, Dummy and the CCSR is obvious. Proportionality is already known by Corollary 1. The theorem is thus proved. Q.E.D.

The Measurement Invariance Axiom in the Discrete Cost Sharing Model

with Yves Sprumont

Abstract

We propose a measurement invariance axiom on cost sharing methods for the discrete model. The axiom is a discrete version of the well-known *scale invariance* axiom in the continuous cost sharing model. We show that the set of measurement invariant and additive methods satisfying the dummy axiom is the set of *simple random order values*. Consequently, the Shapley-Shubik method is the only symmetric method in that set.

1 Introduction

Let us continue to consider the discrete cost sharing model. Instead of focusing on the characterization of the specific discrete Aumann-Shapley method, we try to see if a discrete version of the Scale Invariance axiom can be defined so that it can play a same rule in the characterization of the discrete Aumann-Shapley method as the Scale Invariance in the continuous Aumann-Shapley method. We are interested in this discrete “scale invariance” not only because of this discrete Aumann-Shapley method but also because of the fact that it is an interesting as well as a challenging problem to define a suitable scale invariance axiom in the discrete model. This problem of finding a proper discrete “scale invariance” axiom becomes even more challenging when we require that the discrete Aumann-Shapley method satisfy it²⁶.

As a first step, we formulate a very crude version of it, called Measurement Invariance axiom. The idea is to mimic the traditional scale invariance axiom as much as possible. Since we can not “scale down” an integer vector, we use the following “indirect” way. Informally, for a given problem $(q; C)$ and a given scale vector $r \gg 0$, use r to split $(q; C)$ into two problem $(r \otimes q; C)$ and $(q; C^r)$ (see the next section for detail). We say a method x satisfies Measurement Invariance axiom if for any problem $(q; C)$ and any given scale vector r , the method gives the same solution for the two derived problems, i.e., $x(r \otimes q; C) = x(q; C^r)$.

An immediate consequence (Proposition 1) of this axiom is that it forces us to use the value solution of the stand alone costs game the problem generates. Based on this result, we provide an alternative characterization of an important family of methods, namely the set of *simple random order methods*. The Shapley-Shubik method is the unique symmetric method in this set, and the Aumann-Shapley method unfortunately

²⁶So far we have not found such a scale invariance axiom. Maybe we would never be able to .

is excluded from this set.

By simple random order method we mean a method that uses the same random order value for the same set of active agents. In the context of cooperative game theory, Weber [1988] firstly defined the random order values as being the convex combinations of the incremental values of the stand alone cost game (see the cooperative game theory section in Chapter 1). He characterized the set of random order values by the combination of Additivity and Dummy.

Relevant to this paper is Sprumont [1998b]'s alternative characterization of simple random order methods by Additivity, Dummy, Demand Monotonicity, and a new axiom called Coherence. The Coherence axiom is a weaker version of scale invariance than our Measurement Invariance. It requires that for any given problem and any given scale vector, there always exists a re-scaled problem (called refined problem) which gives the same solution as the original problem. Interestingly, the discrete Aumann-Shapley method still violates this Coherence axiom. See Sprumont [1998b] and our concluding remarks for more detail.

2 The Measurement Invariance and Simple Random Order Values

Given a cost sharing problem $(q; C)$, we associate a cooperative game C_q (on the player set N), called *the stand-alone cost game*, defined by:

$$C_q(S) = C(q_S, 0), \quad S \subseteq N.$$

Let $A(q)$ be the set of active agents of the problem $(q; C)$, i.e., $A(q) = \{i \in N | q_i > 0\}$. Then, with a slight abuse of notation, define a *sub-(stand-alone cost) game* C_q on the

player set $A(q)$ by:

$$C_q(S) = C(q_S, 0), \text{ for all } S \subseteq A(q).$$

Therefore, the set of sub-games are the collections of all games on various subsets of the grand coalition N .

Then, recall the well-known *scale invariance* axiom in the traditional cost sharing problem in which goods are perfectly divisible. It requires that the solution should not depend on the units in which goods are measured. Samet and Tauman [1982] characterized the well-known Aumann-Shapley method as the only scale-invariant additive method prescribing proportional cost shares when the goods enter additively in the cost function.

However, for the discrete cost sharing model, Moulin [1995] noted that the idea of scale invariance can no longer be formulized in any obvious way. Therefore, he dispensed entirely with any requirement of invariance to the measurement conventions. His characterization of the Shapley-Shubik method is hardly comparable with the one obtained in the divisible framework.

We define a discrete version of the traditional scale invariance axiom in the following. Though seemingly very close to its continuous relative, our requirement turns out to be much more demanding.

Let $r \in \mathbb{N}^N$ and $q \in \mathbb{N}_0^N$. Define $r \otimes q$ componentwise by $(r \otimes q)_i = r_i q_i$. Given a cost function C , define $C^r(q) = C(r \otimes q)$ for each $q \in \mathbb{N}_0^N$.

The **Measurement Invariance Axiom** reads as follows:

For every problem $(q; C)$ and every $r \in \mathbb{N}^N$,

$$x(q; C^r) = x(r \otimes q; C) \tag{39}$$

It is obvious that this is not exactly what we want for a scale invariance axiom. We therefore prefer to speak of *measurement invariance*.

It is easy to see that the following well-known Shapley-Shubik method satisfies Measurement Invariance.

Definition 1 *The Shapley-Shubik method x^{SS} assigns to each problem $(q; C)$ the Shapley value of the stand-alone game that it generates:*

$$x_i^{SS}(q; C) = \sum_{S \subset N: i \in S} \frac{(s-1)!(n-s)!}{n!} (C_q(S) - C_q(S \setminus i)), \quad i \in N, \quad (40)$$

where $s = |S|$.

However, the discrete Aumann-Shapley method in the previous paper does not satisfy the Measurement Invariance. An example can be found in Sprumont [1998b]. It can also be inferred from our main theorem in the next section.

In fact, we will see that Measurement Invariance not only rules out the discrete Aumann-Shapley method but also any method using the data beyond the stand-alone costs.

A method x is called *simple* if it solves all problems with the same set of active agents generating a same sub- game in the same way, i.e., $x(q; C) = x(q'; C')$ whenever $A(q) = A(q')$ and $C_q = C'_{q'}$.

Clearly, every simple method is measurement-invariant. Conversely, we now prove the following result.

Proposition 1 *Every method which is measurement-invariant and independent of irrelevant Costs²⁷, is simple.*

Proof. Let the method x be measurement-invariant and independent of irrelevant costs. Let $(q; C)$ and $(q'; C')$ be two problems such that $A(q) = A(q')$ and $C_q = C'_{q'}$.

²⁷See the Appendix in Section 1 of this chapter for this property.

Denote e the unit vector ($e_i = 1, i \in N$). By measurement invariance,

$$\begin{aligned} x(q; C) &= x((q_{A(q)}, 0); C) \\ &= x((q_{A(q)}, e_{-A(q)}) \otimes (e_{A(q)}, 0); C) \\ &= x((e_{A(q)}, 0); C^{(q_{A(q)}, e_{-A(q)})}) \end{aligned}$$

and

$$\begin{aligned} x(q'; C') &= x((q'_{A(q)}, 0); C') \\ &= x((q'_{A(q)}, e_{-A(q)}) \otimes (e_{A(q)}, 0); C') \\ &= x((e_{A(q)}, 0); C'^{(q'_{A(q)}, e_{-A(q)})}). \end{aligned}$$

But $C^{(q_{A(q)}, e_{-A(q)})}(t) = C'^{(q'_{A(q)}, e_{-A(q)})}(t)$ for every demand vector $t \leq (e_{A(q)}, 0)$.

Indeed,

$$\begin{aligned} C^{(q_{A(q)}, e_{-A(q)})}(t) &= C((q_{A(q)}, e_{-A(q)}) \otimes (t_{A(q)}, t_{-A(q)})) \\ &= C((q_{A(q)}, e_{-A(q)}) \otimes (t_{A(q)}, 0)) \\ &= C(q_S, 0_{-S}) (S = A(t) \subseteq A(q)) \\ &= C_q(S) \\ &= C'_{q'}(S) \\ &= C''(q'_S, 0_{-S}) \\ &= C''((q'_{A(q)}, e_{-A(q)}) \otimes (t_{A(q)}, 0)) \\ &= C''((q'_{A(q)}, e_{-A(q)}) \otimes (t_{A(q)}, t_{-A(q)})) \\ &= C'^{(q'_{A(q)}, e_{-A(q)})}(t) \end{aligned}$$

Since x is independent of irrelevant costs, therefore,

$$x(q; C) = x(q'; C'),$$

proving that x is simple. This completes our proof of the Proposition. Q.E.D.

Let $S \subseteq N$. Without loss of generality, assume that $S = \{1, 2, \dots, s\}$. For a given ordering σ^S of $\{1, 2, \dots, s\}$, we define the σ^S -order method x^{σ^S} as follows:

$$x_i^{\sigma^S}(q; C) = C(q_{P(\sigma^S, i)} \cup i, 0) - C(q_{P(\sigma^S, i)}, 0), \quad i \in S \text{ and } 0 \text{ if } i \in N \setminus S,$$

where $P(\sigma^S, i)$ is the coalition of agents preceding i in the ordering σ^S .

A S -random order method is a convex combination of the σ^S -order methods. When $S = N$, it is the standard random order method in the sense of Weber [1988].

Definition 2 *A simple random-order method is defined by a collection of the S -random order methods ($S \subseteq N$) in the way that, for any problem $(q; C)$, the $S = A(q)$ -random order method is applied.*

3 The Characterization of Simple Random Order Methods

Our main result is the following characterization of the simple random order methods by Additivity, Dummy and Measurement Invariance.

Theorem 1 *The measurement-invariant additive methods satisfying the dummy axiom are the simple random order methods. The Shapley-Shubik method is the only symmetric method in that set.*

(Where x is called *symmetric* if $x_i(q; C) = x_j(q; C)$ whenever $q_i = q_j$ and C is a symmetric function of i and j 's demands.)

Proof. Let x be a measurement-invariant additive method satisfying the dummy axiom. Since additivity and dummy imply independence of irrelevant costs (Moulin

[1995]), it follows from our Proposition 1 that x is simple. Clearly, a game v (with $v(\emptyset) = 0$ and the player set $S \subseteq N$) is generated by some cost sharing problem (with active agent set S) if and only if it is monotonic (i.e., $v(S') \leq v(T')$ whenever $S' \subseteq T' \subseteq S$). In fact, given any cost sharing problem $(q; C)$, define a cooperative game C_q on the player set $A(q)$ (recall that $A(q)$ is the set of active agents at q , it is a subset of the universal set N .) as follows:

$$C_q(S) = C(q_S, 0), \text{ for every } S \subseteq A(q).$$

It is clear that C_q is monotonic.

On the other hand, for any monotonic game v with the player set S ($S \subseteq N$), we can construct a cost sharing problem $(q; C)$ which generates the game v . Actually, let $q = (e_S, 0)$ and define C on \mathbb{N}_0^N as follows:

$$\begin{aligned} C(t_S, t_{-S}) &= v(A(t_S)), \text{ if } t_S \leq e_S \text{ (i.e., } t_i \leq 1, i \in S) \\ C(t) &\text{ nondecreasing, otherwise.} \end{aligned}$$

Since x is simple, we can define a collection of values $(\xi^S)_{S \subseteq N}$, one for each subclass of monotonic games on each subset of agents, such that

$$\xi_i^{A(q)}(C_q) = x_i(q; C), \quad i \in A(q). \quad (41)$$

Note that, for each subset S , ξ^S satisfies the additivity and dummy axioms on the set of monotonic games. This follows directly from the additivity and dummy properties of x . Moreover, from the linearity of x (shown in the proof of the representation lemma in the first paper) and the positivity of x , we know that the value ξ^S is also linear and satisfies Weber's monotonicity axiom. From Theorem 4 in Weber [1988], each ξ_i^S ($i \in S$) is a *probabilistic (individual) value*. From Weber's Theorem 13, ξ^S is a *random order value* (on the player set S). Therefore, it follows that x is a simple

random order method. It is easily seen that symmetry pinpoints the Shapley values to each subclass of the monotonic games induced by the cost sharing problems and it coincides with the Shapley value for the (*enlarged*) monotonic games on the universal set N .²⁸ Therefore, it is the Shapley-Shubik method. The part that the simple random order methods do satisfy additivity, dummy and measurement invariance is obvious. This completes our proof. Q.E.D.

4 Concluding Remarks

Besides Measurement Invariance, there have been two alternative scale invariance axioms proposed for the discrete model. The first is Moulin's [1995] (discrete) axiom of *ordinality*. It is the following condition. Fix a cost function C , an agent i , and a demand level t_i^0 for i . Construct the cost function $C_{t_i^0}$ by jumping above t_i^0 : $C_{t_i^0}(t) = C(t)$ if $t_i < t_i^0$ and $C_{t_i^0}(t) = C(t_{N \setminus i}, t_i + 1)$ otherwise. If C is flat at t_i^0 , i.e., $C(t_{N \setminus i}, t_i^0 + 1) = C(t_{N \setminus i}, t_i^0)$ for all $t_{N \setminus i}$, Moulin requires that $x(q; C_{t_i^0}) = x((q_{N \setminus i}, q_i + 1); C)$ for every demand vector q such that $q_i \geq t_i^0 + 1$. From our Proposition 1, it is clear that every measurement-invariant method which is independent of irrelevant costs satisfies Moulin's ordinality axiom. On the other hand, ordinal methods (in Moulin's sense) which are independent of irrelevant costs need not to be measurement-invariant. Here is an example. Let $n = 2$. Define a *path system* π which associates with each demand vector $q \in \mathbb{N}_0^N$ a path to q in the following way: choose the unique path P which passes through the points $(\min\{q_1, 1\}, 0)$, $(\min\{q_1, 1\}, \min\{q_2, 1\})$, $(q_1, \min\{q_2, 1\})$ and (q_1, q_2) . Then, define the π -incremental method by charging each agent the sum of his marginal costs along the path recommended by the system π . It is not hard to see that this method is additive, dummy (therefore independent of irrelevant costs) and sat-

²⁸Let v be a game on $S \subseteq N$. The enlarged game v (we use the same notation without the risk of confusion) with the player set N is defined by: $v(T) = v(T \cap S)$, for each $T \subseteq N$.

isfies Moulin's ordinality axiom. But this method is not measurement-invariant. Indeed, Moulin [1995] noted that there exist ordinal additive methods satisfying dummy but violating demand monotonicity or cross monotonicity. By our Theorem 1, such methods can not be measurement-invariant since all simple random order methods are demand-monotonic and cross-monotonic. Therefore our measurement invariance axiom is more binding than Moulin's ordinality axiom.

The second is the *Coherence* axiom proposed by Sprumont [1998b]. It is based on the following observation. In a discrete problem where agent i demands q_i units of good i , the relevant cost data are described by a cost function defined on $\times_i \{0, 1, \dots, q_i\}$, that is to say, by $(q_1 + 1)(q_2 + 1) \dots (q_n + 1)$ numbers. The relevant information, therefore, gets richer as the demand vector q grows. It means that a cost sharing problem may be "refined" by adding (or learning) information about costs at "intermediate" production vectors. A cost sharing method may use this richer information to revise the cost shares originally decided upon. If the method is coherent, however, there should exist at least one conceivable refinement of the problem at hand for which the cost shares remain unchanged. Formally, Given a problem $(q; C)$ and $r \in \mathbb{N}^N$. A cost function C^r r -refines C if $C^r(r \otimes t) = C(t)$ for every $t \in \mathbb{N}_0^N$. We say x is Coherent if for every problem $(q; C)$ and $r \in \mathbb{N}^N$, there exists a cost function C^r which r -refines C such that

$$x(q; C) = x(r \otimes q; C^r).$$

Compare it with our Measurement Invariance axiom, i.e.,

$$x(q; C^r) = x(r \otimes q; C),$$

where $C^r(t) = C(r \otimes t)$. It is easy to see that these two axioms are very different (however, it is obvious that they are identical in the continuous model). Coherence is more like a discrete counterpart of the classical scale invariance axiom. This is not only

because of its formulation similarity but also the fact that Coherence plays a similar rule as scale invariance in characterizing the Shaply-Shubik method as demonstrated in Sprumont's [1998]. From our Theorem 1, we see that Measurement Invariance combines Coherence and Demand Monotonicity with regard to the additive methods. On one hand, there are additive methods which are Coherent but not Measurement Invariant. Therefore, Measurement Invariance is more binding than Coherence. On the other hand, interestingly, the discrete Auamnn-Shapley method does not satisfy Coherence (see Sprumont [1998b]).

Chapter 3 : The Continuous Model

Ordinal Additive Methods Must Be Simple Random Order Values

with Yves Sprumont

Abstract

We consider the continuous cost sharing model. We provide a characterization for the set of *simple* random order methods by the axioms of Additivity, Dummy, and Ordinality. Ordinality requires that cost shares be invariant with essentially all increasing transformations of the measurement scales of the demands.

1 Introduction

This paper considers the well-studied continuous cost sharing model (Billera et al [1978] [1982], Samet and Tauman [1982], Tauman [1988], Friedman and Moulin [1995], and Sprumont [1998]), for which most of the literature, except Friedman and Moulin [1995] and Sprumont [1998], had focused on the Aumann-Shapley pricing method in the set of additive methods.

A subset of the set of additive methods, namely the set of simple random order methods, has been characterized by Additivity, Dummy, and Measurement Invariance (the third paper in Chapter 2), and alternatively by Additivity, Dummy, Demand Monotonicity, and Coherence (Sprumont [1998b]), in the discrete model.

We consider the set of simple random order methods in the continuous model. We provide a characterization of this set by Additivity, Dummy, and Ordinality (Sprumont [1998]).

Ordinality requires that cost shares should not depend on the conventions used to measure the agent's demands. Formally it says that the cost shares must be invariant under essentially all increasing transformations of the measuring scales (rather than just the linear ones as the scale invariance imposes).

Mathematically speaking, this axiom has much bite on cost sharing methods. However, Sprumont [1998] pointed out that it still allows a lot of flexibility to the cost sharing methods. In fact, on the one hand, it strengthens the classical Scale Invariance and therefore is able to sharpen the characterization of the Shapley-Shubik method. On the other hand, it provides a basic axiom for investigating non-additive methods.

Apart from being a powerful (mathematical) axiom, Ordinality is also a meaningful and compelling axiom in practical situations where the goods to be measured

are non-physical goods, e.g., services. Sprumont [1998] provided two examples in which the conventions used to measure the agent's demands should not affect the cost shares assigned to each agent by a cost sharing method. In his examples he tested the Aumann-Shapley method to two "ordinally equivalent" (see the next section) cost functions using different ways of measuring the demands, where a non-linear relation holds between those two different measurements (rather than the linear relation imposed by the classical Scale Invariance). He showed that the Aumann-Shapley method is not ordinal. In other words, the Aumann-Shapley method depends upon the conventions used to measure the demands. In cases where the goods or the demands to be measured are quality-oriented, such as the services, labor, efforts, Ordinality is a meaningful requirement.

In this paper, we show that the Ordinality axiom together with the Additivity and Dummy axioms characterizes the set of *simple random-order values*. As a corollary, the Shapley-Shubik method is the only *symmetric* ordinal additive method. And the Demand Monotonicity is implied by the combination of Additivity, Dummy and Ordinality.

This conclusion implies that the Ordinality axiom when it combines with Additivity axiom, can be very binding. It forces the methods to use value solutions. This is to say that if we want to use the "Proportional" solutions or the "Serial" solutions (see Sprumont [1998]) *and* require them to be "ordinal", then we must abandon Additivity. In other words, the non-additive sharing methods should be explored.

2 The Model and the Ordinality Axiom

Let $N = \{1, \dots, n\}$ be the set of agents. A demand profile q is a vector in R_+^N . Let $C_0(N)$ be the set of functions $C : R_+^N \rightarrow R_+$ which are non-decreasing ($p \leq$

$q \Rightarrow C(p) \leq C(q)$ for all p, q) and satisfy $C(0) = 0$. A cost function (for N) is an element of some generic domain $\mathcal{C}(N) \subseteq \mathcal{C}_0(N)$. If the first-order partial derivative of $C \in \mathcal{C}_0(N)$ with respect to its i th argument exists at $q \in R_+^N$, we denote it by $\partial_i C(q)$.²⁹ Denote $\mathcal{C}_1(N)$ the domain of all continuously differentiable functions in $\mathcal{C}_0(N)$. A (*cost sharing*) *problem* is a pair $(q; C)$ where q is a demand vector and C is a cost function. Given a problem $(q; C)$, a solution of the problem is a vector $(x_1, \dots, x_n) \in R_+^N$ such that $\sum_1^n x_i = C(q)$. A cost sharing method x is a mapping associating each problem $(q; C)$ a solution $x(q; C)$.

We consider the family of methods satisfying the following two well-known axioms: Additivity and Dummy.

- Additivity

$$x(q; C_1 + C_2) = x(q; C_1) + x(q; C_2) \text{ for each } q \in R_+^N \text{ and } C_1, C_2 \in \mathcal{C}.$$

- Dummy

$$\text{If } \partial_i C(p) = 0, \forall p \in R_+^N, \text{ then } x_i(q; C) = 0.$$

As in the discrete model, the Additivity and Dummy axioms admit the following representation lemma due to Friedman and Moulin [1995], which will be used in our proof of Theorem 1. This representation lemma is exactly the continuous counterpart of the representation lemma in the first paper on the discrete model. It is an application of the Riesz representation theorem.

Lemma 1 *Fix $q \in R_+^N$. Let x be additive and satisfy the dummy axiom. Then, for each $i \in N$, there exists a measure μ_i^q such that*

$$x_i(q; C) = \int_{[0, q]} \partial_i C(p) d\mu_i^q(p), \text{ for each } C \in \mathcal{C}, \quad (42)$$

²⁹If $q_i = 0$, it is understood that $\partial_i C(q)$ stands for the right-hand derivative.

where the measure μ_i^q has the following property: its projection on any interval $[p_i, p'_i]$, $0 \leq p_i < p'_i \leq q_i$, is the Lebesgue measure on R .

Our central axiom in this paper is the following *Ordinality* axiom first introduced by Sprumont [1998]. Fix N and a domain $\mathcal{C}(N)$. Let $f = (f_1, \dots, f_n)$ be a bijection from R_+^n onto itself. For each cost function C in $\mathcal{C}(N)$, define $C^f : R_+^n \rightarrow R_+$ by

$$C^f(t) = C(f(t)) \text{ for all } t \in R_+^n.$$

We call f an ordinal transformation if $\mathcal{C}(N)$ is closed under it, i.e.,

$$C^f \in \mathcal{C}(N) \text{ for all } C \in \mathcal{C}(N).$$

When $\mathcal{C}(N) = \mathcal{C}_1(N)$, a bijection f is an ordinal transformation if and only if it is increasing and continuously differentiable.

Definition 1 *Two problems $(q; C)$ and $(q'; C')$ (N is omitted) are called ordinally equivalent if there exists an ordinal transformation f such that*

$$C' = C^f \text{ and } q = f(q').$$

Now, we state our central axiom as follows.

Ordinality (ORD) axiom: If $(q; C)$ and $(q'; C')$ are two ordinally equivalent problems, then $x(q; C) = x(q'; C')$.

Among the three well-known methods, namely the Shapley-Shubik method, the Aumann-Shapley method, and the Friedman-Moulin method, only the Shapley-Shubik method satisfies Ordinality.

It is easy to see that the following *simple* random order methods satisfy Ordinality.

Let S be a subset of N . Without loss, assume $S = \{i_1, \dots, i_s\}$ where $s = |S|$, the number of elements in S . Let σ_S be an ordering of the elements in S , for example, $\sigma_S = \{i_1, \dots, i_s\}$. Denote π_S the set of all possible such orderings.

A simple incremental method is defined by a set of orderings $\{\sigma_S\}$, one for each subset $S \subseteq N$ such that if for the problem $(q; C)$, $A(q) = S$, then

$$x_i^{inc-S}(q; C) = C(\{i_1, \dots, i_j\}) - C(\{i_1, \dots, i_{j-1}\}), \quad j = 1, \dots, s, \quad (43)$$

and

$$x_i^{inc-S}(q; C) = 0, \quad \text{for } i \notin S. \quad (44)$$

Definition 2 *A simple random order method x is defined by a set of probability distributions $\{p(\pi_S)\}$, one for each π_S such that for any given problem $(q; C)$, if $A(q) = S$, then*

$$x_i(q; C) = \sum_{\sigma_S \in \pi_S} p(\sigma_S) x_i^{inc-S}(q; C) \quad (45)$$

(An equivalent definition can be found in Sprumont [1998b].)

Note that a (classical) incremental method is defined for a given ordering of the elements in the set N . And a random order method (in the sense of Weber [1988]) is a convex combination of the incremental methods.

From these definitions, clearly, every simple random order method is ordinal. In the next section, we will prove that, conversely, every ordinal additive method is a simple random order method.

3 The Characterization Theorem

Our main theorem in this paper is the following characterization result.

Theorem 1 *The ordinal additive methods satisfying the dummy axiom are the simple random order methods. The Shapley-Shubik method is the only symmetric method in that set.*

Proof: It is easy to check that all the simple random-order methods satisfy additivity, dummy and ordinality.

Let x be a method satisfying these axioms. The proof that x is a simple random-order method is provided by modifying the proof of Theorem 1 in Sprumont [1998] as follows. For completeness, we adapt some parts of the proof in that paper for our purpose.

Fix $q \gg 0$. We consider the implication of additivity, dummy and ordinality on the measure $\mu_i^q, i \in N$ in the representation lemma. We will see that it suffices to consider them on the following class of simple problems.

Fix a nonempty set $S \subseteq N$. Let $p = (p_S, 0_{-S}) \in R^N$ and $\epsilon \in R$ be such that $0 \leq p_S \leq q_S$ and $0 < \epsilon < q_i - p_i$ for all $i \in S$. Define the mapping $C_{\epsilon p_S} : R_+^N \rightarrow R_+$ by

$$C_{\epsilon p_S}(t) = \sum_{T \subseteq S} (-1)^{|T|+1} \prod_{i \in T} c_{\epsilon p_i}(t_i)$$

where $c_{\epsilon p_i}(t_i) = \min\{1, \frac{1}{\epsilon} \max\{0, t_i - p_i\}\}$. It is obvious that $C_{\epsilon p_S}(0) = 0$ and we can prove that $C_{\epsilon p_S}$ is non decreasing (see Sprumont [1998] for a proof).

Unfortunately, $C_{\epsilon p_S}$ is not quite a cost function because it is not differentiable. But we can approximate it arbitrarily well by a continuously differentiable function. For each $a = 1, 2, \dots$ and $t \in R_+^N$, define

$$C_{\epsilon p_S}^a(t) = \sum_{T \subseteq S} (-1)^{|T|+1} \prod_{i \in T} c_{\epsilon p_i}^a(t_i)$$

where $c_{\epsilon p_i}^a(t_i)$ is worth $\frac{1}{2}(\frac{2}{\epsilon} \max\{0, t_i - p_i\})^{1+\frac{1}{a}}$ if $0 \leq t_i \leq p_i + \frac{\epsilon}{2}$ and $1 - \frac{1}{2}(\frac{2}{\epsilon} \max\{0, \epsilon - t_i + p_i\})^{1+\frac{1}{a}}$ otherwise. Check that each $C_{\epsilon p_S}^a$ is in $\mathcal{C}_1(N)$ and that their sequence converges uniformly on $[0, q]$ to $C_{\epsilon p_S}$. Also note that $C_{\epsilon p_S}^a(q) = 1$, for each $a = 1, 2, \dots$

Step 1. For each $i \in S$, construct a strictly increasing differentiable mapping f_i such that

$$\begin{aligned} f_i(0) &= 0, \\ f_i(t_i) &= 1 + \frac{(t_i - p_i)}{\epsilon} \quad \text{if } p_i \leq t_i \leq p_i + \frac{\epsilon}{2}, \\ f_i(q_i) &= 3 \end{aligned}$$

And for each $i \in N \setminus S$, simply let

$$f_i(t_i) = \frac{1}{q_i} t_i.$$

The mapping $f = (f_1, \dots, f_n)$ is an ordinal transformation making the problem $(q; C_{\epsilon p S})$ ordinally equivalent to

$$((3e_S, e_{N \setminus S}); C_{1e_{NS}}^a)$$

(Indeed, for each $i \in S$, $f_i(q_i) = 3$ and $c_{11}^a(f_i(t_i)) = c_{\epsilon p_i}^a(t_i)$ for every t_i .) For all $i \in N$, let us define

$$\xi_i^a(S) = x_i((3e_S, e_{N \setminus S}); C_{1e_{NS}}^a).$$

Since $\partial_i C_{1e_{NS}}^a = 0$ for all $i \in N \setminus S$, Dummy implies that $\xi_i^a(S) = 0$ for all $i \in N \setminus S$. By definition of a cost sharing method $\sum_{i \in S} \xi_i^a(S) = 1$ and $0 \leq \xi_i^a(S) \leq 1$ for all $i \in S$. By ordinality

$$x(q; C_{\epsilon p S}^a) = \xi^a(S). \quad (46)$$

For fixed $S \subseteq N$ and each $i \in N$, the set

$$\{\xi_i^a(S) \mid a = 1, 2, \dots\}$$

is compact and therefore, there exists a subsequence $\{a'\} \subset \{a\}$ such that

$$\xi_i^{a'}(S) \longrightarrow \xi_i(S)$$

and the following are also true

$$\xi_i(i) = 1, \xi_i(S) \geq 0, \xi_i(S) = 0 \text{ if } i \notin S, \text{ and } \sum_{i \in S} \xi_i(S) = 1.$$

Step 2. Invoking the integral representation lemma (Lemma 1 in this section) we have

$$\begin{aligned}\xi_i^{a'}(S) &= x_i(q; C_{\epsilon p S}^{a'}) \\ &= \int_{[0, q]} \partial_i C_{\epsilon p S}^{a'} d\mu_i^q, \{a'\} \subseteq \{a\}.\end{aligned}\quad (47)$$

Step 3. Define the set

$$Z_i(\epsilon, p) = \{t \in [0, q] | t_i \in \{p_i, p_i + \epsilon\}\}.$$

Note that for $i \in S$, $\partial_i C_{\epsilon p S}$ does not exist on $Z_i(\epsilon, p)$ but

$$\mu_i^q(Z_i(\epsilon, p)) = 0,$$

because the projection of μ_i^q on the interval $0 \leq t_i \leq q_i$ is the Lebesgue measure.

Then

$$\begin{aligned}\xi_i^{a'}(S) &= x_i(q; C_{\epsilon p S}^{a'}) \\ &= \int_{[0, q] \setminus Z_i(\epsilon, p)} \partial_i C_{\epsilon p S}^{a'} d\mu_i^q, \{a'\} \subseteq \{a\}.\end{aligned}\quad (48)$$

Taking limit $a' \rightarrow \infty$ we get

$$\xi_i(S) = \int_{[0, q] \setminus Z_i(\epsilon, p)} \partial_i C_{\epsilon p S} d\mu_i^q \quad (49)$$

Denote

$$A_i(\epsilon, p, S) = \{t \in [0, q] | p_i < t_i < p_i + \epsilon \text{ and } t_j < p_j + \epsilon \forall j \in S \setminus i\},$$

$$B_i(\epsilon, p, S) = \{t \in [0, q] | p_i < t_i < p_i + \epsilon \text{ and } t_j < p_j \forall j \in S \setminus i\},$$

$$Q_i(S) = [(q_{S \setminus i}, 0_{-S \setminus i}), (q_S, 0_{-S})], i \in S$$

$$Q_i(\epsilon) = \cup_{S:i \in S} Q_i(\epsilon, S), i \in S$$

where

$$Q_i(\epsilon, S) = \{t \in [0, q] | t_j \leq \epsilon \text{ if } j \in N \setminus S \text{ and } t_j \geq q_j - \epsilon \text{ if } j \in S \setminus i\},$$

and

$$E_i(\epsilon, p) = \{t \in [0, q] | p_i \leq t_i \leq p_i + \epsilon\}.$$

Then by calculating (49), we have (see Sprumont [1998])

$$\mu_{iq}(A_i(\epsilon, p, S)) = \xi_i(S)\epsilon.$$

Notice that the right hand side of this equality does not depend on p . Letting S and p vary, it follows that

$$\mu_i^q(Q_i(\epsilon)) = \mu_i^q([0, q]),$$

and further letting ϵ go to zero we get

$$\mu_i^q(Q_i) = \mu_i^q([0, q]).$$

Using the same argument as in Sprumont [1998], then we can get the following equations

$$\sum_{T:T \cap S=i} \mu_i^q(E_i(\epsilon, p) \cap Q_i(T)) = \xi_i(S)\epsilon. \quad (50)$$

Let

$$\mu_i^q(E_i(\epsilon, p) \cap Q_i(T)) = \gamma_i(T)\epsilon \quad (51)$$

Then we have the linear equations

$$\sum_{T:T \cap S=i} \gamma_i(T)\epsilon = \xi_i(S)\epsilon, \quad S \subset N, \quad i \in S. \quad (52)$$

That is

$$\sum_{T:T \cap S=i} \gamma_i(T) = \xi_i(S), \quad S \subset N, \quad i \in S. \quad (53)$$

It has the following unique solution

$$\gamma_i(T) = \sum_{S \supset (N-T) \cup i} (-1)^{s-(n-t+1)} \xi_i(S), \quad T \subset N, \quad i \in T. \quad (54)$$

(Also note that $\gamma_i(T)$ is independent of q .)

In fact,

$$\begin{aligned} \sum_{T: T \cap S = i} \gamma_i(T) &= \sum_{T: T \ni i, T \subset (N-S) \cup i} \gamma_i(T) \\ &= \sum_{T: T \ni i, (N-T) \cup i \supset S} \gamma_i(T) \\ &= \sum_{T: T \ni i, (N-T) \cup i \supset S} \sum_{S \supset (N-T) \cup i} (-1)^{s-(n-t+1)} \xi_i(S) \\ &\quad (s = |S|, n-t+1 = |(N-T) \cup i|) \\ &= \sum_{T' \supset S} \sum_{S' \supset T'} (-1)^{s'-t'} \xi_i(S') \\ &= \sum_{S' \supset S} \left[\sum_{t'=s}^{s'} (-1)^{s'-t'} \binom{s'-s}{t'-s} \right] \xi_i(S') \\ &= \xi_i(S) \end{aligned} \quad (55)$$

(since $\sum_{t'=s}^{s'} (-1)^{s'-t'} \binom{s'-s}{t'-s} = 0$ except $s' = s$).

The uniqueness is guaranteed by the fact that the coefficient matrix of the equations has the unit determinant.

Step 4. By letting $S = \{i\}$ in (53), we get

$$\sum_{T: T \ni i} \gamma_i(T) = 1 \quad (56)$$

Now we show that $\{\gamma_i(T)\}$ is a probability distribution. We need only show that

$$\gamma_i(T) \geq 0.$$

This relies on the fact that

$$\sum_{S \supset (N-T) \cup i} (-1)^{s-(n-t+1)} C_{epS}^a \quad (57)$$

is a cost function. This can be proved by direct checking. We omit the details.

Then by Additivity and hence continuity with respect to the cost functions we have

$$\begin{aligned}
 & x(q; \sum_{S \supset (N-T) \cup i} (-1)^{s-(n-t+1)} C_{\epsilon p S}) \\
 = & \lim_{a' \rightarrow \infty} x(q; \sum_{S \supset (N-T) \cup i} (-1)^{s-(n-t+1)} C_{\epsilon p S}^{a'}) \\
 = & \lim_{a' \rightarrow \infty} \sum_{S \supset (N-T) \cup i} (-1)^{s-(n-t+1)} \xi_i^{a'}(S) \\
 = & \sum_{S \supset (N-T) \cup i} (-1)^{s-(n-t+1)} \xi_i(S) \\
 = & \gamma_i(T) \\
 \geq & 0.
 \end{aligned}$$

Now, from our conclusion on the measure μ_i^q , the representation formula (42) boils down to the following probabilistic values

$$x_i(q; C) = \sum_{T: T \ni i, T \subseteq N} \gamma_i(T) [C(q_T, 0) - C(q_{T \setminus i}, 0)] \quad (58)$$

Step 6. In step 5 we showed that x is a group value on the set of the stand-alone cost games and x is efficient ($\sum_1^n x_i = C(q)$) since x is a cost sharing method. Now we show that x is a random order method. We use Weber's Theorem 13 in Weber [1988]. For this purpose, we only need to demonstrate that the set of the stand-alone cost games includes the following simple games:

For any $T \neq \emptyset$, $T \subseteq N$,

$$v_T(S) = 1 \text{ if } S \supseteq T, \text{ and } 0 \text{ otherwise,}$$

$$\hat{v}_T(S) = 1 \text{ if } S \supseteq (\neq)T, \text{ and } 0 \text{ otherwise.}$$

In fact, for given $T \neq \emptyset$, $T \subseteq N$, let $q = e_N$ be the demand vector. Define a "cost" function

$$\delta_{(e_T, 0_{N \setminus T})}(q') = 1 \text{ if } q' \geq (e_T, 0_{N \setminus T}), 0 \text{ otherwise,}$$

and similarly, define

$$\hat{\delta}_{(e_T, 0_{N \setminus T})}(q') = 1 \text{ if } q' \geq (e_T, 0_{N \setminus T}) \text{ and } A(q') > |T|, 0 \text{ otherwise.}$$

Then it is easy to see that the “problems” $(q; \delta_{(e_T, 0_{N \setminus T})})$ and $(q; \hat{\delta}_{(e_T, 0_{N \setminus T})})$ generate the games v_T and \hat{v}_T , respectively.

Notice that x is not defined on these two kinds of games since x is not defined on the above two kinds of “cost” functions. However, we can use an approximation argument as in the beginning to extend x on these games. Moreover, the linearity of x guarantees that the extension is unique.

Now, all the requirements of Weber’s Theorem 13 are satisfied and we conclude that x is a random order method.

Step 7. We conclude the proof.

Note that in the above argument we assume that $q \gg 0$. For arbitrary $q \in R_+^N$. Consider the set $A(q)$ and carry the above argument on this subset. Assign zero value to the agents not belonging to $A(q)$. Therefore, x is a simple random-order method.

The Shapley-Shubik is the only symmetric method in the class of simple random-order methods. Thus, the theorem is proved. Q.E.D

A method satisfies the *demand monotonicity* axiom if the cost share to any agent never decreases as he increases his demand and others stay put. Formally, let $(q; C)$ and $(q'; C)$ be two problems and let $i \in N$, if $q_i \leq q'_i$ and $q_j = q'_j$ for all $j \in N \setminus i$, then

$$x_i(q; C) \leq x_i(q'; C).$$

Corollary 2 *If x satisfies Additivity, Dummy and Ordinality, then x satisfies Demand Monotonicity.*

This follows from Theorem 1 and the fact that every simple random order method is demand monotonic.

A Note on A Local Independence Axiom in Cost Sharing

with Yves Sprumont

Abstract

This note provides a characterization for a new non-additive method, the so-called *proportionally adjusted marginal pricing method (PAMP)* in terms of a local independence property among others. It is shown that the PAMP is characterized by the axioms of *continuity*, *local independence*, *proportionality*, and *scale invariance*. The *local independence* axiom is new in cost sharing. It replaces the traditional *additivity* axiom and plays an important role in the characterization.

1 Introduction

This paper studies a new non-additive method called the *Proportionally Adjusted Marginal Pricing* method (PAMP) for the continuous model. The PAMP is not derived by Sprumont's Ordinality axiom, but instead, is derived by a new axiom called Local Independence.

Recall the following two independence axioms in the literature. The first is the Independence of Irrelevant Costs, i.e., $x(q; C^1) = x(q; C^2)$ whenever $C^1(p) = C^2(p), \forall p \leq q$ (Friedman and Moulin [1995]). The second is the *marginality* (the cost share imputed to good i depends only upon the marginal cost function w.r.t. good i (Young [1985d])). These two axioms, in one way or another, convey an "informational efficiency" property.

Our Local Independence axiom requires that $x(q; C^1) = x(q; C^2)$ whenever $C^1(q) = C^2(q)$ and $\nabla C^1(q) = \nabla C^2(q)$. Obviously, Local Independence is a very strong requirement. But we will show that this axiom still leaves room for meaningful methods, among which the following PAMP is an example. Combined with other relevant axioms (see below) it helps effectively to nail down the PAMP.

Similar "*independence of irrelevant alternatives*" axioms or properties have been used in social choice theory (Arrow [1951]), bargaining (Thomson [1996]). The closest is R. Nagahisa [1991]'s *local independence* condition in his characterization of the Walrasian allocation rule. He demonstrated that his local independence condition has far reaching implications relating to Nash implementation.

The paper is organized as follows. In section 2 we set up the model. In section 3 we introduce the Local Independence axiom. In section 4 we prove the characterization theorem. In the last section we demonstrate some properties of the PAMP and its relation to other methods.

2 The Model

Our model here is slightly different from Sprumont [1998]'s models, where cost functions have bounded derivatives. To be precise, we define our model as follows.

Let $N = \{1, \dots, n\}$ be the set of agents (or goods). Denote $C_0(N)$ the set of functions $C : R_+^n \rightarrow R_+$ which are non-decreasing and $C(0) = 0$. Denote $\partial_i C(\cdot) = \partial C / \partial q_i$ ($i = 1, \dots, n$) when C is differentiable and $\nabla C(q) = (\partial_1 C, \dots, \partial_n C)$. Denote $C_1(N)$ the subset of $C_0(N)$, consisting of all continuously differentiable functions in $C_0(N)$ which have $\nabla C(q) \neq 0, \forall q \in R_+^n$. Let $C_1^+(N)$ be the subset of $C_1(N)$ which consists of all those cost functions with $\nabla C(q) \gg 0$. Let $q \in R_+^n$ be a demand vector for N . Assume that $\bar{q} \in R_+^n$ and $\bar{q} \gg q$. Denote $C_1(N)(\bar{q})$ the set of cost functions which are defined and continuously differentiable on $[0, \bar{q}]$ (the set $\{p \in R_+^n \mid 0 \leq p \leq \bar{q}\}$). For certain domain \mathcal{C} of cost functions, if $q \in R_+^n$ and $C \in \mathcal{C}$, then call the list $(N; q; C)$ (or simply $(q; C)$ if N is fixed) a cost sharing problem.

A cost sharing method x is a mapping which assigns to each problem $(q; C)$ a vector $x(q; C)$ in R_+^n such that

$$\sum_1^n x_i(q; C) = C(q).$$

3 Axioms

Except the *continuity* and the *local independence* axioms, the following *dummy*, *proportionality* and *scale invariance* are well-known.

- Dummy

For a problem $(q; C)$, if $\partial_i C(\cdot) \equiv 0$ for some $i \in N$, then $x_i(q; C) = 0$.

- Continuity in Demand

$x(q; C)$ is continuous with respect to q .

Remark: The continuity is usually a property of the characterization. That is, it is often implied by the combination of the other axioms. However in the following characterizations it is hardly dispensable.

The following continuity property is stronger than the last one.

- Continuity

$x(q; C)$ is continuous in both q and C , and the continuity with C is defined for the topology on the Banach space $C_1(N)(\bar{q})$ with the following norm:

$$\|C\| = \max_{p \in [0; \bar{q}]} |C(p)| + \max_{i=1, \dots, n} \max_{p \in [0; \bar{q}]} |\partial_i C(p)|.$$

- Proportionality

If $C \in C_0(N)$ is homogeneous, i.e., there is a mapping $c : R_+ \rightarrow R_+$ such that

$$C(q) = c\left(\sum_N q_i\right), \quad \forall q \in R_+^n,$$

then

$$x_i(q; C) = \frac{q_i}{\sum_N q_j} c\left(\sum_N q_j\right), \quad i = 1, \dots, n.$$

- Scale Invariance

Let $\lambda \in R_+^n$, $\lambda \gg 0$, $q \in R_+^n$. Denote $\lambda^{-1} = (\lambda_1^{-1}, \dots, \lambda_n^{-1})$ and $\lambda^{-1} \otimes q = (\lambda_1^{-1} q_1, \dots, \lambda_n^{-1} q_n)$. Then $\forall \lambda \gg 0$, $q \in R_+^n$,

$$x(q; C) = x(\lambda^{-1} \otimes q; C^\lambda)$$

where

$$C^\lambda(p) = C(\lambda \otimes p), \quad \forall p.$$

Now, we introduce our new axiom, i.e., Local Independence.

- Local Independence

For any $C^1, C^2 \in \mathbf{C}_1(N)$ and given $q \in R_+^n$, if $C^1(q) = C^2(q)$ and $\nabla C^1(q) = \nabla C^2(q)$, then

$$x(q; C^1) = x(q; C^2).$$

A similar axiom has been used by Nagahisa [1991] to characterize the Walrasian allocation rule, in which the corresponding social choice function (correspondence associating subsets of feasible allocations with utility profiles that describe consumer preferences) is assumed to be locally independent with respect to the preference profile.

Here, since we are in the context of cost sharing our Local Independence axiom is different from Nagahisa's axiom. Roughly speaking, this axiom states that the cost sharing method only requires the information of the cost function around the demand profile up to a first-order approximation and any other information are irrelevant. Obviously this is a very restrictive axiom.

Recall the *independence of irrelevant costs* (IIC) axiom. It has been shown that *additivity* and *dummy* imply IIC (Friedman and Moulin [1995]). This means all the *additive* methods (and *dummy*) satisfy the IIC. A corollary of *additivity* and *dummy* is the *marginality* property. This is from a representation lemma due to Friedman and Moulin [1995]. In our search for non-additive methods we do not have the IIC at prior.

Our *local independence* (LI) axiom is different from all of them. The LI does not imply any of them and neither do they imply LI. Indeed, it rules out all the well-known methods proposed so far. Nevertheless, it is still flexible enough to allow for other methods in addition to the PAMP (see Section 5). This axiom is information-efficient in the sense that for the given realized output vector only the total cost and

the corresponding marginal costs are relevant in the distribution of the costs and there is no need to know such information at each and every conceivable lower output level regarding the distribution of the costs at the realized output level.

4 Characterization of the PAMP

We begin our characterization on the domain $C_1^+(N)$.

Theorem 1 *There exists a unique cost sharing method satisfying Continuity in Demand, Local Independence, Proportionality and Scale Invariance on the domain $C_1^+(N)$.*

It is the following so-called Proportionally Adjusted Marginal Pricing method:

$$x_i(q; C) = \frac{\partial_i C(q) q_i}{\sum_N \partial_j C(q) q_j} C(q), \quad i = 1, \dots, n. \quad (59)$$

Proof. It is easy to check that the method x defined by (59) satisfies these axioms. Now we demonstrate that any method \tilde{x} which satisfies the axioms will coincide with x .

By the axiom of Continuity in Demand, it is no loss of generality to assume that $q \gg 0$.

Consider the subclass of cost functions like:

$$C(p) = \left(\sum_{i=1}^n p_i \right)^\alpha, \quad \alpha > 0.$$

By Proportionality \tilde{x} coincides with x on this class of functions.

By Scale Invariance, it further coincides with x for every cost function like:

$$C_{(\lambda; \alpha)}(p) := \left(\sum_{i=1}^n \lambda_i p_i \right)^\alpha, \quad \lambda_i > 0, i = 1, \dots, n, \quad \alpha > 0.$$

For any given cost function C in $C_1^+(N)$ and demand vector $q \gg 0$, we will find a vector $(\lambda; \alpha)$ where $\lambda = (\lambda_1, \dots, \lambda_n) \gg 0$, $\alpha > 0$ such that

$$C(q) = C_{(\lambda; \alpha)}(q) \text{ and } \nabla C(q) = \nabla C_{(\lambda; \alpha)}(q), \quad (60)$$

then by Local Independence Axiom

$$\tilde{x}(q; C) = \tilde{x}(q; C_{(\lambda; \alpha)}).$$

However \tilde{x} coincides with x on the problem $(q; C_{(\lambda; \alpha)})$, therefore

$$\tilde{x}(q; C_{(\lambda; \alpha)}) = x(q; C_{(\lambda; \alpha)}) = x(q; C)$$

so

$$\tilde{x}(q; C) = x(q; C).$$

Taking into account the argument at the beginning of the proof, we can conclude that \tilde{x} coincides with x at any problem $(q; C)$, in other words

$$\tilde{x} = x.$$

Now we find the vector $(\lambda; \alpha)$ which meets the requirement (60). This is equivalent to the existence of solutions to the following equations:

$$\begin{aligned} \left(\sum_{i=1}^n \lambda_i q_i \right)^\alpha &= C(q) \\ \alpha \left(\sum_{i=1}^n \lambda_i q_i \right)^{\alpha-1} \lambda_i &= \partial_i C(q); \quad i = 1, \dots, n. \end{aligned}$$

Plugging the first equation into the second group of equations, we get the following linear equations:

$$\frac{\alpha \lambda_i}{\sum_{j=1}^n \lambda_j q_j} = \frac{\partial_i C(q)}{C(q)}; \quad i = 1, \dots, n.$$

Multiplying each by q_i respectively and summing up, we get

$$\alpha = \frac{\sum_{i=1}^n \partial_i C(q) q_i}{C(q)}.$$

Therefore we have the following simplified linear equations:

$$\lambda_i = \frac{\partial_i C(q)}{\sum_{j=1}^n \partial_j C(q) q_j} \sum_{j=1}^n \lambda_j q_j; \quad i = 1, \dots, n.$$

Denote

$$\delta_i = \frac{\partial_i C(q)}{\sum_{j=1}^n \partial_j C(q) q_j}; i = 1, \dots, n.$$

Then we have the following homogeneous linear equations:

$$\sum_{j=1}^n \delta_i q_j \lambda_j = \lambda_i; i = 1, \dots, n. \quad (61)$$

They can have a nontrivial solution if and only if the coefficient matrix $(\delta_i q_j - \mathbf{I})$ has a vanishing determinant.

In fact, it is easy to calculate that

$$|(\delta_i q_j - \mathbf{I})| = (-1)^n (\delta_1 q_1 + \dots + \delta_n q_n - 1) = 0.$$

To conclude that there exists the vector (positive) $(\lambda; \alpha)$ as required in (60), reconsider the system of linear equations (61).

It is easy to see that the rank of the coefficient matrix $(\delta_i q_j - \mathbf{I})$ is $n - 1$ (in fact, the sub-matrix $(\delta_i q_j - \mathbf{I})_{(n-1)(n-1)}$ has determinant $(-1)^{n-1} (\sum_{j=1}^{n-1} \delta_j q_j - 1) \neq 0$ by assumption). Therefore the solution set of (61) is a one-dimensional line. By invoking the equation

$$\left(\sum_{i=1}^n \lambda_i q_i \right)^\alpha = C(q)$$

and regarding to (61), it is obvious that these λ_i ($i = 1, \dots, n$) are unique and positive.

The theorem is proved.

Q.E.D.

Now we extend our characterization to the larger domain $C_1(N)$ either by introducing the Dummy axiom or by strengthening the continuity axiom.

First, with the Dummy axiom we have the following characterization.

Theorem 2 *The Proportionally Adjusted Marginal Pricing method is the unique cost sharing method satisfying Dummy, Continuity in Demand, Local Independence, Proportionality and Scale Invariance on the domain $C_1(N)$.*

Proof. Similarly we need to check that on the domain $C_1(N)$ any method \tilde{x} satisfying the above axioms must coincide with x as in (59). By the theorem 1 we only need to show that for any cost function C and q , if there exists a $i \in N$ such that $\partial_i C(q) = 0$, then $x_i(q; C) = 0$. In fact, construct a new cost function C' (this is always possible) which satisfies:

$$C'(q) = C(q) \text{ and } \nabla C'(q) = \nabla C(q)$$

but

$$\partial_1 C'(\cdot) \equiv 0.$$

i.e., $i = 1$ is *dummy* for $(q; C')$. Then by Local Independence and Dummy we have

$$\tilde{x}_1(q; C) = 0,$$

so \tilde{x} coincides with x . This completes the proof. Q.E.D.

Finally, with the stronger Continuity axiom, we have the following characterization.

Theorem 3 *The Proportionally Adjusted Marginal Pricing method is the unique cost sharing method satisfying Continuity, Local Independence, Proportionality and Scale Invariance on the domain $C_1(N)$.*

Proof. The PAMP is uniquely determined by the Local Independence, Proportionality and Scale Invariance on the subdomain $C_1^+(N)$. Then by Continuity, it is uniquely extended to the domain $C_1(N)$. The theorem is proved. Q.E.D.

5 The Tightness of the Characterizations

It is always desirable to have a tight axiomatic characterization for the concerned cost sharing method. An axiomatic characterization is tight if we can not drop anyone

of the axioms without allowing new solution. Here, we point out that we have not talked about the tightness of all the previous characterizations simply because almost all of them are tight (this can be checked). Certainly, tightness is an important issue in axiomatic cost sharing.

Theorem 1 and 2 are almost tight.

- Dropping Dummy (for theorem 2)

Define x as follows:

If C is homogeneous, let

$$x_i(q; C) = \frac{q_i}{\sum_{j=1}^n q_j} C(q), \quad \forall i = 1, \dots, n.$$

If C is not homogeneous, then

$$x_i(q; C) = q_i \partial_i C(q) + \frac{1}{n} [C(q) - \sum_{j=1}^n q_j \partial_j C(q)]$$

if $C(q) - \sum_{j=1}^n q_j \partial_j C(q) \geq 0$;

$$x_i(q; C) = \min\{\lambda; q_i \partial_i C(q)\}$$

otherwise, where $\sum_{j=1}^n \min\{\lambda; q_j \partial_j C(q)\} = C(q)$.

It is easy to check that all the other axioms are satisfied by the above method.

- Dropping Continuity in Demand

We have not been able to determine whether or not Continuity in Demand can be dropped without affecting its conclusion. However we can replace it by the No Exploitation axiom or the following stronger axiom called Independence of Null Agents:

For every problem $(q; C)$ and every $i \in N$,

$$\{q_i = 0\} \Rightarrow \{x_j(q; C) = x_{jN \setminus i}(q_{N \setminus i}; C_{N \setminus i}) \forall j \in N \setminus i\},$$

where $C_{N \setminus i}(q_{N \setminus i}) = C(q_{N \setminus i}, 0)$ and $x_{jN \setminus i}$ is the very method x applied to the problem $(q_{N \setminus i}; C_{N \setminus i})$.

The Independence of Null Agents axiom is introduced in Sprumont [1998]. It says that an agent who demands nothing may safely be ignored: counting him or not does not affect the cost shares of those with a positive demand. As it is pointed out by Sprumont [1998], it implies No Exploitation and a limited form of consistency. A counterpart of this axiom called *null player out property* was studied by Derks and Haller [1996] in the cooperative game model.

- Dropping Local Independence

The Aumann-Shapley pricing method obviously satisfies all the remaining axioms in the characterization.

- Dropping Proportionality

For any given C and q , let $N^0 = \{i \mid \partial_i C(\cdot) = 0\}$, $N' = N \setminus N^0$. Define

For $i \in N'$,

$$x_i(q; C) = q_i \partial_i C(q) + \frac{1}{n} [C(q) - \sum_{j=1}^n q_j \partial_j C(q)]$$

if $C(q) - \sum_{j=1}^n q_j \partial_j C(q) \geq 0$;

$$x_i(q; C) = \min\{\lambda; q_i \partial_i C(q)\}$$

otherwise, where $\sum_{j=1}^n \min\{\lambda; q_j \partial_j C(q)\} = C(q)$,

for $i \in N^0$,

$$x_i(q; C) = 0.$$

- Dropping Scale Invariance

For any given C and q , let $N^0 = \{i \mid \partial_i C(\cdot) = 0\}$, $N' = N \setminus N^0$. Define

$$x_i(q; C) = \begin{cases} \frac{q_i}{\sum_{N'} q_j} C(q) & \text{if } i \in N' \\ 0 & \text{if otherwise.} \end{cases}$$

Theorem 3 is tight.

- Dropping Continuity

For given C , q , for each $i \in N$, if C is homogeneous, let

$$x_i(q; C) = \frac{q_i}{\sum_{j=1}^n q_j} C(q);$$

if not, let

$$x_i(q; C) = q_i \partial_i C(q) + \frac{1}{n} [C(q) - \sum_{j=1}^n q_j \partial_j C(q)]$$

if $C(q) - \sum_{j=1}^n q_j \partial_j C(q) \geq 0$;

$$x_i(q; C) = \min\{\lambda; q_i \partial_i C(q)\}$$

otherwise, where $\sum_{j=1}^n \min\{\lambda; q_j \partial_j C(q)\} = C(q)$.

- Dropping Local Independence

The Aumann-Shapley pricing method satisfies the other three axioms.

- Dropping Proportionality

The following so-called uniform allocation rule (see Sprumont [1991]) satisfies the other three axioms:

For $i = 1, \dots, n$,

$$x_i(q; C) = \min\{\lambda; q_i \partial_i C(q)\},$$

if $C(q) - \sum_{j=1}^n q_j \partial_j C(q) \leq 0$; otherwise

$$x_i(q; C) = \max\{\mu; q_i \partial_i C(q)\},$$

where λ solves the equation $\sum_{i \in N} \min\{\lambda; q_i \partial_i C(q)\} = C(q)$ and μ solves the equation $\sum_{i \in N} \max\{\mu; q_i \partial_i C(q)\} = C(q)$.

- Dropping Scale Invariance

The direct proportional method does satisfy all the other axioms:

$$x_i(q; C) = \frac{q_i}{\sum_{j=1}^n q_j} C(q), \quad i = 1, \dots, n.$$

6 A Few More Properties of the PAMP

Recall the following *distributivity* axiom introduced by Moulin and Shenker [1999] (see Section 3.1, Chapter 1).

Distributivity Axiom:

Let $C_1 \in \mathbf{C}_1(N)$, $C_2 \in \mathbf{C}_1(\{1\})$, $q \in \mathbb{R}_+^n$. Then

$$x(q; C_2 \circ C_1) = x(x(q; C_1); C_2)$$

where

$$C_2 \circ C_1(p) := C_2(C_1(p)), \quad \forall p \in \mathbb{R}_+^n.$$

It is easy to see that the PAMP method satisfies this *distributivity*.

To emphasize this axiom, we adapt the interpretation of the *distributivity* axiom by Moulin and Shenker [1999] to the following cost sharing problem. To create a profile of outputs q of the final good, one division must first create an amount z of some homogeneous intermediate good or service with some associate cost $C^2(z)$, and then another division takes that amount z and creates the amount of output q , where

the amount of intermediate good needed in this second stage of the process is given by $C^1(q)$. The total cost function is then the composition of the two functions: $C^2 \circ C^1(q)$. In such a case one could allocate the costs in stages (in other words, vertically), where the method first allocates the costs in terms of the intermediate good according to $w = \phi(C^1)(q)$. Then, these allocations are used as demands in the second stage of the input process, with the final cost allocation being given by $y = \phi(C^2)(w)$. Equivalently, one could just apply ϕ to the composed cost $y = \phi(C^2 \circ C^1)(q)$. The distributivity axiom requires that the result of this multi-stage allocation process be identical to that of the single stage. Note that the three well-known methods the Shapley-Shubik, the Aumann-Shapley and the Friedman-Moulin do not satisfy this *distributivity*.

Now look at the following so-called *Solidarity* property (Sprumont [1998]):

Cost Solidarity: Let C^1 and C^2 be two cost functions. Suppose there exists a mapping $r : R_+ \rightarrow R_+$ such that $C^2 = r \circ C^1$. Then

$$x(q; C^1) \ll x(q; C^2),$$

or

$$x(q; C^1) = x(q; C^2),$$

or

$$x(q; C^1) \gg x(q; C^2).$$

It is easy to check that the PAMP method satisfies the above *Solidarity* property.

PAMP also satisfies the following Consistency Axiom (Thomson [1996]):

Consistency: x is consistent if for all N, N' with $N' \subseteq N$, all $q \in R_+^n$, $C \in C_1(N)$.

If $x = x(q; C)$, we have

$$x_{N'} = x(q_{N'}; r_{N'}^x(C)) \tag{62}$$

where

$$r_{N'}^x(C)(y_{N'}) := \max\{C(y_{N'}, q_{N \setminus N'}) - \sum_{N \setminus N'} x_i(q; C), 0\}, y_{N'} \in R_+^{N'}$$

and we assume that $r_{N'}^x(C)(\cdot)$ is continuously differentiable at $q_{N'}$.

Now we will demonstrate that the PAMP is a special case of the **Ramsey pricing** (Ramsey [1927], Baumol and Bradford [1970]) when demand functions exhibit equal elasticities.

Let $p_i(q_i)$, $i = 1, \dots, n$ be the inverse demand functions for goods $i = 1, \dots, n$ respectively. Let $\epsilon_i = (dq_i/q_i)(p_i/dp_i)$, $i = 1, \dots, n$ is the demand elasticity of good i with respect to its price p_i . Assume that these elasticities are the same (denoted as ϵ) for all $i = 1, \dots, n$. Let $C(q)$ be the cost function of the joint production q . Then total consumer surplus can be defined by

$$\sum_N \left[\int_0^{q_i} p_i(q'_i) dq'_i - p_i(q_i) q_i \right] \quad (63)$$

and firm's profit by

$$\sum_N p_i(q_i) q_i - C(q). \quad (64)$$

Suppose that there is a social planner who sets the production plan q and hence the prices p_i , $i = 1, \dots, n$ to maximize the total consumer surplus subject to the zero profit (budget balance) constraint for the firm. Namely, the social planner solves the following problem:

$$\begin{aligned} \max_q \quad & \sum_N \int_0^{q_i} p_i(q'_i) dq'_i - \sum_N p_i(q_i) q_i \\ \text{s.t.} \quad & \sum_N p_i(q_i) q_i - C(q) = 0. \end{aligned}$$

The first order condition is as follows.

There exists a multiplier λ such that

$$(\lambda - 1)p'_i(q_i)q_i + \lambda[p_i(q_i) - \partial_i C(q)] = 0.$$

That is

$$\begin{aligned} \frac{p_i(q_i) - \partial_i C(q)}{p_i(q_i)} &= \frac{1 - \lambda \frac{p'_i(q_i) q_i}{p_i(q_i)}}{\lambda} \\ &= \frac{1 - \lambda}{\lambda} \frac{1}{\epsilon} \end{aligned}$$

Let $k := [(1 - \lambda)/\lambda](1/\epsilon)$, then

$$p_i(q_i) - \partial_i C(q) = k p_i(q_i), \quad i = 1, \dots, n.$$

and so

$$p_i(q_i) q_i - \partial_i C(q) q_i = k p_i(q_i) q_i, \quad i = 1, \dots, n.$$

By budget balance

$$\sum_N p_i(q_i) q_i = C(q),$$

then

$$1 - k = \frac{\sum_N \partial_i C(q) q_i}{C(q)}.$$

and

$$p_i(q_i) = \frac{1}{1 - k} \partial_i C(q) = \frac{C(q)}{\sum_N \partial_i C(q) q_i} \partial_i C(q).$$

Therefore

$$x_i(q; C) = p_i(q_i) q_i = \frac{\partial_i C(q) q_i}{\sum_N \partial_i C(q) q_i} C(q).$$

This is the PAMP.

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