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Structures algébriques, systèmes superintégrables et polynômes orthogonaux

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Résumé

Cette thèse est divisée en cinq parties portant sur les thèmes suivants: l'interprétation physique et algébrique de familles de fonctions orthogonales multivariées et leurs applications, les systèmes quantiques superintégrables en deux et trois dimensions faisant intervenir des opérateurs de réflexion, la caractérisation de familles de polynômes orthogonaux appartenant au tableau de Bannai–Ito et l'examen des structures algébriques qui leurs sont associées, l'étude de la relation entre le recouplage de représentations irréductibles d'algèbres et de superalgèbres et les systèmes superintégrables, ainsi que l'interprétation algébrique de familles de polynômes multi-orthogonaux matriciels.

Dans la première partie, on développe l'interprétation physico-algébrique des familles de polynômes orthogonaux multivariés de Krawtchouk, de Meixner et de Charlier en tant qu'éléments de matrice des représentations unitaires des groupes $SO(d+1)$, $SO(d,1)$ et $E(d)$ sur les états d'oscillateurs. On détermine les amplitudes de transition entre les états de l'oscillateur singulier associés aux bases cartésienne et polysphérique en termes des polynômes multivariés de Hahn. On examine les coefficients $9j$ de $\mathfrak{su}(1,1)$ par le biais du système superintégrable générique sur la 3-sphère. On caractérise les polynômes de q-Krawtchouk comme éléments de matrices des « q-rotations » de $U_q(sl_2)$. On conçoit un réseau de spin bidimensionnel qui permet le transfert parfait d'états quantiques à l'aide des polynômes de Krawtchouk à deux variables et on construit un modèle discret de l'oscillateur quantique dans le plan à l'aide des polynômes de Meixner bivariés.

Dans la seconde partie, on étudie les systèmes superintégrables de type Dunkl, qui font intervenir des opérateurs de réflexion. On examine l'oscillateur de Dunkl en deux et trois dimensions, l'oscillateur singulier de Dunkl dans le plan et le système générique sur la 2-sphère avec réflexions. On démontre la superintégrabilité de chacun de ces systèmes. On obtient leurs constantes du mouvement, on détermine leurs algèbres de symétrie et leurs représentations, on donne leurs solutions exactes et on détaille leurs liens avec les polynômes orthogonaux du tableau de Bannai–Ito.

Dans la troisième partie, on caractérise deux familles de polynômes du tableau de Bannai–Ito: les polynômes de Bannai–Ito complémentaires et les polynômes de Chihara. On montre également que les polynômes de Bannai–Ito sont les coefficients de Racah de la superalgèbre $\mathfrak{osp}(1|2)$. On détermine l'algèbre de symétrie des polynômes duaux -1 de Hahn dans le cadre du problème de Clebsch-Gordan de $\mathfrak{osp}(1|2)$. On propose une q-généralisation des polynômes de Bannai–Ito en examinant le problème de Racah pour la superalgèbre quantique $\mathfrak{osp}_q(1|2)$. Finalement, on montre que la q-algèbre de Bannai–Ito sert d'algèbre de covariance à $\mathfrak{osp}_q(1|2)$.

Dans la quatrième partie, on détermine le lien entre le recouplage de représentations des algèbres $\mathfrak{su}(1,1)$ et $\mathfrak{osp}(1|2)$ et les systèmes superintégrables du deuxième ordre avec ou sans réflexions. On étudie également les représentations des algèbres de Racah–Wilson et de Bannai–Ito. On montre aussi que l'algèbre de Racah–Wilson sert d'algèbre de covariance quadratique à l'algèbre de Lie $\mathfrak{sl}(2)$.

Dans la cinquième partie, on construit deux familles explicites de polynômes d-orthogonaux basées sur $\mathfrak{su}(2)$. On étudie les états cohérents et comprimés de l'oscillateur fini et on caractérise une famille de polynômes multi-orthogonaux matriciels.

Mot-clefs

- Polynômes orthogonaux
- Systèmes superintégrables
- Algèbres quadratiques
- Tableau de Bannai–Ito
- Opérateurs de Dunkl

Abstract

This thesis is divided into five parts concerned with the following topics: the physical and algebraic interpretation of families of multivariate orthogonal functions and their applications, the study of superintegrable quantum systems in two and three dimensions involving reflection operators, the characterization of families of orthogonal polynomials of the Bannai-Ito scheme and the study of the algebraic structures associated to them, the investigation of the relationship between the recoupling of irreducible representations of algebras and superalgebras and superintegrable systems, as well as the algebraic interpretation of families of matrix multi-orthogonal polynomials.

In the first part, we develop the physical and algebraic interpretation of the Krawtchouk, Meixner and Charlier families of multivariate orthogonal polynomials as matrix elements of unitary representations of the $SO(d+1)$, $SO(d,1)$ and $E(d)$ groups on oscillator states. We determine the transition amplitudes between the states of the singular oscillator associated to the Cartesian and polyspherical bases in terms of the multivariate Hahn polynomials. We examine the $9j$ coefficients of $\mathfrak{su}(1,1)$ through the generic superintegrable system on the 3-sphere. We characterize the q-Krawtchouk polynomials as matrix elements of "q-rotations" of $U_q(sl_2)$. We show how to design a two-dimensional spin network that allows perfect state transfer using the two-variable Krawtchouk polynomials and we construct a discrete model of the two-dimensional quantum oscillator using the two-variable Meixner polynomials.

In the second part, we study superintegrable systems of Dunkl type, which involve reflections. We examine the Dunkl oscillator in two and three dimensions, the singular Dunkl oscillator in the plane and the generic system on the 2-sphere with reflections. We show that each of these systems is superintegrable. We obtain their constants of motion, we find their symmetry algebras as well as their representations, we give their exact solutions and we exhibit their relationship with the orthogonal polynomials of the Bannai–Ito scheme.

In the third part, we characterize two families of polynomials belonging to the Bannai–Ito scheme: the complementary Bannai-Ito polynomials and the Chihara polynomials. We also show that the Bannai–Ito polynomials arise as Racah coefficients for the $\mathfrak{osp}(1|2)$ superalgebra. We determine the symmetry algebra associated with the dual -1 Hahn polynomials in the context of the Clebsch-Gordan problem for $\mathfrak{osp}(1|2)$. We introduce a q-generalization of the Bannai-Ito polynomials by examining the Racah problem for the quantum superalgebra $\mathfrak{osp}_q(1|2)$. Finally, we show that the q-deformed Bannai-Ito algebra serves as a covariance algebra for $\mathfrak{osp}_q(1|2)$.

In the fourth part, we determine the relationship between the recoupling of representations of the $\mathfrak{su}(1,1)$ and $\mathfrak{osp}(1|2)$ algebras and second-order superintegrable systems with or without reflections. We also study representations of Racah–Wilson and Bannai–Ito algebras. Moreover, we show that the Racah–Wilson algebra serves as a quadratic covariance algebra for $\mathfrak{sl}(2)$.

In the fifth part, we explicitly construct two families of d-orthogonal polynomials based on $\mathfrak{su}(2)$. We investigate the squeezed/coherent states of the finite oscillator and we characterize a family of matrix multi-orthogonal polynomials.

Keywords

- Orthogonal polynomials
- Superintegrable systems
- Quadratic algebras
- Bannai–Ito scheme
- Dunkl operators

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Introduction

L'étude des modèles exactement résolubles joue un rôle fondamental dans l'élaboration des théories qui visent à décrire et expliquer les phénomènes naturels. De manière générique, un modèle est dit exactement résoluble s'il est possible d'en exprimer mathématiquement les quantités d'intérêts de manière explicite. Cette notion prend des formes diverses selon le cadre de travail. Par exemple, en mécanique classique, on dira qu'un système formé de deux planètes en interaction gravitationnelle est exactement résoluble car on peut décrire de manière exacte les trajectoires suivies par chacun des corps [1]. En mécanique quantique, le système formé d'un électron et d'un proton (atome d'hydrogène) est également considéré comme exactement résoluble puisque les énergies possibles du système et ses fonctions d'ondes sont explicitement connues, la notion de trajectoire ayant été évacuée [2]. La notion de résolubilité exacte n'est pas l'apanage de la physique théorique. Par exemple, en biologie mathématique, le modèle de Moran, qui décrit la dynamique d'une population de taille constante subissant des mutations aléatoires et dans laquelle deux types d'allèles se font compétition, est aussi vu comme exactement résoluble car on peut obtenir de manière explicite la loi de probabilité du nombre individus ayant un bagage génétique donné [3].

L'importance des modèles exactement résolubles en physique tient à de nombreux éléments; nous en mentionnons quelques-uns. Tout d'abord, ces modèles constituent un outil de choix dans la validation des principes théoriques fondamentaux. En effet, ils permettent de formuler des prévisions très précises qui peuvent être par la suite soumises à l'expérimentation. À ce titre, la description de la structure fine de l'atome d'hydrogène obtenue par le truchement de l'équation de Dirac est éloquente [4]. Ensuite, les modèles ayant des solutions exactes permettent d'accéder à une compréhension plus fine du contenu physique des théories qui les sous-tendent car ils permettent l'analyse détaillée du rôle de tous les paramètres qui y interviennent; c'est d'ailleurs en partie pourquoi l'examen de ces systèmes occupe une place prépondérante dans les cursus de physique.

Un autre élément qui souligne l'importance des systèmes exactement résolubles est que ceux-ci sont constamment utilisés dans l'élaboration de modèles plus raffinés et dont les caractéristiques sont étudiées à partir de celles du modèle original, entre autres en utilisant la théorie des perturbations. On peut penser ici aux nombreux systèmes quantiques basés sur le modèle de l'oscillateur harmonique [2]. Finalement, l'étude des modèles exactement résolubles est un lieu de rencontre privilégié entre la physique théorique et les mathématiques. Ces deux disciplines se sont à de nombreuses reprises fertilisées mutuellement par le passé, conduisant à des avancées significatives dans les deux domaines. Le théorème de Noether, qui relie les symétries aux lois de conservation en est un exemple particulièrement pertinent [5].

Les symétries sont le dénominateur commun des modèles exactement résolubles: empiriquement, on observe qu'il n'y a de solutions exactes qu'en présence de symétries. Celles-ci se présentent sous diverses formes et sont décrites mathématiquement par des structures algébriques variées. Dans bien des cas, les solutions des modèles exactement résolubles s'expriment en termes de fonctions spéciales. Ces fonctions encodent les symétries des systèmes dans lesquels elles apparaissent. Un exemple typique est celui de l'oscillateur quantique en trois dimensions et des harmoniques sphériques. Ce système est invariant sous les rotations, décrites par le groupe $SO(3)$. L'invariance sous les rotations conduit à la séparation de l'équation de Schrödinger en coordonnées sphériques, les harmoniques sphériques apparaissent comme solutions exactes à l'équation angulaire et elles forment une base pour les représentations irréductibles de $\mathfrak{so}(3)$ [6].

La dynamique des modèles exactement résolubles, des symétries, des structures algébriques et des fonctions spéciales peut être inscrite dans le cercle vertueux suivant:

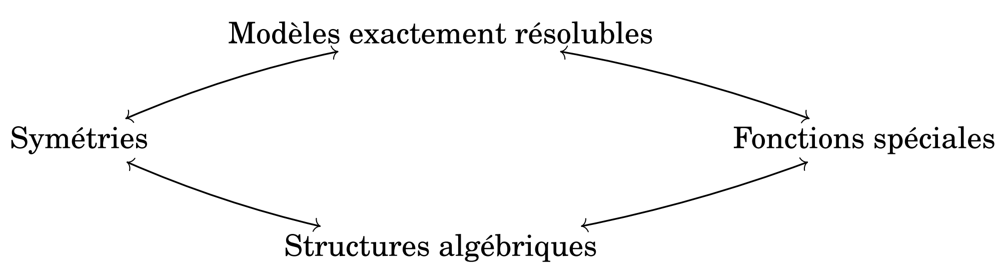

Figure 1: Interactions entre les modèles exactement résolubles, les symétries, les structures algébriques et les fonctions spéciales

Le chemin typique que l'on songe parcourir dans ce schéma est le suivant. On imagine d'abord un modèle d'intérêt. Ensuite, on trouve les symétries de ce modèle et on détermine la structure mathématique qui décrit ces symétries. Puis, on construit les représentations de cette structure algébrique et on établit le lien entre ses représentations et les fonctions spéciales. Finalement on met à profit les fonctions spéciales pour exprimer les solutions du modèle et/ou pour en calculer certaines quantités importantes.

Il s'avère toutefois fructueux de prendre comme point de départ n'importe quel sommet de la figure 1. Par exemple, on peut obtenir et caractériser une nouvelle famille de fonctions spéciales, déterminer la structure algébrique dont ils encodent les propriétés, chercher des modèles dont les symétries sont décrites par cette structure et donner les solutions des modèles obtenus en termes de cette nouvelle famille de fonctions.

Le diagramme 1 reflète l'essence de la recherche qui a mené à la présente thèse, dans laquelle la résolubilité exacte est recherchée et étudiée par le truchement des symétries, des structures algébriques et de leurs représentations ainsi que des fonctions spéciales. La thèse comporte vingt-huit articles qui contribuent à un ou à plusieurs des axes de recherche qui apparaissent sur le diagramme 1. Les résultats originaux qu'elle contient sont en nombre. Ils concernent principalement les polynômes orthogonaux (une classe particulière de fonctions spéciales), les systèmes quantiques superintégrables (une classe de modèles exactement résolubles) et certaines algèbres et superalgèbres quadratiques telles que les algèbres de Bannai–Ito et de Racah.

La thèse se divise en cinq parties comprenant chacune une série d'articles sur un thème commun. Toutes les parties, à l'exception peut-être de la dernière, sont en fort lien les unes avec les autres via le diagramme 1. En outre, plusieurs articles auraient pu se retrouver dans une autre partie que celle où ils sont actuellement.

La partie I de la thèse est intitulée *Polynômes orthogonaux multivariés et applications*. Dans cette partie, on traite des interprétations physique et algébrique de six familles de fonctions orthogonales multivariées et on en détaille trois applications physiques. Dans l'introduction, on explique sommairement le contexte général de l'étude des polynômes orthogonaux multivariés. Dans les chapitres 1 à 3, on montre comment les familles de polynômes orthogonaux à d variables de Krawtchouk, Meixner et Charlier interviennent respectivement en tant qu'éléments de matrice des représentations unitaires des groupes de rotation $SO(d+1)$, du groupe de Lorentz $SO(d,1)$ et du groupe euclidien $E(d)$ sur les états de l'oscillateur harmonique [7, 8, 9]. On illustre de quelle façon cette interprétation conduit à une caractérisation complète de ces familles de polynômes. Dans le chapitre 4, on établit une relation entre les polynômes de Krawtchouk à 2 variables, les coefficients de Clebsch-Gordan de l'algèbre $\mathfrak{su}(1,1)$ donnés par les polynômes de Hahn et les coefficients de transition entre les bases sphérique et cartésienne de l'oscillateur harmonique en trois dimensions [10]. Dans le chapitre 5, on montre que les polynômes de Hahn à d variables de Karlin et McGregor interviennent dans les amplitudes de transition entre les états associés aux bases cartésienne et polysphérique de l'oscillateur singulier en $d+1$ dimensions [11]. On exploite ensuite cette identification pour donner une caractérisation complète de ces polynômes. Dans le chapitre 6, on utilise le lien entre le recouplage de $n+1$ représentations de $\mathfrak{su}(1,1)$ et le modèle superintégrable générique sur la n-sphère obtenu dans la partie IV pour étudier les coefficients $9j$ de $\mathfrak{su}(1,1)$; on montre que ces coefficients sont donnés en termes de fonctions rationnelles orthogonales et on en extrait plusieurs propriétés [12]. Dans le chapitre 7, on met la table pour l'obtention d'une q-généralisation de la relation entre les polynômes de Krawtchouk multivariés et les représentations du groupe des rotations en déterminant le lien entre les polynômes de q-Krawtchouk et les « q-rotations » dans l'algèbre quantique $U_q(sl_2)$ [13]. Dans les chapitres 8 et 9, on présente deux applications des polynômes orthogonaux multivariés. Premièrement, on explique comment les polynômes de Krawtchouk à deux variables peuvent être utilisés pour concevoir un réseau de spins à deux dimensions qui permet le transfert parfait d'états quantiques [14]. Deuxièmement, on élabore un modèle discret de l'oscillateur harmonique quantique en deux dimensions ayant la même algèbre de symétrie $\mathfrak{su}(2)$ que le modèle usuel [15].

La partie II de la thèse est intitulée *systèmes superintégrables avec réflexions*. Dans cette partie, on étudie une série de systèmes quantiques superintégrables en deux et trois dimensions dont les hamiltoniens contiennent des opérateurs de réflexion de la forme $R_i f(x_i) = f(-x_i)$. Dans l'introduction, on rappelle la notion de superintégrabilité et on définit les opérateurs de Dunkl. Dans les chapitres 10 et 11, on examine le modèle de l'oscillateur de Dunkl dans le plan [16, 17]. On montre que ce système est superintégrable, on obtient ses constantes du mouvement et on en donne l'algèbre de symétrie et les solutions exactes. On montre que dans ce modèle les amplitudes de transition entre les états associés aux bases polaire et cartésienne sont exprimées en termes des coefficients de Clebsch-Gordan de la superalgèbre de Lie $\mathfrak{osp}(1|2)$ qui sont donnés en termes des polynômes duaux -1 de Hahn appartenant au tableau de Bannai–Ito discuté dans la partie III. On procède aussi à une analyse détaillée des représentations de l'algèbre de symétrie du modèle, dénommée algèbre de Schwinger–Dunkl. Dans le chapitre 12, on considère une extension du modèle faisant intervenir des termes de potentiel singuliers; on montre que le système demeure superintégrable, on donne ses constantes du mouvement, son algèbre de symétrie et ses solutions exactes [18]. Dans le chapitre 13, on examine l'oscillateur de Dunkl en trois dimensions, lui aussi superintégrable et exactement résoluble [19]. Dans le chapitre 14, on introduit le modèle superintégrable générique sur la 2-sphère avec réflexions [20]. Grâce aux résultats obtenus dans la partie IV, on montre que l'hamiltonien de ce système est lié à l'opérateur de Casimir total intervenant dans la combinaison de trois représentations irréductibles de $\mathfrak{osp}(1|2)$. On détermine conséquemment que l'algèbre de symétrie engendrée par les constantes du mouvement de ce système est l'algèbre de Bannai–Ito. On montre aussi la contraction de ce système vers l'oscillateur de Dunkl dans le plan. Finalement, dans le chapitre 15, on examine l'équation de Dirac–Dunkl sur la 2-sphère [21]. On montre que l'algèbre de symétrie de cette équation est aussi l'algèbre de Bannai–Ito, on construit les représentations de dimension finie de cette algèbre et on construit les solutions exactes du modèle à l'aide de l'extension de Cauchy–Kovalevskaia.

La partie III s'intitule *Tableau de Bannai–Ito et structure algébriques associées*. Dans cette partie, on étudie des familles de polynômes orthogonaux appartenant à la classe des polynômes de Bannai–Ito et on étudie les structures algébriques associées à ces fonctions. Dans l'introduction, on rappelle l'origine des polynômes du tableau de Bannai–Ito, aussi appelés polynômes orthogonaux « -1 », et on explique brièvement la notion de bispectralité. Dans le chapitre 16, on démontre la bispectralité des polynômes complémentaires de Bannai–Ito, c'est-à-dire qu'on obtient l'opérateur duquel ils sont fonctions propres [22]. Dans le chapitre 17, on introduit et on caractérise une famille de polynômes « -1 » appelés polynômes de Chihara [23]. Dans le chapitre 18, on montre que les polynômes de Bannai–Ito interviennent comme coefficients de Racah de la superalgèbre $\mathfrak{osp}(1|2)$, aussi appelée $sl_{-1}(2)$ [24]. Dans le chapitre 19, on obtient la structure algébrique qui sous-tend les polynômes duaux -1 de Hahn et on établit comment cette structure intervient dans le problème de Clebsch-Gordan de $sl_{-1}(2)$ [25]. Le chapitre 20 est le compte-rendu d'une conférence de revue sur l'algèbre de Bannai-Ito et ses applications [26]. Dans le chapitre 21, on introduit une q-généralisation des polynômes de Bannai–Ito et de leur algèbre en considérant les coefficients de Racah de la superalgèbre quantique $\mathfrak{osp}_q(1|2)$ [27]. Dans le chapitre 22, on établit que la q-algèbre de Bannai–Ito est aussi l'algèbre de covariance de $\mathfrak{osp}_q(1|2)$ [28].

La partie IV de la thèse s'intitule *Problème de Racah et systèmes superintégrables*.

Dans cette partie, on détermine le lien entre le recouplage de représentations des algèbres $\mathfrak{su}(1,1)$ et $\mathfrak{osp}(1|2)$ et les systèmes superintégrables dont les constantes du mouvement sont du deuxième ordre. Dans l'introduction on rappelle les bases du problème de Racah, qui advient lors du recouplage de trois représentations. Dans le chapitre 23, on étudie les liens entre le problème de Racah pour l'algèbre de Lie $\mathfrak{su}(1,1)$, l'algèbre de Racah–Wilson et le système superintégrable générique sur la 2-sphère [29]. Dans le chapitre 24, on montre que l'algèbre de Racah peut également être vue comme l'algèbre de covariance quadratique de sl_2 [30]. Le chapitre 25 est le compte-rendu d'une conférence de revue sur l'algèbre de Racah [31]. Finalement, dans le chapitre 26, on établit le lien entre le problème de Racah pour la superalgèbre $\mathfrak{osp}(1|2)$, l'algèbre de Bannai–Ito et le système superintégrable générique sur la 2-sphère avec réflexions [32].

La partie V est intitulée *Polynômes multi-orthogonaux et applications*. Elle est légèrement à la marge des autres parties de la thèse et témoigne de mes premiers travaux. Dans l'introduction, la notion de d-orthogonalité et de multi-orthogonalité matricielle est revue. Dans le chapitre 27, on définit deux nouvelles familles de polynômes d-orthogonaux en utilisant les représentations de $\mathfrak{su}(2)$ [33]. Dans le chapitre 28, on utilise ces résultats pour étudier les états cohérents/comprimés de l'oscillateur fini et pour présenter de manière explicite une famille de polynômes multi-orthogonaux matriciels [34].

Partie I

Polynômes orthogonaux multivariés et applications

Introduction

Les polynômes orthogonaux forment une classe particulièrement importante de fonctions spéciales [35], notamment en raison de leurs nombreuses applications à la physique mathématique, aux probabilités et aux processus stochastiques, à la théorie de l'approximation et aux matrices aléatoires. Une suite de polynômes $\{P_n(x)\}_{n=0}^{\infty}$, où $P_n(x)$ est un polynôme de degré n en x, constitue une famille de polynômes orthogonaux s'il existe une fonctionnelle linéaire $\mathscr{L}$ telle que pour tous les entiers non-négatifs m et n, on a [36]

$$\mathscr{L}[P_m(x)P_n(x)] = 0 \qquad \text{si } m \neq n \quad \text{et} \qquad \mathscr{L}[P_n(x)^2] \neq 0.$$

De tous les polynômes orthogonaux, le sous-ensemble des polynômes orthogonaux hypergéométriques est certainement l'un des plus importants [37]. Il est constitué des familles de polynômes orthogonaux qui peuvent s'écrire de manière explicite en termes de séries ou de q-séries hypergéométriques. Les séries hypergéométriques, dénotées ${}_pF_q$, sont définies ainsi [35]

$${}_pF_q\left(\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix} \,\middle|\, z\right) = \sum_{k\geq 0} \frac{(a_1, a_2, \ldots, a_p)_k}{(b_1, b_2, \ldots, b_q)_k} \frac{z^k}{k!},$$

avec $(a_1, a_2, \ldots, a_p)_k = (a_1)_k(a_2)_k \cdots (a_p)_k$ où $(a)_k$ est le symbole de Pochhammer

$$(a)_k = \prod_{i=0}^{k-1}(a+i) \quad \text{avec} \quad (a)_0 = 1.$$

Les q-séries hypergéométriques, généralement dénotées par ${}_r\phi_s$, sont définies par [38]

$${}_r\phi_s\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix} \,\middle|\, q, z\right) = \sum_{k\geq 0} \frac{(a_1, a_2, \ldots, a_r; q)_k}{(b_1, b_2, \ldots, b_s; q)_k} (-1)^{(1+s-r)k} q^{(1+s-r)\binom{k}{2}} \frac{z^k}{(q;q)_k},$$

avec $(a_1, a_2, \ldots, a_r; q)_k = (a_1; q)_k(a_2; q)_k \cdots (a_r; q)_k$ où $(a; q)_k$ est le symbole de Pochhammer q-déformé

$$(a;q) = \prod_{i=1}^{k}(1 - aq^{i-1}) \quad \text{avec} \quad (a;q)_0 = 1.$$

Les polynômes orthogonaux hypergéométriques sont typiquement organisés au sein d'une hiérarchie connue sous le nom de *Tableau de Askey*[1] [39]. Au sommet de cette hiérarchie trônent les polynômes de Askey–Wilson et les q-polynômes de Racah, qui ont chacun cinq paramètres, incluant q. Tous les polynômes du tableau d'Askey peuvent être obtenus à partir de ces deux familles par des limites, notamment la limite « classique » $q \to 1$, ou alors par des choix particuliers de paramètres. Les polynômes du tableau d'Askey sont ubiquitaires, comme en témoignent les 1500 citations de la monographie de 1998 de Koekoek, Lesky et Swarttouw [39].

Les polynômes du tableau d'Askey ont presque tous une interprétation algébrique. Ils sont tantôt éléments de matrices ou vecteurs de base pour certaines représentations irréductibles d'algèbres de Lie de rang 1, tantôt coefficients de Clebsch-Gordan ou de Racah pour ces mêmes algèbres [40, 41, 42]. Dans tous les cas, les interprétations algébriques des familles de polynômes orthogonaux permettent d'en déduire un grand nombre de propriétés. En fait, le cadre algébrique est lui-même à l'origine de la découvert de certains de ces objets, dont les polynômes de Racah, q-Racah, Wilson et Askey–Wilson.

Il est naturel de chercher à généraliser la hiérarchie du tableau d'Askey aux polynômes orthogonaux multivariés. Il faut savoir toutefois que de manière générale, l'étude des polynômes orthogonaux à plusieurs variables est plus difficile que celle des polynômes univariés, notamment en raison du fait que dans le cas multivarié la mesure d'orthogonalité ne caractérise pas complètement les polynômes associés [43]. Il n'y pas à ce jour de théorie unifiée de tous les polynômes orthogonaux multivariés, à l'exception des polynômes multivariés associés aux systèmes de racines, qui ne sont pas étudiés dans cette thèse [43]. Cependant, nombreuses sont les familles qui sont connues et bien caractérisées.

Les premiers exemples de familles de polynômes multivariés généralisant celles du tableau de Askey ont été proposés dans un cadre probabiliste au début des années 70. C'est Robert Griffiths qui a généralisé à plusieurs variables les familles de polynômes de Krawtchouk et de Meixner en utilisant des fonctions génératrices associées aux distributions multinomiale et multinomiale négative [44, 45]; voir aussi les travaux de Milch qui précèdent ceux de Griffiths [46]. Les polynômes multivariés de Krawtchouk étudiés par Griffiths ont par la suite été redécouverts à quelques reprises, notamment dans [47]. Durant la même période, Karlin et McGregor ont généralisé à plusieurs variables les polynômes de Hahn en considérant le modèle de Moran à plusieurs espèces [48]. On doit

[1] Notons ici que les polynômes -1, bien qu'hypergéométriques, ne se retrouvent pas dans cette hiérarchie.

aussi souligner les nombreux travaux de Koornwinder sur les polynômes à deux variables [49]. Plusieurs années plus tard, Tratnik a proposé une version multivariée du tableau d'Askey à $q = 1$ [50, 51]. Les polynômes multivariés proposés par Tratnik, qui incluent ceux de Karlin et McGregor, sont construits en combinant de manière non triviale des polynômes orthogonaux univariés du tableau d'Askey. De nombreux travaux visant la caractérisation de ces polynômes ont par la suite été publiés [52]. Plus récemment, la même approche a été reprise par Gasper et Rahman pour définir des q-déformations des polynômes proposées par Tratnik [53]; ces familles demeurent toutefois relativement peu étudiées [54].

Cette partie de la thèse porte sur l'interprétation physique et algébrique de certaines familles de polynômes orthogonaux ainsi que sur certaines de leurs applications concrètes à la physique. Tout d'abord, on montre que les polynômes de Krawtchouk, de Meixner (tels que définis par Griffiths) et de Charlier à d-variables correspondent aux éléments de matrices des représentations unitaires des groupes de Lie $SO(d+1)$, $SO(d,1)$ et $E(d)$ sur les états de l'oscillateur harmonique. Les résultats qui concernent les groupes $SO(d+1)$ et $E(d)$ sont directement liés aux propriétés de transformation de systèmes d'oscillateurs harmoniques sous les rotations et les transformations euclidiennes. On illustre également le lien entre les polynômes de Krawtchouk à deux variables et les coefficients de transition entre les états des bases cartésienne et sphérique pour l'oscillateur harmonique en trois dimensions. On montre aussi que les polynômes de Hahn à d variables de Karlin et McGregor interviennent dans les amplitudes de transition entre les états des bases cartésienne et polysphérique de l'oscillateur singulier en $d+1$ dimensions. On examine également les coefficients $9j$ de $\mathfrak{su}(1,1)$ par la lorgnette du système superintégrable générique sur la 3-sphère et on montre que ces coefficients s'expriment non pas en termes de polynômes, mais en termes de fonctions rationnelles. Par ailleurs, on montre que les polynômes de q-Krawtchouk interviennent en tant qu'éléments de matrice des « q-rotations » de l'algèbre quantique $U_q(\mathfrak{sl}_2)$. Les deux applications qui sont présentées concernent respectivement le transfert parfait d'états quantiques à l'aide de réseaux de spins et la discrétisation du modèle de l'oscillateur quantique en deux dimensions.

Chapitre 1

The multivariate Krawtchouk polynomials as matrix elements of the rotation group representations on oscillator states

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Abstract. An algebraic interpretation of the bivariate Krawtchouk polynomials is provided in the framework of the 3-dimensional isotropic harmonic oscillator model. These polynomials in two discrete variables are shown to arise as matrix elements of unitary reducible representations of the rotation group in 3 dimensions. Many of their properties are derived by exploiting the group-theoretic setting. The bivariate Tratnik polynomials of Krawtchouk type are seen to be special cases of the general polynomials that correspond to particular rotations involving only two parameters. It is explained how the approach generalizes naturally to $(d+1)$ dimensions and allows to interpret multivariate Krawtchouk polynomials as matrix elements of $SO(d+1)$ unitary representations. Indications are given on the connection with other algebraic models for these polynomials.

1.1 Introduction

The main objective of this article is to offer a group-theoretic interpretation of the multivariable generalization of the Krawtchouk polynomials and to show how their theory naturally unfolds from this picture. We shall use as framework the space of states of the quantum harmonic oscillator in $d+1$ dimensions. It will be seen that the Krawtchouk polynomials in d variables arise as matrix elements of the reducible unitary representations of the rotation group $SO(d+1)$ on the energy eigenspaces of the $(d+1)$-dimensional oscillator. For simplicity, we shall focus on the $d=2$ case. The bivariate Krawtchouk polynomials will thus appear as matrix elements of $SO(3)$ representations; we will indicate towards the end of the paper how the results directly generalize to an arbitrary finite number of variables.

The ordinary Krawtchouk polynomials in one discrete variable have been obtained by Krawtchouk [19] in 1929 as polynomials orthogonal with respect to the binomial distribution. They possess many remarkable properties [17, 25] (second-order difference equation, duality, explicit expression in terms of Gauss hypergeometric function, etc.) and enjoy numerous applications. The importance of these polynomials in mathematical physics is due, to a large extent, to the fact that the matrix elements of $SU(2)$ irreducible representations known as the Wigner D functions can be expressed in terms of Krawtchouk polynomials [4, 18].

The determination of the multivariable Krawtchouk polynomials goes back at least to 1971 when Griffiths obtained [5] polynomials in several variables that are orthogonal with respect to the multinomial distribution using, in particular, a generating function method. These polynomials, especially the bivariate ones, were subsequently rediscovered by several authors. For instance, the 2-variable Krawtchouk polynomials appear as matrix elements of $U(3)$ group representations in [26] and the same polynomials occur as $9j$ symbols of the oscillator algebra in [29]. An explicit expression in terms of Gel'fand-Aomoto generalized hypergeometric series is given in [24]. Interest was sparked in recent years with the publication by Hoare and Rahman of a paper [11] in which the 2-variable Krawtchouk polynomials were presented, anew, from a probabilistic perspective. This led to bivariate Krawtchouk polynomials being sometimes called Rahman polynomials. A number of papers followed [6, 7, 8, 23]; the approach of [11], related to Markov chains, was extended to the multivariate case in reference [7] to which the reader is directed for an account of the developments at that point in time.

Germane to the present paper are references [13] and [12]. In the first of these papers, Iliev and Terwilliger offer a Lie-algebraic interpretation of the bivariate Krawtchouk polynomials using the algebra $sl_3(\mathbb{C})$. In the second paper, this study was extended by Iliev to the multivariate case by connecting the Krawtchouk polynomials in d variables to $sl_{d+1}(\mathbb{C})$. In these two papers, the Krawtchouk polynomials appear as overlap coefficients between basis elements for two modules of $sl_3(\mathbb{C})$ or $sl_{d+1}(\mathbb{C})$ in general. The basis elements for the representation spaces are defined as eigenvectors of two Cartan subalgebras related by an anti-automorphism specified by the parameters of the polynomials. The interpretation presented here is in a similar spirit. We shall indicate in Section 5 and in the appendix what are the main observations that are required if one wishes to establish the correspondence. In essence, the key is in the recognition that the anti-automorphism used in [12] and [13] can be taken to be a rotation (times i). The analysis is then brought in the realm of the theory of Lie group representations. This entails connecting two parametrizations of the polynomials: the one used in the cited literature and the other that naturally emerges in the interpretation to be presented, in terms of rotation matrix elements. It is noted that the connection with $SO(d+1)$ rotations readily explains the $d(d+1)/2$ parameters of the polynomials.

A major advance in the theory of multivariable orthogonal polynomials was made by Tratnik [27], who defined a family of multivariate Racah polynomials, thereby obtaining a generalization to many variables of the discrete polynomials at the top of the Askey scheme and extending the multivariate Hahn polynomials introduced by Karlin and McGregor in [16] in the context of linear growth models with many types. These Racah polynomials in d variables depend on $d+2$ parameters; they can be expressed as products of single variable Racah polynomials with the parameter arguments depending on the variables. Using limits and specializations, Tratnik further identified multivariate analogs to the various discrete families of the Askey tableau, thus recovering the multidimensional Hahn polynomials of Karlin and McGregor and obtaining in particular an ensemble of Krawtchouk polynomials in d variables depending on only d parameters (in contrast to the $d(d+1)/2$ parameters that we were so far discussing). We shall call these the Krawtchouk-Tratnik polynomials so as to distinguish them from the ones introduced by Griffiths. The bispectral properties of the multivariable Racah-Wilson polynomials defined by Tratnik have been determined in [3]. As a matter of fact, the Krawtchouk-Tratnik polynomials are also orthogonal with respect to the multinomial distribution and have been used in multi-dimensional birth and death processes [22]. It is a natural question then to ask what relation do the Krawtchouk-Tratnik polynomials have with the other family. As will be seen, the former are special cases of the latter corresponding to particular choices of the rotation matrix. This fact had been obscured it seems, by the usual parametrization which is singular in the Tratnik case.

To sum up, we shall see that the multivariable Krawtchouk polynomials are basically the overlap coefficients between the eigenstates of the isotropic harmonic oscillator states in two different Cartesian coordinate systems related to one another by an arbitrary rotation. This will provide a cogent underpinning for the characterization of these functions: simple derivations of known formulas will be given and new identities will come to the fore.

In view of their naturalness, their numerous special properties and especially their connection to the rotation groups, it is to be expected that the multivariable Krawtchouk polynomials will intervene in various additional physical contexts. Let us mention for example two situations where this is so. The bivariate Krawtchouk polynomials have already been shown in [20] to provide the exact solution of the 1-excitation dynamics of a two-dimensional spin lattice with non-homogeneous nearest-neighbor couplings. This allowed for an analysis of quantum state transfer in triangular domains of the plane. The multivariate Krawtchouk polynomials also proved central in the construction [21] of superintegrable finite models of the harmonic oscillator where they arise in the wavefunctions. Let us stress that in this case we have a variant of the relation with group theory as the polynomials are basis vectors for representation spaces of the symmetry group in this application. Indeed, the energy eigenstates of the finite oscillator in d dimensions are given by wavefunctions where the polynomials in the d discrete coordinates have fixed total degrees. This is to say that the Krawtchouk polynomials in d variables, with given degree, span irreducible modules of $SU(d)$.

The paper is structured as follows. In Section 2, we specify the representations of $SO(3)$ on the energy eigensubspaces of the three-dimensional isotropic harmonic oscillator. In Section 3, we show that the matrix elements of these representations define orthogonal polynomials in two discrete variables that are orthogonal with respect to the trinomial distribution. In Section 4, we use the unitarity of the representations to derive the duality property of the polynomials. A generating function is obtained in Section 5 using boson calculus and is identified with that of the multivariate Krawtchouk polynomials. The recurrence relations and difference equations are obtained in section 6. An integral representation of the bivariate Krawtchouk polynomials is given in terms of Hermite polynomials in Section 7. It is determined in Section 8 that the representation matrix elements for rotations in coordinate planes are given in terms of ordinary Krawtchouk polynomials in one variable. In Section 9, the bivariate Krawtchouk-Tratnik polynomials are shown to be a special case of the general polynomials associated to rotations expressible as the product of two rotations in coordinate planes. In the group-theoretic interpretation of special functions, addition formulas are the translation of the group product. This is the object of Section 10, in which a simple derivation of the formula expressing the bivariate Krawtchouk-Tratnik polynomials as a product of two ordinary Krawtchouk polynomials in one variable is given and where an expansion of the general bivariate Krawtchouk polynomials $Q_{m,n}(i,k;N)$ in terms of the Krawtchouk-Tratnik polynomials is provided. We indicate in Section 11 how the analysis presented in details for the two variable case extends straightforwardly to an arbitrary number of variables. It is also explained how the parametrization of [12] is related to the one in terms of rotation matrices. A short conclusion follows. Background on multivariate Krawtchouk polynomials will be found in the Appendix as well as explicit formulas, especially for the bivariate case, relating parametrizations of the polynomials.

1.2 Representations of $SO(3)$ on the quantum states of the harmonic oscillator in three dimensions

In this section, standard results on the Weyl algebra, its representations and the three-dimensional harmonic oscillator are reviewed. Furthermore, the reducible representations of the rotation group $SO(3)$ on the oscillator states that shall be considered throughout the paper are defined.

1.2.1 The Weyl algebra

Consider the Weyl algebra generated by a_i, $a_i^\dagger$, $i = 1,2,3$, and defined by the commutation relations

$$[a_i, a_k] = 0, \qquad [a_i^\dagger, a_k^\dagger] = 0, \qquad [a_i, a_k^\dagger] = \delta_{ik}. \tag{1.1}$$

The algebra (1.1) has a standard representation on the states

$$|\, n_1, n_2, n_3\,\rangle \equiv |\, n_1\,\rangle \otimes |\, n_2\,\rangle \otimes |\, n_3\,\rangle, \tag{1.2}$$

where n_1, n_2 and n_3 are non-negative integers. This representation is defined by the following actions on the factors of the direct product states:

$$a_i |\, n_i \,\rangle = \sqrt{n_i}\, |\, n_i - 1 \,\rangle, \quad a_i^{\dagger} |\, n_i \,\rangle = \sqrt{n_i + 1}\, |\, n_i + 1 \,\rangle. \tag{1.3}$$

It follows from (1.3) that one can write

$$|\, n_1, n_2, n_3 \,\rangle = \frac{(a_1^{\dagger})^{n_1} (a_2^{\dagger})^{n_2} (a_3^{\dagger})^{n_3}}{\sqrt{n_1! n_2! n_3!}} |\, 0,0,0 \,\rangle. \tag{1.4}$$

The algebra (1.1) has a realization in the Cartesian coordinates x_i given by

$$a_i = \frac{1}{\sqrt{2}}(x_i + \partial_{x_i}), \quad a_i^{\dagger} = \frac{1}{\sqrt{2}}(x_i - \partial_{x_i}), \tag{1.5}$$

where ∂_{x_i} denotes differentiation with respect to the variable x_i.

1.2.2 The 3D quantum harmonic oscillator

Consider now the Hamiltonian of the three-dimensional quantum harmonic oscillator

$$\mathscr{H} = -\frac{1}{2}\nabla^2 + \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) - 3/2. \tag{1.6}$$

In terms of the realization (1.5), the Hamiltonian (1.6) reads

$$\mathscr{H} = a_1^{\dagger} a_1 + a_2^{\dagger} a_2 + a_3^{\dagger} a_3. \tag{1.7}$$

From the expression (1.7) and the actions (1.3), it is directly seen that the Hamiltonian $\mathscr{H}$ of the three-dimensional harmonic oscillator is diagonal on the states (1.2) with eigenvalues $N = n_1 + n_2 + n_3$:

$$\mathscr{H} |\, n_1, n_2, n_3 \,\rangle = N |\, n_1, n_2, n_3 \,\rangle, \tag{1.8}$$

The Schrödinger equation

$$\mathscr{H}\Psi = E\Psi,$$

associated to the Hamiltonian (1.6) separates in particular in the Cartesian coordinates x_i and in these coordinates the wavefunctions have the expression

$$\begin{aligned} \langle\, x_1, x_2, x_3 \,|\, n_1, n_2, n_3 \,\rangle &= \Psi_{n_1,n_2,n_3}(x_1,x_2,x_3) \\ &= \frac{1}{\sqrt{2^N \pi^{3/2} n_1! n_2! n_3!}} e^{-(x_1^2+x_2^2+x_3^2)/2} H_{n_1}(x_1) H_{n_2}(x_2) H_{n_3}(x_3), \end{aligned} \tag{1.9}$$

where $H_n(x)$ stands for the Hermite polynomials [17].

1.2.3 The representations of $SO(3) \subset SU(3)$ on oscillator states

It is manifest that the harmonic oscillator Hamiltonian $\mathscr{H}$, given by (1.6) in the coordinate representation, is invariant under rotations. Moreover, it is clear from the expression (1.7) that $\mathscr{H}$ is invariant under $SU(3)$ transformations. We introduce the set of orthonormal basis vectors

$$|\, m,n \,\rangle_N = |\, m,n,N-m-n \,\rangle, \qquad m,n = 0,\ldots,N, \tag{1.10}$$

which span the eigensubspace of energy N which is of dimension $(N+1)(N+2)/2$. For each N, the basis vectors (1.10) support an irreducible representation of the group $SU(3)$, which is generated by the constants of motion of the form $a_i^\dagger a_j$. In the following, we shall however focus on the subgroup $SO(3) \subset SU(3)$, which is generated by the three angular momenta

$$J_i = -i \sum_{j,k=1}^{3} \epsilon_{ijk} a_j^\dagger a_k, \tag{1.11}$$

satisfying the commutation relations

$$[J_i, J_j] = i\epsilon_{ijk} J_k,$$

and shall consider the reducible representations of this $SO(3)$ subgroup that are thus provided. For a given N, this representation decomposes into the multiplicity-free direct sum of every $(2\ell+1)$-dimensional irreducible representation of $SO(3)$ with values $\ell = N, N-2, \ldots, 1/0$. On the basis vectors (1.10), the actions (1.3) take the form

$$a_1 |\, m,n \,\rangle_N = \sqrt{m}\, |\, m-1,n \,\rangle_{N-1}, \qquad a_1^\dagger |\, m,n \,\rangle_N = \sqrt{m+1}\, |\, m+1,n \,\rangle_{N+1}, \tag{1.12a}$$

$$a_2 |\, m,n \,\rangle_N = \sqrt{n}\, |\, m,n-1 \,\rangle_{N-1}, \qquad a_2^\dagger |\, m,n \,\rangle_N = \sqrt{n+1}\, |\, m,n+1 \,\rangle_{N+1}, \tag{1.12b}$$

$$a_3 |\, m,n \,\rangle_N = \sqrt{N-m-n}\, |\, m,n \,\rangle_{N-1}, \tag{1.12c}$$

$$a_3^\dagger |\, m,n \,\rangle_N = \sqrt{N-m-n+1}\, |\, m,n \,\rangle_{N+1}. \tag{1.12d}$$

We use the following notation. Let B be a 3×3 real antisymmetric matrix ($B^T = -B$) and $R \in SO(3)$ be the rotation matrix related to B by

$$R = e^B. \tag{1.13}$$

One has of course

$$R^T R = R R^T = 1,$$

which in components reads

$$\sum_{k=1}^{3} R_{ki} R_{kj} = \delta_{ij}, \qquad \sum_{k=1}^{3} R_{ik} R_{jk} = \delta_{ij}. \tag{1.14}$$

Consider the unitary representation defined by

$$U(R) = \exp\left(\sum_{i,k=1}^{3} B_{ik} a_i^\dagger a_k \right). \tag{1.15}$$

The transformations of the generators $a_i^\dagger$, a_i under the action of $U(R)$ are given by

$$U(R) a_i^\dagger U^\dagger(R) = \sum_{k=1}^{3} R_{ki} a_k^\dagger, \qquad U(R) a_i U^\dagger(R) = \sum_{k=1}^{3} R_{ki} a_k. \tag{1.16}$$

Note that $U(R)$ satisfies

$$U(RV) = U(R)U(V), \qquad R, V \in SO(3), \tag{1.17}$$

as should be for a group representation.

1.3 The representation matrix elements as orthogonal polynomials

In this section, it is shown that the matrix elements of the unitary representations of $SO(3)$ defined in the previous section are expressed in terms of orthogonal polynomials in the two discrete variables i, k.

The matrix elements of the unitary operator (1.15) in the basis (1.10) can be cast in the form

$$_N\langle i, k \,|\, U(R) \,|\, m, n \rangle_N = W_{i,k;N} P_{m,n}(i, k; N), \tag{1.18}$$

where $P_{0,0}(i,k;N) \equiv 1$ and where $W_{i,k;N}$ is defined by

$$W_{i,k;N} = {}_N\langle i, k \,|\, U(R) \,|\, 0, 0 \rangle_N. \tag{1.19}$$

For notational convenience, we shall drop the explicit dependence of the operator U on the rotation R in what follows.

1.3.1 Calculation of the amplitude $W_{i,k;N}$

An explicit expression can be obtained for the amplitude $W_{i,k;N}$. To that end, one notes using (1.12) that on the one hand

$$_{N-1}\langle\, i,k\,|\,Ua_1\,|\,0,0\,\rangle_N = 0,$$

and that on the other hand using (1.16)

$$\begin{aligned}
_{N-1}\langle\, i,k\,|\,Ua_1\,|\,0,0\,\rangle_N &= {}_{N-1}\langle\, i,k\,|\,Ua_1U^{\dagger}U\,|\,0,0\,\rangle_N \\
&= R_{11}\sqrt{i+1}\,{}_N\langle\, i+1,k\,|\,U\,|\,0,0\,\rangle_N + R_{21}\sqrt{k+1}\,{}_N\langle\, i,k+1\,|\,U\,|\,0,0\,\rangle_N \\
&\qquad + R_{31}\sqrt{N-i-k}\,{}_N\langle\, i,k\,|\,U\,|\,0,0\,\rangle_N.
\end{aligned}$$

Making use of the definition (1.19), it follows from the above relations that

$$R_{11}\sqrt{i+1}\,W_{i+1,k;N} + R_{21}\sqrt{k+1}\,W_{i,k+1;N} + R_{31}\sqrt{N-i-k}\,W_{i,k;N} = 0. \tag{1.20}$$

In a similar fashion, starting instead from the relation

$$_{N-1}\langle\, i,k\,|\,Ua_2\,|\,0,0\,\rangle_N = 0,$$

one obtains

$$R_{12}\sqrt{i+1}\,W_{i+1,k;N} + R_{22}\sqrt{k+1}\,W_{i,k+1;N} + R_{32}\sqrt{N-i-k}\,W_{i,k;N} = 0. \tag{1.21}$$

Mindful of (1.14), it is easily verified that the common solution to the difference equations (1.20) and (1.21) is

$$W_{i,k;N} = C\,\frac{R_{13}^{i}R_{23}^{k}R_{33}^{N-i-k}}{\sqrt{i!k!(N-i-k)!}},$$

where C is an arbitrary constant. This constant can be obtained from the normalization of the basis vectors:

$$1 = {}_N\langle\, 0,0\,|\,U^{\dagger}U\,|\,0,0\,\rangle_N = \sum_{i+k\leqslant N} {}_N\langle\, 0,0\,|\,U^{\dagger}\,|\,i,k\,\rangle_N\,{}_N\langle\, i,k\,|\,U\,|\,0,0\,\rangle_N = \sum_{i+k\leqslant N}|W_{i,k;N}|^2.$$

Upon using the trinomial theorem, which reads

$$(z+y+z)^N = \sum_{i+k\leq N}\frac{N!}{i!k!(N-i-k)!}\,x^i y^k z^{N-i-k},$$

and the orthogonality relation (1.14), one finds that $C=\sqrt{N!}$. Hence the explicit expression for $W_{i,k;N}$ is given by

$$W_{i,k;N} = R_{13}^{i}R_{23}^{k}R_{33}^{N-i-k}\sqrt{\frac{N!}{i!k!(N-i-k)!}}. \tag{1.22}$$

1.3.2 Raising relations

We show that the $P_{m,n}(i,k;N)$ appearing in the matrix elements (1.18) are polynomials of total degree $m+n$ in the variables i, k by obtaining raising relations for these polynomials. Consider the matrix element ${}_N\langle i,k \,|\, Ua_1^{\dagger} \,|\, m,n \rangle_{N-1}$. On the one hand, one has

$$ {}_N\langle i,k \,|\, Ua_1^{\dagger} \,|\, m,n \rangle_{N-1} = \sqrt{m+1}\, W_{i,k;N} P_{m+1,n}(i,k;N), $$

and on the other hand, using (1.16), one finds

$$ {}_N\langle i,k \,|\, Ua_1^{\dagger} \,|\, m,n \rangle_{N-1} = {}_N\langle i,k \,|\, Ua_1^{\dagger}U^{\dagger}U \,|\, m,n \rangle_{N-1} = \sum_{m=1}^{3} R_{m,1}\, {}_N\langle i,k \,|\, a_m^{\dagger}U \,|\, m,n \rangle_{N-1}. $$

Upon comparing the two preceding expressions and using (1.22), (1.18), it follows that

$$ \sqrt{N(m+1)}P_{m+1,n}(i,k;N) = \frac{R_{11}}{R_{13}}\, i P_{m,n}(i-1,k;N-1) + \frac{R_{21}}{R_{23}}\, k P_{m,n}(i,k-1;N-1) + \frac{R_{31}}{R_{33}}(N-i-k)P_{m,n}(i,k;N-1). \tag{1.23} $$

In a similar fashion, starting with the matrix element ${}_N\langle i,k \,|\, Ua_2^{\dagger} \,|\, m,n \rangle_{N-1}$, one finds

$$ \sqrt{N(n+1)}P_{m,n+1}(i,k;N) = \frac{R_{12}}{R_{13}}\, i P_{m,n}(i-1,k;N-1) + \frac{R_{22}}{R_{23}}\, k P_{m,n}(i,k-1;N-1) + \frac{R_{32}}{R_{33}}(N-i-k)P_{m,n}(i,k;N-1). \tag{1.24} $$

By definition, we have $P_{0,0}(i,k;N) = 1$. It is then seen that the formulas (1.23) and (1.24) allow to construct any $P_{m,n}(ik;N)$ from $P_{0,0}(i,k;N)$ through a step by step iterative process and it is observed that these functions are polynomials in the two (discrete) variables i, k of total degree $n+m$.

1.3.3 Orthogonality relation

The unitarity of the representation (1.15) can be used to show that the polynomials $P_{m,n}(i,k;N)$ obey an orthogonality relation. Indeed, one has

$$ {}_N\langle m',n' \,|\, U^{\dagger}U \,|\, m,n \rangle_N = \sum_{i+k\leqslant N} {}_N\langle m'n' \,|\, U^{\dagger} \,|\, i,k \rangle_{N\,N}\langle i,k \,|\, U \,|\, m,n \rangle_N = \delta_{m'm}\delta_{n'n}. $$

Upon inserting (1.18), one finds that the polynomials $P_{m,n}(i,k;N)$ are orthonormal

$$ \sum_{i+k\leqslant N} w_{i,k;N} P_{m,n}(i,k;N) P_{m',n'}(i,k;N) = \delta_{m'm}\delta_{n'n}, \tag{1.25} $$

with respect to the discrete weight

$$w_{i,k;N} = W_{i,k;N}^2 = \frac{N!}{i!k!(N-i-k)!} R_{13}^{2i} R_{23}^{2k} R_{33}^{2(N-i-k)}. \tag{1.26}$$

1.3.4 Lowering relations

Lowering relations for the polynomials $P_{m,n}(i,k;N)$ can be obtained in a way similar to how the raising relations were found. One first considers the matrix element

$${}_N\langle\, i,k \,|\, U a_1 \,|\, m,n \,\rangle_{N+1},$$

which leads to the relation

$$\begin{aligned} \sqrt{\frac{m}{N+1}} P_{m-1,n}(i,k;N) = \alpha_1 \left[P_{m,n}(i+1,k;N+1) - P_{m,n}(i,k;N+1) \right] \\ + \beta_1 \left[P_{m,n}(i,k+1;N+1) - P_{m,n}(i,k;N+1) \right]. \end{aligned} \tag{1.27}$$

If one considers instead the matrix element ${}_N\langle\, i,k \,|\, U a_2 \,|\, m,n \,\rangle_{N+1}$, one finds

$$\begin{aligned} \sqrt{\frac{n}{N+1}} P_{m,n-1}(i,k;N) = \alpha_2 \left[P_{m,n}(i+1,k;N+1) - P_{m,n}(i,k;N+1) \right] \\ + \beta_2 \left[P_{m,n}(i,k+1;N+1) - P_{m,n}(i,k;N+1) \right]. \end{aligned} \tag{1.28}$$

In (1.27) and (1.28), the parameters α, β are given by

$$\alpha_1 = R_{11}R_{13}, \quad \beta_1 = R_{21}R_{23}, \quad \alpha_2 = R_{12}R_{13}, \quad \beta_2 = R_{22}R_{23}.$$

1.4 Duality

A duality relation under the exchange of the variables i, k and the degrees m, n is obtained in this section for the polynomials $P_{m,n}(i,k;N)$. This property is seen to take a particularly simple form for a set of polynomials $Q_{m,n}(i,k;N)$ which are obtained from $P_{m,n}(i,k;N)$ by a renormalization.

The duality relation for the polynomials $P_{m,n}(i,k;N)$ is obtained by considering the matrix elements ${}_N\langle\, i,k \,|\, U^{\dagger}(R) \,|\, m,n \,\rangle_N$ from two different points of view. First one writes

$${}_N\langle\, i,k \,|\, U^{\dagger}(R) \,|\, m,n \,\rangle_N = \widetilde{W}_{i,k;N} \widetilde{P}_{m,n;N}, \tag{1.29}$$

where $\widetilde{P}_{0,0}(i,k;N) = 1$ and $\widetilde{W}_{i,k;N} = {}_N\langle\, i,k \mid U^{\dagger} \mid 0,0\,\rangle_N$. Since $U^{\dagger}(R) = U(R^T)$, it follows from (1.22) that

$$\widetilde{W}_{i,k;N} = R_{31}^{i} R_{32}^{k} R_{33}^{N-i-k} \sqrt{\frac{N!}{i!k!(N-i-k)!}},$$

and that $\widetilde{P}_{m,n}(i,k;N)$ are the polynomials corresponding to the rotation matrix R^T. Second, one instead writes

$$\begin{aligned} {}_N\langle\, i,k \mid U^{\dagger}(R) \mid m,n\,\rangle_N &= \overline{{}_N\langle\, m,n \mid U(R) \mid i,k\,\rangle_N} = {}_N\langle\, m,n \mid U(R) \mid i,k\,\rangle_N \\ &= W_{m,n;N} P_{i,k}(m,n;N), \end{aligned} \tag{1.30}$$

where $\overline{x}$ denotes complex conjugation and where the reality of the matrix elements has been used. Upon comparing (1.29), (1.30) and using (1.22), one directly obtains the duality formula

$$P_{i,k}(m,n;N) = \sqrt{\frac{m!n!(N-m-n)!}{i!k!(-i-k)!}} \frac{R_{31}^{i} R_{32}^{k} R_{33}^{n+m}}{R_{13}^{m} R_{23}^{n} R_{33}^{i+k}} \widetilde{P}_{m,n}(i,k;N). \tag{1.31}$$

It is convenient to introduce the two variable polynomials $Q_{m,n}(i,k;N)$ defined by

$$P_{m,n}(i,k;N) = \sqrt{\frac{N!}{m!n!(N-m-n)!}} \left(\frac{R_{31}}{R_{33}}\right)^{m} \left(\frac{R_{32}}{R_{33}}\right)^{n} Q_{m,n}(i,k;N). \tag{1.32}$$

In terms of these polynomials, the duality relation (1.31) reads

$$Q_{i,k}(m,n;N) = \widetilde{Q}_{m,n}(i,k;N), \tag{1.33}$$

where the parameters appearing in the polynomial $\widetilde{Q}_{m,n}(i,k;N)$ correspond to the transpose matrix R^T.

1.5 Generating function

In this section, the generating functions for the multivariate polynomials $P_{m,n}(i,k;N)$ and $Q_{m,n}(i,k;N)$ are derived using boson calculus. The generating function obtained for $Q_{m,n}(i,k;N)$ is shown to coincide with that of the Rahman polynomials, thus establishing the fact that the polynomials $Q_{m,n}(i,k;N)$ are precisely those defined in [11]. The connection with the parameters used in [12] and [13] is established.

Consider the following generating function for the polynomials $P_{m,n}(i,k;N)$:

$$G(\alpha_1,\alpha_2,\alpha_3) = \sum_{m+n+\ell=N} \frac{\alpha_1^m \alpha_2^n \alpha_3^\ell}{\sqrt{m!n!\ell!}} W_{i,k;N} P_{m,n}(i,k;N). \tag{1.34}$$

Given the definition (1.18) of the matrix elements of $U(R)$, one has

$$G(\alpha_1,\alpha_2,\alpha_3) = \sum_{m+n+\ell=N} \frac{\alpha_1^m \alpha_2^n \alpha_3^\ell}{\sqrt{m!n!\ell!}} {}_N\langle\, i,k \,|\, U(R) \,|\, m,n \,\rangle_N.$$

Upon using (1.4) in the above relation, one finds

$$G(\alpha_1,\alpha_2,\alpha_3) = \sum_{m+n+\ell=N} {}_N\langle\, i,k \,|\, U \frac{(\alpha_1 a_1^\dagger)^m}{m!} \frac{(\alpha_2 a_2^\dagger)^n}{n!} \frac{(\alpha_3 a_3^\dagger)^\ell}{\ell!} \,|\, 0,0,0 \,\rangle.$$

Since the rotation operator U preserves any eigenspace with a given energy N and since the states are mutually orthogonal, one can write

$$\begin{aligned} &G(\alpha_1,\alpha_2,\alpha_3) \\ &\quad = {}_N\langle\, i,k \,|\, U\, e^{\alpha_1 a_1^\dagger + \alpha_2 a_2^\dagger + \alpha_3 a_3^\dagger} \,|\, 0,0,0 \,\rangle = {}_N\langle\, i,k \,|\, U\, e^{\alpha_1 a_1^\dagger + \alpha_2 a_2^\dagger + \alpha_3 a_3^\dagger} U^\dagger U \,|\, 0,0,0 \,\rangle. \end{aligned}$$

Since $U|\,0,0,0\,\rangle = |\,0,0,0\,\rangle$ and

$$U e^{\sum_s \alpha_s a_s^\dagger} U^\dagger = e^{\sum_s \alpha_s U a_s^\dagger U^\dagger} = e^{\sum_{st} \alpha_s R_{ts} a_t^\dagger} = e^{\sum_t \beta_t a_t^\dagger},$$

where

$$\beta_t = \sum_{s=1}^{3} R_{ts} \alpha_s,$$

it follows that

$$G(\alpha_1,\alpha_2,\alpha_3) = {}_N\langle\, i,k \,|\, e^{\beta_1 a_1^\dagger + \beta_2 a_2^\dagger + \beta_3 a_3^\dagger} \,|\, 0,0,0 \,\rangle = \sum_{p,q,r} \frac{\beta_1^p \beta_2^q \beta_3^r}{\sqrt{p!q!r!}} \langle\, i,k,j \,|\, p,q,r \,\rangle,$$

with $j = N-i-k$. Using the orthogonality of the basis states, one thus obtains

$$G(\alpha_1,\alpha_2,\alpha_3) = \frac{\beta_1^i \beta_2^k \beta_3^{N-i-k}}{\sqrt{i!k!(N-i-k)!}}. \tag{1.35}$$

Upon comparing the expressions (1.34), (1.35) and recalling the expression for $W_{i,k;N}$ given by (1.22), one finds

$$\frac{\beta_1^i \beta_2^k \beta_3^{N-i-k}}{\sqrt{N!}} = R_{13}^i R_{23}^k R_{33}^{N-i-k} \sum_{m+n\leqslant N} \frac{\alpha_1^m \alpha_2^n \alpha_3^{N-n-m}}{\sqrt{m!n!(N-n-m)!}} P_{m,n}(i,k;N). \tag{1.36}$$

Taking $\alpha_1 = u$, $\alpha_2 = v$ and $\alpha_3 = 1$ in (1.36), one obtains the following generating function for the polynomials $P_{m,n}(i,k;N)$:

$$\left(1+\frac{R_{11}}{R_{13}}u+\frac{R_{12}}{R_{13}}v\right)^i\left(1+\frac{R_{21}}{R_{23}}u+\frac{R_{22}}{R_{23}}v\right)^k\left(1+\frac{R_{31}}{R_{33}}u+\frac{R_{32}}{R_{33}}v\right)^{N-i-k} = \sum_{m+n\leqslant N}\sqrt{\frac{N!}{m!n!(N-m-n)!}}\,P_{m,n}(i,k;N)\,u^m v^n. \tag{1.37}$$

In terms of the polynomials $Q_{m,n}(i,k;N)$, the relation (1.36) reads

$$\left(\frac{R_{11}}{R_{13}}\alpha_1+\frac{R_{12}}{R_{13}}\alpha_2+\alpha_3\right)^i\left(\frac{R_{21}}{R_{23}}\alpha_1+\frac{R_{22}}{R_{23}}\alpha_2+\alpha_3\right)^k\left(\frac{R_{31}}{R_{33}}\alpha_1+\frac{R_{32}}{R_{33}}\alpha_2+\alpha_3\right)^{N-i-k}$$
$$=\sum_{m+n\leqslant N}\frac{N!}{m!n!(N-m-n)!}\left(\frac{R_{31}}{R_{33}}\right)^m\left(\frac{R_{32}}{R_{33}}\right)^n Q_{m,n}(i,k;N)\,\alpha_1^m\alpha_2^n\alpha_3^{N-m-n}.$$

Upon taking instead

$$\alpha_1=\frac{R_{33}}{R_{31}}z_1,\quad \alpha_2=\frac{R_{33}}{R_{32}}z_2,\quad \alpha_3=1,$$

one finds the following generating function for the polynomials $Q_{m,n}(i,k;N)$:

$$\left(1+\frac{R_{11}R_{33}}{R_{13}R_{31}}z_1+\frac{R_{12}R_{33}}{R_{13}R_{32}}z_2\right)^i\left(1+\frac{R_{21}R_{33}}{R_{23}R_{31}}z_1+\frac{R_{22}R_{33}}{R_{23}R_{32}}z_2\right)^k(1+z_1+z_2)^{N-i-k} = \sum_{m+n\leqslant N}\frac{N!}{m!n!(N-m-n)!}\,Q_{m,n}(i,k;N)\,z_1^m z_2^n. \tag{1.38}$$

The formula (1.38) lends itself to comparison with the generating function which can be taken to define the bivariate Krawtchouk polynomials (see (1.78)). Up to an obvious identification of the indices $(i,k)\to(\widetilde{m}_1,\widetilde{m}_2)$ and $(m,n)\to(m_1,m_2)$, the generating function (1.38) coincides with (1.78) and shows that the polynomials $Q_{m,n}(i,k;N)$ are the same as the Rahman polynomials $Q(m,\widetilde{m})$ if one takes

$$u_{11}=\frac{R_{11}R_{33}}{R_{13}R_{31}},\quad u_{12}=\frac{R_{12}R_{33}}{R_{13}R_{32}}, \tag{1.39a}$$
$$u_{21}=\frac{R_{21}R_{33}}{R_{23}R_{31}},\quad u_{22}=\frac{R_{22}R_{33}}{R_{23}R_{32}}. \tag{1.39b}$$

The parametrization (1.39) can be related to the one in terms of the four parameters p_1, p_2, p_3 and p_4 that is customarily used to define the Rahman polynomials. The reader is referred to the Appendix for the precise correspondence and for further details concerning the parametrizations.

1.6 Recurrence relations and difference equations

In this section, the algebraic setting is used to derive the recurrence relations and difference equations that the polynomials $P_{m,n}(i,k;N)$ and $Q_{m,n}(i,k;N)$ satisfy. We note that these have been obtained previously in [6] and [13] (see also [28]). It is interesting to see how easily the recurrence relations and difference equations follow from the group-theoretic interpretation. See also [1, 14, 28].

1.6.1 Recurrence relations

The obtain the recurrence relations for the polynomials $P_{m,n}(i,k;N)$, one begins by considering the matrix element ${}_N\langle i,k \mid a_1^\dagger a_1 U \mid m,n \rangle_N$. One has on the one hand

$$
{}_N\langle i,k \mid a_1^\dagger a_1 U \mid m,n \rangle_N = i \, {}_N\langle i,k \mid U \mid m,n \rangle_N,
$$

and on the other hand

$$
\begin{aligned}
{}_N\langle i,k \mid a_1^\dagger a_1 U \mid m,n \rangle_N &= {}_N\langle i,k \mid U U^\dagger a_1^\dagger a_1 U \mid m,n \rangle_N \\
&= \sum_{r,s=1}^{3} R_{r,1} R_{s,1} \, {}_N\langle i,k \mid U a_r^\dagger a_s \mid m,n \rangle_N.
\end{aligned}
$$

Upon comparing the above equations and using (1.18) and (1.22), one finds

$$
\begin{aligned}
i\,P_{m,n}(i,k;N) &= \left[R_{11}^2 m + R_{12}^2 n + R_{13}^2 (N-m-n)\right] P_{m,n}(i,k;N) \\
&+ R_{11}R_{12}\left[\sqrt{m(n+1)}P_{m-1,n+1}(i,k;N) + \sqrt{n(m+1)}P_{m+1,n-1}(i,k;N)\right] \\
&+ R_{11}R_{13}\left[\sqrt{m(N-m-n+1)}P_{m-1,n}(ik;N) + \sqrt{(m+1)(N-m-n)}P_{m+1,n}(i,k;N)\right] \\
&+ R_{12}R_{13}\left[\sqrt{n(N-m-n+1)}P_{m,n-1}(ik;N) + \sqrt{(n+1)(N-m-n)}P_{m,n+1}(ik;N)\right].
\end{aligned} \tag{1.40}
$$

Proceeding similarly from the matrix element ${}_N\langle i,k \mid a_2^\dagger a_2 U \mid m,n \rangle_N$, one obtains

$$
\begin{aligned}
k\,P_{m,n}(i,k;N) &= \left[R_{21}^2 m + R_{22}^2 n + R_{23}^2 (N-m-n)\right] P_{m,n}(i,k;N) \\
&+ R_{21}R_{22}\left[\sqrt{m(n+1)}P_{m-1,n+1}(i,k;N) + \sqrt{n(m+1)}P_{m+1,n-1}(i,k;N)\right] \\
&+ R_{21}R_{23}\left[\sqrt{m(N-m-n+1)}P_{m-1,n}(i,k;N) + \sqrt{(m+1)(N-m-n)}P_{m+1,n}(i,k;N)\right] \\
&+ R_{22}R_{23}\left[\sqrt{n(N-m-n+1)}P_{m,n-1}(i,k;N) + \sqrt{(n+1)(N-m-n)}P_{m,n+1}(i,k;N)\right].
\end{aligned} \tag{1.41}
$$

In terms of the polynomials $Q_{m,n}(i,k;N)$ defined by (1.32), the recurrence relations (1.40) and (1.41) become

$$\begin{aligned}
i\,Q_{m,n}(i,k;N) &= \left[R_{11}^2\,m + R_{12}^2\,n + R_{13}^2\,(N-m-n)\right] Q_{m,n}(i,k;N) \\
&+ \frac{R_{11}R_{12}R_{32}}{R_{31}}\,m\,Q_{m-1,n+1}(i,k;N) + \frac{R_{11}R_{12}R_{31}}{R_{32}}\,n\,Q_{m+1,n-1}(i,k;N) \\
&+ \frac{R_{11}R_{13}R_{33}}{R_{31}}\,m\,Q_{m-1,n}(i,k;N) + \frac{R_{11}R_{13}R_{31}}{R_{33}}\,(N-m-n)Q_{m+1,n}(i,k;N) \\
&+ \frac{R_{12}R_{13}R_{33}}{R_{32}}\,n\,Q_{m,n-1}(i,k;N) + \frac{R_{12}R_{13}R_{32}}{R_{33}}\,(N-m-n)Q_{m,n+1}(i,k;N),
\end{aligned} \tag{1.42}$$

and

$$\begin{aligned}
k\,Q_{m,n}(i,k;N) &= \left[R_{21}^2\,m + R_{22}^2\,n + R_{23}^2\,(N-m-n)\right] Q_{m,n}(i,k;N) \\
&+ \frac{R_{21}R_{22}R_{32}}{R_{31}}\,m\,Q_{m-1,n+1}(i,k;N) + \frac{R_{21}R_{22}R_{31}}{R_{32}}\,n\,Q_{m+1,n-1}(i,k;N) \\
&+ \frac{R_{21}R_{23}R_{33}}{R_{31}}\,m\,Q_{m-1,n}(i,k;N) + \frac{R_{21}R_{23}R_{31}}{R_{33}}\,(N-m-n)Q_{m+1,n}(i,k;N) \\
&+ \frac{R_{22}R_{23}R_{33}}{R_{32}}\,n\,Q_{m,n-1}(i,k;N) + \frac{R_{22}R_{23}R_{32}}{R_{33}}\,(N-m-n)Q_{m,n+1}(i,k;N).
\end{aligned} \tag{1.43}$$

1.6.2 Difference equations

To obtain the difference equations satisfied by the polynomials $Q_{m,n}(i,k;N)$, one could consider the matrix elements ${}_N\langle\, i,k \mid U a_j^{\dagger} a_j \mid m,n\,\rangle_N$, $j = 1,2$ and proceed along the same lines as for the recurrence relations. It is however easier to proceed directly from the recurrence relations (1.42), (1.43) and use the duality relation (1.33). To illustrate the procedure, consider the left hand side of (1.42). Using the duality (1.33), one may write

$$iQ_{m,n}(i,k;N) = i\,\widetilde{Q}_{i,k}(m,n;N) = m\,\widetilde{Q}_{m,n}(i,k;N),$$

where in the last step the replacements $m \leftrightarrow i$ and $n \leftrightarrow k$ were performed. Since one can obtain $\widetilde{Q}_{m,n}(i,k;N)$ from $Q_{m,n}(i,k;N)$ by replacing the rotation parameters by the elements of the transposed, it is seen that the recurrence relations (1.42) and (1.43) can be turned into difference equations by operating the substitutions $m \leftrightarrow i$, $n \leftrightarrow k$ and replacing the parameters of R by those of R^T. Applying this procedure, it then follows easily that

$$
\begin{aligned}
m\,Q_{m,n}(i,k;N) &= \left[R_{11}^2\,i + R_{21}^2\,k + R_{31}^2\,(N-i-k)\right] Q_{m,n}(i,k;N)\\
&+ \frac{R_{11}R_{21}R_{23}}{R_{13}}\,i\,Q_{m,n}(i-1,k+1;N) + \frac{R_{11}R_{21}R_{13}}{R_{23}}\,k\,Q_{m,n}(i+1,k-1;N)\\
&+ \frac{R_{11}R_{31}R_{33}}{R_{13}}\,i\,Q_{m,n}(i-1,k;N) + \frac{R_{11}R_{31}R_{13}}{R_{33}}\,(N-i-k)Q_{m,n}(i+1,k;N)\\
&+ \frac{R_{21}R_{31}R_{33}}{R_{23}}\,k\,Q_{m,n}(i,k-1;N) + \frac{R_{21}R_{31}R_{23}}{R_{33}}\,(N-i-k)Q_{m,n}(i,k+1;N),
\end{aligned} \tag{1.44}
$$

and

$$
\begin{aligned}
n\,Q_{m,n}(i,k;N) &= \left[R_{12}^2\,i + R_{22}^2\,k + R_{32}^2\,(N-i-k)\right] Q_{m,n}(i,k;N)\\
&+ \frac{R_{12}R_{22}R_{23}}{R_{13}}\,i\,Q_{m,n}(i-1,k+1;N) + \frac{R_{12}R_{22}R_{13}}{R_{23}}\,k\,Q_{m,n}(i+1,k-1;N)\\
&+ \frac{R_{12}R_{32}R_{33}}{R_{13}}\,i\,Q_{m,n}(i-1,k;N) + \frac{R_{12}R_{32}R_{13}}{R_{33}}\,(N-i-k)Q_{m,n}(i+1,k;N)\\
&+ \frac{R_{22}R_{32}R_{33}}{R_{23}}\,k\,Q_{m,n}(i,k-1;N) + \frac{R_{22}R_{32}R_{23}}{R_{33}}\,(N-i-k)Q_{m,n}(i,k+1;N),
\end{aligned} \tag{1.45}
$$

Similar formulas can be obtained straightforwardly for the polynomials $P_{m,n}(i,k;N)$.

1.7 Integral representation

In this section, a relation between the Hermite polynomials and the bivariate Krawtchouk polynomials $P_{m,n}(i,k;N)$ is found. This relation allows for the presentation of an integral formula for these polynomials. To obtain these formulas, one begins by considering the matrix element

$$\langle x_1,x_2,x_3|U(R)|\,m,n\,\rangle_N,$$

from two points of view. By acting with $U(R)$ on the vector $|\,m,n\,\rangle_N$ and using (1.18) and (1.9), one finds

$$
\begin{aligned}
\langle x_1,x_2,x_3|U(R)|\,m,n\,\rangle_N = \sqrt{\frac{N!}{2^N\pi^{3/2}}}\,e^{-(x_1^2+x_2^2+x_3^2)/2} \sum_{i+k\leqslant N} \frac{R_{13}^i R_{23}^k R_{33}^{N-i-k}}{i!k!(N-i-k)!} P_{m,n}(i,k;N)\\
\times H_i(x_1)H_k(x_2)H_{N-i-k}(x_3).
\end{aligned}
$$

By acting with $U^\dagger(R)$ on $\langle\, x_1,x_2,x_3\,||$ and using (1.9), one finds

$$\langle x_1,x_2,x_3|U(R)|\,m,n\,\rangle_N = \frac{e^{-(\tilde{x}_1^2+\tilde{x}_2^2+\tilde{x}_3^2)/2}}{\sqrt{2^N\pi^{3/2}m!n!(N-m-n)!}} H_m(\tilde{x}_1)H_n(\tilde{x}_2)H_{N-m-n}(\tilde{x}_3),$$

where $(\widetilde{x}_1,\widetilde{x}_2,\widetilde{x}_3)^T = R^T(x_1,x_2,x_3)^T$. Since obviously $x_1^2+x_2^2+x_3^2 = \widetilde{x}_1^2+\widetilde{x}_2^2+\widetilde{x}_3^2$, one finds from the above

$$\sqrt{\frac{1}{N!m!n!(N-m-n)!}}H_m(\widetilde{x}_1)H_n(\widetilde{x}_2)H_{N-m-n}(\widetilde{x}_3)$$
$$= \sum_{i+k\leqslant N}\frac{R_{13}^iR_{23}^kR_{33}^{N-i-k}}{i!k!(N-i-k)!}P_{m,n}(i,k;N)H_i(x_1)H_k(x_2)H_{N-i-k}(x_3). \tag{1.46}$$

Using the relation (1.46) and the well-known orthogonality relation satisfied by the Hermite polynomials [17], one obtains the following integral representation for the polynomials $P_{m,n}(i,k;N)$:

$$P_{m,n}(i,k;N) = \frac{R_{13}^{-i}R_{23}^{-k}R_{33}^{i+k-N}}{2^N\pi^{3/2}}\sqrt{\frac{1}{N!m!n!(N-m-n)!}}$$
$$\times\int_{\mathbb{R}^3} e^{-(x_1^2+x_2^2+x_3^2)}H_m(\widetilde{x}_1)H_n(\widetilde{x}_2)H_{N-m-n}(\widetilde{x}_3)H_i(x_1)H_k(x_2)H_{N-i-k}(x_3)\,dx_1dx_2dx_3. \tag{1.47}$$

1.8 Rotations in coordinate planes and univariate Krawtchouk polynomials

It has been assumed so far generically that the entries R_{ik}, $i,k = 1,2,3$, of the rotation matrix R are non-zero. We shall now consider the degenerate cases corresponding to when rotations are restricted to coordinate planes and when the matrix R has thus four zero entries. We shall confirm, as expected, that the representation matrix elements ${}_N\langle\, i,k\,|\,U(R)\,|\,m,n\,\rangle_N$ are then expressed in terms of univariate Krawtchouk polynomials. With J a non-negative integer, the one-dimensional Krawtchouk polynomials $k_n(x;p;J)$ that we shall use are defined by

$$k_n(x;p;J) = (-J)_n\,{}_2F_1\left[\begin{matrix}-n,-x\\ -J\end{matrix};\frac{1}{p}\right] = (-J)_n\sum_{k=0}^{\infty}\frac{(-n)_k(-x)_k}{k!(-J)_k}\left(\frac{1}{p}\right)^k, \tag{1.48}$$

where

$$(a)_k = a(a+1)\cdots(a+k-1),\qquad k\geqslant 1,\quad (a)_0 = 1.$$

These polynomials are orthogonal with respect to the binomial distribution and satisfy the orthogonality relation

$$\sum_{x=0}^{J}\frac{J!}{x!(J-x)!}p^x(1-p)^{J-x}k_m(x;p;J)k_n(x;p;J) = (-1)^n n!(-J)_n\left[\frac{(1-p)}{p}\right]^n\delta_{mn}.$$

They are related as follows to the monic polynomials $q_n(x)$:

$$q_n(x) = p^n k_n(x; p, J).$$

which satisfy the following three-term recurrence relation [17]:

$$xq_n(x) = q_{n+1}(x) + [p(J-n) + n(1-p)]q_n(x) + np(1-p)(J+1-n)q_{n-1}(x). \tag{1.49}$$

Consider the clockwise rotation $R_{(yz)}(\theta)$ by an angle θ in the (yz) plane and the clockwise rotation $R_{(xz)}(\chi)$ by an angle χ in the (xz) plane. They correspond to the matrices

$$R_{(yz)}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}, \quad R_{(xz)}(\chi) = \begin{pmatrix} \cos\chi & 0 & -\sin\chi \\ 0 & 1 & 0 \\ \sin\chi & 0 & \cos\chi \end{pmatrix}. \tag{1.50}$$

Note that their product

$$R_{(yz)}(\theta)R_{(xz)}(\chi) = \begin{pmatrix} \cos\chi & 0 & -\sin\chi \\ \sin\theta\sin\chi & \cos\theta & \sin\theta\cos\chi \\ \cos\theta\sin\chi & -\sin\theta & \cos\theta\cos\chi \end{pmatrix}, \tag{1.51}$$

has one zero entry ($R_{12} = 0$). The rotations $R_{(yz)}(\theta)$ and $R_{(xz)}(\chi)$ are unitarily represented by the operators

$$U_{(yz)}(\theta) = e^{\theta(a_2^\dagger a_3 - a_3^\dagger a_2)}, \quad \text{and} \quad U_{(xz)}(\chi) = e^{\chi(a_3^\dagger a_1 - a_1^\dagger a_3)},$$

respectively. We now wish to obtain the matrix elements ${}_N\langle\, i,k \mid U_{(yz)(\theta)} \mid m,n\,\rangle_N$ and ${}_N\langle\, i,k \mid U_{(xz)}(\chi) \mid m,n\,\rangle_N$ of these operators. This can be done by adopting the same approach as in Section 3. Details shall be given for the rotation about the x axis. Since $U_{(yz)}(\theta)$ leaves a_1 and $a_1^\dagger$ unchanged and acts trivially on the first quantum number, it is readily seen that

$${}_N\langle\, i,k \mid U_{(yz)}(\theta) \mid m,n\,\rangle_N = \delta_{im}\ {}_J\langle\, k \mid U_{(yz)}(\theta) \mid n\,\rangle_J, \tag{1.52}$$

where

$$\mid \ell\,\rangle_J \equiv \mid i,\ell\,\rangle_N,$$

and with $J = N - i$ and ℓ taking the values $0, 1, \ldots, J$. Given that

$$U_{(yz)}^\dagger(\theta) a_2 U_{(yz)}(\theta) = a_2\cos\theta + a_3\sin\theta,$$

and that $a_2^\dagger$ transforms in the same way, the identity

$$ {}_J\langle\, k \mid a_2^\dagger a_2 U_{(yz)}(\theta) \mid n\,\rangle_J = {}_J\langle\, K \mid U_{(yz)}(\theta) U_{(yz)}^\dagger(\theta) a_2^\dagger a_2 U_{(yz)}(\theta) \mid n\,\rangle_J, $$

yields the recurrence relation

$$ \begin{aligned} k\; {}_J\langle\, k \mid U_{(yz)}(\theta) \mid n\,\rangle_J = {} & \left[n\cos^2\theta + (J-n)\sin^2\theta\right] {}_J\langle\, k \mid U_{(yz)}(\theta) \mid n\,\rangle_J \\ & + \cos\theta\sin\theta \Big[\sqrt{(n+1)(J-n)}\; {}_J\langle\, k \mid U_{(yz)}(\theta) \mid n+1\,\rangle_J \\ & + \sqrt{n(J-n+1)}\; {}_J\langle\, k \mid U_{(yz)}(\theta) \mid n-1\,\rangle_J \Big]. \end{aligned} $$

Now write

$$ {}_J\langle\, k \mid U_{(yz)}(\theta) \mid n\,\rangle_J = {}_J\langle\, k \mid U_{(yz)}(\theta) \mid 0\,\rangle_J \sqrt{\frac{(-1)^n}{n!(-J)_n}}\, \frac{q_n(k)}{\cos^n\theta \sin^n\theta}, \tag{1.53} $$

to find that indeed $q_n(k)$ verifies the three-term recurrence relation (1.49) of the monic Krawtchouk polynomials with $p = \sin^2\theta$. Using the identity

$$ {}_{J-1}\langle\, k \mid U_{(yz)}(\theta) a_2 U_{(yz)}^\dagger(\theta) U_{(yz)}(\theta) \mid 0\,\rangle_J = {}_{J-1}\langle\, k \mid U_{(yz)}(\theta) a_2 \mid n\,\rangle_J = 0, $$

we find the prefactor to obey the two-term recurrence relation

$$ \sqrt{k+1}\cos\theta\; {}_J\langle\, k+1 \mid U_{(yz)}(\theta) \mid 0\,\rangle_J = \sqrt{J-k}\,\sin\theta\; {}_J\langle\, k \mid U_{(yz)}(\theta) \mid 0\,\rangle_J, $$

which has for solution

$$ {}_J\langle\, k \mid U_{(yz)}(\theta) \mid 0\,\rangle_J = {}_J\langle\, 0 \mid U_{(yz)}(\theta) \mid 0\,\rangle_J \sqrt{\frac{J!}{k!(J-k)!}}\tan^k\theta. \tag{1.54} $$

The ground state expectation value is again found from the normalization of the state vectors. One has

$$ \begin{aligned} 1 = {}_J\langle\, 0 \mid U_{(yz)}(\theta) U_{(yz)}^\dagger(\theta) \mid 0\,\rangle_J &= \sum_{k=0}^{J} {}_J\langle\, 0 \mid U_{(yz)}(\theta) \mid k\,\rangle_J\, {}_J\langle\, k \mid U_{(yz)}(\theta) \mid 0\,\rangle_J \\ = |{}_J\langle\, 0 \mid U_{(yz)}(\theta) \mid 0\,\rangle_J|^2 & \sum_{k=0}^{J} \frac{J!}{k!(J-k)!}\tan^k\theta, \end{aligned} $$

which gives

$$ {}_J\langle\, 0 \mid U_{(yz)}(\theta) \mid 0\,\rangle_J = \cos^J\theta. \tag{1.55} $$

Putting (1.52), (1.53), (1.54) and (1.55) together, one finds

$$
\begin{aligned}
&{}_N\langle\, i,k \mid U_{(yz)}(\theta) \mid m,n \,\rangle_N = \delta_{im} \\
&\times \sqrt{\frac{(-1)^n (N-i)!}{k!n!(N-i-k)!(i-N)_n}} \cos^{N-i}\theta \tan^{k+n}\theta\, k_n(k;\sin^2\theta;N-i).
\end{aligned}
\tag{1.56}
$$

The matrix elements of $U_{(xz)}(\chi)$ can be obtained in an identical fashion and one finds

$$
\begin{aligned}
&{}_N\langle\, i,k \mid U_{(xz)}(\chi) \mid m,n \,\rangle_N = \delta_{kn} \\
&\times (-1)^{i+m} \sqrt{\frac{(-1)^m (N-n)!}{i!m!(N-n-i)!(n-N)_m}} \cos^{N-n}\chi \tan^{i+m}\chi\, k_m(i;\sin^2\chi;N-n).
\end{aligned}
\tag{1.57}
$$

Note that in this case, the Kronecker delta involves the second quantum numbers as those are the ones that are left unscathed by the rotation about the y axis.

1.9 The bivariate Krawtchouk-Tratnik as special cases

In this section, the recurrence relations derived in Section 6 will be used to show that the Krawtchouk-Tratnik polynomials are specializations of the Rahman or Krawtchouk-Griffiths polynomials. Aspects of the relation between the two sets of polynomials are also discussed in [21].

The bivariate Krawtchouk-Tratnik polynomials, denoted $K_2(m,n;i,k;\mathfrak{p}_1,\mathfrak{p}_2;N)$, are a family of polynomials introduced by Tratnik in [27]. These polynomials are defined in terms of the univariate Krawtchouk polynomials as follows:

$$
K_2(m,n;i,k;\mathfrak{p}_1,\mathfrak{p}_2;N) = \frac{1}{(-N)_{n+m}} k_m(i;\mathfrak{p}_1;N-n) k_n(k;\frac{\mathfrak{p}_2}{1-\mathfrak{p}_1};N-i),
\tag{1.58}
$$

where $k_n(x;p,J)$ is as in (1.48). They are orthogonal with respect to the trinomial distribution

$$
w_{ik} = \frac{N!}{i!k!(N-i-k)!} \mathfrak{p}_1^i \mathfrak{p}_2^k (1-\mathfrak{p}_1-\mathfrak{p}_2)^{N-i-k}.
$$

Their bispectral properties have been determined by Geronimo and Iliev in [3] to which the reader is referred (see the Appendix) for details.

Consider at this point the recurrence relations (1.42), (1.43) and take $R_{12}=0$. It is observed that in this case the recurrence relation (1.42) simplifies considerably. Indeed, one has

$$
\begin{aligned}
i\,Q_{m,n}(i,k;N) &= \left[R_{11}^2 m + R_{31}^2(N-m-n)\right] Q_{m,n}(i,k;N) \\
&+ \frac{R_{11}R_{31}R_{33}}{R_{13}}\, m\, Q_{m-1,n}(i,k;N) + \frac{R_{11}R_{31}R_{13}}{R_{33}}\,(N-n-m)Q_{m+1,n}(i,k;N).
\end{aligned} \tag{1.59}
$$

The condition $R_{12}=0$ implies that the matrix elements now verify the orthogonality relation $R_{11}R_{31}+R_{13}R_{33}=0$; hence (1.59) reduces to

$$
\begin{aligned}
i\,Q_{m,n}(i,k;N) &= R_{11}^2\, m\, \left[Q_{m,n}(i,k;N) - Q_{m-1,n}(i,k;N)\right] \\
&\qquad + R_{13}^2(N-m-n)\left[Q_{m,n}(i,k;N) - Q_{m+1,n}(i,k;N)\right].
\end{aligned} \tag{1.60}
$$

Upon comparing the formula (1.60) and the formula of Geronimo and Iliev (1.82a), it is seen that they coincide provided that

$$
\mathfrak{p}_1 = R_{13}^2, \quad \mathfrak{p}_2 = R_{23}^2. \tag{1.61}
$$

One also checks (see Appendix A for more details) that the second relation (1.82b) is recovered under this identification of the parameters. This shows that the Krawtchouk-Tratnik polynomials are a special case of the general bivariate Krawtchouk polynomials $Q_{m,n}(i,k;N)$ where the rotation matrix has one of its entries equal to zero, namely $R_{12}=0$. The Krawtchouk-Tratnik polynomials thus arise when the rotation matrix can be written as the composition of two rotations: one in the plane (yz) and the other in the plane (xz), this explains why the Krawtchouk-Tratnik polynomials only depend on two parameters. In the next section, the addition formulas for the general polynomials $P_{m,n}(i,k;N)$ provided by the group product will give a direct derivation of the Tratnik formula (1.58).

1.10 Addition formulas

In this section, the group product is used to derive an addition formula for the general bivariate Krawtchouk polynomials $P_{m,n}(i,k;N)$. In the special case where the rotation is a product of plane rotations around the x and y axes, the addition formula is used to recover the explicit expression of the Krawtchouk-Tratnik polynomials. In the general case, in which the rotation is a product of three rotations, the addition formula yields an expansion formula of the general polynomials $Q_{m,n}(i,k;N)$ in terms of the Krawtchouk-Tratnik polynomials.

1.10.1 General addition formula

Let A and B be two arbitrary rotation matrices. Their product $C = AB$ is also a rotation matrix. Denote by $U(C)$, $U(A)$ and $U(B)$ the unitary operators representing the rotations C, A and B as specified by (1.15). For a given N, to each of these rotations is associated a system of bivariate Krawtchouk polynomials that can be designated by $P^{(C)}_{m,n}(i,k;N)$, $P^{(A)}_{m,n}(i,k;N)$ and $P^{(B)}_{m,n}(i,k;N)$. Since U defines a representation, one has $U(C) = U(A)U(B)$ and hence it follows that

$$ {}_N\langle\, i,k \mid U(C) \mid m,n\,\rangle_N = \sum_{q+r\leqslant N} {}_N\langle\, i,k \mid U(A) \mid q,r\,\rangle_N \, {}_N\langle\, q,r \mid U(B) \mid m,n\,\rangle_N. \tag{1.62} $$

In terms of the polynomials, this identity amounts to the general addition formula

$$ \left(\frac{W^{(C)}_{i,k;N}}{W^{(A)}_{i,k;N}}\right) P^{(C)}_{m,n}(i,k;N) = \sum_{q+r\leqslant N} W^{(B)}_{q,r;N} P^{(A)}_{q,r}(i,k;N) P^{(B)}_{m,n}(q,r;N). \tag{1.63} $$

1.10.2 The Tratnik expression

The addition property (1.62) of the matrix elements can be used to recover the explicit expression for the bivariate Krawtchouk-Tratnik polynomials. It has already been established that the general polynomials $Q_{m,n}(i,k;N)$ correspond to the Tratnik ones when $R_{12} = 0$. We saw that this occurs when the rotation is of the form $C = AB$ with $A = R_{(yz)}(\theta)$ and $B = R_{(xz)}(\chi)$. Considering the left hand side of (1.62) and using (1.32), it follows that

$$ \begin{aligned} {}_N\langle\, i,k \mid U(C(\theta,\chi)) \mid m,n\,\rangle_N &= \frac{N!\, C_{33}^N}{\sqrt{i!k!(N-i-k)!m!n!(N-m-n)!}} \\ &\times \left(\frac{C_{13}}{C_{33}}\right)^i \left(\frac{C_{23}}{C_{33}}\right)^k \left(\frac{C_{31}}{C_{33}}\right)^m \left(\frac{C_{32}}{C_{33}}\right)^n K_2(m,n;i,k;\sin^2\chi, \sin^2\theta\cos^2\chi;N), \end{aligned} \tag{1.64} $$

where the parameter identification (1.61) has been used and where the corresponding rotation matrix $C(\theta,\chi)$ is given in (1.51). Considering the right hand side of (1.62) and recalling the expressions (1.56), (1.57) for the one-parameter rotations, one finds

$$ \begin{aligned} &\sum_{p+q\leqslant N} {}_N\langle\, i,k \mid U_{(yz)}(\theta) \mid p,q\,\rangle_N \, {}_N\langle\, p,q \mid U_{(xz)}(\chi) \mid m,n\,\rangle_N \\ &= \sqrt{\frac{(-1)^{m+n}(N-i)!(N-n)!}{k!n!i!m!(N-i-k)!(N-n-i)!}} \frac{(-1)^{i+m}\tan^{k+n}\theta\tan^{i+m}\chi}{\cos^{i-N}\theta\cos^{n-N}\chi\sqrt{(i-N)_n(n-N)_m}} \\ &\qquad \times\, k_m(i;\sin^2\chi;N-n)\, k_n(k;\sin^2\theta;N-i). \end{aligned} \tag{1.65} $$

Comparing the expressions (1.64) and (1.65), a short calculation shows that the parameters conspire to give

$$
\begin{aligned}
&K_2(m,n;i,k;\sin^2\chi;\sin^2\theta\cos^2\chi;N)\\
&=\frac{1}{(-N)_{n+m}}\,k_m(i;\sin^2\chi;N-n)k_n(k;\sin^2\theta;N-i),
\end{aligned}
\tag{1.66}
$$

as expected from (1.58) and (1.61). Thus the addition formula (1.62) leads to the explicit expression for the polynomials $Q_{m,n}(i,k)$ when $R_{12}=0$.

1.10.3 Expansion of the general Krawtchouk polynomials in the Krawtchouk-Tratnik polynomials

It is possible to find from (1.62) an expansion formula of the general Krawtchouk polynomials $Q_{m,n}(i,k;N)$ in terms of the Krawtchouk-Tratnik polynomials. To obtain the expansion, one considers the most general rotation R, which can be taken of the form

$$
R(\phi,\theta,\chi)=R_{(xz)}(\phi)R_{(yz)}(\theta)R_{(xz)}(\chi)=R_{(xz)}(\phi)C(\theta,\chi),
$$

where $C(\theta,\chi)$ is given by (1.51) and where $R_{(xz)}(\phi)$ is as in (1.50). The formula (1.62) then yields

$$
\begin{aligned}
&{}_N\langle\, i,k\mid U(R(\phi,\theta,\chi))\mid m,n\,\rangle_N\\
&\qquad=\sum_{p+q\leqslant N}{}_N\langle\, i,k\mid U_{(xz)}(\phi)\mid p,q\,\rangle_N\,{}_N\langle\, p,q\mid U(C(\theta,\chi))\mid m,n\,\rangle_N.
\end{aligned}
$$

The expressions for ${}_N\langle\, p,q\mid U((C(\theta,\chi))\mid m,n\,\rangle_N$ and ${}_N\langle\, i,k\mid U_{(xz)}(\phi)\mid p,q\,\rangle_N$ are given by (1.64) and (1.57), respectively. Using (1.18) and (1.32) to express the matrix elements of $U(R)$ in terms of the general polynomials $Q_{m,n}(i,k;N)$, one obtains the expansion

$$
\begin{aligned}
&Q_{m,n}(i,k;N)=\Omega_{i,k;m,n;N}(\phi,\theta,\chi)\\
&\times\sum_{p=0}^{N-k}\frac{(\tan\phi\sec\theta\tan\chi)^p}{p!}\,k_p(i;\sin^2\phi;N-k)K_2(m,n;p,k;\mathfrak{p}_1,\mathfrak{p}_2;N),
\end{aligned}
\tag{1.67}
$$

where $\mathfrak{p}_1=\sin^2\chi$, $\mathfrak{p}_2=\sin^2\theta\cos^2\chi$, and

$$
\begin{aligned}
&\Omega_{i,k;m,n;N}(\phi,\theta,\chi)\\
&=(-1)^i\tan^i\phi\cos^{N-k}\phi\left(\frac{C_{33}}{R_{33}}\right)^N\left(\frac{R_{33}}{R_{13}}\right)^i\left(\frac{C_{23}R_{33}}{C_{33}R_{23}}\right)^k\left(\frac{C_{31}R_{33}}{C_{33}R_{31}}\right)^m\left(\frac{C_{32}R_{33}}{C_{33}R_{32}}\right)^n.
\end{aligned}
$$

Substituting the formula (1.66) for $K_2(m,n;i,k;\sin^2\chi,\sin^2\theta\cos^2\chi;N)$ and using (1.48) transforms (1.67) in an expression for $Q_{m,n}(i,k;N)$ in terms of hypergeometric series.

1.11 Multidimensional case

We now show in this section how the results obtained thus far can easily be generalized by considering the state vectors of the $d+1$-dimensional harmonic oscillator so as to obtain an algebraic description of the general multivariate Krawtchouk polynomials in d variables that are orthogonal with respect to the multinomial distribution [12].

Consider the Hamiltonian of the $d+1$-dimensional harmonic oscillator

$$H = a_1^\dagger a_1 + a_2^\dagger a_2 + \cdots + a_{d+1}^\dagger a_{d+1},$$

where the operators a_i, $a_i^\dagger$ obey the commutation relations (1.1) with $i,k = 1,\ldots,d+1$. Let $\mathcal{V}_N$ denote the eigensubspaces of the $d+1$-dimensional Hamiltonian H corresponding to the energy eigenvalues $N = 0,1,2,\ldots$. An orthonormal basis for the space $\mathcal{V}_N$ is provided by the vectors

$$|\, n_1,\ldots,n_d\,\rangle_N = |\, n_1,n_2,\ldots,N-n_1-\ldots-n_d\,\rangle. \tag{1.68}$$

The action of the operators a_i, $a_i^\dagger$ on the basis vectors $|\, n_1,\ldots,n_{d+1}\,\rangle$ is identical to the one given in (1.3). Since the Hamiltonian of the $(d+1)$-dimensional oscillator is clearly invariant under $SU(d+1)$ and hence $SO(d+1)$ transformations, it follows that the states (1.68) provide a reducible representation of the rotation group $SO(d+1)$ in $d+1$ dimensions.

Let B be a real $(d+1)\times(d+1)$ antisymmetric matrix ($B^T = -B$) and let $R \in SO(d+1)$ be the rotation matrix related to B by $e^B = R$. One has evidently $R^T R = 1$. Consider now the unitary representation

$$U(R) = \exp\left(\sum_{j,k=1}^{d+1} B_{jk} a_j^\dagger a_k\right), \tag{1.69}$$

which has for parameters the $d(d+1)/2$ independent matrix elements of the matrix B. The transformations of the operators $a_i^\dagger$, a_i under the action of $U(R)$ are given by

$$U(R) a_i U^\dagger(R) = \sum_{k=1}^{d+1} R_{ki} a_k, \qquad U(R) a_i^\dagger U^\dagger(R) = \sum_{k=1}^{d+1} R_{ki} a_k^\dagger.$$

In the same spirit as in Section 3, one can write the matrix elements of the reducible representations of $SO(d+1)$ on the space $\mathcal{V}_N$ of the $(d+1)$-dimensional oscillator eigenstates as follows:

$${}_N\langle\, i_1,\ldots,i_d \,|\, U(R) \,|\, n_1,\ldots,n_d\,\rangle_N = W_{i_1,\ldots,i_d;N} P_{n_1,\ldots,n_d}(i_1,\ldots,i_d;N), \tag{1.70}$$

where $P_{0,\ldots,0}(i_1,\ldots,i_d;N) \equiv 1$ and where

$$W_{i_1,\ldots,i_d;N} = {}_N\langle\, i_1,\ldots,i_d \,|\, U(R) \,|\, 0,\ldots,0 \,\rangle_N.$$

It is straightforward to show (as in Section 3) that

$$W_{i_1,\ldots,i_d;N} = \sqrt{\frac{N!}{i_1!i_2!\ldots i_d!(N-i_1-\ldots-i_d)!}}\, R_{1,d+1}^{i_1} R_{2,d+1}^{i_2} \ldots R_{d+1,d+1}^{N-i_1-\cdots-i_d}.$$

Note that one has

$$\sum_{i_1+\cdots+i_d \leqslant N} W_{i_1,\ldots,i_d;N}^2 = 1,$$

which follows immediately from the multinomial formula and from the orthogonality relation

$$R_{1,d+1}^2 + R_{2,d+1}^2 + \cdots + R_{d+1,d+1}^2 = 1.$$

It is easily verified that $P_{n_1,\ldots,n_d}(i_1,\ldots,i_d;N)$ are polynomials in the discrete variables $i_1,\ldots,i_d$ that are of total degree $n_1+\cdots+n_d$. These polynomials are orthonormal with respect to the multinomial distribution $W_{i_1,\ldots,i_d;N}^2$

$$\sum_{i_1+\cdots+i_d \leqslant N} W_{i_1,\ldots,i_d;N}^2 P_{m_1,\ldots,m_d}(i_1,\ldots,i_d;N) P_{n_1,\ldots,n_d}(i_1,\ldots,i_d;N) = \delta_{n_1 m_1}\cdots\delta_{n_d m_d}.$$

For the monic polynomials $Q_{n_1,\ldots,n_d}(i_1,\ldots,i_d;N)$, one finds for the generating function

$$\begin{aligned}
&\big(1+\sum_{k=1}^{d} z_k\big)^{N-i_1-\cdots-i_d} \prod_{j=1}^{d} \Big(1+\sum_{k=1}^{d} u_{j,k} z_k\Big)^{i_j} \\
&= \sum_{n_1+\cdots+n_d \leqslant N} \binom{N}{n_1,\ldots,n_d} Q_{n_1,\ldots,n_d}(i_1,\ldots,i_d;N) z_1^{n_1}\cdots z_d^{n_d},
\end{aligned}$$

where $\binom{N}{x_1,\ldots,x_d}$ are the mutinomial coefficients

$$\binom{N}{x_1,\ldots,x_d} = \frac{N!}{x_1!\cdots x_d!(N-x_1-\cdots-x_d)!},$$

and where

$$u_{j,k} = \frac{R_{j,k}R_{d+1,d+1}}{R_{j,d+1}R_{d+1,k}}.$$

Deriving the properties of these polynomials for a general d can be done exactly as for $d=2$.

1.12 Conclusion

To summarize we have considered the reducible representations of the rotation group $SO(d+1)$ on the energy eigenspaces of the $(d+1)$-dimensional harmonic isotropic oscillator. We have specialized for the most our discussion to $d=2$ with the understanding that it extends easily. We have shown that the multivariate Krawtchouk polynomials arise as matrix elements of these $SO(d+1)$ representations. This interpretation has brought much clarity on the general theory of these polynomials and in particular on the relation to the Krawtchouk-Tratnik polynomials.

Our main results can equivalently be described as providing the overlap coefficients between two Cartesian bases, one rotated with respect to the other, in which the Schrödinger equation for the 3-dimensional harmonic oscillator separates. This paper therefore also adds to studies of interbasis expansions for the harmonic oscillator and more general systems that have been carried out for instance in [9, 10, 15] and references therein.

More results can be expected from this group-theoretic picture, some technical, some of a more insightful nature. In the first category, it is clear that the defining formula (1.18) is bound to yield an expression for the general multivariable Krawtchouk polynomials in terms of single-variable Krawtchouk polynomials and appropriate Clebsch-Gordan coefficients when the rotation group representation spanned by the basis vectors $|\, m,n\,\rangle_N$ in three dimensions for example, is decomposed into its irreducible components. This will be the object of a forthcoming publication [2]. To illustrate the possibilities in the second category, let us observe that the analysis presented here puts the properties of the multivariable Krawtchouk polynomials in an interesting light if one has in mind generalizations. One can see for instance a path to a q-extension of the multivariate Krawtchouk polynomials. Moreover, understanding that Lie groups will no longer enter the picture in all likelihood, the analysis offers nevertheless an interesting starting point to explore multivariate analogs of the higher level polynomials in the Askey tableau with more parameters than those defined by Tratnik. We hope to report on these related questions in the near future.

1.A Background on multivariate Krawtchouk polynomials

In order to make the paper self-contained, we shall collect in this appendix a number of properties of the multivariate Krawtchouk polynomials that can be found in the literature. We shall furthermore indicate what is the relation between the parameters that have been used in these references and the rotation matrix elements that arise naturally in the algebraic model presented here. We shall adopt (for the most) the notation of Iliev [12] in the following.

d-variable Krawtchouk polynomials

In order to define d-variable Krawtchouk polynomials, the set of 4-tuples $(\nu, P, \widetilde{P}, \mathscr{U})$ is introduced. Here ν is a non-zero number and P, $\widetilde{P}$, $\mathscr{U}$ are square matrices of size $d+1$ with entries satisfying the following conditions:

1. $P = \mathrm{diag}(\eta_0, \eta_1, \ldots, \eta_d)$ and $\widetilde{P} = \mathrm{diag}(\widetilde{\eta}_0, \widetilde{\eta}_1, \ldots, \widetilde{\eta}_d)$ and $\eta_0 = \widetilde{\eta}_0 = 1/\nu$,

2. $\mathscr{U} = (u_{ij})_{0 \leqslant i,j \leqslant d}$ is such that $u_{0,j} = u_{j,0} = 1$ for all $j = 0, \ldots, d$, i.e.

$$\mathscr{U} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & u_{1,1} & u_{1,2} & \cdots & u_{1,d} \\ \vdots & & & & \\ 1 & u_{1,d} & u_{2,d} & \cdots & u_{d,d} \end{pmatrix},$$

3. The following matrix equation holds

$$\nu P \mathscr{U} \widetilde{P} \mathscr{U}^T = I_{d+1}. \tag{1.71}$$

It follows from this definition that

$$\sum_{j=0}^{d} \eta_j = \sum_{j=0}^{d} \widetilde{\eta}_j = 1.$$

Take N to be a positive integer and let $m = (m_1, \ldots, m_d)$ and $\widetilde{m} = (\widetilde{m}_1, \ldots, \widetilde{m}_d)$ with m_i, $\widetilde{m}_i$, $i = 1, \ldots, d$, non-negative integers such that $m_1 + m_2 + \cdots + m_d \leqslant N$, $\widetilde{m}_1 + \widetilde{m}_2 + \cdots + \widetilde{m}_d \leqslant N$.

Following Griffiths [5], the polynomials $Q(m,\tilde{m})$ in the variables $\tilde{m}_i$ with degrees m_i are obtained from the generating function

$$\prod_{i=0}^{d}\Big(1+\sum_{j=1}^{d} u_{i,j}z_j\Big)^{\tilde{m}_i} = \sum_{m_1+\cdots m_d\leqslant N} \frac{N!}{m_0!m_1!m_2!\cdots m_d!} Q(m,\tilde{m}) z_1^{m_1}\cdots z_m^{m_d},$$

where $m_0 = N - m_1 - m_2 - \cdots m_d$ and $\tilde{m}_0 = N - \tilde{m}_1 - \widetilde{m_2} - \cdots \tilde{m}_d$.

Identifying $(n_1,\ldots,n_d)$ with $(m_1,\ldots,m_d)$ and $(i_1,\ldots,i_d)$ with $(\tilde{m}_1,\ldots,\tilde{m}_d)$, the polynomials $P_{n_1,\ldots,n_d}(i_1,\ldots,i_d;N)$ introduced in (1.70) are the polynomials $Q(m,\tilde{m})$ up to a normalization factor.

An explicit formula for $Q(m,\tilde{m})$ in terms of Gel'fand-Aomoto series has been given by Mizukawa and Tanaka [24]:

$$Q(m,\tilde{m}) = \sum_{\{a_{ij}\}} \frac{\prod_{j=1}^{d}(-m_j)_{\sum_{i=1}^{d} a_{ij}} \prod_{i=1}^{d}(-\tilde{m}_i)_{\sum_{j=1}^{d} a_{i,j}}}{(-N)_{\sum_{i,j}^{d} a_{i,j}}} \prod_{i,j=1}^{d} \frac{\omega_{i,j}^{a_{i,j}}}{a_{i,j}!}, \tag{1.72}$$

where $\omega_{ij} = 1 - u_{ij}$, a_{ij} are non-negative integers such that $\sum_{i,j=1}^{d} a_{i,j} \leqslant N$.

Let

$$S_1^2 = \nu P, \qquad S_2^2 = Q, \tag{1.73}$$

and set

$$V = S_1 \mathscr{U} S_2. \tag{1.74}$$

It then follows that

$$VV^T = S_1\mathscr{U}S_2S_2\mathscr{U}^T S_1 = S_1\mathscr{U}Q\mathscr{U}^T S_1 = 1,$$

V is thus an orthogonal matrix and one has $\det V = \pm 1$. By an appropriate choice of the signs of the entries of the matrices S_1 and S_2, one can ensure that $\det(V) = 1$ so that V corresponds to a proper rotation. Consequently, the rotation matrix R providing the parameters for the general polynomials $Q_{m,n}(i,k;N)$ in our picture can be obtained from V by rearranging the rows and columns.

Bivariate case

When $d = 2$, the above formulas have been specialized as follows. The matrix elements u_{ij} have been taken [13] to be parametrized by four numbers p_1, p_2, p_3, p_4 according to

$$u_{11} = 1 - \frac{(p_1+p_2)(p_1+p_3)}{p_1(p_1+p_2+p_3+p_4)} = \frac{p_1p_4 - p_2p_3}{p_1(p_1+p_2+p_3+p_4)}, \tag{1.75a}$$

$$u_{12} = 1 - \frac{(p_1+p_2)(p_2+p_4)}{p_2(p_1+p_2+p_3+p_4)} = \frac{p_2p_3 - p_1p_4}{p_2(p_1+p_2+p_3+p_4)}, \tag{1.75b}$$

$$u_{21} = 1 - \frac{(p_1+p_3)(p_3+p_4)}{p_3(p_1+p_2+p_3+p_4)} = \frac{p_2p_3 - p_1p_4}{p_3(p_1+p_2+p_3+p_4)}, \tag{1.75c}$$

$$u_{22} = 1 - \frac{(p_2+p_4)(p_3+p_4)}{p_4(p_1+p_2+p_3+p_4)} = \frac{p_1p_4 - p_2p_3}{p_4(p_1+p_2+p_3+p_4)}, \tag{1.75d}$$

with η_i and $\widetilde{\eta}_i$ given by

$$\eta_1 = \frac{p_1p_2(p_1+p_2+p_3+p_4)}{(p_1+p_2)(p_1+p_3)(p_2+p_4)}, \quad \eta_2 = \frac{p_3p_4(p_1+p_2+p_3+p_4)}{(p_1+p_3)(p_2+p_4)(p_3+p_4)}, \tag{1.76a}$$

$$\widetilde{\eta}_1 = \frac{p_1p_3(p_1+p_2+p_3+p_4)}{(p_1+p_2)(p_1+p_3)(p_3+p_4)}, \quad \widetilde{\eta}_2 = \frac{p_2p_4(p_1+p_2+p_3+p_4)}{(p_1+p_2)(p_2+p_4)(p_3+p_4)}, \tag{1.76b}$$

and where $\eta_0 = \widetilde{\eta}_0 = 1 - \eta_1 - \eta_2$. The numbers $p_1, \cdots, p_4$ are assumed to be arbitrary apart for certain combinations that would lead to divisions by 0. It is checked that (1.71) is satisfied with these definitions. It is also observed that these parameters are defined up to an arbitrary common factor since the quadruplet $(\gamma p_1, \gamma p_2, \gamma p_3, \gamma p_4)$ with an arbitrary non-zero γ ($\gamma \neq 0$) leads to the same u_{ij} as (p_1, p_2, p_3, p_4). This means that only three of the four parameters p_i are independent. This is related to the fact that 3-dimensional rotations depend at most on 3 independent parameters like the Euler angles for instance. The explicit formula (1.72) thus reduces to

$$Q_{m,n}(\widetilde{m}, \widetilde{n}) = \sum_{i+j+k+\ell \leqslant N} \frac{(-m)_{i+j}(-n)_{k+\ell}(-\widetilde{m})_{i+k}(-\widetilde{n})_{j+\ell}}{i!j!k!\ell!(-N)_{i+j+k+\ell}} \times (1-u_{11})^i(1-u_{21})^j(1-u_{12})^k(1-u_{22})^\ell. \tag{1.77}$$

As for the generating function, it becomes

$$(1+u_{11}z_1+u_{12}z_2)^{\widetilde{m}_1}(1+u_{21}z_1+u_{22}z_2)^{\widetilde{m}_2}(1+z_1+z_2)^{\widetilde{m}_0} = \sum_{m_1+m_2 \leqslant N} \frac{N!}{m_0!m_1!m_2!} Q(m, \widetilde{m}) z_1^{m_1} z_2^{m_2}. \tag{1.78}$$

From the identification (1.39) that brought the generating function (1.38) into the form (1.78), we can express the parameters $p_1, \ldots, p_4$ in terms of the rotation matrix elements.

One observes from (1.75) that

$$\frac{u_{11}}{u_{12}} = -\frac{p_1}{p_2} = \frac{R_{11}R_{32}}{R_{31}R_{12}}, \quad \frac{u_{21}}{u_{22}} = -\frac{p_4}{p_3} = \frac{R_{21}R_{32}}{R_{31}R_{22}}, \tag{1.79a}$$

$$\frac{u_{12}}{u_{21}} = \frac{p_3}{p_2} = \frac{R_{12}R_{23}R_{31}}{R_{13}R_{32}R_{21}}, \quad \frac{u_{11}}{u_{22}} = \frac{p_4}{p_1} = \frac{R_{11}R_{23}R_{31}}{R_{13}R_{31}R_{21}}, \tag{1.79b}$$

whence it is seen that one possible identification satisfying (1.79) is

$$p_1 = \frac{R_{31}}{R_{11}}, \quad p_2 = -\frac{R_{32}}{R_{12}}, \quad p_3 = -\frac{R_{23}R_{31}}{R_{13}R_{21}}, \quad p_4 = \frac{R_{32}R_{23}}{R_{13}R_{22}}. \tag{1.80}$$

One can use the affine latitude noted above to write down a more symmetric parametrization (by multiplying the above parameters by R_{31}) where

$$p_1 = \frac{R_{31}R_{13}}{R_{11}}, \quad p_2 = -\frac{R_{32}R_{13}}{R_{12}}, \quad p_3 = -\frac{R_{23}R_{31}}{R_{21}}, \quad p_4 = \frac{R_{32}R_{23}}{R_{22}}.$$

The expressions for η_1, η_2, $\widetilde{\eta}_1$ and $\widetilde{\eta}_2$ in terms of the rotation matrix elements R_{ij} can also be determined. For instance, one obtains from (1.76) with the help of (1.75), (1.39) and (1.79)

$$\eta_1 = \frac{p_2}{p_2+p_4}\frac{1}{1+u_{11}} = \frac{R_{13}^2 R_{22}R_{31}}{(R_{13}R_{22}-R_{12}R_{23})(R_{13}R_{31}-R_{11}R_{33})}. \tag{1.81}$$

Now given that $\det R = 1$ and $R^{-1} = R^T$, one has $R^T = \operatorname{adj} R$. This yields in particular,

$$R_{31} = R_{12}R_{23} - R_{13}R_{22}, \quad R_{22} = R_{11}R_{33} - R_{13}R_{31},$$

from where we find from (1.81) that $\eta_1 = R_{13}^2$. Similarly one arrives at

$$\eta_2 = R_{23}^2, \quad \widetilde{\eta}_1 = R_{31}^2, \quad \widetilde{\eta}_2 = R_{32}^2,$$

and $\eta_0 = \widetilde{\eta}_0 = 1 - \eta_1 - \eta_2 = R_{33}^2$. This is in keeping with the fact that the polynomials $Q(m, \widetilde{m})$ are orthogonal with respect to the trinomial weight distribution

$$\omega(\widetilde{m}_1, \widetilde{m}_2) = \frac{N!}{\widetilde{m}_1!\widetilde{m_2}!(N-\widetilde{m}_1-\widetilde{m}_2)!}\eta_1^{\widetilde{m}_1}\eta_2^{\widetilde{m}_2}(1-\eta_1-\eta_2)^{N-\widetilde{m}_1-\widetilde{m}_2},$$

that should be compared with formula (1.26).

In the multivariate case, the parameters u_{ij} of the matrix $\mathscr{U}$ are related to those of $SO(d+1)$ rotations according to formulas (1.73) and (1.74). One may verify these relations in the bivariate case with

$$S_1 = \operatorname{diag}(R_{33}, R_{13}, R_{23}), \quad S_2 = \operatorname{diag}(R_{33}, R_{31}, R_{32}), \quad \nu = R_{33},$$

and the entries of $\mathscr{U}$ given by (1.39). A direct calculation shows that

$$V = \frac{1}{\nu} S_1 \mathscr{U} S_2 = \begin{pmatrix} R_{33} & R_{31} & R_{32} \\ R_{13} & R_{11} & R_{12} \\ R_{23} & R_{21} & R_{22} \end{pmatrix},$$

which is the transpose of R after a cyclic permutation of the rows and columns.

Krawtchouk-Tratnik polynomials

Recall that the Krawtchouk-Tratnik polynomials $K_2(m,n;i,k;\mathfrak{p}_1,\mathfrak{p}_2;N)$ were seen to be a special case of the Krawtchouk polynomials in two variables when $R_{12}=0$ (or $R_{21}=0$ as a matter of fact). We note from (1.80) that the parametrization in terms of p_1, p_2, p_3, p_4 becomes singular in these instances. For completeness, let us record here the recurrence relations that are satisfied by these polynomials. They are obtained directly from the formulas given in Appendix A.3 of [3] and read (suppressing the parameters $\mathfrak{p}_1$, $\mathfrak{p}_2$ and N):

$$\begin{aligned} i K_2(m,n;i,k) &= \mathfrak{p}_1(m+n-N)\Big[K_2(m+1,n;i,k)-K_2(m,n;i,k)\Big] \\ &\quad + (1-\mathfrak{p}_1)m\Big[K_2(m,n;i,k)-K_2(m-1,n;i,k)\Big], \end{aligned} \tag{1.82a}$$

$$\begin{aligned} k K_2(m,n;i,k) &= \Big[\frac{\mathfrak{p}_1\mathfrak{p}_2}{1-\mathfrak{p}_1}m + \frac{(1-\mathfrak{p}_1-\mathfrak{p}_2)}{1-\mathfrak{p}_1}n + \mathfrak{p}_2(N-m-n)\Big]K_2(m,n;i,k), \\ &\quad - \frac{\mathfrak{p}_2}{1-\mathfrak{p}_1} m K_2(m-1,n+1;i,k) - \mathfrak{p}_1\frac{(1-\mathfrak{p}_1-\mathfrak{p}_2)}{1-\mathfrak{p}_1} n K_2(m+1,n-1;i,k) \\ &\quad + \mathfrak{p}_2\, m K_2(m-1,n;i,k) + \frac{\mathfrak{p}_1\mathfrak{p}_2}{1-\mathfrak{p}_1}(N-n-m)K_2(m+1,n;i,k) \\ &\quad - (1-\mathfrak{p}_1-\mathfrak{p}_2)\, n K_2(m,n-1;i,k) - \frac{\mathfrak{p}_2}{1-\mathfrak{p}_1}(N-n-m)K_2(m,n+1;i,k). \end{aligned} \tag{1.82b}$$

To check that (1.43) reduces to (1.82b) when $R_{12}=0$, given (1.61), one uses the relations that correspond to $R^TR=1$ in this case, namely

$$\begin{aligned} &R_{11}R_{21}+R_{13}R_{23} = R_{11}R_{31}+R_{13}R_{33} = 0, \qquad R_{11}^2+R_{13}^2 = R_{22}^2+R_{32}^2 = 1, \\ &R_{22}R_{21}+R_{32}R_{31} = R_{22}R_{23}+R_{32}R_{33} = 0, \qquad R_{21}^2+R_{22}^2+R_{23}^2 = R_{13}^2+R_{23}^2+R_{33}^2 = 1, \end{aligned}$$

to find for instance that

$$\begin{aligned} &R_{32}^2 = \frac{1-\mathfrak{p}_2}{1-\mathfrak{p}_1}, \quad R_{22}^2 = \frac{1-\mathfrak{p}_1-\mathfrak{p}_2}{1-\mathfrak{p}_2}, \quad R_{21}^2 = \frac{\mathfrak{p}_1\mathfrak{p}_2}{1-\mathfrak{p}_1}, \\ &R_{11}^2 = 1-\mathfrak{p}_1, \quad R_{33}^2 = 1-\mathfrak{p}_1-\mathfrak{p}_2, \quad R_{31}^2 = \mathfrak{p}_1\frac{(1-\mathfrak{p}_1-\mathfrak{p}_2)}{1-\mathfrak{p}_1}, \end{aligned}$$

and to see that all the factors in (1.43) simplify then to those of (1.82b).

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Chapitre 2

The multivariate Meixner polynomials as matrix elements of $SO(d,1)$ representations on oscillator states

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Abstract. The multivariate Meixner polynomials are shown to arise as matrix elements of unitary representations of the $SO(d,1)$ group on oscillator states. These polynomials depend on d discrete variables and are orthogonal with respect to the negative multinomial distribution. The emphasis is put on the bivariate case for which the $SO(2,1)$ connection is used to derive the main properties of the polynomials: orthogonality relation, raising/lowering relations, generating function, recurrence relations and difference equations as well as explicit expressions in terms of standard (univariate) Krawtchouk and Meixner polynomials. It is explained how these results generalize directly to d variables.

2.1 Introduction

The objective of this paper is to provide a group theoretical interpretation of the multivariate Meixner polynomials and to show how their properties naturally follow from this picture. The Meixner polynomials in d variables will be shown to arise as matrix elements of the reducible unitary representations of the pseudo rotation group $SO(d,1)$ on oscillator states. For simplicity, the emphasis will be placed on the $d = 2$ case, where the bivariate Meixner polynomials occur as matrix elements of $SO(2,1)$ representations. The extension of these results to an arbitrary finite number of variables is direct and shall be presented at the end of the paper.

The standard Meixner polynomials of single discrete variable were defined by Meixner [18] in 1934 as polynomials orthogonal on the negative binomial distribution

$$w^{(\beta)}(x) = \frac{(\beta)_x}{x!}(1-c)^{\beta}c^x, \qquad x = 0,1,\ldots$$

with $\beta > 0$, $0 < c < 1$ and where $(\beta)_x = (\beta)(\beta+1)\cdots(\beta+x-1)$ stands for the Pochhammer symbol [17]. These polynomials possess a number of interesting features such as a self duality property, an explicit expression in terms of the Gauss hypergeometric function, a second order difference equation, etc. [17] and have found numerous applications in combinatorics [1, 5], stochastic processes [15, 16], probability theory [2, 14] and mathematical physics [4, 13]. They also enjoy an algebraic interpretation as they arise in the matrix elements of unitary irreducible representations of the $SU(1,1)$ group [21].

The multivariate Meixner polynomials were first identified by Griffiths in 1975. In his paper [9], Griffiths defined the polynomials through a generating function and gave a proof of their orthogonality with respect to a multivariate generalization of the negative binomial distribution. The same Meixner polynomials were considered by Iliev in [12]. Using generating function arguments, Iliev established the bispectrality of these polynomials, i.e. he gave the recurrence relations and difference equations they satisfy, and also gave an explicit expression for them in terms of Gel'fand-Aomoto hypergeometric series. In both cases, the multivariate Meixner polynomials came in as generalizations of the multivariate Krawtchouk polynomials [8, 11], which are multivariate polynomials orthogonal on the multinomial distribution (see [3, 7] and references therein for additional background on the Krawtchouk polynomials).

Recently in [7, 6], a group theoretical interpretation of the multivariate Krawtchouk polynomials was found in the framework of the $d+1$-dimensional isotropic quantum harmonic oscillator model. More specifically, it was shown that the multivariate Krawtchouk polynomials in d variables arise as matrix elements of reducible unitary representations of the rotation group $SO(d+1)$ on the eigenstates of the $(d+1)$-dimensional isotropic harmonic oscillator. The group theoretical setting allowed to recover in a simple fashion all known properties of the polynomials and led to addition formulas as well as to an explicit expression in terms of standard (univariate) Krawtchouk polynomials. The approach moreover permitted to determine that the multivariate generalization of the Krawtchouk polynomials introduced by Tratnik in [20] are special cases of the general ones associated to $SO(d+1)$.

The algebraic interpretation of the multivariate Meixner polynomials proposed here in terms of the pseudo-orthogonal group $SO(d,1)$ is in a similar spirit. The relevant unitary reducible representations of $SO(d,1)$ will be defined on the eigensubspaces of a $SU(d,1)$-invariant bilinear expression in the creation/annihilation operators of $d+1$ independent harmonic oscillators. This embedding of $\mathfrak{so}(d,1)$ in the Weyl algebra will allow for simple derivations of the known properties of the polynomials and will also lead to new formulas stemming from the group theoretical context. This will provide a cogent underpinning of the multivariate Meixner polynomials.

The paper is organized as follows. In Section 2, the reducible unitary representations of $SO(2,1)$ on the eigenspaces of a $SU(2,1)$-invariant bilinear expression in the creation/annihilation operators of three independent harmonic oscillators are constructed. In Section 3, it is shown that the matrix elements of these representations are given in terms of polynomials in two discrete variables that are orthogonal on the negative trinomial distribution. The unitarity of the representation is used in section 4 to obtain the duality property satisfied by the polynomials. In Section 5, a generating function is obtained and is identified with that of the multivariate Meixner polynomials. In Section 6, the recurrence relations and the difference equations satisfied by the multivariate Meixner polynomials are derived. In Section 7, the matrix elements of natural one-parameter subgroups of $SO(2,1)$ are related to the standard Meixner and Krawtchouk polynomials. In section 8, addition formulas and a number of special cases of interest related to possible parametrizations of $SO(2,1)$ elements are discussed. In particular, these considerations lead to explicit expressions of the multivariate Meixner polynomials in terms of standard (univariate) Meixner and Krawtchouk polynomials. In Section 9, the analysis presented in details for the bivariate case is extended to an arbitrary number of variables, thus establishing that the d-variable Meixner polynomials occur as matrix elements of reducible unitary representations of the $SO(d,1)$ group. A short conclusion follows.

2.2 Representations of $SO(2,1)$ on oscillator states

In this section, the reducible $SO(2,1)$ representations on oscillator states that shall be used in the paper are defined. These representations will be specified on the infinite-dimensional eigensubspaces of a bilinear expression in the creation/annihilation operators of three independent harmonic oscillators.

Let a_i, $a_i^{\dagger}$, $i=1,2,3$ be the generators of the Weyl algebra satisfying the commutation relations

$$[a_i,a_k]=0, \qquad [a_i^{\dagger},a_k^{\dagger}]=0, \qquad [a_i,a_k^{\dagger}]=\delta_{ik}.$$

This algebra has a standard representation on the vectors

$$|\, n_1,n_2,n_3\,\rangle \equiv |\, n_1\,\rangle \otimes |\, n_2\,\rangle \otimes |\, n_3\,\rangle, \quad n_1,n_2,n_3=0,1,\ldots, \tag{2.1}$$

and is defined by the following actions on the factors of the direct product:

$$a_i|\,n_i\,\rangle=\sqrt{n_i}|\,n_i-1\,\rangle,\quad a_i^{\dagger}|\,n_i\,\rangle=\sqrt{n_i+1}|\,n_i+1\,\rangle. \tag{2.2}$$

Consider the Hermitian operator

$$H=a_1^{\dagger}a_1+a_2^{\dagger}a_2-a_3^{\dagger}a_3. \tag{2.3}$$

It is seen that (2.3) differs by a sign from the standard Hamiltonian of the three-dimensional isotropic harmonic oscillator. As opposed to the latter, H does not have a positive definite spectrum. Indeed, it is easily seen that H is diagonal on the oscillator states (2.1) with eigenvalues $E=n_1+n_2-n_3$, that is

$$H|\,n_1,n_2,n_3\,\rangle=E|\,n_1,n_2,n_3\,\rangle.$$

It is obvious from the expression (2.3) that H is invariant under $SU(2,1)$ transformations. We introduce the set of orthonormal basis vectors

$$|\,m,n\,\rangle_{\beta}=|\,m,n,m+n+\beta-1\,\rangle,\qquad m,n=0,1,\ldots \tag{2.4}$$

where $\beta\geqslant 1$ takes integer values. The vectors (2.4) span the infinite-dimensional eigenspace associated to the eigenvalue $E=1-\beta$ of H. These vectors support an irreducible representation of the $SU(2,1)$ group generated by the symmetries of H which are of the form $a_i^{\dagger}a_j$, $a_3^{\dagger}a_3$, a_ia_3 and $a_i^{\dagger}a_3^{\dagger}$ for $i,j=1,2$. In the following, we shall concentrate on the subgroup $SO(2,1)\subset SU(2,1)$ generated by the Hermitian bilinears

$$K_1=i(a_2a_3-a_2^{\dagger}a_3^{\dagger}),\quad K_2=i(a_1^{\dagger}a_3^{\dagger}-a_1a_3),\quad K_3=i(a_1a_2^{\dagger}-a_1^{\dagger}a_2), \tag{2.5}$$

satisfying the $\mathfrak{so}(2,1)$ commutation relations

$$[K_1,K_2]=-iK_3,\quad [K_2,K_3]=iK_1,\quad [K_3,K_1]=iK_2. \tag{2.6}$$

The reducible representations of the $SO(2,1)$ subgroup provided by the vectors (2.4) will be considered. It will prove convenient to use the operators $b_i^{\dagger}$, b_i defined by

$$b_1=a_1,\qquad b_2=a_2,\qquad b_3=a_3^{\dagger} \tag{2.7}$$

instead of the standard creation/annihilation operators a_i, $a_i^{\dagger}$. On the basis (2.4), it is easily seen that one has the actions

$$b_1|\,m,n\,\rangle_{\beta}=\sqrt{m}|\,m-1,n\,\rangle_{\beta+1},\qquad b_1^{\dagger}|\,m,n\,\rangle_{\beta}=\sqrt{m+1}|\,m+1,n\,\rangle_{\beta-1}, \tag{2.8a}$$

$$b_2|\,m,n\,\rangle_{\beta}=\sqrt{n}|\,m,n-1\,\rangle_{\beta+1},\qquad b_2^{\dagger}|\,m,n\,\rangle_{\beta}=\sqrt{n+1}|\,m,n+1\,\rangle_{\beta-1}, \tag{2.8b}$$

$$b_3|\,m,n\,\rangle_{\beta}=\sqrt{m+n+\beta}|\,m,n\,\rangle_{\beta+1},\qquad b_3^{\dagger}|\,m,n\,\rangle_{\beta}=\sqrt{m+n+\beta-1}|\,m,n\,\rangle_{\beta-1}. \tag{2.8c}$$

It is directly checked that the actions (2.8) define an infinite-dimensional representation of the Lie algebra (2.6) on the oscillator states (2.4). The assertion that this representation is *reducible* follows from the fact that the $\mathfrak{so}(2,1)$ Casimir operator $C = -K_1^2 - K_2^2 + K_3^2$ does not act as a multiple of the identity on (2.4).

We use the following notation. Let Λ be an orthochronous transformation of $SO(2,1)$; this means that

$$\Lambda^t \eta \Lambda = \eta, \qquad \Lambda_{33} \geqslant 1, \tag{2.9}$$

where A^t denotes the transpose matrix of A and where $\eta = \text{diag}(1,1,-1)$. Consider the unitary representation defined by

$$\mathscr{F}(\Lambda) = \exp\left(\sum_{i,k=1}^{3} B_{ik} b_i^{\dagger} b_k\right), \tag{2.10}$$

where $B_{ik} = -B_{ki}$. One has of course $\mathscr{F}(\Lambda)\mathscr{F}^{\dagger}(\Lambda) = 1$. The transformations of the generators $b_i^{\dagger}$, b_i under the action of $\mathscr{F}(\Lambda)$ are given by

$$\mathscr{F}(\Lambda) b_i^{\dagger} \mathscr{F}^{\dagger}(\Lambda) = \sum_{k=1}^{3} \widetilde{\Lambda}_{ik} b_k^{\dagger}, \qquad \mathscr{F}(\Lambda) b_i \mathscr{F}^{\dagger}(\Lambda) = \sum_{k=1}^{3} \widetilde{\Lambda}_{ik} b_k, \tag{2.11}$$

where $\widetilde{\Lambda} = \eta \Lambda^t \eta$ stands for the inverse matrix of Λ: $\Lambda\widetilde{\Lambda} = 1$. It is directly checked that $\mathscr{F}(\Lambda)$ satisfies

$$\mathscr{F}(\Lambda\Delta) = \mathscr{F}(\Lambda)\mathscr{F}(\Delta), \qquad \Lambda, \Delta \in SO(2,1), \tag{2.12}$$

as should be for a group representation.

2.3 The representation matrix elements as orthogonal polynomials

In this section, it is shown that the matrix elements of the $SO(2,1)$ unitary representation defined above are expressed in terms of orthogonal polynomials in two discrete variables.

The matrix elements of the unitary operator (2.10) in the oscillator basis (2.4) can be written as

$${}_{\beta}\langle\, i,k \mid \mathscr{F}(\Lambda) \mid m,n \,\rangle_{\beta} = W_{i,k}^{(\beta)} M_{m,n}^{(\beta)}(i,k), \tag{2.13}$$

where $M_{0,0}^{(\beta)}(i,k) = 1$ and where

$$W_{i,k}^{(\beta)} = {}_{\beta}\langle\, i,k \mid \mathscr{F}(\Lambda) \mid 0,0 \,\rangle_{\beta}. \tag{2.14}$$

For notational ease, the explicit dependence of $\mathscr{F}(\Lambda)$ on Λ will be omitted at times.

2.3.1 Calculation of $W_{i,k}^{(\beta)}$

To derive the explicit expression of the amplitude $W_{i,k}^{(\beta)}$, one first observes that

$$
{}_{\beta+1}\langle\, i,k \,|\, \mathscr{F}\, b_j \,|\, 0,0\,\rangle_{\beta} = 0,
$$

for $j = 1,2$. Since ${}_{\beta+1}\langle\, i,k \,|\, \mathscr{F} b_j \,|\, 0,0\,\rangle_{\beta} = {}_{\beta}\langle\, i,k \,|\, \mathscr{F} b_j \mathscr{F}^{\dagger}\mathscr{F} \,|\, 0,0\,\rangle_{\beta}$, one obtains, using (2.11), the following system of difference equations for $W_{i,k}^{\beta}$:

$$
\widetilde{\Lambda}_{11}\sqrt{i+1}\,W_{i+1,k}^{(\beta)} + \widetilde{\Lambda}_{12}\sqrt{k+1}\,W_{i,k+1}^{(\beta)} + \widetilde{\Lambda}_{13}\sqrt{i+k+\beta}\,W_{i,k}^{(\beta)} = 0, \tag{2.15a}
$$

$$
\widetilde{\Lambda}_{21}\sqrt{i+1}\,W_{i+1,k}^{(\beta)} + \widetilde{\Lambda}_{22}\sqrt{k+1}\,W_{i,k+1}^{(\beta)} + \widetilde{\Lambda}_{23}\sqrt{i+k+\beta}\,W_{i,k}^{(\beta)} = 0. \tag{2.15b}
$$

Using the fact that $\widetilde{\Lambda}\eta\widetilde{\Lambda}^{t}\eta = 1$, it is readily seen that the solution to the system (2.15) is of the form

$$
W_{i,k}^{(\beta)} = \sqrt{\frac{(\beta)_{i+k}}{i!k!}}\left(-\frac{\widetilde{\Lambda}_{31}}{\widetilde{\Lambda}_{33}}\right)^{i}\left(-\frac{\widetilde{\Lambda}_{32}}{\widetilde{\Lambda}_{33}}\right)^{k} W_{0,0}^{(\beta)},
$$

where $W_{0,0}^{(\beta)} = {}_{\beta}\langle\, 0,0 \,|\, \mathscr{F} \,|\, 0,0\,\rangle_{\beta}$. The constant $W_{0,0}^{(\beta)}$ can be obtained from the normalization condition

$$
1 = {}_{\beta}\langle\, 0,0 \,|\, \mathscr{F}^{\dagger}\mathscr{F} \,|\, 0,0\,\rangle_{\beta} = \sum_{i,k\geqslant 0} {}_{\beta}\langle\, i,k \,|\, \mathscr{F} \,|\, 0,0\,\rangle_{\beta\beta}\langle\, 0,0 \,|\, \mathscr{F}^{\dagger} \,|\, i,k\,\rangle_{\beta} = \sum_{i,k\geqslant 0} |W_{i,k}^{(\beta)}|^{2}.
$$

One can then use the formula

$$
(1 - z_1 - z_2)^{-\beta} = \sum_{i,k} \frac{(\beta)_{i+k}}{i!k!} z_1^{i} z_2^{k},
$$

which holds provided that $|z_1| + |z_2| < 1$. It is directly seen from (2.9) that this condition is identically satisfied and hence one finds that $W_{0,0}^{(\beta)} = [\widetilde{\Lambda}_{33}]^{-\beta}$. In terms of the matrix elements of Λ, the complete expression for the amplitude $W_{i,k}^{(\beta)}$ is thus found to be

$$
W_{i,k}^{(\beta)} = \sqrt{\frac{(\beta)_{i+k}}{i!k!}}\,(\Lambda_{33})^{-\beta-i-k}\,\Lambda_{13}^{i}\Lambda_{23}^{k}. \tag{2.16}
$$

2.3.2 Raising relations

To show that the $M_{m,n}^{(\beta)}(i,k)$ appearing in the matrix elements (2.13) are polynomials of total degree $m+n$ in the two variables i, k, one can examine their raising relations, which are obtained as follows. One has on the one hand

$$
{}_{\beta}\langle\, i,k \,|\, \mathscr{F}\, b_1^{\dagger} \,|\, m,n\,\rangle_{\beta+1} = \sqrt{m+1}\,W_{i,k}^{(\beta)} M_{m+1,n}^{(\beta)}(i,k),
$$

$$
{}_{\beta}\langle\, i,k \,|\, \mathscr{F}\, b_2^{\dagger} \,|\, m,n\,\rangle_{\beta+1} = \sqrt{n+1}\,W_{i,k}^{(\beta)} M_{m,n+1}^{(\beta)}(i,k).
$$

On the other hand, using (2.11), one has

$$
{}_\beta\langle i,k\,|\,\mathscr{F}\,b_1^\dagger\,|\,m,n\,\rangle_{\beta+1} = {}_\beta\langle i,k\,|\,\mathscr{F}\,b_1^\dagger\mathscr{F}^\dagger\mathscr{F}\,|\,m,n\,\rangle_{\beta+1} = \sum_{j=1}^{3}\widetilde{\Lambda}_{1j}\,{}_\beta\langle i,k\,|\,b_j^\dagger\mathscr{F}\,|\,m,n\,\rangle_{\beta+1},
$$

$$
{}_\beta\langle i,k\,|\,\mathscr{F}\,b_2^\dagger\,|\,m,n\,\rangle_{\beta+1} = {}_\beta\langle i,k\,|\,\mathscr{F}\,b_2^\dagger\mathscr{F}^\dagger\mathscr{F}\,|\,m,n\,\rangle_{\beta+1} = \sum_{j=1}^{3}\widetilde{\Lambda}_{2j}\,{}_\beta\langle i,k\,|\,b_j^\dagger\mathscr{F}\,|\,m,n\,\rangle_{\beta+1}.
$$

Upon combining the above relations, one obtains the raising relations

$$
\begin{aligned}
\sqrt{\beta(m+1)}\,M^{(\beta)}_{m+1,n}(i,k) = &\frac{\Lambda_{11}}{\Lambda_{13}}\,i\,M^{(\beta+1)}_{m,n}(i-1,k) + \frac{\Lambda_{21}}{\Lambda_{23}}\,k\,M^{(\beta+1)}_{m,n}(i,k-1)\\
&-\frac{\Lambda_{31}}{\Lambda_{33}}(i+k+\beta)M^{(\beta+1)}_{m,n}(i,k),
\end{aligned}
\tag{2.17a}
$$

$$
\begin{aligned}
\sqrt{\beta(n+1)}\,M^{(\beta)}_{m,n+1}(i,k) = &\frac{\Lambda_{12}}{\Lambda_{13}}\,i\,M^{(\beta+1)}_{m,n}(i-1,k) + \frac{\Lambda_{22}}{\Lambda_{23}}\,k\,M^{(\beta+1)}_{m,n}(i,k-1)\\
&-\frac{\Lambda_{32}}{\Lambda_{33}}(i+k+\beta)M^{(\beta+1)}_{m,n}(i,k).
\end{aligned}
\tag{2.17b}
$$

By definition, one has $M^{(\beta)}_{-1,n}(i,k) = M^{(\beta)}_{m,-1}(i,k) = 0$ and $M^{(\beta)}_{0,0}(i,k) = 1$. Therefore, the formulas (2.17) can be used to construct $M^{(\beta)}_{m,n}(i,k)$ from $M^{(\beta)}_{0,0}(i,k)$ iteratively. Furthermore, it is observed that $M^{(\beta)}_{m,n}(i,k)$ are polynomials of total degree $m+n$ in the discrete variables i and k.

2.3.3 Orthogonality Relation

It follows from the unitarity of the representation (2.10) and the orthonormality of the oscillator states that the polynomials $M^{(\beta)}_{m,n}(i,k)$ are orthogonal with respect to the negative trinomial distribution. Indeed, one has

$$
{}_\beta\langle m',n'\,|\,\mathscr{F}^\dagger\mathscr{F}\,|\,m,n\,\rangle_\beta = \sum_{i,k\geqslant 0}{}_\beta\langle i,k\,|\,\mathscr{F}\,|\,m,n\,\rangle_{\beta\,\beta}\langle m',n'\,|\,\mathscr{F}^\dagger\,|\,i,k\,\rangle_\beta = \delta_{mm'}\delta_{nn'}.
$$

Upon using (2.13), one finds that the polynomials satisfy the orthogonality relation

$$
\sum_{i,k\geqslant 0} w^{(\beta)}_{i,k}\,M^{(\beta)}_{m,n}(i,k)\,M^{(\beta)}_{m',n'}(i,k) = \delta_{mm'}\delta_{nn'},
\tag{2.18}
$$

where $w^{(\beta)}_{i,k}$ is the negative trinomial distribution

$$
w^{(\beta)}_{i,k} = \frac{(\beta)_{i+k}}{i!k!}(1-c_1-c_2)^\beta c_1^i c_2^k,
\tag{2.19}
$$

with

$$
c_1 = \frac{\Lambda_{13}^2}{\Lambda_{33}^2}, \qquad c_2 = \frac{\Lambda_{23}^2}{\Lambda_{33}^2}.
$$

Recall that in view of (2.9), the condition $|c_1|+|c_2|<1$ is identically satisfied for orthochronous elements of $SO(2,1)$.

2.3.4 Lowering Relations

Lowering relations for the polynomials $M^{(\beta)}_{m,n}(i,k)$ can also be obtained. To this end, one first considers the matrix elements ${}_{\beta}\langle\, i,k\,|\,\mathscr{F}\, b_j\,|\, m,n\,\rangle_{\beta-1}$ with $j=1,2$. One has on the one hand

$$
\begin{aligned}
{}_{\beta}\langle\, i,k\,|\,\mathscr{F}\, b_1\,|\, m,n\,\rangle_{\beta-1} &= \sqrt{m}\, W^{(\beta)}_{i,k}\, M^{(\beta)}_{m-1,n}(i,k),\\
{}_{\beta}\langle\, i,k\,|\,\mathscr{F}\, b_2\,|\, m,n\,\rangle_{\beta-1} &= \sqrt{n}\, W^{(\beta)}_{i,k}\, M^{(\beta)}_{m,n-1}(i,k),
\end{aligned}
$$

and on the other hand

$$
{}_{\beta}\langle\, i,k\,|\,\mathscr{F}\, b_j\,|\, m,n\,\rangle_{\beta-1} = {}_{\beta}\langle\, i,k\,|\,\mathscr{F}\, b_j\, \mathscr{F}^{\dagger}\mathscr{F}\,|\, m,n\,\rangle_{\beta-1} = \sum_{\ell=1}^{3} \widetilde{\Lambda}_{j\ell}\, {}_{\beta}\langle\, i,k\,|\, b_{\ell}\mathscr{F}\,|\, m,n\,\rangle_{\beta-1}.
$$

Upon comparing the two expressions, a simple calculation yields

$$
\begin{aligned}
\sqrt{\frac{m}{\beta-1}}\, M^{(\beta)}_{m-1,n}(i,k) =& \Lambda_{11}\Lambda_{13}\big[M^{(\beta-1)}_{m,n}(i+1,k) - M^{(\beta-1)}_{m,n}(i,k)\big]\\
&+ \Lambda_{21}\Lambda_{23}\big[M^{(\beta-1)}_{m,n}(i,k+1) - M^{(\beta-1)}_{m,n}(i,k)\big],
\end{aligned}
\tag{2.20a}
$$

$$
\begin{aligned}
\sqrt{\frac{n}{\beta-1}}\, M^{(\beta)}_{m,n-1}(i,k) =& \Lambda_{12}\Lambda_{13}\big[M^{(\beta-1)}_{m,n}(i+1,k) - M^{(\beta-1)}_{m,n}(i,k)\big]\\
&+ \Lambda_{22}\Lambda_{23}\big[M^{(\beta-1)}_{m,n}(i,k+1) - M^{(\beta-1)}_{m,n}(i,k)\big].
\end{aligned}
\tag{2.20b}
$$

2.4 Duality

In this section, a duality relation under the exchange of the degrees m, n and the variables i, k is derived for the multivariate polynomials $M^{(\beta)}_{m,n}(i,k)$. For the monic polynomials $R^{(\beta)}_{m,n}(i,k)$, which are obtained from the $M^{(\beta)}_{m,n}(i,k)$ by a normalization, this duality property is seen to take a particularly simple form.

The duality relation for the polynomials $M^{(\beta)}_{m,n}(i,k)$ is found by considering the matrix elements ${}_{\beta}\langle\, i,k\,|\,\mathscr{F}^{\dagger}(\Lambda)\,|\, m,n\,\rangle_{\beta}$ from two different points of view. First, one writes

$$
{}_{\beta}\langle\, i,k\,|\,\mathscr{F}^{\dagger}(\Lambda)\,|\, m,n\,\rangle_{\beta} = \widetilde{W}^{(\beta)}_{i,k}\widetilde{M}^{(\beta)}_{m,n}(i,k),
\tag{2.21}
$$

where $\widetilde{W}^{(\beta)}_{i,k} = {}_{\beta}\langle\, i,k\,|\,\mathscr{F}^{\dagger}(\Lambda)\,|\, 0,0\,\rangle_{\beta}$ and $\widetilde{M}^{(\beta)}_{0,0}(i,k)=1$. On account of the fact that $\mathscr{F}^{\dagger}(\Lambda) = \mathscr{F}(\Lambda^{-1}) = \mathscr{F}(\eta\Lambda^{t}\eta)$, it follows that

$$
\widetilde{W}^{(\beta)}_{i,k} = \sqrt{\frac{(\beta)_{i+k}}{i!k!}}\big(\Lambda_{33}\big)^{-\beta-i-k}(-\Lambda_{31})^{i}(-\Lambda_{32})^{k},
$$

and that $\widetilde{M}^{(\beta)}_{m,n}(i,k)$ are the polynomials corresponding to the matrix $\widetilde{\Lambda} = \eta \Lambda^t \eta$. Second, one writes

$$\begin{aligned} {}_\beta\langle\, i,k \mid \mathscr{F}^\dagger(\Lambda) \mid m,n \,\rangle_\beta &= \overline{{}_\beta\langle\, m,n \mid \mathscr{F}(\Lambda) \mid i,k \,\rangle_\beta} \\ &= {}_\beta\langle\, m,n \mid \mathscr{F}(\Lambda) \mid i,k \,\rangle_\beta = W^{(\beta)}_{m,n}\, M^{(\beta)}_{i,k}(m,n), \end{aligned} \tag{2.22}$$

where $\overline{x}$ stands for complex conjugation and where the reality of the matrix elements has been used. Upon comparing (2.21) and (2.22), one obtains the duality relation

$$M^{(\beta)}_{i,k}(m,n) = (-1)^{i+k} \sqrt{\frac{(\beta)_{i+k}}{i!k!} \frac{m!n!}{(\beta)_{m+n}}}\, \frac{\Lambda_{33}^{m+n} \Lambda_{31}^{i} \Lambda_{32}^{k}}{\Lambda_{33}^{i+k} \Lambda_{13}^{m} \Lambda_{23}^{n}}\, \widetilde{M}^{(\beta)}_{m,n}(i,k). \tag{2.23}$$

It is opportune here to introduce the monic polynomials $R^{(\beta)}_{m,n}(i,k)$ defined by

$$M^{(\beta)}_{m,n}(i,k) = (-1)^{m+n} \sqrt{\frac{(\beta)_{m+n}}{m!n!}}\, \frac{\Lambda_{31}^{m} \Lambda_{32}^{n}}{\Lambda_{33}^{m+n}}\, R^{(\beta)}_{m,n}(i,k).$$

In terms of these polynomials, the duality relation (2.23) has the attractive expression

$$R^{(\beta)}_{i,k}(m,n) = \widetilde{R}^{(\beta)}_{m,n}(i,k), \tag{2.24}$$

where the parameters appearing in the polynomials $\widetilde{R}^{(\beta)}_{m,n}(i,k)$ are those of the inverse matrix $\widetilde{\Lambda} = \eta \Lambda^t \eta$.

2.5 Generating function and hypergeometric expression

In this section, generating functions for the multivariate polynomials $M^{(\beta)}_{m,n}(i,k)$ and $R^{(\beta)}_{m,n}(i,k)$ are obtained using the group product. The generating function derived for the polynomials $R^{(\beta)}_{m,n}(i,k)$ is shown to coincide with the one defining the multivariate Meixner polynomials, which establishes that the polynomials $R^{(\beta)}_{m,n}(i,k)$ are precisely those defined in [9] and [12]. Using the results of [12], an explicit expression of the polynomials $R^{(\beta)}_{m,n}(i,k)$ in terms of Gel'fand-Aomoto hypergeometric series is given.

2.5.1 Generating function

Let $\Delta \in SO(2,1)$ be an arbitrary group element and consider the following generating function:

$$G(\Delta) = \sum_{m,n \geqslant 0} \sqrt{\frac{(\beta)_{m+n}}{m!n!}}\, \Delta_{33}^{-\beta-m-n} \Delta_{13}^{m} \Delta_{23}^{n}\, W^{(\beta)}_{i,k}\, M^{(\beta)}_{m,n}(i,k). \tag{2.25}$$

Given (2.13), one obviously has

$$G(\Delta)=\sum_{m,n\geqslant 0}\sqrt{\frac{(\beta)_{m+n}}{m!n!}}\Delta_{33}^{-\beta-m-n}\Delta_{13}^{m}\Delta_{23}^{n}\ {}_{\beta}\langle\, i,k\mid \mathscr{F}(\Lambda)\mid m,n\,\rangle_{\beta}.$$

In view of (2.14) and (2.16), one finds that the above expression for $G(\Delta)$ can be written in the form

$$G(\Delta)={}_{\beta}\langle\, i,k\mid \mathscr{F}(\Lambda)\mathscr{F}(\Delta)\mid 0,0\,\rangle_{\beta}={}_{\beta}\langle\, i,k\mid \mathscr{F}(\Lambda\Delta)\mid 0,0\,\rangle_{\beta}.$$

Upon using again the expression (2.16), one arrives at

$$G(\Delta)=\sqrt{\frac{(\beta)_{i+k}}{i!k!}}[(\Lambda\Delta)_{33}]^{-\beta-i-k}[(\Lambda\Delta)_{13}]^{i}[(\Lambda\Delta)_{23}]^{k}, \tag{2.26}$$

where $(\Lambda\Delta)_{ij}$ are the matrix elements of the matrix $\Lambda\cdot\Delta$. Comparing (2.25) with (2.26) using the expression (2.16), one arrives at the expression

$$\begin{aligned}&\left(\Delta_{33}+\frac{\Delta_{13}\Lambda_{11}}{\Lambda_{13}}+\frac{\Delta_{23}\Lambda_{12}}{\Lambda_{13}}\right)^{i}\left(\Delta_{33}+\frac{\Delta_{13}\Lambda_{21}}{\Lambda_{23}}+\frac{\Delta_{23}\Lambda_{22}}{\Lambda_{23}}\right)^{k}\\&\times\left(\Delta_{33}+\frac{\Delta_{13}\Lambda_{31}}{\Lambda_{33}}+\frac{\Delta_{23}\Lambda_{32}}{\Lambda_{33}}\right)^{-\beta-i-k}=\sum_{m,n\geqslant 0}\sqrt{\frac{(\beta)_{m+n}}{m!n!}}(\Delta_{33})^{-\beta-m-n}\Delta_{13}^{m}\Delta_{23}^{n}M_{m,n}^{(\beta)}(i,k).\end{aligned}$$

Since Δ is arbitrary, one can choose the parametrization

$$\Delta_{13}=\delta_1,\qquad \Delta_{23}=\delta_2,\quad \Delta_{33}=1,$$

which gives the generating function for the polynomials $M_{m,n}^{(\beta)}(i,k)$

$$\begin{aligned}&\left(1+\frac{\Lambda_{11}}{\Lambda_{13}}\delta_1+\frac{\Lambda_{12}}{\Lambda_{13}}\delta_2\right)^{i}\left(1+\frac{\Lambda_{21}}{\Lambda_{23}}\delta_1+\frac{\Lambda_{22}}{\Lambda_{23}}\delta_2\right)^{k}\\&\times\left(1+\frac{\Lambda_{31}}{\Lambda_{33}}\delta_1+\frac{\Lambda_{32}}{\Lambda_{33}}\delta_2\right)^{-\beta-i-k}=\sum_{m,n\geqslant 0}\sqrt{\frac{(\beta)_{m+n}}{m!n!}}M_{m,n}^{(\beta)}(i,k)\delta_1^{m}\delta_2^{n}.\end{aligned} \tag{2.27}$$

Upon choosing instead the parametrization

$$\Delta_{13}=-\frac{\Lambda_{33}}{\Lambda_{31}}z_1,\quad \Delta_{23}=-\frac{\Lambda_{33}}{\Lambda_{32}}z_2,\quad \Delta_{33}=1,$$

one finds the following generating function for the monic polynomials $R_{m,n}^{(\beta)}(i,k)$:

$$\begin{aligned}&(1-z_1-z_2)^{-\beta-i-k}(1-u_{11}z_1-u_{12}z_2)^{i}(1-u_{21}z_1-u_{22}z_2)^{k}\\&=\sum_{m,n\geqslant 0}\frac{(\beta)_{m+n}}{m!n!}R_{m,n}^{(\beta)}(i,k)z_1^{m}z_2^{n},\end{aligned} \tag{2.28}$$

where

$$u_{11}=\frac{\Lambda_{11}\Lambda_{33}}{\Lambda_{13}\Lambda_{31}},\qquad u_{12}=\frac{\Lambda_{12}\Lambda_{33}}{\Lambda_{13}\Lambda_{32}},\qquad u_{21}=\frac{\Lambda_{21}\Lambda_{33}}{\Lambda_{23}\Lambda_{31}},\qquad u_{22}=\frac{\Lambda_{22}\Lambda_{33}}{\Lambda_{23}\Lambda_{32}}. \tag{2.29}$$

The generating function (2.28) is identical to the one considered in [9, 12] which is taken to define the bivariate Meixner polynomials. Thus it follows that the monic polynomials $R_{m,n}^{(\beta)}(i,k)$ are the bivariate Meixner polynomials. Note that here $\beta>1$ is an integer, but it is directly seen that the polynomials $R_{m,n}^{(\beta)}(i,k)$ can be defined for $\beta\in\mathbb{R}$.

2.5.2 Hypergeometric expression

In [12], Iliev obtained an explicit expression for the Meixner polynomials $R^{(\beta)}_{m,n}(i,k)$ in terms of a Gel'fand-Aomoto hypergeometric series. Using his result, one writes

$$
\begin{aligned}
R^{(\beta)}_{m,n}(i,k) &= \sum_{\mu,\nu,\rho,\sigma\geqslant 0} \frac{(-m)_{\mu+\nu}(-n)_{\rho+\sigma}(-i)_{\mu+\rho}(-k)_{\nu+\sigma}}{\mu!\nu!\rho!\sigma!(\beta)_{\mu+\nu+\rho+\sigma}} \\
&\times (1-u_{11})^{\mu}(1-u_{21})^{\nu}(1-u_{12})^{\rho}(1-u_{22})^{\sigma},
\end{aligned}
\tag{2.30}
$$

where the parameters u_{ij} are given by (2.29). Since $(-m)_k = 0$ for $k > m$, it is clear that the summation in (2.30) is finite.

2.6 Recurrence relations and difference equations

In this section, the recurrence relations and the difference equations satisfied by the bivariate Meixner polynomials $M^{(\beta)}_{m,n}(i,k)$ and $R^{(\beta)}_{m,n}(i,k)$ are derived. These relations have been obtained by Iliev in [12]. It is however interesting to see how easily these relations follow from the group-theoretical interpretation.

2.6.1 Recurrence relations

To obtain the recurrence relations satisfied by the Meixner polynomials, one considers the matrix elements ${}_{\beta}\langle i,k \,|\, b_j^{\dagger} b_j \mathscr{F} \,|\, m,n \rangle_{\beta}$ for $j = 1,2$. On the one hand one has

$$
\begin{aligned}
{}_{\beta}\langle i,k \,|\, b_1^{\dagger} b_1 \mathscr{F} \,|\, m,n \rangle_{\beta} &= i\; {}_{\beta}\langle i,k \,|\, \mathscr{F} \,|\, m,n \rangle_{\beta}, \\
{}_{\beta}\langle i,k \,|\, b_2^{\dagger} b_2 \mathscr{F} \,|\, m,n \rangle_{\beta} &= k\; {}_{\beta}\langle i,k \,|\, \mathscr{F} \,|\, m,n \rangle_{\beta},
\end{aligned}
\tag{2.31}
$$

and on the other hand

$$
\begin{aligned}
&{}_{\beta}\langle i,k \,|\, b_j^{\dagger} b_j \mathscr{F} \,|\, m,n \rangle_{\beta} = {}_{\beta}\langle i,k \,|\, \mathscr{F}\mathscr{F}^{\dagger} b_j^{\dagger} b_j \mathscr{F} \,|\, m,n \rangle_{\beta} \\
&\sum_{\ell=1}^{3}\sum_{r=1}^{3} \Lambda_{j\ell}\Lambda_{jr}\; {}_{\beta}\langle i,k \,|\, \mathscr{F}\, b_{\ell}^{\dagger} b_r \,|\, m,n \rangle_{\beta}.
\end{aligned}
\tag{2.32}
$$

Upon comparing (2.31) with (2.32), one directly obtains

$$
\begin{aligned}
i\,M^{(\beta)}_{m,n}(i,k) &= \Big[m\Lambda_{11}^2 + n\Lambda_{12}^2 + (m+n+\beta)\Lambda_{13}^2\Big] M^{(\beta)}_{m,n}(i,k) \\
&+ \Lambda_{11}\Lambda_{12}\Big[\sqrt{m(n+1)}M^{(\beta)}_{m-1,n+1}(i,k) + \sqrt{n(m+1)}M^{(\beta)}_{m+1,n-1}(i,k)\Big] \\
&+ \Lambda_{11}\Lambda_{13}\Big[\sqrt{m(m+n+\beta-1)}M^{(\beta)}_{m-1,n}(i,k) + \sqrt{(m+1)(m+n+\beta)}M^{(\beta)}_{m+1,n}(i,k)\Big] \\
&+ \Lambda_{12}\Lambda_{13}\Big[\sqrt{n(m+n+\beta-1)}M^{(\beta)}_{m,n-1}(i,k) + \sqrt{(n+1)(n+m+\beta)}M^{(\beta)}_{m,n+1}(i,k)\Big]
\end{aligned}
\tag{2.33a}
$$

$$
\begin{aligned}
k\,M^{(\beta)}_{m,n}(i,k) &= \Big[m\Lambda_{21}^2 + n\Lambda_{22}^2 + (m+n+\beta)\Lambda_{23}^2\Big]M^{(\beta)}_{m,n}(i,k) \\
&+ \Lambda_{21}\Lambda_{22}\Big[\sqrt{m(n+1)}M^{(\beta)}_{m-1,n+1}(i,k) + \sqrt{n(m+1)}M^{(\beta)}_{m+1,n-1}(i,k)\Big] \\
&+ \Lambda_{21}\Lambda_{23}\Big[\sqrt{m(m+n+\beta-1)}M^{(\beta)}_{m-1,n}(i,k) + \sqrt{(m+1)(m+n+\beta)}M^{(\beta)}_{m+1,n}(i,k)\Big] \\
&+ \Lambda_{22}\Lambda_{23}\Big[\sqrt{n(m+n+\beta-1)}M^{(\beta)}_{m,n-1}(i,k) + \sqrt{(n+1)(n+m+\beta)}M^{(\beta)}_{m,n+1}(i,k)\Big]
\end{aligned}
\tag{2.33b}
$$

For the monic Meixner polynomials $R^{(\beta)}_{m,n}(i,k)$, one finds from (2.33)

$$
\begin{aligned}
i\,R^{(\beta)}_{m,n}(i,k) = \Big[m\Lambda_{11}^2 + n\Lambda_{12}^2 + (m+n+\beta)\Lambda_{13}^2\Big]R^{(\beta)}_{m,n}(i,k) \\
+ \frac{\Lambda_{11}\Lambda_{12}\Lambda_{32}}{\Lambda_{31}}\,m R^{(\beta)}_{m-1,n+1}(i,k) + \frac{\Lambda_{11}\Lambda_{12}\Lambda_{31}}{\Lambda_{32}}\,n R^{(\beta)}_{m+1,n-1}(i,k) \\
- \frac{\Lambda_{11}\Lambda_{13}\Lambda_{33}}{\Lambda_{31}}\,m R^{(\beta)}_{m-1,n}(i,k) - \frac{\Lambda_{11}\Lambda_{13}\Lambda_{31}}{\Lambda_{33}}\,(m+n+\beta) R^{(\beta)}_{m+1,n}(i,k) \\
- \frac{\Lambda_{12}\Lambda_{13}\Lambda_{33}}{\Lambda_{32}}\,n R^{(\beta)}_{m,n-1}(i,k) - \frac{\Lambda_{12}\Lambda_{13}\Lambda_{32}}{\Lambda_{33}}\,(m+n+\beta) R^{(\beta)}_{m,n+1}(i,k),
\end{aligned}
\tag{2.34a}
$$

$$
\begin{aligned}
k\,R^{(\beta)}_{m,n}(i,k) = \Big[m\Lambda_{21}^2 + n\Lambda_{22}^2 + (m+n+\beta)\Lambda_{23}^2\Big]R^{(\beta)}_{m,n}(i,k) \\
+ \frac{\Lambda_{21}\Lambda_{22}\Lambda_{32}}{\Lambda_{31}}\,m R^{(\beta)}_{m-1,n+1}(i,k) + \frac{\Lambda_{21}\Lambda_{22}\Lambda_{31}}{\Lambda_{32}}\,n R^{(\beta)}_{m+1,n-1}(i,k) \\
- \frac{\Lambda_{21}\Lambda_{23}\Lambda_{33}}{\Lambda_{31}}\,m R^{(\beta)}_{m-1,n}(i,k) - \frac{\Lambda_{21}\Lambda_{23}\Lambda_{31}}{\Lambda_{33}}\,(m+n+\beta) R^{(\beta)}_{m+1,n}(i,k) \\
- \frac{\Lambda_{22}\Lambda_{23}\Lambda_{33}}{\Lambda_{32}}\,n R^{(\beta)}_{m,n-1}(i,k) - \frac{\Lambda_{22}\Lambda_{23}\Lambda_{32}}{\Lambda_{33}}\,(m+n+\beta) R^{(\beta)}_{m,n+1}(i,k).
\end{aligned}
\tag{2.34b}
$$

2.6.2 Difference equations

To obtain the difference equations satisfied by the polynomials $R^{(\beta)}_{m,n}(i,k)$, one could consider the matrix elements ${}_\beta\langle\, i,k \mid \mathscr{F} b_j^{\dagger} b_j \mid m,n\,\rangle_\beta$ for $j=1,2$ and proceed along the same lines as for the recurrence relations. It is however more elegant to proceed directly from the recurrence relations (2.34) and to use the duality property (2.24) of the monic bivariate Meixner polynomials. To illustrate the method, consider the left-hand side of (2.34a). Upon using the duality (2.24), one may write

$$
i\,R^{(\beta)}_{m,n}(i,k) = i\,\widetilde{R}^{(\beta)}_{i,k}(m,n) \to m\,\widetilde{R}^{(\beta)}_{m,n}(i,k),
$$

where in the last step the replacements $m \leftrightarrow i$, $n \leftrightarrow k$ were performed. Since $\widetilde{R}^{(\beta)}_{m,n}(i,k)$, is obtained from $R^{(\beta)}_{m,n}(i,k)$ by replacing the parameters of Λ by the parameters of the inverse matrix $\widetilde{\Lambda} = \eta\Lambda^t\eta$, it is seen that the recurrence relations (2.34) can be converted into difference equations by taking $m \leftrightarrow i$, $n \leftrightarrow k$ and replacing the parameters of Λ by those of the inverse. This yields

$$\begin{aligned}
mR^{(\beta)}_{m,n}(i,k) = \Big[i\,\Lambda_{11}^2 + k\,\Lambda_{21}^2 + (i+k+\beta)\,\Lambda_{31}^2\Big] R^{(\beta)}_{m,n}(i,k) \\
+ \frac{\Lambda_{11}\Lambda_{21}\Lambda_{23}}{\Lambda_{13}}\, i R^{(\beta)}_{m,n}(i-1,k+1) + \frac{\Lambda_{11}\Lambda_{21}\Lambda_{13}}{\Lambda_{23}}\, k R^{(\beta)}_{m,n}(i+1,k-1) \\
- \frac{\Lambda_{11}\Lambda_{31}\Lambda_{33}}{\Lambda_{13}}\, i R^{(\beta)}_{m,n}(i-1,k) - \frac{\Lambda_{11}\Lambda_{31}\Lambda_{13}}{\Lambda_{33}}\,(i+k+\beta) R^{(\beta)}_{m,n}(i+1,k) \\
- \frac{\Lambda_{21}\Lambda_{31}\Lambda_{33}}{\Lambda_{23}}\, k R^{(\beta)}_{m,n}(i,k-1) - \frac{\Lambda_{21}\Lambda_{31}\Lambda_{23}}{\Lambda_{33}}\,(i+k+\beta) R^{(\beta)}_{m,n}(i,k+1),
\end{aligned} \tag{2.35a}$$

$$\begin{aligned}
nR^{(\beta)}_{m,n}(i,k) = \Big[i\,\Lambda_{12}^2 + k\,\Lambda_{22}^2 + (i+k+\beta)\,\Lambda_{32}^2\Big] R^{(\beta)}_{m,n}(i,k) \\
+ \frac{\Lambda_{12}\Lambda_{22}\Lambda_{23}}{\Lambda_{13}}\, i R^{(\beta)}_{m,n}(i-1,k+1) + \frac{\Lambda_{12}\Lambda_{22}\Lambda_{13}}{\Lambda_{23}}\, k R^{(\beta)}_{m,n}(i+1,k-1) \\
- \frac{\Lambda_{12}\Lambda_{32}\Lambda_{33}}{\Lambda_{13}}\, i R^{(\beta)}_{m,n}(i-1,k) - \frac{\Lambda_{12}\Lambda_{32}\Lambda_{13}}{\Lambda_{33}}\,(i+k+\beta) R^{(\beta)}_{m,n}(i+1,k) \\
- \frac{\Lambda_{22}\Lambda_{32}\Lambda_{33}}{\Lambda_{23}}\, k R^{(\beta)}_{m,n}(i,k-1) - \frac{\Lambda_{22}\Lambda_{32}\Lambda_{23}}{\Lambda_{33}}\,(i+k+\beta) R^{(\beta)}_{m,n}(i,k+1).
\end{aligned} \tag{2.35b}$$

The same method can be applied to obtain the difference equations satisfied by the polynomials $M^{(\beta)}_{m,n}(i,k)$. Note that the difference equations (2.35) can be combined to give the following nearest neighbour difference equation for the polynomials $R^{(\beta)}_{m,n}(i,k)$:

$$\begin{aligned}
&\left[\frac{m}{\Lambda_{11}\Lambda_{21}} - \frac{n}{\Lambda_{12}\Lambda_{22}}\right] R^{(\beta)}_{m,n}(i,k) = \\
&\left[i\left(\frac{\Lambda_{11}}{\Lambda_{21}} - \frac{\Lambda_{12}}{\Lambda_{22}}\right) + k\left(\frac{\Lambda_{21}}{\Lambda_{11}} - \frac{\Lambda_{22}}{\Lambda_{12}}\right) + (i+k+\beta)\left(\frac{\Lambda_{31}^2}{\Lambda_{11}\Lambda_{21}} - \frac{\Lambda_{32}^2}{\Lambda_{12}\Lambda_{22}}\right)\right] R^{(\beta)}_{m,n}(i,k) \\
&+ i\left[\frac{\Lambda_{32}\Lambda_{33}}{\Lambda_{13}\Lambda_{22}} - \frac{\Lambda_{31}\Lambda_{33}}{\Lambda_{21}\Lambda_{13}}\right] R^{(\beta)}_{m,n}(i-1,k) + (i+k+\beta)\left[\frac{\Lambda_{13}\Lambda_{32}}{\Lambda_{22}\Lambda_{33}} - \frac{\Lambda_{13}\Lambda_{31}}{\Lambda_{21}\Lambda_{33}}\right] R^{(\beta)}_{m,n}(i+1,k) \\
&+ k\left[\frac{\Lambda_{32}\Lambda_{33}}{\Lambda_{12}\Lambda_{23}} - \frac{\Lambda_{31}\Lambda_{33}}{\Lambda_{11}\Lambda_{23}}\right] R^{(\beta)}_{m,n}(i,k-1) + (i+k+\beta)\left[\frac{\Lambda_{23}\Lambda_{32}}{\Lambda_{12}\Lambda_{33}} - \frac{\Lambda_{23}\Lambda_{31}}{\Lambda_{11}\Lambda_{33}}\right] R^{(\beta)}_{m,n}(i,k+1).
\end{aligned}$$

A similar formulas holds for the bivariate Krawtchouk polynomials [10].

2.7 One-parameter subgroups and univariate Meixner & Krawtchouk polynomials

It has been assumed so far that the entries Λ_{ij} of the $SO(2,1)$ parameter matrix for the bivariate Meixner polynomials are non-zero. In this section, degenerate cases corresponding to natural one-parameter subgroups of $SO(2,1)$ shall be considered. In particular, it will be shown that for transformations Λ belonging to hyperbolic subgroups, the matrix elements ${}_\beta\langle\, i,k \mid \mathscr{F}(\Lambda) \mid m,n\,\rangle_\beta$ are given in terms of the standard (univariate) Meixner polynomials and that for transformations Λ belonging to the elliptic subgroup, the matrix elements ${}_\beta\langle\, i,k \mid \mathscr{F}(\Lambda) \mid m,n\,\rangle_\beta$ are expressed in terms of standard (univariate) Krawtchouk polynomials. The one-variable Meixner polynomials $M_n(x;\delta;c)$ are defined by their explicit expression [17]

$$M_n(x;\delta;c) = {}_2F_1\left[\begin{matrix}-n,-x\\ \delta\end{matrix};1-\frac{1}{c}\right], \tag{2.36}$$

where ${}_2F_1$ is the Gauss hypergeometric function. The monic Meixner polynomials $m_n(x)$ defined through $M_n(x;\delta;c) = \frac{1}{(\delta)_n}\left(\frac{c-1}{c}\right)^n m_n(x)$ obey the three term recurrence relation

$$x\,m_n(x) = m_{n+1}(x) + \frac{n+(n+\delta)c}{1-c}\,m_n(x) + \frac{n(n+\delta-1)c}{(1-c)^2}\,m_{n-1}(x), \tag{2.37}$$

with $m_{-1}(x) = 0$ and $m_0(x) = 1$. The one-variable Krawtchouk polynomials are denoted $K_n(x;p;N)$ and have the expression

$$K_n(x;p;N) = {}_2F_1\left[\begin{matrix}-n,-x\\ -N\end{matrix};\frac{1}{p}\right], \tag{2.38}$$

where N is a positive integer.

2.7.1 Hyperbolic subgroups: Meixner polynomials

Consider the two hyperbolic one-parameter subgroups of $SO(2,1)$ which have as representative elements the following matrices:

$$\Xi(\xi) = \begin{pmatrix}\cosh\xi & 0 & \sinh\xi\\ 0 & 1 & 0\\ \sinh\xi & 0 & \cosh\xi\end{pmatrix}, \tag{2.39a}$$

$$\Psi(\psi) = \begin{pmatrix}1 & 0 & 0\\ 0 & \cosh\psi & \sinh\psi\\ 0 & \sinh\psi & \cosh\psi\end{pmatrix}. \tag{2.39b}$$

The pseudo-rotations $\Xi(\xi)$ and $\Psi(\psi)$ are unitarily represented by the operators

$$\mathscr{F}(\Xi(\xi)) = e^{-i\xi K_2}, \qquad \mathscr{F}(\Psi(\psi)) = e^{-i\psi K_1},$$

where K_1, K_2 are given by (2.5). The matrix elements ${}_\beta\langle\, i,k \mid \mathscr{F}(\Xi(\xi)) \mid m,n\,\rangle_\beta$ and ${}_\beta\langle\, i,k \mid \mathscr{F}(\Psi(\psi)) \mid m,n\,\rangle_\beta$ can be evaluated using the approach of Section 3. Let us give the details of the calculation of the matrix elements of $\mathscr{F}(\Xi(\xi))$. Since $\Xi(\xi)$ leaves b_2 and $b_2^\dagger$ unaffected and hence acts in a trivial way on the second quantum number, it is readily seen that

$$_\beta\langle\, i,k \,|\, \mathscr{F}(\Xi) \,|\, m,n \,\rangle_\beta = \delta_{kn}\; _\beta\langle\, i \,|\, \mathscr{F}(\Xi) \,|\, m \,\rangle_\beta,$$

where the dependence on ξ and on the second quantum number have been suppressed for notational convenience. Given that

$$\mathscr{F}^\dagger(\Xi) b_1 \mathscr{F}(\Xi) = b_1 \cosh\xi + b_3 \sinh\xi, \quad \mathscr{F}^\dagger(\Xi) b_1^\dagger \mathscr{F}(\Xi) = b_1^\dagger \cosh\xi + b_3^\dagger \sinh\xi,$$

the identity $_\beta\langle\, i \,|\, b_1^\dagger b_1 \mathscr{F}(\Xi) \,|\, m \,\rangle_\beta = {}_\beta\langle\, i \,|\, \mathscr{F}(\Xi)\mathscr{F}^\dagger(\Xi) b_1^\dagger b_1 \mathscr{F}(\Xi) \,|\, m \,\rangle_\beta$ yields the recurrence relation

$$\begin{aligned} i\; _\beta\langle\, i \,|\, \mathscr{F}(\Xi) \,|\, m \,\rangle_\beta = &\left[m\cosh^2\xi + (m+\gamma)\sinh^2\xi\right] {}_\beta\langle\, i \,|\, \mathscr{F}(\Xi) \,|\, m \,\rangle_\beta \\ &+ \cosh\xi \sinh\xi \sqrt{m(m+\gamma-1)}\; {}_\beta\langle\, i \,|\, \mathscr{F}(\Xi) \,|\, m-1 \,\rangle_\beta \\ &+ \cosh\xi \sinh\xi \sqrt{(m+1)(m+\gamma)}\; {}_\beta\langle\, i \,|\, \mathscr{F}(\Xi) \,|\, m+1 \,\rangle_\beta, \end{aligned}$$

where $\gamma = n + \beta$. Upon taking

$$_\beta\langle\, i \,|\, \mathscr{F}(\Xi) \,|\, m \,\rangle_\beta = {}_\beta\langle\, i \,|\, \mathscr{F}(\Xi) \,|\, 0 \,\rangle_\beta \sqrt{\frac{1}{m!(\gamma)_m}} (\cosh\xi \sinh\xi)^{-n} P_m(i),$$

where $P_0(i) = 1$, it is seen that $P_m(i)$ satisfies the three-term recurrence relation (2.37) of the monic Meixner polynomials with $c = \tanh^2\xi$ and $\delta = \gamma = \beta + n$. One thus has

$$_\beta\langle\, i \,|\, \mathscr{F}(\Xi) \,|\, m \,\rangle_\beta = {}_\beta\langle\, i \,|\, \mathscr{F}(\Xi) \,|\, 0 \,\rangle_\beta (-1)^m \sqrt{\frac{(n+\beta)_m}{m!}} \tanh^m\xi\, M_m(i;\beta+n;\tanh^2\xi),$$

where $M_n(x;\delta;c)$ are the univariate Meixner polynomials. There remains to evaluate the amplitude $_\beta\langle\, i \,|\, \mathscr{F}(\Xi) \,|\, 0 \,\rangle_\beta$. This can be done using the identity $_{\beta+1}\langle\, i \,|\, \mathscr{F}(\Xi) b_1 \,|\, 0 \,\rangle_\beta = 0$ which gives the two-term recurrence relation

$$_\beta\langle\, i+1 \,|\, \mathscr{F}(\Xi) \,|\, 0 \,\rangle_\beta = \tanh\xi \sqrt{\frac{i+\gamma}{i+1}}\; {}_\beta\langle\, i \,|\, \mathscr{F}(\Xi) \,|\, 0 \,\rangle_\beta,$$

that has for solution

$$_\beta\langle\, i \,|\, \mathscr{F}(\Xi) \,|\, 0 \,\rangle_\beta = \tanh^i\xi \sqrt{\frac{(\gamma)_i}{i!}}\; {}_\beta\langle\, 0 \,|\, \mathscr{F}(\Xi) \,|\, 0 \,\rangle_\beta.$$

Since one has

$$1 = {}_\beta\langle 0 \,|\, 0 \rangle_\beta = \sum_{i\geqslant 0} {}_\beta\langle\, 0 \,|\, \mathscr{F}^\dagger(\Xi) \,|\, i \,\rangle_\beta \, {}_\beta\langle\, i \,|\, \mathscr{F}(\Xi) \,|\, 0 \,\rangle_\beta$$
$$= \sum_{i\geqslant 0} \frac{(\gamma)_i}{i!} \tanh^{2i}\xi \, |{}_\beta\langle\, 0 \,|\, \mathscr{F}(\Xi) \,|\, 0 \,\rangle_\beta|^2,$$

it follows that

$${}_\beta\langle\, i \,|\, \mathscr{F}(\Xi) \,|\, 0 \,\rangle_\beta = \sqrt{\frac{(\gamma)_i}{i!}} \cosh^{-\gamma-i}\xi \sinh^i \xi.$$

The matrix elements of the one-parameter hyperbolic elements $\Xi(\xi)$ are thus given by

$$\begin{aligned} &{}_\beta\langle\, i,k \,|\, \mathscr{F}(\Xi(\xi)) \,|\, m,n \,\rangle_\beta \\ &= \delta_{kn} (-1)^m \sqrt{\frac{(k+\beta)_i (k+\beta)_m}{i! m!}} \cosh^{-k-\beta}\xi \tanh^{i+m}\xi \, M_m(i; k+\beta; \tanh^2\xi). \end{aligned} \tag{2.40}$$

In a similar fashion, one obtains for the matrix elements of $\mathscr{F}(\Psi)$

$$\begin{aligned} &{}_\beta\langle\, i,k \,|\, \mathscr{F}(\Psi(\psi)) \,|\, m,n \,\rangle_\beta \\ &= \delta_{im} (-1)^n \sqrt{\frac{(i+\beta)_k (i+\beta)_n}{k! n!}} \cosh^{-i-\beta}\psi \tanh^{k+n}\psi \, M_n(k; i+\beta; \tanh^2\psi). \end{aligned} \tag{2.41}$$

2.7.2 Elliptic subgroup: Krawtchouk polynomials

The group $SO(2,1)$ also has a one-parameter elliptic subgroup which has for representative element the matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.42}$$

which is unitarily represented by $\mathscr{F}(R(\theta)) = e^{i\theta K_3}$. The matrix elements

$${}_\beta\langle\, i,k \,|\, \mathscr{F}(R(\theta)) \,|\, m,n \,\rangle_\beta,$$

can be evaluated using the same approach as the one adopted above. Since the details of similar computations are found in [7], we only give the result which reads

$$\begin{aligned} &{}_\beta\langle\, i,k \,|\, \mathscr{F}(R(\theta)) \,|\, m,n \,\rangle_\beta \\ &= \delta_{i+k,m+n} (-1)^k \sqrt{\binom{i+k}{k}\binom{i+k}{n}} \cos^{i+k}\theta \tan^{k+n}\theta \, K_n(k; \sin^2\theta; i+k), \end{aligned} \tag{2.43}$$

where $K_n(x;p;N)$ is given by (2.38) and where $\binom{N}{i}$ stands for the binomial coefficient. Note that the formula (2.43) does not define proper Krawtchouk polynomials since here N is not an independent parameter as the variable $i+k$ occurs in its place.

2.8 Addition formulas

In this section, the group product is used to derive a general addition formula for the bivariate Meixner polynomials.

2.8.1 General addition formula

Let A, B and C be $SO(2,1)$ elements such that $C = A \cdot B$ with unitary representations $\mathscr{F}(A)$, $\mathscr{F}(B)$ and $\mathscr{F}(C)$. For a given value of β, to each of these elements is associated a system of bivariate Meixner polynomials denoted by $M_{m,n}^{(\beta)}(i,k;A)$, $M_{m,n}^{(\beta)}(i,k;B)$ and $M_{m,n}^{(\beta)}(i,k;C)$. Since $\mathscr{F}(C) = \mathscr{F}(A)\mathscr{F}(B)$, it follows that

$$ {}_{\beta}\langle\, i,k \mid \mathscr{F}(C) \mid m,n\,\rangle_{\beta} = \sum_{\rho,\sigma \geqslant 0} {}_{\beta}\langle\, i,k \mid \mathscr{F}(A) \mid \rho,\sigma\,\rangle_{\beta}\, {}_{\beta}\langle\, \rho,\sigma \mid \mathscr{F}(B) \mid m,n\,\rangle_{\beta}. \tag{2.44} $$

In terms of the polynomials $M_{m,n}^{(\beta)}(i,k)$, this identity translates into the addition formula

$$ \left(\frac{W_{i,k}^{(\beta)}(C)}{W_{i,k}^{(\beta)}(A)}\right) M_{m,n}^{(\beta)}(i,k;C) = \sum_{\rho,\sigma \geqslant 0} W_{\rho,\sigma}^{(\beta)}(B) M_{\rho,\sigma}^{(\beta)}(i,k;A) M_{m,n}^{(\beta)}(\rho,\sigma;B). \tag{2.45} $$

2.8.2 Special case I: product of two hyperbolic elements

Consider the case where the $SO(2,1)$ parameter matrix for the bivariate Meixner polynomials $R_{m,n}^{(\beta)}(i,k)$ is of the form

$$ \Lambda = \Psi(\psi)\cdot\Xi(\xi) = \begin{pmatrix} \cosh\xi & 0 & \sinh\xi \\ \sinh\xi \sinh\psi & \cosh\psi & \cosh\xi\cosh\psi \\ \cosh\psi\sinh\xi & \sinh\psi & \cosh\xi\cosh\psi \end{pmatrix}. \tag{2.46} $$

In this case the decomposition formula (2.44) can be used to obtain an elegant expression for the polynomials $R_{m,n}^{(\beta)}(i,k)$. For the parameter matrix (2.46), one has on the one hand

$$ \begin{gathered} {}_{\beta}\langle\, i,k \mid \mathscr{F}(\Lambda) \mid m,n\,\rangle_{\beta} = \\ \sqrt{\frac{(\beta)_{i+k}(\beta)_{m+n}}{i!k!m!n!}} \left(\frac{\Lambda_{13}}{\Lambda_{33}}\right)^{i} \left(\frac{\Lambda_{23}}{\Lambda_{33}}\right)^{k} \left(\frac{-\Lambda_{31}}{\Lambda_{33}}\right)^{m} \left(\frac{-\Lambda_{32}}{\Lambda_{33}}\right)^{n} \left(\frac{1}{\Lambda_{33}}\right)^{\beta} R_{m,n}^{(\beta)}(i,k), \end{gathered} \tag{2.47} $$

and on the other hand

$$ {}_{\beta}\langle\, i,k\mid \mathscr{F}(\Lambda)\mid m,n\,\rangle_{\beta} = \sum_{\mu,\nu\geqslant 0} {}_{\beta}\langle\, i,k\mid \mathscr{F}(\Psi(\psi))\mid \mu,\nu\,\rangle_{\beta\beta}\langle\, \mu,\nu\mid \mathscr{F}(\Xi(\xi))\mid m,n\,\rangle_{\beta}. \tag{2.48} $$

Upon comparing the formulas (2.47) and (2.48) and using the one-parameter matrix elements (2.40) and (2.41), a direct computation shows that the parameters conspire to yield the expression

$$ R^{(\beta)}_{m,n}(i,k) = \frac{(i+\beta)_n}{(\beta)_n} M_m(i;n+\beta;\tanh^2\xi) M_n(k;i+\beta;\tanh^2\psi). \tag{2.49} $$

The factorization (2.49) of the bivariate Meixner polynomials as a product of two univariate Meixner polynomials is reminiscent of the bivariate Meixner polynomials defined by Tratnik. However, the Meixner polynomials (2.49) do not exactly coincide with those defined in [20]. It is seen from (2.34) that in the special case (2.46) one of the recurrence relations simplifies drastically. Indeed, (2.34) becomes

$$ \begin{aligned} i R^{(\beta)}_{m,n}(i,k) &= \Big[m\cosh^2\xi + (m+n+\beta)\sinh^2\xi\Big] R^{(\beta)}_{m,n}(i,k) \\ &\quad - \cosh^2\xi\, m R^{(\beta)}_{m-1,n}(i,k) - \sinh^2\xi\,(m+n+\beta) R^{(\beta)}_{m+1,n}(i,k), \\ k R^{(\beta)}_{m,n}(i,k) &= \Big[m\sinh^2\xi\sinh^2\psi + n\cosh^2\psi + (m+n+\beta)\cosh^2\xi\sinh^2\psi\Big] R^{(\beta)}_{m,n}(i,k) \\ &\quad + \sinh^2\psi\, m R^{(\beta)}_{m-1,n+1}(i,k) + \cosh^2\psi\sinh^2\xi\, n R^{(\beta)}_{m+1,n-1}(i,k) \\ &\quad - \cosh^2\xi\sinh^2\psi\, m R^{(\beta)}_{m-1,n}(i,k) - \sinh^2\xi\sinh^2\psi\,(m+n+\beta) R^{(\beta)}_{m+1,n}(i,k) \\ &\quad - \cosh^2\xi\cosh^2\psi\, n R^{(\beta)}_{m,n-1}(i,k) - \sinh^2\psi\,(m+n+\beta) R^{(\beta)}_{m,n+1}(i,k). \end{aligned} $$

The generating function (2.28) also has the simplification

$$ \begin{aligned} &(1-z_1-z_2)^{-\beta-i-k}(1-\coth^2\xi\, z_1)^i(1-z_1-\coth^2\psi\, z_2)^k \\ &= \sum_{m,n\geqslant 0} \frac{(\beta)_{m+n}}{m!n!} R^{(\beta)}_{m,n}(i,k) z_1^m z_2^n. \end{aligned} $$

Note that polynomials (2.49) corresponding to the special case (2.46) are orthogonal with respect to the same weight function (2.19) as the generic polynomials.

2.8.3 General case

Let us now give a formula for the general bivariate Meixner polynomials. The most general $SO(2,1)$ pseudo-rotation can be taken of the form

$$ \Lambda = R(\chi)\Psi(\psi)R(\theta), $$

where $\Psi(\psi)$ is given by (2.39a) and where $R(\theta)$, $R(\chi)$ are given by (2.42). For the matrix Λ, one has again on the one hand

$$
{}_{\beta}\langle\, i,k \,|\, \mathscr{F}(\Lambda) \,|\, m,n \,\rangle_{\beta} = \\
\sqrt{\frac{(\beta)_{i+k}(\beta)_{m+n}}{i!k!m!n!}} \left(\frac{\Lambda_{13}}{\Lambda_{33}}\right)^{i} \left(\frac{\Lambda_{23}}{\Lambda_{33}}\right)^{k} \left(\frac{-\Lambda_{31}}{\Lambda_{33}}\right)^{m} \left(\frac{-\Lambda_{32}}{\Lambda_{33}}\right)^{n} \left(\frac{1}{\Lambda_{33}}\right)^{\beta} R^{(\beta)}_{m,n}(i,k), \tag{2.50}
$$

and on the other hand

$$
{}_{\beta}\langle\, i,k \,|\, \mathscr{F}(\Lambda) \,|\, m,n \,\rangle_{\beta} = \\
= \sum_{\mu,\nu,\rho,\sigma \geqslant 0} {}_{\beta}\langle\, i,k \,|\, \mathscr{F}(R(\chi)) \,|\, \mu,\nu \,\rangle_{\beta\beta}\langle\, \mu,\nu \,|\, \mathscr{F}(\Psi(\psi)) \,|\, \rho,\sigma \,\rangle_{\beta\beta}\langle\, \rho,\sigma \,|\, \mathscr{F}(R(\theta)) \,|\, m,n \,\rangle_{\beta}. \tag{2.51}
$$

Upon comparing the formulas (2.50), (2.51) using the expressions (2.41), (2.43) for the one-variable matrix elements, one arrives at the following formula for the general bivariate Meixner polynomials:

$$
R^{(\beta)}_{m,n}(i,k) = \\
(-\tan^2\chi)^k(-\tan^2\theta)^n \sum_{\mu\geqslant 0} \frac{(-i-k)_{\mu}(-n-m)_{\mu}}{\mu!(\beta)_{\mu}} (\tan\chi\tan\theta\sinh\psi\tanh\psi)^{-\mu} \\
\times K_{i+k-\mu}(k;\sin^2\chi;i+k) M_{m+n-\mu}(i+k-\mu;\mu+\beta;\tanh^2\psi) K_n(m+n-\mu;\sin^2\theta;m+n). \tag{2.52}
$$

The formula (2.52) thus gives an explicit expression of the bivariate Meixner polynomials in terms of the Krawtchouk and Meixner polynomials. It is directly seen that the summation appearing in (2.52) is finite. Moreover, the duality property (2.24) of the bivariate Meixner polynomials is manifest in (2.52) in view of the duality property of the univariate Krawtchouk and Meixner polynomials. Note that the comment below (2.43) also applies for (2.52).

2.9 Multivariate case

In this section, it is shown how the results obtained thus far can easily be generalized to d variables by considering the eigenspace of a bilinear expression in the creation/annihilation operators of $d+1$ harmonic oscillators.

Consider $d+1$ pairs of creation and annihilation operators $a_i^{\dagger}$, a_i satisfying the Weyl algebra commutation relations

$$
[a_i, a_k] = 0, \qquad [a_i^{\dagger}, a_k^{\dagger}] = 0, \qquad [a_i, a_k^{\dagger}] = \delta_{ik},
$$

for $i,k=1,\ldots,d+1$ and let H be the Hermitian operator

$$H = a_1^\dagger a_1 + a_2^\dagger a_2 + \cdots - a_{d+1}^\dagger a_{d+1}. \tag{2.53}$$

Let β be a positive integer and denote by $\mathcal{V}_\beta$ be the infinite-dimensional eigenspace associated to the eigenvalue $1-\beta$ of H. An orthonormal basis for the space $\mathcal{V}_\beta$ is provided by the vectors

$$|\, n_1,\ldots,n_d\,\rangle_\beta = |\, n_1,\ldots,n_d, |n|+\beta-1\,\rangle, \tag{2.54}$$

where the notation $|n| = n_1+\ldots+n_d$ was used. The action of the operators $a_i^\dagger$, a_i is identical to the one given in (2.2). Since (2.53) is clearly invariant under $SU(d,1)$ transformations, it follows that $\mathcal{V}_\beta$ provides a reducible representation space for the subgroup $SO(d,1)$. Again, one uses the notation $a_i = b_i$ for $i=1,\ldots,d$ and $a_{d+1} = b_{d+1}^\dagger$.

Let B be a real $(d+1)\times(d+1)$ antisymmetric matrix and let Λ be an orthochronous element of $SO(d,1)$. This means that Λ satisfies

$$\Lambda^t \eta \Lambda = \eta, \qquad \Lambda_{d+1,d+1} \geqslant 1,$$

where $\eta = \mathrm{diag}(1,1,\ldots,-1)$. Consider now the unitary representation

$$\mathscr{F}(\Lambda) = \exp\left(\sum_{ij=1}^{d+1} B_{ij} b_i^\dagger b_j\right), \tag{2.55}$$

which has for parameters the $d(d+1)/2$ independent matrix elements of B. The transformations of the operators b_i, $b_i^\dagger$ under the action of $\mathscr{F}(\Lambda)$ are given by

$$\mathscr{F}(\Lambda) b_i \mathscr{F}^\dagger(\Lambda) = \sum_{k=1}^{d+1} \widetilde{\Lambda}_{ik} b_k, \qquad \mathscr{F}(\Lambda) b_i^\dagger \mathscr{F}^\dagger(\Lambda) = \sum_{k=1}^{d+1} \widetilde{\Lambda}_{ik} b_k^\dagger,$$

where $\widetilde{\Lambda}$ denotes the inverse matrix of Λ: $\widetilde{\Lambda}\Lambda = 1$. Proceeding in as in Section 4, one can write the matrix elements of the reducible representations of $SO(d,1)$ on the space $\mathcal{V}_\beta$ as follows:

$${}_\beta\langle\, x_1,\ldots,x_d \,|\, \mathscr{F}(\Lambda) \,|\, n_1,\ldots,n_d\,\rangle_\beta = W_{x_1,\ldots,x_d}^{(\beta)}\, M_{n_1,\ldots,n_d}^{(\beta)}(x_1,\ldots,x_d),$$

with $M_{0,\ldots,0}^{(\beta)}(x_1,\ldots,x_d) = 1$ and where

$$W_{x_1,\ldots,x_d}^{(\beta)} = {}_\beta\langle\, x_1,\ldots,x_d \,|\, \mathscr{F}(\Lambda) \,|\, 0,\ldots,0\,\rangle_\beta.$$

By considering the identities ${}_{\beta+1}\langle\, x_1,\dots,x_d \,|\, \mathscr{F}(\Lambda) b_i \,|\, 0,\dots,0 \,\rangle_\beta = 0$ for $i = 1,\dots,d$, one finds that $W^{(\beta)}_{x_1,\cdots,x_d}$ is given by

$$W^{(\beta)}_{x_1,\dots,x_d} = \sqrt{\frac{(\beta)_{|x|}}{x_1!\cdots x_d!}}\,\Lambda^{x_1}_{1,d+1}\Lambda^{x_2}_{2,d+1}\cdots\Lambda^{x_d}_{d,d+1}\Lambda^{-\beta-|x|}_{d+1,d+1}.$$

Since

$$\Lambda^2_{d+1,d+1} - \sum_{i=1}^{d}\Lambda^2_{d+1,i} = 1, \tag{2.56}$$

one has

$$|W^{(\beta)}_{i,k}|^2 = w^{(\beta)}_{i,k} = \frac{(\beta)_{|x|}}{x_1!\cdots x_d!}(1-|c|)^\beta c_1^{x_1}\cdots c_d^{x_d}, \tag{2.57}$$

with

$$c_i = \frac{\Lambda^2_{i,d+1}}{\Lambda^2_{d+1,d+1}}, \qquad i = 1,\dots,d$$

In view of (2.56), the condition $|c| < 1$ is identically satisfied and hence the following normalization condition holds:

$$\sum_{x_1,\dots,x_d \geqslant 0} w^{(\beta)}_{x_1,\dots,x_d} = 1.$$

The raising relations (2.17) are readily generalized to d variables and from there it is seen that $M^{(\beta)}_{n_1,\dots,n_d}(x_1,\dots,x_d)$ are polynomials in the variables $x_1,\dots,x_d$ of total degree $|n|$. As a consequence of the unitarity of the operator $\mathscr{F}(\Lambda)$, the polynomials $M^{(\beta)}_{n_1,\dots,n_d}(x_1,\dots,x_d)$ satisfy the orthogonality relation

$$\sum_{x_1,\dots,x_d \geqslant 0} w^{(\beta)}_{x_1,\dots,x_d} M^{(\beta)}_{n_1,\dots,n_d}(x_1,\dots,x_d) M^{(\beta)}_{m_1,\dots,m_d}(x_1,\dots,x_d) = \delta_{n_1,m_1}\cdots\delta_{n_d,m_d}, \tag{2.58}$$

with respect to the negative multinomial distribution (2.57). The calculation of the generating function of section 5 is also easily generalized to an arbitrary finite number of variables. One then obtains the generating function used by Griffiths and Iliev to define the d-variable Meixner polynomials $R^{(\beta)}_{n_1,\dots,n_d}(x_1,\dots,x_d)$:

$$\begin{aligned}&(1-|z|)^{-\beta-|x|}\prod_{i=1}^{d}\left(1-\sum_{j=1}^{d}u_{i,j}z_j\right)^{x_i}\\ &= \sum_{n_1,\dots,n_d \geqslant 0}\frac{(\beta)_{|n|}}{n_1!\cdots n_d!}R^{(\beta)}_{n_1,\dots,n_d}(x_1,\dots,x_d)\, z_1^{n_1}\cdots z_d^{n_d}.\end{aligned}$$

where the parameters $u_{i,j}$ are given by

$$u_{i,j} = \frac{\Lambda_{i,j}\Lambda_{d+1,d+1}}{\Lambda_{i,d+1}\Lambda_{d+1,j}},$$

for $i, j = 1, \ldots, d$. All properties of the multivariate Meixner polynomials can be derived in complete analogy with the $d = 2$ case which has been treated in detail here.

2.10 Conclusion

In summary, we have considered the reducible representations of the $SO(d,1)$ group on the eigenspace of a bilinear expression in the creation/annihilation operators of $d+1$ independent quantum harmonic oscillators. We have shown that the multivariate Meixner polynomials arise as matrix elements of these $SO(d,1)$ representations and we have seen that the main properties of the polynomials can be derived systematically using the group theoretical interpretation.

In [19], the bivariate Krawtchouk polynomials were seen to occur as wavefunctions of a Hamiltonian describing a discrete/finite model of the harmonic oscillator in two dimensions possessing a $SU(2)$ symmetry. This result and the considerations of the present paper suggest that the bivariate Meixner polynomials could also arise as wavefunctions of a discrete Hamiltonian. We hope to report on this issue in the future.

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Chapitre 3

The multivariate Charlier polynomials as matrix elements of the Euclidean group representation on oscillator states

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Abstract. A family of multivariate orthogonal polynomials generalizing the standard (univariate) Charlier polynomials is shown to arise in the matrix elements of the unitary representation of the Euclidean group $E(d)$ on oscillator states. These polynomials in d discrete variables are orthogonal on the product of d Poisson distributions. The accent is put on the $d = 2$ case and the group theoretical setting is used to obtain the main properties of the polynomials: orthogonality and recurrence relations, difference equation, raising/lowering relations, generating function, hypergeometric and integral representations and explicit expression in terms of standard Charlier and Krawtchouk polynomials. The approach is seen to extend straightforwardly to an arbitrary number of variables. The contraction of $SO(3)$ to $E(2)$ is used to show that the bivariate Charlier polynomials correspond to a limit of the bivariate Krawtchouk polynomials.

3.1 Introduction

In this paper, a new family of multi-variable Charlier polynomials that arise as matrix elements of the unitary reducible Euclidean group representation on oscillator states is introduced. The main properties of these polynomials are obtained naturally from the group theoretical context. The focus is put mainly on the bivariate case, for which the two-variable Charlier polynomials occur in the matrix elements of unitary reducible $E(2)$ representations on the eigenstates of a two-dimensional isotropic harmonic oscillator. The extension to an arbitrary number of variables is straightforward and is given towards the end of the paper.

The standard Charlier polynomials $C_n(x;a)$ of degree n in the variable x were introduced in 1905 [2]. These polynomials form one of the most elementary family of orthogonal polynomials (OPs) in the Askey scheme of hypergeometric OPs [17]. They are orthogonal with respect to the Poisson distribution $w_x^{(a)}$ with parameter $a > 0$ which is defined by

$$w_x^{(a)} = \frac{a^x e^{-a}}{x!},$$

and their discrete orthogonality relation reads

$$\sum_{x=0}^{\infty} w_x^{(a)} C_n(x;a) C_m(x;a) = a^{-n} n! \delta_{nm}.$$

They can be defined through their generating function

$$e^t \left(1 - \frac{t}{a}\right)^x = \sum_{n=0}^{\infty} \frac{C_n(x;a)}{n!} t^n, \tag{3.1}$$

and can be expressed in terms of a ${}_2F_0$ hypergeometric function (see [17] for additional properties and references). The Charlier polynomials appear in various fields including combinatorics [18] as well as statistics and probability [9, 19]. In Physics, the importance of these polynomials is mostly due to their appearance in the matrix elements of unitary irreducible representations of the one-dimensional oscillator group [10, 21].

In a recent series of papers [6, 7, 8], we have presented group theoretical interpretations for two families of multivariate orthogonal polynomials: the multi-variable Krawtchouk and Meixner polynomials. These two families, introduced by Griffiths in [11, 12], were seen to occur in the matrix elements of reducible unitary representations of the rotation and pseudo-rotation groups on oscillator states. This algebraic framework led to a number of new identities for these polynomials and allowed for simple derivations of their known properties.

Here we consider the Euclidean group $E(d)$ which is the group generated by the translations and the rotations in d-dimensional Euclidean space. We shall investigate the matrix elements of the unitary reducible representation of this group on the eigenstates of a d-dimensional isotropic harmonic oscillator and show that these are expressed in terms of a new family of multivariate orthogonal polynomials that shall be identified as multivariate extensions of the standard Charlier polynomials. The main properties of these polynomials will be derived in a simple fashion using the group theoretical interpretation.

The paper is organized as follows. In section 2, the unitary representations of the Euclidean group $E(2)$ are defined. In section 3, it is shown that the matrix elements of these representations are given in terms of bivariate polynomials that are orthogonal with respect to the product of two (independent) Poisson distributions. The duality relation satisfied by these polynomials is discussed in section 4. In section 5, a generating function is obtained and the polynomials are identified as multivariate Charlier polynomials. The generating function is used in section 6 to find an explicit expression for these Charlier polynomials in terms of generalized hypergeometric series. The recurrence relations and the difference equations are given in section 7. In section 8, the matrix elements for one-parameter subgroups are considered and used to obtain an explicit expression for the bivariate Charlier polynomials in terms of standard Charlier and Krawtchouk polynomials. In section 9, an integral representation is given. In section 10, it is shown that the bivariate Charlier polynomials can be obtained from the bivariate Krawtchouk polynomials through a limit process. In section 11, the d-dimensional case is treated. A conclusion follows.

3.2 Unitary representations of $E(2)$ on oscillator states

In this section, the reducible unitary representation of the Euclidean group that shall be used throughout the paper is defined. This representation will be specified on the eigenstates of the two-dimensional isotropic harmonic oscillator.

3.2.1 The Heisenberg-Weyl algebra

Let a_i, $a_i^{\dagger}$, $i = 1, 2$, be the generators of the Heisenberg-Weyl algebra satisfying the commutation relations

$$[a_i, a_j^{\dagger}] = \delta_{ij}, \quad [a_i, a_j] = 0, \quad [a_i^{\dagger}, a_j^{\dagger}] = 0. \tag{3.2}$$

This algebra has a standard representation on the basis vectors

$$|\, n_1, n_2\,\rangle \equiv |\, n_1\,\rangle \otimes |\, n_2\,\rangle, \qquad n_1, n_2 = 0, 1, \ldots,$$

defined by the actions of the generators on the factors of the direct product:

$$a_i |\, n_i\,\rangle = \sqrt{n_i}\, |\, n_i - 1\,\rangle, \quad a_i^{\dagger} |\, n_i\,\rangle = \sqrt{n_i + 1}\, |\, n_i + 1\,\rangle. \tag{3.3}$$

In view of the commutation relations (3.2) and the actions (3.3), the basis vectors $|\, n_1, n_2 \,\rangle$ can equivalently be written as

$$|\, n_1, n_2 \,\rangle = \frac{(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}}{\sqrt{n_1! n_2!}} |\, 0, 0 \,\rangle. \tag{3.4}$$

In Cartesian coordinates, the algebra (3.2) has the following realization:

$$a_i = \frac{1}{\sqrt{2}}\left(x_i + \frac{\partial}{\partial x_i}\right), \quad a_i^\dagger = \frac{1}{\sqrt{2}}\left(x_i - \frac{\partial}{\partial x_i}\right), \qquad i = 1, 2. \tag{3.5}$$

3.2.2 The two-dimensional isotropic oscillator

Consider the Hamiltonian $\mathscr{H}$ governing the isotropic harmonic oscillator in the two-dimensional Euclidean space

$$\mathscr{H} = -\frac{1}{2}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) + \frac{1}{2}(x_1^2 + x_2^2). \tag{3.6}$$

Using the realization (3.5), the Hamiltonian (3.6) can be written as

$$\mathscr{H} = a_1^\dagger a_1 + a_2^\dagger a_2 + 1. \tag{3.7}$$

It is seen from (3.3) that the Hamiltonian (3.7) is diagonal on the basis vectors $|\, n_1, n_2 \,\rangle$ with energy eigenvalue E given by:

$$\mathscr{H} |\, n_1, n_2 \,\rangle = E |\, n_1, n_2 \,\rangle, \qquad E = n_1 + n_2 + 1.$$

The Schrödinger equation $\mathscr{H}\Psi = E\Psi$ associated to the Hamiltonian (3.6) separates in the Cartesian coordinates x_1, x_2. In these coordinates, the wavefunctions take the form

$$\langle\, x_1, x_2 \,|\, n_1, n_2 \,\rangle = \Psi_{n_1}(x_1)\Psi_{n_2}(x_2),$$

with

$$\langle\, x_i \,|\, n_i \,\rangle = \Psi_{n_i}(x_i) = \sqrt{\frac{1}{2^{n_i}\, \pi^{1/2}\, n_i!}}\, e^{-x_i^2/2} H_{n_i}(x_i), \tag{3.8}$$

where $H_n(x)$ denotes the Hermite polynomials [17]. The wavefunctions $\Psi_{n_i}(x_i)$ satisfy the orthogonality relation

$$\int_{-\infty}^{\infty} \Psi_{n_i}(x_i)\Psi_{n_i'}(x_i)\,\mathrm{d}x_i = \delta_{n_i n_i'}. \tag{3.9}$$

3.2.3 The representation of $E(2)$ on oscillator states

The eigenstates of the two-dimensional quantum harmonic oscillator support a reducible representation of the Euclidean group $E(2)$. We introduce the following notation for the basis vectors:

$$|\, m,n\,\rangle \equiv |\, n_1,n_2\,\rangle,$$

so that m and n are identified with n_1 and n_2, respectively. The $E(2)$ group is generated by two translation operators in the x_1 and x_2 directions given by

$$P_1 = i(a_1 - a_1^\dagger), \quad P_2 = i(a_2 - a_2^\dagger), \tag{3.10}$$

and by a rotation generator J which has the expression

$$J = i(a_1 a_2^\dagger - a_1^\dagger a_2). \tag{3.11}$$

The generators P_1, P_2 and J satisfy the commutation relations of the Euclidean Lie algebra $\mathfrak{e}(2)$ which read

$$[P_1,P_2] = 0, \quad [P_2,J] = iP_1, \quad [J,P_1] = iP_2. \tag{3.12}$$

Using the formulas (3.3), the actions of the Euclidean generators defined by (3.10) and (3.11) on the eigenstates of the two-dimensional oscillator are easily obtained. The assertion that this representation of the Euclidean group is reducible follows from the observation that the Casimir operator C of $\mathfrak{e}(2)$, which can be written as

$$C = P_1^2 + P_2^2,$$

does not act, as is directly checked, as a multiple of the identity on $|\, m,n\,\rangle$.

We use the following notation. Let $T(\theta,\alpha,\beta)$ be a generic element of the Euclidean group $E(2)$ where θ, α and β are real parameters; $T(\theta,\alpha,\beta)$ can be written as

$$T(\theta,\alpha,\beta) = \begin{pmatrix} \cos\theta & \sin\theta & \alpha/\sqrt{2} \\ -\sin\theta & \cos\theta & \beta/\sqrt{2} \\ 0 & 0 & 1 \end{pmatrix},$$

and represents the Euclidean move

$$(x_1,x_2,1)^\top \to T(\theta,\alpha,\beta)(x_1,x_2,1)^\top,$$

where $z^\top$ stands for transposition. The group multiplication law is provided by the standard matrix product. Consider the unitary representation defined by

$$U(T) = e^{i\alpha P_1} e^{i\beta P_2} e^{i\theta J}. \tag{3.13}$$

It is readily checked that $U(T)U^{\dagger}(T)=1$. The transformations of the generators a_i, $a_i^{\dagger}$ under the action of $U(T)$ is given by

$$\begin{aligned}
U(T)a_1U^{\dagger}(T) &= \cos\theta\, a_1 - \sin\theta\, a_2 - \alpha\cos\theta + \beta\sin\theta,\\
U(T)a_2U^{\dagger}(T) &= \sin\theta\, a_1 + \cos\theta\, a_2 - \alpha\sin\theta - \beta\cos\theta,
\end{aligned} \tag{3.14}$$

Similar formulas involving $a_1^{\dagger}$ and $a_2^{\dagger}$ are obtained by taking the complex conjugate of (3.14). Since one has $X_i = 2^{-1/2}(a_i + a_i^{\dagger})$, $i=1,2$, the transformation laws (3.14) give for the coordinate operator (X_1, X_2)

$$\begin{pmatrix}\widetilde{X}_1\\ \widetilde{X}_2\end{pmatrix} = U(T)\begin{pmatrix}X_1\\ X_2\end{pmatrix}U^{\dagger}(T) = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}X_1\\ X_2\end{pmatrix} + \begin{pmatrix}A\\ B\end{pmatrix}, \tag{3.15}$$

where

$$A = -\frac{2}{\sqrt{2}}\big(\alpha\cos\theta - \beta\sin\theta\big), \quad B = -\frac{2}{\sqrt{2}}\big(\alpha\sin\theta + \beta\cos\theta\big). \tag{3.16}$$

One thus has

$$U^{\dagger}(T)|\, x_1, x_2\,\rangle = |\,\widetilde{x}_1, \widetilde{x}_2\,\rangle = |\,T^{-1}x_1, T^{-1}x_2\,\rangle,$$

where $\widetilde{x}_1$, $\widetilde{x}_2$ are given by

$$\begin{pmatrix}\widetilde{x}_1\\ \widetilde{x}_2\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}A\\ B\end{pmatrix}, \tag{3.17}$$

with A, B given by (3.16). Moreover, one has $U(TT') = U(T)U(T')$ as should be for a group representation. The inverse transformation formulas

$$\begin{aligned}
U^{\dagger}(T)a_1U(T) &= \cos\theta\, a_1 + \sin\theta\, a_2 + \alpha,\\
U^{\dagger}(T)a_2U(T) &= -\sin\theta\, a_1 + \cos\theta\, a_2 + \beta.
\end{aligned} \tag{3.18}$$

and the Glauber formula [3]:

$$e^{\gamma(a_i^{\dagger} - a_i)} = e^{-\gamma^2/2} e^{\gamma a_i^{\dagger}} e^{-\gamma a_i}, \quad i = 1,2, \tag{3.19}$$

shall also prove useful in what follows.

3.3 The representation matrix elements as orthogonal polynomials

In this section it is shown that the matrix elements of the unitary representation of $E(2)$ defined in section 2 are expressed in terms of bivariate orthogonal polynomials. The matrix elements of

$U(T)$ defined by (3.13) can be written as

$$\langle i,k \mid U(T) \mid m,n \rangle = W_{i,k}\, C_{m,n}(i,k), \tag{3.20}$$

where $C_{0,0}(i,k) = 1$ and where $W_{i,k}$ is defined by

$$W_{i,k} = \langle i,k \mid U(T) \mid 0,0 \rangle. \tag{3.21}$$

To ease the notation, the explicit dependence of $U(T)$ on T shall be dropped.

3.3.1 Calculation of $W_{i,k}$

The amplitude $W_{i,k}$ can evaluated by a direct computation. Indeed, since one has $e^{i\theta J} \mid 0,0 \rangle = \mid 0,0 \rangle$, it follows that

$$W_{i,k} = \langle i,k \mid e^{i\alpha P_1} e^{i\beta P_2} e^{i\theta J} \mid 0,0 \rangle = \langle i,k \mid e^{i\alpha P_1} e^{i\beta P_2} \mid 0,0 \rangle.$$

Upon using the Glauber formula (3.19) to write

$$e^{i\alpha P_1} = e^{\alpha(a_1^{\dagger} - a_1)} = e^{-\alpha^2/2} e^{\alpha a_1^{\dagger}} e^{-\alpha a_1}, \quad e^{i\beta P_2} = e^{\beta(a_2^{\dagger} - a_2)} = e^{-\beta^2/2} e^{\beta a_2^{\dagger}} e^{-\beta a_2},$$

and the actions (3.3), one easily finds

$$W_{i,k} = e^{-(\alpha^2+\beta^2)/2} \frac{\alpha^i \beta^k}{\sqrt{i!k!}}. \tag{3.22}$$

3.3.2 Raising relations

It will now be shown that the functions $C_{m,n}(i,k)$ that appear in the matrix elements (3.20) are polynomials of total degree $m+n$ in the discrete variables i,k. This will be done by exhibiting raising relations for $C_{m,n}(i,k)$. Consider the matrix element $\langle i,k \mid U a_1^{\dagger} \mid m,n \rangle$. One has on the one hand

$$\langle i,k \mid U a_1^{\dagger} \mid m,n \rangle = \sqrt{m+1}\, W_{i,k}\, C_{m+1,n}(i,k), \tag{3.23}$$

and on the other hand, using (3.14), one has

$$\begin{aligned} \langle i,k \mid U a_1^{\dagger} \mid m,n \rangle = \langle i,k \mid U a_1^{\dagger} U^{\dagger} U \mid m,n \rangle &= \cos\theta \sqrt{i}\, W_{i-1,k} C_{m,n}(i-1,k) \\ &- \sin\theta \sqrt{k}\, W_{i,k-1} C_{m,n}(i,k-1) + (\beta \sin\theta - \alpha\cos\theta) W_{i,k} C_{m,n}(i,k). \end{aligned} \tag{3.24}$$

Upon comparing (3.23) and (3.24), one obtains using (3.22)

$$\begin{aligned} \sqrt{m+1}\, C_{m+1,n}(i,k) = &\left(\frac{i}{\alpha}\right) \cos\theta\, C_{m,n}(i-1,k) \\ &- \left(\frac{k}{\beta}\right) \sin\theta\, C_{m,n}(i,k-1) + (\beta\sin\theta - \alpha\cos\theta) C_{m,n}(i,k). \end{aligned} \tag{3.25}$$

Considering instead the matrix element $\langle i,k \mid Ua_2^\dagger \mid m,n \rangle$, one similarly finds

$$\begin{aligned}\sqrt{n+1}C_{m,n+1}(i,k) = &\left(\frac{i}{\alpha}\right)\sin\theta\, C_{m,n}(i-1,k)\\ &+\left(\frac{k}{\beta}\right)\cos\theta\, C_{m,n}(i,k-1) - (\alpha\sin\theta+\beta\cos\theta)C_{m,n}(i,k).\end{aligned} \tag{3.26}$$

By definition one has $C_{-1,n}(i,k) = C_{m,-1}(i,k) = 0$ and $C_{0,0}(i,k) = 1$. As a consequence, the formulas (3.25) and (3.26) can be used to construct $C_{m,n}(i,k)$ from $C_{0,0}(i,k)$ iteratively. One then observes that $C_{m,n}(i,k)$ are polynomials of total degree $m+n$ in the (discrete) variables i,k.

3.3.3 Orthogonality relation

The unitarity of the representation (3.13) and the orthonormality of the basis states leads to an orthogonality relation for the polynomials $C_{m,n}(i,k)$. One has

$$\langle m',n' \mid U^\dagger U \mid m,n \rangle = \sum_{i,k=0}^{\infty} \langle i,k \mid U \mid m,n \rangle \langle m',n' \mid U^\dagger \mid i,k \rangle = \delta_{mm'}\delta_{nn'}.$$

Upon using (3.20) and the reality of the matrix elements in the above equation, the following orthogonality relation is obtained:

$$\sum_{i,k=0}^{\infty} w_{i,k}\, C_{m,n}(i,k) C_{m',n'}(i,k) = \delta_{mm'}\delta_{nn'}, \tag{3.27}$$

where $w_{i,k}$ is the product of two independent Poisson distributions with (positive) parameters α^2 and β^2:

$$w_{i,k} = W_{i,k}^2 = e^{-(\alpha^2+\beta^2)} \frac{\alpha^{2i}\beta^{2k}}{i!k!}. \tag{3.28}$$

3.3.4 Lowering relations

Lowering relations for the polynomials $C_{m,n}(i,k)$ can be obtained by considering the matrix elements $\langle i,k \mid Ua_i \mid m,n \rangle$, $i = 1,2$ and proceeding as for the raising relations. From the matrix element $\langle i,k \mid Ua_1 \mid m,n \rangle$, one finds

$$\begin{aligned}\sqrt{m}\, C_{m-1,n}(i,k) = &\alpha\cos\theta\, C_{m,n}(i+1,k)\\ &-\beta\sin\theta\, C_{m,n}(i,k+1) + (\beta\sin\theta - \alpha\cos\theta)C_{m,n}(i,k).\end{aligned}$$

From the matrix element $\langle i,k \mid Ua_2 \mid m,n \rangle$, one obtains

$$\begin{aligned}\sqrt{n}\, C_{m,n-1}(i,k) = &\alpha\sin\theta\, C_{m,n}(i+1,k)\\ &+\beta\cos\theta\, C_{m,n}(i,k+1) - (\alpha\sin\theta+\beta\cos\theta)C_{m,n}(i,k).\end{aligned}$$

3.4 Duality

In this section, a duality relation under the exchange of the variables i,k and the degrees m,n in the polynomials $C_{m,n}(i,k)$ is obtained. Consider the matrix elements $\langle\, i,k \,|\, U^{\dagger} \,|\, m,n \,\rangle$ and write

$$\langle\, i,k \,|\, U^{\dagger} \,|\, m,n \,\rangle = \widetilde{W}_{i,k}\widetilde{C}_{m,n}(i,k), \tag{3.29}$$

where $\widetilde{C}_{0,0}(i,k) = 1$ and $\widetilde{W}_{i,k} = \langle\, i,k \,|\, U^{\dagger} \,|\, 0,0 \,\rangle$. To evaluate the amplitude $\widetilde{W}_{i,k}$, one first observes that the identity $\langle\, i,k \,|\, U^{\dagger} a_i \,|\, 0,0 \,\rangle = 0$ holds for $i = 1,2$. Using the inverse transformation formulas (3.18), one obtains the following system of difference equation

$$\begin{aligned}
\cos\theta\sqrt{i+1}\,\widetilde{W}_{i+1,k} + \sin\theta\sqrt{k+1}\,\widetilde{W}_{i,k+1} + \alpha\,\widetilde{W}_{i,k} &= 0,\\
-\sin\theta\sqrt{i+1}\,\widetilde{W}_{i+1,k} + \cos\theta\sqrt{k+1}\,\widetilde{W}_{i,k+1} + \beta\,\widetilde{W}_{i,k} &= 0.
\end{aligned}$$

It is easily seen that the solution of this system is given by

$$\widetilde{W}_{i,k} = C\,\frac{(\beta\sin\theta - \alpha\cos\theta)^i(-\alpha\sin\theta - \beta\cos\theta)^k}{\sqrt{i!k!}},$$

where C is a constant. The value of C can be determined by the normalization condition

$$1 = \langle\, 0,0 \,|\, U^{\dagger}U \,|\, 0,0 \,\rangle = \sum_{i,k=0}^{\infty} \langle\, i,k \,|\, U^{\dagger} \,|\, 0,0 \,\rangle\langle\, 0,0 \,|\, U \,|\, i,k \,\rangle = \sum_{i,k=0}^{\infty} |\widetilde{W}_{i,k}|^2,$$

which gives $C^2 = e^{-(\alpha\cos\theta - \beta\sin\theta)^2}e^{-(\alpha\sin\theta+\beta\cos\theta)^2} = e^{-(\alpha^2+\beta^2)}$ and thus

$$\widetilde{W}_{i,k} = e^{-(\alpha^2+\beta^2)/2}\,\frac{(\beta\sin\theta - \alpha\cos\theta)^i(-\alpha\sin\theta - \beta\cos\theta)^k}{\sqrt{i!k!}}. \tag{3.30}$$

Note that $\widetilde{W}_{i,k}$ can also be computed directly (see section 5). Since $U^{\dagger}(T) = U(T^{-1})$, the $\widetilde{C}_{m,n}(i,k)$ are the polynomials corresponding to the inverse transformation T^{-1}. For a transformation $T \in E(2)$ specified by the parameters (θ,α,β), the inverse $T^{-1} \in E(2)$ is specified by the parameters $(\widetilde{\theta},\widetilde{\alpha},\widetilde{\beta})$ given by

$$\widetilde{\theta} = -\theta, \quad \widetilde{\alpha} = (\beta\sin\theta - \alpha\cos\theta), \quad \widetilde{\beta} = -(\alpha\sin\theta + \beta\cos\theta). \tag{3.31}$$

One can also obtain the matrix element (3.29) by

$$\langle\, i,k \,|\, U^{\dagger} \,|\, m,n \,\rangle = \langle\, m,n \,|\, U \,|\, i,k \,\rangle^* = \langle\, m,n \,|\, U \,|\, i,k \,\rangle = W_{m,n}C_{i,k}(m,n), \tag{3.32}$$

where the reality of the matrix elements (3.20) has been used. Upon combining (3.29) and (3.32), one finds that

$$C_{i,k}(m,n) = \sqrt{\frac{m!n!}{i!k!}}\left(\frac{\widetilde{\alpha}^i\widetilde{\beta}^k}{\alpha^m\beta^n}\right)\widetilde{C}_{m,n}(i,k), \tag{3.33}$$

where $\widetilde{C}_{m,n}(i,k)$ corresponds to the polynomial $C_{m,n}(i,k)$ with parameters $(\widetilde{\theta},\widetilde{\alpha},\widetilde{\beta})$ given by (3.31). For the two variable polynomials $S_{m,n}(i,k)$ defined by

$$C_{m,n}(i,k) = \frac{(-1)^{n+m}}{\sqrt{m!n!}}(\alpha\cos\theta - \beta\sin\theta)^m(\alpha\sin\theta + \beta\cos\theta)^n S_{m,n}(i,k), \tag{3.34}$$

the duality relation (3.33) takes the elegant form

$$S_{i,k}(m,n) = \widetilde{S}_{m,n}(i,k).$$

3.5 Generating function

In this section, a generating function for the bivariate orthogonal polynomials $C_{m,n}(i,k)$ is obtained and is seen to correspond to a multivariate extension of that of the standard Charlier polynomials. Consider the generating series

$$F(x,y) = \sum_{m,n=0}^{\infty} W_{i,k}C_{m,n}(i,k)\frac{x^m y^n}{\sqrt{m!n!}} = \sum_{m,n=0}^{\infty}\langle\, i,k \mid U \mid m,n\,\rangle\frac{x^m y^n}{\sqrt{m!n!}}. \tag{3.35}$$

Using the expression (3.4) for the basis vectors $|\, n_1,n_2\,\rangle$ and the transformation formulas (3.14), one has

$$\begin{aligned}
F(x,y) &= \sum_{m,n=0}^{\infty}\frac{x^m y^n}{\sqrt{m!n!}}\langle\, i,k \mid U\frac{(a_1^\dagger)^m}{\sqrt{m!}}\frac{(a_2^\dagger)^n}{\sqrt{n!}}\mid 0,0\,\rangle = \sum_{m,n=0}^{\infty}\langle\, i,k \mid U\frac{(xa_1^\dagger)^m(ya_2^\dagger)^n}{m!n!}\mid 0,0\,\rangle\\
&= \langle\, i,k \mid Ue^{xa_1^\dagger}e^{ya_2^\dagger}\mid 0,0\,\rangle = \langle\, i,k \mid Ue^{xa_1^\dagger}U^\dagger Ue^{ya_2^\dagger}U^\dagger U\mid 0,0\,\rangle\\
&= \langle\, i,k \mid e^{xUa_1^\dagger U^\dagger}e^{yUa_2^\dagger U^\dagger}U\mid 0,0\,\rangle\\
&= e^{-x(\alpha\cos\theta-\beta\sin\theta)}e^{-y(\alpha\sin\theta+\beta\cos\theta)}\langle\, i,k \mid e^{a_1^\dagger(x\cos\theta+y\sin\theta)}e^{a_2^\dagger(y\cos\theta-x\sin\theta)}U\mid 0,0\,\rangle.
\end{aligned} \tag{3.36}$$

Since one has $U|\,0,0\,\rangle = e^{-(\alpha^2+\beta^2)/2}e^{\alpha a_1^\dagger}e^{\beta a_2^\dagger}|\,0,0\,\rangle$ by the Glauber formula (3.19) and by the actions (3.3), one finds

$$\begin{aligned}
F(x,y) &= e^{-(\alpha^2+\beta^2)/2}\\
&\quad\times e^{-x(\alpha\cos\theta-\beta\sin\theta)}e^{-y(\alpha\sin\theta+\beta\cos\theta)}\langle\, i,k \mid e^{a_1^\dagger(\alpha+x\cos\theta+y\sin\theta)}e^{a_2^\dagger(\beta+y\cos\theta-x\sin\theta)}\mid 0,0\,\rangle,
\end{aligned}$$

which gives

$$\begin{aligned}
F(x,y) &= e^{-(\alpha^2+\beta^2)/2}\\
&\quad\times e^{-x(\alpha\cos\theta-\beta\sin\theta)}e^{-y(\alpha\sin\theta+\beta\cos\theta)}\frac{(\alpha+x\cos\theta+y\sin\theta)^i(\beta+y\cos\theta-x\sin\theta)^k}{\sqrt{i!k!}}.
\end{aligned}$$

Recalling the expression (3.22) for $W_{i,k}$, the following generating function for the polynomials $C_{m,n}(i,k)$ is obtained:

$$
\begin{aligned}
e^{-x(\alpha\cos\theta-\beta\sin\theta)}e^{-y(\alpha\sin\theta+\beta\cos\theta)}\left[1+\frac{x}{\alpha}\cos\theta+\frac{y}{\alpha}\sin\theta\right]^{i}\left[1-\frac{x}{\beta}\sin\theta+\frac{y}{\beta}\cos\theta\right]^{k}\\
=\sum_{m,n=0}^{\infty}C_{m,n}(i,k)\frac{x^m y^n}{\sqrt{m!n!}}.
\end{aligned}
\tag{3.37}
$$

For the polynomials $S_{m,n}(i,k)$ given by (3.34), defining

$$
z_1=-x(\alpha\cos\theta-\beta\sin\theta)x,\quad z_2=-y(\alpha\sin\theta+\beta\cos\theta),
$$

yields the generating function

$$
e^{z_1+z_2}[1+u_{11}z_1+u_{12}z_2]^{i}[1+u_{21}z_1+u_{22}z_2]^{k}=\sum_{m,n=0}^{\infty}\frac{S_{m,n}(i,k)}{m!n!}z_1^m z_2^n
\tag{3.38}
$$

where the parameters u_{ij} are of the form

$$
\begin{aligned}
u_{11}&=\frac{-\cos\theta}{\alpha^2\cos\theta-\alpha\beta\sin\theta},\quad u_{12}=\frac{-\sin\theta}{\alpha^2\sin\theta+\alpha\beta\cos\theta},\\
u_{21}&=\frac{-\sin\theta}{\beta^2\sin\theta-\alpha\beta\cos\theta},\quad u_{22}=\frac{-\cos\theta}{\beta^2\cos\theta+\alpha\beta\sin\theta}.
\end{aligned}
\tag{3.39}
$$

The expression (3.38) for the generating function of the polynomials $S_{m,n}(i,k)$ lends itself to comparison with that of the Charlier polynomials (3.1). It is clear from this that $S_{m,n}(i,k)$ can be identified with multivariate Charlier polynomials.

3.6 Explicit expression in hypergeometric series

In this section, an explicit expression for the bivariate Charlier polynomials $S_{m,n}(i,k)$ in terms of a Gelfan'd-Aomoto hypergeometric series is obtained. Consider the generating relation (3.38). Upon denoting by $F(z_1,z_2)$ the left-hand side of (3.38) and using the trinomial expansion, one finds

$$
F(z_1,z_2)=e^{z_1+z_2}\sum_{\rho,\sigma,\mu,\nu}\binom{i}{\rho}\binom{i-\rho}{\sigma}\binom{k}{\mu}\binom{k-\mu}{\nu}u_{11}^{\rho}u_{12}^{\sigma}u_{21}^{\mu}u_{22}^{\nu}\,z_1^{\rho+\mu}z_2^{\sigma+\nu},
\tag{3.40}
$$

where the summation runs over all non-negative values of the indices and where binomial coefficients with negative entries are taken to be zero. Upon expanding the exponential in (3.40), gathering the terms in $z_1^m z_2^n$ and using the identity $\frac{(-1)^n m!}{(m-n)!}=(-m)_n$ where $(a)_n$ stands for the Pochhammer symbol

$$
(a)_n=a(a+1)\cdots(a+n-1),\quad (a)_0=1,
$$

one finds that the bivariate Charlier polynomials can be written as

$$S_{m,n}(i,k) = \sum_{\rho,\sigma,\mu,\nu} \frac{(-m)_{\rho+\mu}(-n)_{\nu+\sigma}(-i)_{\rho+\sigma}(-k)_{\mu+\nu}}{\rho!\sigma!\mu!\nu!} u_{11}^{\rho} u_{12}^{\sigma} u_{21}^{\mu} u_{22}^{\nu}, \tag{3.41}$$

where the u_{ij} are given by (3.39). The series appearing in (3.41) is a special case of Gelfan'd-Aomoto hypergeometric series [1, 5]. The multivariate orthogonal polynomials of Krawtchouk and Meixner type are also known to admit explicit expressions in terms of these multi-variable generalized hypergeometric series (see [13] and [14]).

3.7 Recurrence relations and difference equations

In this section, the group theoretical framework is exploited to obtain the bispectral properties of the bivariate Charlier polynomials $C_{m,n}(i,k)$.

3.7.1 Recurrence relations

A pair of recurrence relations for the bivariate Charlier polynomials can be obtained by examining the matrix elements $\langle\, i,k \mid a_i^{\dagger}a_i U \mid m,n\,\rangle$ for $i = 1,2$. Consider the case $i = 1$ first, one has on the one hand

$$\langle\, i,k \mid a_1^{\dagger}a_1 U \mid m,n\,\rangle = i\, W_{i,k}\, C_{m,n}(i,k). \tag{3.42}$$

On the other hand, using the unitary of U, one has

$$\langle\, i,k \mid a_1^{\dagger}a_1 U \mid m,n\,\rangle = \langle\, i,k \mid U\, U^{\dagger} a_1^{\dagger} U\, U^{\dagger} a_1 U \mid m,n\,\rangle. \tag{3.43}$$

Comparing (3.42) with (3.43) using the transformation rules (3.18), one obtains the recurrence relation

$$\begin{aligned} i\, C_{m,n}(i,k) &= \left[m\cos^2\theta + n\sin^2\theta + \alpha^2\right] C_{m,n}(i,k) \\ &+ \alpha\sin\theta\left[\sqrt{n+1}C_{m,n+1}(i,k) + \sqrt{n}C_{m,n-1}(i,k)\right] + \sin\theta\cos\theta\sqrt{n(m+1)}C_{m+1,n-1}(i,k) \\ &\quad + \alpha\cos\theta\left[\sqrt{m+1}C_{m+1,n}(i,k) + \sqrt{m}C_{m-1,n}(i,k)\right] + \sin\theta\cos\theta\sqrt{m(n+1)}C_{m-1,n+1}(i,k). \end{aligned}$$

In a similar fashion, one finds from the matrix element $\langle\, i,k \mid a_2^{\dagger}a_2 U \mid m,n\,\rangle$ a second recurrence relation

$$\begin{aligned} k\, C_{m,n}(i,k) &= \left[m\sin^2\theta + n\cos^2\theta + \beta^2\right] C_{m,n}(i,k) \\ &+ \beta\cos\theta\left[\sqrt{n+1}C_{m,n+1}(i,k) + \sqrt{n}C_{m,n-1}(i,k)\right] - \sin\theta\cos\theta\sqrt{n(m+1)}C_{m+1,n-1}(i,k) \\ &\quad - \beta\sin\theta\left[\sqrt{m+1}C_{m+1,n}(i,k) + \sqrt{m}C_{m-1,n}(i,k)\right] - \sin\theta\cos\theta\sqrt{m(n+1)}C_{m-1,n+1}(i,k). \end{aligned}$$

3.7.2 Difference equations

A pair of difference equations can be obtained by considering instead the matrix elements

$$\langle i,k \mid U a_i^\dagger a_i \mid m,n \rangle$$

for $i = 1,2$. Taking $i = 1$, one has

$$\langle i,k \mid U a_1^\dagger a_1 \mid m,n \rangle = m\, W_{i,k}\, C_{m,n}(i,k).$$

Comparing the above relation with $\langle i,k \mid U a_1^\dagger a_1 \mid m,n \rangle = \langle i,k \mid U a_1^\dagger U^\dagger\, U a_1 U^\dagger U \mid m,n \rangle$ using the transformation formulas (3.14), one obtains

$$\begin{aligned} m\, C_{m,n}(i,k) &= \left[i \cos^2\theta + k \sin^2\theta + \omega^2\right] C_{m,n}(i,k) \\ &- \omega \cos\theta \left[\frac{i}{\alpha} C_{m,n}(i-1,k) + \alpha\, C_{m,n}(i+1,k)\right] - \frac{i\beta}{\alpha} \cos\theta \sin\theta\, C_{m,n}(i-1,k+1) \\ &+ \omega \sin\theta \left[\frac{k}{\beta} C_{m,n}(i,k-1) + \beta\, C_{m,n}(i,k+1)\right] - \frac{k\alpha}{\beta} \cos\theta \sin\theta\, C_{m,n}(i+1,k-1), \end{aligned}$$

where $\omega = \alpha \cos\theta - \beta \sin\theta$. Starting instead from $\langle i,k \mid U a_2^\dagger a_2 \mid m,n \rangle$, one similarly finds

$$\begin{aligned} n\, C_{m,n}(i,k) &= \left[i \sin^2\theta + k \cos^2\theta + \zeta^2\right] C_{m,n}(i,k) \\ &- \zeta \sin\theta \left[\frac{i}{\alpha} C_{m,n}(i-1,k) + \alpha\, C_{m,n}(i+1,k)\right] + \frac{i\beta}{\alpha} \cos\theta \sin\theta\, C_{m,n}(i-1,k+1) \\ &- \zeta \cos\theta \left[\frac{k}{\beta} C_{m,n}(i,k-1) + \beta\, C_{m,n}(i,k+1)\right] + \frac{k\alpha}{\beta} \cos\theta \sin\theta\, C_{m,n}(i+1,k-1), \end{aligned}$$

with $\zeta = \alpha \sin\theta + \beta \cos\theta$.

3.8 Explicit expression in standard Charlier and Krawtchouk polynomials

In this section, an explicit expression for the bivariate Charlier polynomials $C_{m,n}(i,k)$ involving the standard (univariate) Charlier and Krawtchouk polynomials. To this end, one first notes that the matrix elements $\langle i,k \mid U \mid m,n \rangle$ can be decomposed as follows:

$$\langle i,k \mid U \mid m,n \rangle = \sum_{r,s,u,v=0}^{\infty} \langle i,k \mid e^{iP_1} \mid r,s \rangle \langle r,s \mid e^{i\beta P_2} \mid u,v \rangle \langle u,v \mid e^{i\theta J} \mid m,n \rangle. \tag{3.44}$$

We shall now examine individually the matrix elements appearing in the right hand side of the above equation.

Consider the matrix element $\langle\, i,k \mid e^{i\alpha P_1} \mid r,s \,\rangle$ which corresponds to a translation in the x_1 direction. Since P_1 acts only on the first quantum number, one has

$$\langle\, i,k \mid e^{i\alpha P_1} \mid r,s \,\rangle = \delta_{ks} \langle\, i,k \mid e^{\alpha P_1} \mid r,k \,\rangle.$$

The expression for the above matrix element is well known (see for example [22] or [21]) and can be obtained directly by taking $\beta = 0$ in the formula (3.22) for the amplitude $W_{i,k}$ and by taking $\theta = \beta = y = 0$ in the generating function (3.37). Comparing with (3.1), one finds that

$$\langle\, i,k \mid e^{i\alpha P_1} \mid r,s \,\rangle = \delta_{ks} \frac{(-1)^r \alpha^{r+i}}{\sqrt{i!r!}} e^{-\alpha^2/2} C_r(i;\alpha^2), \tag{3.45}$$

where $C_n(x;a)$ are the standard Charlier polynomials [17]. Similarly, one has

$$\langle\, r,s \mid e^{i\beta P_2} \mid u,v \,\rangle = \delta_{ru} \frac{(-1)^v \beta^{v+s}}{\sqrt{s!v!}} e^{-\beta^2/2} C_v(s;\beta^2). \tag{3.46}$$

The matrix element $\langle\, u,v \mid e^{i\theta J} \mid m,n \,\rangle$ can be evaluated straightforwardly using the methods of [6, 7]. The result reads

$$\langle\, u,v \mid e^{i\theta J} \mid m,n \,\rangle = \delta_{u+v,m+n} (-1)^v \binom{N}{v}^{1/2} \binom{N}{n}^{1/2} \cos^N\theta \tan^{v+n}\theta\, K_n(v;\sin^2\theta;N) \tag{3.47}$$

where $u + v = m + n = N$ and where $K_n(x;p;N)$ stands for the standard Krawtchouk polynomials [17]. Upon using the matrix elements (3.45), (3.46) and (3.47) in the decomposition formula (3.44) and using the formula (3.22), one finds that the bivariate Charlier polynomials $C_{m,n}(i,k)$ are given by

$$C_{m,n}(i,k) = \frac{(-1)^{n+m} \alpha^{m+n} \cos^m\theta \sin^n\theta}{\sqrt{m!n!}} \sum_{v=0}^{m+n} \binom{m+n}{v} \left(\frac{-\beta \sin\theta}{\alpha\cos\theta}\right)^v C_v(k;\beta^2) C_{m+n-v}(i;\alpha^2) K_n(v;\sin^2\theta, n+m), \tag{3.48}$$

where $C_n(x;a)$ and $K_n(x;p;N)$ are the standard Charlier and Krawtchouk polynomials.

3.9 Integral representation

An integral representation for the bivariate Charlier polynomials can be obtained by considering the matrix element $\langle\, x_1, x_2 \mid U \mid m,n \,\rangle$ from two different points of view. First, consider the action of U on $\mid m,n \,\rangle$. Using the definition (3.20) of the matrix elements, one can write

$$\langle\, x_1, x_2 \mid U \mid m,n \,\rangle = \sum_{i,k=0}^{\infty} W_{i,k} C_{m,n}(i,k) \Psi_i(x_1) \Psi_k(x_2),$$

where $\Psi_{n_i}(x_i)$, $i=1,2$, is given by (3.8). Second, take the action of U on the bra $\langle x_1,x_2 \|$. In view of (3.17), one may write

$$\langle x_1,x_2 \,|\, U \,|\, m,n \rangle = \langle \widetilde{x}_1,\widetilde{x}_2 \,|\, m,n \rangle = \Psi_m(\widetilde{x}_1)\Psi_n(\widetilde{x}_2),$$

where $(\widetilde{x}_1,\widetilde{x}_2)$ is given by (3.17). Combining the two previous relations, there comes

$$\Psi_m(\widetilde{x}_1)\Psi_n(\widetilde{x}_2) = \sum_{i,k=0}^{\infty} W_{i,k}\, C_{m,n}(i,k)\, \Psi_i(x_1)\Psi_k(x_2).$$

Upon multiplying both sides of the above equation by $\Psi_{i'}(x_1)\Psi_{k'}(x_2)$, integrating over the whole Euclidean plane and using the orthogonality relation (3.9) for the wavefunctions, one finds that

$$C_{m,n}(i,k) = \frac{1}{W_{i,k}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi_i(x_1)\Psi_k(x_2)\Psi_m(\widetilde{x}_1)\Psi_n(\widetilde{x_2})\,\mathrm{d}x_1\mathrm{d}x_2,$$

with $(\widetilde{x}_1,\widetilde{x}_2)$ given by (3.17). In view of (3.8), this gives a formula for the bivariate Charlier polynomials $C_{m,n}(i,k)$ in terms of a double integral of a product of four Hermite polynomials.

3.10 Charlier polynomials as limits of Krawtchouk polynomials

In this section, it is shown that the bivariate Charlier polynomials $C_{m,n}(i,k)$ can be obtained from the bivariate Krawtchouk polynomials by a limit process. We begin by providing some background information on the bivariate Krawtchouk polynomials.

3.10.1 Bivariate Krawtchouk polynomials

The bivariate Krawtchouk polynomials $P_{m,n}(i,k;N)$ of two discrete variables i, k arise as matrix elements of the rotation group $SO(3)$ on the energy $E = N + 3/2$ eigenspace of a three-dimensional isotropic harmonic oscillator [7]. In addition to the non-negative integer N, the bivariate Krawtchouk polynomials have for parameters the entries of a rotation matrix $R \in SO(3)$. Hence for each N and $R \in SO(3)$, one has a finite set of bivariate Krawtchouk polynomials. The polynomials $P_{m,n}(i,k;N)$ can be defined through the following generating function:

$$\begin{aligned} G(u,v) &= \left(1+\frac{R_{11}}{R_{13}}u+\frac{R_{12}}{R_{13}}v\right)^{i}\left(1+\frac{R_{21}}{R_{23}}u+\frac{R_{22}}{R_{23}}v\right)^{k}\left(1+\frac{R_{31}}{R_{33}}u+\frac{R_{32}}{R_{33}}v\right)^{N-i-k} \\ &= \sum_{\substack{m,n=0\\ m+n\leqslant N}}^{N} \binom{N}{m,n}^{1/2} P_{m,n}(i,k;N)\,u^m v^n, \end{aligned} \tag{3.49}$$

where $\binom{N}{m,n}$ stands for the trinomial coefficients

$$\binom{N}{m,n} = \frac{N!}{m!n!(N-m-n)!}.$$

They satisfy the orthogonality relation

$$\sum_{\substack{i,k=0\\ i+k\leqslant N}}^{N} w_{i,k;N} P_{m,n}(i,k;N) P_{m',n'}(i,k;N) = \delta_{mm'}\delta_{nn'},$$

with respect to the discrete weight

$$w_{i,k;N} = \binom{N}{i,k} R_{13}^{2i} R_{23}^{2k} R_{33}^{2(N-i-k)}.$$

The normalization condition $\sum_{i+k\leqslant N}^{N} w_{i,k;N} = 1$ is ensured by the fact that R is an orthogonal matrix, i.e. $RR^T = 1$.

3.10.2 The $N \to \infty$ limit of the bivariate Krawtchouk polynomials

It is well known that the $E(2)$ group can be obtained from the $SO(3)$ group by a contraction [16]. Consider the Lie algebra $\mathfrak{so}(3)$ defined by the commutation relations

$$[J_1, J_2] = iJ_3, \quad [J_2, J_3] = iJ_1, \quad [J_3, J_1] = iJ_2.$$

Upon redefining $J_1 = \epsilon P_1$, $J_2 = \epsilon P_2$ and $J_3 = J$, it is easily seen that in the limit as $\epsilon \to 0$, the $\mathfrak{so}(3)$ commutation relations contract to those of the Euclidean Lie algebra $\mathfrak{e}(2)$ given in (3.12). In view of the connection between $SO(3)$ and bivariate Krawtchouk polynomials, this relation can be used to obtain the bivariate Charlier polynomials $C_{m,n}(i,k)$ defined here as limits of the bivariate Krawtchouk polynomials.

Let R be a general $SO(3)$ rotation matrix. One can take the parametrization

$$R = r_{x_2}(\delta) r_{x_1}(\gamma) r_{x_3}(\theta), \tag{3.50}$$

where r_{x_i}, $i = 1,2,3$, are the rotation matrices around the x_1, x_2 and x_3 axes:

$$r_{x_2}(\delta) = \begin{pmatrix} \cos\delta & 0 & \sin\delta \\ 0 & 1 & 0 \\ -\sin\delta & 0 & \cos\delta \end{pmatrix}, \quad r_{x_1}(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & \sin\gamma \\ 0 & -\sin\gamma & \cos\gamma \end{pmatrix},$$

$$r_{x_3}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Upon taking the parametrization

$$\delta \to \frac{\alpha}{\sqrt{N}}, \quad \gamma \to \frac{\beta}{\sqrt{N}}, \tag{3.51}$$

in the generating function (3.49) of the bivariate Krawtchouk polynomials, a direct computation shows that

$$\lim_{N\to\infty} G\left(\frac{x}{\sqrt{N}}, \frac{y}{\sqrt{N}}\right) =$$
$$e^{-x(\alpha\cos\theta-\beta\sin\theta)}e^{-y(\alpha\sin\theta-\beta\cos\theta)}\left(1+\frac{x}{\alpha}\cos\theta+\frac{y}{\alpha}\sin\theta\right)^{i}\left(1-\frac{x}{\beta}\sin\theta+\frac{y}{\beta}\cos\theta\right)^{k}, \tag{3.52}$$

and also that

$$\lim_{N\to\infty} G\left(\frac{x}{\sqrt{N}}, \frac{y}{\sqrt{N}}\right) = \sum_{m,n=0}^{\infty} \frac{x^m y^n}{\sqrt{m!n!}} \lim_{N\to\infty} P_{m,n}(i,k;N). \tag{3.53}$$

Combining (3.52) and (3.53), it is directly seen that the resulting generating function coincides with that of the bivariate Charlier polynomials given by (3.37). Consequently, under the parametrizations (3.50) and (3.51), the Charlier polynomials $C_{m,n}(i,k)$ can be obtained by a $N\to\infty$ limit of the bivariate Krawtchouk polynomials $P_{m,n}(i,k;N)$:

$$\lim_{N\to\infty} P_{m,n}(i,k;N) = C_{m,n}(i,k). \tag{3.54}$$

This limiting procedure can be applied to the raising/lowering relations, difference equations, recurrence relations and explicit expression derived in [7] for the bivariate Krawtchouk polynomials and it is verified that these yield the corresponding relations obtained here for the bivariate Charlier polynomials.

3.11 Multidimensional case

In this section, it is shown how the results obtained so far can be generalized by considering the space of state vectors of a d-dimensional isotropic harmonic oscillator to obtain an algebraic description of the multivariate Charlier polynomials in d variables. Consider the Hamiltonian of an isotropic d-dimensional harmonic oscillator

$$\mathcal{H} = \sum_{k=1}^{d} a_k^{\dagger} a_k + d/2,$$

where the creation/annihilation operators $a_k^{\dagger}$, a_k satisfy the Weyl algebra commutation relations (3.2). An eigenbasis for $\mathcal{H}$ is provided by the state vectors $|\, n_1, n_2, \ldots, n_d\,\rangle$ where

n_k, $k = 1,\ldots,d$, are non-negative integers and the action of the creation/annihilation operators is given by (3.3). The states $|\, n_1, n_2, \ldots, n_d\,\rangle$ provide a reducible representation of the Euclidean group $E(d)$. The elements of the Euclidean group $E(d)$ are specified by a $d \times d$ orthogonal matrix R and a real vector $(\alpha_1, \alpha_2, \ldots, \alpha_d)$ with $d+1$ components. These elements denoted by $T(R,\alpha)$ hence depend on $\frac{d(d+1)}{2}$ independent parameters and they can be represented by the $(d+1)\times(d+1)$ matrix

$$T(R,\alpha) = \begin{pmatrix} R & \begin{pmatrix} \alpha_1/\sqrt{2} \\ \vdots \\ \alpha_d/\sqrt{2} \end{pmatrix} \\ \mathbf{0} & 1 \end{pmatrix}.$$

The group law is provided by matrix multiplication. Consider now the unitary representation $U(T)$ defined by

$$U(T) = \prod_{k=1}^{d} e^{\alpha_k (a_k^\dagger - a_k)}\, e^{\sum_{j,k=1} B_{jk} a_j^\dagger a_k},$$

where the rotation matrix R is related to the antisymmetric matrix B by $e^B = R$. The transformations of the generators a_k, $a_k^\dagger$ under the action $U(T)$ are

$$U(T) a_k U^\dagger(T) = \sum_{j=1}^{d} R_{jk} \left(a_j^\dagger + \alpha_j \right), \tag{3.55}$$

and similarly for $a_k^\dagger$. In the same spirit as in section 3, one can write the matrix elements of this reducible $E(d)$ representation on the eigenstates of the d-dimensional isotropic oscillator as follows

$$\langle\, i_1, i_2, \ldots, i_d \,|\, U(T) \,|\, n_1, n_2, \ldots, n_d \,\rangle = W_{i_1,\ldots,i_d} C_{n_1,\ldots,n_d}(i_1, \ldots, i_d),$$

where

$$W_{i_1,\ldots,i_d} = \langle\, i_1, \ldots, i_d \,|\, U(T) \,|\, 0, \ldots, 0 \,\rangle.$$

Since $U(T)|\, 0,\ldots,0\,\rangle = \prod_{k=1}^{d} e^{\alpha_k (a_k^\dagger - a_k)} |\, 0, \ldots, 0\,\rangle$, the amplitude $W_{i_1,\ldots,i_d}$ is directly evaluated to

$$W_{i_1,\ldots,i_d} = e^{-\sum_{k=1}^{d} \alpha_k^2/2} \prod_{k=1}^{d} \frac{\alpha_k^{i_k}}{\sqrt{i_k!}}.$$

It is easily verified by deriving the raising relations as in section 3, that the functions $C_{n_1,\dots,n_d}(i_1,\dots,i_d)$ are polynomials in the discrete variables $i_1,\dots,i_d$ of total degree $n_1+n_2+\dots+n_d$. These polynomials are orthogonal with respect to the product of d independent Poisson distributions

$$\sum_{i_1,\dots i_d=0}^{\infty} W^2_{i_1,\dots,i_d} C_{n_1,\dots,n_d}(i_1,\dots,i_d) C_{m_1,\dots,m_d}(i_1,\dots,i_d) = \delta_{n_1m_1}\delta_{n_2m_2}\cdots\delta_{n_dm_d}.$$

The generating function can be obtained following the method of section 5 and one finds

$$e^{-\sum_{i,j} R_{ij}\alpha_i x_j} \prod_{k=1}^{d} \left(1+\sum_{\ell=1}^{d} R_{k\ell}\frac{x_\ell}{\alpha_k}\right)^{i_k} = \sum_{n_1,\dots,n_d=0}^{\infty} C_{n_1,\dots,n_d}(i_1,\dots,i_k)\frac{x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k}}{\sqrt{n_1!n_2!\cdots n_k!}}$$

Deriving the properties of the d-variable polynomials $C_{n_1,\dots,n_d}(i_1,\dots,i_d)$ can be done exactly as for $d=2$.

3.12 Conclusion

In this paper we have considered the matrix elements of the unitary representation of the Euclidean group $E(2)$ on the states of the two-dimensional isotropic harmonic oscillator and showed that these matrix elements can be expressed in terms of new bivariate orthogonal polynomials that generalize the standard Charlier polynomials. Using the group theoretical setting, the main properties of the polynomials were derived. Furthermore, it was shown that the approach easily extends to d dimensions giving the d-variate Charlier polynomials as matrix elements of unitary representations of the Euclidean group $E(d)$ on oscillator states. Let us now offer some comments.

As a first remark, we note that the approach proposed here could be modified straightforwardly to obtain a different family of multivariate Charlier polynomials associated to the pseudo-Euclidean group $E(d-1,1)$. In the bivariate case, the $\mathfrak{e}(1,1)$ generators can be realized with the creation/annihilation operators in the following way:

$$\widetilde{P}_1 = i(a_1 - a_1^\dagger), \quad \widetilde{P}_2 = i(a_2 - a_2^\dagger), \quad K = i(a_1^\dagger a_2^\dagger - a_1 a_2).$$

Using this realization, the unitary representation of the group pseudo-Euclidean group $E(1,1)$ on the states $|\, n_1, n_2\,\rangle$ can be constructed as in section 2 and the matrix elements can be expressed in terms of multivariate orthogonal polynomials.

As a second remark, it is worth mentioning that the results presented here can be combined with those of [6] to construct unitary representations of the Poincaré group on oscillator states whose matrix elements are given in terms of both multivariate Charlier and Meixner polynomials. In two dimensions the Poincaré group generators are realized as follows with the operators of three harmonic oscillators (see [6]):

Space translations: $P_1 = \frac{i}{\sqrt{2}}(a_1^\dagger - a_1), \quad P_2 = \frac{i}{\sqrt{2}}(a_2^\dagger - a_2),$

Time translation: $P_0 = \frac{i}{\sqrt{2}}(a_3 - a_3^\dagger),$

Lorentz Boosts: $K_1 = i(a_1^\dagger a_3^\dagger - a_1 a_3), \quad K_2 = i(a_2^\dagger a_3^\dagger - a_2 a_3),$

Rotation: $J = i(a_1 a_2^\dagger - a_1^\dagger a_2).$

For this case, it would be of interest to proceed as in [8] and decompose the unitary representations in its irreducible components.

As a last remark, it is observed that our approach offers a path to defining q-extensions of the multi-variable Charlier polynomials considered here. Indeed, one could consider the realization of the quantum group $E_q(2)$ with two q-oscillators and construct the matrix elements of q-exponentials in the $E_q(2)$ generators. A comparison with the multivariate q-Charlier polynomials defined in [4] would be of interest. We hope to report on this in the future.

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Chapitre 4

Interbasis expansions for the isotropic 3D harmonic oscillator and bivariate Krawtchouk polynomials

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Abstract. An explicit expression for the general bivariate Krawtchouk polynomials is obtained in terms of the standard Krawtchouk and dual Hahn polynomials. The bivariate Krawtchouk polynomials occur as matrix elements of the unitary reducible representations of $SO(3)$ on the energy eigenspaces of the 3-dimensional isotropic harmonic oscillator and the explicit formula is obtained from the decomposition of these representations into their irreducible components. The decomposition entails expanding the Cartesian basis states in the spherical bases that span irreducible $SO(3)$ representations. The overlap coefficients are obtained from the Clebsch-Gordan problem for the $\mathfrak{su}(1,1)$ Lie algebra.

4.1 Introduction

The standard Krawtchouk polynomials orthogonal with respect to the binomial distribution are known to enter the expression of the Wigner $\mathscr{D}$-functions which give the matrix elements of the irreducible representations of $SU(2)$ in the standard bases. The multivariate polynomials that generalize them are orthogonal with respect to the multinomial distribution. Although the definition of the multivariate polynomials goes back to 1971 when it was given in a Statistics context [6], to our knowledge their introduction in the study of Mathematical Physics problems is much more recent (see [4] for more background). For instance, the bivariate Krawtchouk polynomials have been seen to occur in the wavefunctions of a superintegrable finite oscillator model with $SU(2)$ symmetry [11]. They have also been shown to arise as the $9j$-symbol of the oscillator algebra [15]. As well, the 2-variable Krawtchouk polynomials have been used to design a two-dimensional spin lattice with remarkable quantum state transfer properties [10].

Lately, the Krawtchouk polynomials in n discrete variables have been interpreted as matrix elements of the reducible representations of $SO(n+1)$ on the energy eigenspaces of the $(n+1)$-dimensional isotropic harmonic oscillator [4]. This has provided a natural setting within which the various properties of these polynomials could be straightforwardly derived. It is the purpose of this paper to further exploit this group theoretical connection and to obtain a new expansion formula that emerges from the irreducible decomposition of the relevant rotation group representations. The overlap coefficients between the Cartesian and spherical bases [9] will be needed and it shall also be indicated how these can be recovered using a correspondence with the Clebsch-Gordan problem of the $\mathfrak{su}(1,1)$ algebra. The focus here is on the bivariate case.

4.1.1 Three-dimensional isotropic harmonic oscillator

The isotropic 3-dimensional harmonic oscillator is described by the Hamiltonian

$$\mathscr{H} = -\frac{1}{2}\nabla^2 + \frac{1}{2}(x^2+y^2+z^2), \tag{4.1}$$

where ∇^2 denotes the Laplacian. The Schrödinger equation $\mathscr{H}\Psi = \mathscr{E}\Psi$ associated to (4.1) separates in particular in Cartesian, polar (cylindrical) and spherical coordinates. In each of these coordinate systems, the exact solutions are known [3] and the eigenstates of (4.1) are labeled by three quantum numbers. One has the following bases and the corresponding wavefunctions for the states of the oscillator:

1. The Cartesian basis denoted by $|\, n_x, n_y, n_z\, \rangle_C$ where $n_x, n_y, n_z \in \mathbb{N}$ and with energy eigenvalue $\mathscr{E} = n_x + n_y + n_y + 3/2 = N + 3/2$. The wavefunctions are denoted by $\Psi_{n_x,n_y,n_z}(x,y,z)$ and given by

$$\Psi_{n_x,n_y,n_z}(x,y,z) = \sqrt{\frac{1}{2^N \pi^{3/2} n_x! n_y! n_z!}}\, e^{-(x^2+y^2+z^2)/2} H_{n_x}(x) H_{n_y}(y) H_{n_z}(z), \tag{4.2}$$

   where $H_n(x)$ stands for the Hermite polynomials [7].

2. The polar basis denoted by $|\, n_\rho, m, n_z\, \rangle_P$ where $n_\rho \in \mathbb{N}$, $m \in \mathbb{Z}$, $n_z \in \mathbb{N}$ and with energy eigenvalue $\mathscr{E} = 2n_\rho + |m| + n_z + 3/2 = N + 3/2$. The associated wavefunctions are denoted

$\Psi_{n_\rho,m,n_z}(\rho,\phi,z)$ and given by

$$\Psi_{n_\rho,m,n_z}(\rho,\phi,z) = \frac{(-1)^{n_\rho}}{\pi^{3/4}}\sqrt{\frac{n_\rho!}{2^{n_z}n_z!\Gamma(n_\rho+|m|+1)}}\,e^{-(\rho^2+z^2)/2}\rho^{|m|}L_{n_\rho}^{(|m|)}(\rho^2)H_{n_z}(z)e^{im\phi}, \tag{4.3}$$

where $L_n^{(\alpha)}(x)$ are the Laguerre polynomials [7].

3. The spherical basis $|\, n_r,\ell,m\,\rangle_S$ where $n_r \in \mathbb{N}$, $\ell \in \mathbb{N}$, $m=-\ell,\ldots,\ell$ and with energy eigenvalue $\mathscr{E} = 2n_r+\ell+3/2 = N+3/2$. The wavefunctions are denoted $\Psi_{n_r,\ell,m}(r,\theta,\phi)$ and given by

$$\Psi_{n_r,\ell,m}(r,\theta,\phi) = (-1)^{n_r}\,e^{-r^2/2}\,r^\ell\sqrt{\frac{2n_r!}{\Gamma(n_r+\ell+3/2)}}L_{n_r}^{(\ell+1/2)}(r^2)Y_\ell^m(\theta,\phi), \tag{4.4}$$

where $Y_\ell^m(\theta,\phi)$ are the spherical harmonics [3].

It is directly seen that the energy level N has degeneracy $(N+1)(N+2)/2$. The creation/annihilation operators

$$a_{x_i} = \frac{1}{\sqrt{2}}(x_i+\partial_{x_i}), \quad a_{x_i}^\dagger = \frac{1}{\sqrt{2}}(x_i-\partial_{x_i}), \quad i=1,2,3,$$

with $x_1 = x$, $x_2 = y$, $x_3 = z$ obey the commutation relations

$$[a_{x_i},a_{x_j}^\dagger] = \delta_{ij}, \quad [a_{x_i},a_{x_j}] = 0, \qquad i,j=1,2,3,$$

and have the following actions on the Cartesian basis states:

$$a_{x_i}|\, n_{x_i}\,\rangle_C = \sqrt{n_{x_i}}\,|\, n_{x_i}-1\,\rangle_C,\ a_{x_i}^\dagger|\, n_{x_i}\,\rangle = \sqrt{n_{x_i}+1}\,|\, n_{x_i}+1\,\rangle_C.$$

It follows that $a_{x_i}^\dagger a_{x_i}|\, n_{x_i}\,\rangle = n_{x_i}|\, n_{x_i}\,\rangle_C$. In terms of these operators, (4.1) takes the form

$$\mathscr{H} = a_x^\dagger a_x + a_y^\dagger a_y + a_z^\dagger a_z + 3/2, \tag{4.5}$$

and one has indeed $\mathscr{H}|\, n_x,n_y,n_z\,\rangle_C = (N+3/2)|\, n_x,n_y,n_z\,\rangle_C$.

4.1.2 $SO(3) \subset SU(3)$ and oscillator states

The Hamiltonian (4.5) of the 3-dimensional isotropic Harmonic oscillator is clearly invariant under $SU(3)$ transformations, which are generated by the constants of motion of the form $a_i^\dagger a_j$. For each value of N, the Cartesian basis states $|\, n_x,n_y,n_z\,\rangle_C$ support the completely symmetric irreducible representation of $SU(3)$. The Hamiltonian (4.1) is also manifestly invariant under $SO(3)\subset SU(3)$ transformations. These rotations are generated by the three angular momentum generators

$$L_x = -i(a_y^\dagger a_z - a_z^\dagger a_y),\ L_y = -i(a_z^\dagger a_x - a_x^\dagger a_z),\ L_z = -i(a_x^\dagger a_y - a_y^\dagger a_x), \tag{4.6}$$

obeying the commutation relations

$$[L_x, L_y] = iL_z, \quad [L_y, L_z] = iL_x, \quad [L_z, L_y] = iL_x.$$

The representation of $SO(3)$ on the oscillator states with a given energy is reducible. The irreducible content of this representation can be found by examining the states $|\, n_r, \ell, m\,\rangle_S$ of the spherical basis. These states are the common eigenstates of the $\mathfrak{so}(3)$ Casimir operator $\vec{L}^2 = L_x^2 + L_y^2 + L_z^2$ and of L_z with eigenvalues

$$\vec{L}^2 |\, n_r, \ell, m\,\rangle_S = \ell(\ell+1) |\, n_r, \ell, m\,\rangle_S, \quad L_z |\, n_r, \ell, m\,\rangle_S = m |\, n_r, \ell, m\,\rangle_S.$$

For each value of n_r, these states provide a basis for the $(2\ell+1)$-dimensional irreducible representation of $SO(3)$. Since $N = 2n_r + \ell$, it follows that for a given N the $SO(3)$ representation on the eigenstates of the isotropic oscillator contains once, each and every $(2\ell+1)$-dimensional irreducible representation of $SO(3)$ with $\ell = N, N-2, \ldots, 1$ or 0, depending on the parity of N. One notes that in the polar basis $|\, n_r, m, n_z\,\rangle_P$, the following operators are diagonal:

$$L_z |\, n_\rho, m, n_z\,\rangle_P = m |\, n_\rho, m, n_z\,\rangle_P, \quad a_z^\dagger a_z |\, n_\rho, m, n_z\,\rangle_P = n_z |\, n_\rho, m, n_z\,\rangle_P.$$

4.1.3 Unitary representations of $SO(3)$ and bivariate Krawtchouk polynomials

Let $R \in SO(3)$ and consider the unitary representation provided by

$$U(R) = \exp\left(\sum_{i,j=1}^{3} B_{ij} a_i^\dagger a_j\right), \tag{4.7}$$

where $B^\top = -B$ and $R = e^B$. It has been shown in [4] that the matrix elements of this unitary operator in the Cartesian basis have the expression

$${}_C\langle\, i, k, l \,|\, U(R) \,|\, r, s, t\,\rangle_C = W_{i,k;N} P_{r,s}(i, k; N),$$

where $i + k + l = N = r + s + t$ and where

$$W_{i,k;N} = \binom{N}{i,k}^{1/2} R_{33}^N \left(\frac{R_{13}}{R_{33}}\right)^i \left(\frac{R_{23}}{R_{33}}\right)^k, \tag{4.8}$$

with $\binom{N}{i,k}$ denoting the trinomial coefficients

$$\binom{N}{i,k} = \frac{N!}{i!k!(N-i-k)!}.$$

The $P_{r,s}(i,k;N)$ are the general bivariate Krawtchouk polynomials which have for parameters the entries R_{ij} of the 3×3 rotation matrix $R\in SO(3)$. The polynomials $P_{r,s}(i,k;N)$ enjoy many interesting properties. They are orthonormal with respect to the trinomial distribution

$$\sum_{i+k\leqslant N} W_{i,k;N}^{2} P_{r,s}(i,k;N) P_{r',s'}(i,k;N) = \delta_{rr'}\delta_{ss'}.$$

and have for generating relation

$$\left(1+\frac{R_{11}}{R_{13}}u+\frac{R_{12}}{R_{13}}v\right)^{i}\left(1+\frac{R_{21}}{R_{23}}u+\frac{R_{22}}{R_{23}}v\right)^{k}\left(1+\frac{R_{31}}{R_{33}}u+\frac{R_{32}}{R_{33}}v\right)^{N-i-k} = \sum_{r+s\leqslant N}\binom{N}{r,s}^{1/2} P_{r,s}(i,k;N)\,u^{r}v^{s}.$$

The polynomials $P_{r,s}(i,k;N)$ have an explicit formula in terms of Gel'fand-Aomoto hypergeometric series

$$P_{r,s}(i,k;N)=\binom{N}{r,s}^{1/2}\left(\frac{R_{31}}{R_{33}}\right)^{r}\left(\frac{R_{32}}{R_{33}}\right)^{s} \times \sum_{\alpha+\beta+\gamma+\delta\leqslant N}\frac{(-r)_{\alpha+\beta}(-s)_{\gamma+\delta}(-i)_{\alpha+\gamma}(-k)_{\beta+\delta}}{\alpha!\beta!\gamma!\delta!(-N)_{\alpha+\beta+\gamma+\delta}}(1-u_{11})^{\alpha}(1-u_{21})^{\beta}(1-u_{12})^{\gamma}(1-u_{22})^{\delta},$$

where $(a)_n=(a)(a+1)\cdots(a+n-1)$ stands for the Pochammer symbol and where

$$u_{11}=\frac{R_{11}R_{33}}{R_{13}R_{31}},\ u_{12}=\frac{R_{12}R_{33}}{R_{13}R_{32}},\ u_{21}=\frac{R_{21}R_{33}}{R_{23}R_{31}},\ u_{22}=\frac{R_{22}R_{33}}{R_{23}R_{32}}.$$

The polynomials $P_{r,s}(i,k;N)$ have the following integral representation involving the Hermite polynomials:

$$P_{r,s}(i,k;N)=\frac{R_{13}^{-i}R_{23}^{-k}R_{33}^{-l}}{2^{N}\pi^{3/2}N!}\binom{N}{r,s}^{1/2} \times \int_{\mathbb{R}^3} e^{-(x_1^2+x_2^2+x_3^2)}H_r(\widetilde{x}_1)H_s(\widetilde{x}_2)H_t(\widetilde{x}_3)H_i(x_1)H_k(x_2)H_l(x_3)\,dx_1dx_2dx_3.$$

where $N=i+k+l=r+s+t$ and $(\widetilde{x}_1,\widetilde{x}_2,\widetilde{x}_3)^{\top}=R^{\top}(x_1,x_2,x_3)^{\top}$. They can also be expressed as a sum over products of three standard Krawtchouk polynomials (4.9).

In the case $R_{12}=0$, the general bivariate Krawtchouk polynomials $P_{r,s}(i,k;N)$ reduce to the bivariate Krawtchouk polynomials $K_2(m,n;i,k;\mathfrak{p}_1,\mathfrak{p}_2;N)$ introduced by Tratnik in [12] (see also [5] for their bispectral properties). These polynomials have the explicit expression

$$K_2(m,n;i,k;\mathfrak{p}_1,\mathfrak{p}_2;N)=\frac{(n-N)_m(i-N)_n}{(-N)_{m+n}}K_m(i;\mathfrak{p}_1;N-n)K_n(k;\frac{\mathfrak{p}_2}{1-\mathfrak{p}_1};N-i),$$

where $K_n(x;p;N)$ stands for the standard Krawtchouk polynomials

$$K_n(x;p;N) = {}_2F_1\left[\begin{matrix}-n,-n\\-N\end{matrix};\frac{1}{p}\right], \tag{4.9}$$

and where ${}_pF_q$ denotes the generalized hypergeometric function [7]. The condition $R_{12} = 0$ is ensured if R is taken to be a product of two successive clockwise rotations $R = R_x(\theta)R_y(\chi)$ around the x and y axes, respectively. This rotation is unitarily represented by $U(R) = e^{i\theta L_x}e^{i\chi L_y}$ and one has [4]

$${}_C\langle\, i,k,l \,|\, e^{i\theta L_x}e^{i\chi L_y} \,|\, r,s,t \,\rangle_C = R_{33}^{-N}\, W_{i,k;N}\widetilde{W}_{r,s;N}\, K_2(r,s;i,k;\mathfrak{p}_1,\mathfrak{p}_2;N), \tag{4.10}$$

where $\widetilde{W}_{m,n;N}$ is given by (4.8) with the parameters of the rotation matrix R replaced by their transpose. One has again $r+s+t = N = i+k+l$ and furthermore $\mathfrak{p}_1 = R_{13}^2$ and $\mathfrak{p}_2 = R_{23}^2$. The polynomials of Tratnik thus depend only on two parameters, as opposed to three parameters for the general polynomials $P_{r,s}(i,k;N)$. The reader is referred to [4] for the group theoretical characterization of the polynomials $P_{r,s}(i,k;N)$ and references on the multivariate Krawtchouk polynomials.

4.1.4 The main result

The stage has now been set for the statement of the main formula of this paper. The most general rotation $R \in SO(3)$, which depends on three parameters, can be taken of the form

$$R = \begin{pmatrix} c_\alpha c_\beta c_\gamma - s_\alpha s_\gamma & -s_\alpha c_\beta c_\gamma - c_\alpha s_\gamma & s_\beta c_\gamma \\ c_\alpha c_\beta s_\gamma + s_\alpha c_\gamma & c_\alpha c_\gamma - s_\alpha c_\beta s_\gamma & s_\beta s_\gamma \\ -c_\alpha s_\beta & s_\alpha s_\beta & c_\beta \end{pmatrix},$$

where $c_\theta = \cos\theta$ and $s_\theta = \sin\theta$. This rotation is unitarily represented by the operator

$$U(R) = e^{-i\gamma L_z}e^{-i\beta L_y}e^{-i\alpha L_z}.$$

The parameters α, β and γ thus correspond to the Euler angles. The decomposition of the $SO(3)$ representation on the energy eigenspaces of the isotropic 3D harmonic oscillator in irreducible components amounts to the expansion of the Cartesian basis states $|\, n_x, n_y, n_z \,\rangle_C$ in the spherical basis states $|\, n_r, \ell, m \,\rangle_S$:

$$\begin{aligned} &{}_C\langle\, i,k,l \,|\, U(R) \,|\, r,s,t \,\rangle_C \\ &= \sum_{n_r,\ell,m}\sum_{n_r',\ell',m'} {}_C\langle i,k,l \,|\, n_r',\ell',m' \,\rangle_{SS}\langle\, n_r',\ell',m' \,|\, U(R) \,|\, n_r,\ell,m \,\rangle_{SS}\langle n_r,\ell,m \,|\, r,s,t \,\rangle_C, \end{aligned}$$

where $i+k+l=N=r+s+t$. The following expression for the bivariate Krawtchouk polynomials $P_{r,s}(i,k;N)$ stems from this decomposition:

$$P_{r,s}(i,k;N) = W_{i,k;N}^{-1} \times \sum_{\substack{n_r,\ell \\ 2n_r+\ell=N}} \sum_{m,m'=-\ell}^{\ell} \mathscr{D}_{mm'}^{(\ell)}(R)\, {}_C\langle i,k,l \mid n_r,\ell,m'\rangle_S\, {}_S\langle n_r,\ell,m \mid r,s,t\rangle_C. \tag{4.11}$$

The matrix elements $\mathscr{D}_{m'm}^{(\ell)}(R) = {}_S\langle\, n_r',\ell',m' \mid U(R) \mid n_r,\ell,m\,\rangle_S$ of the $\mathfrak{so}(3)$ Wigner $\mathscr{D}$-matrix are given by [8]

$$\mathscr{D}_{m'm}^{(\ell)} = \delta_{n_r n_r'}\delta_{\ell\ell'}\, e^{-i(\gamma m'+\alpha m)} \times (-1)^{m'+\ell}\sin^{2\ell}\Big(\frac{\beta}{2}\Big)\tan^{m+m'}\Big(\frac{\beta}{2}\Big)\left[\binom{2\ell}{m+\ell}\binom{2\ell}{m'+\ell}\right]^{1/2} K_{m+\ell}\Big(m'+\ell;\sin^2\frac{\beta}{2};2\ell\Big).$$

The overlap coefficients between the Cartesian and spherical bases are obtained by using the intermediary decomposition over the polar basis states and read

$${}_S\langle n_r,\ell,m \mid r,s,t\rangle_C = \sum_{n_\rho} \frac{(-1)^{\widetilde{n}_r+n_\rho}(-i)^{m+|m|}(-\sigma_m i)^s}{\sqrt{2}}\, \mathscr{C}_{\widetilde{r},\widetilde{s},n_\rho}^{\frac{1/2+q_r}{2},\frac{1/2+q_s}{2},\frac{1+|m|}{2}}\, \mathscr{C}_{n_\rho,\widetilde{t},n_r}^{\frac{1+|m|}{2},\frac{1/2+q_t}{2},\frac{\ell+3/2}{2}}, \tag{4.12}$$

where $2n_\rho+|m|=r+s$, $2n_r+\ell=r+s+t$ and $w=2\widetilde{w}+q_w$ with $w=r,s,t$ and $q_w=0,1$. In (4.12), the square root factor should be omitted for $m=0$ and $\sigma_m=1$ if $m\geqslant 0$ and -1 otherwise. The coefficients $\mathscr{C}$ are given by

$$\mathscr{C}_{n_1,n_2,n_{12}}^{\nu_1,\nu_2,\nu_{12}} = \delta_{\nu_{12},\nu_1+\nu_2+x}\left[\frac{(2\nu_1)_{n_1}(2\nu_2)_{n_2}(2\nu_1)_x}{n_1!n_2!n_{12}!x!(2\nu_2)_x(2\nu_1+2\nu_2+2x)_{n_{12}}(2\nu_1+2\nu_2+x-1)_x}\right]^{1/2} \times (x+n_{12})!\,R_{n_1}(\lambda(x);2\nu_1-1,2\nu_2-1;n_1+n_2),$$

with $x=n_1+n_2-n_{12}$, where $R_n(\lambda(x);\gamma,\delta;N)$ are the dual Hahn polynomials [7] (see (4.28)). One has ${}_S\langle n_r,\ell,m \mid r,s,t\rangle_C = {}_C\langle r,s,t \mid n_r,\ell,m\rangle_S^*$, where x^* denotes the complex conjugate of x. Note that in (4.11), the dependence of the polynomials $P_{r,s}(i,k;N)$ on the parameters is all contained in the Wigner function.

The main formula (4.11) can also be used for the special case $R_{12}=0$ corresponding to the Tratnik polynomials. Indeed, since one has $e^{i\theta L_x}e^{i\chi L_y} = e^{-i\frac{\pi}{2}L_y}e^{i\theta L_z}e^{i\chi L_y}e^{i\frac{\pi}{2}L_y}$, it follows that

$${}_C\langle\, i,k,l \mid e^{i\theta L_x}e^{i\chi L_y} \mid r,s,t\,\rangle_C = (-1)^{l+t}\, {}_C\langle\, l,k,i \mid e^{i\theta L_z}e^{i\chi L_y} \mid t,s,r\,\rangle_C, \tag{4.13}$$

where $i+k+l=N=r+s+t$. The LHS of (4.13) is given by (4.10) in terms of the Tratnik polynomials and the RHS of (4.13) is given by (4.11) with the Euler angles values $\gamma=-\theta$, $\beta=-\chi$, $\alpha=0$. The following relations have been used to obtain (4.13):

$$ {}_C\langle\, a',b',c' \,|\, e^{i\frac{\pi}{2}L_y} \,|\, r,s,t \,\rangle_C = (-1)^t \delta_{a't}\delta_{b's}, \quad {}_C\langle\, i,k,l \,|\, e^{-i\frac{\pi}{2}L_y} \,|\, a,b,c \,\rangle_C = (-1)^l \delta_{ic}\delta_{kb}. $$

These relations are special cases of the formulas derived in [4] (see section 8).

4.1.5 Outline

The remainder of the paper is organized in a straightforward manner. In section 2, the essentials of the $\mathfrak{su}(1,1)$ Lie algebra and its Clebsch-Gordan problems are reviewed. In section 3, the explicit expressions for the overlap coefficients between the Cartesian, polar and spherical bases are derived using their identification as Clebsch-Gordan coefficients of $\mathfrak{su}(1,1)$. A discussion of the generalization to d variables is found in the conclusion.

4.2 The $\mathfrak{su}(1,1)$ Lie algebra and the Clebsch-Gordan problem

In this section, the essential results on the $\mathfrak{su}(1,1)$ Lie algebra that shall be needed are reviewed. In particular, the Clebsch-Gordan coefficients for the positive discrete series of irreducible representations are derived by a recurrence method. These coefficients are known (see for example [13]) and are presented here to make the paper self-contained.

4.2.1 The $\mathfrak{su}(1,1)$ algebra and its positive-discrete series of representations

The $\mathfrak{su}(1,1)$ algebra has for generators J_0, $J_\pm$ which satisfy the commutation relations

$$ [J_0,J_\pm]=\pm J_\pm, \quad [J_+,J_-]=-2J_0. $$

The Casimir operator, which commutes with all generators, is given by

$$ Q=J_0^2-J_+J_--J_0. \tag{4.14} $$

The positive-discrete series of irreducible representations of $\mathfrak{su}(1,1)$ are labeled by a positive number $\nu>0$ and are infinite-dimensional. They can be defined by the following actions of the generators on a canonical basis $|\,\nu,n\,\rangle$, where $n \in \mathbb{N}$:

$$J_0|\,\nu,n\,\rangle = (n+\nu)|\,\nu,n\,\rangle, \tag{4.15a}$$

$$J_+|\,\nu,n\,\rangle = \sqrt{(n+1)(n+2\nu)}|\,\nu,n+1\,\rangle, \tag{4.15b}$$

$$J_-|\,\nu,n\,\rangle = \sqrt{n(n+2\nu-1)}|\,\nu,n-1\,\rangle. \tag{4.15c}$$

The $\mathfrak{su}(1,1)$-modules spanned by the basis vectors $|\,\nu,n\,\rangle$, $n \in \mathbb{N}$, with actions (4.15) will be denoted by $V^{(\nu)}$. As expected from Schur's lemma, the Casimir operator (4.14) acts as a multiple of the identity on $V^{(\nu)}$:

$$Q|\,\nu,n\,\rangle = \nu(\nu-1)|\,\nu,n\,\rangle. \tag{4.16}$$

4.2.2 The Clebsch-Gordan problem

The vector space $V^{(\nu_1)} \otimes V^{(\nu_2)}$ is a module for the $\mathfrak{su}(1,1)$ algebra generated by

$$J_0^{(12)} = J_0^{(1)} + J_0^{(2)}, \quad J_\pm^{(12)} = J_\pm^{(1)} + J_\pm^{(2)}, \tag{4.17}$$

where the superscripts indicate on which vector space the generators act, for example $J_\pm^{(2)} = 1 \otimes J_\pm$. In general, this module is not irreducible. From the addition rule (4.17), it is easy to see that each irreducible representation occurs only once and hence that one has the irreducible decomposition

$$V^{(\nu_1)} \otimes V^{(\nu_2)} = \bigoplus_{\nu_{12}} V^{(\nu_{12})}. \tag{4.18}$$

The admissible values of ν_{12}, which give the irreducible content in the decomposition (4.18), correspond to the eigenvalues of the combined Casimir operator

$$Q^{(12)} = [J_0^{(12)}]^2 - J_+^{(12)}J_-^{(12)} - J_0^{(12)},$$

which commutes with $J_0^{(12)}$, $J_\pm^{(12)}$, $Q^{(1)}$ and $Q^{(2)}$. Upon using (4.17), the combined Casimir operator can be cast in the form

$$Q^{(12)} = 2J_0^{(1)}J_0^{(2)} - (J_+^{(1)}J_-^{(2)} + J_-^{(1)}J_+^{(2)}) + Q^{(1)} + Q^{(2)}. \tag{4.19}$$

The Clebsch-Gordan coefficients relate two possible bases for the module $V^{(\nu_1)} \otimes V^{(\nu_2)}$. On the one hand the direct product basis with vectors

$$|\,\nu_1,n_1\,\rangle \otimes |\,\nu_2,n_2\,\rangle \equiv |\,\nu_1,n_1;\nu_2,n_2\,\rangle, \tag{4.20}$$

and on the other hand, the "coupled" basis with vectors $|\,\nu_{12}, n_{12}\,\rangle$ defined by

$$Q^{(12)}|\,\nu_{12},n_{12}\,\rangle = \nu_{12}(\nu_{12}-1)|\,\nu_{12},n_{12}\,\rangle, \quad J_0^{(12)}|\,\nu_{12},n_{12}\,\rangle = (n_{12}+\nu_{12})|\,\nu_{12},n_{12}\,\rangle. \tag{4.21}$$

In both bases, the Casimir operators $Q^{(1)}$, $Q^{(2)}$ act as multiples of the identity. The two bases are orthonormal and span the representation space $V^{(\nu_1)} \otimes V^{(\nu_2)}$. Hence it follows that they are related by a unitary transformation

$$|\,\nu_{12},n_{12}\,\rangle = \sum_{n_1,n_2} \mathscr{C}^{\nu_1,\nu_2,\nu_{12}}_{n_1,n_2,n_{12}}|\,\nu_1,n_1;\nu_2,n_2\,\rangle. \tag{4.22}$$

By virtue of (4.17) and (4.21), it is clear that the condition

$$n_{12}+\nu_{12} = n_1+n_2+\nu_1+\nu_2,$$

holds in the decomposition (4.22). Since n_{12} is an integer, it follows that

$$\nu_{12} = \nu_1+\nu_2+x, \quad n_{12}+x = n_1+n_2, \tag{4.23}$$

where $x \in \{0,\ldots,N\}$ for a given value of $N = n_1+n_2$. The coefficients $\mathscr{C}^{\nu_1,\nu_2,\nu_{12}}_{n_1,n_2,n_{12}}$, which can be written

$$\mathscr{C}^{\nu_1,\nu_2,\nu_{12}}_{n_1,n_2,n_{12}} = \langle\,\nu_1,n_1;\nu_2,n_2\,|\,\nu_{12},n_{12}\,\rangle, \tag{4.24}$$

are the Clebsch-Gordan coefficients for the positive-discrete series of irreducible representations $\mathfrak{su}(1,1)$.

4.2.3 Explicit expression for the Clebsch-Gordan coefficients

The explicit expression for the Clebsch-Gordan coefficients (4.24) is known [13], hence only a short derivation using a recurrence relation is presented. By definition of the coupled basis states (4.21), one has

$$\nu_{12}(\nu_{12}-1)\mathscr{C}^{\nu_1,\nu_2,\nu_{12}}_{n_1,n_2,n_{12}} = \langle\,\nu_1,n_1;\nu_2,n_2\,|\,Q^{(12)}\,|\,\nu_{12},n_{12}\,\rangle. \tag{4.25}$$

On the other hand, upon using (4.19) and the actions (4.15), one finds

$$\begin{aligned}
&\langle\,\nu_1,n_1;\nu_2,n_2\,|\,Q^{(12)}\,|\,\nu_{12},n_{12}\,\rangle = \{2(n_1+\nu_1)(n_2+\nu_2)\}\,\mathscr{C}^{\nu_1,\nu_2,\nu_{12}}_{n_1,n_2,n_{12}}\\
&-\sqrt{n_1(n_1+2\nu_1-1)(n_2+1)(n_2+2\nu_2)}\,\mathscr{C}^{\nu_1,\nu_2,\nu_{12}}_{n_1-1,n_2+1,n_{12}} + \nu_1(\nu_1-1)\mathscr{C}^{\nu_1,\nu_2,\nu_{12}}_{n_1,n_2,n_{12}}\\
&-\sqrt{n_2(n_2+2\nu_2-1)(n_1+1)(n_1+2\nu_1)}\,\mathscr{C}^{\nu_1,\nu_2,\nu_{12}}_{n_1+1,n_2-1,n_{12}} + \nu_2(\nu_2-1)\mathscr{C}^{\nu_1,\nu_2,\nu_{12}}_{n_1,n_2,n_{12}}.
\end{aligned} \tag{4.26}$$

For a given value of $N = n_1 + n_2$, taking $n_1 = n$ and $n_2 = N - n$, one can use the conditions (4.23) to make explicit the dependence of $\mathscr{C}$ on x:

$$\mathscr{C}_{n_1,n_2,n_{12}}^{\nu_1,\nu_2,\nu_{12}} = \omega P_n(x;\nu_1,\nu_2;N),$$

where $\omega = \mathscr{C}_{0,N,N-x}^{\nu_1,\nu_2,\nu_1+\nu_2+x}$ and $P_0(x) = 1$. With these definitions, it follows from (4.25) and (4.26) that $P_n(x)$ satisfies the three-term recurrence relation

$$\begin{aligned}\lambda(x)P_n(x;\nu_1,\nu_2;N) = 2\{n(N-n)+\nu_2 n+\nu_1(N-n)\}P_n(x;\nu_1,\nu_2;N)\\ + W_n P_{n-1}(x;\nu_1,\nu_2;N) + W_{n+1}P_{n+1}(x;\nu_1,\nu_2;N).\end{aligned}$$

where

$$\lambda(x) = x(x+2\nu_1+2\nu_2-1)$$

and where

$$W_n = -[n(N-n+1)(n+2\nu_1-1)(N-n+2\nu_2)]^{1/2}.$$

Upon taking $P_n(x;\nu_1,\nu_2;N) = [W_1\ldots W_n]^{-1}\widehat{P}_n(x;\nu_1,\nu_2;N)$, one finds

$$\lambda(x)\widehat{P}_n(x) = \widehat{P}_{n+1}(x) - (A_n + C_n)\widehat{P}_n(x) + A_{n-1}C_n\widehat{P}_{n-1}(x), \tag{4.27}$$

where

$$A_n = (n-N)(n+2\nu_1), \quad C_n = n(n-2\nu_2-N).$$

It is directly seen from (4.27) that the polynomials $\widehat{P}_n(x)$ correspond to the monic dual Hahn polynomials $R_n(\lambda(x);\gamma,\delta;N)$ with parameters $\gamma = 2\nu_1 - 1$ and $\delta = 2\nu_2 - 1$. The dual Hahn polynomials are defined by [7]

$$R_n(\lambda(x);\gamma,\delta;N) = {}_3F_2\left[\begin{matrix}-n,-x,x+\gamma+\delta+1\\ \gamma+1,-N\end{matrix};1\right]. \tag{4.28}$$

Since the orthonormality condition

$$\sum_{\substack{\nu_{12},n_{12}\\ \nu_{12}+n_{12}=n_1+n_2+\nu_1+\nu_2}} \mathscr{C}_{n_1,n_2,n_{12}}^{\nu_1,\nu_2,\nu_{12}}\mathscr{C}_{n_1',n_2',n_{12}}^{\nu_1,\nu_2,\nu_{12}} = \delta_{n_1n_1'}\delta_{n_2n_2'},$$

must hold, one can use the orthogonality relation of the dual Hahn polynomials to completely determine the coefficients $\mathscr{C}_{n_1,n_2,n_{12}}^{\nu_1,\nu_2,\nu_{12}}$ up to a phase factor. One finds

$$\begin{aligned}\mathscr{C}_{n_1,n_2,n_{12}}^{\nu_1,\nu_2,\nu_{12}} = \left[\frac{(2\nu_1)_{n_1}(2\nu_2)_{n_2}(2\nu_1)_x}{n_1!n_2!n_{12}!x!(2\nu_2)_x(2\nu_1+2\nu_2+2x)_{n_{12}}(2\nu_1+2\nu_2+x-1)_x}\right]^{1/2}\\ \times(x+n_{12})!\,R_{n_1}(\lambda(x);2\nu_1-1,2\nu_2-1;n_1+n_2),\end{aligned} \tag{4.29}$$

which is valid provided that the conditions (4.23) hold. Note that one also has

$$\sum_{n_1,n_2} \mathscr{C}^{\nu_1,\nu_2,\nu_{12}}_{n_1,n_2,n_{12}} \mathscr{C}^{\nu_1,\nu_2,\nu'_{12}}_{n_1,n_2,n'_{12}} = \delta_{n_{12}n'_{12}} \delta_{\nu_{12}\nu'_{12}},$$

where the sum is restricted by $n_1 + n_2 = n_{12} + \nu_{12} - \nu_1 - \nu_2$.

4.3 Overlap coefficients for the isotropic 3D harmonic oscillator

In this section, the explicit expressions for the overlap coefficients between the Cartesian, polar and spherical bases for the states of the isotropic 3D harmonic oscillator are given. Again, these expressions are not new and can be found in [9]. Since these results are not so readily accessible however, we rederive them here using an interpretation in terms of the Clebsch-Gordan coefficients given in (4.29).

4.3.1 The Cartesian/polar overlaps

The overlap coefficients between the Cartesian $| n_x, n_y, n_z \rangle_C$ and polar $| n_\rho, m, n'_z \rangle_P$ basis states of the oscillator are defined by

$${}_C\langle n_x, n_y, n_z \,|\, n_\rho, m, n'_z \rangle_P.$$

It is obvious that

$${}_C\langle n_x, n_y, n_z \,|\, n_\rho, m, n'_z \rangle_P = \delta_{n_z, n'_z}\, {}_C\langle n_x, n_y, n_z \,|\, n_\rho, m, n_z \rangle_P.$$

One has the expansion

$$| n_\rho, m, n_z \rangle_P = \sum_{n_x, n_y} {}_C\langle n_x, n_y, n_z \,|\, n_\rho, m, n_z \rangle_P | n_x, n_y, n_z \rangle_C, \tag{4.30}$$

where the condition $n_x + n_y = 2n_\rho + |m|$ holds since only the states in the same energy eigenspace can be related to one another. The states $| n_x, n_y \rangle_C = | n_x \rangle \otimes | n_y \rangle$ can be identified with vectors $| \nu_1, n_1; \nu_2, n_2 \rangle$ of the direct product basis for a $\mathfrak{su}(1,1)$-module $V^{(\nu_x)} \otimes V^{(\nu_y)}$. Indeed, it is directly checked that the operators

$$J_0^{(x_i)} = \frac{1}{2}(a^\dagger_{x_i} a_{x_i} + 1/2), \quad J_+^{(x_i)} = \frac{1}{2}(a^\dagger_{x_i})^2, \quad J_-^{(x_i)} = \frac{1}{2} a^2_{x_i}, \tag{4.31}$$

with $i = 1,2$, realize the $\mathfrak{su}(1,1)$ algebra and that the Cartesian states $| n_{x_i} \rangle$, with the quantum number n_{x_i} either even or odd, are basis vectors for an irreducible module $V^{(\nu_{x_i})}$ with representation parameters $\nu_{x_i} = 1/4$ if n_{x_i} is even and $\nu_{x_i} = 3/4$ if n_{x_i} is odd. Hence we have the identification

$$| 2\tilde{n}_x + q_x, 2\tilde{n}_y + q_y \rangle_C \sim | 1/4 + q_x/2, \tilde{n}_x; 1/4 + q_y/2, \tilde{n}_y \rangle \equiv | \nu_1, n_1; \nu_2, n_2 \rangle, \tag{4.32}$$

where $q_x, q_y \in \{0,1\}$ and where the third quantum number n_z as been suppressed from the Cartesian states in (4.32) to facilitate the correspondence with the notation used in the previous section.

The polar basis states $| n_\rho, m, n_z \rangle_P$ can be identified with vectors of the "coupled" basis. Indeed, consider the realization of the $\mathfrak{su}(1,1)$ algebra obtained by taking

$$J_0^{(xy)} = J_0^{(x)} + J_0^{(y)}, \quad J_\pm^{(xy)} = J_\pm^{(x)} + J_\pm^{(y)}.$$

By definition, the states $| n_\rho, m, n_z \rangle_P$ satisfy

$$L_z | n_\rho, m, n_z \rangle_P = m | n_\rho, m, n_z \rangle_P.$$

Furthermore, a direct computation shows that the coupled Casimir $Q^{(xy)}$ operator can be expressed in terms of L_z in the following way:

$$Q^{(xy)} = \frac{1}{4}(L_z^2 - 1).$$

Hence it follows that the polar basis states are eigenvectors of the combined Casimir operator $Q^{(xy)}$ with eigenvalue

$$Q^{(xy)} | n_\rho, m \rangle_P = \frac{1}{4}(m^2 - 1) | n_\rho, m \rangle_P. \tag{4.33}$$

Since from (4.30), (4.31) and $n_x + n_y = 2n_\rho + |m|$ one also has

$$J_0^{(xy)} | n_\rho, m, n_z \rangle_P = \left(n_\rho + \frac{|m|}{2} + 1/2 \right) | n_\rho, m, n_z \rangle_P,$$

it is seen that the polar basis states $| n_\rho, m, n_z \rangle_P$ correspond to coupled $\mathfrak{su}(1,1)$ basis states of $V^{(\nu_{xy})}$ with representation parameter $\nu_{xy} = (|m| + 1)/2$. One thus writes

$$| n_\rho, m, n_z \rangle_P \sim | \frac{1 + |m|}{2}, n_\rho \rangle \equiv | \nu_{12}, n_{12} \rangle. \tag{4.34}$$

The correspondence (4.32), (4.34) can now be used to recover the overlap coefficients between the Cartesian and polar bases of the 3D isotropic harmonic oscillator. One needs to keep in mind that for $m \neq 0$, there is a sign ambiguity in (4.34) which has to be taken into account to ensure the orthonormality conditions for the overlap coefficients. One finds

$$ {}_C\langle n_x, n_y, n_z \mid n_\rho, 0, n_z' \rangle_P = e^{i\phi}\, \delta_{n_z, n_z'}\, \mathscr{C}^{\nu_1,\nu_2,\nu_{12}}_{n_1,n_2,n_{12}}, $$

for $m = 0$

$$ {}_C\langle n_x, n_y, n_z \mid n_\rho, m, n_z' \rangle_P = \frac{e^{i\phi}}{\sqrt{2}}\, \delta_{n_z, n_z'}\, \mathscr{C}^{\nu_1,\nu_2,\nu_{12}}_{n_1,n_2,n_{12}}, $$

for $m \neq 0$, where $e^{i\phi}$ is a phase factor that remains to be evaluated. The correspondence between the quantum numbers and representation parameters is given by

$$ \nu_1 = 1/4 + q_x/2, \quad \nu_2 = 1/4 + q_y/2, \quad \nu_{12} = (1 + |m|)/2, \tag{4.35a} $$

$$ n_1 = \widetilde{n_x}, \quad n_2 = \widetilde{n_y}, \quad n_{12} = n_\rho, \tag{4.35b} $$

where $n_{x_i} = 2\widetilde{n}_x + q_x$ with $q_{x_i} \in \{0, 1\}$. The remaining phase factor can be evaluated by requiring that the expansion

$$ \Psi_{n_\rho, m, n_z'}(\rho, \phi, z) = \sum_{n_x, n_y} {}_C\langle n_x, n_y, n_z \mid n_\rho, m, n_z' \rangle_P\, \Psi_{n_x, n_y, n_z}(x, y, z), $$

holds for the wavefunctions. By inspection of (4.2) and (4.3), one finds

$$ e^{i\phi} = (-1)^{\widetilde{n}_x + n_\rho} (\sigma_m\, i)^{n_y}, \quad \text{with} \quad \sigma_m = \begin{cases} 1 & m \geqslant 0, \\ -1 & m < 0. \end{cases} $$

The complete expression for the overlaps is therefore given by

$$ {}_C\langle n_x, n_y, n_z \mid n_\rho, m, n_z' \rangle_P = \delta_{n_z n_z'} \left(\frac{(-1)^{\widetilde{n}_x + n_\rho} (\sigma_m\, i)^{n_y}}{\sqrt{2}} \right) \mathscr{C}^{\nu_1,\nu_2,\nu_{12}}_{n_1,n_2,n_{12}}, \tag{4.36} $$

with the identification (4.35) and where it is understood that the $\sqrt{2}$ factor is to be omitted when $m = 0$.

4.3.2 The polar/spherical overlaps

The overlap coefficients between the polar and spherical bases are defined by

$$ {}_P\langle n_\rho, m', n_z \mid n_r, \ell, m \rangle_S. $$

Since both set of basis states are eigenstates of L_z, it follows that one can write

$$ {}_P\langle n_\rho, m', n_z \mid n_r, \ell, m \rangle_S = \delta_{mm'}\, {}_P\langle n_\rho, m, n_z \mid n_r, \ell, m \rangle_S. $$

One has the decomposition

$$|\, n_r, \ell, m\,\rangle_S = \sum_{n_\rho, n_z} {}_P\langle n_\rho, m, n_z \,|\, n_r, \ell, m\,\rangle_S \,|\, n_\rho, m, n_z\,\rangle_P, \tag{4.37}$$

where the condition $2n_\rho + |m| + n_z = 2n_r + \ell$ holds since only the states with identical energies can be related to one another. The states $|\, n_\rho, m\,\rangle$ and $|\, n_z\,\rangle$ have already been identified with basis vectors of irreducible $\mathfrak{su}(1,1)$ representations. We thus write the polar basis states $|\, n_\rho, m, n_z\,\rangle_P = |\, n_\rho, m\,\rangle \otimes |\, n_z\,\rangle$ as direct product vectors

$$|\, n_\rho, m, 2\tilde{n}_z + q_z\,\rangle_P \sim |\,(1+|m|)/2, n_\rho; 1/4 + q_z/2, \tilde{n}_z\,\rangle \equiv |\, \nu_1, n_1; \nu_2, n_2\,\rangle.$$

The spherical basis states $|\, n_r, \ell, m\,\rangle_S$ can be identified with those of the "coupled" basis. Indeed, consider the $\mathfrak{su}(1,1)$ algebra obtained by taking

$$J_0^{((xy)z)} = J_0^{(xy)} + J_0^{(z)}, \quad J_\pm^{((xy)z)} = J_\pm^{(xy)} + J_\pm^{(z)}. \tag{4.38}$$

By definition, the states $|\, n_r, \ell, m\,\rangle_S$ satisfy

$$\vec{L}^2 |\, n_r, \ell, m\,\rangle_S = \ell(\ell+1) |\, n_r, \ell, m\,\rangle_S,$$

where $\vec{L}^2 = L_x^2 + L_y^2 + L_z^2$. Furthermore, a direct computation shows that $\vec{L}^2$ and the coupled Casimir operator $Q^{((xy)z)}$ are related by

$$Q^{((xy)z)} = \frac{1}{4}\left(\vec{L}^2 - \frac{3}{4}\right).$$

Hence one may write

$$Q^{((xy)z)} |\, n_r, \ell, m\,\rangle_S = (\ell/2 + 3/4)(\ell/2 - 1/4) |\, n_r, \ell, m\,\rangle_S.$$

Since from (4.37), (4.38) and the condition $2n_r + \ell = 2n_\rho + |m| + n_z$ one has

$$J_0^{((xy)z)} |\, n_r, \ell, m\,\rangle_S = \{n_r + (\ell+3)/2\} |\, n_r, \ell, m\,\rangle_S,$$

it follows that the states of the spherical basis correspond to coupled $\mathfrak{su}(1,1)$ states

$$|\, n_r, \ell, m\,\rangle_S \sim |\, \frac{\ell + 3/2}{2}, n_r\,\rangle \sim |\, \nu_{12}, n_{12}\,\rangle.$$

Using this identification, one writes

$${}_P\langle n_\rho, m, n_z \,|\, n_r, \ell, m\,\rangle_S = e^{i\psi} \delta_{mm'} \mathscr{C}_{n_1, n_2, n_{12}}^{\nu_1, \nu_2, \nu_{12}},$$

where

$$\nu_1 = \frac{1+|m|}{2}, \quad \nu_2 = \frac{1/2+q_z}{2}, \quad \nu_{12} = \frac{\ell+3/2}{2}, \tag{4.39a}$$

$$n_1 = n_\rho, \quad n_2 = \widetilde{n}_z, \quad n_{12} = n_r, \tag{4.39b}$$

and with $n_z = 2\widetilde{n}_z + q_z$. The phase factor $e^{i\psi}$ can be determined by requiring that the expansion formula

$$\Psi_{n_r,\ell,m}(\rho,\theta,\phi) = \sum_{n_\rho,n_z} {}_P\langle n_\rho, m, n_z \mid n_r, \ell, m\rangle_S \, \Psi_{n_\rho,m,n_z}(\rho,\phi,z),$$

holds for the wavefunctions. Upon inspecting (4.3) and (4.4), one finds that $e^{i\psi} = i^{m+|m|}$. Hence the following expression holds

$$_P\langle n_\rho, m, n_z \mid n_r, \ell, m'\rangle_S = \delta_{mm'}\, i^{m+|m|}\, \mathscr{C}^{\nu_1,\nu_2,\nu_{12}}_{n_1,n_2,n_{12}}, \tag{4.40}$$

with the identification (4.39). Note that one has also

$$_S\langle n_r, \ell, m' \mid n_\rho, m, n_z\rangle_P = \delta_{mm'}\, (-i)^{m+|m|}\, \mathscr{C}^{\nu_1,\nu_2,\nu_{12}}_{n_1,n_2,n_{12}},$$

since the Clebsch-Gordan coefficients $\mathscr{C}^{\nu_1,\nu_2,\nu_{12}}_{n_1,n_2,n_{12}}$ are real.

4.4 Conclusion

To sum up, we have obtained a new explicit formula for the bivariate Krawtchouk polynomials in terms of the standard (univariate) Krawtchouk and dual Hahn polynomials. Furthermore, the explicit expressions for the overlap coefficients of the isotropic oscillator have been rederived using a correspondence with the Clebsch-Gordan problem of $\mathfrak{su}(1,1)$.

In [4], the results obtained using $SO(3)$ were seen to extend directly to higher dimensions and indeed the d-variable Krawtchouk polynomials can be interpreted as matrix elements of unitary reducible $SO(d+1)$ representations on (Cartesian) oscillator states. The main result (4.11) obtained here for the bivariate Krawtchouk polynomials can also be generalized to d variables. The derivation is similar in spirit to the one presented here but is quite technical. We now outline how this generalization proceeds.

Let $R \in SO(d+1)$. The matrix elements of the reducible $SO(d+1)$ unitary representation (4.7) in the Cartesian basis of the $\mathscr{E} = N + d/2$ energy eigenspace of the $(d+1)$-dimensional isotropic harmonic oscillator are expressed as follows [4]:

$$\langle i_1,\ldots,i_{d+1} \mid U(R) \mid n_1,\ldots,n_{d+1}\rangle = W_{i_1,\ldots,i_d;N}\, P_{n_1,\ldots,n_d}(i_1,\ldots,i_d;N),$$

where $P_{n_1,\ldots,n_d}(i_1,\ldots,i_d;N)$ are the multivariate Krawtchouk polynomials and where

$$W_{i_1,\ldots,i_d;N} = \binom{N}{i_1,\ldots,i_d}^{1/2} R_{1,d+1}^{i_1}\ldots R_{d,d+1}^{i_d} R_{d+1,d+1}^{N-i_1-\ldots-i_d},$$

with $\sum_{k=1}^{d+1} i_k = \sum_{k=1}^{d+1} n_k = N$. The decomposition of this $SO(d+1)$ representation in irreducible components can be accomplished by a passage to a canonical basis which corresponds to the separation of variables of the Schrödinger equation in hyperspherical coordinates [2]. These basis states are denoted by $|\, n_r, \lambda, \mu_1, \ldots \mu_{d-1}\,\rangle$ with $n_r, \in \mathbb{N}$ and $\lambda \geqslant \mu_1 \geqslant \cdots \geqslant |\mu_{d-1}| \geqslant 0$. They are eigenstates of the $(d+1)$-dimensional harmonic oscillator Hamiltonian with energy $\mathscr{E} = 2n_r + \lambda + d/2$ and the corresponding wavefunctions can be expressed in terms of Laguerre polynomials and hyperspherical harmonics [1, 14]. These states form a basis for (class 1) irreducible representations of $SO(d+1)$ [13]. They are eigenvectors of the quadratic Casimir operator of $SO(d+1)$ with eigenvalue $\lambda(\lambda+d-1)$ and of the quadratic Casimir operators of each element in the canonical subgroup chain $SO(d+1) \supset SO(d) \supset \cdots \supset SO(2)$ with eigenvalues $\mu_1(\mu_1+d-2), \mu_2(\mu_2+d-3)\ldots, \mu_{d-1}^2$ [2]. This is the origin of the parameters μ_i, $i = 1,\ldots,d-1$. For a given N, the $SO(d+1)$ representation on the eigenstates of the $(d+1)$-dimensional oscillator contains once, each and every (class one) irreducible representation of $SO(d+1)$ with $\lambda = N, N-2, \ldots, 0, 1$ depending on the parity. This decomposition is equivalent to the decomposition of the quasi-regular representation of $SO(d+1)$ [13]. Upon introducing the states corresponding to separation in hyperspherical coordinates, one is led to the decomposition formula

$$\begin{aligned} &P_{n_1,\ldots,n_d}(i_1,\ldots,i_d;N) = W^{-1}_{i_1,\ldots,i_d;N} \\ &\times \sum_{n_r,\lambda} \sum_{\mu,\mu'} \langle\, n_r, \lambda, \mu'_1, \ldots, \mu'_{d-1} \,|\, U(R) \,|\, n_r, \lambda, \mu_1, \ldots, \mu_{d-1}\,\rangle \\ &\qquad\qquad \times \langle\, i_1, \ldots, i_{d+1} \,|\, n_r, \lambda, \mu'_1, \ldots, \mu'_{d-1}\,\rangle \langle\, n_r, \lambda, \mu_1, \ldots, \mu_{d-1} \,|\, n_1, \ldots, n_{d+1}\,\rangle, \end{aligned}$$

where $2n_r + \lambda = N = i_1 + \cdots + i_{d+1} = n_1 + \cdots + n_{d+1}$ and where μ denotes the multi-index $(\mu_1, \ldots, \mu_{d-1})$ with $\lambda \geqslant \mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant |\mu_{d-1}| \geqslant 0$. The overlap coefficients can be evaluated as sums of products of d $\mathfrak{su}(1,1)$ Clebsch-Gordan coefficients using successive recouplings of the quantum numbers, as was done in Section 3. The matrix elements $\langle\, n_r, \lambda, \mu'_1, \ldots, \mu'_{d-1} \,|\, U(R) \,|\, n_r, \lambda, \mu_1, \ldots, \mu_{d-1}\,\rangle$ are very involved. They can be evaluated only recursively using the canonical subgroup chain of $SO(d+1)$. See [13] for details.

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Chapitre 5

The multivariate Hahn polynomials and the singular oscillator

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Abstract. Karlin and McGregor's d-variable Hahn polynomials are shown to arise in the $(d+1)$-dimensional singular oscillator model as the overlap coefficients between bases associated to the separation of variables in Cartesian and hyperspherical coordinates. These polynomials in d discrete variables depend on $d+1$ real parameters and are orthogonal with respect to the multidimensional hypergeometric distribution. The focus is put on the $d=2$ case for which the connection with the three-dimensional singular oscillator is used to derive the main properties of the polynomials: forward/backward shift operators, orthogonality relation, generating function, recurrence relations, bispectrality (difference equations) and explicit expression in terms of the univariate Hahn polynomials. The extension of these results to an arbitrary number of variables is presented at the end of the paper.

5.1 Introduction

The objective of this article is to show that the multidimensional Hahn polynomials arise in the quantum singular oscillator model as the overlap coefficients between energy eigenstate bases associated to the separation of variables in Cartesian and hyperspherical coordinates and to obtain their properties from this framework. This offers an algebraic analysis of the multivariate Hahn polynomials which is resting on their interpretation as overlap coefficients and on the special properties of the functions arising in the basis wavefunctions. For definiteness and ease of notation, the emphasis shall be put on the case where the Hahn polynomials in two variables appear as the Cartesian vs. spherical interbasis expansion coefficients for the three-dimensional singular oscillator. It shall be indicated towards the end of the paper how these results can be extended directly to an arbitrary number of variables.

The Hahn polynomials in one variable, which shall be denoted by $h_n(x;\alpha,\beta;N)$, are the polynomials of degree n in the variable x defined by [28, 30]

$$h_n(x;\alpha,\beta;N) = (\alpha+1)_n(-N)_n\, {}_3F_2\left[\begin{matrix}-n, n+\alpha+\beta+1, -x\\ \alpha+1, -N\end{matrix};1\right],$$

where ${}_pF_q$ is the generalized hypergeometric function [2] and where $(a)_n$ stands for the Pochhammer symbol (or shifted factorial)

$$(a)_n = (a)(a+1)\cdots(a+n-1), \qquad (a)_0 \equiv 1.$$

These polynomials belong to the discrete part of the Askey scheme of hypergeometric orthogonal polynomials [28]. They satisfy the orthogonality relation

$$\sum_{x=0}^{N} \rho(x;\alpha,\beta;N)\, h_n(x;\alpha,\beta;N)\, h_m(x;\alpha,\beta;N) = \lambda_n(\alpha,\beta;N)\, \delta_{nm},$$

with respect to the hypergeometric distribution [21]

$$\rho(x;\alpha,\beta;N) = \binom{N}{x}\frac{(\alpha+1)_x(\beta+1)_{N-x}}{(\alpha+\beta+2)_N}, \tag{5.1}$$

where $\binom{N}{x}$ are the binomial coefficients. The weight function (5.1) is positive provided that $\alpha,\beta > -1$ or $\alpha,\beta < -N$. The normalization factor λ_n reads

$$\lambda_n(\alpha,\beta;N) = \frac{\alpha+\beta+1}{2n+\alpha+\beta+1}\,\frac{N!\,n!}{(N-n)!}\,\frac{(\alpha+1)_n(\beta+1)_n(N+\alpha+\beta+2)_n}{(\alpha+\beta+1)_n}. \tag{5.2}$$

The polynomials $h_n(x;\alpha,\beta;N)$ can be obtained from the generating function [28]

$$ {}_1F_1\left[\begin{matrix}-x\\ \alpha+1\end{matrix};-t\right] {}_1F_1\left[\begin{matrix}x-N\\ \beta+1\end{matrix};t\right] = \sum_{n=0}^{N}\frac{h_n(x;\alpha,\beta;N)}{(\alpha+1)_n(\beta+1)_n}\frac{t^n}{n!}, \tag{5.3}$$

or from the dual generating function [24]

$$(-N)_n\, n!\,(1+t)^N P_n^{(\alpha,\beta)}\left(\frac{1-t}{1+t}\right) = \sum_{x=0}^{N}\binom{N}{x} h_n(x;\alpha,\beta;N)\, t^x, \tag{5.4}$$

where $P_n^{(\alpha,\beta)}(z)$ stands for the classical Jacobi polynomials [28]. In mathematical physics, the Hahn polynomials are mostly known for their appearance in the Clebsch-Gordan coefficients of the $\mathfrak{su}(2)$ or $\mathfrak{su}(1,1)$ algebras (see for example [36]). However, these polynomials have also been used in the designing of spin chains allowing perfect quantum state transfer [1, 3, 37] and moreover, they occur as exact solutions of certain discrete Markov processes [21].

The multivariable extension of the Hahn polynomials is due to Karlin and McGregor who obtained these polynomials in [25] as exact solutions of a multidimensional genetics model. This family of multidimensional polynomials is a member of the multivariate analogue of the discrete Askey scheme proposed by Tratnik in [35] and generalized to the basic (q-deformed) case by Gasper and Rahman in [9]. One of the key features of the polynomials in this multivariate scheme is their bispectrality (in the sense of Duistermaat and Grünbaum [6]), which was established by Geronimo and Iliev in [15] and by Iliev [19] in the q-deformed case. Since their introduction, the multivariate Hahn polynomials have been studied from different points of view by a number of authors [20, 31, 40, 38] and used in particular for applications in probability [18, 26]. Of particular relevance to the present article are the papers of Dunkl [7], Scarabotti [34] and Rosengren [33], where the multivariate Hahn polynomials occur in an algebraic framework.

Here we give a physical interpretation of the multivariate Hahn polynomials by establishing that they occur in the overlap coefficients between wavefunctions of the singular oscillator model separated in Cartesian and hyperspherical coordinates. It will be seen that this framework provides a cogent foundation for the characterization of these polynomials: new derivations of known formulas will be given and new identities will come to the fore. The results presented here are in line with the physico-algebraic models that were exhibited in [13, 14], [11] and [10] where the multivariate Krawtchouk, Meixner and Charlier polynomials were identified and characterized as matrix elements of the representations of the rotation, Lorentz and Euclidean groups on oscillator states. However the approach and techniques used in the present paper differ from the ones used in [13], [11] and [10] as the multivariate Hahn polynomials do not arise as matrix elements of Lie group representations.

The outline of the paper is the following. In section 2, the singular oscillator model in three-dimensions is reviewed. The wavefunctions separated in Cartesian and spherical coordinates are explicitly written and the corresponding constants of motion are given. In section 3, it is shown that the expansion coefficients are expressed in terms of orthogonal polynomials in two discrete variables that are orthogonal with respect to a two-variable generalization of the hypergeometric distribution. This is accomplished by bringing intertwining operators that raise/lower the appropriate quantum numbers. In section 4, a generating function is derived by examining the asymptotic behavior of the wavefunctions and this generating function is identified with the one derived by Karlin and McGregor for the multivariate Hahn polynomials. Backward and forward structure relations are obtained in section 5 and are seen to provide a factorization of the pair of recurrence relations satisfied by the bivariate Hahn polynomials. In section 6, the two difference equations are derived: one by factorization and the other by a direct computation involving one of the symmetry operators responsible for the separation of variable in spherical coordinates. In section 7, the explicit expression of the bivariate Hahn polynomials in terms of univariate Hahn polynomials is obtained by combining the Cartesian vs. cylindrical and cylindrical vs. spherical interbasis expansion coefficients for the singular oscillator. The connection with the recoupling of $\mathfrak{su}(1,1)$ representations is explained in section 8. In section 9, the multivariate case is considered. A conclusion follows with perspectives on the multivariate Racah polynomials. A compendium of formulas for the bivariate Hahn polynomials has been included in the appendix.

5.2 The three-dimensional singular oscillator

In this section, the 3-dimensional singular oscillator model is reviewed. The two bases for the energy eigenstates associated to the separation of variable in Cartesian and spherical coordinates are presented in terms of Laguerre and Jacobi polynomials. For each basis, the symmetry operators that are diagonalized and their eigenvalues are given. The main object of the paper, the interbasis expansion coefficients between these two bases, is defined and shown to exhibit an exchange symmetry.

5.2.1 Hamiltonian and spectrum

The singular oscillator model in three dimensions is governed by the Hamiltonian

$$\mathscr{H} = \frac{1}{4}\sum_{i=1}^{3}\left(-\partial_{x_i}^2 + x_i^2 + \frac{\alpha_i^2 - \frac{1}{4}}{x_i^2}\right), \tag{5.5}$$

where $\alpha_i > -1$ are real parameters. The energy eigenvalues of $\mathscr{H}$, labeled by the non-negative integer N, have the form

$$\mathscr{E}_N = N + \alpha_1/2 + \alpha_2/2 + \alpha_3/2 + 3/2,$$

and exhibit a $\frac{(N+1)(N+2)}{2}$-fold degeneracy. The Schrödinger equation associated to the Hamiltonian (5.5) can be exactly solved by separation of variables in Cartesian and spherical coordinates (separation also occurs in other coordinate systems), thus providing two distinct bases to describe the states of the system.

5.2.2 The Cartesian basis

Let i and k be non-negative integers such that $i+k \leq N$. We shall denote by $|\,\alpha_1,\alpha_2,\alpha_3;i,k;N\,\rangle_C$ the basis vectors for the $\mathscr{E}_N$-energy eigenspace associated to the separation of variables in Cartesian coordinates. The corresponding wavefunctions read

$$\begin{aligned}\langle\, x_1,x_2,x_3\,|\,\alpha_1,\alpha_2,\alpha_3;i,k;N\,\rangle_C &= \Psi_{i,k;N}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2,x_3)\\ &= \xi_i^{(\alpha_1)}\xi_k^{(\alpha_2)}\xi_{N-i-k}^{(\alpha_3)}\,\mathscr{G}^{(\alpha_1,\alpha_2,\alpha_3)}\,L_i^{(\alpha_1)}(x_1^2)L_k^{(\alpha_2)}(x_2^2)L_{N-i-k}^{(\alpha_3)}(x_3^2),\end{aligned} \tag{5.6}$$

where $L_n^{(\alpha)}(x)$ are the standard Laguerre polynomials [28] and where the gauge factor $\mathscr{G}^{(\alpha_1,\alpha_2,\alpha_3)}$ has the form

$$\mathscr{G}^{(\alpha_1,\alpha_2,\alpha_3)} = e^{-(x_1^2+x_2^2+x_3^2)/2} \prod_{j=1}^{3} x_j^{\alpha_j+1/2}.$$

The normalization factors

$$\xi_n^{(\alpha)} = \sqrt{\frac{2n!}{\Gamma(n+\alpha+1)}}, \tag{5.7}$$

where $\Gamma(x)$ is the gamma function [2], ensure that the wavefunctions satisfy the orthogonality relation

$$\begin{aligned} {}_C\langle \alpha_1,\alpha_2,\alpha_3;i,k;N \mid \alpha_1,\alpha_2,\alpha_3;i',k';N' \rangle_C = \\ \int_{\mathbb{R}_+^3} \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \left[\Psi_{i,k;N}^{(\alpha_1,\alpha_2,\alpha_3)}\right]^* \Psi_{i',k';N'}^{(\alpha_1,\alpha_2,\alpha_3)} = \delta_{ii'}\delta_{kk'}\delta_{NN'}, \end{aligned}$$

where $\mathbb{R}_+$ stands for the non-negative real line and where z^* stands for complex conjugation. The Cartesian basis states are completely determined by the set of eigenvalue equations

$$\begin{aligned} K_0^{(1)} | \alpha_1,\alpha_2,\alpha_3;i,k;N \rangle_C &= (i+\alpha_1/2+1/2) | \alpha_1,\alpha_2,\alpha_3;i,k;N \rangle_C, \\ K_0^{(2)} | \alpha_1,\alpha_2,\alpha_3;i,k;N \rangle_C &= (k+\alpha_2/2+1/2) | \alpha_1,\alpha_2,\alpha_3;i,k;N \rangle_C, \\ \mathscr{H} | \alpha_1,\alpha_2,\alpha_3;i,k;N \rangle_C &= \mathscr{E}_N | \alpha_1,\alpha_2,\alpha_3;i,k;N \rangle_C, \end{aligned}$$

where $K_0^{(i)}$, $i = 1,2$, are the constants of motion ($[\mathscr{H},K_0^{(i)}] = 0$) associated to the separation of variables in Cartesian coordinates. These (Hermitian) operators have the expression

$$K_0^{(i)} = \frac{1}{4}\left(-\partial_{x_i}^2 + x_i^2 + \frac{\alpha_i^2 - \frac{1}{4}}{x_i^2}\right), \tag{5.8}$$

and correspond to the one-dimensional singular oscillator Hamiltonian. For convenience, the Cartesian basis states $| \alpha_1,\alpha_2,\alpha_3;i,k;N \rangle_C$ shall sometimes be written simply as $| i,k;N \rangle_C$ when the explicit dependence on the parameters α_i is not needed.

5.2.3 The spherical basis

Let m and n be non-negative integers such that $m+n \leq N$. We shall denote by $| \alpha_1,\alpha_2,\alpha_3;m,n;N \rangle_S$ the basis vectors for the $\mathscr{E}_N$-energy eigenspace associated to the separation of variables in spherical coordinates

$$x_1 = r\sin\theta\cos\phi, \quad x_2 = r\sin\theta\sin\phi, \quad x_3 = r\cos\theta.$$

In this case the corresponding wavefunctions are given by

$$
\begin{aligned}
\langle r,\theta,\phi \mid \alpha_1,\alpha_2,\alpha_3;m,n;N\rangle_S &= \Xi^{(\alpha_1,\alpha_2,\alpha_3)}_{m,n;N}(r,\theta,\phi)\\
&= \eta^{(\alpha_1,\alpha_2)}_m \eta^{(2m+\alpha_{12}+1,\alpha_3)}_n \xi^{(2m+2n+\alpha_{123}+2)}_{N-m-n}\mathscr{G}^{(\alpha_1,\alpha_2,\alpha_3)}\\
&\quad\times P^{(\alpha_1,\alpha_2)}_m(-\cos 2\phi)(\sin^2\theta)^m P^{(2m+\alpha_{12}+1,\alpha_3)}_n(\cos 2\theta)(r^2)^{m+n}L^{(2m+2n+\alpha_{123}+2)}_{N-m-n}(r^2),
\end{aligned}
\tag{5.9}
$$

where $P^{(\alpha,\beta)}_n(z)$ are the Jacobi polynomials and where we have introduced the notation

$$
\alpha_{ij}=\alpha_i+\alpha_j,\quad \alpha_{ijk}=\alpha_i+\alpha_j+\alpha_k.
$$

The normalization factors

$$
\eta^{(\alpha,\beta)}_n=\sqrt{\frac{2(2n+\alpha+\beta+1)n!\,\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}},
\tag{5.10}
$$

ensure that the wavefunctions $\Xi^{(\alpha_1,\alpha_2,\alpha_3)}_{m,n;N}$ satisfy the orthogonality relation

$$
\begin{aligned}
&{}_S\langle \alpha_1,\alpha_2,\alpha_3;m,n;N \mid \alpha_1,\alpha_2,\alpha_3;m',n',N'\rangle_S =\\
&\qquad\int_0^\infty\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi\,\left[\Xi^{(\alpha_1,\alpha_2,\alpha_3)}_{m,n;N}\right]^*\,\Xi^{(\alpha_1,\alpha_2,\alpha_3)}_{m',n';N'}=\delta_{mm'}\delta_{nn'}\delta_{NN'}.
\end{aligned}
$$

The spherical basis states $\mid \alpha_1,\alpha_2,\alpha_3;m,n;N\rangle_S$ are completely determined by the set of eigenvalue equations

$$
\begin{aligned}
Q^{(12)}\mid \alpha_1,\alpha_2,\alpha_3;m,n;N\rangle_S &= \lambda^{(12)}_m\mid \alpha_1,\alpha_2,\alpha_3;m,n;N\rangle_S,\\
Q^{(123)}\mid \alpha_1,\alpha_2,\alpha_3;m,n;N\rangle_S &= \lambda^{(123)}_{m,n}\mid \alpha_1,\alpha_2,\alpha_3;m,n;N\rangle_S,\\
\mathscr{H}\mid \alpha_1,\alpha_2,\alpha_3;m,n;N\rangle_S &= \mathscr{E}_N\mid \alpha_1,\alpha_2,\alpha_3;m,n;N\rangle_S,
\end{aligned}
\tag{5.11}
$$

where the eigenvalues $\lambda^{(12)}_m$ and $\lambda^{(123)}_{m,n}$ are given by

$$
\begin{aligned}
\lambda^{(12)}_m &= (m+\alpha_{12}/2+1)(m+\alpha_{12}/2),\\
\lambda^{(123)}_{m,n} &= (m+n+\alpha_{123}/2+3/2)(m+n+\alpha_{123}/2+1/2).
\end{aligned}
$$

The (Hermitian) operators $Q^{(12)}$ and $Q^{(123)}$, which can be seen to commute with the Hamiltonian, have the expressions

$$
Q^{(12)}=\frac{1}{4}\Big\{-\partial^2_\phi+\frac{\alpha_1^2-1/4}{\cos^2\phi}+\frac{\alpha_2^2-1/4}{\sin^2\phi}-1\Big\},
\tag{5.12a}
$$

$$
Q^{(123)}=\frac{1}{4}\Big\{-\partial^2_\theta-\operatorname{ctg}\theta\,\partial_\theta+\frac{\alpha_3^2-1/4}{\cos^2\theta}+\frac{1}{\sin^2\theta}\left(-\partial^2_\phi+\frac{\alpha_1^2-1/4}{\cos^2\phi}+\frac{\alpha_2^2-1/4}{\sin^2\phi}\right)-\frac{3}{4}\Big\}.
\tag{5.12b}
$$

In Cartesian coordinates, the spherical basis wavefunctions read

$$
\begin{aligned}
\langle x_1,x_2,x_3 \mid \alpha_1,\alpha_2,\alpha_3;m,n;N\rangle_S &= \Xi^{(\alpha_1,\alpha_2,\alpha_3)}_{m,n;N}(x_1,x_2,x_3)\\
&= \eta^{(\alpha_1,\alpha_2)}_m \eta^{(2m+\alpha_{12}+1,\alpha_3)}_n \xi^{(2m+2n+\alpha_{123}+2)}_{N-m-n} \mathscr{G}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1^2+x_2^2)^m P^{(\alpha_1,\alpha_2)}_m\left(\frac{x_2^2-x_1^2}{x_1^2+x_2^2}\right)\\
&\quad\times (x_1^2+x_2^2+x_3^2)^n P^{(2m+\alpha_{12}+1,\alpha_3)}_n\left(\frac{x_3^2-x_1^2-x_2^2}{x_1^2+x_2^2+x_3^2}\right) L^{(2m+2n+\alpha_{123}+2)}_{N-m-n}(x_1^2+x_2^2+x_3^2),
\end{aligned}
\tag{5.13}
$$

and the operators $Q^{(12)}$, $Q^{(123)}$ have the form

$$
\begin{aligned}
Q^{(12)} &= \frac{1}{4}\left\{J_3^2+(x_1^2+x_2^2)\left(\frac{\alpha_1^2-1/4}{x_1^2}+\frac{\alpha_2^2-1/4}{x_2^2}\right)-1\right\},\\
Q^{(123)} &= \frac{1}{4}\left\{J_1^2+J_2^2+J_3^2+(x_1^2+x_2^2+x_3^2)\left(\frac{\alpha_1^2-1/4}{x_1^2}+\frac{\alpha_2^2-1/4}{x_2^2}+\frac{\alpha_3^2-1/4}{x_3^2}\right)-\frac{3}{4}\right\},
\end{aligned}
\tag{5.14}
$$

where the J_j are the familiar angular momentum operators

$$
J_1 = \frac{1}{i}(x_2\partial_{x_3}-x_3\partial_{x_2}),\quad J_2 = \frac{1}{i}(x_3\partial_{x_1}-x_1\partial_{x_3}),\quad J_3 = \frac{1}{i}(x_1\partial_{x_2}-x_2\partial_{x_1}).
$$

For notational convenience, the spherical basis vectors $\mid \alpha_1,\alpha_2,\alpha_3;m,n;N\rangle_S$ shall sometimes be written simply as $\mid m,n;N\rangle_S$ when the explicit dependence on the parameters α_i is not needed.

5.2.4 The main object

In this paper, we shall be concerned with the overlap coefficients between the Cartesian and spherical bases. These coefficients are given by the integral

$$
{}_C\langle i,k;N \mid m,n;N\rangle_S = \int_{\mathbb{R}^3_+} \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\,[\Psi^{(\alpha_1,\alpha_2,\alpha_3)}_{i,k;N}(x_1,x_2,x_3)]^*\,\Xi^{(\alpha_1,\alpha_2,\alpha_3)}_{m,n;N}(x_1,x_2,x_3). \tag{5.15}
$$

Since the wavefunctions are real one has

$$
{}_C\langle i,k;N \mid m,n;N\rangle_S = {}_S\langle m,n;N \mid i,k;N\rangle_C.
$$

One can write the expansion formulas

$$
\mid i,k;N\rangle_C = \sum_{\substack{m,n\\ m+n\leq N}} {}_S\langle m,n;N \mid i,k;N\rangle_C \mid m,n;N\rangle_S, \tag{5.16a}
$$

$$
\mid m,n;N\rangle_S = \sum_{\substack{i,k\\ i+k\leq N}} {}_C\langle i,k;N \mid m,n;N\rangle_S \mid i,k;N\rangle_C, \tag{5.16b}
$$

relating the states of the Cartesian and spherical bases. Since these states are orthonormal, the expansion coefficients satisfy the pair of orthogonality relations

$$\sum_{i+k\leq N} {}_S\langle m,n;N \mid i,k;N\rangle_C {}_C\langle i,k;N \mid m',n';N\rangle_S = \delta_{mm'}\delta_{nn'}, \tag{5.17a}$$

$$\sum_{m+n\leq N} {}_C\langle i,k;N \mid m,n;N\rangle_S {}_S\langle m,n;N \mid i',k';N\rangle_C = \delta_{ii'}\delta_{kk'}. \tag{5.17b}$$

Upon using the explicit expressions (5.6) and (5.13) of the wavefunctions in Cartesian coordinates and the property $P_n^{(\alpha,\beta)}(-z) = (-1)^n P_n^{(\beta,\alpha)}(z)$ satisfied by the Jacobi polynomials, it is directly seen from (5.15) that the expansion coefficients obey the symmetry relation

$$\begin{aligned} &{}_C\langle \alpha_1,\alpha_2,\alpha_3;i,k;N \mid \alpha_1,\alpha_2,\alpha_3;m,n;N\rangle_S \\ &\qquad = (-1)^m {}_C\langle \alpha_2,\alpha_1,\alpha_3;k,i;N \mid \alpha_2,\alpha_1,\alpha_3;m,n;N\rangle_S, \end{aligned} \tag{5.18}$$

which allows one to interchange the pairs (i,α_1) and (k,α_2). This symmetry shall prove useful in what follows.

5.3 The expansion coefficients as orthogonal polynomials in two variables

In this section, it is shown that the overlap coefficients between the Cartesian and spherical basis states defined in the previous section are expressed in terms of orthogonal polynomials in the two discrete variables i,k.

The expansion coefficients (5.15) can be cast in the form

$${}_C\langle \alpha_1,\alpha_2,\alpha_3;i,k;N \mid \alpha_1,\alpha_2,\alpha_3;m,n;N\rangle_S = W_{i,k;N}^{(\alpha_1,\alpha_2,\alpha_3)} Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N), \tag{5.19}$$

where $Q_{0,0}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N) \equiv 1$ and where we have defined

$$W_{i,k;N}^{(\alpha_1,\alpha_2,\alpha_3)} = {}_C\langle \alpha_1,\alpha_2,\alpha_3;i,k;N \mid \alpha_1,\alpha_2,\alpha_3;0,0;N\rangle_S.$$

5.3.1 Calculation of $W_{i,k;N}^{(\alpha_1,\alpha_2,\alpha_3)}$

The coefficient $W_{i,k;N}^{(\alpha_1,\alpha_2,\alpha_3)}$ in (5.19) can be evaluated explicitly using the definition (5.15) of the overlap coefficients. Indeed, upon taking $m = n = 0$ in (5.15) with the expressions

(5.6) and (5.13) for the wavefunctions, one finds

$$W_{i,k;N}^{(\alpha_1,\alpha_2,\alpha_3)} = \xi_i^{(\alpha_1)}\xi_k^{(\alpha_2)}\xi_{N-i-k}^{(\alpha_3)}\,\eta_0^{(\alpha_1,\alpha_2)}\eta_0^{(\alpha_{12}+1,\alpha_3)}\xi_N^{(\alpha_{123}+2)}\int_0^\infty\int_0^\infty\int_0^\infty \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$$
$$e^{-(x_1^2+x_2^2+x_3^2)}\prod_{j=1}^{3}(x_j^2)^{\alpha_j+1/2}\,L_i^{(\alpha_1)}(x_1^2)L_k^{(\alpha_2)}(x_2^2)L_{N-i-k}^{(\alpha_3)}(x_3^2)L_N^{(\alpha_{123}+2)}(x_1^2+x_2^2+x_3^2). \quad (5.20)$$

Upon using twice the addition formula for the Laguerre polynomials [2]

$$L_n^{(\alpha+\beta+1)}(x+y) = \sum_{\ell+k\leq n} L_\ell^{(\alpha)}(x)L_k^{(\beta)}(y),$$

one obtains the relation

$$L_N^{(\alpha_{123}+2)}(x_1^2+x_2^2+x_3^2) = \sum_{i'+k'\leq N} L_{i'}^{(\alpha_1)}(x_1^2)L_{k'}^{(\alpha_2)}(x_2^2)L_{N-i'-k'}^{(\alpha_3)}(x_3^2).$$

The use of the above identity in (5.20) along with the orthogonality relation for the Laguerre polynomials directly yields the explicit formula

$$W_{i,k;N}^{(\alpha_1,\alpha_2,\alpha_3)} = \frac{\eta_0^{(\alpha_1,\alpha_2)}\eta_0^{(\alpha_{12}+1,\alpha_3)}\xi_N^{(\alpha_{123}+2)}}{\xi_i^{(\alpha_1)}\xi_k^{(\alpha_2)}\xi_{N-i-k}^{(\alpha_3)}}.$$

With the help of the identity $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ for the Pochhammer symbol, the above expression is easily cast in the form

$$W_{i,k;N}^{(\alpha_1,\alpha_2,\alpha_3)} = \sqrt{\frac{N!}{x!y!(N-x-y)!}\,\frac{(\alpha_1+1)_i(\alpha_2+1)_k(\alpha_3+1)_{N-i-k}}{(\alpha_1+\alpha_2+\alpha_3+3)_N}}. \quad (5.21)$$

5.3.2 Raising relations

We shall now show that the functions $Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)$ appearing in (5.19) are polynomials of degree $m+n$ in the variable i,k by obtaining their raising relations.

Raising relation in m

Consider the operator $C_+^{(\alpha_1,\alpha_2)}$ having the following expression in spherical coordinates

$$C_+^{(\alpha_1,\alpha_2)} = \frac{1}{2}\left[-\partial_\phi + \mathrm{tg}\,\phi(\alpha_1+1/2) - \frac{(\alpha_2+1/2)}{\mathrm{tg}\,\phi}\right]. \quad (5.22)$$

Using the structure relation (5.106) for the Jacobi polynomials, it can be directly checked that one has

$$C_+^{(\alpha_1,\alpha_2)}\,\Xi_{m,n;N}^{(\alpha_1+1,\alpha_2+1,\alpha_3)} = \sqrt{(m+1)(m+\alpha_{12}+2)}\,\Xi_{m+1,n;N+1}^{(\alpha_1,\alpha_2,\alpha_3)}. \quad (5.23)$$

Consider the matrix element ${}_C\langle\, \alpha_1,\alpha_2,\alpha_3;i,k;N \mid C_{+}^{(\alpha_1,\alpha_2)} \mid \alpha_1+1,\alpha_2+1,\alpha_3;m,n;N-1\,\rangle_S$. On the one hand, the action (5.23) and the definition (5.19) give

$$
\begin{aligned}
&{}_C\langle\, \alpha_1,\alpha_2,\alpha_3;i,k;N \mid C_{+}^{(\alpha_1,\alpha_2)} \mid \alpha_1+1,\alpha_2+1,\alpha_3;m,n;N-1\,\rangle_S \\
&\qquad\qquad = \sqrt{(m+1)(m+\alpha_{12}+2)}\, W_{i,k;N}^{(\alpha_1,\alpha_2,\alpha_3)}\, Q_{m+1,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N).
\end{aligned}
\tag{5.24}
$$

To obtain a raising relation, one needs to compute ${}_{\alpha_1,\alpha_2,\alpha_3;i,k;N}\langle\, C\,| C_{+}^{(\alpha_1,\alpha_2)}$ or equivalently (recalling that the wavefunctions are real) $(C_{+}^{(\alpha_1,\alpha_2)})^{\dagger} |\, \alpha_1,\alpha_2,\alpha_3;i,k;N\,\rangle_C$. This computation can be performed in a straightforward fashion by writing (5.22) in Cartesian coordinates, acting on the wavefunctions (5.6) and using identities of the Laguerre polynomials (see appendix of [12] for the details of a similar computation). One finds as a result

$$
\begin{aligned}
&(C_{+}^{(\alpha_1,\alpha_2)})^{\dagger}\Psi_{i,k;N}^{(\alpha_1,\alpha_2,\alpha_3)} \\
&\qquad = \sqrt{i(k+\alpha_2+1)}\,\Psi_{i-1,k;N-1}^{(\alpha_1+1,\alpha_2+1,\alpha_3)} - \sqrt{(i+\alpha_1+1)k}\,\Psi_{i,k-1;N-1}^{(\alpha_1+1,\alpha_2+1,\alpha_3)}.
\end{aligned}
\tag{5.25}
$$

Upon combining (5.24) and (5.25) and using (5.21), one arrives at the following contiguity relation for the functions $Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)$:

$$
\begin{aligned}
c_{m,n;N}^{(\alpha_1,\alpha_2,\alpha_3)}\, Q_{m+1,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N) &= i(k+\alpha_2+1)\,Q_{m,n}^{(\alpha_1+1,\alpha_2+1,\alpha_3)}(i-1,k;N-1) \\
&\quad - k(i+\alpha_1+1)\,Q_{m,n}^{(\alpha_1+1,\alpha_2+1,\alpha_3)}(i,k-1;N-1).
\end{aligned}
\tag{5.26}
$$

where $c_{m,n;N}^{(\alpha_1,\alpha_2,\alpha_3)}$ are the coefficients given by the expression

$$
c_{m,n;N}^{(\alpha_1,\alpha_2,\alpha_3)} = \sqrt{\tfrac{N(\alpha_1+1)(\alpha_2+1)(N+\alpha_{123}+3)(m+1)(m+\alpha_{12}+2)}{(\alpha_{123}+3)(\alpha_{123}+4)}}.
$$

Raising relation in n

Consider the operator $D_{+}^{(\alpha_1,\alpha_2,\alpha_3)}$ defined as follows in spherical coordinates:

$$
D_{+}^{(\alpha_1,\alpha_2,\alpha_3)} = \left\{ Q^{(123)} - Q^{(12)} + \frac{\alpha_3+1}{2}\left[\operatorname{tg}\theta\,\partial_\theta - \frac{\alpha_3-1/2}{\cos^2\theta} + \frac{\alpha_3+2}{2}\right]\right\},
\tag{5.27}
$$

where $Q^{(12)}$ and $Q^{(123)}$ are given by (5.12a) and (5.12b), respectively. Using the structure relation (5.107) for the Jacobi polynomials as well as the eigenvalue equations (5.11), one finds that the action of the operator $D_{+}^{(\alpha_1,\alpha_2,\alpha_3)}$ on the spherical basis states is

$$
\begin{aligned}
&D_{+}^{(\alpha_1,\alpha_2,\alpha_3)}\,\Xi_{m,n;N}^{(\alpha_1,\alpha_2,\alpha_3+2)} = \\
&\qquad \sqrt{(n+1)(n+\alpha_3+2)(n+2m+\alpha_{12}+2)(n+2m+\alpha_{123}+3)}\,\Xi_{m,n+1;N+1}^{(\alpha_1,\alpha_2,\alpha_3)}
\end{aligned}
\tag{5.28}
$$

Consider the matrix element ${}_C\langle\, \alpha_1,\alpha_2,\alpha_3;i,k;N \mid D_{+}^{(\alpha_1,\alpha_2,\alpha_3)} \mid \alpha_1,\alpha_2,\alpha_3+2;m,n;N-1\,\rangle_S$. Upon using the action (5.28) and the definition (5.15) of the overlap coefficients, one finds on the one hand

$$
\begin{aligned}
{}_C\langle\, \alpha_1,\alpha_2,\alpha_3;i,k;N \mid D_{+}^{(\alpha_1,\alpha_2,\alpha_3)} \mid \alpha_1,\alpha_2,\alpha_3+2;m,n;N-1\,\rangle_S &= \sqrt{(n+1)(n+\alpha_3+2)} \\
\times \sqrt{(n+2m+\alpha_{12}+2)(n+2m+\alpha_{123}+3)}\, W_{i,k;N}^{(\alpha_1,\alpha_2,\alpha_3)}\, Q_{m,n+1}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N) \quad (5.29)
\end{aligned}
$$

On the other hand, a direct computation shows that

$$
\begin{aligned}
(D_{+}^{(\alpha_1,\alpha_2,\alpha_3)})^{\dagger}\Psi_{i,k;N}^{(\alpha_1,\alpha_2,\alpha_3)} &= \sqrt{(i+1)(i+\alpha_1+1)(N-i-k)(N-i-k-1)}\,\Psi_{i+1,k;N-1}^{(\alpha_1,\alpha_2,\alpha_3+2)} \\
&+ \sqrt{(k+1)(k+\alpha_2+1)(N-i-k)(N-i-k-1)}\,\Psi_{i,k+1;N-1}^{(\alpha_1,\alpha_2,\alpha_3+2)} \\
&+ \sqrt{i(i+\alpha_1)(N-i-k+\alpha_3+1)(N-i-k+\alpha_3+2)}\,\Psi_{i-1,k;N-1}^{(\alpha_1,\alpha_2,\alpha_3+2)} \\
&+ \sqrt{k(k+\alpha_2)(N-i-k+\alpha_3+1)(N-i-k+\alpha_3+2)}\,\Psi_{i,k-1;N-1}^{(\alpha_1,\alpha_2,\alpha_3+2)} \\
&- (2i+2k+\alpha_{12}+2)\sqrt{(N-i-k)(N-i-k+\alpha_3+1)}\,\Psi_{i,k;N-1}^{(\alpha_1,\alpha_2,\alpha_3+2)}. \quad (5.30)
\end{aligned}
$$

Upon combining (5.29) and (5.30) and using (5.21), one obtains another contiguity relation of the form

$$
\begin{aligned}
& d_{m,n;N}^{(\alpha_1,\alpha_2,\alpha_3)}\, Q_{m,n+1}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N) = \\
& (i+\alpha_1+1)(N-i-k)(N-i-k-1)\,Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3+2)}(i+1,k;N-1) \\
& + (k+\alpha_2+1)(N-i-k)(N-i-k-1)\,Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3+2)}(i,k+1;N-1) \\
& + i(N-i-k+\alpha_3+1)(N-i-k+\alpha_3+2)\,Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3+2)}(i-1,k;N-1) \\
& + k(N-i-k+\alpha_3+1)(N-i-k+\alpha_3+2)\,Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3+2)}(i,k-1;N-1) \\
& \qquad - (N-i-k)(N-i-k+\alpha_3+1)(2i+2k+\alpha_{12}+2)\,Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3+2)}(i,k;N-1). \quad (5.31)
\end{aligned}
$$

where $d_{m,n;N}^{(\alpha_1,\alpha_2,\alpha_3)}$ are the coefficients given by the expression

$$
d_{m,n;N}^{(\alpha_1,\alpha_2,\alpha_3)} = \sqrt{\frac{N(N+\alpha_{123}+3)(\alpha_3+1)(\alpha_3+2)(n+1)(n+\alpha_3+2)(n+2m+\alpha_{12}+2)(n+2m+\alpha_{123}+3)}{(\alpha_{123}+3)(\alpha_{123}+4)}}.
$$

Since by definition $Q_{0,0}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)=1$, the relations (5.26) and (5.31) allow to construct any $Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)$ iteratively. Writing up the first few cases, one observes that the $Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)$ are polynomials of total degree $m+n$ in the variables i,k.

5.3.3 Orthogonality relation

It is easy to see that the orthogonality relation (5.17a) satisfied by the transition coefficients ${}_C\langle i,k;N \mid m,n;N\,\rangle_S$ implies that the polynomials $Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)$ are orthogonal. Indeed, upon inserting (5.19) in the relation (5.17a), one finds that the polynomials $Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)$ are orthonormal

$$\sum_{i+k\leq N} w_{i,k;N}^{(\alpha_1,\alpha_2,\alpha_3)} Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N) Q_{m',n'}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N) = \delta_{mm'}\delta_{nn'},$$

with respect to the discrete weight function

$$w_{i,k;N}^{(\alpha_1,\alpha_2,\alpha_3)} = \left[W_{i,k;N}^{(\alpha_1,\alpha_2,\alpha_3)}\right]^2 = \binom{N}{i,k} \frac{(\alpha_1+1)_i(\alpha_2+1)_k(\alpha_3+1)_{N-i-k}}{(\alpha_1+\alpha_2+\alpha_3+3)_N}, \tag{5.32}$$

where $\binom{N}{x,y}$ are the trinomial coefficients. It is clear that the weight (5.32) is a bivariate extension of the Hahn weight function (5.1).

5.3.4 Lowering relations

It is also possible to obtain lowering relations for the polynomials $Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)$ using the operators that are conjugate to $C_+^{(\alpha_1,\alpha_2)}$ and $D_+^{(\alpha_1,\alpha_2,\alpha_3)}$.

Lowering relation in m

Let us first examine the operator

$$C_-^{(\alpha_1,\alpha_2)} = \frac{1}{2}\left[\partial_\phi + \mathrm{tg}\phi(\alpha_1+1/2) - \frac{(\alpha_2+1/2)}{\mathrm{tg}\phi}\right]. \tag{5.33}$$

It is obvious from the definitions (5.22) and (5.33) that $(C_\pm^{(\alpha_1,\alpha_2)})^\dagger = C_\mp^{(\alpha_1,\alpha_2)}$. Furthermore, it is directly verified with the help of (5.104) that (5.33) has the action

$$C_-^{(\alpha_1,\alpha_2)}\,\Xi_{m,n;N}^{(\alpha_1,\alpha_2,\alpha_3)} = \sqrt{m(m+\alpha_{12}+1)}\,\Xi_{m-1,n;N-1}^{(\alpha_1+1,\alpha_2+1,\alpha_3)}. \tag{5.34}$$

Consider the matrix element ${}_C\langle\, \alpha_1+1,\alpha_2+1,\alpha_3; i,k:N \mid C_-^{(\alpha_1,\alpha_2)} \mid \alpha_1,\alpha_2,\alpha_3; m,n; N+1\,\rangle_S$. Upon using (5.34) and the definition (5.19), one finds on the one hand

$$\begin{aligned} &{}_C\langle\, \alpha_1+1,\alpha_2+1,\alpha_3; i,k:N \mid C_-^{(\alpha_1,\alpha_2)} \mid \alpha_1,\alpha_2,\alpha_3; m,n; N+1\,\rangle_S \\ &\qquad = \sqrt{m(m+\alpha_{12}+1)}\,W_{i,k;N}^{(\alpha_1+1,\alpha_2+1,\alpha_3)}\,Q_{m-1,n}^{(\alpha_1+1,\alpha_2+1,\alpha_3)}(i,k;N). \end{aligned} \tag{5.35}$$

Upon writing $(C_-^{(\alpha_1,\alpha_2)})^\dagger$ in Cartesian coordinates and acting on the wavefunctions (5.6), one finds on the other hand

$$\begin{aligned} &(C_-^{(\alpha_1,\alpha_2)})^\dagger \Psi_{i,k;N}^{(\alpha_1+1,\alpha_2+1,\alpha_3)} = \\ &\qquad \sqrt{(i+1)(k+\alpha_2+1)}\,\Psi_{i+1,k;N+1}^{(\alpha_1,\alpha_2,\alpha_3)} - \sqrt{(i+\alpha_1+1)(k+1)}\,\Psi_{i,k+1;N+1}^{(\alpha_1,\alpha_2,\alpha_3)}. \end{aligned} \tag{5.36}$$

Combining (5.35) and (5.36) using (5.19) and (5.21), the following lowering relation for the polynomials $Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)$ is obtained

$$e_{m,n;N}^{(\alpha_1,\alpha_2,\alpha_3)} Q_{m-1,n}^{(\alpha_1+1,\alpha_2+1,\alpha_3)}(i,k;N) = Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i+1,k;N+1) - Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k+1;N+1),$$

where

$$e_{m,n;N}^{(\alpha_1,\alpha_2,\alpha_3)} = \sqrt{\tfrac{m(m+\alpha_{12}+1)(\alpha_{123}+3)(\alpha_{123}+4)}{(\alpha_1+1)(\alpha_2+1)(N+1)(N+\alpha_{123}+4)}}.$$

Lowering relation in n

Let us now consider the operator

$$D_-^{(\alpha_1,\alpha_2,\alpha_3)} = \left\{ Q^{(123)} - Q^{(12)} - \frac{\alpha_3+1}{2}\left[\operatorname{tg}\theta\,\partial_\theta + \frac{\alpha_3+1/2}{\cos^2\theta} - \frac{\alpha_3}{2} \right] \right\}. \tag{5.37}$$

Taking into account that $(Q^{(123)})^\dagger = Q^{(123)}$ and $(Q^{(12)})^\dagger = Q^{(12)}$, it can be seen from the definitions (5.27) and (5.37) that $(D_\pm^{(\alpha_1,\alpha_2,\alpha_3)})^\dagger = D_\mp^{(\alpha_1,\alpha_2,\alpha_3)}$. In view of the relation (5.105) and using the eigenvalue equations (5.11), it follows that the action of the operator $D_-^{(\alpha_1,\alpha_2,\alpha_3)}$ is given by

$$D_-^{(\alpha_1,\alpha_2,\alpha_3)}\,\Xi_{m,n;N}^{(\alpha_1,\alpha_2,\alpha_3)} = \sqrt{n(n+\alpha_3+1)(n+2m+\alpha_{12}+1)(n+2m+\alpha_{123}+2)}\,\Xi_{m,n-1;N-1}^{(\alpha_1,\alpha_2,\alpha_3+2)}. \tag{5.38}$$

Consider the matrix element ${}_C\langle\, \alpha_1,\alpha_2,\alpha_3+2;i,k;N \mid D_-^{(\alpha_1,\alpha_2,\alpha_3)} \mid \alpha_1,\alpha_2,\alpha_3;m,n;N+1\,\rangle_S$. Using (5.38) and (5.15), one can write

$$\begin{aligned}{}_C\langle\, \alpha_1,\alpha_2,\alpha_3+2;i,k;N \mid D_-^{(\alpha_1,\alpha_2,\alpha_3)} \mid \alpha_1,\alpha_2,\alpha_3;m,n;N+1\,\rangle_S &= \sqrt{n(n+\alpha_3+1)} \\ \times \sqrt{(n+2m+\alpha_{12}+1)(n+2m+\alpha_{123}+2)}\,W_{i,k;N}^{(\alpha_1,\alpha_2,\alpha_3+2)} &Q_{m,n-1}^{(\alpha_1,\alpha_2,\alpha_3+2)}(i,k;N).\end{aligned} \tag{5.39}$$

The action of $(D_-^{(\alpha_1,\alpha_2,\alpha_3)})^\dagger$ on the Cartesian basis wavefunctions (5.6) can be computed with the result

$$\begin{aligned}(D_-^{(\alpha_1,\alpha_2,\alpha_3)})^\dagger\,\Psi_{i,k;N}^{(\alpha_1,\alpha_2,\alpha_3+2)} &= \\ \sqrt{(i+1)(i+\alpha_1+1)(N-i-k+\alpha_3+1)(N-i-k+\alpha_3+2)}\,&\Psi_{i+1,k;N+1}^{(\alpha_1,\alpha_2,\alpha_3)} \\ + \sqrt{(k+1)(k+\alpha_2+1)(N-i-k+\alpha_3+1)(N-i-k+\alpha_3+2)}\,&\Psi_{i,k+1;N+1}^{(\alpha_1,\alpha_2,\alpha_3)} \\ + \sqrt{i(i+\alpha_1)(N-i-k+1)(N-i-k+2)}\,&\Psi_{i-1,k;N+1}^{(\alpha_1,\alpha_2,\alpha_3)} \\ + \sqrt{k(k+\alpha_2)(N-i-k+1)(N-i-k+2)}\,&\Psi_{i,k-1;N+1}^{(\alpha_1,\alpha_2,\alpha_3)} \\ -(2i+2k+\alpha_{12}+2)\sqrt{(N-i-k+1)(N-i-k+\alpha_3+2)}\,&\Psi_{i,k;N+1}^{(\alpha_1,\alpha_2,\alpha_3)}.\end{aligned} \tag{5.40}$$

Combining (5.39) and (5.40) and making use of (5.21) yields

$$
\begin{aligned}
&f_{m,n;N}^{(\alpha_1,\alpha_2,\alpha_3)} Q_{m,n-1}^{(\alpha_1,\alpha_2,\alpha_3+2)}(i,k;N) = (i+\alpha_1+1) Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i+1,k;N+1) \\
&+ (k+\alpha_2+1) Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k+1;N+1) + i\, Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i-1,;N+1) \\
&\qquad\qquad + k\, Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k-1;N+1) - (2i+2k+\alpha_{12}+2)\, Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N+1),
\end{aligned}
$$

with

$$
f_{m,n;N}^{(\alpha_1,\alpha_2,\alpha_3)} = \sqrt{\frac{n(n+\alpha_3+1)(n+2m+\alpha_{12}+1)(n+2m+\alpha_{123}+2)(\alpha_{123}+3)(\alpha_{123}+4)}{(\alpha_3+1)(\alpha_3+2)(N+1)(N+\alpha_{123}+4)}}.
$$

5.4 Generating function

In this section, a generating function for the bivariate polynomials $Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)$ is derived by examining the asymptotic behavior of the wavefunctions. The generating function is then seen to coincide with that of the Hahn polynomials, thus establishing that the polynomials $Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}$ are precisely the bivariate Hahn polynomials introduced by Karlin and McGregor in [25].

Consider the interbasis expansion formula (5.16b). Using spherical coordinates, it is easily seen from (5.6), (5.9) and (5.19) that this formula can be cast in the form

$$
\begin{aligned}
&\eta_m^{(\alpha_1,\alpha_2)} \eta_n^{(2m+\alpha_{12}+1,\alpha_3)} \xi_{N-m-n}^{(2m+2n+\alpha_{123}+2)} \\
&\times P_m^{(\alpha_1,\alpha_2)}(-\cos 2\phi)(\sin^2\theta)^m P_n^{(2m+\alpha_{12}+1,\alpha_3)}(\cos 2\theta)(r^2)^{m+n} L_{N-m-n}^{(2m+2n+\alpha_{123}+2)}(r^2) \\
&= \sum_{i+k\leq N} W_{i,k;N}^{(\alpha_1,\alpha_2,\alpha_3)} Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N) \\
&\quad \times \xi_i^{(\alpha_1)} \xi_k^{(\alpha_2)} \xi_{N-i-k}^{(\alpha_3)} L_i^{(\alpha_1)}(r^2\sin^2\theta\cos^2\phi) L_k^{(\alpha_2)}(r^2\sin^2\theta\sin^2\phi) L_{N-i-k}^{(\alpha_3)}(r^2\cos^2\theta). \qquad (5.41)
\end{aligned}
$$

In (5.41), the expansion coefficients

$$
{}_S\langle m,n;N \mid i,k;N\rangle_C = W_{i,k;N}^{(\alpha_1,\alpha_2,\alpha_3)} Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N),
$$

are independent of the expansion point specified by the values of the coordinates (r,θ,ϕ). Let us consider the case where the value of the radial coordinate r is large. Since the asymptotic behavior of the Laguerre polynomials is of the form

$$
L_n^{(\alpha)}(x) \sim \frac{(-1)^n}{n!} x^n + \mathscr{O}(x^{n-1}).
$$

it follows that the asymptotic form of expansion formula (5.41) is

$$
\begin{aligned}
&\eta_m^{(\alpha_1,\alpha_2)}\eta_n^{(2m+\alpha_{12}+1,\alpha_3)}\xi_{N-m-n}^{(2m+2n+\alpha_{123}+2)}\,P_m^{(\alpha_1,\alpha_2)}(-\cos 2\phi)(\sin^2\theta)^m P_n^{(2m+\alpha_{12}+1,\alpha_3)}(\cos 2\theta)\\
&=(-1)^{n+m}(N-m-n)!\sum_{i+k\le N} W_{i,k;N}^{(\alpha_1,\alpha_2,\alpha_3)}\,Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)\\
&\qquad\times\frac{\xi_i^{(\alpha_1)}\xi_k^{(\alpha_2)}\xi_{N-i-k}^{(\alpha_3)}}{i!k!(N-i-k)!}\,(\sin^2\theta\cos^2\phi)^i(\sin^2\theta\sin^2\phi)^k(\cos^2\theta)^{N-i-k}. \qquad (5.42)
\end{aligned}
$$

In terms of the variables $z_1=\mathrm{tg}^2\,\theta\cos^2\phi$ and $z_2=\mathrm{tg}^2\,\theta\sin^2\phi$, the formula (5.42) reads

$$
\begin{aligned}
&\left\{\frac{(-1)^{m+n}}{(N-m-n)!}\,\eta_m^{(\alpha_1,\alpha_2)}\eta_n^{(2m+\alpha_{12}+1,\alpha_3)}\xi_{N-m-n}^{(2m+2n+\alpha_{123}+2)}\right\}\\
&\times(1+z_1+z_2)^{N-m}(z_1+z_2)^m\,P_m^{(\alpha_1,\alpha_2)}\left(\frac{z_2-z_1}{z_1+z_2}\right)P_n^{(2m+\alpha_{12}+1,\alpha_3)}\left(\frac{1-z_1-z_2}{1+z_1+z_2}\right)\\
&\qquad=\sum_{i+k\le N}W_{i,k;N}^{(\alpha_1,\alpha_2,\alpha_3)}Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)\left\{\frac{\xi_i^{(\alpha_1)}\xi_k^{(\alpha_2)}\xi_{N-i-k}^{(\alpha_3)}}{i!k!(N-i-k)!}\right\}z_1^i z_2^k, \qquad (5.43)
\end{aligned}
$$

which has the form of a generating relation for the polynomials $Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)$. Let $H_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)$ denote the polynomials

$$
Q_{m,n;N}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)=\frac{m!n!}{\sqrt{\Lambda_{m,n;N}}}(-N)_{m+n}H_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N), \qquad (5.44)
$$

where

$$
\begin{aligned}
&\Lambda_{m,n;N}^{(\alpha_1,\alpha_2,\alpha_3)}=\\
&\left\{\frac{N!m!n!}{(N-m-n)!}\,\frac{(\alpha_1+1)_m(\alpha_2+1)_m(\alpha_3+1)_n(\alpha_{12}+1)_{2m}}{(\alpha_{12}+1)_m(\alpha_{123}+3)_N}\,\frac{(2m+\alpha_{12}+2)_n(2m+\alpha_{123}+2)_{2n}(m+n+\alpha_{123}+3)_N}{(2m+\alpha_{123}+2)_n(m+n+\alpha_{123}+3)_{m+n}}\right\}. \qquad (5.45)
\end{aligned}
$$

that differ from $Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)$ only by a normalization factor. Performing elementary simplifications, it follows from (5.43) that the generating relation for the polynomials $H_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)$ has the expression

$$
\begin{aligned}
&(1+z_1+z_2)^{N-m}(z_1+z_2)^m\,P_m^{(\alpha_1,\alpha_2)}\left(\frac{z_2-z_1}{z_1+z_2}\right)P_n^{(2m+\alpha_{12}+1,\alpha_3)}\left(\frac{1-z_1-z_2}{1+z_1+z_2}\right)\\
&\qquad=\sum_{i+k\le N}\binom{N}{i,k}H_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)z_1^i z_2^k. \qquad (5.46)
\end{aligned}
$$

The generating function (5.46) is a bivariate generalization of the dual generating function (5.4) for the Hahn polynomials of a single variable. Comparing the generating function (5.46) with the one used in [25] to define the bivariate Hahn polynomials, it is not hard to see that the two generating functions coincide. Hence one may conclude that the polynomials $H_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)$ (and equivalently $Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)$) are precisely the bivariate Hahn polynomials of Karlin and McGregor. Note that on the L.H.S of (5.46) are essentially the Jacobi polynomials on the 2-simplex [8], as observed by Xu in [39].

5.5 Recurrence relations

In this section, backward and forward structure relations for the bivariate Hahn polynomials are obtained using the raising/lowering relations of the Laguerre polynomials. These structure relations are then used to derive by factorization the recurrence relations of the polynomials $Q_{m,n;N}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)$ and $H_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)$.

5.5.1 Forward structure relation in the variable i

To obtain a forward structure relation in the variable i, consider the first order operator

$$A_{+}^{(\alpha_1)} = \frac{1}{2}\left[\partial_{x_1} + \frac{(\alpha_1+1/2)}{x_1} - x_1\right].$$

With the help of the relation (5.109) for the Laguerre polynomials, it is verified that the action of this operator on the Cartesian basis wavefunctions (5.6) is

$$A_{+}^{(\alpha_1)}\Psi_{i,k;N}^{(\alpha_1+1,\alpha_2,\alpha_3)} = \sqrt{i+1}\,\Psi_{i+1,k;N+1}^{(\alpha_1,\alpha_2,\alpha_3)}. \tag{5.47}$$

Consider the matrix element ${}_S\langle\, \alpha_1,\alpha_2,\alpha_3;m,n;N \mid A_{+}^{(\alpha_1)} \mid \alpha_1+1,\alpha_2,\alpha_3;i,k;N-1\,\rangle_C$. The action (5.47) gives on the one hand

$$\begin{aligned}{}_S\langle\, \alpha_1,\alpha_2,\alpha_3;m,n;N \mid A_{+}^{(\alpha_1)} \mid \alpha_1+1,\alpha_2,\alpha_3;i,k;N-1\,\rangle_C \\ = \sqrt{i+1}\,W_{i+1,k;N}^{(\alpha_1,\alpha_2,\alpha_3)}\,Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N).\end{aligned} \tag{5.48}$$

Upon writing $A_{+}^{(\alpha_1)}$ in spherical coordinates and acting on (5.9), one finds

$$\begin{aligned}(A_{+}^{(\alpha_1)})^{\dagger}\,\Xi_{m,n;N}^{(\alpha_1,\alpha_2,\alpha_3)} = {} & \alpha_{m,n;N}^{(\alpha_1,\alpha_2,\alpha_3)}\,\Xi_{m,n;N-1}^{(\alpha_1+1,\alpha_2,\alpha_3)} + \beta_{m,n;N}^{(\alpha_1,\alpha_2,\alpha_3)}\,\Xi_{m-1,n;N-1}^{(\alpha_1+1,\alpha_2,\alpha_3)} \\ & + \gamma_{m,n;N}^{(\alpha_1,\alpha_2,\alpha_3)}\,\Xi_{m,n-1;N-1}^{(\alpha_1+1,\alpha_2,\alpha_3)} + \delta_{m,n+1;N}^{(\alpha_1,\alpha_2,\alpha_3)}\,\Xi_{m-1,n+1;N-1}^{(\alpha_1+1,\alpha_2,\alpha_3)}.\end{aligned} \tag{5.49}$$

where the coefficients α, β, γ and δ are given by

$$\begin{aligned}
\alpha^{(\alpha_1,\alpha_2,\alpha_3)}_{m,n;N} &= \sqrt{\frac{(m+\alpha_1+1)(m+\alpha_{12}+1)(n+2m+\alpha_{12}+2)(n+2m+\alpha_{123}+2)(N-m-n)}{(2m+\alpha_{12}+1)(2m+\alpha_{12}+2)(2n+2m+\alpha_{123}+2)(2n+2m+\alpha_{123}+3)}},\\
\beta^{(\alpha_1,\alpha_2,\alpha_3)}_{m,n;N} &= \sqrt{\frac{m(m+\alpha_2)(n+2m+\alpha_{12}+1)(n+2m+\alpha_{123}+1)(N+m+n+\alpha_{123}+2)}{(2m+\alpha_{12})(2m+\alpha_{12}+1)(2n+2m+\alpha_{123}+1)(2n+2m+\alpha_{123}+2)}},\\
\gamma^{(\alpha_1,\alpha_2,\alpha_3)}_{m,n;N} &= \sqrt{\frac{n(n+\alpha_3)(m+\alpha_1+1)(m+\alpha_{12}+1)(N+m+n+\alpha_{123}+2)}{(2m+\alpha_{12}+1)(2m+\alpha_{12}+2)(2n+2m+\alpha_{123}+2)(2n+2m+\alpha_{123}+3)}},\\
\delta^{(\alpha_1,\alpha_2,\alpha_3)}_{m,n;N} &= \sqrt{\frac{mn(m+\alpha_2)(n+\alpha_3)(N-m-n+1)}{(2m+\alpha_{12})(2m+\alpha_{12}+1)(2n+2m+\alpha_{123})(2n+2m+\alpha_{123}+1)}}.
\end{aligned} \tag{5.50}$$

Upon combining (5.48) with (5.49) and using (5.21), one obtains the forward structure relation in the variable i for the polynomials $Q^{(\alpha_1,\alpha_2,\alpha_3)}_{m,n}(i,k;N)$:

$$\begin{aligned}
&\sqrt{\tfrac{N(\alpha_1+1)}{(\alpha_{123}+3)}}\,Q^{(\alpha_1,\alpha_2,\alpha_3)}_{m,n}(i+1,k;N) =\\
&\alpha^{(\alpha_1,\alpha_2,\alpha_3)}_{m,n;N}\,Q^{(\alpha_1+1,\alpha_2,\alpha_3)}_{m,n}(i,k;N-1)+\beta^{(\alpha_1,\alpha_2,\alpha_3)}_{m,n;N}\,Q^{(\alpha_1+1,\alpha_2,\alpha_3)}_{m-1,n}(i,k;N-1)\\
&\quad+\gamma^{(\alpha_1,\alpha_2,\alpha_3)}_{m,n;N}\,Q^{(\alpha_1+1,\alpha_2,\alpha_3)}_{m,n-1}(i,k;N-1)+\delta^{(\alpha_1,\alpha_2,\alpha_3)}_{m,n+1;N}\,Q^{(\alpha_1+1,\alpha_2,\alpha_3)}_{m-1,n+1}(i,k;N-1).
\end{aligned} \tag{5.51}$$

5.5.2 Backward structure relation in the variable i

To obtain the backward structure relation in i, one considers the operator

$$A^{(\alpha_1)}_{-} = \frac{1}{2}\left[-\partial_{x_1}+\frac{(\alpha_1+1/2)}{x_1}-x_1\right]. \tag{5.52}$$

It is clear that $(A^{(\alpha_1)}_{\pm})^{\dagger} = A^{(\alpha_1)}_{\mp}$. In view of (5.108), it follows that the action of $A^{(\alpha_1)}_{-}$ on the Cartesian basis wavefunctions is simply

$$A^{(\alpha_1)}_{-}\Psi^{(\alpha_1,\alpha_2,\alpha_3)}_{i,k;N} = \sqrt{i}\,\Psi^{(\alpha_1+1,\alpha_2,\alpha_3)}_{i-1,k;N-1}. \tag{5.53}$$

Consider the matrix element ${}_S\langle\, \alpha_1+1,\alpha_2,\alpha_3;m,n;N-1 \mid A^{(\alpha_1)}_{-} \mid \alpha_1,\alpha_2,\alpha_3;i,k;N\,\rangle_C$. The action (5.53) implies that

$$\begin{aligned}
&{}_S\langle\, \alpha_1+1,\alpha_2,\alpha_3;m,n;N-1 \mid A^{(\alpha_1)}_{-} \mid \alpha_1,\alpha_2,\alpha_3;i,k;N\,\rangle_C\\
&\qquad = \sqrt{i}\,W^{(\alpha_1+1,\alpha_2,\alpha_3)}_{i-1,k;N-1}\,Q^{(\alpha_1+1,\alpha_2,\alpha_3)}_{m,n}(i-1,k;N-1).
\end{aligned} \tag{5.54}$$

The action of $(A^{(\alpha_1)}_{-})^{\dagger}$ on the states of the spherical basis is given by

$$\begin{aligned}
(A^{(\alpha_1)}_{-})^{\dagger}\,\Xi^{(\alpha_1+1,\alpha_2,\alpha_3)}_{m,n;N-1} &= \alpha^{(\alpha_1,\alpha_2,\alpha_3)}_{m,n;N}\Xi^{(\alpha_1,\alpha_2,\alpha_3)}_{m,n;N}+\beta^{(\alpha_1,\alpha_2,\alpha_3)}_{m+1,n;N}\Xi^{(\alpha_1,\alpha_2,\alpha_3)}_{m+1,n;N}\\
&\quad+\gamma^{(\alpha_1,\alpha_2,\alpha_3)}_{m,n+1;N}\Xi^{(\alpha_1,\alpha_2,\alpha_3)}_{m,n+1;N}+\delta^{(\alpha_1,\alpha_2,\alpha_3)}_{m+1,n;N}\Xi^{(\alpha_1,\alpha_2,\alpha_3)}_{m+1,n-1},
\end{aligned} \tag{5.55}$$

where the coefficients are given by (5.50). Combining (5.54) and (5.55), we obtain the following backward structure relation in the variable i for the polynomials $Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)$:

$$
\begin{aligned}
&i\sqrt{\tfrac{(\alpha_{123}+3)}{N(\alpha_1+1)}}\,Q_{m,n}^{(\alpha_1+1,\alpha_2,\alpha_3)}(i-1,k;N-1)=\\
&\alpha_{m,n;N}^{(\alpha_1,\alpha_2,\alpha_3)}\,Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)+\beta_{m+1,n;N}^{(\alpha_1,\alpha_2,\alpha_3)}\,Q_{m+1,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)\\
&\qquad+\gamma_{m,n+1;N}^{(\alpha_1,\alpha_2,\alpha_3)}\,Q_{m,n+1}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)+\delta_{m+1,n;N}^{(\alpha_1,\alpha_2,\alpha_3)}\,Q_{m+1,n-1}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N). \quad (5.56)
\end{aligned}
$$

5.5.3 Forward and backward structure relations in the variable k

To obtain the forward and backward structure relations analogous to (5.51) and (5.56), one could consider the operators

$$
B_{\pm}^{(\alpha_2)}=\frac{1}{2}\left[\pm\partial_{x_2}+\frac{(\alpha_2+1/2)}{x_2}-x_2\right],
$$

and follow the same steps as in subsections (5.1) and (5.2). Alternatively, one can effectively use the symmetry relation (5.18) to derive these relations directly from (5.51) and (5.56) without additional computations. Upon using (5.18) on (5.51), one finds the forward structure relation

$$
\begin{aligned}
&\sqrt{\tfrac{N(\alpha_2+1)}{(\alpha_{123}+3)}}\,Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k+1;N)=\\
&\alpha_{m,n;N}^{(\alpha_2,\alpha_1,\alpha_3)}\,Q_{m,n}^{(\alpha_1,\alpha_2+1,\alpha_3)}(i,k;N-1)-\beta_{m,n;N}^{(\alpha_2,\alpha_1,\alpha_3)}\,Q_{m-1,n}^{(\alpha_1,\alpha_2+1,\alpha_3)}(i,k;N-1)\\
&\qquad+\gamma_{m,n;N}^{(\alpha_2,\alpha_1,\alpha_3)}\,Q_{m,n-1}^{(\alpha_1,\alpha_2+1,\alpha_3)}(i,k;N-1)-\delta_{m,n+1;N}^{(\alpha_2,\alpha_1,\alpha_3)}\,Q_{m-1,n+1}^{(\alpha_1,\alpha_2+1,\alpha_3)}(i,k;N-1). \quad (5.57)
\end{aligned}
$$

Note the permutation of the parameters (α_1,α_2) in the coefficients α, β, γ, δ and the sign differences. With the help of (5.18), one obtains from (5.56) the second backward structure relation

$$
\begin{aligned}
&k\sqrt{\tfrac{(\alpha_{123}+3)}{N(\alpha_2+1)}}\,Q_{m,n}^{(\alpha_1,\alpha_2+1,\alpha_3)}(i,k-1;N-1)=\\
&\alpha_{m,n;N}^{(\alpha_2,\alpha_1,\alpha_3)}\,Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)-\beta_{m+1,n;N}^{(\alpha_2,\alpha_1,\alpha_3)}\,Q_{m+1,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)\\
&\qquad+\gamma_{m,n+1;N}^{(\alpha_2,\alpha_1,\alpha_3)}\,Q_{m,n+1}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)-\delta_{m+1,n;N}^{(\alpha_2,\alpha_1,\alpha_3)}\,Q_{m+1,n-1}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N). \quad (5.58)
\end{aligned}
$$

The backward and forward structure relations (5.51), (5.56), (5.57) and (5.58) are of a different kind than those found in [32], which do not involve a change in the parameters.

5.5.4 Recurrence relations for the polynomials $Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_2)}(i,k;N)$

The operators $A_{\pm}^{(\alpha_1)}$ and the symmetry relation (5.18) can be used to construct the recurrence relations satisfied by the bivariate Hahn polynomials $Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)$. To that end, consider the matrix element ${}_S\langle\, \alpha_1,\alpha_2,\alpha_3;m,n;N \mid A_{+}^{(\alpha_1)}A_{-}^{(\alpha_1)} \mid \alpha_1,\alpha_2,\alpha_3;i,k;N\,\rangle_C$. The actions (5.47) and (5.53) give

$$
\begin{aligned}
&{}_S\langle\, \alpha_1,\alpha_2,\alpha_3;m,n;N \mid A_{+}^{(\alpha_1)}A_{-}^{(\alpha_1)} \mid \alpha_1,\alpha_2,\alpha_3;i,k;N\,\rangle_C \\
&\qquad = i\, W_{i,k;N}^{(\alpha_1,\alpha_2,\alpha_3)}\, Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N). \qquad (5.59)
\end{aligned}
$$

Note that $A_{+}^{(\alpha_1)}A_{-}^{(\alpha_1)}$ is essentially the Hermitian symmetry operator $K_0^{(1)}$ defined in (5.8) since $K_0^{(1)} = A_{+}^{(\alpha_1)}A_{-}^{(\alpha_1)} + (\alpha_1+1)/2$. Upon combining (5.49) and (5.55) with (5.59), one finds that the bivariate Hahn polynomials $Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)$ satisfy the 9-point recurrence relation

$$
\begin{aligned}
i\,Q_{m,n}(i,k) &= (\alpha_{m,n}\beta_{m+1,n})Q_{m+1,n}(i,k) + (\alpha_{m,n}\gamma_{m,n+1} + \beta_{m,n+1}\delta_{m,n+1})Q_{m,n+1}(i,k) \\
&+ (\gamma_{m-1,n+2}\delta_{m,n+1})Q_{m-1,n+2}(i,k) + (\alpha_{m,n}\delta_{m+1,n} + \beta_{m+1,n-1}\gamma_{m,n})Q_{m+1,n-1}(i,k) \\
&+ (\alpha_{m,n}^2 + \beta_{m,n}^2 + \gamma_{m,n}^2 + \delta_{m,n+1}^2)Q_{m,n}(i,k) + (\alpha_{m-1,n+1}\delta_{m,n+1} \\
&+ \beta_{m,n}\gamma_{m-1,n+1})Q_{m-1,n+1}(i,k) \\
&+ (\gamma_{m,n}\delta_{m+1,n-1})Q_{m+1,n-2}(i,k) + (\alpha_{m,n-1}\gamma_{m,n} + \beta_{m,n}\delta_{m,n})Q_{m,n-1}(i,k) \\
&\qquad + (\alpha_{m-1,n}\beta_{m,n})Q_{m-1,n}(i,k), \qquad (5.60)
\end{aligned}
$$

where the coefficients are given by (5.50); the explicit dependence on the parameters α_i and N has been dropped to facilitate the reading. The second recurrence relation in k is directly obtained using the symmetry (5.18). One finds

$$
\begin{aligned}
k\,Q_{m,n}(i,k) &= -(\tilde{\alpha}_{m,n}\tilde{\beta}_{m+1,n})Q_{m+1,n}(i,k) + (\tilde{\alpha}_{m,n}\tilde{\gamma}_{m,n+1} + \tilde{\beta}_{m,n+1}\tilde{\delta}_{m,n+1})Q_{m,n+1}(i,k) \\
&- (\tilde{\gamma}_{m-1,n+2}\tilde{\delta}_{m,n+1})Q_{m-1,n+2}(i,k) - (\tilde{\alpha}_{m,n}\tilde{\delta}_{m+1,n} + \tilde{\beta}_{m+1,n-1}\tilde{\gamma}_{m,n})Q_{m+1,n-1}(i,k) \\
&+ (\tilde{\alpha}_{m,n}^2 + \tilde{\beta}_{m,n}^2 + \tilde{\gamma}_{m,n}^2 + \tilde{\delta}_{m,n+1}^2)Q_{m,n}(i,k) - (\tilde{\alpha}_{m-1,n+1}\tilde{\delta}_{m,n+1} \\
&+ \tilde{\beta}_{m,n}\tilde{\gamma}_{m-1,n+1})Q_{m-1,n+1}(i,k) \\
&- (\tilde{\gamma}_{m,n}\tilde{\delta}_{m+1,n-1})Q_{m+1,n-2}(i,k) + (\tilde{\alpha}_{m,n-1}\tilde{\gamma}_{m,n} + \tilde{\beta}_{m,n}\tilde{\delta}_{m,n})Q_{m,n-1}(i,k) \\
&\qquad - (\tilde{\alpha}_{m-1,n}\tilde{\beta}_{m,n})Q_{m-1,n}(i,k), \qquad (5.61)
\end{aligned}
$$

where the $\tilde{x}$ coefficients correspond to (5.50) with $\alpha_1 \leftrightarrow \alpha_2$. For reference purposes, it is useful to explicitly show the coefficients appearing in the recurrence relations (5.60) and

(5.61). The recurrence relation (5.60) can be written as

$$
\begin{aligned}
i\,Q_{m,n}(i,k) &= a_{m+1,n}\,Q_{m+1,n}(i,k) + a_{m,n}\,Q_{m-1,n}(i,k) + b_{m,n+1}\,Q_{m,n+1}(i,k)\\
&+ b_{m,n}\,Q_{m,n-1}(i,k) + c_{m,n+2}\,Q_{m-1,n+2}(i,k) + c_{m+1,n}\,Q_{m+1,n-2}(i,k)\\
&\qquad + d_{m+1,n}\,Q_{m+1,n-1}(i,k) + d_{m,n+1}\,Q_{m-1,n+1}(i,k) + e_{m,n}\,Q_{m,n}(i,k),
\end{aligned}
$$

with $a_{m,n}$ and $c_{m,n}$ given by

$$
\begin{aligned}
a_{m,n} &= \sqrt{\frac{m(m+\alpha_1)(m+\alpha_2)(m+\alpha_{12})(n+2m+\alpha_{12})_2(n+2m+\alpha_{123})_2(N+m+n+\alpha_{123}+2)(N-m-n+1)}{(2m+\alpha_{12}-1)_2(2m+\alpha_{12})_2(2n+2m+\alpha_{123})_2(2n+2m+\alpha_{123}+1)_2}},\\
c_{m,n} &= \sqrt{\frac{m\,n(n-1)(m+\alpha_1)(m+\alpha_2)(m+\alpha_{12})(n+\alpha_3-1)_2(N+m+n+\alpha_{123}+1)(N-m-n+2)}{(2m+\alpha_{12}-1)_2(2m+\alpha_{12})_2(2n+2m+\alpha_{123}-2)_2(2n+2m+\alpha_{123}-1)_2}},
\end{aligned}
$$

where $b_{m,n}$ and $d_{m,n}$ have the expression

$$
\begin{aligned}
b_{m,n} &= \sqrt{\frac{n(n+\alpha_3)(n+2m+\alpha_{12}+1)(n+2m+\alpha_{123}+1)(N+m+n+\alpha_{123}+2)(N-m-n+1)}{(2m+\alpha_{12}+1)^2(2m+2n+\alpha_{123})_2(2n+2m+\alpha_{123}+1)_2}}\\
&\qquad\qquad \times \left\{\frac{m(m+\alpha_2)}{2m+\alpha_{12}} + \frac{(m+\alpha_1+1)(m+\alpha_{12}+1)}{2m+\alpha_{12}+2}\right\},\\
d_{m,n} &= \sqrt{\frac{m\,n(m+\alpha_1)(m+\alpha_2)(m+\alpha_{12})(n+\alpha_3)(n+2m+\alpha_{12})(n+2m+\alpha_{123})}{(2m+\alpha_{12}-1)_2(2m+\alpha_{12})_2}}\\
&\qquad\qquad \times \left\{\frac{(2N+\alpha_{123}+3)}{(2n+2m+\alpha_{123}-1)(2n+2m+\alpha_{123}+1)}\right\},
\end{aligned}
$$

and where $e_{m,n}$ reads

$$
\begin{aligned}
e_{m,n} &= \frac{(m+\alpha_1+1)(m+\alpha_{12}+1)n(n+\alpha_3)(N+m+n+\alpha_{123}+2)}{(2m+\alpha_{12}+1)_2(2n+2m+\alpha_{123}+1)_2} + \frac{m(m+\alpha_2)(n+1)(n+\alpha_3+1)(N-m-n)}{(2m+\alpha_{12})_2(2n+2m+\alpha_{123}+2)_2}\\
&+ \frac{m(m+\alpha_2)(n+2m+\alpha_{12}+1)(n+2m+\alpha_{123}+1)(N+m+n+\alpha_{123}+2)}{(2m+\alpha_{12})_2(2m+2n+\alpha_{123}+1)_2}\\
&\qquad + \frac{(m+\alpha_1+1)(m+\alpha_{12}+1)(n+2m+\alpha_{12}+2)(n+2m+\alpha_{123}+2)(N-m-n)}{(2m+\alpha_{12}+1)_2(2n+2m+\alpha_{123}+2)_2}.
\end{aligned}
$$

As for the relation (5.61), it can be written as

$$
\begin{aligned}
k\,Q_{m,n}(i,k) &= -\tilde{a}_{m+1,n}\,Q_{m+1,n}(i,k) - \tilde{a}_{m,n}\,Q_{m-1,n}(i,k) + \tilde{b}_{m,n+1}\,Q_{m,n+1}(i,k)\\
&+ \tilde{b}_{m,n}\,Q_{m,n-1}(i,k) - \tilde{c}_{m,n+2}\,Q_{m-1,n+2}(i,k) - \tilde{c}_{m+1,n}\,Q_{m+1,n-2}(i,k)\\
&\qquad - \tilde{d}_{m+1,n}\,Q_{m+1,n-1}(i,k) - \tilde{d}_{m,n+1}\,Q_{m-1,n+1}(i,k) + \tilde{e}_{m,n}\,Q_{m,n}(i,k),
\end{aligned}
$$

where $\tilde{x}_{m,n}$ is obtained from $x_{m,n}$ by the permutation $\alpha_1 \leftrightarrow \alpha_2$.

5.6 Difference equations

In this section, the difference equations satisfied by the Hahn polynomials are obtained. The first one is obtained by factorization using the intertwining operators that raise/lower the first degree m. The second difference equation is found by a direct computation of the matrix elements of one of the symmetry operators associated to the spherical basis.

5.6.1 First difference equation

To obtain a first difference equation for the bivariate Hahn polynomials, we start from the matrix element ${}_C\langle\, \alpha_1,\alpha_2,\alpha_3;i,k : N \mid C_{+}^{(\alpha_1,\alpha_2)} C_{-}^{(\alpha_1,\alpha_2)} \mid \alpha_1,\alpha_2,\alpha_3;m,n;N\,\rangle_S$ where $C_{\pm}^{(\alpha_1,\alpha_2)}$ are the operators defined by (5.22) and (5.33). In view of the actions (5.23) and (5.34), it follows that

$$
\begin{aligned}
{}_C\langle\, \alpha_1,\alpha_2,\alpha_3;i,k : N \mid C_{+}^{(\alpha_1,\alpha_2)} C_{-}^{(\alpha_1,\alpha_2)} &\mid \alpha_1,\alpha_2,\alpha_3;m,n;N\,\rangle_S \\
&= m(m+\alpha_{12}+1) W_{i,k;N}^{(\alpha_1,\alpha_2,\alpha_3)} Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N).
\end{aligned} \tag{5.62}
$$

Note that $C_{+}^{(\alpha_1,\alpha_2)} C_{-}^{(\alpha_1,\alpha_2)}$ is related to the operator $Q^{(12)}$ defined in (5.12a) since one has $C_{+}^{(\alpha_1,\alpha_2)} C_{-}^{(\alpha_1,\alpha_2)} = Q^{(12)} - \alpha_{12}(\alpha_{12}+2)/4$. Upon using the formulas (5.25) and (5.36) giving the actions of $C_{\pm}^{(\alpha_1,\alpha_2)}$ on the Cartesian basis wavefunctions, one finds

$$
\begin{aligned}
(C_{+}^{(\alpha_1,\alpha_2)} C_{-}^{(\alpha_1,\alpha_2)})^{\dagger} \Psi_{i,k;N}^{(\alpha_1,\alpha_2,\alpha_3)} &= [i(k+\alpha_2+1) + k(i+\alpha_1+1)] \Psi_{i,k;N}^{(\alpha_1,\alpha_2,\alpha_3)} \\
&- \sqrt{i(i+\alpha_1)(k+1)(k+\alpha_2+1)}\, \Psi_{i-1,k+1;N}^{(\alpha_1,\alpha_2,\alpha_3)} \\
&- \sqrt{k(i+1)(i+\alpha_1+1)(k+\alpha_2)}\, \Psi_{i+1,k-1;N}^{(\alpha_1,\alpha_2,\alpha_3)}.
\end{aligned} \tag{5.63}
$$

Combining (5.62) with (5.63) and using the explicit expression (5.21) for the amplitude $W_{i,k;N}^{(\alpha_1,\alpha_2,\alpha_3)}$, one finds that the bivariate Hahn polynomials satisfy the difference equation

$$
\begin{aligned}
m(m+\alpha_{12}+1) Q_{m,n}(i,k) &= [i(k+\alpha_2+1) + k(i+\alpha_1+1)] Q_{m,n}(i,k) \\
&- i(k+\alpha_2+1) Q_{m,n}(i-1,k+1) \\
&- k(i+\alpha_1+1) Q_{m,n}(i+1,k-1),
\end{aligned} \tag{5.64}
$$

where the explicit dependence on the parameters α_i and N was omitted to ease the notation. Defining the operator $\mathscr{L}_1$ as

$$
\mathscr{L}_1 = \Upsilon_1(i,k) T_i^{-} T_k^{+} + \Upsilon_2(i,k) T_i^{+} T_k^{-} - [\Upsilon_1(i,k) + \Upsilon_2(i,k)]\mathbb{I}, \tag{5.65}
$$

with coefficients

$$
\Upsilon_1(i,k) = i(k+\alpha_2+1), \quad \Upsilon_2(i,k) = k(i+\alpha_1+1),
$$

and where $T_i^{\pm} f(i,k) = f(i\pm 1,k)$ (and similarly for $T_k^{\pm}$) are the shift operators and $\mathbb{I}$ stands for the identity operator, the difference equation (5.64) can be written as the eigenvalue equation

$$
\mathscr{L}_1 Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N) = -m(m+\alpha_{12}+1) Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N).
$$

5.6.2 Second difference equation

It is possible to derive a second difference equation for the bivariate Hahn polynomials. To that end, consider the matrix element ${}_C\langle\, \alpha_1,\alpha_2,\alpha_3;i,k;N \mid Q \mid \alpha_1,\alpha_2,\alpha_3;m,n;N\,\rangle_S$, where Q is defined by

$$Q = Q^{(123)} - (\alpha_{123}+1)(\alpha_{123}+3)/4,$$

with $Q^{(123)}$ given by (5.12b). It follows from (5.11) that

$$\begin{aligned} &{}_C\langle\, \alpha_1,\alpha_2,\alpha_3;i,k;N \mid Q \mid \alpha_1,\alpha_2,\alpha_3;m,n;N\,\rangle_S \\ &\qquad\qquad = (n+m)(n+m+\alpha_{123}+2)\, W_{i,k;N}^{(\alpha_1,\alpha_2,\alpha_3)}\, Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N). \end{aligned} \tag{5.66}$$

Upon writing $Q^{(123)}$ in Cartesian coordinates (see (5.14)) and acting on the Cartesian basis wavefunctions, a straightforward calculation yields

$$\begin{aligned} Q\,\Psi_{i,k;N}^{(\alpha_1,\alpha_2,\alpha_3)} &= \tilde{\kappa}_{i,k}\Psi_{i,k;N}^{\alpha_1,\alpha_2,\alpha_3} - \tilde{\sigma}_{i,k}\,\Psi_{i-1,k+1;N}^{(\alpha_1,\alpha_2,\alpha_3)} - \tilde{\rho}_{i,k}\Psi_{i+1,k-1;N}^{(\alpha_1,\alpha_2,\alpha_3)} \\ &\quad - \tilde{\mu}_{i+1,k}\Psi_{i+1,k;N}^{(\alpha_1,\alpha_2,\alpha_3)} - \tilde{\mu}_{i,k}\Psi_{i-1,k;N}^{(\alpha_1,\alpha_2,\alpha_3)} - \tilde{\nu}_{i,k+1}\Psi_{i,k+1;N}^{(\alpha_1,\alpha_2,\alpha_3)} - \tilde{\nu}_{i,k}\Psi_{i,k-1;N}^{(\alpha_1,\alpha_2,\alpha_3)}, \end{aligned} \tag{5.67}$$

where the coefficients are of the form

$$\begin{aligned} &\tilde{\kappa}_{i,k} = i\,\alpha_{23} + k\,\alpha_{13} + (N-i-k)\,\alpha_{12} - 2(i^2+k^2+ik-iN-kN-N), \\ &\tilde{\sigma}_{i,k} = \sqrt{i(i+\alpha_1)(k+1)(k+\alpha_2+1)}, \quad \tilde{\rho}_{i,k} = \sqrt{(i+1)(i+\alpha_1+1)k(k+\alpha_2)}, \\ &\tilde{\mu}_{i,k} = \sqrt{i(i+\alpha_1)(N-i-k+1)(N-i-k+\alpha_3+1)}, \\ &\tilde{\nu}_{i,k} = \sqrt{k(k+\alpha_2)(N-i-k+1)(N-i-k+\alpha_3+1)}. \end{aligned} \tag{5.68}$$

Combining (5.62) and (5.67) with the formula (5.21), one finds that the bivariate Hahn polynomials $Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)$ satisfy the following difference equation:

$$\begin{aligned} &-(n+m)(n+m+\alpha_{123}+2)Q_{m,n}(i,k) = -\tilde{\kappa}_{i,k}\,Q_{m,n}(i,k) \\ &+ i(k+\alpha_2+1)Q_{m,n}(i-1,k+1) + k(i+\alpha_1+1)Q_{m,n}(i+1,k-1) \\ &+ (k+\alpha_2+1)(N-i-k)Q_{m,n}(i,k+1) + k(N-i-k+\alpha_3+1)Q_{m,n}(i,k-1) \\ &\qquad + (i+\alpha_1+1)(N-i-k)Q_{m,n}(i+1,k) + i(N-i-k+\alpha_3+1)Q_{m,n}(i-1,k), \end{aligned} \tag{5.69}$$

where the explicit dependence on N and α_i was again dropped for convenience. One can present the difference equation (5.69) as an eigenvalue equation in the following way. We define the operator

$$\mathscr{L}_2 = \Omega_1(i,k)T_i^+ + \Omega_2(i,k)T_k^+ + \Omega_3(i,k)T_i^- + \Omega_4(i,k)T_k^- \\ + \Omega_5(i,k)T_i^+T_k^- + \Omega_6(i,k)T_i^-T_k^+ - \Big(\sum_{j=1}^{6}\Omega_j(i,k)\Big)\mathbb{I}, \quad (5.70)$$

with coefficients

$$\begin{aligned} &\Omega_1(i,k) = (i+\alpha_1+1)(N-i-k), && \Omega_2(i,k) = (k+\alpha_2+1)(N-i-k), \\ &\Omega_3(i,k) = i(N-i-k+\alpha_3+1), && \Omega_4(i,k) = k(N-i-k+\alpha_3+1), \\ &\Omega_5(i,k) = k(i+\alpha_1+1), && \Omega_6(i,k) = i(k+\alpha_2+1). \end{aligned} \quad (5.71)$$

Then (5.69) assumes the form

$$\mathscr{L}_2 Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N) = -(n+m)(n+m+\alpha_{123}+2)Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N).$$

5.7 Expression in hypergeometric series

In this section, the explicit expression for the bivariate Hahn polynomials $Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}$ in terms of the Hahn polynomials in one variable is derived. This is done by introducing an ancillary basis of states corresponding to the separation of variables in cylindrical coordinates and by evaluating explicitly the Cartesian vs. cylindrical and cylindrical vs. spherical interbasis expansion coefficients in terms of the univariate Hahn polynomials.

5.7.1 The cylindrical-polar basis

Let p and q be non-negative integers such that $p \leq q \leq N$. We shall denote by

$$|\,\alpha_1,\alpha_2,\alpha_3;p,q;N\,\rangle_P,$$

the basis vectors for the $\mathscr{E}_N$-energy eigenspace associated to the separation of variables in cylindrical-polar coordinates

$$x_1 = \rho\cos\varphi, \quad x_2 = \rho\sin\varphi, \quad x_3 = x_3.$$

In these coordinates, the wavefunctions have the expression

$$\langle\,\rho,\varphi,x_3\,|\,\alpha_1,\alpha_2,\alpha_3;p,q;N\,\rangle_P = \mathscr{A}_{p,q;N}^{(\alpha_1,\alpha_2,\alpha_3)}(\rho,\varphi,x_3) = \\ \eta_p^{(\alpha_1,\alpha_2)}\xi_{q-p}^{(2p+\alpha_{12}+1)}\xi_{N-q}^{(\alpha_3)}\,\mathscr{G}^{(\alpha_1,\alpha_2,\alpha_3)}\,P_p^{(\alpha_1,\alpha_2)}(-\cos 2\varphi)(\rho^2)^p L_{q-p}^{(2p+\alpha_{12}+1)}(\rho^2)L_{N-q}^{(\alpha_3)}(x_3^2), \quad (5.72)$$

where the normalization factors (5.7) and (5.10) ensure that the wavefunctions (5.72) satisfy the orthogonality condition

$$\int_0^\infty \int_0^{\pi/2} \int_0^\infty \left[\mathscr{A}^{(\alpha_1,\alpha_2,\alpha_3)}_{p,q;N}(\rho,\varphi,x_3)\right]^* \mathscr{A}^{(\alpha_1,\alpha_2,\alpha_3)}_{p',q';N'}(\rho,\varphi,x_3)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}x_3 = \delta_{pp'}\delta_{qq'}\delta_{NN'}.$$

In Cartesian coordinates, the wavefunctions of the cylindrical basis take the form

$$\langle\, x_1,x_2,x_3 \mid \alpha_1,\alpha_2,\alpha_3;p,q;N\,\rangle_P = \eta_p^{(\alpha_1,\alpha_2)}\xi_{q-p}^{(2p+\alpha_{12}+1)}\xi_{N-q}^{(\alpha_3)}\,\mathscr{G}^{(\alpha_1,\alpha_2,\alpha_3)}$$
$$(x_1^2+x_2^2)^p P_p^{(\alpha_1,\alpha_2)}\left(\frac{x_2^2-x_1^2}{x_1^2+x_2^2}\right) L_{q-p}^{(2m+\alpha_{12}+1)}(x_1^2+x_2^2) L_{N-q}^{(\alpha_3)}(x_3^2). \quad (5.73)$$

5.7.2 The cylindrical/Cartesian expansion

Let us obtain the explicit expression for the expansion coefficients ${}_P\langle p,q;N \mid i,k;N\,\rangle_C$ between the states of the cylindrical-polar and Cartesian bases. These expressions are already known (see for example [27]) but we give here a new derivation of these coefficients using a generating function technique [4, 12, 17].

Upon comparing the formulas (5.6) and (5.73) for the Cartesian and cylindrical-polar wavefunctions, it is clear that one can write

$${}_P\langle p,q;N \mid i,k;N\,\rangle_C = \delta_{q,i+k}\;{}_P\langle p;q \mid i;q\,\rangle_C,$$

where ${}_P\langle p;q \mid i;q\,\rangle_C$ are the coefficients appearing in the expansion formula

$$\xi_i^{(\alpha_1)}\xi_{q-i}^{(\alpha_2)}L_i^{(\alpha_1)}(x_1^2)L_{q-i}^{(\alpha_2)}(x_2^2) = \sum_{p=0}^{q} {}_P\langle p;q \mid i;q\,\rangle_C$$
$$\times\, \eta_p^{(\alpha_1,\alpha_2)}\xi_{q-p}^{(2p+\alpha_{12}+1)}(x_1^2+x_2^2)^p P_p^{(\alpha_1,\alpha_2)}\left(\frac{x_2^2-x_1^2}{x_1^2+x_2^2}\right) L_{q-p}^{(2p+\alpha_{12}+1)}(x_1^2+x_2^2). \quad (5.74)$$

Since the coefficients ${}_P\langle p;q \mid i;q\,\rangle_C$ are independent of x_1, x_2, the expansion formula (5.74) holds regardless of the value taken by these coordinates, i.e. (5.74) is a formal expansion. Let us set $x_1^2+x_2^2=0$. Upon using the formula [2]

$$(x+y)^m P_m^{(\alpha,\beta)}\left(\frac{x-y}{x+y}\right) = \frac{(\alpha+1)_m}{m!}\,x^m\,{}_2F_1\left[\begin{matrix}-m,-m-\beta\\ \alpha+1\end{matrix};-\frac{y}{x}\right],$$

and Gauss's summation formula [2] as well as taking $x_2^2=u$, one finds that the expansion formula (5.74) reduces to the generating relation

$$\xi_i^{(\alpha_1)}\xi_{q-i}^{(\alpha_2)}L_i^{(\alpha_1)}(u)L_{q-i}^{(\alpha_2)}(-u)$$
$$= \sum_{p=0}^{q} {}_P\langle p;q \mid i;q\,\rangle_C\;\eta_p^{(\alpha_1,\alpha_2)}\xi_{q-p}^{(2p+\alpha_{12}+1)}\left\{\frac{(p+\alpha_{12}+1)_p(2p+\alpha_{12}+2)_{q-p}}{p!(q-p)!}\right\}u^p.$$

The above relation can be written as

$$
{}_1F_1\left[\begin{matrix}-i\\ \alpha_1+1\end{matrix};-u\right]{}_1F_1\left[\begin{matrix}i-q\\ \alpha_2+1\end{matrix};u\right]=\left\{\frac{i!(q-i)!}{(\alpha_1+1)_i(\alpha_2+1)_{q-i}}\frac{1}{\xi_i^{(\alpha_1)}\xi_{q-i}^{(\alpha_2)}}\right\}
$$

$$
\times\sum_{p=0}^{q}{}_P\langle p;q\mid i;q\rangle_C\,\eta_p^{(\alpha_1,\alpha_2)}\xi_{q-p}^{(2p+\alpha_{12}+1)}\left\{\frac{(p+\alpha_{12}+1)_p(2p+\alpha_{12}+2)_{q-p}}{p!(q-p)!}\right\}u^p. \quad (5.75)
$$

Comparing (5.75) with the generating function (5.3) of the one-variable Hahn polynomials, it is easily seen that

$$
{}_P\langle p;q\mid i;q\rangle_C=\sqrt{\frac{\rho(i;\alpha_1,\alpha_2;q)}{\lambda_p(\alpha_1,\alpha_2;q)}}\,h_p(i;\alpha_1,\alpha_2;q),
$$

where $\rho(x;\alpha,\beta;N)$ and $\lambda_n(\alpha,\beta;N)$ are respectively given by (5.1) and (5.2). The complete expression for the overlap coefficients ${}_P\langle\alpha_1,\alpha_2,\alpha_3;p,q;N\mid\alpha_1,\alpha_2,\alpha_3;i,k;N\rangle_C$ between the states of the cylindrical and Cartesian bases is thus expressed in terms of the Hahn polynomials $h_n(x;\alpha,\beta;N)$ in the following way:

$$
{}_P\langle\alpha_1,\alpha_2,\alpha_3;p,q;N\mid\alpha_1,\alpha_2,\alpha_3;i,k;N\rangle_C=\delta_{q,i+k}\sqrt{\frac{\rho(i;\alpha_1,\alpha_2;q)}{\lambda_p(\alpha_1,\alpha_2;q)}}\,h_p(i;\alpha_1,\alpha_2;q). \quad (5.76)
$$

5.7.3 The spherical/cylindrical expansion

Upon comparing the expressions (5.13) and (5.73) giving the wavefunctions of the spherical and cylindrical-polar bases in Cartesian coordinates, it is easy to see that the overlap coefficients ${}_S\langle m,n;N\mid p,q;N\rangle_P$ between these two bases is of the form

$$
{}_S\langle m,n;N\mid p,q;N\rangle_P=\delta_{mp}\,{}_S\langle n;N\mid q;N\rangle_P,
$$

where ${}_S\langle n;N\mid q;N\rangle_P$ are the coefficients arising in the expansion

$$
\begin{aligned}
&\xi_{q-m}^{(2m+\alpha_{12}+1)}\xi_{N-q}^{(\alpha_3)}L_{q-m}^{(2m+\alpha_{12}+1)}(x_1^2+x_2^2)\,L_{N-q}^{(\alpha_3)}(x_3^2)\\
&=\sum_{n=0}^{N-m}{}_S\langle n;N\mid q;N\rangle_P\,\eta_n^{(2m+\alpha_{12}+1,\alpha_3)}\xi_{N-m-n}^{(2m+2n+\alpha_{123}+2)}\\
&\qquad\times(x_1^2+x_2^2+x_3^2)^n P_n^{(2m+\alpha_{12}+1,\alpha_3)}\left(\frac{x_3^2-x_1^2-x_2^2}{x_1^2+x_2^2+x_3^2}\right)L_{N-m-n}^{(2m+2n+\alpha_{123}+2)}(x_1^2+x_2^2+x_3^2). \quad (5.77)
\end{aligned}
$$

Taking $x_1^2=0$ in (5.77) and comparing with (5.74), it is easily seen that the complete expression for the overlap coefficients between the spherical and the cylindrical-polar bases are given by

$$
{}_S\langle \alpha_1,\alpha_2,\alpha_3;m,n;N \mid \alpha_1,\alpha_2,\alpha_3;p,q;N\rangle_P = \delta_{mp}
\sqrt{\frac{\rho(q-m;2m+\alpha_{12}+1,\alpha_3;N-m)}{\lambda_n(2m+\alpha_{12}+1,\alpha_3;N-m)}}\, h_n(q-m;2m+\alpha_{12}+1,\alpha_3;N-m). \quad (5.78)
$$

5.7.4 Explicit expression for $Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)$

The expansion formulas (5.76) and (5.78) can be combined to obtain the explicit expression for the bivariate Hahn polynomials in terms of the univariate Hahn polynomials. Indeed, one can write

$$
\begin{aligned}
W_{i,k;N}^{(\alpha_1,\alpha_2,\alpha_3)} Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N) &= {}_S\langle \alpha_1,\alpha_2,\alpha_3;m,n;N \mid \alpha_1,\alpha_2,\alpha_3;i,k;N\rangle_C \\
&= \sum_{p=0}^{N}\sum_{q=p}^{N} {}_S\langle m,n;N \mid p,q;N\rangle_P\, {}_P\langle p,q;N \mid i,k;N\rangle_C \\
&= \sqrt{\frac{\rho(i;\alpha_1,\alpha_2;i+k)}{\lambda_m(\alpha_1,\alpha_2;i+k)}\,\frac{\rho(i+k-m;2m+\alpha_{12}+1,\alpha_3;N-m)}{\lambda_n(i+k-m;2m+\alpha_{12}+1,\alpha_3;N-m)}} \\
&\qquad \times h_m(i;\alpha_1,\alpha_2;i+k)\,h_n(i+k-m;2m+\alpha_{12}+1,\alpha_3;N-m). \quad (5.79)
\end{aligned}
$$

With the expression (5.21), one finds the following expression for $Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)$:

$$
Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N) = \left(\Lambda_{m,n;N}^{(\alpha_1,\alpha_2,\alpha_3)}\right)^{-1/2} h_m(i;\alpha_1,\alpha_2;i+k)\,h_n(i+k-m;2m+\alpha_{12}+1,\alpha_3;N-m), \quad (5.80)
$$

where $\Lambda_{m,n;N}^{(\alpha_1,\alpha_2,\alpha_3)}$ are the normalization coefficients defined in (5.45). The explicit expression (5.80) for the bivariate Hahn polynomials $Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)$ corresponds to Karlin and McGregor's [25]. From the results of this section, it is clear that the complete theory of the univariate Hahn polynomials could also be worked out from their interpretation as interbasis expansion coefficients for the two-dimensional singular oscillator.

5.8 Algebraic interpretation

In this section, an algebraic interpretation of the overlap coefficients between the Cartesian and spherical bases is presented in terms of $\mathfrak{su}(1,1)$ representations. It is seen that these overlap coefficients can be assimilated to generalized Clebsch-Gordan coefficients, a result that entails a connection with the work of Rosengren [33].

5.8.1 Generalized Clebsch-Gordan problem for $\mathfrak{su}(1,1)$

The $\mathfrak{su}(1,1)$ algebra has for generators the elements K_0 and $K_\pm$ that satisfy the commutation relations [16, 36]

$$[K_0, K_\pm] = \pm K_\pm, \quad [K_-, K_+] = 2K_0. \tag{5.81}$$

The Casimir operator C, which commutes with every generator, is of the form

$$C = K_0^2 - K_+ K_- - K_0. \tag{5.82}$$

Let $\nu > 0$ be a real number and let $V^{(\nu)}$ denote the infinite-dimensional vector space spanned by the basis vectors $e_n^{(\nu)}$, $n \in \{0, 1, \ldots, \}$. If $V^{(\nu)}$ is endowed with the actions

$$\begin{aligned} K_0\, e_n^{(\nu)} &= (n+\nu)\, e_n^{(\nu)}, \\ K_+\, e_n^{(\nu)} &= \sqrt{(n+1)(n+2\nu)}\, e_{n+1}^{(\nu)}, \\ K_-\, e_n^{(\nu)} &= \sqrt{n(n+2\nu-1)}\, e_{n-1}^{(\nu)}, \end{aligned} \tag{5.83}$$

then $V^{(\nu)}$ becomes an irreducible $\mathfrak{su}(1,1)$-module; the representation (5.83) belongs to the positive discrete series [36]. On this module the Casimir operator acts as a multiple of the identity

$$C\, e_n^{(\nu)} = \nu(\nu-1)\, e_n^{(\nu)},$$

as expected from Schur's lemma. Consider three mutually commuting sets $\{K_0^{(i)}, K_\pm^{(i)}\}$, $i = 1, 2, 3$, of $\mathfrak{su}(1,1)$ generators. These generators can be combined as follows to produce a fourth set of generators:

$$K_0^{(123)} = K_0^{(1)} + K_0^{(2)} + K_0^{(3)}, \quad K_\pm^{(123)} = K_\pm^{(1)} + K_\pm^{(2)} + K_\pm^{(3)}.$$

There is a natural representation for this realization of $\mathfrak{su}(1,1)$ on the tensor product space $V^{(\nu_1)} \otimes V^{(\nu_2)} \otimes V^{(\nu_3)}$; in this representation each set of generators $\{K_0^{(i)}, K_\pm^{(i)}\}$ acts on $V^{(\nu_i)}$ only. A convenient basis for this module is the direct product basis spanned by the vectors $e_{n_1}^{(\nu_1)} \otimes e_{n_2}^{(\nu_2)} \otimes e_{n_3}^{(\nu_3)}$ with the actions of the generators $\{K_0^{(i)}, K_\pm^{(i)}\}$ on the vectors $e_{n_i}^{(\nu_i)}$ as prescribed by (5.83). In general, this representation is not irreducible and it can be completely decomposed in a direct sum of irreducible representations $V^{(\nu)}$ also belonging to the positive-discrete series. To perform this decomposition, one can proceed in two steps by first decomposing $V^{(\nu_1)} \otimes V^{(\nu_2)}$ in irreducible modules $V^{(\nu_{12})}$ and then decomposing $V^{(\nu_{12})} \otimes V^{(\nu_3)}$ in irreducible modules $V^{(\nu)}$ for each occurring values of ν_{12}. A natural basis associated to this decomposition scheme, which we shall call the "coupled" basis, is provided by the vectors $e_{n_{123}}^{(\nu_{12},\nu)}$, $n_{123} \in \{0,1,\ldots\}$, satisfying

$$
\begin{aligned}
C^{(12)}\, e_{n_{123}}^{(\nu_{12},\nu)} &= \nu_{12}(\nu_{12}-1)\, e_{n_{123}}^{(\nu_{12},\nu)},\\
C^{(123)}\, e_{n_{123}}^{(\nu_{12},\nu)} &= \nu(\nu-1)\, e_{n_{123}}^{(\nu_{12},\nu)},\\
K_0^{(123)}\, e_{n_{123}}^{(\nu_{12},\nu)} &= (n_{123}+\nu)\, e_{n_{123}}^{(\nu_{12},\nu)},
\end{aligned}
\tag{5.84}
$$

where $C^{(12)}$ is the Casimir operator associated to the decomposition of $V^{(\nu_1)} \otimes V^{(\nu_2)}$:

$$
C^{(12)} = [K_0^{(12)}]^2 - K_+^{(12)} K_-^{(12)} - K_0^{(12)},
\tag{5.85}
$$

with $K_0^{(ij)} = K_0^{(i)} + K_0^{(j)}$, $K_\pm^{(ij)} = K_\pm^{(i)} + K_\pm^{(j)}$ and where $C^{(123)}$ is the Casimir operator associated to the decomposition of $V^{(\nu_{12})} \otimes V^{(\nu_3)}$:

$$
C^{(123)} = [K_0^{(123)}]^2 - K_+^{(123)} K_-^{(123)} - K_0^{(123)}.
\tag{5.86}
$$

It is well known (see for example [5]) that the occurring values of ν_{12} and ν are given by

$$
\nu_{12}(m) = m + \nu_1 + \nu_2, \quad \nu(m,n) = n + m + \nu_1 + \nu_2 + \nu_3,
\tag{5.87}
$$

where m,n are non-negative integers. The direct product and coupled bases span the same representation space and the corresponding basis vectors are thus related by a linear transformation. Furthermore, since these vectors are both eigenvectors of $K_0^{(123)}$ the transformation is non-trivial if and only if the involved vectors correspond to the same eigenvalue of $K_0^{(123)}$. Let $\lambda_{K_0} = N + \nu_1 + \nu_2 + \nu_3$, $N \in \{0,\ldots,N\}$, be the eigenvalues of $K_0^{(123)}$, then for each N one has

$$
e_i^{(\nu_1)} \otimes e_k^{(\nu_2)} \otimes e_{N-i-k}^{(\nu_3)} = \sum_{\substack{m,n\\ m+n\leq N}} C_{m,n}^{(\nu_1,\nu_2,\nu_3)}(i,k;N)\, e_{N-m-n}^{(\nu_{12}(m),\,\nu(m,n))},
\tag{5.88}
$$

where i,k are positive integers such that $i+k \leq N$. The coefficients $C_{m,n}^{(\nu_1,\nu_2,\nu_3)}(i,k;N)$ are generalized Clebsch-Gordan coefficients for the positive-discrete series of irreducible representations of $\mathfrak{su}(1,1)$; the reader is referred to [5, 36] for the standard Clebsch-Gordan problem, which involves only two representations of $\mathfrak{su}(1,1)$.

5.8.2 Connection with the singular oscillator

The connection between the singular oscillator model and the combination of three $\mathfrak{su}(1,1)$ representations can be established as follows. Consider the following coordinate realizations of the $\mathfrak{su}(1,1)$ algebra

$$K_0^{(i)} = \frac{1}{4}\left(-\partial_{x_i}^2 + x_i^2 + \frac{\alpha_i^2 - 1/4}{x_i^2}\right), \tag{5.89}$$

$$K_\pm^{(i)} = \frac{1}{4}\left((x_i \mp \partial_{x_i})^2 - \frac{\alpha_i^2 - 1/4}{x_i^2}\right), \tag{5.90}$$

where $i = 1,2,3$. A direct computation shows that in the realization (5.89), the Casimir operator $C^{(i)}$ takes the value $\nu_i(\nu_i - 1)$ with

$$\nu_i = \frac{\alpha_i + 1}{2}, \quad i = 1,2,3. \tag{5.91}$$

It is easily seen from (5.5) that $\mathscr{H} = K_0^{(123)}$. One can check using (5.6) and (5.89) that the states $|\, i,k;N\,\rangle_C$ of the Cartesian basis provide, up to an inessential phase factor, a realization of the tensor product basis in the addition of three irreducible modules $V^{(\nu_i)}$ of the positive-discrete series. Hence we have the identification

$$|\, i,k;N\,\rangle_C \sim e_i^{(\nu_1)} \otimes e_k^{(\nu_2)} \otimes e_{N-i-k}^{(\nu_3)}, \tag{5.92}$$

with ν_i given by (5.91). Upon computing the Casimir operators $C^{(12)}$ and $C^{(123)}$ in the realization (5.89) from their definitions (5.85) and (5.86) and comparing with the operators $Q^{(12)}$ and $Q^{(123)}$ given in Cartesian coordinates by (5.14), it is directly checked that

$$C^{(12)} \sim Q^{(12)}, \quad C^{(123)} \sim Q^{(123)}. \tag{5.93}$$

It is also checked that the eigenvalues (5.11) correspond to (5.87) and thus we have the following identification between the spherical basis states and the coupled basis vectors:

$$|\, m,n;N\,\rangle_S \sim e_{N-m-n}^{(\nu_{12}(m),\,\nu(m,n))}. \tag{5.94}$$

In view of (5.88), (5.92) and (5.94), the interbasis expansion coefficients between the spherical and Cartesian bases

$${}_S\langle m,n;N \,|\, i,k;N\,\rangle_C = W_{i,k;N}^{(\alpha_1,\alpha_2,\alpha_3)} Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N),$$

given in terms of the bivariate Hahn polynomials $Q_{m,n}^{(\alpha_1,\alpha_2,\alpha_3)}(i,k;N)$ correspond to the generalized Clebsch-Gordan coefficients

$$C_{m,n;N}^{(\nu_1,\nu_2,\nu_3)}(i,k;N) \simeq W_{i,k;N}^{(2\nu_1-1,2\nu_2-1,2\nu_3-1)} Q_{m,n}^{(2\nu_1-1,2\nu_2-1,2\nu_3-1)}(i,k;N),$$

where the $\simeq$ symbol is used to account for the possible phase factors coming from the choices of phase factors in the basis states.

5.9 Multivariate case

In this section, it shown how the results of the previous sections can be directly generalized so as to find the Hahn polynomials in d-variables as the interbasis expansion coefficients between the Cartesian and hyperspherical eigenbases for the singular oscillator model in $(d+1)$ dimensions.

5.9.1 Cartesian and hyperspherical bases

Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{d+1})$ with $\alpha_i > -1$ be the parameter vector and consider the following Hamiltonian describing the $(d+1)$-dimensional singular oscillator:

$$H = \frac{1}{4}\sum_{i=1}^{d+1}\left(-\partial_{x_i}^2 + x_i^2 + \frac{\alpha_i^2 - \frac{1}{4}}{x_i^2}\right).$$

The energy spectrum E_N of this Hamiltonian is of the form

$$E_N = N + |\boldsymbol{\alpha}|/2 + (d+1)/2, \qquad |\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_{d+1},$$

and exhibits a $\binom{N+d}{d}$-fold degeneracy. Let $\boldsymbol{i} = (i_1, \ldots, i_{d+1})$ with $i_{d+1} = N - \sum_{j=1}^{d} i_j$ and let $|\, \boldsymbol{\alpha}; \boldsymbol{i}\,\rangle_C$ denote the states spanning the Cartesian basis. In Cartesian coordinates, the corresponding wavefunctions have the expression

$$\langle\, \boldsymbol{x} \mid \boldsymbol{\alpha}; \boldsymbol{i}\,\rangle_C = \Psi_{\boldsymbol{i}}^{(\boldsymbol{\alpha})}(\boldsymbol{x}) = \mathscr{G}^{(\boldsymbol{\alpha})}(\boldsymbol{x}) \prod_{k=1}^{d+1} \xi_{i_k}^{(\alpha_k)} L_{i_k}^{(\alpha_k)}(x_k^2), \tag{5.95}$$

where $\boldsymbol{x} = (x_1, \ldots, x_{d+1})$ is the coordinate vector and where the gauge factor $\mathscr{G}^{(\boldsymbol{\alpha})}(\boldsymbol{x})$ is

$$\mathscr{G}^{(\boldsymbol{\alpha})}(\boldsymbol{x}) = e^{-|\boldsymbol{x}|^2/2} \prod_{k=1}^{d+1} x_k^{\alpha_k + 1/2},$$

with $|\boldsymbol{x}|^2 = x_1^2 + \cdots x_{d+1}^2$. With the normalization coefficients $\xi_n^{(\alpha)}$ as in (5.7) one has

$$\int_{\mathbb{R}_+^{d+1}} {}_C\langle\, \boldsymbol{\alpha}; \boldsymbol{i}' \mid \boldsymbol{x}\,\rangle \langle\, \boldsymbol{x} \mid \boldsymbol{\alpha}; \boldsymbol{i}\,\rangle_C \, \mathrm{d}\boldsymbol{x} = \delta_{\boldsymbol{i}\boldsymbol{i}'}.$$

Let $\boldsymbol{n} = (n_1, \ldots, n_{d+1})$ with $n_{d+1} = N - \sum_{k=1}^{d} n_d$ and let $|\, \boldsymbol{\alpha}; \boldsymbol{n}\,\rangle_S$ denote the states spanning the hyperspherical basis. In Cartesian coordinates, the corresponding wavefunctions are given by

$$\langle\, \boldsymbol{x} \mid \boldsymbol{\alpha}; \boldsymbol{n}\,\rangle_S = \Xi_{\boldsymbol{n}}^{(\boldsymbol{\alpha})}(\boldsymbol{x}) = \mathscr{G}^{(\boldsymbol{\alpha})}(\boldsymbol{x})$$
$$\times \left\{ \prod_{k=1}^{d} \eta_{n_k}^{(a_k, \alpha_{k+1})} \left(|\boldsymbol{x}_{k+1}|^2\right)^{n_k} P_{n_k}^{(a_k, \alpha_{k+1})}\left(\frac{x_{k+1}^2 - |\boldsymbol{x}_k|^2}{|\boldsymbol{x}_{k+1}|^2}\right)\right\} \xi_{n_{d+1}}^{(a_{d+1})} L_{n_{d+1}}^{(a_{d+1})}\left(|\boldsymbol{x}|^2\right), \tag{5.96}$$

where the following notations were used:

$$|\boldsymbol{y}_k| = y_1 + \cdots + y_k, \quad a_k = a_k(\boldsymbol{\alpha},\boldsymbol{n}) = 2|\boldsymbol{n}_{k-1}| + |\boldsymbol{\alpha}_k| + k - 1, \tag{5.97a}$$

$$|\boldsymbol{y}_k|^2 = y_1^2 + \cdots y_k^2, \quad |\boldsymbol{y}_0| = 0. \tag{5.97b}$$

The normalization factors $\xi_n^{(\alpha)}$ given by (5.7) and $\eta_m^{(\alpha,\beta)}$ given by (5.10) ensure that one has

$$\int_{\mathbb{R}_+^{d+1}} {}_S\langle\, \boldsymbol{\alpha};\boldsymbol{n}' \mid \boldsymbol{x}\,\rangle \langle\, \boldsymbol{x} \mid \boldsymbol{\alpha};\boldsymbol{n}\,\rangle_S \,\mathrm{d}\boldsymbol{x} = \delta_{\boldsymbol{n},\boldsymbol{n}'}.$$

It is directly seen that the wavefunctions of the hyperspherical basis are separated in the hyperspherical coordinates

$$\begin{aligned}
x_1 &= r\cos\theta_1\sin\theta_2\cdots\sin\theta_d,\\
x_2 &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_d,\\
&\vdots\\
x_k &= r\cos\theta_{k-1}\sin\theta_k\cdots\sin\theta_d,\\
&\vdots\\
x_{d+1} &= r\cos\theta_d,
\end{aligned}$$

The operators that are diagonal on (5.96) and their eigenvalues are obtained through the correspondence (5.93) with the combining of $d+1$ copies of $\mathfrak{su}(1,1)$; they correspond to the Casimir operators $C^{(12)}$, $C^{(123)}$, $C^{(1234)}$, etc.

The overlap coefficients between the Cartesian and hyperspherical bases are denoted ${}_C\langle \boldsymbol{\alpha};\boldsymbol{i} \mid \boldsymbol{\alpha};\boldsymbol{n}\,\rangle_S$ and are defined by the integral

$${}_C\langle \boldsymbol{\alpha};\boldsymbol{i} \mid \boldsymbol{\alpha};\boldsymbol{n}\,\rangle_S = \int_{\mathbb{R}_+^{d+1}} \left[\Xi_{\boldsymbol{n}}^{(\boldsymbol{\alpha})}(\boldsymbol{x})\right]^* \Psi_{\boldsymbol{i}}^{(\boldsymbol{i})}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \tag{5.98}$$

from which one easily sees that

$${}_C\langle \boldsymbol{\alpha};\boldsymbol{i} \mid \boldsymbol{\alpha};\boldsymbol{n}\,\rangle_S = {}_S\langle \boldsymbol{\alpha};\boldsymbol{n} \mid \boldsymbol{\alpha};\boldsymbol{i}\,\rangle_C.$$

The overlap coefficients provide the expansion formulas

$$\begin{aligned}
|\, \boldsymbol{\alpha};\boldsymbol{n}\,\rangle_S &= \sum_{|\boldsymbol{i}|=N} {}_C\langle \boldsymbol{\alpha};\boldsymbol{i} \mid \boldsymbol{\alpha};\boldsymbol{n}\,\rangle_S \,|\, \boldsymbol{\alpha};\boldsymbol{i}\,\rangle_C,\\
|\, \boldsymbol{\alpha};\boldsymbol{i}\,\rangle_C &= \sum_{|\boldsymbol{n}|=N} {}_S\langle \boldsymbol{\alpha};\boldsymbol{n} \mid \boldsymbol{\alpha};\boldsymbol{i}\,\rangle_C \,|\, \boldsymbol{\alpha};\boldsymbol{n}\,\rangle_S,
\end{aligned}$$

between the hyperspherical and Cartesian bases. Since the Cartesian and hyperspherical basis vectors are orthonormal, the interbasis expansions coefficients satisfy the discrete orthogonality relations

$$\sum_{|\boldsymbol{i}|=N} {}_S\langle \boldsymbol{\alpha};\boldsymbol{n}' \mid \boldsymbol{\alpha};\boldsymbol{i}\,\rangle_C\, {}_C\langle \boldsymbol{\alpha};\boldsymbol{i} \mid \boldsymbol{\alpha};\boldsymbol{n}\,\rangle_S = \delta_{\boldsymbol{n}\boldsymbol{n}'},$$
$$\sum_{|\boldsymbol{n}|=N} {}_C\langle \boldsymbol{\alpha};\boldsymbol{i}' \mid \boldsymbol{\alpha};\boldsymbol{n}\,\rangle_S\, {}_S\langle \boldsymbol{\alpha};\boldsymbol{n} \mid \boldsymbol{\alpha};\boldsymbol{i}\,\rangle_C = \delta_{\boldsymbol{i}\boldsymbol{i}'}.$$

5.9.2 Interbasis expansion coefficients as orthogonal polynomials

The interbasis expansion coefficients can be cast in the form

$$ {}_C\langle \boldsymbol{\alpha};\boldsymbol{i} \mid \boldsymbol{\alpha};\boldsymbol{n}\,\rangle_S = W_{\boldsymbol{i}}^{(\boldsymbol{\alpha})} Q_{\boldsymbol{n}}^{(\boldsymbol{\alpha})}(\boldsymbol{i}), \tag{5.99}$$

where $W_{\boldsymbol{i}}^{(\boldsymbol{\alpha})}$ is defined by

$$W_{\boldsymbol{i}}^{(\boldsymbol{\alpha})} = {}_C\langle \boldsymbol{\alpha};\boldsymbol{i} \mid \boldsymbol{\alpha};\boldsymbol{0}\,\rangle_S, \tag{5.100}$$

with $\boldsymbol{0} = (0,\cdots,0,N)$. The explicit expression for (5.100) is easily found by repeatedly using the addition formula for the Laguerre polynomials on the hyperspherical wavefunctions (5.96) in the integral expression (5.98). One then finds

$$W_{\boldsymbol{i}}^{(\boldsymbol{\alpha})} = \sqrt{\binom{N}{i_1,\ldots,i_d} \frac{(\alpha_1+1)_{i_1}\cdots(\alpha_{d+1}+1)_{i_{d+1}}}{(|\boldsymbol{\alpha}|+d+1)_N}}, \tag{5.101}$$

where $\binom{N}{x_1,\ldots,x_d}$ are the multinomial coefficients. The explicit formula for the complete interbasis expansion coefficients (5.99) in terms of the univariate Hahn polynomials can be obtained by introducing a sequence of "cylindrical" coordinate systems corresponding to the coordinate couplings (x_1,x_2), (x_1,x_2,x_3), etc.. Upon using (5.100), one finds in this way that the $Q_{\boldsymbol{n}}^{(\boldsymbol{\alpha})}(\boldsymbol{i})$ appearing in (5.99) are of the form

$$Q_{\boldsymbol{n}}^{(\boldsymbol{\alpha})}(\boldsymbol{i}) = \left[\Lambda_{\boldsymbol{m}}^{(\boldsymbol{\alpha})}\right]^{-1/2} \prod_{k=1}^{d} h_{n_k}(|\boldsymbol{i}_k|-|\boldsymbol{n}_{k-1}|;a_k(\boldsymbol{\alpha},\boldsymbol{n});\alpha_{k+1};|\boldsymbol{i}_{k+1}|-|\boldsymbol{n}_{k-1}|), \tag{5.102}$$

where $\Lambda_{\boldsymbol{m}}^{(\boldsymbol{\alpha})}$ is an easily obtained normalization factor and where the notations (5.97) have been used. It is directly seen from (5.102) that the functions $Q_{\boldsymbol{m}}^{(\boldsymbol{\alpha})}(\boldsymbol{i})$ are polynomials of total degree $|\boldsymbol{n}|$ in the variables $\boldsymbol{i}$ that satisfy the orthogonality relation

$$\sum_{|\boldsymbol{i}|=N} w_{\boldsymbol{i}}^{(\boldsymbol{\alpha})} Q_{\boldsymbol{n}'}^{(\boldsymbol{\alpha})}(\boldsymbol{i}) Q_{\boldsymbol{n}}^{(\boldsymbol{\alpha})}(\boldsymbol{i}) = \delta_{\boldsymbol{n}\boldsymbol{n}'},$$

with respect to the multivariate hypergeometric distribution

$$w_{\boldsymbol{i}}^{(\boldsymbol{\alpha})} = \left[W_{\boldsymbol{i}}^{(\boldsymbol{\alpha})}\right]^2 = \frac{\prod_{k=1}^{d+1}\binom{i_k+\alpha_k}{i_k}}{\binom{N+|\boldsymbol{\alpha}|+d}{N}}.$$

The properties of the multivariate Hahn polynomials $Q_{\boldsymbol{n}}^{(\boldsymbol{\alpha})}(\boldsymbol{i})$ can be derived using the same methods as in the previous sections.

5.10 Conclusion

In this paper, we have shown that Karlin and McGregor's d-variable Hahn polynomials arise as interbasis expansion coefficients in the $(d+1)$-dimensional singular oscillator model. Using the framework provided by this interpretation, the main properties of the bivariate polynomials were obtained: explicit expression in univariate Hahn polynomials, recurrence relations, difference equations, generating function, raising/lowering relations, etc. The connection between our approach and the combining of $\mathfrak{su}(1,1)$ representations was also established.

A natural question that arises from our considerations is whether a similar interpretation can be given for the multivariate Racah polynomials, which have one parameter more than the multivariate Hahn polynomials. The answer to that question is in the positive. Indeed, the multivariate Racah polynomials can be seen to occur as interbasis expansion coefficients in the so-called generic $(d+1)$-parameter model on the d-sphere. With the usual embedding $x_1^2+\cdots+x_{d+1}^2 = 1$ of the d-sphere in the $(d+1)$-dimensional Euclidean plane, this model is described by the Hamiltonian

$$H = \sum_{0\leq i<j\leq d+1}\left[\frac{1}{i}\left(x_i\partial_{x_j} - x_j\partial_{x_i}\right)\right]^2 + \sum_{k=1}^{d+1}\frac{\alpha_k^2-1/4}{x_k^2},$$

and the $(d-1)$-variate Racah polynomials arise as the overlap coefficients between bases associated to the separation of variables different hyperspherical coordinate systems. In the $d=2$ and $d=3$ cases, this result is contained (in a hidden way) in the papers [23] and [22] of Kalnins, Miller and Post; these papers focus on the representations of the symmetry algebra. We shall soon report on the characterization of the multivariate Racah polynomials using their interpretation as interbasis expansion coefficients for the generic $(d+1)$-parameter system on the d-sphere.

5.A A compendium of formulas for the bivariate Hahn polynomials

In this appendix we give for reference a compendium of formulas for the bivariate Hahn polynomials; some of them can be found in the literature, others not as far as we know. Recall that the univariate Hahn polynomials $h_n(x;\alpha,\beta;N)$ are defined by

$$h_n(x;\alpha,\beta;N) = (\alpha+1)_n(-N)_n\,{}_3F_2\left[\begin{matrix}-n, n+\alpha+\beta+1, -x\\ \alpha+1, -N\end{matrix};1\right],$$

where ${}_pF_q$ is the generalized hypergeometric function [2].

5.A.1 Definition

The bivariate Hahn polynomials $\mathscr{P}_{n_1,n_2}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2;N)$ are defined by

$$\mathscr{P}_{n_1,n_2}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2;N) = \frac{1}{(-N)_{n_1+n_2}}\,h_{n_1}(x_1;\alpha_1,\alpha_2;x_1+x_2)\,h_{n_2}(x_1+x_2-n_1;2n_1+\alpha_1+\alpha_2+1,\alpha_3;N-n_1).$$

It is checked that $\mathscr{P}_{n_1,n_2}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2;N)$ are polynomials of total degree n_1+n_2 in the variables x_1 and x_2.

5.A.2 Orthogonality

The polynomials $\mathscr{P}_{n_1,n_2}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2;N)$ satisfy the orthogonality relation

$$\sum_{\substack{x_1,x_2\\ x_1+x_2\leq N}} \omega_{x_1,x_2;N}^{(\alpha_1,\alpha_2,\alpha_3)}\,\mathscr{P}_{n_1,n_2}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2;N)\,\mathscr{P}_{m_1,m_2}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2;N) = \lambda_{n_1,n_2;N}^{(\alpha_1,\alpha_2,\alpha_3)}\delta_{m_1,n_1}\delta_{m_2,n_2}.$$

The orthogonality weight $\omega_{x_1,x_2;N}^{(\alpha_1,\alpha_2,\alpha_3)}$ is given by

$$\omega_{x_1,x_2;N}^{(\alpha_1,\alpha_2,\alpha_3)} = \frac{\binom{x_1+\alpha_1}{x_1}\binom{x_2+\alpha_2}{x_2}\binom{N-x_1-x_2+\alpha_3}{N-x_1-x_2}}{\binom{N+\alpha_1+\alpha_2+\alpha_3+2}{N}},$$

and the normalization factor $\lambda_{n_1,n_2;N}^{(\alpha_1,\alpha_2,\alpha_3)}$ reads

$$\lambda_{n_1,n_2;N}^{(\alpha_1,\alpha_2,\alpha_3)} = \frac{n_1!n_2!(N-n_1-n_2)!}{N!}\,\frac{(\alpha_1+1)_{n_1}(\alpha_2+1)_{n_1}(\alpha_3+1)_{n_2}(\alpha_1+\alpha_2+1)_{2n_1}}{(\alpha_1+\alpha_2+1)_{n_1}(\alpha_1+\alpha_2+\alpha_3+3)_N}$$
$$\times\frac{(2n_1+\alpha_1+\alpha_2+2)_{n_2}(2n_1+\alpha_1+\alpha_2+\alpha_3+2)_{2n_2}(n_1+n_2+\alpha_1+\alpha_2+\alpha_3+3)_N}{(2n_1+\alpha_1+\alpha_2+\alpha_3+2)_{n_2}(n_1+n_2+\alpha_1+\alpha_2+\alpha_3+3)_{n_1+n_2}}.$$

5.A.3 Recurrence relations

The bivariate Hahn polynomials $\mathscr{P}_{n_1,n_2}(x_1,x_2)$ satisfy the recurrence relation

$$
\begin{aligned}
x_1\mathscr{P}_{n_1,n_2}(x_1,x_2) &= a_{n_1,n_2}\mathscr{P}_{n_1+1,n_2}(x_1,x_2)+b_{n_1,n_2}\mathscr{P}_{n_1,n_2+1}(x_1,x_2)\\
&+c_{n_1,n_2}\mathscr{P}_{n_1-1,n_2+2}(x_1,x_2)+d_{n_1,n_2}\mathscr{P}_{n_1-1,n_2+1}(x_1,x_2)+e_{n_1,n_2}\mathscr{P}_{n_1,n_2}(x_1,x_2)\\
&+f_{n_1,n_2}\mathscr{P}_{n_1+1,n_2-1}(x_1,x_2)-g_{n_1,n_2}\mathscr{P}_{n_1+1,n_2-2}(x_1,x_2)\\
&\qquad -h_{n_1,n_2}\mathscr{P}_{n_1,n_2-1}(x_1,x_2)-i_{n_1,n_2}\mathscr{P}_{n_1-1,n_2}(x_1,x_2),
\end{aligned}
$$

where the coefficients are given by

$$
\begin{aligned}
a_{n_1,n_2} &= \frac{(n_1+\alpha_1+\alpha_2+1)(2n_1+n_2+\alpha_1+\alpha_2+\alpha_3+2)(2n_1+n_2+\alpha_1+\alpha_2+\alpha_3+3)(n_1+n_2-N)}{(2n_1+\alpha_1+\alpha_2+1)(2n_1+\alpha_1+\alpha_2+2)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+2)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+3)},\\
b_{n_1,n_2} &= \frac{(2n_1+n_2+\alpha_1+\alpha_2+\alpha_3+2)[2n_1^2+2n_1(\alpha_1+\alpha_2+1)+(\alpha_1+1)(\alpha_1+\alpha_2)](n_1+n_2-N)}{(2n_1+\alpha_1+\alpha_2)(2n_1+\alpha_1+\alpha_2+2)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+2)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+3)},\\
c_{n_1,n_2} &= \frac{n_1(n_1+\alpha_1)(n_1+\alpha_2)(n_1+n_2-N)}{(2n_1+\alpha_1+\alpha_2)(2n_1+\alpha_1+\alpha_2+1)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+2)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+3)},\\
d_{n_1,n_2} &= \frac{n_1(n_1+\alpha_1)(n_1+\alpha_2)(2n_1+n_2+\alpha_1+\alpha_2+1)(2N+\alpha_1+\alpha_2+\alpha_3+3)}{(2n_1+\alpha_1+\alpha_2)(2n_1+\alpha_1+\alpha_2+1)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+1)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+3)},\\
f_{n_1,n_2} &= \frac{n_2(n_2+\alpha_3)(n_1+\alpha_1+\alpha_2+1)(2n_1+n_2+\alpha_1+\alpha_2+\alpha_3+2)(2N+\alpha_1+\alpha_2+\alpha_3+3)}{(2n_1+\alpha_1+\alpha_2+1)(2n_1+\alpha_1+\alpha_2+2)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+1)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+3)},\\
g_{n_1,n_2} &= \frac{n_2(n_2-1)(n_2+\alpha_3)(n_2+\alpha_3-1)(n_1+\alpha_1+\alpha_2+1)(N+n_1+n_2+\alpha_1+\alpha_2+\alpha_3+2)}{(2n_1+\alpha_1+\alpha_2+1)(2n_1+\alpha_1+\alpha_2+2)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+1)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+2)},\\
h_{n_1,n_2} &= \frac{n_2(n_2+\alpha_3)(2n_1^2+2n_1(\alpha_1+\alpha_2+1)+(\alpha_1+1)(\alpha_1+\alpha_2))(2n_1+n_2+\alpha_1+\alpha_2+1)(N+n_1+n_2+\alpha_1+\alpha_2+\alpha_3+2)}{(2n_1+\alpha_1+\alpha_2)(2n_1+\alpha_1+\alpha_2+2)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+1)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+2)},\\
i_{n_1,n_2} &= \frac{n_1(n_1+\alpha_1)(n_1+\alpha_2)(2n_1+n_2+\alpha_1+\alpha_2)(2n_1+n_2+\alpha_1+\alpha_2+1)(N+n_1+n_2+\alpha_1+\alpha_2+\alpha_3+2)}{(2n_1+\alpha_1+\alpha_2)(2n_1+\alpha_1+\alpha_2+1)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+1)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+2)},
\end{aligned}
$$

and by

$$
\begin{aligned}
e_{n_1,n_2} &= \frac{n_2(n_1+\alpha_1+1)(n_1+\alpha_1+\alpha_2+1)(n_2+\alpha_3)(N+n_1+n_2+\alpha_1+\alpha_2+\alpha_3+2)}{(2n_1+\alpha_1+\alpha_2+1)(2n_1+\alpha_1+\alpha_2+2)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+1)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+2)}\\
&+\frac{n_1(n_1+\alpha_2)(n_2+1)(n_2+\alpha_3+1)(N-n_1-n_2)}{(2n_1+\alpha_1+\alpha_2)(2n_1+\alpha_1+\alpha_2+1)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+2)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+3)}\\
&+\frac{n_1(n_1+\alpha_2)(2n_1+n_2+\alpha_1+\alpha_2+1)(2n_1+n_2+\alpha_1+\alpha_2+\alpha_3+1)(N+n_1+n_2+\alpha_1+\alpha_2+\alpha_3+2)}{(2m+\alpha_{12})_2(2m+2n+\alpha_{123}+1)_2}\\
&\qquad +\frac{(n_1+\alpha_1+1)(n_1+\alpha_1+\alpha_2+1)(2n_1+n_2+\alpha_1+\alpha_2+2)(2n_1+n_2+\alpha_1+\alpha_2+\alpha_3+2)(N-n_1-n_2)}{(2n_1+\alpha_1+\alpha_2+1)(2n_1+\alpha_1+\alpha_2+2)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+2)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+3)}.
\end{aligned}
$$

The bivariate Hahn polynomials also satisfy the recurrence relation

$$
\begin{aligned}
x_2\mathscr{P}_{n_1,n_2}(x_1,x_2) &= -\tilde{a}_{n_1,n_2}\mathscr{P}_{n_1+1,n_2}(x_1,x_2)+\tilde{b}_{n_1,n_2}\mathscr{P}_{n_1,n_2+1}(x_1,x_2)\\
&-\tilde{c}_{n_1,n_2}\mathscr{P}_{n_1-1,n_2+2}(x_1,x_2)-\tilde{d}_{n_1,n_2}\mathscr{P}_{n_1-1,n_2+1}(x_1,x_2)+\tilde{e}_{n_1,n_2}\mathscr{P}_{n_1,n_2}(x_1,x_2)\\
&-\tilde{f}_{n_1,n_2}\mathscr{P}_{n_1+1,n_2-1}(x_1,x_2)+\tilde{g}_{n_1,n_2}\mathscr{P}_{n_1+1,n_2-2}(x_1,x_2)\\
&\qquad -\tilde{h}_{n_1,n_2}\mathscr{P}_{n_1,n_2-1}(x_1,x_2)+\tilde{i}_{n_1,n_2}\mathscr{P}_{n_1-1,n_2}(x_1,x_2),
\end{aligned}
$$

where the coefficients $\tilde{y}_{n_1,n_2}$ are obtained from y_{n_1,n_2} by the permutation $\alpha_1 \leftrightarrow \alpha_2$

5.A.4 Difference equations

The bivariate Hahn polynomials $\mathscr{P}_{n_1,n_2}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2;N)$ satisfy the eigenvalues equation

$$\mathscr{L}_1 \mathscr{P}_{n_1,n_2}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2;N) = -n_1(n_1+\alpha_1+\alpha_2+1)\mathscr{P}_{n_1,n_2}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2;N)$$

where

$$\mathscr{L}_1 = x_1(x_2+\alpha_2+1)T_{x_1}^{-}T_{x_2}^{+} + x_2(x_1+\alpha_1+1)T_{x_1}^{+}T_{x_2}^{-} \\ -(x_1(x_2+\alpha_2+1)+x_2(x_1+\alpha_1+1))\mathbb{I},$$

where $T_{x_i}^{\pm}$ are the usual forward and backward shift operators in the variable x_i. The bivariate Hahn polynomials also satisfy

$$\mathscr{L}_2 \mathscr{P}_{n_1,n_2}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2;N) = \\ -(n_1+n_2)(n_1+n_2+\alpha_1+\alpha_2+\alpha_3+2)\mathscr{P}_{n_1,n_2}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2;N),$$

where $\mathscr{L}_2$ is the difference operator

$$\mathscr{L}_2 = (N-x_1-x_2)\left[(x_1+\alpha_1+1)T_{x_1}^{+} + (x_2+\alpha_2+1)T_{x_2}^{+}\right] + x_1(x_2+\alpha_2+1)T_{x_1}^{-}T_{x_2}^{+} \\ +(N-x_1-x_2+\alpha_3+1)\left[x_1T_{x_1}^{-}+x_2T_{x_2}^{-}\right] + x_2(x_1+\alpha_1+1)T_{x_1}^{+}T_{x_2}^{-} \\ -\Big[(N-x_1-x_2)(x_1+x_2+\alpha_1+\alpha_2+2)+(x_1+x_2)(N-x_1-x_2+\alpha_3+1) \\ +x_1(x_2+\alpha_2+1)+x_2(x_1+\alpha_1+1)\Big]\mathbb{I}.$$

5.A.5 Generating Function

The polynomials $\mathscr{P}_{n_1,n_2}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2;N)$ have for generating function

$$(1+z_1+z_2)^{N-n_1}(z_1+z_2)^{n_1}P_{n_1}^{(\alpha_1,\alpha_2)}\left(\frac{z_2-z_1}{z_1+z_2}\right)P_{n_2}^{(2n_1+\alpha_1+\alpha_2+1,\alpha_3)}\left(\frac{1-z_1-z_2}{1+z_1+z_2}\right) \\ = \sum_{\substack{x_1,x_2\\ x_1+x_2\leq N}} \frac{N!}{x_1!x_2!(N-x_1-x_2)!}\frac{\mathscr{P}_{n_1,n_2}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2;N)}{n_1!n_2!}z_1^{x_1}z_2^{x_2}.$$

5.A.6 Forward shift operators

One has the forward relation

$$-N\mathscr{P}_{n_1+1,n_2}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2;N) = x_1(x_2+\alpha_2+1)\mathscr{P}_{n_1,n_2}^{(\alpha_1+1,\alpha_2+1,\alpha_3)}(x_1-1,x_2;N-1) \\ -x_2(x_1+\alpha_1+1)\mathscr{P}_{n_1,n_2}^{(\alpha_1+1,\alpha_2+1,\alpha_3)}(x_1,x_2-1;N-1),$$

and

$$
\begin{aligned}
&-N(n_2+\alpha_3+2)\mathscr{P}_{n_1,n_2+1}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2;N)=\\
&(x_1+\alpha_1+1)(N-x_1-x_2)(N-x_1-x_2-1)\mathscr{P}_{n_1,n_2}^{(\alpha_1,\alpha_2,\alpha_3+2)}(x_1+1,x_2;N-1)\\
&+(x_2+\alpha_2+1)(N-x_1-x_2)(N-x_1-x_2-1)\mathscr{P}_{n_1,n_2}^{(\alpha_1,\alpha_2,\alpha_3+2)}(x_1,x_2+1;N-1)\\
&+x_1(N-x_1-x_2+\alpha_3+1)(N-x_1-x_2+\alpha_3+2)\mathscr{P}_{n_1,n_2}^{(\alpha_1,\alpha_2,\alpha_3+2)}(x_1-1,x_2;N-1)\\
&+x_2(N-x_1-x_2+\alpha_3+1)(N-x_1-x_2+\alpha_3+2)\mathscr{P}_{n_1,n_2}^{(\alpha_1,\alpha_2,\alpha_3+2)}(x_1,x_2-1;N-1)\\
&-(N-x_1-x_2)(N-x_1-x_2+\alpha_3+1)(2x_1+2x_2+\alpha_1+\alpha_2+2)\mathscr{P}_{n_1,n_2}^{(\alpha_1,\alpha_2,\alpha_3+2)}(x_1,x_2;N-1).
\end{aligned}
$$

These relations can be used to generate the polynomials recursively.

5.A.7 Backward shift operators

The backward relations are given by

$$
\begin{aligned}
&-\tfrac{n_1(n_1+\alpha_1+\alpha_2+1)}{N+1}\mathscr{P}_{n_1-1,n_2}^{(\alpha_1+1,\alpha_2+1,\alpha_3)}(x_1,x_2;N)=\\
&\qquad\mathscr{P}_{n_1,n_2}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1+1,x_2;N+1)-\mathscr{P}_{n_1,n_2}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2+1;N+1),
\end{aligned}
$$

and

$$
\begin{aligned}
&-\tfrac{n_2(2n_1+n_2+\alpha_1+\alpha_2+1)(2n_1+n_2+\alpha_1+\alpha_2+\alpha_3+2)}{N+1}\mathscr{P}_{n_1,n_2-1}^{(\alpha_1,\alpha_2,\alpha_3+2)}(x_1,x_2;N)=\\
&(x_1+\alpha_1+1)\mathscr{P}_{n_1,n_2}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1+1,x_2;N+1)+x_1\mathscr{P}_{n_1,n_2}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1-1,x_2;N+1)\\
&+(x_2+\alpha_2+1)\mathscr{P}_{n_1,n_2}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2+1;N+1)+x_2\mathscr{P}_{n_1,n_2}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2-1;N+1)\\
&\qquad-(2x_1+2x_2+\alpha_1+\alpha_2+2)\mathscr{P}_{n_1,n_2}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2;N+1).
\end{aligned}
$$

5.A.8 Structure relations

One has

$$
\begin{aligned}
&N\mathscr{P}_{n_1,n_2}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1+1,x_2;N)=\\
&\tfrac{(n_1+\alpha_1+\alpha_2+1)(2n_1+n_2+\alpha_1+\alpha_2+\alpha_3+2)(N-n_1-n_2)}{(2n_1+\alpha_1+\alpha_2+1)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+2)}\mathscr{P}_{n_1,n_2}^{(\alpha_1+1,\alpha_2,\alpha_3)}(x_1,x_2;N-1)\\
&-\tfrac{n_1(n_1+\alpha_2)(2n_1+n_2+\alpha_1+\alpha_2+1)(N+n_1+n_2+\alpha_1+\alpha_2+\alpha_3+2)}{(2n_1+\alpha_1+\alpha_2+1)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+2)}\mathscr{P}_{n_1-1,n_2}^{(\alpha_1+1,\alpha_2,\alpha_3)}(x_1,x_2;N-1)\\
&-\tfrac{n_2(n_2+\alpha_3)(n_1+\alpha_1+\alpha_2+1)(N+n_1+n_2+\alpha_1+\alpha_2+\alpha_3+2)}{(2n_1+\alpha_1+\alpha_2+1)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+2)}\mathscr{P}_{n_1,n_2-1}^{(\alpha_1+1,\alpha_2,\alpha_3)}(x_1,x_2;N-1)\\
&\qquad+\tfrac{n_1(n_1+\alpha_2)(N-n_1-n_2)}{(2n_1+\alpha_1+\alpha_2+1)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+2)}\mathscr{P}_{n_1-1,n_2+1}^{(\alpha_1+1,\alpha_2,\alpha_3)}(x_1,x_2;N-1).
\end{aligned}
$$

Since $\mathscr{P}_{n_1,n_2}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2;N) = (-1)^{n_1}\mathscr{P}_{n_1,n_2}^{(\alpha_2,\alpha_1,\alpha_3)}(x_2,x_1;N)$, we also have

$$
\begin{aligned}
&N\mathscr{P}_{n_1,n_2}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2+1;N) = \\
&\tfrac{(n_1+\alpha_1+\alpha_2+1)(2n_1+n_2+\alpha_1+\alpha_2+\alpha_3+2)(N-n_1-n_2)}{(2n_1+\alpha_1+\alpha_2+1)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+2)}\mathscr{P}_{n_1,n_2}^{(\alpha_1,\alpha_2+1,\alpha_3)}(x_1,x_2;N-1) \\
&+\tfrac{n_1(n_1+\alpha_1)(2n_1+n_2+\alpha_1+\alpha_2+1)(N+n_1+n_2+\alpha_1+\alpha_2+\alpha_3+2)}{(2n_1+\alpha_1+\alpha_2+1)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+2)}\mathscr{P}_{n_1-1,n_2}^{(\alpha_1,\alpha_2+1,\alpha_3)}(x_1,x_2;N-1) \\
&-\tfrac{n_2(n_2+\alpha_3)(n_1+\alpha_1+\alpha_2+1)(N+n_1+n_2+\alpha_1+\alpha_2+\alpha_3+2)}{(2n_1+\alpha_1+\alpha_2+1)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+2)}\mathscr{P}_{n_1,n_2-1}^{(\alpha_1,\alpha_2+1,\alpha_3)}(x_1,x_2;N-1) \\
&\qquad-\tfrac{n_1(n_1+\alpha_1)(N-n_1-n_2)}{(2n_1+\alpha_1+\alpha_2+1)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+2)}\mathscr{P}_{n_1-1,n_2+1}^{(\alpha_1,\alpha_2+1,\alpha_3)}(x_1,x_2;N-1).
\end{aligned}
$$

Another set of structure relations is the following:

$$
\begin{aligned}
&\frac{x_1}{N}\mathscr{P}_{n_1,n_2}^{(\alpha_1+1,\alpha_2,\alpha_3)}(x_1-1,x_2;N-1) = \\
&\tfrac{(n_1+\alpha_1+1)(2n_1+n_2+\alpha_1+\alpha_2+2)}{(2n_1+\alpha_1+\alpha_2+2)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+3)}\mathscr{P}_{n_1,n_2}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2;N) \\
&-\tfrac{(2n_1+n_2+\alpha_1+\alpha_2+\alpha_3+3)}{(2n_1+\alpha_1+\alpha_2+2)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+3)}\mathscr{P}_{n_1+1,n_2}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2;N) \\
&-\tfrac{(n_1+\alpha_1+1)}{(2n_1+\alpha_1+\alpha_2+2)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+3)}\mathscr{P}_{n_1,n_2+1}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2;N) \\
&\qquad+\tfrac{n_2(n_2+\alpha_3)}{(2n_1+\alpha_1+\alpha_2+2)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+3)}\mathscr{P}_{n_1+1,n_2-1}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2;N).
\end{aligned}
$$

$$
\begin{aligned}
&\frac{x_2}{N}\mathscr{P}_{n_1,n_2}^{(\alpha_1,\alpha_2+1,\alpha_3)}(x_1,x_2-1;N-1) = \\
&\tfrac{(n_1+\alpha_2+1)(2n_1+n_2+\alpha_1+\alpha_2+2)}{(2n_1+\alpha_1+\alpha_2+2)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+3)}\mathscr{P}_{n_1,n_2}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2;N) \\
&+\tfrac{(2n_1+n_2+\alpha_1+\alpha_2+\alpha_3+3)}{(2n_1+\alpha_1+\alpha_2+2)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+3)}\mathscr{P}_{n_1+1,n_2}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2;N) \\
&-\tfrac{(n_1+\alpha_2+1)}{(2n_1+\alpha_1+\alpha_2+2)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+3)}\mathscr{P}_{n_1,n_2+1}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2;N) \\
&\qquad-\tfrac{n_2(n_2+\alpha_3)}{(2n_1+\alpha_1+\alpha_2+2)(2n_1+2n_2+\alpha_1+\alpha_2+\alpha_3+3)}\mathscr{P}_{n_1+1,n_2-1}^{(\alpha_1,\alpha_2,\alpha_3)}(x_1,x_2;N).
\end{aligned}
$$

5.B Structure relations for Jacobi polynomials

The Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ are defined by [28]

$$P_n^{(\alpha,\beta)}(z) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left[\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix};\frac{1-z}{2}\right]. \tag{5.103}$$

The following structure relations hold for the Jacobi polynomials [29]:

$$\partial_z P_n^{(\alpha,\beta)}(z) = \frac{(n+\alpha+\beta+1)}{2}P_{n-1}^{(\alpha+1,\beta+1)}(z), \tag{5.104}$$

$$\left[(z-1)\partial_z^2 + (\alpha+1)\partial_z\right]P_n^{(\alpha,\beta)}(z) = \frac{(n+\alpha)(n+\alpha+\beta+1)}{2}P_{n-1}^{(\alpha,\beta+2)}(z), \tag{5.105}$$

$$\left[(1-z^2)\partial_z + [(\beta-\alpha)-(\alpha+\beta)z]\right] P_n^{(\alpha,\beta)}(z) = -2(n+1)P_{n+1}^{(\alpha-1,\beta-1)}(z), \tag{5.106}$$

$$\Big\{(1+z)(z^2-1)\partial_z^2 + (1+z)[1+\alpha-2\beta+(1+\alpha+2\beta)z]\partial_z + \beta[2+\alpha(1+z)+\beta(z-1)]\Big\} P_n^{(\alpha,\beta)}(z) = 2(n+1)(n+\beta)P_{n+1}^{(\alpha,\beta-2)}(z). \tag{5.107}$$

5.C Structure relations for Laguerre polynomials

The Laguerre polynomials $L_n^{(\alpha)}(z)$ are defined by

$$L_n^{(\alpha)}(z) = \frac{(\alpha+1)_n}{n!}\, {}_1F_1\left[\begin{matrix}-n\\ \alpha+1\end{matrix}; z\right].$$

The following structure relations hold for the Laguerre polynomials [28]:

$$\partial_z L_n^{(\alpha)}(z) = -L_{n-1}^{(\alpha+1)}(z), \tag{5.108}$$

$$[z\partial_z + (\alpha - z)]L_n^{(\alpha)}(z) = (n+1)L_{n+1}^{(\alpha-1)}(z). \tag{5.109}$$

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Chapitre 6

The generic superintegrable system on the 3-sphere and the $9j$ symbols of $\mathfrak{su}(1,1)$

V. X. Genest et L. Vinet (2014). The generic superintegrable system on the 3-sphere and the $9j$ symbols of $\mathfrak{su}(1,1)$. *SIGMA* **10**, 108

Abstract. The $9j$ symbols of $\mathfrak{su}(1,1)$ are studied within the framework of the generic superintegrable system on the 3-sphere. The canonical bases corresponding to the binary coupling schemes of four $\mathfrak{su}(1,1)$ representations are constructed explicitly in terms of Jacobi polynomials and are seen to correspond to the separation of variables in different cylindrical coordinate systems. A triple integral expression for the $9j$ coefficients exhibiting their symmetries is derived. A double integral formula is obtained by extending the model to the complex three-sphere and taking the complex radius to zero. The explicit expression for the vacuum coefficients is given. Raising and lowering operators are constructed and are used to recover the relations between contiguous coefficients. It is seen that the $9j$ symbols can be expressed as the product of the vacuum coefficients and a rational function. The recurrence relations and the difference equations satisfied by the $9j$ coefficients are derived.

6.1 Introduction

The objective of this paper is to show how the framework provided by the generic superintegrable system on the 3-sphere can be used to study the $9j$ coefficients of $\mathfrak{su}(1,1)$. In addition to providing a new interpretation for these coefficients, this approach, which can be viewed as a treatment of the problem in the position representation, allows for an explicit construction of the canonical bases involved in the $9j$ problem and a direct derivation of the properties of the $9j$ symbols without reference to Clebsch-Gordan or Racah coefficients.

The $9j$ symbols arise as recoupling coefficients in the combination of four irreducible $\mathfrak{su}(1,1)$ representations of the positive-discrete series. These coefficients and their equivalent $\mathfrak{su}(2)$ analogues have traditionally found applications in molecular [30] and nuclear [31] physics but have also appeared in the study of spin networks related to quantum gravity [26]. Over the past years, they have been the object of a number of publications, many of which study the $9j$ coefficients from the point of view of special functions. For example, the connection between $9j$ coefficients and orthogonal polynomials in two variables has been studied by Van der Jeugt [7], Suslov [24] and more recently by Hoare and Rahman [15] who used the $9j$ coefficients as a starting point to their study of bivariate Krawtchouk polynomials [9, 12]. A number of explicit multi-sums expressions have also been investigated by Jucys and Ališauskas [1, 2], Rosengren [27, 28, 29] and Rao and Rajeswari [25]. Also worth mentioning is the original approach of Granovskii and Zhedanov [13] which opened a path to a new method for deriving generating functions and convolution identities for orthogonal polynomials [6, 20].

In the present paper, we shall indicate how the $9j$ problem can be studied in the position representation using the connection between the coupling of four $\mathfrak{su}(1,1)$ representations and the generic superintegrable system on the three-sphere. This system has been discussed by Kalnins, Kress and Miller in [16]. It is governed by the Hamiltonian

$$H=\sum_{1\leqslant i<k\leqslant 4} J_{ik}^2+r^2\sum_{1\leqslant \ell\leqslant 4}\frac{a_\ell}{s_\ell^2},\qquad a_\ell=\alpha_\ell^2-1/4, \tag{6.1}$$

where $\alpha_\ell>-1$ and is defined on the 3-sphere of square radius $r^2=s_1^2+s_2^2+s_3^2+s_4^2$. Here the operators J_{ik} stand for the familiar angular momentum generators

$$J_{ik}=i(s_i\partial_{s_k}-s_k\partial_{s_i}),\qquad 1\leqslant i<k\leqslant 4. \tag{6.2}$$

The system described by (6.1) is both superintegrable and exactly solvable. It has five algebraically independent second order constants of motion that generate a quadratic algebra [18].

It will first be shown that the Hamiltonian (6.1) coincides with the total Casimir operator for the combination of four $\mathfrak{su}(1,1)$ representations and that its constants of motion correspond to the intermediate Casimir operators associated to each possible pairing of the four representations; these results extend the author's previous work [11]. Using this framework, the canonical orthonormal bases of the $9j$ problem, which correspond to the joint diagonalization of different pairs of commuting intermediate Casimir operators, will be constructed as solutions of the Schrödinger equation associated to (6.1) separated in different cylindrical coordinate systems; these solutions will be given in terms of Jacobi polynomials. The coordinate realization of the canonical bases and the underlying quantum mechanical framework will yield an expression for the $9j$ coefficients in terms of an integral on the 3-sphere exhibiting their symmetries. By extending the model to the complex 3-sphere and taking the complex radius to zero, the expression for the $9j$ symbols in terms of a double integral found by Granovskii and Zhedanov [13] shall be recovered. This formula will be used to obtain an explicit hypergeometric formula for the special case corresponding to the "vacuum" $9j$ coefficients. The coordinate realization will also allow for the construction of raising and lowering operators based on the structure relations of the Jacobi polynomials. These operators will then be used to derive directly the relations between contiguous $9j$ symbols, which are usually obtained by manipulations of Clebsch-Gordan or Racah coefficients (given in terms of the Hahn or the Racah polynomials [7]). From these relations, it will be possible to conclude that the $9j$ coefficients can be expressed as a product of the vacuum coefficients and of functions that are rational (and not polynomial as stated in [15]). The fact that the raising and lowering operators factorize the corresponding intermediate Casimir operators shall be used to obtain the action of the intermediate Casimirs on the basis states. This will also lead to both the difference equations and the recurrence relations satisfied by the $9j$ coefficients.

The organization of the paper is as follows.

- Section 1: Generic system on the 3-sphere from four $\mathfrak{su}(1,1)$ representations, Exact solutions, Canonical basis vectors of the $9j$ problem, Triple integral representation, Symmetries of the $9j$ coefficients.

- Section 2: Double integral formula, Explicit vacuum $9j$ coefficients.

- Section 3: Raising/Lowering operators, Relations between contiguous $9j$ symbols.

- Section 4: Difference equations and recurrence relations for $9j$ symbols.

6.2 The $9j$ problem for $\mathfrak{su}(1,1)$ in the position representation

In this section the $9j$ problem for the positive-discrete series of $\mathfrak{su}(1,1)$ representations is examined in the position representation. The total Casimir operator for the addition of four representations is identified with the Hamiltonian of the generic superintegrable system on S^3 and the intermediate Casimir operators are identified with its symmetries. The canonical basis vectors of the $9j$ problem are constructed as wavefunctions separated in different coordinate systems and the $9j$ coefficients are expressed as the overlap coefficients between these bases.

6.2.1 The addition of four representations and the generic system on S^3

Consider the operators

$$K_0^{(i)} = \frac{1}{4}\left(-\partial_{s_i}^2 + s_i^2 + \frac{\alpha_i}{s_i^2}\right), \quad K_\pm^{(i)} = \frac{1}{4}\left((s_i \mp \partial_{s_i})^2 - \frac{\alpha_i}{s_i^2}\right), \qquad i = 1, \ldots, 4, \tag{6.3}$$

which form four mutually commuting sets of generators satisfying the $\mathfrak{su}(1,1)$ commutation relations

$$[K_0^{(i)}, K_\pm^{(i)}] = \pm K_\pm^{(i)}, \quad [K_-^{(i)}, K_+^{(i)}] = 2K_0^{(i)}.$$

The operators (6.3) provide a realization of the positive-discrete series of $\mathfrak{su}(1,1)$ representations on the space of square-integrable functions on the positive real line. A set of basis vectors $e_{n_i}^{(\nu_i)}$, $n_i = 0, 1, \ldots$, for these representations specified by a positive real number ν_i taking the value

$$\nu_i = \frac{\alpha_i + 1}{2}, \tag{6.4}$$

is given in terms of Laguerre polynomials [19] according to

$$e_{n_i}^{(\nu_i)}(s_i) = (-1)^{n_i} \sqrt{\frac{2\,\Gamma(n_i+1)}{\Gamma(n_i+\alpha_i+1)}}\, e^{-s_i^2/2} s_i^{\alpha_i+1/2}\, L_{n_i}^{(\alpha_i)}(s_i^2), \quad n_i = 0, 1, \ldots, \tag{6.5}$$

where $\Gamma(z)$ is the gamma function [4]. These basis vectors are orthonormal with respect to the scalar product [19]

$$\int_0^\infty e_{n_i}^{(\nu_i)}(s_i)\, e_{n_i'}^{(\nu_i)}(s_i)\, \mathrm{d}s_i = \delta_{n_i n_i'},$$

and the action of the generators on the basis vectors is given by

$$\begin{aligned}
K_+^{(i)} e_{n_i}^{(\nu_i)}(s_i) &= \sqrt{(n_i+1)(n_i+2\nu_i)}\, e_{n_i+1}^{(\nu_i)}(s_i),\\
K_-^{(i)} e_{n_i}^{(\nu_i)}(s_i) &= \sqrt{n_i(n_i+2\nu_i-1)}\, e_{n_i-1}^{(\nu_i)}(s_i),\\
K_0^{(i)} e_{n_i}^{(\nu_i)}(s_i) &= (n_i+\nu_i) e_{n_i}^{(\nu_i)}(s_i),
\end{aligned}$$

which corresponds to the usual action defining the irreducible representations of the positive-discrete series [33]. In the realization (6.3), it is easily verified that the Casimir operator of $\mathfrak{su}(1,1)$ which has the expression

$$Q^{(i)} = [K_0^{(i)}]^2 - K_+^{(i)} K_-^{(i)} - K_0^{(i)},$$

acts as a multiple of the identity, i.e.

$$Q^{(i)} = \nu_i(\nu_i - 1),$$

for $i=1,\ldots,4$. The four sets (6.3) can be used to define a fifth set of $\mathfrak{su}(1,1)$ generators through

$$K_0 = \sum_{1\leqslant i\leqslant 4} K_0^{(i)}, \quad K_\pm = \sum_{1\leqslant i\leqslant 4} K_\pm^{(i)}. \tag{6.6}$$

The above operators realize a representation of $\mathfrak{su}(1,1)$ on the space $\bigotimes_{i=1}^4 V^{(\nu_i)}$ where $V^{(\nu_i)}$ is the space spanned by the basis vectors (6.5). It is directly checked that the total Casimir operator associated to this realization is

$$Q = H/4, \tag{6.7}$$

where H is the Hamiltonian of the generic superintegrable system on the 3-sphere given by (6.1).

When considering the tensor product of several representations, it is natural to consider the intermediate Casimir operators associated to each possible pairing of representations. These intermediate Casimir operators are defined by

$$Q^{(ij)} = [K_0^{(i)}+K_0^{(j)}]^2 - [K_+^{(i)}+K_+^{(j)}][K_-^{(i)}+K_-^{(j)}] - [K_0^{(i)}+K_0^{(j)}], \qquad 1\leqslant i<j\leqslant 4,$$

and have the expression

$$Q^{(ij)} = \frac{1}{4}\left(J_{ij}^2 + \frac{a_i s_j^2}{s_i^2} + \frac{a_j s_i^2}{s_j^2} + a_i + a_j - 1\right), \qquad 1\leqslant i<j\leqslant 4, \tag{6.8}$$

where J_{ij} are the angular momentum operators (6.2). By construction, the intermediate Casimir operators $Q^{(ij)}$ commute with the total Casimir operator Q and hence the intermediate Casimir operators (6.8) are the symmetries of the Hamiltonian (6.1). It is directly checked that the intermediate Casimir operators $Q^{(ij)}$, $Q^{(k\ell)}$ commute only when i,j,k,ℓ are all different and hence the largest set of commuting intermediate Casimir operators has two elements. Note that the intermediate Casimir operators are linearly related to the total Casimir operator as per the relation

$$Q = \sum_{1\leqslant i<j\leqslant 4} Q^{(ij)} - 2\sum_{1\leqslant i\leqslant 4} Q^{(i)}.$$

In considering the total Casimir operator (6.7), one can take the value of the square radius r^2 to be fixed since the operator

$$2K_0 + K_+ + K_- = r^2,$$

commutes with Q and all the intermediate Casimir operators $Q^{(ij)}$. We shall take $r^2=1$, thus considering the Hamiltonian (6.1) on the unit 3-sphere.

6.2.2 The $9j$ symbols

In general, the representation $\bigotimes_{i=1}^{4} V^{(\nu_i)}$ is not irreducible, but it is known to be completely reducible in representations of the positive-discrete series. In this context, the $9j$ symbols arise as the overlap coefficients between natural bases associated to two different decomposition schemes.

- In the first scheme, one first decomposes $V^{(\nu_1)}\otimes V^{(\nu_2)}$ and $V^{(\nu_3)}\otimes V^{(\nu_4)}$ in irreducible components $V^{(\nu_{12})}$, $V^{(\nu_{34})}$ and then decomposes $V^{(\nu_{12})}\otimes V^{(\nu_{34})}$ in irreducible components $V^{(\nu)}$ for each occurring values of (ν_{12},ν_{34}). The natural (orthonormal) basis vectors for this scheme are denoted $|\,\vec{\nu};\nu_{12},\nu_{34};\nu\,\rangle$ and defined by

$$\begin{aligned} Q^{(12)}|\,\vec{\nu};\nu_{12},\nu_{34};\nu\,\rangle &= \nu_{12}(\nu_{12}-1)|\,\vec{\nu};\nu_{12},\nu_{34};\nu\,\rangle,\\ Q^{(34)}|\,\vec{\nu};\nu_{12},\nu_{34};\nu\,\rangle &= \nu_{34}(\nu_{34}-1)|\,\vec{\nu};\nu_{12},\nu_{34};\nu\,\rangle,\\ Q|\,\vec{\nu};\nu_{12},\nu_{34};\nu\,\rangle &= \nu(\nu-1)|\,\vec{\nu};\nu_{12},\nu_{34};\nu\,\rangle, \end{aligned} \tag{6.9}$$

  where $\vec{\nu}=(\nu_1,\nu_2,\nu_3,\nu_4)$.

- In the second scheme, one first decomposes $V^{(\nu_1)}\otimes V^{(\nu_3)}$ and $V^{(\nu_2)}\otimes V^{(\nu_4)}$ in irreducible components $V^{(\nu_{13})}$, $V^{(\nu_{24})}$ and then decomposes $V^{(\nu_{13})}\otimes V^{(\nu_{24})}$ in irreducible components $V^{(\nu)}$ for each occurring values of (ν_{13},ν_{24}). The natural (orthonormal) basis vectors for this scheme are denoted $|\,\vec{\nu};\nu_{13},\nu_{24};\nu\,\rangle$ and defined by

$$\begin{aligned} Q^{(13)}|\,\vec{\nu};\nu_{13},\nu_{24};\nu\,\rangle &= \nu_{13}(\nu_{13}-1)|\,\vec{\nu};\nu_{13},\nu_{24};\nu\,\rangle,\\ Q^{(24)}|\,\vec{\nu};\nu_{13},\nu_{24};\nu\,\rangle &= \nu_{24}(\nu_{24}-1)|\,\vec{\nu};\nu_{13},\nu_{24};\nu\,\rangle,\\ Q|\,\vec{\nu};\nu_{13},\nu_{24};\nu\,\rangle &= \nu(\nu-1)|\,\vec{\nu};\nu_{13},\nu_{24};\nu\,\rangle. \end{aligned} \tag{6.10}$$

The $9j$ symbols are defined as the overlap coefficients between these two bases, i.e.

$$|\,\vec{\nu};\nu_{12},\nu_{34};\nu\,\rangle = \sum_{\nu_{13},\nu_{24}} \begin{Bmatrix} \nu_1 & \nu_2 & \nu_{12}\\ \nu_3 & \nu_4 & \nu_{34}\\ \nu_{13} & \nu_{24} & \nu \end{Bmatrix} |\,\vec{\nu};\nu_{13},\nu_{24};\nu\,\rangle.$$

For the $9j$ symbols to be non-vanishing, one must have

$$\begin{aligned} &\nu_{12}=\nu_1+\nu_2+m, \quad \nu_{34}=\nu_3+\nu_4+n,\\ &\nu_{13}=\nu_1+\nu_3+x, \quad \nu_{24}=\nu_2+\nu_4+y,\\ &\nu=\nu_1+\nu_2+\nu_3+\nu_4+N, \end{aligned} \tag{6.11}$$

where m,n,x,y and N are non-negative integers such that $m+n\leqslant N$ and $x+y\leqslant N$.

In view of the coordinate realization stemming from the previous subsection, the bases (6.9) and (6.10) can be constructed explicitly by solving the corresponding eigenvalue equations: these bases correspond to the diagonalization of the Hamiltonian (6.1) together with the pairs of commuting intermediate Casimir operators (symmetries) $(Q^{(12)}, Q^{(34)})$ or $(Q^{(13)}, Q^{(24)})$. In view of the conditions (6.4), (6.11) and for notational convenience, the basis corresponding to the scheme (6.9) shall be simply denoted by $|\,m,n\,\rangle_N$, the basis corresponding to (6.10) by $|\,x,y\,\rangle_N$ and the $9j$ coefficients will be written as

$$|\,m,n\,\rangle_N = \sum_{\substack{x,y \\ x+y\leqslant N}} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} |\,x,y\,\rangle_N, \tag{6.12}$$

or equivalently as

$$\begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} = {}_N\langle\,x,y\,|\,m,n\,\rangle_N. \tag{6.13}$$

The $9j$ coefficients are taken to be real. Since they are transition coefficients between two orthonormal bases, it follows from elementary linear algebra that

$$\sum_{\substack{x,y \\ x+y\leqslant N}} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} \begin{Bmatrix} \alpha_1 & \alpha_2 & m' \\ \alpha_3 & \alpha_4 & n' \\ x & y & N \end{Bmatrix} = \delta_{mm'}\delta_{nn'}, \tag{6.14}$$

and similarly

$$\sum_{\substack{m,n \\ m+n\leqslant N}} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x' & y' & N \end{Bmatrix} = \delta_{xx'}\delta_{yy'}.$$

6.2.3 The canonical bases by separation of variables

Let us now obtain the explicit realizations for the bases $|\,m,n\,\rangle_N$ and $|\,x,y\,\rangle_N$ corresponding to the coupling schemes (6.9) and (6.10). As shall be seen, these bases correspond to the separation of variables in the equation $H\Upsilon = \Lambda\Upsilon$ using different cylindrical coordinate systems. Note that this eigenvalue equation has been studied by Kalnins, Miller and Tratnik in [17].

The basis for $\{Q^{(12)}, Q^{(34)}\}$

To obtain the coordinate realization of the basis corresponding to the first coupling scheme (6.9), we look for functions $\Psi_{m,n;N}$ on the 3-sphere that satisfy

$$Q^{(12)}\Psi_{m,n;N} = \lambda_m^{(12)}\Psi_{m,n;N}, \quad Q^{(34)}\Psi_{m,n;N} = \lambda_n^{(34)}\Psi_{m,n;N}, \quad Q\Psi_{m,n;N} = \Lambda_N\Psi_{m,n;N},$$

with eigenvalues

$$\begin{aligned}
\lambda_m^{(12)} &= (m+\alpha_1/2+\alpha_2/2)(m+\alpha_1/2+\alpha_2/2+1),\\
\lambda_n^{(34)} &= (n+\alpha_3/2+\alpha_4/2)(n+\alpha_3/2+\alpha_4/2+1),\\
\Lambda_N &= (N+|\alpha|/2+1)(N+|\alpha|/2+2),
\end{aligned}$$

where $|\alpha| = \sum_i \alpha_i$. The expressions for the spectra follow directly from the fact that the operators are intermediate Casimir operators in the addition of $\mathfrak{su}(1,1)$ representations of the positive-discrete series [8]. Consider the set of cylindrical coordinates $\{\theta,\phi_1,\phi_2\}$ defined by

$$s_1=\cos\theta\cos\phi_1,\quad s_2=\cos\theta\sin\phi_1,\quad s_3=\sin\theta\cos\phi_2,\quad s_4=\sin\theta\sin\phi_2. \tag{6.15}$$

In these coordinates, one finds from (6.8) that the operators $Q^{(12)}$, $Q^{(34)}$ read

$$\begin{aligned}
Q^{(12)} &= \frac{1}{4}\left(-\partial_{\phi_1}^2+\alpha_1\mathrm{tg}^2\phi_1+\frac{\alpha_2}{\mathrm{tg}^2\phi_1}+(\alpha_1+\alpha_2-1)\right),\\
Q^{(34)} &= \frac{1}{4}\left(-\partial_{\phi_2}^2+\alpha_3\mathrm{tg}^2\phi_2+\frac{\alpha_4}{\mathrm{tg}^2\phi_2}+(\alpha_3+\alpha_4-1)\right),
\end{aligned}$$

and that Q takes the form

$$\begin{aligned}
Q = \frac{1}{4}\Big[-\partial_\theta^2+\left(\mathrm{tg}\theta-\frac{1}{\mathrm{tg}\theta}\right)\partial_\theta&\\
+\frac{1}{\cos^2\theta}\left(-\partial_{\phi_1}^2+\frac{\alpha_1}{\cos^2\phi_1}+\frac{\alpha_2}{\sin^2\phi_1}\right)&+\frac{1}{\sin^2\theta}\left(-\partial_{\phi_2}^2+\frac{\alpha_3}{\cos^2\phi_2}+\frac{\alpha_4}{\sin^2\phi_2}\right)\Big].
\end{aligned}$$

It is directly seen from the above expressions that $\Psi_{m,n;N}$ will separate in the coordinates (6.15). Using standard techniques, one finds that the wavefunctions have the expression

$$\begin{aligned}
\langle\,\theta,\phi_1,\phi_2\,|\,m,n\,\rangle_N &= \Psi_{m,n;N}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}(\theta,\phi_1,\phi_2)=\eta_m^{(\alpha_2,\alpha_1)}\eta_n^{(\alpha_4,\alpha_3)}\\
&\times\eta_{N-m-n}^{(2n+\alpha_3+\alpha_4+1,2m+\alpha_1+\alpha_2+1)}(\cos\theta\cos\phi_1)^{\alpha_1+1/2}(\cos\theta\sin\phi_1)^{\alpha_2+1/2}\\
&\times(\sin\theta\cos\phi_2)^{\alpha_3+1/2}(\sin\theta\sin\phi_2)^{\alpha_4+1/2}\cos^{2m}\theta\sin^{2n}\theta\,P_m^{(\alpha_2,\alpha_1)}(\cos2\phi_1)\\
&\times P_n^{(\alpha_4,\alpha_3)}(\cos2\phi_2)\,P_{N-m-n}^{(2n+\alpha_3+\alpha_4+1,2m+\alpha_1+\alpha_2+1)}(\cos2\theta),
\end{aligned}\tag{6.16}$$

where $P_n^{(\alpha,\beta)}(x)$ are the classical Jacobi polynomials (see appendix A). The normalization factor

$$\eta_n^{(\alpha,\beta)}=\sqrt{\frac{2\,\Gamma(m+1)\Gamma(m+\alpha+\beta+1)\Gamma(2m+\alpha+\beta+2)}{\Gamma(m+\alpha+1)\Gamma(m+\beta+1)\Gamma(2m+\alpha+\beta+1)}}, \tag{6.17}$$

ensures that the following orthonormality condition holds:

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\pi/2}{}_N\langle\,m',n'\,|\,\theta,\phi_1,\phi_2\,\rangle\langle\,\theta,\phi_1,\phi_2\,|\,m,n\,\rangle_N\,\mathrm{d}\Omega=\delta_{mm'}\delta_{nn'}\delta_{NN'}, \tag{6.18}$$

where $\mathrm{d}\Omega = \cos\theta \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi_1 \,\mathrm{d}\phi_2$. In Cartesian coordinates, $\Psi_{m,n;N}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}$ assumes the form

$$\begin{aligned}\langle s_1,s_2,s_3,s_4 \mid m,n\rangle_N = \Psi_{m,n;N}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}(s_1,s_2,s_3,s_4) = \eta_m^{(\alpha_2,\alpha_1)}\eta_n^{(\alpha_4,\alpha_3)}\\ \times \eta_{N-m-n}^{(2n+\alpha_3+\alpha_4+1,2m+\alpha_1+\alpha_2+1)}\left(\prod_{i=1}^{4} s_i^{\alpha_i+1/2}\right)(s_1^2+s_2^2)^m(s_3^2+s_4^2)^n\, P_m^{(\alpha_2,\alpha_1)}\left(\frac{s_1^2-s_2^2}{s_1^2+s_2^2}\right)\\ \times P_n^{(\alpha_4,\alpha_3)}\left(\frac{s_3^2-s_4^2}{s_3^2+s_4^2}\right)P_{N-m-n}^{(2n+\alpha_3+\alpha_4+1,2m+\alpha_1+\alpha_2+1)}(s_1^2+s_2^2-s_3^2-s_4^2).\end{aligned} \tag{6.19}$$

The wavefunctions $\Psi_{m,n;N}$ thus provide a concrete realization in the position representation of the basis state $|\, m,n\rangle_N$ corresponding to the first coupling scheme. A different realization of this state is given by Lievens and Van der Jeugt in [22], who examined realizations of coupled vectors in the coherent state representation for general tensor products.

The basis for $\{Q^{(13)},Q^{(24)}\}$

To obtain the coordinate realization of the basis corresponding to the second coupling scheme (6.10), we look for functions $\Xi_{x,y;N}$ on the 3-sphere that satisfy

$$Q^{(13)}\,\Xi_{x,y;N} = \lambda_x^{(13)}\,\Xi_{x,y;N}, \quad Q^{(24)}\,\Xi_{x,y;N} = \lambda_y^{(24)}\,\Xi_{x,y;N}, \quad Q\,\Xi_{x,y;N} = \Lambda_N\,\Xi_{x,y;N},$$

where

$$\begin{aligned}\lambda_x^{(13)} &= (x+\alpha_1/2+\alpha_3/2)(x+\alpha_1/2+\alpha_3/2+1),\\ \lambda_y^{(24)} &= (y+\alpha_2/2+\alpha_4/2)(y+\alpha_2/2+\alpha_4/2+1),\\ \Lambda_N &= (N+|\alpha|/2+1)(N+|\alpha|/2+2),\end{aligned}$$

and $|\alpha| = \sum_{i=1}^{4}\alpha_i$. Consider the set of cylindrical coordinates $\{\vartheta,\varphi_1,\varphi_2\}$ defined by

$$s_1 = \cos\vartheta\cos\varphi_1, \quad s_2 = \sin\vartheta\cos\varphi_2, \quad s_3 = \cos\vartheta\sin\varphi_1, \quad s_4 = \sin\vartheta\sin\varphi_2. \tag{6.20}$$

In these coordinates, the operators $Q^{(13)}$, $Q^{(24)}$ have the expressions

$$\begin{aligned}Q^{(13)} &= \frac{1}{4}\left(-\partial_{\varphi_1}^2 + a_1\mathrm{tg}^2\varphi_1 + \frac{a_3}{\mathrm{tg}^2\varphi_1} + (a_1+a_3-1)\right),\\ Q^{(24)} &= \frac{1}{4}\left(-\partial_{\varphi_2}^2 + a_2\mathrm{tg}^2\varphi_2 + \frac{a_4}{\mathrm{tg}^2\varphi_2} + (a_2+a_4-1)\right),\end{aligned}$$

and the total Casimir operator Q reads

$$\begin{aligned}Q = \frac{1}{4}\Big[-\partial_\vartheta^2 + \left(\mathrm{tg}\,\vartheta + \frac{1}{\mathrm{tg}\,\vartheta}\right)\partial_\vartheta\\ + \frac{1}{\cos^2\vartheta}\left(-\partial_{\varphi_1}^2 + \frac{a_1}{\cos^2\varphi_1} + \frac{a_3}{\sin^2\varphi_1}\right) + \frac{1}{\sin^2\vartheta}\left(-\partial_{\varphi_2} + \frac{a_2}{\cos^2\varphi_2} + \frac{a_4}{\sin^2\varphi_2}\right)\Big].\end{aligned}$$

It is clear from the above that the functions $\Xi_{x,y;N}$ will separate in the coordinates (6.20). The wavefunctions $\Xi_{x,y;N}$ have the expression

$$\begin{aligned}
\langle\, \vartheta,\varphi_1,\varphi_2 \mid x,y\,\rangle_N &= \Xi_{x,y;N}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}(\vartheta,\varphi_1,\varphi_2) = \eta_x^{(\alpha_3,\alpha_1)}\eta_y^{(\alpha_4,\alpha_2)} \\
&\times \eta_{N-x-y}^{(2y+\alpha_2+\alpha_4+1,2x+\alpha_1+\alpha_3+1)}(\cos\vartheta\cos\varphi_1)^{\alpha_1+1/2}\,(\sin\vartheta\cos\varphi_2)^{\alpha_2+1/2} \\
&\times\,(\cos\vartheta\sin\varphi_1)^{\alpha_3+1/2}\,(\sin\vartheta\sin\varphi_2)^{\alpha_4+1/2}\,\cos^{2x}\vartheta\sin^{2y}\vartheta\,P_x^{(\alpha_3,\alpha_1)}(\cos 2\varphi_1) \\
&\qquad\times\,P_y^{(\alpha_4,\alpha_2)}(\cos 2\varphi_2)\,P_{N-x-y}^{(2y+\alpha_2+\alpha_4+1,2x+\alpha_1+\alpha_3+1)}(\cos 2\vartheta),
\end{aligned}\tag{6.21}$$

where $\eta_n^{(\alpha,\beta)}$ is given by (6.17) and where $P_n^{(\alpha,\beta)}(x)$ are again the classical Jacobi polynomials. The wavefunctions obey the orthonormality condition

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\pi/2} {}_N\langle\, x',y' \mid \vartheta,\varphi_1,\varphi_2\,\rangle\,\langle\,\vartheta,\varphi_1,\varphi_2 \mid x,y\,\rangle_N\,\mathrm{d}\Omega = \delta_{xx'}\delta_{yy'}\delta_{NN'},\tag{6.22}$$

where $\mathrm{d}\Omega = \cos\vartheta\sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi_1\,\mathrm{d}\varphi_2$. In Cartesian coordinates, one has

$$\begin{aligned}
\langle\, s_1,s_2,s_3,s_4 \mid x,y\,\rangle_N &= \Xi_{x,y;N}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}(s_1,s_2,s_3,s_4) = \eta_x^{(\alpha_3,\alpha_1)}\eta_y^{(\alpha_4,\alpha_2)} \\
&\times \eta_{N-x-y}^{(2y+\alpha_2+\alpha_4+1,2x+\alpha_1+\alpha_3+1)}\left(\prod_{i=1}^4 s_i^{\alpha_i+1/2}\right)(s_1^2+s_3^2)^x(s_2^2+s_4^2)^y\,P_x^{(\alpha_3,\alpha_1)}\left(\frac{s_1^2-s_3^2}{s_1^2+s_3^2}\right) \\
&\qquad\times P_y^{(\alpha_4,\alpha_2)}\left(\frac{s_2^2-s_4^2}{s_2^2+s_4^2}\right)\,P_{N-x-y}^{(2y+\alpha_2+\alpha_4+1,2x+\alpha_1+\alpha_3+1)}(s_1^2+s_3^2-s_2^2-s_4^2).
\end{aligned}\tag{6.23}$$

Note that (6.23) can be obtained directly from (6.19) by permuting the indices 2 and 3. The wavefunctions $\Xi_{x,y;N}$ thus provide a concrete realization of the basis states $|\,x,y\,\rangle_N$ corresponding to the coupling scheme (6.10) in the position representation.

6.2.4 $9j$ symbols as overlap coefficients, integral representation and symmetries

In view of (6.12), the $9j$ coefficients for the positive-discrete series of $\mathfrak{su}(1,1)$ representations can be expressed as the expansion coefficients between the wavefunctions $\Psi_{m,n;N}$ and $\Xi_{x,y;N}$ at a given point, i.e.:

$$\Psi_{m,n;N} = \sum_{\substack{x,y\\ x+y\leq N}} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} \Xi_{x,y;N}.\tag{6.24}$$

The orthogonality relation (6.22) immediately yields the integral formula

$$\begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} = \eta_m^{(\alpha_2,\alpha_1)} \eta_n^{(\alpha_4,\alpha_3)} \eta_{N-m-n}^{(2n+\alpha_3+\alpha_4+1,2m+\alpha_1+\alpha_2+1)}$$

$$\times \eta_x^{(\alpha_3,\alpha_1)} \eta_y^{(\alpha_4,\alpha_2)} \eta_{N-x-y}^{(2y+\alpha_2+\alpha_4+1,2x+\alpha_1+\alpha_3+1)} \int_{S_+^3} \prod_{i=1}^{4} (s_i^2)^{\alpha_i+1/2} \mathrm{d}s_i \, (s_1^2+s_2^2)^m (s_3^2+s_4^2)^n$$

$$\times (s_1^2+s_3^2)^x (s_2^2+s_4^2)^y \, P_m^{(\alpha_2,\alpha_1)}\left(\frac{s_1^2-s_2^2}{s_1^2+s_2^2}\right) P_n^{(\alpha_4,\alpha_3)}\left(\frac{s_3^2-s_4^2}{s_3^2+s_4^2}\right) P_x^{(\alpha_3,\alpha_1)}\left(\frac{s_1^2-s_3^2}{s_1^2+s_3^2}\right)$$

$$\times P_y^{(\alpha_4,\alpha_2)}\left(\frac{s_2^2-s_4^2}{s_2^2+s_4^2}\right) P_{N-m-n}^{(2n+\alpha_3+\alpha_4+1,2m+\alpha_1+\alpha_2+1)}(s_1^2+s_2^2-s_3^2-s_4^2)$$

$$\times P_{N-x-y}^{(2y+\alpha_2+\alpha_4+1,2x+\alpha_1+\alpha_3+1)}(s_1^2+s_3^2-s_2^2-s_4^2), \quad (6.25)$$

where S_+^3 stands for the totally positive octant of the 3-sphere described by $\sum_{i=1}^4 s_i^2 = 1$ with $s_i \geqslant 0$. The integral expression (6.25) looks rather complicated and shall be simplified in the next section. However, the formula (6.25) and the elementary properties of the Jacobi polynomials can be used to efficiently obtain the symmetry relations satisfied by the $9j$ symbols (6.12). As a first example, one can read off directly from (6.25) the symmetry relation

$$\begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} = \begin{Bmatrix} \alpha_1 & \alpha_3 & x \\ \alpha_2 & \alpha_4 & y \\ m & n & N \end{Bmatrix}. \quad (6.26)$$

which we shall refer to as the “duality property” of $9j$ symbols. As a second example, using the well-known identity $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$, one finds that

$$\begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} = (-1)^{N+m+n-x-y} \begin{Bmatrix} \alpha_2 & \alpha_1 & m \\ \alpha_4 & \alpha_3 & n \\ y & x & N \end{Bmatrix} = (-1)^{N+x+y-m-n} \begin{Bmatrix} \alpha_3 & \alpha_4 & n \\ \alpha_1 & \alpha_2 & m \\ x & y & N \end{Bmatrix}. \quad (6.27)$$

A number of other symmetries can be derived by combining the above. Let us note that the formula (6.25) can also be found from the results of [21].

6.3 Double integral formula and the vacuum $9j$ coefficients

In this section, a double integral formula for the $9j$ symbols is obtained by extending the wave-functions to the complex three-sphere and taking the complex radius to zero. The formula is then used to compute the vacuum $9j$ coefficients explicitly.

6.3.1 Extension of the wavefunctions

The wavefunctions $\Psi_{m,n;N}$ and $\Xi_{x,y;N}$ can easily be extended to the complex three-sphere of radius r^2 using their expressions in Cartesian coordinates. The extended wavefunctions $\widetilde{\Psi}_{m,n;N}$, $\widetilde{\Xi}_{x,y;N}$ have the expressions

$$\begin{aligned}\widetilde{\Psi}_{m,n;N} &= \eta_m^{(\alpha_2,\alpha_1)}\eta_n^{(\alpha_4,\alpha_3)}\eta_{N-m-n}^{(2n+\alpha_3+\alpha_4+1,2m+\alpha_1+\alpha_2+1)}\left(\prod_{i=1}^{4} s_i^{\alpha_i+1/2}\right)\\ &\times (s_1^2+s_2^2)^m(s_3^2+s_4^2)^n(s_1^2+s_2^2+s_3^2+s_4^2)^{N-m-n}\,P_m^{(\alpha_2,\alpha_1)}\left(\frac{s_1^2-s_2^2}{s_1^2+s_2^2}\right)\\ &\times P_n^{(\alpha_4,\alpha_3)}\left(\frac{s_3^2-s_4^2}{s_3^2+s_4^2}\right)P_{N-m-n}^{(2n+\alpha_3+\alpha_4+1,2m+\alpha_1+\alpha_2+1)}\left(\frac{s_1^2+s_2^2-s_3^2-s_4^2}{s_1^2+s_2^2+s_3^2+s_4^2}\right),\end{aligned} \tag{6.28}$$

and

$$\begin{aligned}\widetilde{\Xi}_{x,y;N} &= \eta_x^{(\alpha_3,\alpha_1)}\eta_y^{(\alpha_4,\alpha_2)}\eta_{N-x-y}^{(2y+\alpha_2+\alpha_4+1,2x+\alpha_1+\alpha_3+1)}\left(\prod_{i=1}^{4} s_i^{\alpha_i+1/2}\right)\\ &\times (s_1^2+s_3^2)^x(s_2^2+s_4^2)^y(s_1^2+s_2^2+s_3^2+s_4^2)^{N-x-y}\,P_x^{(\alpha_3,\alpha_1)}\left(\frac{s_1^2-s_3^2}{s_1^2+s_3^2}\right)\\ &\times P_y^{(\alpha_4,\alpha_2)}\left(\frac{s_2^2-s_4^2}{s_2^2+s_4^2}\right)P_{N-x-y}^{(2y+\alpha_2+\alpha_4+1,2x+\alpha_1+\alpha_3+1)}\left(\frac{s_1^2+s_3^2-s_2^2-s_4^2}{s_1^2+s_2^2+s_3^2+s_4^2}\right),\end{aligned} \tag{6.29}$$

with $s_i \in \mathbb{C}$ for $i = 1,\ldots,4$. The expressions (6.28) and (6.29) correspond to the bases constructed by Lievens and Van der Jeugt in [21] in their examination of $3nj$ symbols for $\mathfrak{su}(1,1)$. The basis vectors (6.28) and (6.29) also resemble the harmonic functions on S^3 of Dunkl and Xu [10], but do not correspond to the same separation of variables.

When the coordinates satisfy $s_1^2+s_2^2+s_3^2+s_4^2 = 1$, the wavefunctions (6.28) and (6.29) coincide with (6.19) and (6.23), respectively. When $r^2 \neq 1$, $\widetilde{\Psi}_{m,n;N}$ and $\widetilde{\Xi}_{x,y;N}$ differ from $\Psi_{m,n;N}$ and $\Xi_{x,y;N}$ by a constant factor of $r^{N+|\alpha|+2}$. Since the parameters N and α_i are fixed, the expansion (6.24) is not affected by this common multiplicative factor and one can write

$$\widetilde{\Psi}_{m,n;N}(s_1,s_2,s_3,s_4) = \sum_{\substack{x,y\\ x+y\leq N}} \begin{Bmatrix} \alpha_1 & \alpha_2 & m\\ \alpha_3 & \alpha_4 & n\\ x & y & N\end{Bmatrix} \widetilde{\Xi}_{x,y;N}(s_1,s_2,s_3,s_4), \tag{6.30}$$

for a given point (s_1,s_2,s_3,s_4) satisfying $s_1^2+s_2^2+s_3^2+s_4^2 = r^2$. Let us now impose the condition

$$s_1^2+s_2^2 = -(s_3^2+s_4^2),$$

which corresponds to taking the radius of the complex three-sphere to zero. Upon introducing the new variables u and v defined by

$$u = \frac{s_1^2-s_3^2}{s_1^2+s_3^2}, \qquad v = \frac{s_3^2+2s_4^2+s_1^2}{s_1^2+s_3^2},$$

and using the identity

$$(x+y)^m\, P_m^{(\alpha,\beta)}\left(\frac{x-y}{x+y}\right) = \frac{(\alpha+1)_m}{m!}\, x^m\, {}_2F_1\left[\begin{matrix}-m, -\beta-m\\ \alpha+1\end{matrix}; -\frac{y}{x}\right],$$

in (6.28) and (6.29), one finds that the expansion (6.30) reduces to

$$c_{m;n;N}\left(\frac{u-v}{2}\right)^N P_m^{(\alpha_2,\alpha_1)}\left(\frac{u+v+2}{u-v}\right) P_n^{(\alpha_4,\alpha_3)}\left(\frac{2-u-v}{v-u}\right)$$
$$= \sum_{\substack{x,y\\ x+y\leqslant N}} \begin{Bmatrix}\alpha_1 & \alpha_2 & m\\ \alpha_3 & \alpha_4 & n\\ x & y & N\end{Bmatrix} d_{x,y;N}\, P_x^{(\alpha_3,\alpha_1)}(u) P_y^{(\alpha_4,\alpha_2)}(v),$$

where the coefficients $c_{m,n;N}$ and $d_{x,y;N}$ read

$$c_{m,n;N} = \eta_m^{(\alpha_2,\alpha_1)}\eta_n^{(\alpha_4,\alpha_3)}\eta_{N-m-n}^{(2n+\alpha_3+\alpha_4+1,2m+\alpha_1+\alpha_2+1)}\,\frac{(-1)^n(N+m+n+|\alpha|+3)_{N-m-n}}{(N-m-n)!},$$
$$d_{x,y;N} = \eta_x^{(\alpha_3,\alpha_1)}\eta_y^{(\alpha_4,\alpha_2)}\eta_{N-x-y}^{(2y+\alpha_2+\alpha_4+1,2x+\alpha_1+\alpha_3+1)}\,\frac{(-1)^y(N+x+y+|\alpha|+3)_{N-x-y}}{(N-x-y)!}.$$

Here $(a)_n$ stands for the Pochhammer symbol defined by

$$(a)_n = (a)(a+1)\cdots(a+n-1), \quad (a)_0 = 1.$$

The orthogonality relation (6.60) for the Jacobi polynomials then leads to the integral representation

$$\begin{Bmatrix}\alpha_1 & \alpha_2 & m\\ \alpha_3 & \alpha_4 & n\\ x & y & N\end{Bmatrix} = \left[\frac{c_{m,n;N}}{d_{x,y;N}}\,\frac{2^{-N}}{h_x^{(\alpha_3,\alpha_1)}h_y^{(\alpha_4,\alpha_2)}}\right]$$
$$\times \int_{-1}^{1}\int_{-1}^{1} \mathrm{d}u\,\mathrm{d}v\,(1-u)^{\alpha_3}(1+u)^{\alpha_1}(1-v)^{\alpha_4}(1+v)^{\alpha_2}$$
$$\times P_x^{(\alpha_3,\alpha_1)}(u)\left[P_m^{(\alpha_2,\alpha_1)}\left(\frac{u+v+2}{u-v}\right)(u-v)^N P_n^{(\alpha_4,\alpha_3)}\left(\frac{2-u-v}{v-u}\right)\right] P_y^{(\alpha_4,\alpha_2)}(v), \quad (6.31)$$

where $h_n^{(\alpha,\beta)}$ is given by (6.61). The integral formula (6.31) coincides with the one found by Granovskii and Zhedanov [13] using a related approach. The formula (6.31) is one of the most simple expressions for $9j$ symbols. Given the wealth of results on the asymptotic behavior of Jacobi polynomials, one can expect the formula (6.31) to be useful in the examination of the asymptotic behavior of the $9j$ symbols, an active field [5, 34] of interest in particular for the study of spin networks related to quantum gravity [14].

6.3.2 The vacuum $9j$ coefficients

The integral expression (6.31) will now be used to obtain the explicit expression for the "vacuum" $9j$ coefficients, which correspond to the special case $m = n = 0$. These shall be used in the next section to further characterize the $9j$ symbols. Upon using the binomial expansion, the formula (6.31) gives the following expression for the vacuum $9j$ coefficients:

$$\begin{Bmatrix} \alpha_1 & \alpha_2 & 0 \\ \alpha_3 & \alpha_4 & 0 \\ x & y & N \end{Bmatrix} = \left[\frac{c_{0,0;N}}{d_{x,y;N}} \frac{2^{-N}}{h_x^{(\alpha_3,\alpha_1)} h_y^{(\alpha_4,\alpha_2)}} \right] \sum_{k=0}^{N} \binom{N}{k} (-1)^{N-k} \int_{-1}^{1}\int_{-1}^{1} \mathrm{d}u\, \mathrm{d}v$$

$$\times\ (1-u)^{\alpha_3}(1+u)^{\alpha_1} P_x^{(\alpha_3,\alpha_1)}(u)\, u^k\ (1-v)^{\alpha_4}(1+v)^{\alpha_2} P_y^{(\alpha_4,\alpha_2)}(v)\, v^{N-k}, \quad (6.32)$$

where $\binom{N}{k}$ is the binomial coefficient. To evaluate the integrals, one can use the expansion of the power function in series of Jacobi polynomials [3] which reads

$$x^k = \sum_{j=0}^{k} \left\{ \frac{2^j k!}{(k-j)!} \frac{\Gamma(j+\alpha+\beta+1)}{\Gamma(2j+\alpha+\beta+1)} {}_2F_1\left[{j-k, j+\alpha+1 \atop 2j+\alpha+\beta+2} ; 2 \right] \right\} P_j^{(\alpha,\beta)}(x),$$

where ${}_pF_q$ stands for the generalized hypergeometric function [3]. Upon inserting the above expansion in (6.32) and using the orthogonality relation (6.60) for the Jacobi polynomials, one finds

$$\begin{Bmatrix} \alpha_1 & \alpha_2 & 0 \\ \alpha_3 & \alpha_4 & 0 \\ x & y & N \end{Bmatrix} = \left[\frac{\eta_0^{(\alpha_2,\alpha_1)} \eta_0^{(\alpha_4,\alpha_3)} \eta_N^{(\alpha_3+\alpha_4+1,\alpha_1+\alpha_2+1)}}{\eta_x^{(\alpha_3,\alpha_1)} \eta_y^{(\alpha_4,\alpha_2)} \eta_{N-x-y}^{(2y+\alpha_2+\alpha_4+1,2x+\alpha_1+\alpha_3+1)}} \right]$$

$$\times \left[\frac{(-1)^{N+x+y}}{2^{N-x-y}} \frac{(N+|\alpha|+3)_N}{(N+x+y+|\alpha|+3)_{N-x-y}} \right] \left[\frac{\Gamma(x+\alpha_1+\alpha_3+1)\Gamma(y+\alpha_2+\alpha_4+1)}{\Gamma(2x+\alpha_1+\alpha_3+1)\Gamma(2y+\alpha_2+\alpha_4+1)} \right]$$

$$\times \sum_{k=0}^{N-x-y} (-1)^k \binom{N-x-y}{k} {}_2F_1\left[{-k, x+\alpha_3+1 \atop 2x+\alpha_1+\alpha_3+2} ; 2 \right] {}_2F_1\left[{-(N-x-y-k), y+\alpha_4+1 \atop 2y+\alpha_2+\alpha_4+2} ; 2 \right].$$

The summation in the above relation can be evaluated by means of the formula

$$\sum_{\ell=0}^{M} \frac{(-N)_\ell}{\ell!} {}_2F_1\left[{-\ell, a_1 \atop b_1} ; x \right] {}_2F_1\left[{\ell-N, a_2 \atop b_2} ; x \right] = x^N \frac{(a_1)_N}{(b_1)_N} {}_3F_2\left[{-N, a_2, 1-b_1-N \atop b_2, 1-a_1-N} ; 1 \right].$$

Then using identity $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$, the following expression is obtained:

$$\begin{Bmatrix} \alpha_1 & \alpha_2 & 0 \\ \alpha_3 & \alpha_4 & 0 \\ x & y & N \end{Bmatrix} = \binom{N}{x,y}^{1/2} [(\alpha_1+1)_x(\alpha_2+1)_y(\alpha_3+1)_x(\alpha_4+1)_y]^{1/2}$$

$$\times \left[\frac{(N+|\alpha|+3)_{x+y}}{(\alpha_1+\alpha_2+2)_N(\alpha_3+\alpha_4+2)_N} \right]^{1/2} \left[\frac{(\alpha_1+\alpha_3+1)_x(\alpha_1+\alpha_3+2)_{N+x-y}}{(\alpha_1+\alpha_3+1)_{2x}(\alpha_1+\alpha_3+2)_{2x}} \right]^{1/2}$$

$$\times \left[\frac{(\alpha_2+\alpha_4+1)_y(\alpha_2+\alpha_4+2)_{2y}}{(\alpha_2+\alpha_4+1)_{2y}(\alpha_2+\alpha_4+2)_{N-x+y}} \right]^{1/2} (y+\alpha_4+1)_{N-x-y}$$

$$\times\ {}_3F_2\left[{-(N-x-y), -(N-x+y+\alpha_2+\alpha_4+1), x+\alpha_3+1 \atop -(N-x+\alpha_4), 2x+\alpha_1+\alpha_3+2} ; 1 \right], \quad (6.33)$$

where $\binom{N}{x,y} = \frac{N!}{x!y!(N-x-y)!}$ stands for the trinomial coefficients. The analogous formula for the $9j$ coefficients of $\mathfrak{su}(2)$ has been given by Hoare and Rahman [15]. The duality formula (6.26) can be used to obtain a similar expression for the case where $x = y = 0$.

6.4 Raising, lowering operators and contiguity relations

In this section, raising and lowering operators are introduced and are called upon to obtain the relations between contiguous $9j$ symbols by direct computation. These relations are used to show that the $9j$ symbols can be expressed as the product of the vacuum $9j$ coefficients and a rational function of the variables x, y.

6.4.1 Raising, lowering operators and factorization

Let $A_{\pm}^{(\alpha_1,\alpha_2)}$ be defined as

$$A_{\pm}^{(\alpha_1,\alpha_2)} = \frac{1}{2}\left[\pm\partial_{\phi_1} - \operatorname{tg}\phi_1\,(\alpha_1+1/2) + \frac{1}{\operatorname{tg}\phi_1}\,(\alpha_2+1/2)\right], \tag{6.34}$$

and let $B_{\pm}^{(\alpha_3,\alpha_4)}$ have the expression

$$B_{\pm}^{(\alpha_3,\alpha_4)} = \frac{1}{2}\left[\pm\partial_{\phi_2} - \operatorname{tg}\phi_2\,(\alpha_3+1/2) + \frac{1}{\operatorname{tg}\phi_2}\,(\alpha_4+1/2)\right], \tag{6.35}$$

where the coordinates (6.15) have been used. It is directly checked that with respect to the scalar product in (6.18), one has $(A_{\pm}^{(\alpha_1,\alpha_2)})^{\dagger} = A_{\mp}^{(\alpha_1,\alpha_2)}$ and $(B_{\pm}^{(\alpha_3,\alpha_4)})^{\dagger} = B_{\mp}^{(\alpha_3,\alpha_4)}$, where $x^{\dagger}$ stands for the adjoint of x. With the help of the relations (6.63) and (6.62), it is easily verified that one has on the one hand

$$\begin{aligned}
A_{+}^{(\alpha_1,\alpha_2)}\Psi_{m,n;N}^{(\alpha_1+1,\alpha_2+1,\alpha_3,\alpha_4)}(\theta,\phi_1,\phi_2) &= \\
&\sqrt{(m+1)(m+\alpha_1+\alpha_2+2)}\,\Psi_{m+1,n;N+1}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}(\theta,\phi_1,\phi_2),\\
A_{-}^{(\alpha_1,\alpha_2)}\Psi_{m,n;N}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}(\theta,\phi_1,\phi_2) &= \\
&\sqrt{m(m+\alpha_1+\alpha_2+1)}\,\Psi_{m-1,n;N-1}^{(\alpha_1+1,\alpha_2+1,\alpha_3,\alpha_4)}(\theta,\phi_1,\phi_2),
\end{aligned} \tag{6.36}$$

and on the other hand

$$\begin{aligned}
B_{+}^{(\alpha_3,\alpha_4)}\Psi_{m,n;N}^{(\alpha_1,\alpha_2,\alpha_3+1,\alpha_4+1)}(\theta,\phi_1,\phi_2) &= \\
&\sqrt{(n+1)(n+\alpha_3+\alpha_4+2)}\,\Psi_{m,n+1;N+1}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}(\theta,\phi_1,\phi_2),\\
B_{-}^{(\alpha_3,\alpha_4)}\Psi_{m,n;N}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}(\theta,\phi_1,\phi_2) &= \\
&\sqrt{n(n+\alpha_3+\alpha_4+1)}\,\Psi_{m,n-1;N-1}^{(\alpha_1,\alpha_2,\alpha_3+1,\alpha_4+1)}(\theta,\phi_1,\phi_2),
\end{aligned}$$

where $\Psi_{m,n;N}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}$ is given by (6.16). The operators (6.34) and (6.35) provide a factorization of the intermediate Casimir operators $Q^{(12)}$ and $Q^{(34)}$, respectively. Indeed, it is directly checked that

$$\begin{aligned}
A_{+}^{(\alpha_1,\alpha_2)}A_{-}^{(\alpha_1,\alpha_2)} &= Q^{(12)} - (\alpha_1/2+\alpha_2/2)(\alpha_1/2+\alpha_2/2+1),\\
B_{+}^{(\alpha_3,\alpha_4)}B_{-}^{(\alpha_3,\alpha_4)} &= Q^{(34)} - (\alpha_3/2+\alpha_4/2)(\alpha_3/2+\alpha_4/2+1).
\end{aligned} \tag{6.37}$$

6.4.2 Contiguity relations

The raising/lowering operators (6.34) and (6.35) can be used to obtain the relations satisfied by contiguous $9j$ symbols. To facilitate the computations, let us make explicit the dependence of the canonical basis vectors $|\, m,n\,\rangle_N$, $|\, x,y\,\rangle_N$ on the parameters α_i by writing

$$|\, m,n\,\rangle_N \equiv |\,\alpha_1,\alpha_2,\alpha_3,\alpha_4;m,n\,\rangle_N, \quad |\, x,y\,\rangle_N \equiv |\,\alpha_1,\alpha_2,\alpha_3,\alpha_4;x,y\,\rangle_N.$$

With this notation the $9j$ symbols are written as

$$\begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} = {}_N\langle \alpha_1,\alpha_2,\alpha_3,\alpha_4;x,y\,|\,\alpha_1,\alpha_2,\alpha_3,\alpha_4;m,n\,\rangle_N.$$

To obtain the first contiguity relation for $9j$ symbols, one considers the matrix element

$${}_N\langle\,\alpha_1,\alpha_2,\alpha_3,\alpha_4;x,y\,|\,A_+^{(\alpha_1,\alpha_2)}\,|\,\alpha_1+1,\alpha_2+1,\alpha_3,\alpha_4;m,n\,\rangle_{N-1}.$$

By acting with $A_+^{(\alpha_1,\alpha_2)}$ on $|\,\alpha_1+1,\alpha_2+1,\alpha_3,\alpha_4;m,n\,\rangle_{N-1}$ using (6.36), one finds

$$\sqrt{(m+1)(m+\alpha_1+\alpha_2+2)}\begin{Bmatrix} \alpha_1 & \alpha_2 & m+1 \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} = \\ {}_N\langle\,\alpha_1,\alpha_2,\alpha_3,\alpha_4;x,y\,|\,A_+^{(\alpha_1,\alpha_2)}\,|\,\alpha_1+1,\alpha_2+1,\alpha_3,\alpha_4;m,n\,\rangle_{N-1}. \quad (6.38)$$

To obtain the desired relation, one must determine ${}_N\langle\,\alpha_1,\alpha_2,\alpha_3,\alpha_4;x,y\,|A_+^{(\alpha_1,\alpha_2)}$ or equivalently

$$(A_+^{(\alpha_1,\alpha_2)})^\dagger|\,\alpha_1,\alpha_2,\alpha_3,\alpha_4;x,y\,\rangle_N = A_-^{(\alpha_1,\alpha_2)}|\,\alpha_1,\alpha_2,\alpha_3,\alpha_4;x,y\,\rangle_N,$$

where the reality of the basis functions $\Xi_{x,y;N}$ has been used. This can be done directly by writing $A_-^{(\alpha_1,\alpha_2)}$ in the coordinates $\{\vartheta,\varphi_1,\varphi_2\}$ defined in (6.20), acting with this operator on the wavefunctions $\Xi_{x,y;N}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}(\vartheta,\varphi_1,\varphi_2)$ and using the properties of the Jacobi polynomials. Since this step represents no fundamental difficulties, the details of the computation are relegated to appendix B. One finds that

$$\begin{aligned} &A_-^{(\alpha_1,\alpha_2)}\,\Xi_{x,y;N}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)} = \\ &\sqrt{\frac{(x+\alpha_1+1)(x+\alpha_{13}+1)(y+\alpha_2+1)(y+\alpha_{24}+1)(N-x-y)(N+x+y+|\alpha|+3)}{(2x+\alpha_{13}+1)(2x+\alpha_{13}+2)(2y+\alpha_{24}+1)(2y+\alpha_{24}+2)}} \times \Xi_{x,y;N-1}^{(\alpha_1+1,\alpha_2+1,\alpha_3,\alpha_4)} \\ &+\sqrt{\frac{x(x+\alpha_3)(y+\alpha_2+1)(y+\alpha_{24}+1)(N-x+y+\alpha_{24}+2)(N+x-y+\alpha_{13}+1)}{(2x+\alpha_{13})(2x+\alpha_{13}+1)(2y+\alpha_{24}+1)(2y+\alpha_{24}+2)}} \times \Xi_{x-1,y;N-1}^{(\alpha_1+1,\alpha_2+1,\alpha_3,\alpha_4)} \\ &-\sqrt{\frac{(x+\alpha_1+1)(x+\alpha_{13}+1)y(y+\alpha_4)(N+x-y+\alpha_{13}+2)(N-x+y+\alpha_{24}+1)}{(2x+\alpha_{13}+1)(2x+\alpha_{13}+2)(2y+\alpha_{24})(2y+\alpha_{24}+1)}} \times \Xi_{x,y-1;N-1}^{(\alpha_1+1,\alpha_2+1,\alpha_3,\alpha_4)} \\ &\qquad -\sqrt{\frac{x(x+\alpha_3)y(y+\alpha_4)(N-x-y+1)(N+x+y+|\alpha|+2)}{(2x+\alpha_{13})(2x+\alpha_{13}+1)(2y+\alpha_{24})(2y+\alpha_{24}+1)}} \times \Xi_{x-1,y-1;N-1}^{(\alpha_1+1,\alpha_2+1,\alpha_3,\alpha_4)}, \quad (6.39) \end{aligned}$$

where the shorthand notation $\alpha_{ij} = \alpha_i + \alpha_j$ was used. Combining (6.38) with (6.39), one finds

$$\sqrt{(m+1)(m+\alpha_{12}+2)} \begin{Bmatrix} \alpha_1 & \alpha_2 & m+1 \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} =$$

$$\sqrt{\frac{(x+\alpha_1+1)(x+\alpha_{13}+1)(y+\alpha_2+1)(y+\alpha_{24}+1)(N-x-y)(N+x+y+|\alpha|+3)}{(2x+\alpha_{13}+1)(2x+\alpha_{13}+2)(2y+\alpha_{24}+1)(2y+\alpha_{24}+2)}} \times \begin{Bmatrix} \alpha_1+1 & \alpha_2+1 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N-1 \end{Bmatrix}$$

$$+\sqrt{\frac{x(x+\alpha_3)(y+\alpha_2+1)(y+\alpha_{24}+1)(N-x+y+\alpha_{24}+2)(N+x-y+\alpha_{13}+1)}{(2x+\alpha_{13})(2x+\alpha_{13}+1)(2y+\alpha_{24}+1)(2y+\alpha_{24}+2)}} \times \begin{Bmatrix} \alpha_1+1 & \alpha_2+1 & m \\ \alpha_3 & \alpha_4 & n \\ x-1 & y & N-1 \end{Bmatrix}$$

$$-\sqrt{\frac{(x+\alpha_1+1)(x+\alpha_{13}+1)y(y+\alpha_4)(N+x-y+\alpha_{13}+2)(N-x+y+\alpha_{24}+1)}{(2x+\alpha_{13}+1)(2x+\alpha_{13}+2)(2y+\alpha_{24})(2y+\alpha_{24}+1)}} \times \begin{Bmatrix} \alpha_1+1 & \alpha_2+1 & m \\ \alpha_3 & \alpha_4 & n \\ x & y-1 & N-1 \end{Bmatrix}$$

$$-\sqrt{\frac{x(x+\alpha_3)y(y+\alpha_4)(N-x-y+1)(N+x+y+|\alpha|+2)}{(2x+\alpha_{13})(2x+\alpha_{13}+1)(2y+\alpha_{24})(2y+\alpha_{24}+1)}} \times \begin{Bmatrix} \alpha_1+1 & \alpha_2+1 & m \\ \alpha_3 & \alpha_4 & n \\ x-1 & y-1 & N-1 \end{Bmatrix}. \quad (6.40)$$

To obtain the second contiguity relation, we could consider the matrix element

$${}_N\langle\, \alpha_1,\alpha_2,\alpha_3,\alpha_4;x,y\,|\,B_+^{(\alpha_3,\alpha_4)}\,|\,\alpha_1,\alpha_2,\alpha_3+1,\alpha_4+1;m,n\,\rangle_{N-1},$$

and proceed similarly by direct computation. However, it is easier to use the symmetry relation (6.27) to permute the first two rows of the relation (6.40) and then take $\alpha_1 \leftrightarrow \alpha_3$, $\alpha_2 \leftrightarrow \alpha_4$, $m \leftrightarrow n$. This directly leads to the second contiguity relation

$$\sqrt{(n+1)(n+\alpha_{34}+2)} \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n+1 \\ x & y & N \end{Bmatrix} =$$

$$\sqrt{\frac{(x+\alpha_3+1)(x+\alpha_{13}+1)(y+\alpha_4+1)(y+\alpha_{24}+1)(N-x-y)(N+x+y+|\alpha|+3)}{(2x+\alpha_{13}+1)(2x+\alpha_{13}+2)(2y+\alpha_{24}+1)(2y+\alpha_{24}+2)}} \times \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3+1 & \alpha_4+1 & n \\ x & y & N-1 \end{Bmatrix}$$

$$-\sqrt{\frac{x(x+\alpha_1)(y+\alpha_4+1)(y+\alpha_{24}+1)(N-x+y+\alpha_{24}+2)(N+x-y+\alpha_{13}+1)}{(2x+\alpha_{13})(2x+\alpha_{13}+1)(2y+\alpha_{24}+1)(2y+\alpha_{24}+2)}} \times \begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3+1 & \alpha_4+1 & n \\ x-1 & y & N-1 \end{Bmatrix} \quad (6.41)$$

$$+\sqrt{\frac{(x+\alpha_3+1)(x+\alpha_{13}+1)y(y+\alpha_2)(N+x-y+\alpha_{13}+2)(N-x+y+\alpha_{24}+1)}{(2x+\alpha_{13}+1)(2x+\alpha_{13}+2)(2y+\alpha_{24})(2y+\alpha_{24}+1)}}\times\begin{Bmatrix}\alpha_1 & \alpha_2 & m\\ \alpha_3+1 & \alpha_4+1 & n\\ x & y-1 & N-1\end{Bmatrix}$$

$$-\sqrt{\frac{x(x+\alpha_1)y(y+\alpha_2)(N-x-y+1)(N+x+y+|\alpha|+2)}{(2x+\alpha_{13})(2x+\alpha_{13}+1)(2y+\alpha_{24})(2y+\alpha_{24}+1)}}\times\begin{Bmatrix}\alpha_1 & \alpha_2 & m\\ \alpha_3+1 & \alpha_4+1 & n\\ x-1 & y-1 & N-1\end{Bmatrix}.$$

A third contiguity relation can be found by considering the matrix element

$$_N\langle\, \alpha_1,\alpha_2,\alpha_3,\alpha_4;x,y\,|\,A_-^{(\alpha_1-1,\alpha_2-1)}\,|\,m,n;\alpha_1-1,\alpha_2-1,\alpha_3,\alpha_4\,\rangle_{N+1}.$$

Upon using the action (6.36), one has

$$\sqrt{m(m+\alpha_1+\alpha_2-1)}\begin{Bmatrix}\alpha_1 & \alpha_2 & m-1\\ \alpha_3 & \alpha_4 & n\\ x & y & N\end{Bmatrix}= {}_N\langle\, \alpha_1,\alpha_2,\alpha_3,\alpha_4;x,y\,|\,A_-^{(\alpha_1-1,\alpha_2-1)}\,|\,m,n;\alpha_1-1,\alpha_2-1,\alpha_3,\alpha_4\,\rangle_N.$$

To obtain the relation, one needs to compute $_N\langle\, \alpha_1,\alpha_2,\alpha_3,\alpha_4;x,y\,|A_-^{(\alpha_1-1,\alpha_2-1)}$ or equivalently

$$(A_-^{(\alpha_1-1,\alpha_2-1)})^{\dagger}|\,\alpha_1,\alpha_2,\alpha_3,\alpha_4;x,y\,\rangle_N=A_+^{(\alpha_1-1,\alpha_2-1)}|\,\alpha_1,\alpha_2,\alpha_3,\alpha_4;x,y\,\rangle_N. \tag{6.42}$$

Following the calculations of appendix C, one arrives at

$$\begin{aligned}
&A_+^{(\alpha_1-1,\alpha_2-1)}\Xi_{x,y;N}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}=\\
&\sqrt{\frac{(x+\alpha_1)(x+\alpha_{13})(y+\alpha_2)(y+\alpha_{24})(N-x-y+1)(N+x+y+|\alpha|+2)}{(2x+\alpha_{13})(2x+\alpha_{13}+1)(2y+\alpha_{24})(2y+\alpha_{24}+1)}}\times\Xi_{x,y;N+1}^{(\alpha_1-1,\alpha_2-1,\alpha_3,\alpha_4)}\\
&+\sqrt{\frac{(x+1)(x+\alpha_3+1)(y+\alpha_2)(y+\alpha_{24})(N-x+y+\alpha_{24}+1)(N+x-y+\alpha_{13}+2)}{(2x+\alpha_{13}+1)(2x+\alpha_{13}+2)(2y+\alpha_{24})(2y+\alpha_{24}+1)}}\times\Xi_{x+1,y;N+1}^{(\alpha_1-1,\alpha_2-1,\alpha_3,\alpha_4)}\\
&-\sqrt{\frac{(x+\alpha_1)(x+\alpha_{13})(y+1)(y+\alpha_4+1)(N+x-y+\alpha_{13}+1)(N-x+y+\alpha_{24}+2)}{(2x+\alpha_{13})(2x+\alpha_{13}+1)(2y+\alpha_{24}+1)(2y+\alpha_{24}+2)}}\times\Xi_{x,y+1;N+1}^{(\alpha_1-1,\alpha_2-1,\alpha_3,\alpha_4)}\\
&-\sqrt{\frac{(x+1)(x+\alpha_3+1)(y+1)(y+\alpha_4+1)(N-x-y)(N+x+y+|\alpha|+3)}{(2x+\alpha_{13}+1)(2x+\alpha_{13}+2)(2y+\alpha_{24}+1)(2y+\alpha_{24}+2)}}\times\Xi_{x+1,y+1;N+1}^{(\alpha_1-1,\alpha_2-1,\alpha_3,\alpha_4)}.
\end{aligned}\tag{6.43}$$

Combining the above relation with (6.42), there comes

$$\sqrt{m(m+\alpha_{12}-1)}\begin{Bmatrix}\alpha_1 & \alpha_2 & m-1\\ \alpha_3 & \alpha_4 & n\\ x & y & N\end{Bmatrix}=$$

$$\sqrt{\frac{(x+\alpha_1)(x+\alpha_{13})(y+\alpha_2)(y+\alpha_{24})(N-x-y+1)(N+x+y+|\alpha|+2)}{(2x+\alpha_{13})(2x+\alpha_{13}+1)(2y+\alpha_{24})(2y+\alpha_{24}+1)}}\times\begin{Bmatrix}\alpha_1-1 & \alpha_2-1 & m\\ \alpha_3 & \alpha_4 & n\\ x & y & N+1\end{Bmatrix}$$

$$+\sqrt{\frac{(x+1)(x+\alpha_3+1)(y+\alpha_2)(y+\alpha_{24})(N-x+y+\alpha_{24}+1)(N+x-y+\alpha_{13}+2)}{(2x+\alpha_{13}+1)(2x+\alpha_{13}+2)(2y+\alpha_{24})(2y+\alpha_{24}+1)}}\times\begin{Bmatrix}\alpha_1-1 & \alpha_2-1 & m\\ \alpha_3 & \alpha_4 & n\\ x+1 & y & N+1\end{Bmatrix}$$

$$-\sqrt{\frac{(x+\alpha_1)(x+\alpha_{13})(y+1)(y+\alpha_4+1)(N+x-y+\alpha_{13}+1)(N-x+y+\alpha_{24}+2)}{(2x+\alpha_{13})(2x+\alpha_{13}+1)(2y+\alpha_{24}+1)(2y+\alpha_{24}+2)}}\times\begin{Bmatrix}\alpha_1-1 & \alpha_2-1 & m\\ \alpha_3 & \alpha_4 & n\\ x & y+1 & N+1\end{Bmatrix}$$

$$-\sqrt{\frac{(x+1)(x+\alpha_3+1)(y+1)(y+\alpha_4+1)(N-x-y)(N+x+y+|\alpha|+3)}{(2x+\alpha_{13}+1)(2x+\alpha_{13}+2)(2y+\alpha_{24}+1)(2y+\alpha_{24}+2)}}\times\begin{Bmatrix}\alpha_1-1 & \alpha_2-1 & m\\ \alpha_3 & \alpha_4 & n\\ x+1 & y+1 & N+1\end{Bmatrix}. \quad (6.44)$$

Upon applying the symmetry relation (6.27) on (6.44) and then performing the substitutions $\alpha_1 \leftrightarrow \alpha_3$, $\alpha_2 \leftrightarrow \alpha_4$ and $m \leftrightarrow n$, one finds a fourth contiguity relation

$$\sqrt{n(n+\alpha_{34}-1)}\begin{Bmatrix}\alpha_1 & \alpha_2 & m\\ \alpha_3 & \alpha_4 & n-1\\ x & y & N\end{Bmatrix}=$$

$$\sqrt{\frac{(x+\alpha_3)(x+\alpha_{13})(y+\alpha_4)(y+\alpha_{24})(N-x-y+1)(N+x+y+|\alpha|+2)}{(2x+\alpha_{13})(2x+\alpha_{13}+1)(2y+\alpha_{24})(2y+\alpha_{24}+1)}}\times\begin{Bmatrix}\alpha_1 & \alpha_2 & m\\ \alpha_3-1 & \alpha_4-1 & n\\ x & y & N+1\end{Bmatrix}$$

$$-\sqrt{\frac{(x+1)(x+\alpha_1+1)(y+\alpha_4)(y+\alpha_{24})(N-x+y+\alpha_{24}+1)(N+x-y+\alpha_{13}+2)}{(2x+\alpha_{13}+1)(2x+\alpha_{13}+2)(2y+\alpha_{24})(2y+\alpha_{24}+1)}}\times\begin{Bmatrix}\alpha_1 & \alpha_2 & m\\ \alpha_3-1 & \alpha_4-1 & n\\ x+1 & y & N+1\end{Bmatrix}$$

$$+\sqrt{\frac{(x+\alpha_3)(x+\alpha_{13})(y+1)(y+\alpha_2+1)(N+x-y+\alpha_{13}+1)(N-x+y+\alpha_{24}+2)}{(2x+\alpha_{13})(2x+\alpha_{13}+1)(2y+\alpha_{24}+1)(2y+\alpha_{24}+2)}}\times\begin{Bmatrix}\alpha_1 & \alpha_2 & m\\ \alpha_3-1 & \alpha_4-1 & n\\ x & y+1 & N+1\end{Bmatrix}$$

(6.45)

$$
-\sqrt{\frac{(x+1)(x+\alpha_1+1)(y+1)(y+\alpha_2+1)(N-x-y)(N+x+y+|\alpha|+3)}{(2x+\alpha_{13}+1)(2x+\alpha_{13}+2)(2y+\alpha_{24}+1)(2y+\alpha_{24}+2)}}\times\begin{Bmatrix}\alpha_1 & \alpha_2 & m\\ \alpha_3-1 & \alpha_4-1 & n\\ x+1 & y+1 & N+1\end{Bmatrix}.
$$

The relations (6.40), (6.41), (6.44) and (6.45) are usually obtained by writing the $9j$ symbols in terms of Clebsch-Gordan coefficients (given in terms of the Hahn polynomials) and using the properties of the latter. In our presentation however, these relations emerge from a direct computation involving Jacobi polynomials.

6.4.3 $9j$ symbols and rational functions

It will now be shown that the $9j$ symbols of $\mathfrak{su}(1,1)$ can be expressed as the product of the vacuum coefficients and a rational function. To this end, let us write the $9j$ symbols as

$$
\begin{Bmatrix}\alpha_1 & \alpha_2 & m\\ \alpha_3 & \alpha_4 & n\\ x & y & N\end{Bmatrix}=\begin{Bmatrix}\alpha_1 & \alpha_2 & 0\\ \alpha_3 & \alpha_4 & 0\\ x & y & N\end{Bmatrix}R^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}_{m,n;N}(x,y),
$$

where $R_{0,0;N}(x,y)\equiv 1$, $R_{-1,n;N}(x,y)=R_{m,-1;N}(x,y)=R_{m,n;-1}(x,y)=0$. Since the vacuum $9j$ coefficients are known explicitly, the contiguity relations (6.40), (6.41) can be used to generate the functions $R_{m,n;N}(x,y)$. Using the expression (6.33) for the vacuum coefficients, the relations (6.40) and (6.41) become

$$
\begin{aligned}
&\sqrt{\frac{(m+1)(m+\alpha_{12}+2)N(N+\alpha_{12}+2)(N+|\alpha|+3)(\alpha_1+1)(\alpha_2+1)}{(\alpha_{12}+2)(\alpha_{12}+3)(N+\alpha_{34}+1)}}\\
&\times R^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}_{m+1,n;N}(x,y)=\frac{G^{(\alpha_1+1,\alpha_2+1,\alpha_3,\alpha_4)}_{x,y;N-1}}{G^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}_{x,y;N}}R^{(\alpha_1+1,\alpha_2+1,\alpha_3,\alpha_4)}_{m,n;N-1}(x,y)\\
&\times\left[\frac{(x+\alpha_1+1)(x+\alpha_{13}+1)(y+\alpha_2+1)(y+\alpha_{24}+1)(N-x-y)(N+x+y+|\alpha|+3)}{(2x+\alpha_{13}+1)(2x+\alpha_{13}+2)(2y+\alpha_{24}+1)(N-x+\alpha_4)}\right]\\
&+\left[\frac{x(y+\alpha_2+1)(y+\alpha_2+\alpha_4+1)}{(2y+\alpha_{24}+1)}\right]\frac{G^{(\alpha_1+1,\alpha_2+1,\alpha_3,\alpha_4)}_{x-1,y;N-1}}{G^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}_{x,y;N}}R^{(\alpha_1+1,\alpha_2+1,\alpha_3,\alpha_4)}_{m,n;N-1}(x-1,y)\\
&-\left[\frac{(x+\alpha_1+1)(x+\alpha_{13}+1)(N+x-y+\alpha_{13}+2)y(y+\alpha_4)(N-x+y+\alpha_{24}+1)}{(N-x+\alpha_4)(2x+\alpha_{13}+1)(2x+\alpha_{13}+2)(2y+\alpha_{24}+1)}\right]\\
&\times\frac{G^{(\alpha_1+1,\alpha_2+1,\alpha_3,\alpha_4)}_{x,y-1;N-1}}{G^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}_{x,y;N}}R^{(\alpha_1+1,\alpha_2+1,\alpha_3,\alpha_4)}_{m,n;N-1}(x,y-1)\\
&\qquad-\left[\frac{xy(y+\alpha_4)}{(2y+\alpha_{24}+1)}\right]\frac{G^{(\alpha_1+1,\alpha_2+1,\alpha_3,\alpha_4)}_{x-1,y-1;N-1}}{G^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}_{x,y;N}}R^{(\alpha_1+1,\alpha_2+1,\alpha_3,\alpha_4)}_{m,n;N-1}(x-1,y-1),
\end{aligned}\tag{6.46}
$$

and

$$\sqrt{\frac{(n+1)(n+\alpha_{34}+2)N(N+\alpha_{34}+2)(N+|\alpha|+3)(\alpha_3+1)(\alpha_4+1)}{(\alpha_{34}+2)(\alpha_{34}+3)(N+\alpha_{12}+1)}}$$

$$\times R^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}_{m,n+1;N} = \frac{G^{(\alpha_1,\alpha_2,\alpha_3+1,\alpha_4+1)}_{x,y;N-1}}{G^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}_{x,y;N}} R^{(\alpha_1,\alpha_2,\alpha_3+1,\alpha_4+1)}_{m,n;N-1}(x,y)$$

$$\times \left[\frac{(x+\alpha_3+1)(x+\alpha_{13}+1)(y+\alpha_{24}+1)(N-x-y)(N+x+y+|\alpha|+3)}{(2x+\alpha_{13}+1)(2x+\alpha_{13}+2)(2y+\alpha_{24}+1)}\right]$$

$$-\left[\frac{x(y+\alpha_{24}+1)(N-x+\alpha_4+1)}{(2y+\alpha_{24}+1)}\right]\frac{G^{(\alpha_1,\alpha_2,\alpha_3+1,\alpha_4+1)}_{x-1,y;N-1}}{G^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}_{x,y;N}} R^{(\alpha_1,\alpha_2,\alpha_3+1,\alpha_4+1)}_{m,n;N-1}(x-1,y)$$

$$+\left[\frac{(x+\alpha_3+1)(x+\alpha_{13}+1)(y)(N+x-y+\alpha_{13}+2)(N-x+y+\alpha_{24}+1)}{(2x+\alpha_{13}+1)(2x+\alpha_{13}+2)(2y+\alpha_{24}+1)}\right]$$

$$\times \frac{G^{(\alpha_1,\alpha_2,\alpha_3+1,\alpha_4+1)}_{x,y-1;N-1}}{G^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}_{x,y;N}} R^{(\alpha_1,\alpha_2,\alpha_3+1,\alpha_4+1)}_{m,n;N-1}(x,y-1)$$

$$-\left[\frac{xy(N-x+\alpha_4+1)}{(2y+\alpha_{24}+1)}\right]\frac{G^{(\alpha_1,\alpha_2,\alpha_3+1,\alpha_4+1)}_{x-1,y-1;N-1}}{G^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}_{x,y;N}} R^{(\alpha_1,\alpha_2,\alpha_3+1,\alpha_4+1)}_{m,n;N-1}(x-1,y-1), \quad (6.47)$$

where

$$G^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}_{x,y;N} = {}_3F_2\left[\begin{matrix} -(N-x-y), -(N-x+y+\alpha_2+\alpha_4+1), x+\alpha_3+1 \\ -(N-x+\alpha_4), 2x+\alpha_1+\alpha_3+2 \end{matrix};1\right].$$

From (6.46) and (6.47), one can generate the functions $R_{m,n:N}(x,y)$ recursively. Writing the first few cases, one sees that the $R_{m,n}(x,y)$ are rational functions of the variables x,y. This is in contradiction with the assertion of ref. [15], where the functions $R_{m,n}(x,y)$ are claimed to be polynomials in the variables x,y. In view of the orthogonality relation (6.14), the rational functions $R_{m,n}(x,y)$ satisfy the orthogonality relation

$$\sum_{\substack{x,y \\ x+y\leqslant N}} t_{x,y;N} R_{m,n;N}(x,y) R_{m',n'}(x,y) = \delta_{mm'}\delta_{nn'}$$

where the weight function is of the form

$$t_{x,y;N} = \begin{Bmatrix} \alpha_1 & \alpha_2 & 0 \\ \alpha_3 & \alpha_4 & 0 \\ x & y & N \end{Bmatrix}^2. \quad (6.48)$$

It is possible to express the $9j$ symbols of $\mathfrak{su}(1,1)$ in terms of polynomials in the two variables x,y as was done by Van der Jeugt in [7]. However the involved family of polynomials $P_{m,n;N}(x,y)$ is of degree $(N-m,N-n)$ the variables $x(x+\alpha_{13}+1)$ and $y(y+\alpha_{24}+1)$ and hence do not include polynomials whose total degree is less then N.

6.5 Difference equations and recurrence relations

In this section, it is shown that the factorization property of the intermediate Casimir operators and the contiguity relations can be used to exhibit difference equations and recurrence relations for the $9j$ symbols.

A first difference equation can be obtained by considering the matrix element

$$_N\langle\, \alpha_1,\alpha_2,\alpha_3,\alpha_4;x,y \,|\, A_{+}^{(\alpha_1,\alpha_2)} A_{-}^{(\alpha_1,\alpha_2)} \,|\, \alpha_1,\alpha_2,\alpha_3,\alpha_4;m,n\,\rangle_N.$$

Using (6.36), one has on the one hand

$$_N\langle\, \alpha_1,\alpha_2,\alpha_3,\alpha_4;x,y \,|\, A_{+}^{(\alpha_1,\alpha_2)} A_{-}^{(\alpha_1,\alpha_2)} \,|\, \alpha_1,\alpha_2,\alpha_3,\alpha_4;m,n\,\rangle_N = m(m+\alpha_1+\alpha_2+1)\begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix}.$$

Using on the other hand (6.39) and (6.43) to compute $_N\langle\, \alpha_1,\alpha_2,\alpha_3,\alpha_4;x,y \,|A_{+}^{(\alpha_1,\alpha_2)} A_{-}^{(\alpha_1,\alpha_2)}$, one arrives at the difference equation

$$\begin{aligned}
m(m+\alpha_{12}+1)\begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix} &= E_{x,y}\begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x-1 & y-1 & N \end{Bmatrix} \\
&+ E_{x+1,y+1}\begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x+1 & y+1 & N \end{Bmatrix} + D_{x,y}\begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y-1 & N \end{Bmatrix} \\
&+ D_{x,y+1}\begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y+1 & N \end{Bmatrix} + C_{x,y}\begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x-1 & y & N \end{Bmatrix} \\
&+ C_{x+1,y}\begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x+1 & y & N \end{Bmatrix} + B_{x+1,y}\begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x+1 & y-1 & N \end{Bmatrix} \\
&+ B_{x,y+1}\begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x-1 & y+1 & N \end{Bmatrix} + A_{x,y}\begin{Bmatrix} \alpha_1 & \alpha_2 & m \\ \alpha_3 & \alpha_4 & n \\ x & y & N \end{Bmatrix}.
\end{aligned} \tag{6.49}$$

The coefficients are given by

$$E_{x,y} = -\sqrt{(N+x+y+|\alpha|+1)(N+x+y+|\alpha|+2)}$$
$$\times \sqrt{(N-x-y+1)(N-x-y+2)}\sqrt{\frac{x(x+\alpha_1)(x+\alpha_3)(x+\alpha_{13})}{(2x+\alpha_{13}-1)(2x+\alpha_{13})^2(2x+\alpha_{13}+1)}}$$
$$\times \sqrt{\frac{y(y+\alpha_2)(y+\alpha_4)(y+\alpha_{24})}{(2y+\alpha_{24}-1)(2y+\alpha_{24})^2(2y+\alpha_{24}+1)}}, \quad (6.50)$$

$$D_{x,y} = -\sqrt{(N+x-y+\alpha_{13}+2)(N-x+y+\alpha_{24}+1)}$$
$$\times \sqrt{(N-x-y+1)(N+x+y+|\alpha|+2)}\sqrt{\frac{y(y+\alpha_2)(y+\alpha_4)(y+\alpha_{24})}{(2y+\alpha_{24}-1)(2y+\alpha_{24})^2(2y+\alpha_{24}+1)}}$$
$$\left[\frac{x(x+\alpha_3)}{(2x+\alpha_{13})(2x+\alpha_{13}+1)} + \frac{(x+\alpha_1+1)(x+\alpha_{13}+1)}{(2x+\alpha_{13}+1)(2x+\alpha_{13}+2)}\right], \quad (6.51)$$

$$C_{x,y} = \sqrt{(N+x-y+\alpha_{13}+1)(N-x+y+\alpha_{24}+2)}$$
$$\times \sqrt{(N-x-y+1)(N+x+y+|\alpha|+2)}\sqrt{\frac{x(x+\alpha_1)(x+\alpha_3)(x+\alpha_{13})}{(2x+\alpha_{13}-1)(2x+\alpha_{13})^2(2x+\alpha_{13}+1)}}$$
$$\left[\frac{y(y+\alpha_4)}{(2y+\alpha_{24})(2y+\alpha_{24}+1)} + \frac{(y+\alpha_2+1)(y+\alpha_{24}+1)}{(2y+\alpha_{24}+1)(2y+\alpha_{24}+2)}\right], \quad (6.52)$$

$$B_{x,y} = -\sqrt{(N+x-y+\alpha_{13}+1)(N+x-y+\alpha_{13}+2)(N-x+y+\alpha_{24}+1)}$$
$$\sqrt{(N-x+y+\alpha_{24}+2)}\sqrt{\frac{x(x+\alpha_1)(x+\alpha_3)(x+\alpha_{13})}{(2x+\alpha_{13}-1)(2x+\alpha_{13})^2(2x+\alpha_{13}+1)}}$$
$$\times \sqrt{\frac{y(y+\alpha_2)(y+\alpha_4)(y+\alpha_{24})}{(2y+\alpha_{24}-1)(2y+\alpha_{24})^2(2y+\alpha_{24}+1)}}, \quad (6.53)$$

$$A_{x,y} =$$
$$\left[\frac{(x+\alpha_1+1)(x+\alpha_{13}+1)y(y+\alpha_4)(N+x-y+\alpha_{13}+2)(N-x+y+\alpha_{24}+1)}{(2x+\alpha_{13}+1)(2x+\alpha_{13}+2)(2y+\alpha_{24})(2y+\alpha_{24}+1)}\right.$$
$$+\frac{x(x+\alpha_3)(y+\alpha_2+1)(y+\alpha_{24}+1)(N+x-y+\alpha_{13}+1)(N-x+y+\alpha_{24}+2)}{(2x+\alpha_{13})(2x+\alpha_{13}+1)(2y+\alpha_{24}+1)(2y+\alpha_{24}+2)}$$
$$+\frac{(x+\alpha_1+1)(x+\alpha_{13}+1)(y+\alpha_2+1)(y+\alpha_{24}+1)(N-x-y)(N+x+y+|\alpha|+3)}{(2x+\alpha_{13}+1)(2x+\alpha_{13}+2)(2y+\alpha_{24}+1)(2y+\alpha_{24}+2)}$$
$$\left.+\frac{x(x+\alpha_3)y(y+\alpha_4)(N-x-y+1)(N+x+y+|\alpha|+2)}{(2x+\alpha_{13})(2x+\alpha_{13}+1)(2y+\alpha_{24})(2y+\alpha_{24}+1)}\right]. \quad (6.54)$$

A second difference equation is found with the help of the symmetry relation (6.27). It reads

$$n(n+\alpha_{34}+1)\begin{Bmatrix}\alpha_1 & \alpha_2 & m\\ \alpha_3 & \alpha_4 & n\\ x & y & N\end{Bmatrix} = \widetilde{E}_{x,y}\begin{Bmatrix}\alpha_1 & \alpha_2 & m\\ \alpha_3 & \alpha_4 & n\\ x-1 & y-1 & N\end{Bmatrix} + \widetilde{E}_{x+1,y+1}\begin{Bmatrix}\alpha_1 & \alpha_2 & m\\ \alpha_3 & \alpha_4 & n\\ x+1 & y+1 & N\end{Bmatrix}$$
$$-\widetilde{D}_{x,y}\begin{Bmatrix}\alpha_1 & \alpha_2 & m\\ \alpha_3 & \alpha_4 & n\\ x & y-1 & N\end{Bmatrix} - \widetilde{D}_{x,y+1}\begin{Bmatrix}\alpha_1 & \alpha_2 & m\\ \alpha_3 & \alpha_4 & n\\ x & y+1 & N\end{Bmatrix} - \widetilde{C}_{x,y}\begin{Bmatrix}\alpha_1 & \alpha_2 & m\\ \alpha_3 & \alpha_4 & n\\ x-1 & y & N\end{Bmatrix}$$
$$-\widetilde{C}_{x+1,y}\begin{Bmatrix}\alpha_1 & \alpha_2 & m\\ \alpha_3 & \alpha_4 & n\\ x+1 & y & N\end{Bmatrix} + \widetilde{B}_{x+1,y}\begin{Bmatrix}\alpha_1 & \alpha_2 & m\\ \alpha_3 & \alpha_4 & n\\ x+1 & y-1 & N\end{Bmatrix}$$
$$+\widetilde{B}_{x,y+1}\begin{Bmatrix}\alpha_1 & \alpha_2 & m\\ \alpha_3 & \alpha_4 & n\\ x-1 & y+1 & N\end{Bmatrix} + \widetilde{A}_{x,y}\begin{Bmatrix}\alpha_1 & \alpha_2 & m\\ \alpha_3 & \alpha_4 & n\\ x & y & N\end{Bmatrix}, \quad (6.55)$$

where the coefficients $\widetilde{E}_{x,y}$, $\widetilde{D}_{x,y}$, ..., etc. are obtained from $E_{x,y}$, $D_{x,y}$, ... by taking $\alpha_1 \leftrightarrow \alpha_3$ and $\alpha_2 \leftrightarrow \alpha_4$. Given the factorization property (6.37), the RHS of equations (6.49), (6.55) give the action of the intermediate Casimir operators $Q^{(12)}$, $Q^{(34)}$ on the basis where $Q^{(13)}$, $Q^{(24)}$ are diagonal. Using the duality relation (6.26), it possible to write recurrence relations for the $9j$ symbols which give the action of the intermediate Casimir operators $Q^{(13)}$, $Q^{(24)}$ on the basis where $Q^{(12)}$, $Q^{(34)}$ are diagonal. These relations read

$$x(x+\alpha_{13}+1)\begin{Bmatrix}\alpha_1 & \alpha_2 & m\\ \alpha_3 & \alpha_4 & n\\ x & y & N\end{Bmatrix} = \widehat{E}_{m,n}\begin{Bmatrix}\alpha_1 & \alpha_2 & m-1\\ \alpha_3 & \alpha_4 & n-1\\ x & y & N\end{Bmatrix} + \widehat{E}_{m+1,n+1}\begin{Bmatrix}\alpha_1 & \alpha_2 & m+1\\ \alpha_3 & \alpha_4 & n+1\\ x & y & N\end{Bmatrix}$$
$$+\widehat{D}_{m,n}\begin{Bmatrix}\alpha_1 & \alpha_2 & m\\ \alpha_3 & \alpha_4 & n-1\\ x & y & N\end{Bmatrix} + \widehat{D}_{m,n+1}\begin{Bmatrix}\alpha_1 & \alpha_2 & m\\ \alpha_3 & \alpha_4 & n+1\\ x & y & N\end{Bmatrix} + \widehat{C}_{m,n}\begin{Bmatrix}\alpha_1 & \alpha_2 & m-1\\ \alpha_3 & \alpha_4 & n\\ x & y & N\end{Bmatrix}$$
$$+\widehat{C}_{m+1,n}\begin{Bmatrix}\alpha_1 & \alpha_2 & m+1\\ \alpha_3 & \alpha_4 & n\\ x & y & N\end{Bmatrix} + \widehat{B}_{m+1,n}\begin{Bmatrix}\alpha_1 & \alpha_2 & m+1\\ \alpha_3 & \alpha_4 & n-1\\ x & y & N\end{Bmatrix}$$
$$+\widehat{B}_{m,n+1}\begin{Bmatrix}\alpha_1 & \alpha_2 & m-1\\ \alpha_3 & \alpha_4 & n+1\\ x & y & N\end{Bmatrix} + \widehat{A}_{m,n}\begin{Bmatrix}\alpha_1 & \alpha_2 & m\\ \alpha_3 & \alpha_4 & n\\ x & y & N\end{Bmatrix}, \quad (6.56)$$

where $\widehat{E}_{m,n}$, $\widehat{D}_{m,n}$, ... are obtained from $E_{m,n}$, $D_{m,n}$, ... by taking $\alpha_2 \leftrightarrow \alpha_3$. The second recurrence relation is

$$y(y+\alpha_{24}+1)\begin{Bmatrix}\alpha_1 & \alpha_2 & m\\ \alpha_3 & \alpha_4 & n\\ x & y & N\end{Bmatrix} = \check{E}_{m,n}\begin{Bmatrix}\alpha_1 & \alpha_2 & m-1\\ \alpha_3 & \alpha_4 & n-1\\ x & y & N\end{Bmatrix} + \check{E}_{m+1,n+1}\begin{Bmatrix}\alpha_1 & \alpha_2 & m+1\\ \alpha_3 & \alpha_4 & n+1\\ x & y & N\end{Bmatrix}$$
$$-\check{D}_{m,n}\begin{Bmatrix}\alpha_1 & \alpha_2 & m\\ \alpha_3 & \alpha_4 & n-1\\ x & y & N\end{Bmatrix} - \check{D}_{m,n+1}\begin{Bmatrix}\alpha_1 & \alpha_2 & m\\ \alpha_3 & \alpha_4 & n+1\\ x & y & N\end{Bmatrix} - \check{C}_{m,n}\begin{Bmatrix}\alpha_1 & \alpha_2 & m-1\\ \alpha_3 & \alpha_4 & n\\ x & y & N\end{Bmatrix}$$
$$-\check{C}_{m+1,n}\begin{Bmatrix}\alpha_1 & \alpha_2 & m+1\\ \alpha_3 & \alpha_4 & n\\ x & y & N\end{Bmatrix} + \check{B}_{m+1,n}\begin{Bmatrix}\alpha_1 & \alpha_2 & m+1\\ \alpha_3 & \alpha_4 & n-1\\ x & y & N\end{Bmatrix}$$
$$+\check{B}_{m,n+1}\begin{Bmatrix}\alpha_1 & \alpha_2 & m-1\\ \alpha_3 & \alpha_4 & n+1\\ x & y & N\end{Bmatrix} + \check{A}_{m,n}\begin{Bmatrix}\alpha_1 & \alpha_2 & m\\ \alpha_3 & \alpha_4 & n\\ x & y & N\end{Bmatrix}, \quad (6.57)$$

where $\check{E}_{m,n}$, $\check{D}_{m,n}$, etc. are obtained from $E_{m,n}$, $D_{m,n}$, etc, by effecting the permutation $\sigma = (1243)$ on the parameters $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. Writing once again the $9j$ symbols as

$$\begin{Bmatrix}\alpha_1 & \alpha_2 & m\\ \alpha_3 & \alpha_4 & n\\ x & y & N\end{Bmatrix} = \begin{Bmatrix}\alpha_1 & \alpha_2 & 0\\ \alpha_3 & \alpha_4 & 0\\ x & y & N\end{Bmatrix} R_{m,n}(x,y),$$

and defining

$$\mathbb{R}_0(x,y) = \begin{pmatrix}1\end{pmatrix}, \quad \mathbb{R}_1(x,y) = \begin{pmatrix}R_{1,0}(x,y)\\ R_{0,1}(x,y)\end{pmatrix}, \quad \mathbb{R}_2(x,y) = \begin{pmatrix}R_{2,0}(x,y)\\ R_{1,1}(x,y)\\ R_{0,2}(x,y)\end{pmatrix}, \quad \cdots$$

the recurrence relations (6.56) and (6.57) can be written in matrix form as follows

$$x(x+\alpha_{13}+1)\mathbb{R}_n(x,y) = q^{(1)}_{n+2}\,\mathbb{R}_{n+2}(x,y) + r^{(1)}_{n+1}\,\mathbb{R}_{n+1}(x,y) + s^{(1)}_n\,\mathbb{R}_n(x,y) + r^{(1)}_n\,\mathbb{R}_{n-1}(x,y) + q^{(1)}_n\,\mathbb{R}_{n-2}(x,y), \quad (6.58)$$

$$y(y+\alpha_{24}+1)\mathbb{R}_n(x,y) = q^{(2)}_{n+2}\,\mathbb{R}_{n+2}(x,y) + r^{(2)}_{n+1}\,\mathbb{R}_{n+1}(x,y) + s^{(2)}_n\,\mathbb{R}_n(x,y) + r^{(2)}_n\,\mathbb{R}_{n-1}(x,y) + q^{(2)}_n\,\mathbb{R}_{n-2}(x,y), \quad (6.59)$$

where the matrices $q^{(i)}_n$, $r^{(i)}_n$ and $s^{(i)}_n$ are easily found from the coefficients in (6.56) and (6.57). It is apparent from (6.58) and (6.59) that the vector functions $\mathbb{R}_m(x,y)$ satisfy a five term recurrence relation. In view of the multivariate extension of Favard's theorem [10], this confirms that the functions $\mathbb{R}_m(x,y)$ are not orthogonal polynomials.

6.6 Conclusion

In this paper, we have used the connection between the addition of four $\mathfrak{su}(1,1)$ representations of the positive discrete series and the generic superintegrable model on the 3-sphere to study the $9j$ coefficients in the position representation. We constructed the canonical basis vectors of the $9j$ problem explicitly and related them to the separation of variables in cylindrical coordinates. Moreover, we have obtained by direct computation the contiguity relations, the difference equations and the recurrence relations satisfied by the $9j$ symbols. The properties of the $9j$ coefficients as bivariate functions have thus been clarified.

The present work suggests many avenues for further investigations. For example Lievens and Van der Jeugt [21] have constructed explicitly the coupled basis vectors arising in the tensor product of an arbitrary number of $\mathfrak{su}(1,1)$ representations in the coherent state representation. Given this result, it would be of interest to give the realization of these vectors in the position representation by examining the generic superintegrable system on the n-sphere. Another interesting question is that of the orthogonal polynomials in two variables connected with the $9j$ problem. With the observations of the present work and those made by Van der Jeugt in ref [6], one must conclude that the study of $9j$ symbols do not naturally lead to families of bivariate orthogonal polynomials that would be two-variable extensions of the Racah polynomials. However, the results obtained by Kalnins, Miller and Post [18] and the connection between the generic model on the three-sphere and the $9j$ problem exhibited here suggest that an algebraic interpretation for the bivariate extension of the Racah polynomials, as defined by Tratnik [32], could be given in the framework of the addition of four $\mathfrak{su}(1,1)$ algebras by investigating the overlap coefficients between bases which are different from the canonical ones. We plan to follow up on this.

6.A Properties of Jacobi polynomials

The Jacobi polynomials, denoted by $P_n^{(\alpha,\beta)}(z)$, are defined as follows [19]:

$$P_n^{(\alpha,\beta)}(z) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left[\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix};\frac{1-z}{2}\right],$$

where ${}_pF_q$ stands for the generalized hypergeometric function [3]. The polynomials satisfy

$$\int_{-1}^{1}(1-z)^{\alpha}(1+z)^{\beta}P_n^{(\alpha,\beta)}(z)P_m^{(\alpha,\beta)}(z)\,\mathrm{d}z = h_n^{(\alpha,\beta)}\,\delta_{nm}, \tag{6.60}$$

where the normalization coefficient is

$$h_n^{(\alpha,\beta)} = 2^{\alpha+\beta+1}\frac{\Gamma(2n+\alpha+\beta+1)\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)\Gamma(n+\alpha+\beta+1)\Gamma(n+1)}. \tag{6.61}$$

The derivatives of the Jacobi polynomials give [23]

$$\partial_z P_n^{(\alpha,\beta)}(z) = \left[\frac{n+\alpha+\beta+1}{2}\right]P_{n-1}^{(\alpha+1,\beta+1)}(z), \tag{6.62}$$

$$\partial_z\left((1-z)^{\alpha}(1+z)^{\beta}P_n^{(\alpha,\beta)}(z)\right) = -2(n+1)\,(1-z)^{\alpha-1}(1+z)^{\beta-1}\,P_{n+1}^{(\alpha-1,\beta-1)}(z). \tag{6.63}$$

One has

$$P_n^{(\alpha,\beta)}(z) = \left(\frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1}\right)P_n^{(\alpha,\beta+1)}(z) + \left(\frac{n+\alpha}{2n+\alpha+\beta+1}\right)P_{n-1}^{(\alpha,\beta+1)}(z). \tag{6.64}$$

and

$$\left(\frac{1-z}{2}\right)P_{n-1}^{(\alpha,\beta)}(z) = \left(\frac{n+\alpha-1}{2n+\alpha+\beta-1}\right)P_{n-1}^{(\alpha_1-1,\beta)}(z) - \left(\frac{n}{2n+\alpha+\beta-1}\right)P_n^{(\alpha-1,\beta)}(z). \tag{6.65}$$

Since $P_n^{(\alpha,\beta)}(-z) = (-1)^n P_n^{(\beta,\alpha)}(z)$, one has also

$$\left(\frac{1+z}{2}\right)P_n^{(\alpha,\beta)}(z) = \left(\frac{n+\beta}{2n+\alpha+\beta+1}\right)P_n^{(\alpha,\beta-1)}(z) + \left(\frac{n+1}{2n+\alpha+\beta+1}\right)P_{n+1}^{(\alpha,\beta-1)}(z), \tag{6.66}$$

and

$$P_n^{(\alpha,\beta)}(z) = \left(\frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1}\right)P_n^{(\alpha+1,\beta)}(z) - \left(\frac{n+\beta}{2n+\alpha+\beta+1}\right)P_{n-1}^{(\alpha+1,\beta)}(z). \tag{6.67}$$

6.B Action of $A_-^{(\alpha_1,\alpha_2)}$ on $\Xi_{x,y;N}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}$

In Cartesian coordinates, the operator $A_-^{(\alpha_1,\alpha_2)}$ reads

$$A_-^{(\alpha_1,\alpha_2)} = -\frac{1}{2}(s_1\partial_{s_2} - s_2\partial_{s_1}) + \frac{s_1}{2s_2}(\alpha_2+1/2) - \frac{s_2}{2s_1}(\alpha_1+1/2).$$

The action of $A_-^{(\alpha_1,\alpha_2)}$ on the wavefunctions $\Xi_{x,y;N}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}$ can be written as

$$\begin{aligned}&\mathscr{F}\,\eta_x^{(\alpha_3,\alpha_1)}\eta_y^{(\alpha_4,\alpha_2)}\eta_{N-x-y}^{(2y+\alpha_{24}+1,2x+\alpha_{13}+1)}\\ &\quad\times\left[\mathscr{F}^{-1}A_-^{(\alpha_1,\alpha_2)}\mathscr{F}\right]\left[P_{N-x-y}^{(2y+\alpha_{24}+1,2x+\alpha_{13}+1)}(\cos 2\vartheta)P_x^{(\alpha_3,\alpha_1)}\left(\cos 2\varphi_1\right)P_y^{(\alpha_4,\alpha_2)}\left(\cos 2\varphi_2\right)\right],\end{aligned}$$

where

$$\mathscr{F} = (s_1^2+s_3^2)^x(s_2^2+s_4^2)^y\prod_{i=1}^{4}s_i^{\alpha_i+1/2}$$

One has

$$[\mathscr{F}^{-1}A_-^{(\alpha_1,\alpha_2)}\mathscr{F}] = -\frac{1}{2}(s_1\partial_{s_2} - s_2\partial_{s_1}) + x\frac{s_1s_2}{s_1^2+s_3^2} - y\frac{s_1s_2}{s_2^2+s_4^2}.$$

In the cylindrical coordinates (6.20), the operator reads

$$\begin{aligned}[\mathscr{F}^{-1}A_-^{(\alpha_1,\alpha_2)}\mathscr{F}] &= x\left[\operatorname{tg}\vartheta\cos\varphi_1\cos\varphi_2\right] - y\left[\frac{\cos\varphi_1\cos\varphi_2}{\operatorname{tg}\vartheta}\right]\\ &\quad -\frac{1}{2}\left[\cos\varphi_1\cos\varphi_2\,\partial_\vartheta + \operatorname{tg}\vartheta\sin\varphi_1\cos\varphi_2\,\partial_{\varphi_1} - \frac{\cos\varphi_1\sin\varphi_2}{\operatorname{tg}\vartheta}\partial_{\varphi_2}\right].\end{aligned}$$

Using the relation (6.62), one finds

$$\begin{aligned}&A_-^{(\alpha_1,\alpha_2)}\Xi_{x,y;N}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)} = \upsilon_{x,y,N}^{(\alpha_1,\alpha_2,\alpha_3)}\Big[(N+x+y+|\alpha|+3)(\cos^2\vartheta)^x(\sin^2\vartheta)^y\\ &\times P_{N-x-y-1}^{(2y+\alpha_{24}+2,2x+\alpha_{13}+2)}(\cos 2\vartheta)P_x^{(\alpha_3,\alpha_1)}(\cos 2\varphi_1)P_y^{(\alpha_4,\alpha_2)}(\cos 2\varphi_2)\\ &+(x+\alpha_{13}+1)(\cos^2\vartheta)^{x-1}(\sin^2\vartheta)^y\sin^2\varphi_1\\ &\times P_{N-x-y}^{(2y+\alpha_{24}+1,2x+\alpha_{13}+1)}(\cos 2\vartheta)P_{x-1}^{(\alpha_3+1,\alpha_1+1)}(\cos 2\varphi_1)P_y^{(\alpha_4,\alpha_2)}(\cos 2\varphi_2)\\ &-(y+\alpha_{24}+1)(\cos^2\vartheta)^x(\sin^2\vartheta)^{y-1}\sin^2\varphi_2\\ &\times P_{N-x-y}^{(2y+\alpha_{24}+1,2x+\alpha_{13}+1)}(\cos 2\vartheta)P_x^{(\alpha_3,\alpha_1)}(\cos 2\varphi_1)P_{y-1}^{(\alpha_4+1,\alpha_2+1)}(\cos 2\varphi_2)\\ &+\left[x(\cos^2\vartheta)^{x-1}(\sin^2\vartheta)^y - y(\cos^2\vartheta)^x(\sin^2\vartheta)^{y-1}\right]\\ &\qquad\qquad\times P_{N-x-y}^{(2y+\alpha_{24}+1,2x+\alpha_{13}+1)}(\cos 2\vartheta)P_x^{(\alpha_3,\alpha_1)}(\cos 2\varphi_1)P_y^{(\alpha_4,\alpha_2)}(\cos 2\varphi_2)\Big],\end{aligned}$$

where

$$v_{x,y,N}^{(\alpha_1,\alpha_2,\alpha_3)} = \eta_x^{(\alpha_3,\alpha_1)} \eta_y^{(\alpha_4,\alpha_2)} \eta_{N-x-y}^{(2y+\alpha_{24}+1,2x+\alpha_{13}+1)} (s_1)^{\alpha_1+3/2} (s_2)^{\alpha_2+3/2} (s_3)^{\alpha_3+1/2} (s_4)^{\alpha_4+1/2}.$$

The identities (6.64) and (6.65) can then be used to write the result in a form involving only terms of the type $P_k^{(\alpha_3,\alpha_1+1)}$ and $P_{k'}^{(\alpha_4,\alpha_2+1)}$. Regrouping the terms, one finds

$$\begin{aligned}
A_-^{(\alpha_1,\alpha_2)} \Xi_{x,y;N}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)} &= v_{x,y,N}^{(\alpha_1,\alpha_2,\alpha_3)} \Bigg[\left\{ \frac{(x+\alpha_{13}+1)(y+\alpha_{24}+1)(N+x+y+|\alpha|+3)}{(2x+\alpha_{13}+1)(2y+\alpha_{24}+1)} \right\} \\
&\times (\cos^2\vartheta)^x (\sin^2\vartheta)^y P_x^{(\alpha_3,\alpha_1+1)}(\cos 2\varphi_1) P_y^{(\alpha_4,\alpha_2+1)}(\cos 2\varphi_2) P_{N-x-y-1}^{(2y+\alpha_{24}+2,2x+\alpha_{13}+2)}(\cos 2\vartheta) \\
&+ \left\{ \frac{(x+\alpha_3)(y+\alpha_{24}+1)(\cos^2\vartheta)^{x-1}(\sin^2\vartheta)^y}{(2x+\alpha_{13}+1)(2y+\alpha_{24}+1)} \right\} P_{x-1}^{(\alpha_3,\alpha_1+1)}(\cos 2\varphi_1) P_y^{(\alpha_4,\alpha_2+1)}(\cos 2\varphi_2) \\
&\times \Big((N+x+y+|\alpha|+3) \cos^2\vartheta \, P_{N-x-y-1}^{(2y+\alpha_{24}+2,2x+\alpha_{13}+2)}(\cos 2\vartheta) \\
&\qquad + (2x+\alpha_{13}+1) P_{N-x-y}^{(2y+\alpha_{24}+1,2x+\alpha_{13}+1)}(\cos 2\vartheta) \Big) \\
&+ \left\{ \frac{(y+\alpha_4)(x+\alpha_{13}+1)(\cos^2\vartheta)^x(\sin^2\vartheta)^{y-1}}{(2x+\alpha_{13}+1)(2y+\alpha_{24}+1)} \right\} P_x^{(\alpha_3,\alpha_1+1)}(\cos 2\varphi_1) P_{y-1}^{(\alpha_4,\alpha_2+1)}(\cos 2\varphi_2) \\
&\times \Big((N+x+y+|\alpha|+3) \sin^2\vartheta \, P_{N-x-y-1}^{(2y+\alpha_{24}+2,2x+\alpha_{13}+2)}(\cos 2\vartheta) \\
&\qquad - (2y+\alpha_{24}+1) P_{n-x-y}^{(2y+\alpha_{24}+1,2x+\alpha_{13}+1)}(\cos 2\vartheta) \Big) \\
&+ \left\{ \frac{(y+\alpha_4)(x+\alpha_3)(\cos^2\vartheta)^{x-1}(\sin^2\vartheta)^{y-1}}{(2x+\alpha_{13}+1)(2y+\alpha_{24}+1)} \right\} P_{x-1}^{(\alpha_3,\alpha_1+1)}(\cos 2\varphi_1) P_{y-1}^{(\alpha_4,\alpha_2+1)}(\cos 2\varphi_2) \\
&\times \Big((N+x+y+|\alpha|+3) \cos^2\vartheta \sin^2\vartheta P_{N-x-y-1}^{(2y+\alpha_{24}+2,2x+\alpha_{13}+2)}(\cos 2\vartheta) \\
&+ (2x+\alpha_{13}+1) \sin^2\vartheta P_{N-x-y}^{(2y+\alpha_{24}+1,2x+\alpha_{13}+1)}(\cos 2\vartheta) \\
&\qquad - (2y+\alpha_{24}+1) \cos^2\vartheta \, P_{N-x-y}^{(2y+\alpha_{24}+1,2x+\alpha_{13}+1)}(\cos 2\vartheta) \Big) \Bigg]
\end{aligned}$$

The terms between parentheses in the above expression are easily evaluated and found to be

$$\begin{aligned}
&(N+x+y+|\alpha|+3) \cos^2\vartheta \, P_{N-x-y-1}^{(2y+\alpha_{24}+2,2x+\alpha_{13}+2)}(\cos 2\vartheta) \\
&+ (2x+\alpha_{13}+1) P_{N-x-y}^{(2y+\alpha_{24}+1,2x+\alpha_{13}+1)}(\cos 2\vartheta) \\
&\qquad\qquad = (N+x-y+\alpha_{13}+1) P_{N-x-y}^{(2y+\alpha_{24}+2,2x+\alpha_{13})}(\cos 2\vartheta),
\end{aligned}$$

$$\begin{aligned}
&(N+x+y+|\alpha|+3) \sin^2\vartheta \, P_{N-x-y-1}^{(2y+\alpha_{24}+2,2x+\alpha_{13}+2)}(\cos 2\vartheta) \\
&- (2y+\alpha_{24}+1) P_{n-x-y}^{(2y+\alpha_{24}+1,2x+\alpha_{13}+1)}(\cos 2\vartheta) \\
&\qquad\qquad = -(N-x+y+\alpha_{24}+1) P_{N-x-y}^{(2y+\alpha_{24},2x+\alpha_{13}+2)}(\cos 2\vartheta),
\end{aligned}$$

$$
\begin{aligned}
&(N+x+y+|\alpha|+3)\cos^2\vartheta\sin^2\vartheta P_{N-x-y-1}^{(2y+\alpha_{24}+2,2x+\alpha_{13}+2)}(\cos 2\vartheta)\\
&+(2x+\alpha_{13}+1)\sin^2\vartheta P_{N-x-y}^{(2y+\alpha_{24}+1,2x+\alpha_{13}+1)}(\cos 2\vartheta)\\
&-(2y+\alpha_{24}+1)\cos^2\vartheta\, P_{N-x-y}^{(2y+\alpha_{24}+1,2x+\alpha_{13}+1)}(\cos 2\vartheta)\\
&\qquad\qquad\qquad\qquad=-(N-x-y+1)P_{N-x-y+1}^{(2y+\alpha_{24},2x+\alpha_{13})}(\cos 2\vartheta).
\end{aligned}
$$

Adjusting the normalization factors then yields the result (6.39).

6.C Action of $A_+^{(\alpha_1-1,\alpha_2-1)}$ on $\Xi_{x,y;N}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)}$

In Cartesian coordinates, the operator $A_+^{(\alpha_1-1,\alpha_2-1)}$ reads

$$
A_+^{(\alpha_1-1,\alpha_2-1)}=\frac{1}{2}(s_1\partial_{s_2}-s_2\partial_{s_1})+\frac{s_1}{2s_2}(\alpha_2-1/2)-\frac{s_2}{2s_1}(\alpha_1-1/2).
$$

The action of $A_+^{(\alpha_1-1,\alpha_2-1)}$ on the wavefunctions $\Xi_{x,y;N}$ can be expressed as

$$
\begin{aligned}
&\mathscr{G}^{-1}\eta_x^{(\alpha_3,\alpha_1)}\eta_y^{(\alpha_4,\alpha_2)}\eta_{N-x-y}^{(2y+\alpha_{24}+1,2x+\alpha_{13}+1)}\\
&\times\left[\mathscr{G}A_+^{(\alpha_1-1,\alpha_2-1)}\mathscr{G}^{-1}\right]\left[(\sin^2\varphi_1)^{\alpha_3}(\cos^2\varphi_1)^{\alpha_1}P_x^{(\alpha_3,\alpha_1)}(\cos 2\varphi_1)\right]\\
&\times\left[(\sin^2\varphi_2)^{\alpha_4}(\cos^2\varphi_2)^{\alpha_2}P_y^{(\alpha_4,\alpha_2)}(\cos 2\varphi_2)\right]\\
&\qquad\qquad\times\left[(\sin^2\vartheta)^{2x+\alpha_{24}+1}(\cos^2\vartheta)^{2x+\alpha_{13}+1}P_{N-x-y}^{(2y+\alpha_{24}+1,2x+\alpha_{13}+1)}(\cos 2\vartheta)\right],
\end{aligned}
$$

where

$$
\mathscr{G}=(s_1^2+s_3^2)^{x+1}(s_2^2+s_4^2)^{y+1}\prod_{i=1}^{4}s_i^{\alpha_i-1/2}.
$$

One has

$$
[\mathscr{G}A_+^{(\alpha_1-1,\alpha_2-1)}\mathscr{G}^{-1}]=\frac{1}{2}(s_2\partial_{s_1}-s_1\partial_{s_2})+(x+1)\frac{s_1s_2}{s_1^2+s_3^2}-(y+1)\frac{s_1s_2}{s_2^2+s_4^2}.
$$

In the cylindrical coordinates (6.20), this operator reads

$$
\begin{aligned}
[\mathscr{G}A_+^{(\alpha_1-1,\alpha_2-1)}\mathscr{G}^{-1}]=\frac{1}{2}\Big[&\cos\varphi_1\cos\varphi_2\,\partial_\vartheta+\operatorname{tg}\vartheta\sin\varphi_1\cos\varphi_2\,\partial_{\varphi_1}\\
&-\frac{\cos\varphi_1\sin\varphi_2}{\operatorname{tg}\vartheta}\partial_{\varphi_2}\Big]+(x+1)[\operatorname{tg}\vartheta\cos\varphi_1\cos\varphi_2]-(y+1)\,[\operatorname{ctg}\vartheta\cos\varphi_1\cos\varphi_2].
\end{aligned}
$$

Using the identity (6.63), one finds

$$
\begin{aligned}
A_+^{(\alpha_1-1,\alpha_2-1)}\Xi_{x,y;N}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)} &= \kappa_{x,y;N}^{(\alpha_1,\alpha_2,\alpha_3)}\Big[(N-x-y+1)\cos^2\varphi_1\cos^2\varphi_2\\
&\times(\cos^2\vartheta)^x(\sin^2\vartheta)^y P_x^{(\alpha_3,\alpha_1)}(\cos 2\varphi_1)P_y^{(\alpha_4,\alpha_2)}(\cos 2\varphi_2)P_{N-x-y+1}^{(2y+\alpha_{24},2x+\alpha_{13})}(\cos 2\vartheta)\\
&+(x+1)\cos^2\varphi_2(\cos^2\vartheta)^x(\sin^2\vartheta)^{y+1}\\
&\times P_{x+1}^{(\alpha_3-1,\alpha_1-1)}(\cos 2\varphi_1)P_y^{(\alpha_4,\alpha_2)}(\cos 2\varphi_2)P_{N-x-y}^{(2y+\alpha_{24}+1,2x+\alpha_{13}+1)}(\cos 2\vartheta)\\
&-(y+1)\cos^2\varphi_1(\cos^2\vartheta)^{x+1}(\sin^2\vartheta)^y\\
&\times P_x^{(\alpha_3,\alpha_1)}(\cos 2\varphi_1)P_{y+1}^{(\alpha_4-1,\alpha_2-1)}(\cos 2\varphi_2)P_{N-x-y}^{(2y+\alpha_{24}+1,2x+\alpha_{13}+1)}(\cos 2\vartheta)\\
&+(x+1)\cos^2\varphi_1\cos^2\varphi_2(\cos^2\vartheta)^x(\sin^2\vartheta)^{y+1}\\
&\times P_x^{(\alpha_3,\alpha_1)}(\cos 2\varphi_1)P_y^{(\alpha_4,\alpha_2)}(\cos 2\varphi_2)P_{N-x-y}^{(2y+\alpha_{24}+1,2x+\alpha_{13}+1)}(\cos 2\vartheta)\\
&-(y+1)\cos^2\varphi_1\cos^2\varphi_2(\cos^2\vartheta)^{x+1}(\sin^2\vartheta)^y\\
&\qquad\times P_x^{(\alpha_3,\alpha_1)}(\cos 2\varphi_1)P_y^{(\alpha_4,\alpha_2)}(\cos 2\varphi_2)P_{N-x-y}^{(2y+\alpha_{24}+1,2x+\alpha_{13}+1)}(\cos 2\vartheta)\Big],
\end{aligned}
$$

where

$$
\kappa_{x,y;N}^{(\alpha_1,\alpha_2,\alpha_3)} = \eta_x^{(\alpha_3,\alpha_1)}\eta_y^{(\alpha_4,\alpha_2)}\eta_{N-x-y}^{(2y+\alpha_{24}+1,2x+\alpha_{13}+1)}(s_1)^{\alpha_1-1/2}(s_2)^{\alpha_2-1/2}(s_3)^{\alpha_3+1/2}(s_4)^{\alpha_4+1/2}.
$$

Then using (6.66) and (6.67), one finds

$$
\begin{aligned}
A_+^{(\alpha_1-1,\alpha_2-1)}\Xi_{x,y;N}^{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)} &= \kappa_{x,y;N}^{(\alpha_1,\alpha_2,\alpha_3)}\Bigg[\left\{\frac{(x+\alpha_1)(y+\alpha_2)(N-x-y+1)}{(2x+\alpha_{13}+1)(2y+\alpha_{24}+1)}\right\}\\
&\times(\cos^2\vartheta)^x(\sin^2\vartheta)^y P_x^{(\alpha_3,\alpha_1-1)}(\cos 2\varphi_1)P_y^{(\alpha_4,\alpha_2-1)}(\cos 2\varphi_2)P_{N-x-y+1}^{(2y+\alpha_{24},2x+\alpha_{13})}(\cos 2\vartheta)\\
&+\left\{\frac{(x+1)(y+\alpha_2)(\cos^2\vartheta)^{x+1}(\sin^2\vartheta)^y}{(2x+\alpha_{13}+1)(2y+\alpha_{24}+1)}\right\}P_{x+1}^{(\alpha_3,\alpha_1-1)}(\cos 2\varphi_1)P_y^{(\alpha_4,\alpha_2-1)}(\cos 2\varphi_2)\\
&\times\Bigg([2x+\alpha_{13}+1]\frac{\sin^2\vartheta}{\cos^2\vartheta}P_{N-x-y}^{(2y+\alpha_{24}+1,2x+\alpha_{13}+1)}(\cos 2\vartheta)\\
&+[N-x-y+1]\frac{1}{\cos^2\vartheta}P_{N-x-y+1}^{(2y+\alpha_{24},2x+\alpha_{13})}(\cos 2\vartheta)\Bigg)\\
&+\left\{\frac{(x+\alpha_1)(y+1)(\cos^2\vartheta)^x(\sin^2\vartheta)^{y+1}}{(2x+\alpha_{13}+1)(2y+\alpha_{24}+1)}\right\}P_x^{(\alpha_3,\alpha_1-1)}(\cos 2\varphi_1)P_{y+1}^{(\alpha_4,\alpha_2-1)}(\cos 2\varphi_2)\\
&\qquad\times\Bigg([N-x-y+1]\frac{1}{\sin^2\vartheta}P_{N-x-y+1}^{(2y+\alpha_{24},2x+\alpha_{13})}(\cos 2\vartheta)
\end{aligned}
$$

$$
\begin{aligned}
&-[2y+\alpha_{24}+1]\frac{\cos^2\vartheta}{\sin^2\vartheta}P_{N-x-y}^{2y+\alpha_{24}+1,2x+\alpha_{13}+1)}(\cos 2\vartheta)\Big)\\
&+\left\{\frac{(x+1)(y+1)(\cos^2\vartheta)^{x+1}(\sin^2\vartheta)^{y+1}}{(2x+\alpha_{13}+1)(2y+\alpha_{24}+1)}\right\}P_{x+1}^{(\alpha_3,\alpha_1)}(\cos 2\varphi_1)P_{y+1}^{(\alpha_4,\alpha-2-1)}(\cos 2\varphi_2)\\
&\times\Big([N-x-y+1]\frac{1}{\cos^2\vartheta\sin^2\vartheta}P_{N-x-y+1}^{(2y+\alpha_{24},2x+\alpha_{13})}(\cos 2\vartheta)\\
&-[2y+\alpha_{24}+1]\frac{1}{\sin^2\vartheta}P_{N-x-y}^{(2y+\alpha_{24}+1,2x+\alpha_{13}+1)}(\cos 2\vartheta)\\
&\qquad\qquad+[2x+\alpha_{13}+1]\frac{1}{\cos^2\vartheta}P_{N-x-y}^{(2y+\alpha_{24}+1,2x+\alpha_{13}+1)}(\cos 2\vartheta)\Big).
\end{aligned}
$$

The term between the parentheses are easily determined to be the following

$$
\begin{aligned}
&[2x+\alpha_{13}+1]\frac{\sin^2\vartheta}{\cos^2\vartheta}P_{N-x-y}^{(2y+\alpha_{24}+1,2x+\alpha_{13}+1)}(\cos 2\vartheta)\\
&+[N-x-y+1]\frac{1}{\cos^2\vartheta}P_{N-x-y+1}^{(2y+\alpha_{24},2x+\alpha_{13})}(\cos 2\vartheta)\\
&\qquad\qquad=(N-x+y+\alpha_{24}+1)P_{N-x-y}^{(2y+\alpha_{24},2x+\alpha_{13}+2)}(\cos 2\vartheta)
\end{aligned}
$$

$$
\begin{aligned}
&[N-x-y+1]\frac{1}{\sin^2\vartheta}P_{N-x-y+1}^{(2y+\alpha_{24},2x+\alpha_{13})}(\cos 2\vartheta)\\
&-[2y+\alpha_{24}+1]\frac{\cos^2\vartheta}{\sin^2\vartheta}P_{N-x-y}^{2y+\alpha_{24}+1,2x+\alpha_{13}+1)}(\cos 2\vartheta)\\
&\qquad\qquad=-(N+x-y+\alpha_{13}+1)P_{N-x-y}^{(2y+\alpha_{24}+2,2x+\alpha_{13})}(\cos 2\vartheta)
\end{aligned}
$$

$$
\begin{aligned}
&[N-x-y+1]\frac{1}{\cos^2\vartheta\sin^2\vartheta}P_{N-x-y+1}^{(2y+\alpha_{24},2x+\alpha_{13})}(\cos 2\vartheta)\\
&-[2y+\alpha_{24}+1]\frac{1}{\sin^2\vartheta}P_{N-x-y}^{(2y+\alpha_{24}+1,2x+\alpha_{13}+1)}(\cos 2\vartheta)\\
&+[2x+\alpha_{13}+1]\frac{1}{\cos^2\vartheta}P_{N-x-y}^{(2y+\alpha_{24}+1,2x+\alpha_{13}+1)}(\cos 2\vartheta)\\
&\qquad\qquad=-(N+x+y+|\alpha|+3)P_{N-x-y-1}^{(2y+\alpha_{24}+2,2x+\alpha_{13}+2)}(\cos 2\vartheta).
\end{aligned}
$$

Adjusting the normalization factors then yields the result (6.43).

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Chapitre 7

q-Rotations and Krawtchouk polynomials

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Abstract. An algebraic interpretation of the one-variable quantum q-Krawtchouk polynomials is provided in the framework of the Schwinger realization of $\mathscr{U}_q(sl_2)$ involving two independent q-oscillators. The polynomials are shown to arise as matrix elements of unitary "q-rotation" operators expressed as q-exponentials in the $\mathscr{U}_q(sl_2)$ generators. The properties of the polynomials (orthogonality relation, generating function, structure relations, recurrence relation, difference equation) are derived by exploiting the algebraic setting. The results are extended to another family of polynomials, the affine q-Krawtchouk polynomials, through a duality relation.

7.1 Introduction

This paper is concerned with the algebraic interpretation and characterization of two families of univariate basic orthogonal polynomials: the quantum and the affine q-Krawtchouk polynomials.

Recently, some of us have offered [11, 12] a remarkably simple description of the multivariate Krawtchouk orthogonal polynomials introduced by Griffiths [15] as matrix elements of the unitary representations of the orthogonal groups on multi-oscillator quantum states. These polynomials of Griffiths have as special cases the polynomials of Krawtchouk type proposed by Tratnik [26] which correspond to particular rotations. q-analogs of these Tratnik polynomials were offered by Gasper and Rahman [10]; their bispectrality was established by Geronimo and Iliev [13] in the $q = 1$ case and by Iliev [16] in the basic case. It would be desirable to obtain the algebraic underpinning of the multivariate q-Krawtchouk polynomials that parallels the fruitful framework developed in the $q = 1$ case, that is to relate the polynomials to matrix elements of "q-rotations" on q-oscillator states.

As a first essential step towards that goal, we elaborate here this picture in the one-variable case. The univariate quantum q-Krawtchouk polynomials will be shown to arise as matrix elements of products of q-exponentials in $\mathscr{U}_q(sl_2)$ generators realized à la Schwinger with two independent q-oscillators. By conjugation, these operators effect non-linear automorphisms of the quantum algebra $\mathscr{U}_q(sl_2)$. While this connection between $\mathscr{U}_q(sl_2)$ and q-analogs of the Krawtchouk polynomials is mentioned in [29], the detailed characterization needed for an extension to an arbitrary number of variables is carried out here in full. Using the algebraic interpretation, the main properties of the quantum q-Krawtchouk polynomials such as the orthogonality relation, the generating function, the structure relations, the difference equation, and the recurrence relation will be derived. Our approach will be seen to entail similar results for the univariate affine q-Krawtchouk polynomials through a duality relation. The novelty of the results presented here does not lie of course in the characteristic formulas for the polynomials but in their detailed algebraic interpretation.

Let us point out that an algebraic interpretation of the q-Krawtchouk was originally obtained by Koornwinder some time ago [20] in a quantum group setting, that is using the quantum function algebra dual to $\mathscr{U}_q(sl_2)$. A comparison and a detailed connection between the quantum group approach and the quantum algebra one favored here was given in [8]. While the two methods are fundamentally equivalent, the latter closely follows the representations of Lie groups via the exponentiation of algebra generators and reveals simplicity advantages that shall be helpful in higher dimensional generalizations. In that vein, the embedding of $\mathscr{U}_q(sl_2)$ in the two-dimensional q-Weyl algebra via the use of q-oscillators and the Schwinger realization offers a refined structure that entails, as shall be seen, forward and backward relations for the polynomials. Let us further note that a number of other different algebraic treatments of the various q-generalizations of the Krawtchouk polynomials can be found in [1, 4, 5, 6, 19, 23, 24, 25].

The paper is organized as follows. In Section I, some elements of q-analysis are reviewed, the Schwinger realization of $\mathscr{U}_q(sl_2)$ is revisited and the unitary q-rotation operators are constructed. In section II, the matrix elements of the q-rotation operators are calculated directly and expressed in terms of the quantum q-Krawtchouk polynomials. In section III, the structure relations for the polynomials are derived. In section IV, two types of generating functions are obtained. In section VI, the recurrence relation and the difference equation are recovered. In section VII, the duality relation between the quantum q-Krawtchouk and the affine q-Krawtchouk is examined. A conclusion follows.

7.2 The Schwinger model for $\mathscr{U}_q(sl_2)$ and q-rotations

In this section, the necessary elements of q-analysis are presented, the Schwinger realization of $\mathscr{U}_q(sl_2)$ is reviewed, and the unitary q-rotation operators are constructed.

7.2.1 Elements of q-analysis

We adopt the notation and conventions of [9]. The basic hypergeometric series is defined by

$$ {}_p\phi_q\left(\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix}\,\middle|\, q,z\right)=\sum_{n=0}^{\infty}\frac{(a_1;q)_n\cdots(a_r;q)_n}{(q;q)_n(b_1;q)_n\cdots(b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n, \tag{7.1}$$

with $\binom{n}{2}=n(n-1)/2$ and where $(a;q)_n$ stands for the q-shifted factorial

$$(a;q)_n=\begin{cases}1, & n=0,\\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & n=1,2,\dots.\end{cases}$$

The q-shifted factorials satisfy a number of identities (see Appendix I of [9]); for example, a direct expansion shows that

$$(a;q)_{n-k}=\frac{(a;q)_n}{(q^{1-n}/a;q)_k}(-q/a)^k\,q^{\binom{k}{2}-nk}, \tag{7.2}$$

where n and k are integers. The q-binomial coefficients are defined by

$$\begin{bmatrix}a\\ b\end{bmatrix}_q=\frac{(q;q)_a}{(q;q)_b(q;q)_{a-b}}. \tag{7.3}$$

It is seen that in the limit $q\uparrow 1$, the coefficients (7.3) tend to the ordinary binomial coefficients. The q-exponential functions will play an important role in what follows. The little q-exponential, denoted by $e_q(z)$, is defined as

$$e_q(z)={}_1\phi_0\left(\begin{matrix}0\\ -\end{matrix}\,\middle|\,q,z\right)=\sum_{n=0}^{\infty}\frac{z^n}{(q;q)_n}=\frac{1}{(z;q)_\infty}, \tag{7.4}$$

for $|z|<1$ and the big q-exponential, denoted $E_q(z)$, is given by

$$E_q(z)={}_0\phi_0\left(\begin{matrix}-\\ -\end{matrix}\,\middle|\,q,-z\right)=\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}}{(q;q)_n}z^n=(-z;q)_\infty. \tag{7.5}$$

It follows that $e_q(z)E_q(-z)=1$. The q-extensions of the Baker-Campbell-Hausdorff formula [7, 17] shall be needed. The first relation reads

$$E_q(\lambda X)Y e_q(-\lambda X)=\sum_{n=0}^{\infty}\frac{\lambda^n}{(q;q)_n}[X,Y]_n, \tag{7.6}$$

where

$$[X,Y]_0 = Y, \qquad [X,Y]_{n+1} = q^n X[X,Y]_n - [X,Y]_n X, \qquad n = 0,1,2,\ldots$$

The second relation is of the form

$$e_q(\lambda X) Y E_q(-\lambda X) = \sum_{n=0}^{\infty} \frac{\lambda^n}{(q;q)_n} [X,Y]'_n, \tag{7.7}$$

where

$$[X,Y]'_0 = Y, \qquad [X,Y]'_{n+1} = X[X,Y]'_n - q^n [X,Y]'_n X, \qquad n = 0,1,2,\ldots$$

One has also the identities

$$e_q(X+Y) = e_q(Y)e_q(X), \qquad \text{and} \qquad E_q(X+Y) = E_q(X)E_q(Y), \tag{7.8}$$

for $XY = qYX$.

7.2.2 The Schwinger model for $\mathscr{U}_q(sl_2)$

Consider two mutually commuting sets $\{A_\pm, A_0\}$ and $\{B_\pm, B_0\}$ of q-oscillator algebra generators that satisfy the commutation relations

$$[A_0, A_\pm] = \pm A_\pm, \qquad [A_-, A_+] = q^{A_0}, \qquad A_- A_+ - q A_+ A_- = 1, \tag{7.9a}$$

$$[B_0, B_\pm] = \pm B_\pm, \qquad [B_-, B_+] = q^{B_0}, \qquad B_- B_+ - q B_+ B_- = 1, \tag{7.9b}$$

and $[A_\cdot, B_\cdot] = 0$. It follows from (7.9) that

$$A_+ A_- = \frac{1 - q^{A_0}}{1-q}, \qquad B_+ B_- = \frac{1 - q^{B_0}}{1-q}.$$

The algebra (7.9) has a standard representation on the orthonormal states

$$|\, n_A, n_B \,\rangle \equiv |\, n_A \,\rangle \otimes |\, n_B \,\rangle, \qquad n_A, n_B = 0,1,2,\ldots \tag{7.10}$$

defined by the following actions of the generators on the factors of the tensor product states:

$$\begin{aligned} &X_- |\, n_X \,\rangle = \sqrt{\frac{1 - q^{n_x}}{1-q}} |\, n_X - 1 \,\rangle, \quad X_+ |\, n_X \,\rangle = \sqrt{\frac{1 - q^{n_X+1}}{1-q}} |\, n_X + 1 \,\rangle, \\ &X_0 |\, n_X \,\rangle = n_X |\, n_X \,\rangle, \end{aligned} \tag{7.11}$$

with $X = A$ or B. It is seen that when $q \uparrow 1$, the representation (7.11) goes to the standard oscillator representation (see for example Chap. 5 of [3]). Moreover, one has $X_\pm^\dagger = X_\mp$ in this representation. The Schwinger realization of the quantum algebra $\mathscr{U}_q(sl_2)$ is obtained by taking [2]

$$J_+ = q^{-\frac{A_0+B_0-1}{4}} A_+ B_-, \qquad J_- = q^{-\frac{A_0+B_0-1}{4}} A_- B_+, \qquad J_0 = \frac{A_0 - B_0}{2}. \tag{7.12}$$

It can be verified, using the commutation relations (7.9), that the generators (7.12) satisfy the defining relations of $U_q(sl_2)$ which read

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = \frac{q^{J_0} - q^{-J_0}}{q^{1/2} - q^{-1/2}}.$$

Upon taking $k = q^{2J_0}$, $e = J_+$ and $f = J_-$, the Chevalley presentation is obtained [28]:

$$kk^{-1} = k^{-1}k = 1, \quad k^{1/2}e = qek^{1/2}, \quad k^{1/2}f = q^{-1}fk^{1/2}, \quad [e,f] = \frac{k^{1/2} - k^{-1/2}}{q^{1/2} - q^{-1/2}}.$$

The representation of the oscillator algebra (7.11) on the states (7.10) can be used to construct a representation of $\mathscr{U}_q(sl_2)$. Let N be a non-negative integer and consider the $(N+1)$-dimensional vector space spanned by the states

$$|\, n\,\rangle_N \equiv |\, n, N-n\,\rangle, \qquad n = 0, \ldots, N. \tag{7.13}$$

The states (7.13) are orthonormal, i.e.

$${}_{N'}\langle n' \,|\, n\,\rangle_N = \delta_{nn'}\delta_{NN'}.$$

It follows from (7.11) that the action of the $\mathscr{U}_q(sl_2)$ generators (7.12) on the basis vectors (7.13) is given by

$$\begin{aligned} J_+|\, n\,\rangle_N &= q^{(1-N)/4}\sqrt{\frac{(1-q^{n+1})}{1-q}\frac{(1-q^{N-n})}{1-q}}\,|\, n+1\,\rangle_N, \\ J_-|\, n\,\rangle_N &= q^{(1-N)/4}\sqrt{\frac{(1-q^{n})}{1-q}\frac{(1-q^{N-n+1})}{1-q}}\,|\, n-1\,\rangle_N, \\ J_0|\, n\,\rangle_N &= (n - N/2)|\, n\,\rangle_N. \end{aligned} \tag{7.14}$$

The actions (7.14) correspond to the finite-dimensional irreducible representations of $\mathscr{U}_q(sl_2)$ [28]. Hence the direct product states (7.13) of two independent q-oscillators with fixed sums of the quantum numbers n_A, n_B support the irreducible representations of the quantum algebra $\mathscr{U}_q(sl_2)$.

7.2.3 Unitary q-rotation operators and matrix elements

Let us now construct, as in [29], the unitary q-rotation operators. In analogy with Lie theory, we seek to construct these operators as q-exponentials in the generators. Upon using the conjugation formula (7.7) and the commutation relations (7.9), a straightforward calculation shows that

$$e_q(\alpha A_- B_+) A_+ B_- E_q(-\alpha A_- B_+) = A_+B_- + \frac{\alpha}{(1-q)^2}\, q^{A_0} - \frac{\alpha}{(1-q)^2}\frac{1}{1-\alpha A_- B_+} q^{B_0},$$

and

$$e_q(\alpha A_- B_+) q^{B_0} E_q(-\alpha A_- B_+) = \frac{1}{1-\alpha A_- B_+} q^{B_0},$$

where the formal substitution $\sum_n X^n = \frac{1}{1-X}$ was made. Combining the above identities, one finds

$$e_q(\alpha A_- B_+)\left[A_+B_- + \frac{\alpha}{(1-q)^2}q^{B_0}\right]E_q(-\alpha A_- B_+) = A_+B_- + \frac{\alpha}{(1-q)^2}q^{A_0}.$$

and hence one has

$$e_q(\alpha A_- B_+)e_q\left(\beta A_+B_- + \frac{\alpha\beta}{(1-q)^2}q^{B_0}\right) = e_q\left(\beta A_+B_- + \frac{\alpha\beta}{(1-q)^2}q^{A_0}\right)e_q(\alpha A_- B_+).$$

Since

$$A_+B_-q^{B_0} = q q^{B_0}A_+B_-, \qquad \text{and} \qquad q^{A_0}A_+B_- = qA_+B_-q^{A_0},$$

it follows from the identities (7.8) that

$$e_q(\alpha A_- B_+)e_q\left(\frac{\alpha\beta}{(1-q)^2}q^{B_0}\right)e_q(\beta A_+B_-) = e_q(\beta A_+B_-)e_q\left(\frac{\alpha\beta}{(1-q)^2}q^{A_0}\right)e_q(\alpha A_- B_+). \tag{7.15}$$

Inverting the relation (7.15), one finds a similar relation involving big q-exponentials

$$E_q(\gamma A_+B_-)E_q\left(-\frac{\gamma\delta}{(1-q)^2}q^{B_0}\right)E_q(\delta A_-B_+) = E_q(\delta A_-B_+)E_q\left(-\frac{\gamma\delta}{(1-q)^2}q^{A_0}\right)E_q(\gamma A_+B_-). \tag{7.16}$$

Let θ be a real number such that $|\theta| < 1$ and consider the unitary operator

$$U(\theta) = e_q^{1/2}(\theta^2 q^{B_0})e_q(\theta(1-q)A_+B_-)E_q(-\theta(1-q)A_-B_+)E_q^{1/2}(-\theta^2 q^{A_0}). \tag{7.17}$$

The relation $U^\dagger U = 1$ follows from (7.15) and the relation $UU^\dagger = 1$ follows from (7.16). Acting by conjugation on the generators (7.12) in the Schwinger realization, the operator (7.17) generates automorphisms of $\mathscr{U}_q(sl_2)$ (see [29] for details). In light of the $2:1$ homomorphism between $SU(2)$ and $SO(3)$, the unitary operator (7.17) will be referred to as a "q-rotation"' as it is a q-extension of a $SU(2)$ element obtained via the exponential map from the algebra to the group. Indeed, upon using the relations

$$\lim_{q\to 1}\frac{e_q(\theta^2 q^{B_0})}{e_q(\theta^2)} = (1-\theta^2)^{B_0} = \exp\left(\log(1-\theta^2)\widetilde{B}_0\right),$$

and

$$\lim_{q\to 1}\frac{E_q(-\theta^2 q^{A_0})}{E_q(-\theta^2)} = (1-\theta^2)^{-A_0} = \exp\left(-\log(1-\theta^2)\widetilde{A}_0\right),$$

as well as the standard Baker-Campbell-Hausdorff relation [14], the limit as $q \uparrow 1$ of the unitary operator (7.17) is seen to be

$$\lim_{q\to 1}U(\theta) = \exp\left(\frac{\theta}{\sqrt{1-\theta^2}}\widetilde{A}_+\widetilde{B}_-\right)\exp\left(-\log(1-\theta^2)\frac{\widetilde{A}_0-\widetilde{B}_0}{2}\right)\exp\left(-\frac{\theta}{\sqrt{1-\theta^2}}\widetilde{A}_-\widetilde{B}_+\right), \tag{7.18}$$

where $\widetilde{A}_\pm$ and $\widetilde{B}_\pm$ satisfy the standard oscillator commutation relations [3]. Since the operators

$$\widetilde{J}_0 = \frac{\widetilde{A}_0 - \widetilde{B}_0}{2}, \qquad \widetilde{J}_+ = \widetilde{A}_+ \widetilde{B}_-, \qquad \widetilde{J}_- = \widetilde{A}_- \widetilde{B}_+,$$

satisfy the $\mathfrak{su}(2)$ commutation relations

$$[\widetilde{J}_0, \widetilde{J}_\pm] = \pm \widetilde{J}\pm, \qquad [\widetilde{J}_+, \widetilde{J}_-] = 2\widetilde{J}_0,$$

one finds that upon taking $\theta = \sin\tau$, the operator (7.18) has the expression

$$\lim_{q\to 1} U(\sin\tau) = \exp(\tan\tau \widetilde{J}_+) \exp(-2\log(\cos\tau)\widetilde{J}_0) \exp(-\tan\tau \widetilde{J}_-).$$

From the disentangling formulas for $SU(2)$ [27], one finally obtains

$$\lim_{q\to 1} U(\sin\tau) = \exp(\tau(\widetilde{J}_+ - \widetilde{J}_-)),$$

which corresponds to a $SU(2)$ group element.

In the following we will focus on the matrix elements of the unitary operator $U(\theta)$ given in (7.17) in the basis (7.13) of irreducible representations of $\mathcal{U}_q(sl_2)$; these matrix elements will be denoted by

$$\chi^{(N)}_{n,x} = {}_N\langle\, n \,|\, U(\theta) \,|\, x \,\rangle_N, \tag{7.19}$$

where $n, x \in \{0, 1, \ldots, N\}$.

7.3 Matrix elements and self-duality

In this section, the matrix elements (7.19) of the q-rotation operators (7.17) are obtained by a direct calculation and are shown to involve the quantum q-Krawtchouk polynomials. The weight function and the orthogonality relation satisfied by these polynomials are derived from the properties of $U(\theta)$. A self-duality relation for the matrix elements is also obtained and the $q \uparrow 1$ limit is examined.

7.3.1 Matrix elements and quantum q-Krawtchouk polynomials

To obtain the explicit expression of the matrix elements (7.19) one can proceed directly by expanding the q-exponentials in (7.17) according to (7.4), (7.5) and use the actions (7.11) of the generators on the basis vectors (7.13). To perform this calculation, it is useful to note that

$$(A_- B_+)^\alpha \,|\, x\,\rangle_N = (1-q)^{-\alpha} \sqrt{\frac{(q;q)_x}{(q;q)_{x-\alpha}} \frac{(q;q)_{N-x+\alpha}}{(q;q)_{N-x}}} \,|\, x - \alpha\,\rangle_N, \tag{7.20a}$$

and

$$(A_+B_-)^\beta |\, x\,\rangle_N = (1-q)^{-\beta} \sqrt{\frac{(q;q)_{x+\beta}}{(q;q)_x} \frac{(q;q)_{N-x}}{(q;q)_{N-x-\beta}}}\, |\, x+\beta\,\rangle_N. \tag{7.20b}$$

Upon expanding the operator (7.17) according to (7.4) and (7.5), using the actions (7.20), reversing the order of the first summation and exchanging the summation order, one finds

$$U(\theta)|\, x\,\rangle_N = \sum_{n=0}^{N} \left\{ E_q(-\theta^2 q^x)\, e_q(\theta^2 q^{N-n})\, \frac{(q;q)_x (q;q)_n}{(q;q)_{N-x}(q;q)_{N-n}} \right\}^{1/2} (-1)^x (\theta)^{x+n}$$
$$\times \sum_{\gamma=0}^{x} \frac{(-1/\theta^2)^\gamma}{(q;q)_\gamma}\, q^{\binom{x-\gamma}{2}}\, \frac{(q;q)_{N-\gamma}}{(q;q)_{x-\gamma}(q;q)_{n-\gamma}}\, |\, n\,\rangle_N.$$

Upon using the identity (7.2) for the q-shifted factorials, the formulas (7.4), (7.5) and (7.3) for the q-exponentials and the q-binomial coefficients as well as the definition (7.1) for the basic hypergeometric series, one finds from the above the following expression for the matrix elements:

$$\chi_{n,x}^{(N)} = (-1)^x \theta^{n+x} q^{\binom{x}{2}} \begin{bmatrix} N \\ x \end{bmatrix}_q^{1/2} \begin{bmatrix} N \\ n \end{bmatrix}_q^{1/2} \frac{(\theta^2;q)_{N-n}^{1/2}}{(\theta^2;q)_x^{1/2}}\; {}_2\phi_1\left(\begin{matrix} q^{-n}, q^{-x} \\ q^{-N} \end{matrix} \,\middle|\, q, \frac{q^{n+1}}{\theta^2 q^N} \right). \tag{7.21}$$

The quantum q-Krawtchouk $K_n^{\text{Qtm}}(q^{-x};p,N;q)$ of degree n in the variable q^{-x} are defined by [18]

$$K_n^{\text{Qtm}}(q^{-x};p,N;q) = {}_2\phi_1\left(\begin{matrix} q^{-n}, q^{-x} \\ q^{-N} \end{matrix} \,\middle|\, q, p\, q^{n+1} \right). \tag{7.22}$$

Comparing the definition (7.22) with the formula (7.21), it follows that the matrix elements (7.19) of the unitary q-rotation operator (7.17) can be written as

$$\chi_{n,x}^{(N)} = (-1)^x \theta^{n+x} q^{\binom{x}{2}} \begin{bmatrix} N \\ n \end{bmatrix}_q^{1/2} \begin{bmatrix} N \\ x \end{bmatrix}_q^{1/2} \frac{(\theta^2;q)_{N-n}^{1/2}}{(\theta^2;q)_x^{1/2}}\; K_n^{\text{Qtm}}\left(q^{-x}; \frac{1}{\theta^2 q^N}, N; q \right). \tag{7.23}$$

This result can be compared with those of [22]. The matrix elements can be cast in the form

$$\chi_{n,x}^{(N)} = (-1)^x \sqrt{w_x^{(N)}}\; \widehat{K}_n^{\text{Qtm}}\left(q^{-x}; \frac{1}{\theta^2 q^N}, N; q \right), \tag{7.24}$$

where $w_x^{(N)}$ is a q-analog of the binomial distribution

$$w_x^{(N)} = \left[\chi_{0,x}^{(N)}\right]^2 = \begin{bmatrix} N \\ x \end{bmatrix}_q \frac{(\theta^2;q)_N}{(\theta^2;q)_x} \theta^{2x} q^{x(x-1)}, \tag{7.25}$$

and where $\widehat{K}_n^{\text{Qtm}}\left(q^{-x}; \frac{1}{\theta^2 q^N}, N; q \right)$ are the normalized quantum q-Krawtchouk polynomials

$$\widehat{K}_n^{\text{Qtm}}\left(q^{-x}; \frac{1}{\theta^2 q^N}, N; q \right) = \sqrt{\begin{bmatrix} N \\ n \end{bmatrix}_q \frac{\theta^{2n}}{(\theta^2 q^{N-n};q)_n}}\; K_n^{\text{Qtm}}\left(q^{-x}; \frac{1}{\theta^2 q^N}, N; q \right). \tag{7.26}$$

The orthonormality of the basis states (7.13) and the unitarity of the operator (7.17) directly lead to a pair of orthogonality relations for the quantum q-Krawtchouk polynomials. In fact, one has

$$
{}_N\langle\, n' \,|\, UU^{\dagger} \,|\, n \,\rangle_N = \sum_{x=0}^{N} {}_N\langle\, n \,|\, U \,|\, x \,\rangle_N \, {}_N\langle\, x \,|\, U^{\dagger} \,|\, n \,\rangle_N = \sum_{k=0}^{N} \chi_{n,x}^{(N)} \,[\chi_{n',x}^{(N)}]^{*} = \delta_{nn'},
$$

$$
{}_N\langle\, x' \,|\, U^{\dagger}U \,|\, x \,\rangle_N = \sum_{n=0}^{N} {}_N\langle\, x' \,|\, U^{\dagger} \,|\, n \,\rangle_N {}_N\langle\, n \,|\, U \,|\, x \,\rangle_N = \sum_{n=0}^{N} \chi_{n,x}^{(N)}[\chi_{n,x'}^{(N)}]^{*} = \delta_{xx'},
$$

where z^* stands for complex conjugation. Since the matrix elements are real, it follows from the above that the quantum q-Krawtchouk polynomials (7.26) satisfy the orthogonality relation

$$
\sum_{x=0}^{N} w_x^{(N)}\, K_n^{\mathrm{Qtm}}\left(q^{-x};\frac{1}{\theta^2 q^N},N;q\right) K_{n'}^{\mathrm{Qtm}}\left(q^{-x};\frac{1}{\theta^2 q^N},N;q\right) = \delta_{nn'}\frac{(q;q)_n(q;q)_{N-n}(\theta^2 q^{N-n};q)_n}{(q;q)_N\,\theta^{2n}}, \tag{7.27}
$$

with respect to the weight function (7.25) and the dual orthogonality relation

$$
\sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix}_q \frac{\theta^{2n}}{(\theta^2 q^{N-n};q)_n}\, K_n^{\mathrm{Qtm}}\left(q^{-x};\frac{1}{\theta^2 q^N},N;q\right) K_n^{\mathrm{Qtm}}\left(q^{-x'};\frac{1}{\theta^2 q^N},N;q\right) = \frac{\delta_{xx'}}{\omega_x^{(N)}}. \tag{7.28}
$$

7.3.2 Duality

The matrix elements (7.19) have a self-duality property which can be obtained as follows. Using the reality of the matrix elements (7.19) and the unitary of the q-rotation operator (7.17), one can write

$$
\chi_{n,x}^{(N)} = [{}_N\langle\, n \,|\, U \,|\, x \,\rangle_N]^{*} = {}_N\langle\, x \,|\, U^{\dagger} \,|\, n \,\rangle_N = \langle\, x,N-x \,|\; U^{-1} \;|\, n,N-n \,\rangle,
$$

where the direct product notation (7.10) was used for the last equality. It is easily seen from (7.17) that the inverse operator U^{-1} is obtained from U by permuting the algebra generators $\{A_{\pm},A_0\}$ with $\{B_{\pm},B_0\}$. In view of the definition (7.13) for the states $|\, n\,\rangle_N$, this observation leads to the duality relation

$$
\chi_{n,x}^{(N)} = \chi_{N-x,N-n}^{(N)}. \tag{7.29}
$$

The relation (7.29) allows to exchange the roles of the variable x and the degree n.

7.3.3 The $q \uparrow 1$ limit

The $q \uparrow 1$ limit can be taken in a straightforward fashion in the matrix elements (7.21) using the formula (7.1) for the basic hypergeometric series. With $\theta = \sin\tau$, one finds

$$\begin{aligned}\lim_{q\to 1} \chi_{n,x}^{(N)} &= \binom{N}{x}^{1/2}\binom{N}{n}^{1/2}(-1)^x \tan^{n+x}\tau \cos^N\tau \; {}_2F_1\left(\begin{matrix}-n,-x\\ -N\end{matrix}\;\middle|\; \frac{1}{\sin^2\tau}\right)\\ &= \binom{N}{n}^{1/2}\binom{N}{x}^{1/2}(-1)^x \tan^{n+x}\tau\cos^N\tau \, K_n(x;\sin^2\tau;N),\end{aligned}$$

where $K_n(x;p;N)$ are the standard Krawtchouk polynomials [18].

7.4 Structure relations

In this section, it is shown how the algebraic setting can be used to derive structure relations for the quantum q-Krawtchouk polynomials.

7.4.1 Backward relation

Consider the matrix element ${}_{N-1}\langle n \,|\, A_- U \,|\, x \rangle_N$. The action (7.11) gives

$$ {}_{N-1}\langle n \,|\, A_- U \,|\, x \rangle_N = \sqrt{\frac{1-q^{n+1}}{1-q}}\, \chi_{n+1,x}^{(N)}. \tag{7.30}$$

Using the unitarity of U, one has also

$$ {}_{N-1}\langle n \,|\, A_- U \,|\, x \rangle_N = {}_{N-1}\langle n \,|\, U U^\dagger A_- U \,|\, x \rangle_N. \tag{7.31}$$

To obtain a backward relation, one needs to calculate $U^\dagger A_- U$. Making use of (7.15), one has

$$\begin{aligned} U^\dagger A_- U = e_q^{1/2}(\theta^2 q^{A_0}) e_q(\theta(1-q)A_-B_+) E_q(-\theta(1-q)A_+B_-)\\ \times A_-\, e_q(\theta(1-q)A_+B_-)E_q(-\theta(1-q)A_-B_+)E_q^{1/2}(-\theta^2 q^{A_0}).\end{aligned}$$

With the help of formula (7.6), one easily finds

$$E_q(-\theta(1-q)A_+B_-)A_-\, e_q(\theta(1-q)A_+B_-) = A_- + \theta q^{A_0} B_-,$$

and thus

$$U^\dagger A_- U = e_q^{1/2}(\theta^2 q^{A_0}) e_q(\theta(1-q)A_-B_+)\left[A_- + \theta q^{A_0}B_-\right]E_q(-\theta(1-q)A_-B_+)E_q^{1/2}(-\theta^2 q^{A_0}).$$

The conjugation formula (7.7) gives

$$e_q(\theta(1-q)A_-B_+)q^{A_0}B_-E_q(-\theta(1-q)A_-B_+) = q^{A_0}B_- - \theta q^{A_0}A_-,$$

and consequently

$$U^\dagger A_- U = e_q^{1/2}(\theta^2 q^{A_0})\left[(1-\theta^2 q^{A_0})A_- + \theta q^{A_0}B_-\right]E_q^{1/2}(-\theta^2 q^{A_0}).$$

Formally, one has

$$e_q^{1/2}(\theta^2 q^{A_0})A_- E_q^{1/2}(-\theta^2 q^{A_0}) = \sqrt{\frac{1}{1-\theta^2 q^{A_0}}}A_-,$$

and thus one finally obtains

$$U^\dagger A_- U = \sqrt{1-\theta^2 q^{A_0}}\,A_- + \theta q^{A_0}B_-. \tag{7.32}$$

Upon inserting the result (7.32) in (7.31) and using the actions (7.11), one finds

$${}_{N-1}\langle n \,|\, A_- U \,|\, x\rangle_N = \sqrt{\frac{(1-q^x)(1-\theta^2 q^{x-1})}{1-q}}\,\chi_{n,x-1}^{(N-1)} + \theta q^x \sqrt{\frac{1-q^{N-x}}{1-q}}\,\chi_{n,x}^{(N-1)}.$$

Combining the above relation with (7.30), one obtains the backward relation

$$\sqrt{1-q^{n+1}}\,\chi_{n+1,x}^{(N)} = \sqrt{(1-q^x)(1-\theta^2 q^{x-1})}\,\chi_{n,x-1}^{(N-1)} + \theta\, q^x \sqrt{1-q^{N-x}}\,\chi_{n,x}^{(N-1)}. \tag{7.33}$$

Using the expression (7.24) and the formula (7.25) for the weight function, the relation (7.33) gives for the quantum q-Krawtchouk polynomials

$$\begin{aligned}(1-q^N)K_{n+1}^{\mathrm{Qtm}}\left(q^{-x};\frac{1}{\theta^2 q^N},N;q\right) &= (q^x - q^N)K_n^{\mathrm{Qtm}}\left(q^{-x};\frac{1}{\theta^2 q^{N-1}},N-1;q\right)\\ &\quad + \frac{q}{\theta^2}(1-q^{-x})(1-\theta^2 q^{x-1})K_n^{\mathrm{Qtm}}\left(q^{-(x-1)};\frac{1}{\theta^2 q^{N-1}},N-1;q\right). \end{aligned} \tag{7.34}$$

The backward relation (7.34) can be used to generate polynomials recursively and coincides with the one given in [18].

7.4.2 Forward relation

Consider the matrix element ${}_N\langle n \,|\, A_+ U \,|\, x\rangle_{N-1}$. The action (7.11) gives

$${}_N\langle n \,|\, A_+ U \,|\, x\rangle_{N-1} = \sqrt{\frac{1-q^n}{1-q}}\,\chi_{n-1,x}^{(N-1)}. \tag{7.35}$$

Taking the conjugate of (7.32), one has

$$U^\dagger A_+ U = A_+\sqrt{1-\theta^2 q^{A_0}} + \theta q^{A_0}B_+,$$

which upon using (7.11) yields

$$ {}_N\langle\, n \mid U\, U^{\dagger} A_{+} U \mid x\,\rangle_{N-1} = \sqrt{\frac{(1-q^{x+1})(1-\theta^2 q^x)}{1-q}}\; \chi^{(N)}_{n,x+1} + \theta\, q^x \sqrt{\frac{1-q^{N-x}}{1-q}}\; \chi^{(N)}_{n,x}. \tag{7.36} $$

Comparing (7.36) with (7.35), one obtains the forward relation

$$ \sqrt{1-q^n}\; \chi^{(N-1)}_{n-1,x} = \sqrt{(1-q^{x+1})(1-\theta^2 q^x)}\; \chi^{(N)}_{n,x+1} + \theta\, q^x \sqrt{1-q^{N-x}}\; \chi^{(N)}_{n,x}. \tag{7.37} $$

For the quantum q-Krawtchouk polynomials, the relation (7.37) translates into

$$ \left(\frac{1-q^n}{1-q^N}\right) K^{\mathrm{Qtm}}_{n-1}\left(q^{-x}; \frac{1}{\theta^2 q^{N-1}}, N-1; q\right) = \\ \theta^2\, q^x\, K^{\mathrm{Qtm}}_{n}\left(q^{-x}; \frac{1}{\theta^2 q^{N}}, N; q\right) - \theta^2 q^x\, K^{\mathrm{Qtm}}_{n}\left(q^{-(x+1)}; \frac{1}{\theta^2 q^{N}}, N; q\right). \tag{7.38} $$

It is verified that (7.38) corresponds to the one found in [18].

7.4.3 Dual backward and forward relations

The self-duality property (7.29) can be exploited to derive additional relations from the backward and forward relations (7.33) and (7.37) satisfied by the matrix elements. From (7.29), one finds that

$$ \sqrt{1-q^{N-x}}\; \chi^{(N)}_{n,x} = \sqrt{(1-q^{N-n})(1-\theta^2 q^{N-n-1})}\; \chi^{(N-1)}_{n,x} + \theta q^{N-n} \sqrt{1-q^n}\; \chi^{(N-1)}_{n-1,x}, $$

$$ \sqrt{1-q^{N-x}}\; \chi^{(N-1)}_{n-1,x} = \sqrt{(1-q^{N-n+1})(1-\theta^2 q^{N-n})}\; \chi^{(N)}_{n-1,x} + \theta q^{N-n} \sqrt{1-q^n}\; \chi^{(N)}_{n,x}, $$

which translate into other type of identities for the quantum q-Krawtchouk polynomials. Equivalently, one can consider matrix elements of the form ${}_N\langle\, n \mid U B_{\pm} \mid x\,\rangle_N$ and use the identities

$$ U B_{+} U^{\dagger} = B_{+}\sqrt{1-\theta^2 q^{B_0}} + \theta q^{B_0} A_{+}, \quad \text{and} \quad U B_{-} U^{\dagger} = \sqrt{1-\theta^2 q^{B_0}}\, B_{-} + \theta q^{B_0} A_{-}. \tag{7.39} $$

7.5 Generating function

In this section, two generating functions for the quantum q-Krawtchouk are derived. The first one generates the polynomials with respect to the degrees and the other with respect to the variables.

7.5.1 Generating function with respect to the degrees

Consider the matrix element ${}_N\langle\, 0 \mid V(t) U(\theta) \mid x\,\rangle_N$ where $V(t)$ is the operator

$$ V(t) = E_q(t(1-q)A_{-}B_{+}) E_q^{1/2}(-\theta^2 q^{B_0}). $$

Upon expanding the big q-exponentials according to (7.5), using the action (7.20a) and the definition (7.19) of the matrix elements of $U(\theta)$, it is directly checked that

$$ {}_N\langle 0\,|\,V(t)U(\theta)\,|\,x\,\rangle_N = \sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix}_q^{1/2} E_q^{1/2}(-\theta^2 q^{N-n})\, q^{n(n-1)/2}\, \chi_{n,x}^{(N)}\, t^n. $$

With the identity

$$ E_q(-\lambda\, q^n) = \frac{E_q(-\lambda)}{(\lambda;q)_n}, $$

and the explicit expression (7.23) for the matrix elements $\chi_{n,x}^{(N)}$, one can write

$$ {}_N\langle 0\,|\,V(t)U(\theta)\,|\,x\,\rangle_N = \frac{(-\theta)^x q^{x(x-1)/2}}{(\theta^2;q)_x^{1/2}} \begin{bmatrix} N \\ x \end{bmatrix}_q^{1/2} E_q^{1/2}(-\theta^2) \sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix}_q q^{n(n-1)/2} K_n^{\text{Qtm}}\left(q^{-x}; \frac{1}{\theta^2 q^N}, N; q\right) (\theta\, t)^n, \tag{7.40} $$

which has the form of a generating function for the quantum q-Krawtchouk polynomials. Let us compute the matrix element ${}_N\langle 0\,|\,V(t)U(\theta)\,|\,x\,\rangle_N$ in a different way. It follows from (7.16) that

$$ E_q(\gamma A_- B_+) e_q(-\delta A_+ B_-) = e_q\left(\frac{\gamma\delta}{(1-q)^2} q^{B_0}\right) e_q(-\delta A_+ B_-) E_q(\gamma A_- B_+) E_q\left(\frac{-\gamma\delta}{(1-q)^2} q^{A_0}\right), \tag{7.41} $$

With $\gamma = t(1-q)$ and $\delta = -\theta(1-q)$, the above identity gives

$$ V(t)U(\theta) = e_q(-\theta\, t q^{B_0}) e_q(\theta(1-q)A_+B_-) E_q(t(1-q)A_-B_+) E_q(\theta\, t q^{A_0}) E_q(-\theta(1-q)A_-B_+) E_q^{1/2}(-\theta^2 q^{A_0}), $$

which leads to the expression

$$ {}_N\langle 0\,|\,V(t)U(\theta)\,|\,x\,\rangle_N = e_q(-\theta\, t q^N) E_q^{1/2}(-\theta^2 q^x) \\ {}_N\langle 0\,|\,E_q(t(1-q)A_-B_+) E_q(\theta t q^{A_0}) E_q(-\theta(1-q)A_-B_+)\,|\,x\,\rangle_N. $$

Upon using the identity

$$ e_q(\lambda q^n) = e_q(\lambda)(\lambda;q)_n, $$

and the orthonormality of the states, one easily obtains

$$ {}_N\langle 0\,|\,V(t)U(\theta)\,|\,x\,\rangle_N = \frac{(-\theta)^x q^{x(x-1)/2}}{(\theta^2;q)_x^{1/2}} \begin{bmatrix} N \\ x \end{bmatrix}_q^{1/2} E_q^{1/2}(-\theta^2)\, (-\theta t;q)_N \sum_{\gamma=0}^{x} \frac{(t/\theta)^\gamma q^{\gamma(\gamma+1)/2}}{(q;q)_\gamma} \frac{(q^{-x};q)_\gamma}{(-\theta t;q)_\gamma}. \tag{7.42} $$

Comparing (7.40) with (7.42), using (7.1) and taking $z = \theta t$, one finds the following generating function for the quantum q-Krawtchouk polynomials

$$(-z;q)_N \,{}_1\phi_1\left(\begin{matrix} q^{-x} \\ -z \end{matrix}\,\middle|\, q, -\frac{qz}{\theta^2}\right) = \sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix}_q q^{n(n-1)/2} K_n^{(\mathrm{Qtm})}\left(q^{-x}; \frac{1}{\theta^2 q^N}, N; q\right) z^n. \tag{7.43}$$

Using the identity

$$(q^{-N};q)_n = \frac{(q;q)_N}{(q;q)_{N-n}} (-1)^n q^{\binom{n}{2}-Nn}, \tag{7.44}$$

defining $v = -q^N z$ and taking $p = \frac{1}{\theta^2 q^N}$, the relation (7.43) takes the form

$$(vq^{-N};q)_N \,{}_1\phi_1\left(\begin{matrix} q^{-x} \\ vq^{-N} \end{matrix}\,\middle|\, q,\, pqv\right) = \sum_{n=0}^{N} \frac{(q^{-N};q)_n}{(q;q)_n} K_n^{\mathrm{Qtm}}\left(q^{-x}; p, N; q\right) v^n. \tag{7.45}$$

The right-hand side of (7.45) corresponds to one of the generating functions given in [18]. In the latter reference however, the left-hand side is given in terms of a ${}_2\phi_1$ basic hypergeometric series. The results of [18] can be recovered as follows. Consider the identity ([9] Appendix III):

$${}_2\phi_1\left(\begin{matrix} q^{-n}, b \\ c \end{matrix}\,\middle|\, q, z\right) = \frac{(c/b;q)_n}{(c;q)_n} \,{}_3\phi_2\left(\begin{matrix} q^{-n}, bzq^{-n}/c \\ bq^{1-n}/c, 0 \end{matrix}\,\middle|\, q, q\right).$$

With $b \to \lambda b$ and $z \to z/\lambda$, taking the limit as $\lambda \to \infty$ gives the transformation formula

$${}_1\phi_1\left(\begin{matrix} q^{-n} \\ c \end{matrix}\,\middle|\, q, t\right) = \frac{1}{(c;q)_n} \,{}_2\phi_1\left(\begin{matrix} q^{-n}, tq^{-n}/c \\ 0 \end{matrix}\,\middle|\, q, cq^n\right).$$

Upon using the above identity and the relation $\frac{(aq^{-n};q)_n}{(aq^{-n};q)_k} = (aq^{k-n};q)_{n-k}$, the generating relation (7.45) becomes

$$(vq^{x-N};q)_{N-x} \,{}_2\phi_1\left(\begin{matrix} q^{-x}, pq^{N-x+1} \\ 0 \end{matrix}\,\middle|\, q, vq^{x-N}\right) = \sum_{n=0}^{N} \frac{(q^{-N};q)_n}{(q;q)_n} K_n^{\mathrm{Qtm}}\left(q^{-x}; p, N; q\right) v^n,$$

which coincides with the generating function given in [18].

7.5.2 Generating function with respect to the variables

To obtain a generating function where the sum is performed on the variables, one can consider the matrix element ${}_N\langle\, n \,|\, U(\theta)W(t) \,|\, 0 \,\rangle_x$ where

$$W(t) = e_q^{1/2}(\theta^2 q^{A_0})\, e_q(t(1-q)A_+B_-).$$

On the one hand, expanding the q-exponentials and using (7.23) yields

$$\begin{aligned} &{}_N\langle\, n \,|\, U(\theta)W(t) \,|\, 0 \,\rangle_N \\ &\quad = e_q^{1/2}(\theta^2 q^{N-n})\, \theta^n \begin{bmatrix} N \\ n \end{bmatrix}_q^{1/2} \sum_{x=0}^{N} \begin{bmatrix} N \\ x \end{bmatrix}_q (-\theta t)^x q^{x(x-1)/2} K_n^{\mathrm{Qtm}}\left(q^{-x}; \frac{1}{\theta^2 q^N}, N; q\right), \end{aligned} \tag{7.46}$$

which has the form of a generating function. On the other hand, the identity (7.41) gives

$$ {}_N\langle\, n\,|\,U(\theta)W(t)\,|\,x\,\rangle_N = \\ e_q^{1/2}(\theta^2 q^{N-n})\,E_q(-\theta t)\,{}_N\langle\, n\,|\,e_q(\theta(1-q)A_+B_-)\,e_q(\theta t q^{B_0})\,e_q(t(1-q)A_+B_-)\,|\,0\,\rangle_N. $$

With the identity (7.2), one directly finds from the above

$$ {}_N\langle\, n\,|\,U(\theta)W(t)\,|\,0\,\rangle_x = e_q^{1/2}(\theta^2 q^{N-n})\,\theta^n \begin{bmatrix} N \\ n \end{bmatrix}_q^{1/2} (\theta t;q)_N \sum_{\gamma=0}^{N} \frac{\left(\frac{q^{n-N+1}}{\theta^2}\right)^{\gamma}}{(q;q)_\gamma} \frac{(q^{-n};q)_\gamma}{(\frac{q^{1-N}}{\theta t};q)_\gamma}. \tag{7.47} $$

Upon comparing (7.47) with (7.46) and taking $z=-\theta t$, one obtains the generating relation

$$ (-z;q)_N\, {}_2\phi_1\!\left(\begin{matrix} q^{-n},0 \\ -\frac{q^{1-N}}{z} \end{matrix}\,\middle|\, q,\frac{q^{n+1}}{\theta^2 q^N}\right) = \sum_{x=0}^{N} \begin{bmatrix} N \\ x \end{bmatrix}_q q^{x(x-1)/2}\, K_n^{\mathrm{Qtm}}\!\left(q^{-x};\frac{1}{\theta^2 q^N},N;q\right) z^x. \tag{7.48} $$

Using (7.44), defining $w=-q^N z$ and taking $p=\frac{1}{\theta^2 q^N}$, one writes (7.48) as

$$ (wq^{-N};q)_N\, {}_2\phi_1\!\left(\begin{matrix} q^{-n},0 \\ \frac{q}{w} \end{matrix}\,\middle|\, q,p\,q^{n+1}\right) = \sum_{x=0}^{N} \frac{(q^{-N};q)_x}{(q;q)_x}\, K_n^{\mathrm{Qtm}}(q^{-x};p,N;q)\, w^x. \tag{7.49} $$

7.6 Recurrence relation and difference equation

In this section, the recurrence relation and the difference equation satisfied by the matrix elements $\chi_{n,x}^{(N)}$ of the unitary q-rotation operators $U(\theta)$ are obtained and the corresponding relations for the quantum q-Krawtchouk polynomials are recovered.

7.6.1 Recurrence relation

To obtain a recurrence relation for the matrix elements $\chi_{n,x}^{(N)}$, one may consider the matrix element ${}_N\langle\, n\,|\,UB_+B_-\,|\,x\,\rangle_N$. On the one hand, one has

$$ {}_N\langle\, n\,|\,UB_+B_-\,|\,x\,\rangle_N = \left(\frac{1-q^{N-x}}{1-q}\right)\chi_{n,x}^{(N)}. \tag{7.50} $$

On the other hand, the conjugation identities (7.39) give

$$ UB_+B_-U^\dagger = \\ \left(B_+(1-\theta^2 q^{B_0})B_- + \theta B_+\sqrt{1-\theta^2 q^{B_0}}\,q^{B_0}A_- + \theta q^{B_0}A_+\sqrt{1-\theta^2 q^{B_0}}\,B_- + \theta^2 q^{2B_0}A_+A_-\right), $$

and thus

$$
\begin{aligned}
{}_N\langle\, n \mid U B_+ B_- \mid x\,\rangle_N &= \left(\frac{(1-q^{N-n})(1-\theta^2 q^{N-n-1})}{1-q}\right)\chi^{(N)}_{n,x} \\
&+ \theta\, q^{N-n-1}\sqrt{\frac{(1-q^{n+1})(1-q^{N-n})(1-\theta^2 q^{N-n-1})}{(1-q)^2}}\;\chi^{(N)}_{n+1,x} \\
&\qquad + \theta q^{N-n}\sqrt{\frac{(1-q^{n})(1-\theta^2 q^{N-n})(1-q^{N-n+1})}{(1-q)^2}}\;\chi^{(N)}_{n-1,x} + \theta^2 q^{2(N-n)}\left(\frac{1-q^n}{1-q}\right)\chi^{(N)}_{n,x}. \qquad (7.51)
\end{aligned}
$$

Comparing (7.51) and (7.50), one finds that the matrix elements satisfy the recurrence relation

$$
\begin{aligned}
(1-q^{N-x})\,\chi^{(N)}_{n,x} &= (1-q^{N-n})(1-\theta^2 q^{N-n-1})\,\chi^{(N)}_{n,x} \\
&+ \theta q^{N-n-1}\sqrt{(1-q^{n+1})(1-q^{N-n})(1-\theta^2 q^{N-n-1})}\;\chi^{(N)}_{n+1,x} \\
&\qquad + \theta q^{N-n}\sqrt{(1-q^{n})(1-\theta^2 q^{N-n})(1-q^{N-n+1})}\;\chi^{(N)}_{n-1,x} + \theta^2 q^{2(N-n)}(1-q^n)\,\chi^{(N)}_{n,x}. \qquad (7.52)
\end{aligned}
$$

Using the expression (7.23), one finds that the recurrence relation for the quantum q-Krawtchouk polynomials is of the form

$$
\begin{aligned}
(1-q^{N-x})\,K^{\mathrm{Qtm}}_{n}\left(q^{-x};\frac{1}{\theta^2 q^N},N;q\right) &= \theta^2 q^{N-n-1}(1-q^{N-n})\,K^{\mathrm{Qtm}}_{n+1}\left(q^{-x};\frac{1}{\theta^2 q^N},N;q\right) \\
&+ \left[(1-\theta^2 q^{N-n-1})(1-q^{N-n}) - \theta^2 q^{2(N-n)}(1-q^n)\right] K^{\mathrm{Qtm}}_{n}\left(q^{-x};\frac{1}{\theta^2 q^N},N;q\right) \\
&\qquad + q^{N-n}(1-\theta^2 q^{N-n})(1-q^n)\,K^{\mathrm{Qtm}}_{n-1}\left(q^{-x};\frac{1}{\theta^2 q^N},N;q\right). \qquad (7.53)
\end{aligned}
$$

It can be checked that the recurrence relation (7.53) coincides with the one given in [18].

7.6.2 Difference equation

To obtain the difference equation satisfied by the matrix elements $\chi^{(N)}_{n,x}$ and consequently by the quantum q-Krawtchouk polynomials, one could consider the matrix element ${}_N\langle\, n \mid A_+ A_- U \mid x\,\rangle_N$ and use the conjugation identities (7.6) and (7.7) to compute $U^{\dagger} A_+ A_- U$. Alternatively, one can start from the recurrence relation (7.52) and use the duality relation (7.29). Applying the duality on (7.52), one finds

$$
\begin{aligned}
(1-q^{N-x})\,\chi^{(N)}_{N-x,N-n} &= (1-q^{N-n})(1-\theta^2 q^{N-n-1})\,\chi^{(N)}_{N-x,N-n} \\
&+ \theta q^{N-n-1}\sqrt{(1-q^{n+1})(1-q^{N-n})(1-\theta^2 q^{N-n-1})}\;\chi^{(N)}_{N-x,N-n-1} \\
&\qquad + \theta q^{N-n}\sqrt{(1-q^{n})(1-\theta^2 q^{N-n})(1-q^{N-n+1})}\;\chi^{(N)}_{N-x,N-n+1} + \theta^2 q^{2(N-n)}(1-q^n)\,\chi^{(N)}_{N-x,N-n}.
\end{aligned}
$$

Upon taking $x \to N-n$ and $n \to N-x$, one obtains the following difference equation for the matrix elements $\chi_{n,x}^{(N)}$:

$$
\begin{aligned}
(1-q^n)\,\chi_{n,x}^{(N)} &= (1-q^x)(1-\theta^2 q^{x-1})\,\chi_{n,x}^{(N)} \\
&+\theta q^{x-1}\sqrt{(1-q^{N-x+1})(1-q^x)(1-\theta^2 q^{x-1})}\,\chi_{n,x-1}^{(N)} \\
&\qquad +\theta q^x\sqrt{(1-q^{N-x})(1-\theta^2 q^x)(1-q^{x+1})}\,\chi_{n,x+1}^{(N)}+\theta^2 q^{2x}(1-q^{N-x})\,\chi_{n,x}^{(N)}.
\end{aligned}
\tag{7.54}
$$

Using the expression (7.23), the relation (7.54) gives

$$
\begin{aligned}
(1-q^n)\,K_n^{\mathrm{Qtm}}\left(q^{-x};\frac{1}{\theta^2 q^N},N;q\right) &= (1-q^x)(1-\theta^2 q^{x-1})\,K_n^{\mathrm{Qtm}}\left(q^{-x};\frac{1}{\theta^2 q^N},N;q\right) \\
&-(1-q^x)(1-\theta^2 q^{x-1})\,K_n^{\mathrm{Qtm}}\left(q^{-(x-1)};\frac{1}{\theta^2 q^N},N;q\right) \\
&\quad -\theta^2 q^{2x}(1-q^{N-x})\,K_n^{\mathrm{Qtm}}\left(q^{-(x+1)};\frac{1}{\theta^2 q^N},N;q\right)+\theta^2 q^{2x}(1-q^{N-x})\,K_n^{\mathrm{Qtm}}\left(q^{-x};\frac{1}{\theta^2 q^N},N;q\right),
\end{aligned}
\tag{7.55}
$$

which can be seen to coincide with the one given in [18].

7.7 Duality relation with affine q-Krawtchouk polynomials

In the preceding sections, the properties of the matrix elements $\chi_{n,x}^{(N)}$ of the unitary q-operator (7.17) have been derived algebraically. Through the explicit expression (7.23) of the matrix elements in terms of the quantum q-Krawtchouk polynomials, the properties of these polynomials have been obtained. It is possible to express the matrix elements $\chi_{n,x}^{(N)}$ in terms of another family of orthogonal functions: the affine q-Krawtchouk polynomials. These polynomials are defined as [18]

$$
K_n^{\mathrm{Aff}}(q^{-x};p,N;q) = {}_3\phi_2\left(\begin{matrix} q^{-n},0,q^{-x} \\ pq,q^{-N}\end{matrix}\,\middle|\,q,q\right) = \frac{(-pq)^n q^{\binom{n}{2}}}{(pq;q)_n}\,{}_2\phi_1\left(\begin{matrix} q^{-n},q^{x-N} \\ q^{-N}\end{matrix}\,\middle|\,q,\frac{q^{-x}}{p}\right). \tag{7.56}
$$

By inspection of the hypergeometric formula (7.21) for the matrix elements, it is easily seen comparing with (7.56) that they can be written as

$$
\chi_{n,x}^{(N)} = \theta^{n-x}\begin{bmatrix} N \\ x\end{bmatrix}_q^{1/2}\begin{bmatrix} N \\ n\end{bmatrix}_q^{1/2}(\theta^2;q)_x^{1/2}(\theta^2;q)_{N-n}^{1/2}\,K_x^{\mathrm{Aff}}\left(q^{-(N-n)};\frac{\theta^2}{q},N;q\right). \tag{7.57}
$$

Note that here x appears as the degree. The properties of the affine q-Krawtchouk polynomials can be obtained from those of the matrix elements. For example, with the help of the identification (7.57), the generating relation (7.48) gives a generating function for the affine q-Krawtchouk polynomials

$$(-p\,q\,q^{-N}v;q)_N\,{}_2\phi_1\left(\begin{matrix}q^{-x},0\\ -\frac{1}{pv}\end{matrix}\Big|\,q,\frac{q^{x-N}}{p}\right)$$
$$=\sum_{n=0}^{N}\frac{(q^{-N};q)_n(p\,q;q)_n}{(q;q)_n}q^{-\binom{n}{2}}\,K_n^{\text{Aff}}(q^{-(N-x)};p,N;q)\,v^n. \quad (7.58)$$

The right-hand side of (7.58) corresponds to one of the generating functions for the affine q-Krawtchouk polynomials given in [18]. However in the latter, the left-hand side is expressed in terms of a ${}_2\phi_0$ basic hypergeometric series. The two expressions can be reconciled as follows. Consider the transformation identity

$${}_2\phi_1\left(\begin{matrix}q^{-n},b\\ c\end{matrix}\Big|\,q,z\right)=\frac{b^n(c/b;q)_n}{(c;q)_n}\,{}_3\phi_1\left(\begin{matrix}q^{-n},b,q/z\\ bq^{1-n}/c\end{matrix}\Big|\,q,\frac{z}{c}\right),$$

given in Appendix III of [9]. Taking the limit as $b\to 0$ in the above, one finds

$${}_2\phi_1\left(\begin{matrix}q^{-n},0\\ c\end{matrix}\Big|\,q,z\right)=\frac{(-c)^nq^{\binom{n}{2}}}{(c;q)_n}\,{}_2\phi_0\left(\begin{matrix}q^{-n},q/z\\ -\end{matrix}\Big|\,q,\frac{z}{c}\right).$$

Using the above relation in the left-hand side of (7.58), one easily finds

$$(-pvq^{1-N};q)_{N-x}\,{}_2\phi_0\left(\begin{matrix}q^{-x},pq^{N-x+1}\\ -\end{matrix}\Big|\,q,-vq^{-(N-x)}\right)$$
$$=\sum_{n=0}^{N}\frac{(q^{-N};q)_n(p\,q;q)_n}{(q;q)_n}q^{-\binom{n}{2}}\,K_n^{\text{Aff}}(q^{-(N-x)};p,N;q)\,v^n,$$

which coincides with the generating function given in [18]. The expression (7.57) also implies a duality relation between the affine and the quantum q-Krawtchouk polynomials. Indeed, comparing (7.57) with (7.23), it follows that

$$K_n^{\text{Qtm}}\left(q^{-x};\frac{1}{\theta^2q^N},N;q\right)=\frac{(-1)^x(\theta^2;q)_x}{\theta^{2x}q^{\binom{x}{2}}}\,K_x^{\text{Aff}}\left(q^{-(N-n)};\frac{\theta^2}{q},N;q\right). \quad (7.59)$$

The relation (7.59), given in [1], can be also be obtained straightforwardly by comparing the defining expressions (7.22) and (7.56) of the quantum and affine q-Krawtchouk polynomials. Since the affine q-Krawtchouk polynomials are self-dual, i.e. $K_n^{\text{Aff}}(q^{-x};p,N;q)=K_x^{\text{Aff}}(q^{-n};p,N;q)$ for $x,n\in\{0,1,\ldots,N\}$, one can rewrite (7.59) as

$$\frac{(-1)^x(\theta^2;q)_x}{\theta^{2x}q^{\binom{x}{2}}}\,K_{N-n}^{\text{Aff}}\left(q^{-x};\frac{\theta^2}{q},N;q\right)=\frac{K_{N-n}^{\text{Aff}}\left(q^{-x};\frac{\theta^2}{q},N;q\right)}{K_N^{\text{Aff}}\left(q^{-x};\frac{\theta^2}{q},N;q\right)}=K_n^{\text{Qtm}}\left(q^{-x};\frac{1}{\theta^2q^N},N;q\right), \quad (7.60)$$

an identity which is also found in [21]. There is another relation between the quantum and the affine q-Krawtchouk polynomials involving the transformation $q\to q^{-1}$. This relation is of the form [21]

$$\frac{K_n^{\text{Qtm}}\left(q^{-x};p,N;q^{-1}\right)}{K_n^{\text{Qtm}}\left(q^{-N};p,N;q^{-1}\right)}=K_n^{\text{Aff}}\left(q^{x-N};p^{-1},N;q\right). \quad (7.61)$$

In view of (7.61), one could also take $q\to q^{-1}$ in every formula to have the matrix elements of the q^{-1}-rotation operator (7.17) in terms of the affine q-Krawtchouk polynomials.

7.8 Conclusion

In this paper, it was shown that the quantum q-Krawtchouk polynomials arise as the matrix elements of unitary q-rotation operators expressed as q-exponentials in the $\mathscr{U}_q(sl_2)$ generators in the Schwinger realization. This algebraic interpretation was used to provide a full characterization of these orthogonal functions, as well as of the affine q-Krawtchouk polynomials. We now plan to use the results obtained in this paper to arrive at an algebraic characterization of the multivariate quantum (and affine) q-Krawtchouk polynomials introduced by Gasper and Rahman [10].

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Chapitre 8

Spin lattices, state transfer and bivariate Krawtchouk polynomials

V. X. Genest, H. Miki, L. Vinet et A. Zhedanov (2015). Spin lattices, state transfer and bivariate Krawtchouk polynomials. *Canadian Journal of Physics*.

Abstract. The quantum state transfer properties of a class of two-dimensional spin lattices on a triangular domain are investigated. Systems for which the 1-excitation dynamics is exactly solvable are identified. The exact solutions are expressed in terms of the bivariate Krawtchouk polynomials that arise as matrix elements of the unitary representations of the rotation group on the states of the three-dimensional harmonic oscillator.

8.1 Introduction

The transfer of quantum states between distant locations is an important task in quantum information processing [2, 13]. To perform this task, one needs to design quantum devices that effect this transfer, i.e. devices such that an input state at one location is produced as output state at another location. A desirable property is that the transfer be realized with a high fidelity. When the input state is recovered with probability 1, one has perfect state transfer (PST). One idea to attain perfect state transfer is to exploit the intrinsic dynamics of quantum systems so as to minimize the need of external controls and reduce noise.

Dynamical PST can for instance be achieved using one-dimensional spin chains [1]. In the simplest examples, one considers chains consisting of $N+1$ spins with states

$$|1\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \quad |0\rangle = \begin{pmatrix}0\\1\end{pmatrix},$$

and nearest-neighbor non-homogeneous couplings. These spin chains are governed by Hamiltonians of the form

$$H = \sum_{i=0}^{N} \left[\frac{J_{i+1}}{2} \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y \right) + \frac{B_i}{2} \left(\sigma_i^z + 1 \right) \right], \tag{8.1}$$

where σ_i^x, σ_i^y and σ_i^z are the Pauli matrices

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

acting on the spin located at the site i, where $i \in \{0, \ldots, N\}$. The coefficients J_i are the coupling strengths between nearest neighbor sites and B_i is the magnetic field strength at the site i. The state $|\, 0, \ldots, 0 \,\rangle = |\, 0 \,\rangle^{\otimes(N+1)}$ is the ground state of H with

$$H|\, 0 \,\rangle^{\otimes(N+1)} = 0.$$

The transfer properties of the chain defined by (8.1) are exhibited as follows. Introduce the unknown state $|\, \psi \,\rangle = \alpha |\, 0 \,\rangle + \beta |\, 1 \,\rangle$ at the site $i = 0$. One would like to recuperate $|\, \psi \,\rangle$ on the last site $i = N$ after some time. Since the component $|\, 0 \,\rangle^{\otimes(N+1)}$ is stationary, this amounts to finding the transition probability from the state $|\, 1 \,\rangle \otimes |\, 0 \,\rangle^{\otimes N}$ to the state $|\, 0 \,\rangle^{\otimes N} \otimes |\, 1 \,\rangle$. Thus, one only needs to consider the states with a single excitation; this can be done since the dynamics preserve the number of excitations. Perfect state transfer will be effected by the spin chains (8.1) if there is a finite time T such that

$$U(T)|\, 1 \,\rangle \otimes |\, 0 \,\rangle^{\otimes N} = e^{i\phi} |\, 0 \,\rangle^{\otimes N} \otimes |\, 1 \,\rangle,$$

where $U(T) = e^{-iH}$. This is found to happen under appropriate choices of J_i and B_i [3, 18, 19].

Here we shall be concerned with the study of state transfer in two dimensions. We shall consider two-dimensional spin lattices with non-homogeneous nearest-neighbor couplings on a triangular domain and identify the systems for which the 1-excitation dynamics is exactly solvable and exhibits interesting quantum state transfer properties. This study will take us to introduce and characterize orthogonal polynomials in two discrete variables by looking at matrix elements of reducible representations of $O(3)$ on the states of the three-dimensional harmonic oscillator. These polynomials will be identified with the bivariate Krawtchouk polynomials [7] .

The outline of the paper is as follows. In section 2, the two-dimensional spin lattices are introduced and their 1-excitation dynamics is discussed. In section 3, the connection between representations of the rotation group on oscillator states and bivariate Krawtchouk polynomials is made explicit. In section 4, the recurrence relations of the bivariate Krawtchouk polynomials are derived and are shown to provide exact solutions of the 1-excitation dynamics of a particular class of spin lattices. In section 5, the generating function of the bivariate Krawtchouk polynomials is derived and is used to study the transfer properties of the spin lattices. A short conclusion follows.

8.2 Triangular spin lattices and one-excitation dynamics

We consider a uniform two-dimensional lattice on a triangular domain [16, 15]. The vertices of the

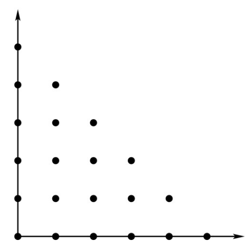

Figure 8.1: Uniform two-dimensional lattice of triangular shape

lattice are labeled by the non-negative integers (i,j) such that $i,j \in \{0,\dots,N\}$ with $i+j \le N$, where N is also a non-negative integer. On each of the $(N+1)(N+2)/2$ sites of the lattice, there is a spin coupled to its nearest neighbors and to a local magnetic field. The Hamiltonian of the system is of the form

$$\mathcal{H} = \sum_{\substack{0\le i,j\le N \\ i+j\le N}} \Bigg[\frac{I_{i+1,j}}{2}\left(\sigma^x_{i,j}\sigma^x_{i+1,j} + \sigma^y_{i,j}\sigma^y_{i+1,j}\right) + \frac{J_{i,j+1}}{2}\left(\sigma^x_{i,j}\sigma^x_{i,j+1} + \sigma^y_{i,j}\sigma^y_{i,j+1}\right) + \frac{B_{i,j}}{2}\left(\sigma^z_{i,j}+1\right)\Bigg], \quad (8.2)$$

where

$$I_{0,j} = J_{i,0} = 0 \text{ and } I_{i,j} = J_{i,j} = 0 \text{ if } i+j > N.$$

The coefficients $I_{i,j}$ and $J_{i,j}$ are the coupling strengths between the sites $(i-1,j)$ and (i,j) and between the sites $(i,j-1)$ and (i,j), respectively. The total number of spins that are up (in state $|\,1\,\rangle$) over the lattice is a conserved quantity. Indeed, it is directly verified that

$$[\mathcal{H}, \sum_{\substack{i,j \\ i+j\le N}} \sigma^z_{i,j}] = 0.$$

Consequently, one can restrict the analysis of the Hamiltonian (8.2) to the 1-excitation sector. A natural basis for the states of the lattice with only one spin up is provided by the vectors $|\,i,j\,\rangle$ labeled by the coordinates (i,j) of the site where the spin up is located. One has thus

$$|\,i,j\,\rangle = E_{i,j}, \quad i,j = 0,\dots,N,$$

where $E_{i,j}$ is the $(N+1)\times(N+1)$ matrix that has a 1 in the (i,j) entry and zeros everywhere else. The 1-excitation eigenstates of $\mathscr{H}$ are denoted by $|\, x_{s,t}\,\rangle$ and are defined by

$$\mathscr{H}|\, x_{s,t}\,\rangle = x_{s,t}|\, x_{s,t}\,\rangle, \tag{8.3}$$

where $x_{s,t}$ is the energy eigenvalue. The expansion of the states $|\, x_{s,t}\,\rangle$ in the $|\, i,j\,\rangle$ basis is written as

$$|\, x_{s,t}\,\rangle = \sum_{\substack{0\leq i,j\leq N\\ i+j\leq N}} M_{i,j}(s,t)|\, i,j\,\rangle.$$

Since both bases $|\, x_{s,t}\,\rangle$ and $|\, i,j\,\rangle$ are orthonormal, the transition matrix $M_{i,j}(s,t)$ is unitary. The energy eigenvalue equation (8.3) imposes that the expansion coefficients $M_{i,j}(s,t)$ satisfy the 5-term recurrence relation

$$\begin{aligned} x_{s,t}M_{i,j}(s,t) = I_{i+1,j}M_{i+1,j}(s,t) + J_{i,j+1}M_{i,j+1}(s,t)\\ + B_{i,j}M_{i,j}(s,t) + I_{i,j}M_{i-1,j}(s,t) + J_{i,j}M_{i,j-1}(s,t). \end{aligned} \tag{8.4}$$

In the following, we shall identity systems specified by the coupling strengths $I_{i,j}$, $J_{i,j}$, and $B_{i,j}$ for which the spectrum $x_{s,t}$ and coefficients $M_{i,j}(s,t)$ can be exactly determined.

8.3 Representations of $O(3)$ on oscillator states and orthogonal polynomials

Consider the eigenstates

$$|\, n_1,n_2,n_3\,\rangle = |\, n_1\,\rangle \otimes |\, n_2\,\rangle \otimes |\, n_3\,\rangle, \quad n_1,n_2,n_3 = 0,1,\ldots,$$

of the three-dimensional isotropic oscillator Hamiltonian

$$H_{\text{osc}} = a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3,$$

with $H_{\text{osc}}|\, n_1,n_2,n_3\,\rangle = N|\, n_1,n_2,n_3\,\rangle$ where the eigenvalue is $N = n_1+n_2+n_3$. Recall that

$$a_i|\, n_i\,\rangle = \sqrt{n_i}|\, n_i\,\rangle, \quad a_i^\dagger|\, n_i\,\rangle = \sqrt{n_i+1}|\, n_i+1\,\rangle.$$

Consider $R \in O(3)$, a rotation matrix. Define $U(R)$ the unitary representation of $O(3)$ by

$$\begin{aligned} U(R)a_iU^\dagger(R) &= \sum_{k=1}^{3} R_{ki}a_k,\\ U(R)a_i^\dagger U^\dagger(R) &= \sum_{k=1}^{3} R_{ki}a_k^\dagger. \end{aligned} \tag{8.5}$$

It is directly seen from the above that $U(RS) = U(R)U(S)$ for R and S in $O(3)$, as should be for a representation. Furthermore, one has $U(R)U^{\dagger}(R) = U^{\dagger}(R)U(R) = 1$. The oscillator Hamiltonian H_{osc} is obviously invariant under rotations, i.e.

$$U(R)H_{\text{osc}}U^{\dagger}(R) = H_{\text{osc}},$$

and thus any rotation stabilizes the energy eigenspaces of H_{osc}. The basis vectors for the eigen-subspaces of H_{osc} with a fixed value of the energy N denoted by

$$|\, i,j\,\rangle_N = |\, i,j,N-i-j\,\rangle,$$

transform reducibly among themselves under the action of the rotations. Consider the matrix elements of $U(R)$ in the basis $\{|\, i,j\,\rangle_N \mid i,j = 0,\ldots,N; i+j \le N\}$. These matrix elements can be cast in the form

$$_N\langle\, s,t \,|\, U(R) \,|\, i,j\,\rangle_N = W_{s,t;N} P_{i,j}(s,t;N), \tag{8.6}$$

where $P_{0,0}(s,t;N) \equiv 1$ and $W_{s,t;N} = {}_N\langle\, s,t \,|\, U(R) \,|\, 0,0\,\rangle_N$. When no confusion can arise, we shall drop the explicit dependence of $U(R)$ on R to ease the notation.

8.3.1 Calculation of $W_{s,t;N}$

Let us first calculate the amplitude $W_{s,t;N}$. To that end, consider the matrix element

$$_{N-1}\langle\, s,t \,|\, Ua_1 \,|\, 0,0\,\rangle_N$$

One has on the one hand

$$_{N-1}\langle\, s,t \,|\, Ua_1 \,|\, 0,0\,\rangle_N = 0.$$

On the other hand, one can write

$$\begin{aligned}
_{N-1}\langle\, s,t \,|\, Ua_1 \,|\, 0,0\,\rangle_N &= {}_{N-1}\langle\, s,t \,|\, Ua_1U^{\dagger}U \,|\, 0,0\,\rangle_N \\
&= R_{11}\sqrt{s+1}\,{}_N\langle\, s+1,t \,|\, U \,|\, 0,0\,\rangle_N + R_{21}\sqrt{t+1}\,{}_N\langle\, s,t+1 \,|\, U \,|\, 0,0\,\rangle_N \\
&\qquad + R_{31}\sqrt{N-s-t}\,{}_N\langle\, s,t \,|\, U \,|\, 0,0\,\rangle_N.
\end{aligned}$$

Combining the two equations above, one obtains

$$R_{11}\sqrt{s+1}\,W_{s+1,t;N} + R_{21}\sqrt{t+1}\,W_{s,t+1;N} + R_{31}\sqrt{N-s-t}\,W_{s,t;N} = 0.$$

Similarly, using $_{N-1}\langle\, s,t \,|\, U(R)a_2 \,|\, 0,0\,\rangle_N = 0$, one finds

$$R_{12}\sqrt{s+1}\,W_{s+1,t;N} + R_{22}\sqrt{t+1}\,W_{s,t+1;N} + R_{32}\sqrt{N-s-t}\,W_{s,t;N} = 0.$$

Recalling that $\sum_{k=1}^{3} R_{ks} R_{kt} = \delta_{st}$, i.e. that $W_{s,t;N}$ is "essentially orthogonal" to the 1^{st} and 2^{nd} column of R, one obtains

$$W_{s,t;N} = C \frac{R_{13}^{s} R_{23}^{t} R_{33}^{N-s-t}}{\sqrt{s!t!(N-s-t)!}}.$$

The constant C can be found from the normalization condition

$$\begin{aligned} 1 &= {}_N\langle\, 0,0 \,|\, U^{\dagger} U \,|\, 0,0 \,\rangle_N \\ &= \sum_{s+t \leq N} {}_N\langle\, 0,0 \,|\, U^{\dagger} \,|\, s,t \,\rangle_N {}_N\langle\, s,t \,|\, U \,|\, 0,0 \,\rangle_N \\ &= \sum_{s+t \leq N} |W_{s,t;N}|^2, \end{aligned}$$

and the trinomial theorem

$$(x+y+z)^N = \sum_{i+j \leq N} \frac{N!}{i!j!(N-i-j)!} x^i y^j z^{N-i-j},$$

giving $C = \sqrt{N!}$ and thus

$$W_{s,t;N} = \binom{N}{s,t}^{1/2} R_{13}^{s} R_{23}^{t} R_{33}^{N-s-t}, \tag{8.7}$$

where

$$\binom{N}{s,t} = \frac{N!}{s!t!(N-s-t)!}.$$

8.3.2 Raising relations

One can show that the functions $P_{i,j}(s,t;N)$ appearing in the matrix elements (8.6) are polynomials of the discrete variables s and t. One can write

$${}_N\langle\, s,t \,|\, U a_1^{\dagger} \,|\, i,j \,\rangle_{N-1} = \sqrt{i+1}\, W_{s,t;N}\, P_{i+1,j}(s;t,N),$$

and also

$${}_N\langle\, s,t \,|\, U a_1^{\dagger} \,|\, i,j \,\rangle_{N-1} = {}_N\langle\, s,t \,|\, U a_1^{\dagger} U^{\dagger} U \,|\, i,j \,\rangle_{N-1} = \sum_{\ell=1}^{3} R_{\ell,1}\, {}_N\langle\, s,t \,|\, a_{\ell}^{\dagger} U \,|\, i,j \,\rangle_{N-1}.$$

Using (8.6) and (8.7), the two equations above yield

$$\begin{aligned} \sqrt{N(i+1)}\, P_{i+1,j}(s,t;N) = \frac{R_{11}}{R_{13}}\, s P_{i,j}(s-1,t;N-1) \\ + \frac{R_{21}}{R_{23}}\, t P_{i,j}(s,t-1;N-1) + \frac{R_{31}}{R_{33}}\, (N-s-t) P_{i,j}(s,t;N-1). \end{aligned}$$

A similar relation is obtained starting instead from the matrix element ${}_N\langle s,t\,|\,Ua_2^\dagger\,|\,i,j\,\rangle_{N-1}$:

$$\sqrt{N(j+1)}\,P_{i,j+1}(s,t;N)=\frac{R_{12}}{R_{13}}\,s\,P_{i,j}(s-1,t;N-1)\\
+\frac{R_{22}}{R_{23}}\,t\,P_{i,j}(s,t-1;N-1)+\frac{R_{32}}{R_{33}}\,(N-s-t)P_{i,j}(s,t;N-1).$$

The two equations above show that the functions $P_{i,j}(s,t;N)$ are polynomials of total degree $i+j$ in the two variables s,t. Indeed, they allow to construct the $P_{i,j}(s,t;N)$ step by step from $P_{0,0}=1$ by iterations that only involve multiplications by the variables s and t.

8.3.3 Orthogonality relation

The fact that the polynomials $P_{i,j}(s,t;N)$ are orthogonal follows from the unitarity of the representation $U(R)$ and from the fact that the states $|\,i,j\,\rangle_N$ are orthonormal. The relation

$$\begin{aligned}&{}_N\langle\,i',j'\,|\,U^\dagger U\,|\,i,j\,\rangle_N\\
&=\sum_{s+t\le N}{}_N\langle\,i',j'\,|\,U^\dagger\,|\,s,t\,\rangle_N\,{}_N\langle\,s,t\,|\,U\,|\,i,j\,\rangle_N=\delta_{ii'}\delta_{jj'},\end{aligned}$$

translates into

$$\sum_{\substack{0\le s,t\le N\\ s+t\le N}}\omega_{s,t;N}\,P_{i,j}(s,t;N)P_{i',j'}(s,t;N)=\delta_{ii'}\delta_{jj'}.$$

Thus the $P_{i,j}(s,t;N)$ are polynomials of two discrete variables that are orthogonal on the finite grid $s+t\le N$ with respect to the trinomial distribution

$$\omega_{s,t;N}=W^2_{s,t;N}=\binom{N}{s,t}R_{13}^{2s}R_{23}^{2t}R_{33}^{2(N-s-t)}.$$

They provide a two-variable generalization of the one-variable Krawtchouk polynomials which are orthogonal with respect to the binomial distribution [14, 11, 10, 9, 4, 12].

8.4 Recurrence relations and exact solutions of 1-excitation dynamics

We shall now derive the recurrence relations satisfied by the polynomials $P_{i,j}(s,t;N)$ and compare them with (8.4). Consider the matrix element ${}_N\langle\,s,t\,|\,a_1^\dagger a_1U\,|\,i,j\,\rangle_N$. One has

$${}_N\langle\,s,t\,|\,a_1^\dagger a_1U\,|\,i,j\,\rangle_N=s\,{}_N\langle\,s,t\,|\,U\,|\,i,j\,\rangle_N.$$

Using (8.5), one has also

$$_N\langle s,t\,|\,a_1^\dagger a_1 U\,|\,i,j\rangle_N = \sum_{m,n=1}^{3} R_{1m}R_{1n}\;{}_N\langle s,t\,|\,Ua_m^\dagger a_n\,|\,i,j\rangle_N.$$

Equating the RHS of the two above equations and using the expression (8.6) for the matrix elements, one finds

$$\begin{aligned}
&sP_{i,j}(s,t;N) = \left[R_{11}^2\, i + R_{12}^2\, j + R_{13}^2(N-i-j)\right]P_{i,j}(s,t;N)\\
&+R_{11}R_{13}\Big[\alpha_{i+1,j}P_{i+1,j}(s,t;N) + \alpha_{i,j}P_{i-1,j}(s,t;N)\Big]\\
&+R_{12}R_{13}\Big[\beta_{i,j+1}P_{i,j+1}(s,t;N) + \beta_{i,j}P_{i,j-1}(s,t;N)\Big]\\
&+R_{11}R_{12}\Big[\gamma_{i,j+1}P_{i-1,j+1}(s,t;N) + \gamma_{i+1,j}P_{i+1,j-1}(s,t;N)\Big],
\end{aligned} \tag{8.8}$$

where

$$\begin{aligned}
&\alpha_{i,j} = \sqrt{i(N-i-j+1)},\ \beta_{i,j} = \sqrt{j(N-i-j+1)},\\
&\gamma_{i,j} = \sqrt{i\,j}.
\end{aligned}$$

Proceeding likewise with $_N\langle s,t\,|\,a_2^\dagger a_2 U\,|\,i,j\rangle_N$, one obtains

$$\begin{aligned}
&tP_{i,j}(s,t;N) = \left[R_{21}^2\, i + R_{22}^2\, j + R_{23}^2(N-i-j)\right]P_{i,j}(s,t;N)\\
&+R_{21}R_{23}\Big[\alpha_{i+1,j}P_{i+1,j}(s,t;N) + \alpha_{i,j}P_{i-1,j}(s,t;N)\Big]\\
&+R_{22}R_{23}\Big[\beta_{i,j+1}P_{i,j+1}(s,t;N) + \beta_{i,j}P_{i,j-1}(s,t;N)\Big]\\
&+R_{21}R_{22}\Big[\gamma_{i,j+1}P_{i-1,j+1}(s,t;N) + \gamma_{i+1,j}P_{i+1,j-1}(s,t;N)\Big].
\end{aligned} \tag{8.9}$$

Upon combining the recurrence relations (8.8) and (8.9), one can eliminate the non nearest-neighbor terms $P_{i-1,j+1}(s,t;N)$ and $P_{i+1,j-1}(s,t;N)$ to find

$$\begin{aligned}
&(R_{21}R_{22}s - R_{11}R_{12}t)P_{i,j}(s,t;N) =\\
&\Big\{[R_{21}R_{22}(R_{11}^2 - R_{13}^2) - R_{11}R_{12}(R_{21}^2 - R_{23}^2)]\,i + [R_{21}R_{22}(R_{12}^2 - R_{13}^2) - R_{11}R_{12}(R_{22}^2 - R_{23}^2)]\,j\\
&+[R_{21}R_{22}R_{13}^2 - R_{11}R_{12}R_{23}^2]N\Big\}P_{i,j}(s,t;N)\\
&+\Big\{R_{21}R_{22}R_{11}R_{13} - R_{11}R_{12}R_{21}R_{23}\Big\}\\
&\times\Big[\alpha_{i,j}P_{i-1,j}(s,t;N) + \alpha_{i+1,j}P_{i+1,j}(s,t;N)\Big]\\
&\qquad\qquad+\Big\{R_{21}R_{22}R_{12}R_{13} - R_{11}R_{12}R_{22}R_{23}\Big\}\Big[\beta_{i,j}P_{i,j-1}(s,t;N) + \beta_{i,j+1}P_{i,j+1}(s,t;N)\Big].
\end{aligned}$$

It is readily noted that the above relation is of the same form as the 5-term recurrence equation (8.4) that one has to solve to obtain the 1-excitation dynamics of the spin lattices governed by the

Hamiltonian (8.2). Take

$$\begin{aligned} I_{i,j} &= (R_{21}R_{22}R_{11}R_{13} - R_{11}R_{12}R_{21}R_{23})\alpha_{i,j}, \\ J_{i,j} &= (R_{21}R_{22}R_{12}R_{13} - R_{11}R_{12}R_{22}R_{23})\beta_{i,j}, \end{aligned} \tag{8.10}$$

and

$$\begin{aligned} B_{i,j} = \Big\{ &[R_{21}R_{22}(R_{11}^2 - R_{13}^2) - R_{11}R_{12}(R_{21}^2 - R_{23}^2)]i \\ &+ [R_{21}R_{22}(R_{12}^2 - R_{13}^2) - R_{11}R_{12}(R_{22}^2 - R_{23}^2)]j + [R_{21}R_{22}R_{13}^2 - R_{11}R_{12}R_{23}^2]N \Big\}. \end{aligned} \tag{8.11}$$

Our polynomial analysis shows that the spectrum of the Hamiltonian (8.2) with couplings (8.10), (8.11) is given by

$$x_{s,t} = R_{21}R_{22}s - R_{11}R_{12}t, \quad s,t \in \{0,\dots,N\},$$

with $s+t \leq N$ and that the unitary expansion coefficients are

$$M_{i,j}(s,t) = {}_N\langle s,t \,|\, U(R) \,|\, i,j \rangle_N = W_{s,t;N} P_{i,j}(s,t;N).$$

The rotation matrix elements R_{ij} are parameters. If one takes for instance

$$R = \begin{pmatrix} \frac{1}{2} - \frac{\sqrt{2}}{4} & -\frac{1}{2} - \frac{\sqrt{2}}{4} & \frac{1}{2} \\ -\frac{1}{2} - \frac{\sqrt{2}}{4} & \frac{1}{2} - \frac{\sqrt{2}}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}, \tag{8.12}$$

one has in particular

$$R_{21}R_{22} = R_{11}R_{12} = -\frac{1}{8}, \quad R_{13} = R_{23} = \frac{1}{2},$$

and

$$\begin{aligned} x_{s,t} &= \frac{1}{8}(t-s), & I_{i,j} &= -\frac{1}{16}\sqrt{i(N-i-j+1)}, \\ B_{i,j} &= \frac{-1}{8\sqrt{2}}(j-i), & J_{i,j} &= \frac{1}{16}\sqrt{j(N-i-j+1)}. \end{aligned}$$

Note that the rotation R specified by (8.12) is improper since $\det R = -1$.

8.5 State transfer

Knowing the 1-excitation dynamics for the particular class of spin lattices, one can determine the transition amplitudes. Let $f_{(i,j),(k,\ell)}(T)$ denote the transition amplitude for the excitation at site

(i,j) to be found at the site (k,ℓ) after some time T. One can write

$$\begin{aligned}
f_{(i,j),(k,\ell)}(T) &= \langle\, i,j \,|\, e^{-iT\mathscr{H}} \,|\, k,\ell \,\rangle \\
&= \sum_{s+t\leq N} \langle\, i,j \,|\, e^{-iT\mathscr{H}} \,|\, x_{s,t} \,\rangle \langle\, x_{s,t} \,|\, k,\ell \,\rangle \\
&= \sum_{s+t\leq N} M_{i,j}(s,t) M_{k,\ell}(s,t) e^{-iTx_{s,t}} \\
&= \sum_{s+t\leq N} {}_N\langle\, s,t \,|\, U(R) \,|\, i,j \,\rangle_N \, {}_N\langle\, s,t \,|\, U(R) \,|\, k,\ell \,\rangle_N \, e^{-iTx_{s,t}},
\end{aligned}$$

with $x_{s,t} = R_{21}R_{22}s - R_{11}R_{12}t$. Typically one wishes to transfer state from a given site taken to be $(0,0)$. Using the expression (8.7) for $W_{s,t;N}$, the transition amplitude from the site $(0,0)$ to an arbitrary site (i,j) is seen to be of the form

$$f_{(0,0),(i,j)} = R_{33}^N \sum_{s+t\leq N} \sqrt{\binom{N}{s,t}} \left(\frac{R_{13}z_1}{R_{33}}\right)^s \left(\frac{R_{23}z_2}{R_{33}}\right)^t {}_N\langle\, s,t \,|\, U(R) \,|\, i,j \,\rangle_N$$

where we have taken

$$z_1 = e^{-iR_{21}R_{22}T}, \quad z_2 = e^{iR_{11}R_{12}T}. \tag{8.13}$$

Introduce another variable u such that $s+t+u = N$ as well as an auxiliary variable z_3. Let

$$\alpha_1 = R_{13}z_1, \quad \alpha_2 = R_{23}z_2, \quad \alpha_3 = R_{33}z_3.$$

and define

$$G_{i,j;N}(\alpha_1,\alpha_2,\alpha_3) = \sum_{\substack{s,t,u \\ s+t+u=N}} \sqrt{\frac{N!}{s!t!u!}} \langle\, s,t,u \,|\, U(R) \,|\, i,j,k \,\rangle\, \alpha_1^s \alpha_2^t \alpha_3^u, \tag{8.14}$$

with $i+j+k = N$. It is seen that $G_{i,j;N}(\alpha_1,\alpha_2,\alpha_3)$ is a generating function for ${}_N\langle\, s,t \,|\, U(R) \,|\, i,j \,\rangle_N$ and that

$$f_{(0,0),(i,j)} = G_{i,j;N}(R_{13}z_1, R_{23}z_2, R_{33}) \qquad z_3 = 1. \tag{8.15}$$

The generating function $G_{i,j;N}(\alpha_1,\alpha_2,\alpha_3)$ is readily computed in the representation framework. Using (8.14), one writes

$$\begin{aligned}
G_{i,j;N}(\alpha_1,\alpha_2,\alpha_3) &= \sqrt{N!} \sum_{s+t+u=N} \langle\, 0,0,0 \,|\, \frac{(\alpha_1 a_1)^s}{s!} \frac{(\alpha_2 a_2)^t}{t!} \frac{(\alpha_3 a_3)^u}{u!} U \,|\, i,j,k \,\rangle \\
&= \sqrt{N!}\, \langle\, 0,0,0 \,|\, U U^\dagger e^{(\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3)} U \,|\, i,j,k \,\rangle,
\end{aligned}$$

since U keeps N fixed and since the states are orthonormal. Because $U|\,0,0,0\,\rangle = |\,0,0,0\,\rangle$ and

$$U^\dagger e^{\sum_\ell \alpha_\ell a_\ell} U = e^{\sum_\ell \alpha_\ell U a_\ell U^\dagger} = e^{\sum_p \beta_p a_p}$$

with $\beta_p = \sum_\ell R_{\ell p}\alpha_\ell$, one can write

$$G_{i,j;N}(\alpha_1,\alpha_2,\alpha_3) = \sqrt{N!}\langle 0,0,0 \,|\, e^{\beta_1\alpha_1+\beta_2\alpha_2+\beta_3\alpha_3} \,|\, i,j,k \rangle$$
$$= \sqrt{N!}\sum_{\ell,m,n}\frac{\beta_1^\ell\beta_2^m\beta_3^n}{\sqrt{\ell!m!n!}}\langle \ell,m,n \,|\, i,j,k \rangle,$$

which gives

$$G_{i,j;N}(\alpha_1,\alpha_2,\alpha_3) = \binom{N}{i,j}^{1/2}\beta_1^i\beta_2^j\beta_3^{N-i-j},$$

since $i+j+k=N$. Consequently, we have

$$G_{i,j;N}(\alpha_1,\alpha_2,\alpha_3) = \sqrt{\binom{N}{i,j}}(R_{11}\alpha_1+R_{21}\alpha_2+R_{31}\alpha_3)^i$$
$$\times (R_{12}\alpha_1+R_{22}\alpha_2+R_{32}\alpha_3)^j(R_{13}\alpha_1+R_{23}\alpha_2+R_{33}\alpha_3)^{N-i-j}.$$

In view of (8.15), we have obtained the following formula for the transition amplitude

$$f_{(0,0),(i,j)}(T) = \sqrt{\binom{N}{i,j}}(R_{11}R_{13}z_1+R_{21}R_{23}z_2+R_{31}R_{33})^i$$
$$\times (R_{12}R_{13}z_1+R_{22}R_{23}z_2+R_{32}R_{33})^j(R_{13}^2z_1+R_{23}^2z_2+R_{33}^2)^{N-i-j},$$

with z_1 and z_2 given by (8.13). Let $R_{21}R_{22} = R_{11}R_{12}$ and take $T = \frac{\pi}{R_{11}R_{12}}$ so that $z_1 = z_2 = -1$. We have

$$f_{(0,0),(i,j)}\left(\frac{\pi}{R_{11}R_{12}}\right) = \sqrt{\binom{N}{i,j}}(-R_{11}R_{13}-R_{21}R_{23}+R_{31}R_{33})^i$$
$$\times (-R_{12}R_{13}-R_{22}R_{23}+R_{32}R_{33})^j(-R_{13}^2-R_{23}^2+R_{33}^2)^{N-i-j}.$$

If one adds to $R_{21}R_{22} = R_{11}R_{12}$ the condition $R_{33} = \sqrt{2}/2$, this implies that $f_{(0,0),(i,j)}\left(\frac{\pi}{R_{11}R_{12}}\right) = 0$ unless $i+j=N$ since $(-R_{13}^2-R_{23}^2+R_{33}^2) = 0$. These conditions were met by the rotation matrix considered in (8.12). With these conditions, the amplitude reads

$$f_{(0,0),(i,j)}\left(\frac{\pi}{R_{11}R_{12}}\right) = \sqrt{\binom{N}{i,j}}(\sqrt{2}R_{31})^i(\sqrt{2}R_{32})^j\delta_{i+j,N},$$

and the output excitation distributes binomially on the site of the boundary hypotenuse. Hence for the values of the parameters such that $R_{21}R_{22} = R_{11}R_{12}$ and $R_{33} = \sqrt{2}/2$, the Hamiltonian $\mathscr{H}$ with non-homogeneous couplings (8.10) and (8.11) will dynamically evolve the state $|\,0,0\,\rangle$ in time $\frac{\pi}{R_{11}R_{12}}$ to any one of the states $|\,i,N-i\,\rangle$ with probability 1. As a consequence

$$\left|f_{(0,0),(i,j)}\left(\frac{\pi}{R_{11}R_{12}}\right)\right|^2 = 0, \text{ when } i+j<N,$$

which is akin to perfect transfer. It can be shown that the bivariate Krawtchouk polynomials are symmetric for these values of the parameters [16].

8.6 Conclusion

We have shown that the solutions of the 1-excitation dynamics for a particular class of spin networks with inhomogeneous couplings is tied to multivariate orthogonal polynomials and we have provided an illustration of the theory of multivariate Krawtchouk polynomials based on the representations of $O(n)$ on oscillator states. For more details on the connection between orthogonal polynomials and perfect state transfer, the reader may wish to consult [19, 16, 17]. For a detailed account of the relation between multivariate orthogonal polynomials and Lie group representations, the reader is referred to [7, 8, 6, 5].

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Chapitre 9

A superintegrable discrete oscillator and two-variable Meixner polynomials

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Abstract. A superintegrable, discrete model of the quantum isotropic oscillator in two-dimensions is introduced. The system is defined on the regular, infinite-dimensional $\mathbb{N} \times \mathbb{N}$ lattice. It is governed by a Hamiltonian expressed as a seven-point difference operator involving three parameters. The exact solutions of the model are given in terms of the two-variable Meixner polynomials orthogonal with respect to the negative trinomial distribution. The constants of motion of the system are constructed using the raising and lowering operators for these polynomials. They are shown to generate an $\mathfrak{su}(2)$ invariance algebra. The two-variable Meixner polynomials are seen to support irreducible representations of this algebra. In the continuum limit, where the lattice constant tends to zero, the standard isotropic quantum oscillator in two dimensions is recovered. The limit process from the two-variable Meixner polynomials to a product of two Hermite polynomials is carried out by involving the bivariate Charlier polynomials.

9.1 Introduction

The purpose of this paper is to present a discrete model of the two-dimensional quantum oscillator that is both superintegrable and exactly solvable. The wavefunctions of this system will be given in terms of the two-variable Meixner polynomials and the constants of motion will be seen to satisfy the $\mathfrak{su}(2)$ algebra.

A considerable amount of literature can be found on superintegrable systems and there is a sustained interest in enlarging the documented set of models with that property. Recall that a quantum system with d degrees of freedom governed by a Hamiltonian H is said to be superintegrable if it possesses, including H itself, $2d-1$ algebraically independent constants of motion, that is operators that commute with the Hamiltonian. The quintessential example of a quantum superintegrable system is the two-dimensional harmonic oscillator, whose constants of motion generate the $\mathfrak{su}(2)$ algebra. One of the motivating observations behind the study of superintegrable systems is that they are exactly solvable, which makes them prime candidates for modeling purposes. Also of importance is the fact that these systems form a bedrock for the analysis of symmetries, of the associated algebraic structures and their representations, and of special functions. The majority of quantum superintegrable models cataloged so far comprises continuous systems, but there has also been some progress in the study of discrete systems [18, 19]; for a review on superintegrable systems (mostly continuous ones) and their applications, see [20].

In the past years, several discrete models of the one-dimensional quantum oscillator, either finite or infinite, were introduced [2, 5, 14, 15]. The most studied system, originally proposed in [6] as a model of multimodal waveguides with a finite number of sensor points, has $\mathfrak{su}(2)$ as its dynamical algebra. In this model, the Hamiltonian, the position and momentum operators are expressed in terms of $\mathfrak{su}(2)$ generators, the eigenstates of the system are the basis vectors of unitary irreducible representations of $\mathfrak{su}(2)$ and the wavefunctions are expressed in terms of the one-variable Krawtchouk polynomials. As a result, the Hamiltonian has a finite number of eigenvalues and the spectra of the position and momentum operators are both discrete and finite. Germane to the present paper is also the discrete oscillator model based on the univariate Meixner polynomials and related to the $\mathfrak{su}(1,1)$ algebra considered in [2]. See also [16], where instead the Meixner-Pollaczek polynomials are involved.

The Krawtchouk one-dimensional finite/discrete oscillator has been exploited to construct finite/discrete systems in two dimensions. Two approaches have been used. The first approach consist in taking the direct product of two one-dimensional $\mathfrak{su}(2)$ systems to obtain a system defined on a square grid with $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ as its dynamical algebra [3]. In the second approach [4], the isomorphism $\mathfrak{so}(4)\cong\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ is exploited to obtain a description of the finite oscillator on the square grid in terms of discrete radial and angular coordinates. In the continuum limit, both of these models tend to the standard two-dimensional oscillator. However, they do not exhibit the $\mathfrak{su}(2)$ invariance, or symmetry algebra, of the standard two-dimensional oscillator.

Recently another discrete and finite model of the two-dimensional oscillator was proposed [19]. This model is defined on a triangular lattice of a given size and, like the standard oscillator in two dimensions, it is superintegrable and has $\mathfrak{su}(2)$ for symmetry algebra. The wavefunctions of the model, which support irreducible representations of $\mathfrak{su}(2)$ at fixed energy, are given in terms of the two-variable Krawtchouk polynomials introduced by Griffiths [10]. These polynomials of two discrete variables are orthogonal with respect to the trinomial distribution. As required, this model tends to the standard two-dimensional oscillator in the continuum limit.

Shortly after [19] appeared, it was recognized that the d-variable Krawtchouk polynomials of Griffiths arise as matrix elements of the unitary representations of the rotation group $SO(d+1)$ on oscillator states [9]. This interpretation has provided a cogent framework for the characterization of these orthogonal functions and has led to a number of new identities. The group theoretical interpretation was also extended to two other families of discrete multivariate polynomials: the multivariate Meixner and Charlier polynomials, orthogonal with respect to the negative multinomial and multivariate Poisson distributions, respectively. The multi-variable Meixner polynomials, also introduced by Griffiths [11], were shown to arise as matrix elements of unitary representations of the pseudo-rotation group $SO(d,1)$ on oscillator states [8]. As for the multivariate Charlier polynomials, they were first introduced as matrix elements of unitary representations of the Euclidean group on oscillator states [7]. Let us note that these family of multivariate polynomials also arise in probability theory in connection with the so-called Lancaster distributions [12].

In this paper, we present a new discrete oscillator model in two-dimensions based on the two-variable Meixner polynomials. The model is defined on the regular infinite-dimensional $\mathbb{N}\times\mathbb{N}$ lattice. It is governed by a Hamiltonian involving three independent parameters expressed as a 7-point difference operator. This operator is obtained by combining the two independent difference equations satisfied by the bivariate Meixner polynomials. By construction, the wavefunctions of the model are given in terms of these two-variable polynomials. The energies of the system are given by the non-negative integers $N=0,1,2,\ldots$ and exhibit a $(N+1)$-fold degeneracy. Using the raising and lowering relations for the two-variable Meixner polynomials, the constants of motion of the system are constructed and are shown to close onto the $\mathfrak{su}(2)$ commutation relations. In the continuum limit, in which the lattice parameter tends to zero, the model contracts to the standard quantum harmonic oscillator, as required for a discrete oscillator model. The contraction process is illustrated at the wavefunction level using the two-variable Charlier polynomials in an intermediary step. The continuum limit is also displayed at the level of operators.

Here is the outline of the paper. In Section two, the essential properties of the two-variable Meixner polynomials are reviewed. In Section three, the Hamiltonian of the model is defined, the constants of motion are constructed, and the wavefunctions are illustrated. In Section four, the continuum limit of the model and wavefunctions is examined. We conclude with an outlook.

9.2 The two-variable Meixner polynomials

We now review the properties of the two-variable Meixner polynomials using the formalism and notation developed in [8]. Let $\beta \geqslant 0$ be a positive real number and let $\Lambda \in O(2,1)$ be a 3×3 pseudo-rotation matrix. This implies that Λ satisfies

$$\Lambda^{\top}\eta\,\Lambda = \eta,$$

where $\eta = \text{diag}(1,1,-1)$ and where $\Lambda^{\top}$ denotes the transposed matrix. In general, Λ can be parametrized by three real numbers akin to the Euler angles. The two-variable Meixner polynomials, denoted by $M^{(\beta)}_{n_1,n_2}(x_1,x_2)$, are defined by the generating function

$$\left(1+\frac{\Lambda_{11}}{\Lambda_{13}}z_1+\frac{\Lambda_{12}}{\Lambda_{13}}z_2\right)^{x_1}\left(1+\frac{\Lambda_{21}}{\Lambda_{23}}z_1+\frac{\Lambda_{22}}{\Lambda_{23}}z_2\right)^{x_2}\left(1+\frac{\Lambda_{31}}{\Lambda_{33}}z_1+\frac{\Lambda_{32}}{\Lambda_{33}}z_2\right)^{-x_1-x_2-\beta}$$
$$=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sqrt{\frac{(\beta)_{n_1+n_2}}{n_1!n_2!}}\,M^{(\beta)}_{n_1,n_2}(x_1,x_2)\,z_1^{n_1}z_2^{n_2}, \tag{9.1}$$

where the Λ_{ij} are the entries of the parameter matrix Λ and where $(\beta)_n$ stands for the Pochhammer symbol defined as

$$(\beta)_n = \begin{cases} 1 & n=0 \\ \prod_{k=0}^{n-1}(\beta+k) & n=1,2,3,\ldots \end{cases}$$

It can be seen from (9.1) that $M^{(\beta)}_{n_1,n_2}(x_1,x_2)$ are polynomials of total degree n_1+n_2 in the variables x_1 and x_2. The functions $M^{(\beta)}_{n_1,n_2}(x_1,x_2)$ satisfy the orthogonality relation

$$\sum_{x_1=0}^{\infty}\sum_{x_2=0}^{\infty}\omega(x_1,x_2)\,M^{(\beta)}_{n_1,n_2}(x_1,x_2)\,M^{(\beta)}_{n_1',n_2'}(x_1,x_2)=\delta_{n_1,n_1'}\delta_{n_2,n_2'}, \tag{9.2}$$

where $\omega(x_1,x_2)$ is the negative trinomial distribution

$$\omega(x_1,x_2)=\frac{(\beta)_{x_1+x_2}}{x_1!x_2!}\,(1-c_1-c_2)^{\beta}\,c_1^{x_1}c_2^{x_2}, \tag{9.3}$$

and where the parameters c_1, c_2 are given by

$$c_1=\left(\frac{\Lambda_{13}}{\Lambda_{33}}\right)^2,\qquad c_2=\left(\frac{\Lambda_{23}}{\Lambda_{33}}\right)^2.$$

The polynomials $M^{(\beta)}_{n_1,n_2}(x_1,x_2)$ have an explicit expression in terms of Aomoto–Gelfand hypergeometric series [13]. This expression reads

$$M^{(\beta)}_{n_1,n_2}(x_1,x_2)=(-1)^{n_1+n_2}\sqrt{\frac{(\beta)_{n_1+n_2}}{n_1!n_2!}}\left(\frac{\Lambda_{31}}{\Lambda_{33}}\right)^{n_1}\left(\frac{\Lambda_{32}}{\Lambda_{33}}\right)^{n_2}$$
$$\times\sum_{\mu,\nu,\rho,\sigma\geqslant 0}\frac{(-n_1)_{\mu+\nu}(-n_2)_{\rho+\sigma}(-x_1)_{\mu+\rho}(-x_2)_{\nu+\sigma}}{\mu!\nu!\rho!\sigma!\,(\beta)_{\mu+\nu+\rho+\sigma}}\,(1-u_{11})^{\mu}(1-u_{21})^{\nu}(1-u_{12})^{\rho}(1-u_{22})^{\sigma}, \tag{9.4}$$

where the parameters u_{ij} are given by

$$u_{11}=\frac{\Lambda_{11}\Lambda_{33}}{\Lambda_{13}\Lambda_{31}},\quad u_{12}=\frac{\Lambda_{12}\Lambda_{33}}{\Lambda_{13}\Lambda_{32}},\quad u_{21}=\frac{\Lambda_{21}\Lambda_{33}}{\Lambda_{23}\Lambda_{31}},\quad u_{22}=\frac{\Lambda_{22}\Lambda_{33}}{\Lambda_{23}\Lambda_{32}}.$$

Let $T_{x_i}^{\pm}f(x_i)=f(x_i\pm 1)$ for $i=1,2$ be the discrete shift operators in the variables x_1 and x_2. Introduce the two intertwining operators $A_{+}^{(i)}$ defined as

$$A_{+}^{(i)}=\frac{\Lambda_{1i}}{\Lambda_{13}}\,x_1T_{x_1}^{-}+\frac{\Lambda_{2i}}{\Lambda_{23}}\,x_2T_{x_2}^{-}-\frac{\Lambda_{3i}}{\Lambda_{33}}\,(x_1+x_2+\beta)\mathbb{I},\qquad i=1,2, \tag{9.5}$$

where $\mathbb{I}$ stands for the identity operator. On the bivariate polynomials $M_{n_1,n_2}^{(\beta)}(x_1,x_2)$, these operators have the actions

$$\begin{aligned}A_{+}^{(1)}M_{n_1,n_2}^{(\beta+1)}(x_1,x_2)&=\sqrt{\beta(n_1+1)}\,M_{n_1+1,n_2}^{(\beta)}(x_1,x_2),\\ A_{+}^{(2)}M_{n_1,n_2}^{(\beta+1)}(x_1,x_2)&=\sqrt{\beta(n_2+1)}\,M_{n_1,n_2+1}^{(\beta)}(x_1,x_2).\end{aligned} \tag{9.6}$$

Introduce also the two intertwining operators $A_{-}^{(i)}$ defined in the following way:

$$A_{-}^{(i)}=\Lambda_{1i}\Lambda_{13}\,T_{x_1}^{+}+\Lambda_{2i}\Lambda_{23}\,T_{x_2}^{+}-(\Lambda_{1i}\Lambda_{13}+\Lambda_{2i}\Lambda_{23})\mathbb{I},\qquad i=1,2. \tag{9.7}$$

These operators act as follows on the bivariate Meixner polynomials

$$\begin{aligned}A_{-}^{(1)}M_{n_1,n_2}^{(\beta)}(x_1,x_2)&=\sqrt{\frac{n_1}{\beta}}\,M_{n_1-1,n_2}^{(\beta+1)}(x_1,x_2),\\ A_{-}^{(2)}M_{n_1,n_2}^{(\beta)}(x_1,x_2)&=\sqrt{\frac{n_2}{\beta}}\,M_{n_1,n_2-1}^{(\beta+1)}(x_1,x_2).\end{aligned} \tag{9.8}$$

The intertwining operators (9.5) and (9.7) can be combined to produce the two commuting difference operators Y_1 and Y_2 that are diagonalized by the bivariate Meixner polynomials. These operators are defined as

$$Y_i=A_{+}^{(i)}A_{-}^{(i)},\qquad i=1,2. \tag{9.9}$$

Explicitly, one finds

$$\begin{aligned}Y_i=&\left(\frac{\Lambda_{1i}\Lambda_{2i}\Lambda_{23}}{\Lambda_{13}}\right)x_1T_{x_1}^{-}T_{x_2}^{+}+\left(\frac{\Lambda_{1i}\Lambda_{2i}\Lambda_{13}}{\Lambda_{23}}\right)x_2T_{x_1}^{+}T_{x_2}^{-}-\left(\frac{\Lambda_{1i}\Lambda_{3i}\Lambda_{33}}{\Lambda_{13}}\right)x_1T_{x_1}^{-}\\ &-\left(\frac{\Lambda_{2i}\Lambda_{3i}\Lambda_{33}}{\Lambda_{23}}\right)x_2T_{x_2}^{-}-\left(\frac{\Lambda_{1i}\Lambda_{3i}\Lambda_{13}}{\Lambda_{33}}\right)(x_1+x_2+\beta)T_{x_1}^{+}\\ &\qquad-\left(\frac{\Lambda_{2i}\Lambda_{3i}\Lambda_{23}}{\Lambda_{33}}\right)(x_1+x_2+\beta)T_{x_2}^{+}+\left[\Lambda_{1i}^2\,x_1+\Lambda_{2i}^2\,x_2+\Lambda_{3i}^2\,(x_1+x_2+\beta)\right]\mathbb{I}.\end{aligned} \tag{9.10}$$

The eigenvalue equations read

$$Y_1M_{n_1,n_2}^{(\beta)}(x_1,x_2)=n_1M_{n_1,n_2}^{(\beta)}(x_1,x_2),\qquad Y_2M_{n_1,n_2}^{(\beta)}(x_1,x_2)=n_2M_{n_1,n_2}^{(\beta)}(x_1,x_2), \tag{9.11}$$

where n_1, n_2 are non-negative integers. Let us note that the operators Y_1 and Y_2 fully characterize the polynomials $M_{n_1,n_2}^{(\beta)}(x_1,x_2)$.

9.3 A discrete and superintegrable Hamiltonian

We now consider the Hamiltonian obtained by taking the sum of the operators Y_1 and Y_2 involved in the eigenvalue equations (9.11). We hence define

$$\mathscr{H} = Y_1 + Y_2. \tag{9.12}$$

In principle the Hamiltonian (9.12) involves four independent parameters including β, as any matrix $\Lambda \in O(2,1)$ depends on three independent parameters. However, it turns out that $\mathscr{H}$ essentially depends only on three parameters, including β. This can be seen explicitly as follows. Consider the following parametrization of Λ in terms of the "Euler angles" ψ, ξ and ϕ:

$$\Lambda(\psi,\xi,\phi) = \begin{pmatrix} \cosh\xi & 0 & \sinh\xi \\ 0 & 1 & 0 \\ \sinh\xi & 0 & \cosh\xi \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\psi & \sinh\psi \\ 0 & \sinh\psi & \cos\psi \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{9.13}$$

Upon using the expressions (9.10), it is seen that in the parametrization (9.13) the Hamiltonian (9.12) has the expression

$$\begin{aligned}
\mathscr{H}(\psi,\xi,\phi) &= \sinh^2\psi\, x_1 T^{-}_{x_1} T^{+}_{x_2} + \cosh^2\psi \sinh^2\xi\, x_2 T^{+}_{x_1} T^{-}_{x_2} \\
&- \cosh^2\psi \sinh^2\xi\, (x_1 + x_2 + \beta) T^{+}_{x_1} - \sinh^2\psi\, (x_1 + x_2 + \beta) T^{+}_{x_2} \\
&- \cosh^2\xi \cosh^2\psi \left[x_1 T^{-}_{x_1} + x_2 T^{-}_{x_2}\right] + \Big[\cosh 2\xi \cosh^2\psi\, x_1 \\
&\qquad + (\cosh^2\psi + \sinh^2\xi + \cosh^2\xi \sinh^2\psi)\, x_2 + \beta\,(\sinh^2\xi + \cosh^2\xi \sinh^2\psi)\Big]\, \mathbb{I},
\end{aligned}$$

hence the parameter ϕ does not explicitly appear in $\mathscr{H}$. The fact that the Hamiltonian (9.12) can be presented in terms three independent parameters is a manifestation of its superintegrability. Let J_X, J_Y and J_Z be the operators defined as follows.

$$\begin{aligned}
J_X &= \frac{1}{2}\left(A^{(1)}_{+} A^{(2)}_{-} + A^{(2)}_{+} A^{(1)}_{-}\right), \\
J_Y &= \frac{1}{2i}\left(A^{(1)}_{+} A^{(2)}_{-} - A^{(2)}_{+} A^{(1)}_{-}\right), \\
J_Z &= \frac{1}{2}\left(A^{(1)}_{+} A^{(1)}_{-} - A^{(2)}_{+} A^{(2)}_{-}\right).
\end{aligned} \tag{9.14}$$

The operators J_X, J_Y and J_Z are constants of motion. Indeed, one can verify by a direct calculation that these operators commute with the Hamiltonian (9.12)

$$[\mathscr{H}, J_X] = 0, \qquad [\mathscr{H}, J_Y] = 0, \qquad [\mathscr{H}, J_Z] = 0.$$

The symmetry operators J_X, J_Y and J_Z satisfy the defining relations of the $\mathfrak{su}(2)$ algebra. One has

$$[J_X, J_Y] = i J_Z, \qquad [J_Y, J_Z] = i J_X, \qquad [J_Z, J_X] = i J_Y. \tag{9.15}$$

In the realization (9.14), the $\mathfrak{su}(2)$ Casimir operator is related to the Hamiltonian (9.12) through

$$J_X^2 + J_Y^2 + J_Z^2 = \frac{1}{2}\mathscr{H}\left(\frac{\mathscr{H}}{2}+1\right). \tag{9.16}$$

The realization (9.14) of the $\mathfrak{su}(2)$ algebra (9.15) and the expression (9.16) of the Casimir operator in terms of the Hamiltonian is very close to the Schwinger realization of $\mathfrak{su}(2)$ that one finds when considering the standard two-dimensional quantum harmonic oscillator. The $SU(2)$ symmetry (9.15) of the Hamiltonian (9.12) explains why $\mathscr{H}$ depends on three parameters instead of four: the ϕ parameter in (9.12) has been "rotated away" from the Hamiltonian by the choice of parametrization (9.13).

By construction, the eigenfunctions of the Hamiltonian (9.12) are expressed in terms of the two-variable Meixner polynomials $M_{n_1,n_2}^{(\beta)}(x_1,x_2)$. These eigenfunctions $\Psi_{N,n}^{(\beta)}(x_1,x_2)$ are labeled by the two non-negative integers N and n and read

$$\Psi_{N,n}^{(\beta)}(x_1,x_2) = M_{n,N-n}^{(\beta)}(x_1,x_2),$$

where $n = 0,1,\ldots,N$ and where $N = 0,1,2,\ldots$. One has

$$\mathscr{H}\,\Psi_{N,n}^{(\beta)}(x_1,x_2) = N\,\Psi_{N,n}^{(\beta)}(x_1,x_2), \qquad J_Z\,\Psi_{N,n}^{(\beta)}(x_1,x_2) = (n-N/2)\,\Psi_{N,n}^{(\beta)}(x_1,x_2). \tag{9.17}$$

Hence the eigenvalues of $\mathscr{H}$ are the non-negative integers $N = 0,1,2,\ldots$ and are $(N+1)$-times degenerate. The states $\Psi_{N,n}(x_1,x_2)$ support $(N+1)$-dimensional irreducible representations of $\mathfrak{su}(2)$. Upon introducing the generators

$$J_\pm = J_X \pm iJ_Y,$$

it is seen from (9.6) and (9.8) that these operators have the actions

$$J_+\,\Psi_{N,n}^{(\beta)}(x_1,x_2) = \sqrt{(n+1)(N-n)}\,\Psi_{N,n+1}^{(\beta)}(x_1,x_2), \tag{9.18}$$

$$J_-\,\Psi_{N,n}^{(\beta)}(x_1,x_2) = \sqrt{n(N-n+1)}\,\Psi_{N,n-1}^{(\beta)}(x_1,x_2). \tag{9.19}$$

It thus follows that the two-variable Meixner polynomials $M_{n_1,n_2}^{(\beta)}(x_1,x_2)$ support $(K+1)$-dimensional unitary representations of $\mathfrak{su}(2)$ were $K = n_1+n_2$.

In view of (9.2) wavefunctions $\Psi_{N,n}(x_1,x_2)$ are not normalized on the infinite grid $(x_1,x_2) \in \mathbb{N}\times\mathbb{N}$ with respect to the standard uniform measure of quantum mechanics. Properly normalized wavefunctions $\Upsilon_{N,n}^{(\beta)}(x_1,x_2)$ are obtained by taking

$$\Upsilon_{N,n}^{(\beta)}(x_1,x_2) = \sqrt{\omega(x_1,x_2)}\,M_{n,N-n}^{(\beta)}(x_1,x_2), \tag{9.20}$$

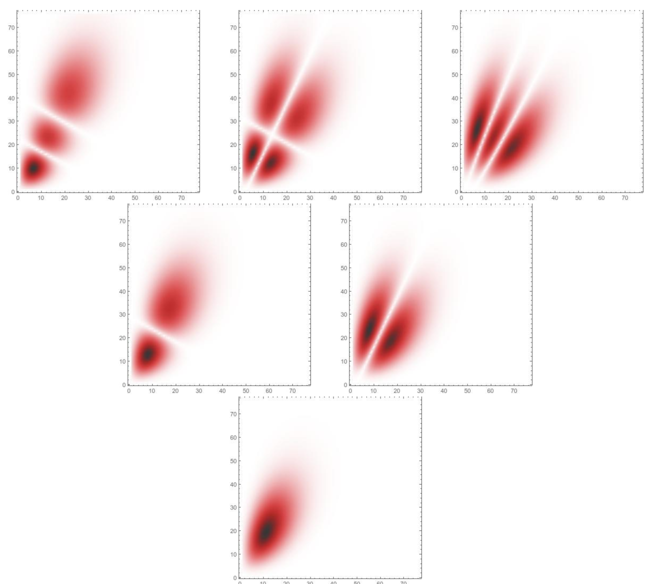
where $\omega(x_1,x_2)$ is given by (9.3). One then has the orthogonality and completeness relations [8]

$$\sum_{x_1=0}^{\infty}\sum_{x_2=0}^{\infty} \Upsilon_{N,n}^{(\beta)}(x_1,x_2)\Upsilon_{N',n'}^{(\beta)}(x_1,x_2) = \delta_{nn'}\delta_{NN'},$$

$$\sum_{N=0}^{\infty}\sum_{n=0}^{N} \Upsilon_{N,n}^{(\beta)}(x_1,x_2)\Upsilon_{N,n}^{(\beta)}(x_1',x_2') = \delta_{x_1,x_1'}\delta_{x_2,x_2'}.$$

The actions (9.17) and (9.18) on the non-normalized wavefunctions $\Psi_{N,n}^{(\beta)}(x_1,x_2)$ can be recovered on the normalized wavefunctions $\Upsilon_{N,n}^{(\beta)}(x_1,x_2)$ by applying the gauge transformation $\mathscr{O} \to \omega^{1/2}(x_1,x_2)\,\mathscr{O}\,\omega^{-1/2}(x_1,x_2)$, where $\mathscr{O}$ is either $\mathscr{H}$ or any one of the symmetries J_X, J_Y, J_Z.

Below are illustrated some of the wavefunctions amplitude $|\Upsilon_{N,n}^{(\beta)}(x_1,x_2)|$ for various values of the parameters ξ, ψ, θ and β. The model is defined on the $\mathbb{N}\times\mathbb{N}$ grid but only the grid 75×75 is shown. It is seen that the particle is localized close to the origin. The energy level N prescribes the number of "nodes" in the wavefunction amplitudes and the parameter β is related to the spatial spreading of the amplitude.

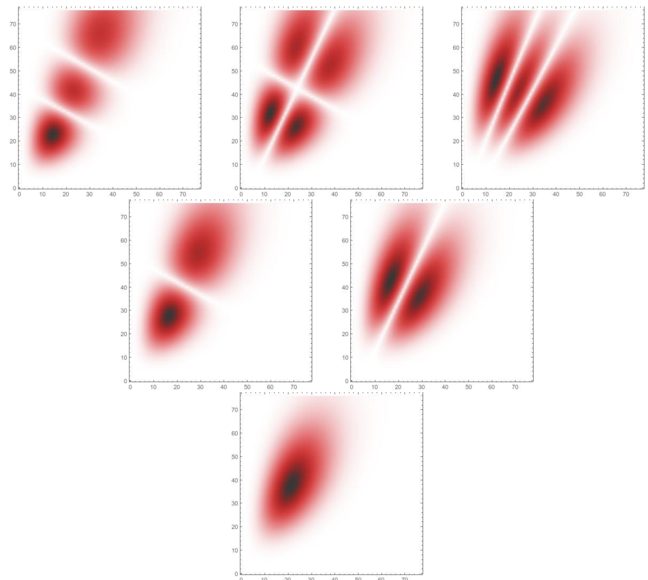


$\rightarrow, \uparrow$: $|\Upsilon_{0,0}^{(\beta)}|, |\Upsilon_{1,0}^{(\beta)}|, |\Upsilon_{1,1}^{(\beta)}|, |\Upsilon_{2,0}^{(\beta)}|, |\Upsilon_{2,1}^{(\beta)}|, |\Upsilon_{2,2}^{(\beta)}|$

Figure 9.1: Wavefunction amplitudes for $\xi = 0.8$, $\psi = 0.8$, $\phi = \pi/4$ and $\beta = 15$

9.4 Continuum limit to the standard oscillator

It will now be shown that in the continuum limit, the model governed by the Hamiltonian (9.12) tends to the standard isotropic quantum oscillator in two dimensions. The continuum limit from the two-variable Meixner polynomials to a product of two Hermite polynomials will be considered first. The explicit limit of the Hamiltonian (9.12) and of the constants of motion (9.14) will then be investigated.



$\rightarrow, \uparrow$: $|\Upsilon_{0,0}^{(\beta)}|, |\Upsilon_{1,0}^{(\beta)}|, |\Upsilon_{1,1}^{(\beta)}|, |\Upsilon_{2,0}^{(\beta)}|, |\Upsilon_{2,1}^{(\beta)}|, |\Upsilon_{2,2}^{(\beta)}|$

Figure 9.2: Wavefunction amplitudes for $\xi = 0.8$, $\psi = 0.8$, $\phi = \pi/4$ and $\beta = 28$

9.4.1 Continuum limit of the two-variable Meixner polynomials

Consider the generating function (9.1) of the two-variable Meixner polynomials (9.4) in the parametrization (9.13). Upon writing

$$\xi \to \frac{a}{\sqrt{\beta}}, \qquad \psi \to \frac{b}{\sqrt{\beta}}, \qquad z_1 \to \frac{z_1}{\sqrt{\beta}}, \qquad z_2 \to \frac{z_2}{\sqrt{\beta}}, \tag{9.21}$$

in left-hand side of (9.1) and taking the limit as $\beta \to \infty$ using the standard result

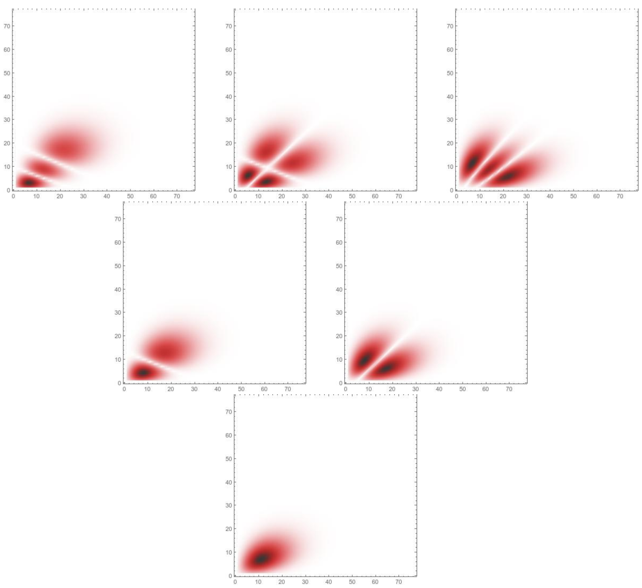
$$\lim_{k\to\infty}\left(1+\frac{x}{k}\right)^k = e^x,$$

one finds that

$$\begin{aligned}
&\lim_{\beta\to\infty}\left[\left(1+\frac{\Lambda_{31}}{\Lambda_{33}}\frac{z_1}{\sqrt{\beta}}+\frac{\Lambda_{32}}{\Lambda_{33}}\frac{z_2}{\sqrt{\beta}}\right)^{-x_1-x_2-\beta}\right.\\
&\left.\times\left(1+\frac{\Lambda_{11}}{\Lambda_{13}}\frac{z_1}{\sqrt{\beta}}+\frac{\Lambda_{12}}{\Lambda_{13}}\frac{z_2}{\sqrt{\beta}}\right)^{x_1}\left(1+\frac{\Lambda_{21}}{\Lambda_{23}}\frac{z_1}{\sqrt{\beta}}+\frac{\Lambda_{22}}{\Lambda_{23}}\frac{z_2}{\sqrt{\beta}}\right)^{x_2}\right]\\
&=\exp\left[-z_1(a\cos\phi-b\sin\phi)\right]\\
&\qquad\qquad\times\exp\left[-z_2(a\sin\phi+b\cos\phi)\right]\left(1+\frac{\cos\phi}{a}z_1+\frac{\sin\phi}{a}z_2\right)^{x_1}\left(1-\frac{\sin\phi}{b}z_1+\frac{\cos\phi}{b}z_2\right)^{x_2}.
\end{aligned}$$

Upon defining

$$C_{n_1,n_2}(x_1,x_2) = \lim_{\beta\to\infty} M_{n_1,n_2}^{(\beta)}(x_1,x_2),$$



$\rightarrow, \uparrow$: $|\Upsilon_{0,0}^{(\beta)}|, |\Upsilon_{1,0}^{(\beta)}|, |\Upsilon_{1,1}^{(\beta)}|, |\Upsilon_{2,0}^{(\beta)}|, |\Upsilon_{2,1}^{(\beta)}|, |\Upsilon_{2,2}^{(\beta)}|$

Figure 9.3: Wavefunction amplitudes for $\xi = 0.5$, $\psi = 0.8$, $\phi = \pi/4$ and $\beta = 15$.

and taking the limit as $\beta \to \infty$ with (9.21) in the right-hand side of (9.1), one finds that

$$\exp\left[-z_1(a\cos\phi - b\sin\phi)\right]\exp\left[-z_2(a\sin\phi + b\cos\phi)\right]\left(1 + \frac{\cos\phi}{a}z_1 + \frac{\sin\phi}{a}z_2\right)^{x_1} \\ \times\left(1 - \frac{\sin\phi}{b}z_1 + \frac{\cos\phi}{b}z_2\right)^{x_2} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} C_{n_1,n_2}(x_1,x_2)\,\frac{z_1^{n_1}z_2^{n_2}}{\sqrt{n_1!n_2!}}. \quad (9.22)$$

It is seen that the polynomials $C_{n_1,n_2}(x_1,x_2)$ correspond to the two-variable Charlier polynomials [7]. If one uses the parametrization (9.21) and takes the limit $\beta \to \infty$ in the weight function (9.3), one finds the two-variable Poisson distribution

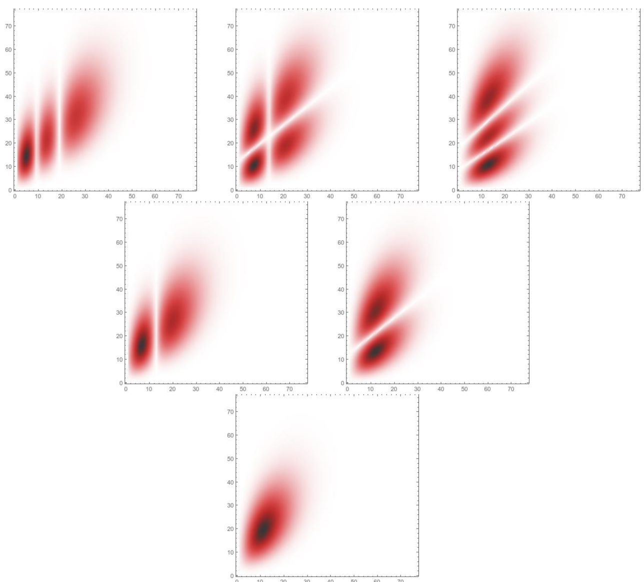
$$\lim_{\beta\to\infty} \omega(x_1,x_2) = e^{-(a^2+b^2)}\frac{(a^2)^{x_1}(b^2)^{x_2}}{x_1!x_2!}, \quad (9.23)$$

and the orthogonality relation (9.2) becomes

$$\sum_{x_1=0}^{\infty}\sum_{x_2=0}^{\infty}\left[e^{-(a^2+b^2)}\frac{(a^2)^{x_1}(b^2)^{x_2}}{x_1!x_2!}\right] C_{n_1,n_2}(x_1,x_2)C_{n_1',n_2'}(x_1,x_2) = \delta_{n_1,n_1'}\delta_{n_2,n_2'}.$$

It is directly seen from (9.22) and the standard generating function for the one-variable Charlier polynomials [17] that when $\phi = 0$, one has

$$C_{n_1,n_2}(x_1,x_2)\Big|_{\phi=0} = \frac{(-1)^{n_1+n_2}}{a^{n_1}b^{n_2}}C_{n_1}(x_1;a^2)C_{n_2}(x_2;b^2), \quad (9.24)$$



$\rightarrow, \uparrow$: $|\Upsilon^{(\beta)}_{0,0}|, |\Upsilon^{(\beta)}_{1,0}|, |\Upsilon^{(\beta)}_{1,1}|, |\Upsilon^{(\beta)}_{2,0}|, |\Upsilon^{(\beta)}_{2,1}|, |\Upsilon^{(\beta)}_{2,2}|$

Figure 9.4: Wavefunction amplitudes for $\xi = 0.8$, $\psi = 0.8$, $\phi = 0$ and $\beta = 15$.

where $C_n(x;a)$ is the one-variable Charlier polynomials. Thus, using the standard limit from the one-variable Charlier polynomials to the one-variable Hermite polynomials, one can set

$$x_1 = \sqrt{2}a\widetilde{x}_1 + a^2, \qquad x_2 = \sqrt{2}b\widetilde{x}_2 + b^2, \qquad \phi = 0, \tag{9.25}$$

in (9.22) and take the limit as $a \to \infty$ and $b \to \infty$ to find

$$\lim_{a\to\infty}\lim_{b\to\infty} e^{-az_1}\left(1+\frac{z_1}{a}\right)^{x_1} e^{-bz_2}\left(1+\frac{z_2}{b}\right)^{x_2} = e^{-\frac{z_1^2}{2}+\sqrt{2}\,\widetilde{x}_1\,z_1} e^{-\frac{z_2^2}{2}+\sqrt{2}\,\widetilde{x}_2\,z_2}.$$

Upon comparing with the well-known generating function for the Hermite polynomials [17], one finds that

$$\lim_{a\to\infty}\lim_{b\to\infty} C_{n_1,n_2}(x_1,x_2)\Big|_{\phi=0} = \sqrt{2^{n_1+n_2}n_1!n_2!}\,H_{n_1}(\widetilde{x}_1)H_{n_2}(\widetilde{x}_2),$$

where $H_n(x)$ are the standard Hermite polynomials. With the parametrization (9.25), it is easily shown using Stirling's approximation that the bivariate Poisson distribution appearing in (9.24) converges to the normal distribution

$$\lim_{a\to\infty} e^{-a^2}\frac{(a^2)^{x_1}}{x_1!} = \frac{e^{-\widetilde{x}_1^2}}{\sqrt{\pi}}.$$

In summary, the wavefunctions (9.20) of the discrete two-dimensional system governed by the Hamiltonian (9.12) tend to a separated product of two univariate Hermite polynomials in the combined limiting process (9.21) and (9.25). This motivates calling (9.12) a discrete oscillator. For other limits of bivariate orthogonal polynomials, see [1].

Remark

It is not needed to take $\phi = 0$ in the second limiting process involving the two-variable Charlier polynomials. If one keeps ϕ arbitrary and performs the change of variable (9.25) and takes the limit as $a \to \infty$ and $b \to \infty$, one simply finds a product of Hermite polynomials in the rotated coordinates $\widehat{x}_1 = \cos\phi\, \widetilde{x}_1 - \sin\phi\, \widetilde{x}_2$ and $\widehat{x}_2 = \sin\phi\, \widetilde{x}_1 - \cos\phi\, \widetilde{x}_2$.

9.4.2 Continuum limit of the raising/lowering operators

Let us now examine the combined limiting procedures of the preceding Subsection and its effect on the defining operators of the discrete oscillator model defined by (9.12). We first consider the raising and lowering operators (9.5) and (9.7) of the bivariate Meixner polynomials. Under the gauge transformation $\widetilde{A}_{\pm}^{(i)} = \omega^{1/2}(x_1,x_2)\, A_{\pm}^{(i)}\, \omega^{-1/2}(x_1,x_2)$, these operators have the expressions

$$\begin{aligned}
\widetilde{A}_{+}^{(i)} &= \Lambda_{1i}\sqrt{x_1}\, T_{x_1}^{-} + \Lambda_{2i}\sqrt{x_2}\, T_{x_2}^{-} - \Lambda_{3i}\sqrt{x_1+x_2+\beta}\,\mathbb{I},\\
\widetilde{A}_{-}^{(i)} &= \Lambda_{1i}\sqrt{x_1+1}\, T_{x_1}^{+} + \Lambda_{2i}\sqrt{x_2+1}\, T_{x_2}^{+} - \Lambda_{3i}\sqrt{x_1+x_2+\beta}\,\mathbb{I},
\end{aligned}$$

for $i = 1,2$. On the wavefunctions (9.20), one has

$$\begin{aligned}
\widetilde{A}_{+}^{(1)}\, \Upsilon_{N,n}^{(\beta+1)}(x_1,x_2) &= \sqrt{n+1}\, \Upsilon_{N+1,n+1}^{(\beta)}(x_1,x_2),\\
\widetilde{A}_{+}^{(2)}\, \Upsilon_{N,n}^{(\beta+1)}(x_1,x_2) &= \sqrt{N-n+1}\, \Upsilon_{N+1,n}^{(\beta)}(x_1,x_2),
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{A}_{-}^{(1)}\, \Upsilon_{N,n}^{(\beta)}(x_1,x_2) &= \sqrt{n}\, \Upsilon_{n-1,N-1}^{(\beta+1)}(x_1,x_2),\\
\widetilde{A}_{-}^{(2)}\, \Upsilon_{N,n}^{(\beta)}(x_1,x_2) &= \sqrt{N-n}\, \Upsilon_{n,N-1}^{(\beta+1)}(x_1,x_2).
\end{aligned}$$

Upon taking the parametrization (9.21) and taking the limit as $\beta \to \infty$, the raising operators become

$$\begin{aligned}
a_{+}^{(1)} &= \lim_{\beta\to\infty} \widetilde{A}_{+}^{(1)} = \cos\phi\, \sqrt{x_1}\, T_{x_1}^{-} - \sin\phi\, \sqrt{x_2}\, T_{x_2}^{-} - (a\cos\phi - b\sin\phi)\,\mathbb{I},\\
a_{+}^{(2)} &= \lim_{\beta\to\infty} \widetilde{A}_{+}^{(2)} = \sin\phi\, \sqrt{x_1}\, T_{x_1}^{-} + \cos\phi\, \sqrt{x_2}\, T_{x_2}^{-} - (a\sin\phi + b\cos\phi)\,\mathbb{I},
\end{aligned} \tag{9.26}$$

and the lowering operators become

$$\begin{aligned}
a_{-}^{(1)} &= \lim_{\beta\to\infty} \widetilde{A}_{-}^{(1)} = \cos\phi\, \sqrt{x_1+1}\, T_{x_1}^{+} - \sin\phi\, \sqrt{x_2+1}\, T_{x_2}^{+} - (a\cos\phi - b\sin\phi)\,\mathbb{I},\\
a_{-}^{(2)} &= \lim_{\beta\to\infty} \widetilde{A}_{-}^{(1)} = \sin\phi\, \sqrt{x_1+1}\, T_{x_1}^{+} + \cos\phi\, \sqrt{x_2+1}\, T_{x_2}^{+} - (a\sin\phi + b\cos\phi)\,\mathbb{I}.
\end{aligned} \tag{9.27}$$

A direct calculation shows that these operators satisfy the commutation relations

$$[a_-^{(i)}, a_+^{(j)}] = \delta_{ij}, \qquad [a_-^{(i)}, a_-^{(j)}] = 0, \qquad [a_+^{(i)}, a_+^{(j)}] = 0.$$

Upon setting $x_1 = \sqrt{2}a\widetilde{x}_1 + a^2$ and $x_2 = \sqrt{2}b\widetilde{x}_2 + b^2$ as in (9.25) and taking the limit as $a \to \infty$ and $b \to \infty$, it is easily seen that the operators (9.26) and (9.27) become the rotated creation/annihilation operators

$$a_+^{(1)} \to \cos\phi\, a_1^\dagger - \sin\phi\, a_2^\dagger, \quad a_+^{(2)} \to \sin\phi\, a_1^\dagger + \cos\phi\, a_2^\dagger,$$
$$a_-^{(1)} \to \cos\phi\, a_1 - \sin\phi\, a_2, \quad a_-^{(2)} \to \sin\phi\, a_1 + \cos\phi\, a_2,$$

where

$$a_i = \frac{x_i + \partial_{x_i}}{\sqrt{2}}, \qquad a_i^\dagger = \frac{x_i - \partial_{x_i}}{\sqrt{2}},$$

are the usual creation/annihilation operators. It immediately follows that in the continuum limit described above in a two-step process, the gauge-transformed Hamiltonian (9.12) and constants of motion (9.14) obtained through $\omega(x_1,x_2)^{1/2}\mathcal{O}\omega^{-1/2}(x_1,x_2)$ tend to the standard two-dimensional oscillator Hamiltonian and the $\mathfrak{su}(2)$ generators in the Schwinger realization.

9.5 Conclusion

In this paper, we have introduced and described a discrete model of the oscillator in two-dimensions based on the bivariate Meixner polynomials. We have shown that this system is superintegrable and that it has the same symmetry algebra as its continuum limit, the standard isotropic oscillator in two dimensions. We have established that the two-variable Meixner polynomials form bases for irreducible representations of $\mathfrak{su}(2)$. We have detailed the limiting processes by which the two-variable Meixner polynomials tend to the bivariate Charlier polynomials and by which the latter tend to a product of standard Hermite polynomials.

In the present paper we have considered for simplicity the two-dimensional case. However, since the theory of multi-variable Meixner is now well established, it is clear that the model can be generalized to any dimensions to give a d-dimensional model of the harmonic oscillator with the same $\mathfrak{su}(d)$ symmetry as the standard quantum oscillator in d-dimensions. Another possible generalization would be to consider, instead of (9.12), a discrete anisotropic oscillator with a Hamiltonian of the form $\mathcal{H} = Y_1 + \alpha Y_2$. It is clear that for rational values of α this model would still be superintegrable, but would exhibit a higher order symmetry algebra.

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Partie II

Systèmes superintégrables avec réflexions

Introduction

Un système quantique possédant d degrés de liberté décrit par un hamiltonien H est dit *maximalement superintégrable* s'il admet $2d-1$ opérateurs de symétrie algébriquement indépendants qui satisfont aux conditions

$$[S_i, H] = 0, \qquad i = 1, 2, \ldots, 2d-1,$$

où l'un des opérateurs de symétrie est l'hamiltonien lui-même [55]. Pour un système quantique superintégrable gouverné par un hamiltonien de la forme

$$H = \Delta + V,$$

où Δ est l'opérateur de Laplace–Beltrami

$$\Delta = \frac{1}{\sqrt{g}} \sum_{ij} \partial_{x_i} (\sqrt{g}\, g^{ij}) \partial_{x_j},$$

et où g^{ij} est la métrique et V est le potentiel, les symétries seront exprimées en termes d'opérateurs différentiels. On dira que le système est superintégrable de degré ℓ si ℓ est le degré maximal des opérateurs de symétrie, excluant cette fois l'hamiltonien. Pour un système maximalement superintégrable, il est impossible que tous les opérateurs de symétrie soient en involution les uns avec les autres; ces derniers engendrent donc une algèbre d'invariance non abélienne.

Les systèmes superintégrables sont d'une grande importance, notamment parce qu'ils peuvent être résolus de manière exacte à la fois analytiquement et algébriquement. Un des exemples classiques de système maximalement superintégrable ($\ell = 1$) est celui de l'oscillateur harmonique en deux dimensions, dont les constantes du mouvement peuvent être obtenues par la construction de Schwinger et engendrent l'algèbre $\mathfrak{su}(2)$ [56].

Lorsque $\ell = 1$, les symétries sont de nature géométrique et engendrent une algèbre de Lie. Les systèmes de ce type sont très bien connus. Lorsque $\ell = 2$, l'algèbre d'invariance est généralement quadratique. En deux dimensions, tous les systèmes superintégrables de ce type ont été identifiés [57]. Le plus général d'entre eux est connu sous le nom de *système générique sur la 2-sphère*: tous les systèmes superintégrables de second ordre en deux dimensions peuvent être obtenus à partir de ce système [58]. On soupçonne également que le système générique sur la 3-sphère joue le même rôle en trois dimensions.

Dans cette partie de la thèse, on étudie des systèmes superintégrables en deux et trois dimensions qui font intervenir des opérateurs de Dunkl, qui contiennent des réflexions. On utilise les opérateurs de Dunkl de rang 1. Ces opérateurs dépendent d'un paramètre réel $\mu \geq 0$ et sont définis par [59]

$$D_i = \partial_{x_i} + \frac{\mu_i}{x_i}(1 - R_i),$$

où R_i est l'opérateur de réflexion $R_i f(x_i) = f(-x_i)$. Il est clair que les opérateurs D_i sont une généralisation à un paramètre de la dérivée partielle usuelle, que l'on retrouve lorsque $\mu_i = 0$. Dans cette partie de la thèse, on considère les systèmes suivants.

- L'oscillateur de Dunkl dans le plan

$$H = -\frac{1}{2}\left[D_1^2 + D_2^2\right] + \frac{1}{2}(x_1^2 + x_2^2).$$

- L'oscillateur singulier de Dunkl dans le plan

$$H = -\frac{1}{2}\left[D_1^2 + D_2^2\right] + \frac{1}{2}(x_1^2 + x_2^2) + \frac{(\alpha_1 + \beta_1 R_1)}{2x_1^2} + \frac{(\alpha_2 + \beta_2 R_2)}{2x_2^2}.$$

- L'oscillateur de Dunkl en trois dimensions

$$H = -\frac{1}{2}\left[D_1^2 + D_2^2 + D_3^2\right] + \frac{1}{2}(x_1^2 + x_2^2 + x_3^2).$$

- Le système générique sur la 2-sphère avec réflexions

$$H = J_1^2 + J_2^2 + J_3^2 + \frac{\mu_1}{x_1^2}(\mu_1 - R_1) + \frac{\mu_2}{x_2^2}(\mu_2 - R_2) + \frac{\mu_3}{x_3^2}(\mu_3 - R_3),$$

  où J_1, J_2 et J_3 sont les opérateurs de moment angulaire

$$J_1 = \frac{1}{i}(x_2\partial_{x_3} - x_3\partial_{x_2}), \quad J_2 = \frac{1}{i}(x_3\partial_{x_1} - x_1\partial_{x_3}), \quad J_3 = \frac{1}{i}(x_1\partial_{x_2} - x_2\partial_{x_1}).$$

On montre que tous ces systèmes sont superintégrables et exactement résolubles, malgré la présence des opérateurs de réflexion. On obtient dans chaque cas leurs symétries, les algèbres d'invariance qu'elles engendrent et leurs représentations. On illustre aussi en quoi ces modèles sont une vitrine pour les polynômes du tableau de Bannai–Ito.

Chapitre 10

The Dunkl oscillator in the plane I : superintegrability, separated wavefunctions and overlap coefficients

V. X. Genest, M. E. H. Ismail, Luc Vinet et A. Zhedanov (2013). The Dunkl oscillator in the plane I : superintegrability, separated wavefunctions and overlap coefficients. *Journal of Physics A: Mathematical and Theoretical* **46** 145201

Abstract. The isotropic Dunkl oscillator model in the plane is investigated. The model is defined by a Hamiltonian constructed from the combination of two independent parabosonic oscillators. The system is superintegrable and its symmetry generators are obtained by the Schwinger construction using parabosonic creation/annihilation operators. The algebra generated by the constants of motion, which we term the Schwinger-Dunkl algebra, is an extension of the Lie algebra $\mathfrak{u}(2)$ with involutions. The system admits separation of variables in both Cartesian and polar coordinates. The separated wavefunctions are respectively expressed in terms of generalized Hermite polynomials and products of Jacobi and Laguerre polynomials. Moreover, the so-called Jacobi-Dunkl polynomials appear as eigenfunctions of the symmetry operator responsible for the separation of variables in polar coordinates. The expansion coefficients between the Cartesian and polar bases (overlap coefficients) are given as linear combinations of dual -1 Hahn polynomials. The connection with the Clebsch-Gordan problem of the $sl_{-1}(2)$ algebra is explained.

10.1 Introduction

This series of two papers is concerned with the analysis of the isotropic Dunkl oscillator model in the plane. The system will be shown to be superintegrable and the representations of its symmetry algebra will be related to different families of -1 orthogonal polynomials [8, 41, 42, 43, 44, 45, 46].

A quantum system defined by a Hamiltonian $\mathcal{H}$ in d dimensions is maximally *superintegrable* if it admits $2d-1$ algebraically independent symmetry operators S_i, $1 \leqslant i \leqslant 2d-1$, that commute with the Hamiltonian

$$[\mathcal{H}, S_i] = 0,$$

where one of the operators is the Hamiltonian itself, e.g. $S_1 \equiv \mathcal{H}$. For a superintegrable system described by a Hamiltonian of the form

$$\mathcal{H} = \Delta + V(x), \quad \Delta = \frac{1}{\sqrt{g}} \sum_{ij} \partial_{x_i} (\sqrt{g} g^{ij}) \partial_{x_j},$$

where Δ is the Laplace–Beltrami operator, the symmetries S_i will be differential operators. In this case, the system is said to be superintegrable of order ℓ if ℓ is the maximum order of the symmetry generators S_i (other than $\mathcal{H}$). One of the most important features of superintegrable models is that they can be exactly solved.

When $\ell = 1$, the constants of motion form a Lie algebra. When $\ell = 2$, the symmetry algebra is quadratic [11, 12, 13, 24, 47]. Substantial work has been done on these systems which are now well understood and classified (see [4, 37, 16, 17, 18, 19, 20, 21, 30] and references therein). Further developments in the study of integrable systems include progress in the classification of superintegrable systems with higher order symmetry [22, 38, 39], the examination of discrete/finite superintegrable models [25] and the exploration of systems involving reflection operators [7, 14, 15, 27, 28, 29, 31, 32, 34, 35].

We here examine the Dunkl oscillator in the plane. This model is possibly the simplest 2D system described by a Hamiltonian involving reflections and corresponds to the combination of two independent parabosonic oscillators. As will be shown, this system possesses many interesting properties. It is second-order superintegrable. Its symmetry algebra, which we term the Schwinger-Dunkl algebra, is obtained using parabosonic creation/annihilation operators in a way that parallels the Schwinger $\mathfrak{su}(2)$ realization in the case of the ordinary 2-dimensional isotropic oscillator; the Schwinger-Dunkl algebra is an extension of the Lie algebra $\mathfrak{u}(2)$ with involutions. The system admits separation of variables in both Cartesian and polar coordinates and its separated wavefunctions can be obtained explicitly in terms of the generalized Hermite, Jacobi and Laguerre polynomials. Furthermore, the study of this model and of the representations of its symmetry algebra will show remarkable occurrences of -1 orthogonal polynomials (OPs) families. The present paper is concerned with the exact solutions of the model, its superintegrability and the calculation of the overlap coefficients between the Cartesian and polar bases. The second paper of the series will focus on the representations of the symmetry algebra and the connections with -1 OPs.

Here is the outline of the paper. In Section 2, we define the Hamiltonian of the Dunkl oscillator and obtain its separated wavefunctions in Cartesian and polar coordinates. We also show that the symmetry operator responsible for the separation of variables in polar coordinates has the so-called Jacobi-Dunkl polynomials as eigenfunctions. In Section 3, we obtain the symmetry algebra of the model in terms of the parabosonic creation/annihilation operators. In Section 4, we show that the overlap coefficients between the Cartesian and polar bases are given by linear combinations of the dual -1 Hahn polynomials. In section 5, we exhibit the relationship between the Dunkl oscillator model and the Clebsch-Gordan problem of $sl_{-1}(2)$ [9, 40].

10.2 The model and exact solutions

The isotropic Dunkl oscillator model in the plane is defined by the Hamiltonian

$$\mathscr{H} = -\frac{1}{2}\left[(\mathscr{D}_x^{\mu_x})^2 + (\mathscr{D}_y^{\mu_y})^2\right] + \frac{1}{2}\left[x^2 + y^2\right], \tag{10.1}$$

where the operator $\mathscr{D}_{x_i}^{\mu_i}$ is the Dunkl derivative

$$\mathscr{D}_{x_i}^{\mu_{x_i}} = \partial_{x_i} + \frac{\mu_{x_i}}{x_i}\left(\mathbb{I} - R_{x_i}\right), \quad x_i \in \{x, y\}, \tag{10.2}$$

with $\mathbb{I}$ the identity operator and $\partial_{x_i} = \frac{\partial}{\partial x_i}$. The operator R_{x_i} is the reflection operator with respect to the plane $x_i = 0$. Hence the reflections in (10.1) have the action

$$R_x f(x,y) = f(-x,y), \qquad R_y f(x,y) = f(x,-y).$$

In connection with the nomenclature of the standard harmonic oscillator, the model is called isotropic because the quadratic potential is $SO(2)$ invariant. For the full Hamiltonian (10.1) to have this symmetry requires of course $\mu_x = \mu_y$. Expanding the square of the Dunkl derivative, one finds

$$(\mathscr{D}_{x_i}^{\mu_{x_i}})^2 = \partial_{x_i}^2 + 2\frac{\mu_{x_i}}{x_i}\partial_{x_i} - \frac{\mu_{x_i}}{x_i^2}\left[\mathbb{I} - R_{x_i}\right].$$

The Schrödinger equation

$$\mathscr{H}\Psi = \mathscr{E}\Psi, \tag{10.3}$$

is manifestly separable in Cartesian coordinates. As shall be seen, even in the presence of reflections, (10.3) also admits separation in polar coordinates. Separation of variables in more than one coordinate system is a signal of superintegrability. This occurs for the Dunkl oscillator because reflections can be viewed as rotations. We provide below the exact separated solutions of (10.3). Note that when $\mu_x = \mu_y = 0$, the Hamiltonian (10.1) corresponds to the standard quantum Harmonic oscillator in the plane.

10.2.1 Solutions in Cartesian coordinates

Since the Hamiltonian (10.1) has the form

$$\mathcal{H} = \mathcal{H}_x + \mathcal{H}_y,$$

where $\mathcal{H}_x$ is the Hamiltonian of the one-dimensional Dunkl oscillator, it is obvious that the solutions to (10.3) in Cartesian coordinates will be given by

$$\Psi(x,y) = \psi(x)\psi(y), \qquad \mathcal{E} = \mathcal{E}_x + \mathcal{E}_y,$$

where $\psi(x_i)$ is an eigenfunction of the 1D Hamiltonian with energy eigenvalue $\mathcal{E}_{x_i}$. For the 1D oscillator $\mathcal{H}_x$, the Schrödinger equation reads

$$\psi''(x) + \frac{2\mu_x}{x}\psi'(x) + (2\mathcal{E}_x - x^2)\psi(x) - \frac{\mu_x}{x^2}(\mathbb{I} - R_x)\psi(x) = 0. \tag{10.4}$$

Since $[\mathcal{H}_x, R_x] = 0$, the eigenfunctions $\psi(x)$ may be chosen to have a definite parity $R_x\psi(x) = s_x\psi(x)$ with $s_x = \pm 1$.

When $s_x = +1$, we have $R_x\psi^+(x) = \psi^+(x)$ and the equation (10.4) has for (admissible) solutions

$$\psi_n^+(x) = \sqrt{\frac{n!}{\Gamma(n+\mu_x+1/2)}}\, e^{-x^2/2} L_n^{(\mu_x-1/2)}(x^2),$$

where $L_n^{(\alpha)}(x)$ are the Laguerre polynomials [23] and where $\Gamma(x)$ denotes the gamma function. The eigenvalues are given by

$$\mathcal{E}_x = 2n + \mu_x + 1/2, \qquad n \in \{0,1,2,\ldots\}.$$

When $s_x = -1$, we have $R_x\psi^-(x) = -\psi^-(x)$ and the solutions to (10.4) are then

$$\psi_m^-(x) = \sqrt{\frac{m!}{\Gamma(m+\mu_x+3/2)}}\, e^{-x^2/2}\, x L_m^{(\mu_x+1/2)}(x^2),$$

with eigenvalues

$$\mathcal{E}_x = 2m + 1 + \mu_x + 1/2, \qquad m \in \{0,1,2,\ldots\}.$$

From the orthogonality relation of the Laguerre polynomials (10.34), it is easily seen that for $\mu_x > -1/2$, the eigenfunction $\psi_n^{\pm}$ obey

$$\int_{-\infty}^{\infty} \psi_n^{s_x}(x)[\psi_m^{s'_x}]^* \,|x|^{2\mu_x} dx = \delta_{nm}\delta_{s_x s'_x},$$

where x^* denotes complex conjugation. From the above considerations, it is clear that the eigenstates of $\mathscr{H}_x$ can be labeled by a single integer n_x whose parity is that of the corresponding wavefunction. For this purpose, we introduce the generalized Hermite polynomials [1, 32, 33]

$$H_{2n+p}^{\mu_x}(x) = (-1)^n \sqrt{\frac{n!}{\Gamma(n+p+\mu_x+1/2)}}\, x^p\, L_n^{(\mu_x-1/2+p)}(x^2),$$

where $p = 0,1$. With this definition, the eigenfunctions of $\mathscr{H}_x$ can be expressed as

$$\psi_{n_x}(x) = e^{-x^2/2} H_{n_x}^{\mu_x}(x), \qquad n_x \in \mathbb{N},$$

with energy eigenvalues $\mathscr{E}_x = n_x + \mu_x + 1/2$.

The eigenfunctions of the one-dimensional Dunkl oscillator are thus normalized and orthogonal on the weighted L^2 space endowed with the scalar product

$$\langle g \mid f \rangle = \int_{-\infty}^{\infty} g^*(x) f(x) |x|^{2\mu_x}\, dx. \tag{10.5}$$

It is directly checked (see Appendix B) that the Dunkl derivative (10.2) is anti-Hermitian with respect to the scalar product (10.5). This establishes that the Hamiltonian (10.1) is Hermitian.

Using the above results for the one-dimensional Dunkl oscillator, it follows that the eigenstates of the full Hamiltonian (10.1) in the Cartesian basis satisfy

$$\mathscr{H}| n_x, n_y \rangle = \mathscr{E}| n_x, n_y \rangle, \qquad \mathscr{E} = n_x + n_y + \mu_x + \mu_y + 1, \tag{10.6}$$

where n_x, n_y are non-negative integers. The wavefunctions have the expression

$$\Psi_{n_x,n_y}(x,y) = e^{-(x^2+y^2)/2} H_{n_x}^{\mu_x}(x) H_{n_y}^{\mu_y}(y),$$

and they satisfy the orthonormality condition

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi_{n_x,n_y}(x,y)\Psi^*_{n'_x,n'_y}(x,y)|x|^{2\mu_x}|y|^{2\mu_y}\, dx\, dy = \delta_{n_x n'_x}\delta_{n_y n'_y},$$

provided that $\mu_x > -1/2$ and $\mu_y > -1/2$. For the 1D case see also [26, 32].

10.2.2 Solutions in polar coordinates

In the polar coordinate system

$$x = \rho\cos\phi, \qquad y = \rho\sin\phi,$$

the Hamiltonian (10.1) can be written as

$$\mathscr{H} = \mathscr{A}_\rho + \frac{1}{\rho^2}\mathscr{B}_\phi,$$

where $\mathscr{A}_\rho$ has the expression

$$\mathscr{A}_\rho = -\frac{1}{2}\left[\partial_\rho^2 + \frac{1}{\rho}\partial_\rho\right] - \frac{1}{\rho}(\mu_x+\mu_y)\partial_\rho + \frac{1}{2}\rho^2,$$

and where $\mathscr{B}_\phi$ is given by

$$\mathscr{B}_\phi = -\frac{1}{2}\partial_\phi^2 + (\mu_x \tan\phi - \mu_y \cot\phi)\partial_\phi + \frac{\mu_x(\mathbb{I}-R_x)}{2\cos^2\phi} + \frac{\mu_y(\mathbb{I}-R_y)}{2\sin^2\phi}. \tag{10.7}$$

For separation of the Dunkl Laplacian in higher dimensions see [3, 5, 6]. The actions of the reflection operators are easily seen to be

$$R_x f(\rho,\phi) = f(\rho,\pi-\phi), \quad R_y f(\rho,\phi) = f(\rho,-\phi).$$

Upon substitution of the separated wavefunction $\Psi(\rho,\phi) = P(\rho)\Phi(\phi)$ in (10.3), one obtains the pair of equations

$$\mathscr{A}_\rho P(\rho) - \mathscr{E}P(\rho) + \frac{m^2}{2\rho^2}P(\rho) = 0, \tag{10.8a}$$

$$\mathscr{B}_\phi \Phi(\phi) - \frac{m^2}{2}\Phi(\phi) = 0, \tag{10.8b}$$

where $m^2/2$ is the separation constant.

We start by examining the angular equation (10.8b); it has the explicit form

$$\Phi'' - 2(\mu_x \tan\phi - \mu_y\cot\phi)\Phi' - \frac{\mu_x(\mathbb{I}-R_x)}{\cos^2\phi}\Phi - \frac{\mu_y(\mathbb{I}-R_y)}{\sin^2\phi}\Phi + m^2\Phi = 0. \tag{10.9}$$

Since $[\mathscr{H},R_x] = [\mathscr{H},R_y] = 0$, we shall label the eigenstates by the eigenvalues $s_x, s_y = \pm 1$ of the reflection operators R_x and R_y.

When $s_x = s_y = +1$, the equation (10.9) has the (admissible) solution

$$\Phi_n^{++} = \sqrt{\frac{(2n+\mu_x+\mu_y)\Gamma(n+\mu_x+\mu_y)n!}{2\Gamma(n+\mu_x+1/2)\Gamma(n+\mu_y+1/2)}}\, P_n^{(\mu_x-1/2,\mu_y-1/2)}(x),$$

with $x = -\cos 2\phi$ and where $P_n^{(\alpha,\beta)}(x)$ denotes the Jacobi polynomials [23]. This solution corresponds to the eigenvalue $m^2 = 4n(n+\mu_x+\mu_y)$ with $n \in \mathbb{N}$.

When $s_x = s_y = -1$, the solutions reads

$$\Phi_n^{--} = \sqrt{\frac{(2n+\mu_x+\mu_y)\Gamma(n+\mu_x+\mu_y+1)(n-1)!}{2\Gamma(n+\mu_x+1/2)\Gamma(n+\mu_y+1/2)}}\, \sin\phi\cos\phi\, P_{n-1}^{(\mu_x+1/2,\mu_y+1/2)}(x),$$

with variable $x = -\cos 2\phi$ and eigenvalue $m^2 = 4n(n+\mu_x+\mu_y)$, $n \in \mathbb{N}$. It is understood that $P_{-1}^{(\alpha,\beta)}(x) = 0$ and hence that $\Phi_0^{--} = 0$.

When $s_x = +1$ and $s_y = -1$, the solution to equation (10.9) is given by

$$\Phi_n^{+-} = \sqrt{\frac{(2n+\mu_x+\mu_y)\Gamma(n+\mu_x+\mu_y+1/2)(n-1/2)!}{2\Gamma(n+\mu_x)\Gamma(n+\mu_y+1)}}\, \sin\phi\, P_{n-1/2}^{(\mu_x-1/2,\mu_y+1/2)}(x),$$

with variable $x = -\cos 2\phi$, eigenvalue $m^2 = 4n(n+\mu_x+\mu_y)$ and where n takes only positive half-integer values $n \in \{1/2, 3/2, 5/2, \ldots\}$.

Lastly, when $s_x = -1$ and $s_y = 1$, the solution to the angular equation has the expression

$$\Phi_n^{-+} = \sqrt{\frac{(2n+\mu_x+\mu_y)\Gamma(n+\mu_x+\mu_y+1/2)(n-1/2)!}{2\Gamma(n+\mu_x+1)\Gamma(n+\mu_y)}} \cos\phi P_{n-1/2}^{(\mu_x+1/2,\mu_y-1/2)}(x),$$

with variable $x = -\cos 2\phi$, eigenvalue $m^2 = 4n(n+\mu_x+\mu_y)$ and where n takes only positive half-integer values $n \in \{1/2, 3/2, 5/2, \ldots\}$.

From the orthogonality relation of the Jacobi polynomials (10.35), it is directly seen that the wavefunctions obey the orthogonality relation

$$\int_0^{2\pi} \Phi_n^{s_x s_y}(\phi)\Phi_m^{s_x' s_y'}(\phi)|\cos\phi|^{2\mu_x}|\sin\phi|^{2\mu_y} d\phi = \delta_{nm}\delta_{s_x s_x'}\delta_{s_y s_y'}.$$

As seen from the above considerations, the value of the separation constant is always $m^2 = 4n(n+\mu_x+\mu_y)$. When the product $s_x s_y = +1$ is positive, n is a non-negative integer. When the product $s_x s_y = -1$ is negative, n is a positive half-integer.

We now examine the radial equation (10.8a). It reads

$$P''(\rho) + \frac{1}{\rho}(1+2\mu_x+2\mu_y)P'(\rho) + \left(2\mathcal{E} - \rho^2 - \frac{m^2}{\rho^2}\right)P(\rho) = 0.$$

This equation has for solutions

$$P_k(\rho) = \sqrt{\frac{2k!}{\Gamma(k+2n+\mu_x+\mu_y+1)}} e^{-\rho^2/2}\rho^{2n} L_k^{(2n+\mu_x+\mu_y)}(\rho^2),$$

with the energy eigenvalues

$$\mathcal{E} = 2(k+n) + \mu_x + \mu_y + 1, \qquad k \in \mathbb{N}.$$

Using the orthogonality relation of the Laguerre polynomials, one finds that the radial wavefunction obeys

$$\int_0^{\infty} P_k(\rho)P_{k'}(\rho)\rho^{1+2\mu_x+2\mu_y} d\rho = \delta_{kk'}.$$

Hence the eigenstates of the Hamiltonian (10.1) in the polar basis can be denoted $|k, n; s_x, s_y\rangle$ and satisfy

$$\mathcal{H}|k,n;s_x,s_y\rangle = \mathcal{E}|k,n;s_x,s_y\rangle, \quad \mathcal{E} = 2(k+n)+\mu_x+\mu_y+1, \tag{10.10}$$

where $k \in \mathbb{N}$ is a non-negative integer and where n is a non-negative integer whenever the product $s_x s_y = +1$ is positive and a positive half-integer whenever the product $s_x s_y = -1$ is negative.

From the equations (10.6) and (10.10), it is seen the states with a given energy $\mathcal{E} = N+\mu_x+\mu_y+1$ exhibit a $N+1$-fold degeneracy. Here are the first few eigenstates :

$\mathcal{E}$	$\vert n_x, n_y\rangle$	$\vert k,n;s_x,s_y\rangle$
$\mathcal{E}_0 = 1+\mu_x+\mu_y$	$\vert 0,0\rangle$	$\vert 0,0;++\rangle$
$\mathcal{E}_1 = 2+\mu_x+\mu_y$	$\vert 1,0\rangle, \vert 0,1\rangle$	$\vert 0,1/2;+-\rangle, \vert 0,1/2;-+\rangle$
$\mathcal{E}_2 = 3+\mu_x+\mu_y$	$\vert 2,0\rangle, \vert 1,1\rangle, \vert 0,2\rangle$	$\vert 1,0;++\rangle, \vert 0,1;++\rangle, \vert 0,1;--\rangle$
$\mathcal{E}_3 = 4+\mu_x+\mu_y$	$\vert 3,0\rangle, \vert 2,1\rangle, \vert 1,2\rangle, \vert 0,3\rangle$	$\vert 1,1/2;+-\rangle, \vert 1,1/2;-+\rangle, \vert 0,3/2;+-\rangle, \vert 0,3/2;-+\rangle$
$\mathcal{E}_4 = 5+\mu_x+\mu_y$	$\vert 4,0\rangle \vert 3,1\rangle, \vert 2,2\rangle, \vert 1,3\rangle, \vert 0,4\rangle$	$\vert 2,0;++\rangle, \vert 1,1;++\rangle, \vert 1,1;--\rangle, \vert 0,2;++\rangle, \vert 0,2;--\rangle$

The presence of these degeneracies can be attributed to the existence of a symmetry algebra that will be identified in Section 3.

10.2.3 Separation of variables and Jacobi-Dunkl polynomials

As is seen from (10.8b), the separation of variables of the Schrödinger equation in polar coordinates is equivalent to the diagonalization of the operator $\mathcal{B}_\phi$. We thus have the following eigenvalue equation:

$$\mathcal{B}_\phi |k,n;s_x,s_y\rangle = \frac{m^2}{2}|k,n;s_x,s_y\rangle, \quad m^2 = 4n(n+\mu_x+\mu_y), \tag{10.11}$$

where $n \in \mathbb{N}$ when $s_x s_y = 1$ and $n \in \{1/2, 3/2, \ldots\}$ when $s_x s_y = -1$. We shall consider the operator

$$\mathcal{J}_2 = i(x\mathcal{D}_y^{\mu_y} - y\mathcal{D}_x^{\mu_x}),$$

which in polar coordinates reads

$$\mathcal{J}_2 = i\left[\partial_\phi + \mu_y \cot\phi(\mathbb{I} - R_y) - \mu_x \tan\phi(\mathbb{I} - R_x)\right].$$

A simple computation shows that the square of the operator $\mathcal{J}_2$ is related to $\mathcal{B}_\phi$ in the following way:

$$\mathcal{J}_2^2 = 2\mathcal{B}_\phi + 2\mu_x\mu_y(\mathbb{I} - R_xR_y). \tag{10.12}$$

Instead of the eigenvalue equation (10.11), we shall consider the one corresponding to the diagonalization of $\mathcal{J}_2$:

$$\mathcal{J}_2 F_\epsilon(\phi) = \lambda_\epsilon F_\epsilon(\phi), \tag{10.13}$$

where $\epsilon = s_x s_y = \pm 1$; this extra label on the eigenvalues λ_ϵ is allowed since R_xR_y commutes with $\mathcal{J}_2$. It follows from (10.11) and (10.12) that the square of the eigenvalues λ_ϵ are given by

$$\lambda_+^2 = 4n(n+\mu_x+\mu_y), \qquad \lambda_-^2 = 4(n+\mu_x)(n+\mu_y), \tag{10.14}$$

where $n \in \mathbb{N}$ when $\epsilon = 1$ and $n = \{1/2, 3/2, \ldots\}$ when $\epsilon = -1$. Moreover, since $s_x s_y = \epsilon$, we have $R_x = \epsilon R_y$. To solve (10.13), we consider the decomposition

$$F_\epsilon(\phi) = f_\epsilon^+(\phi) + f_\epsilon^-(\phi), \tag{10.15}$$

where $R_y f_\epsilon^\pm(\phi) = \pm f_\epsilon^\pm(\phi)$. It is directly seen that given the decomposition (10.15), the eigenvalue equation (10.13) is equivalent to the system of differential equations

$$\begin{aligned}
&\partial_\phi \left[f_\epsilon^+ + f_\epsilon^-\right] + 2\mu_y \cot\phi\, f_\epsilon^- - \mu_x \tan\phi[(1-\epsilon) f_\epsilon^+ + (1+\epsilon) f_\epsilon^-] = -i\lambda_\epsilon[f_\epsilon^+ + f_\epsilon^-],\\
&\partial_\phi \left[-f_\epsilon^+ + f_\epsilon^-\right] + 2\mu_y \cot\phi\, f_\epsilon^- - \mu_x \tan\phi[(\epsilon-1) f_\epsilon^+ + (1+\epsilon) f_\epsilon^-] = -i\lambda_\epsilon[f_\epsilon^+ - f_\epsilon^-],
\end{aligned}$$

where the second equation was obtained from the first one by applying R_y. These equations are easily seen to be equivalent to

$$\begin{aligned}
&\partial_\phi f_\epsilon^- + 2\mu_y \cot\phi f_\epsilon^- - \mu_x \tan\phi(1+\epsilon) f_\epsilon^- = -i\lambda_\epsilon f_\epsilon^+,\\
&\partial_\phi f_\epsilon^+ - \mu_x \tan\phi(1-\epsilon) f_\epsilon^+ = -i\lambda_\epsilon f_\epsilon^-.
\end{aligned}$$

The case $\epsilon = 1$

When $\epsilon = +1$, one has

$$\partial_\phi f_+^- + 2\mu_y \cot\phi f_+^- - 2\mu_x \tan\phi f_+^- = -i\lambda_+ f_+^+ \tag{10.16a}$$

$$\partial_\phi f_+^+ = -i\lambda_+ f_+^-, \tag{10.16b}$$

Substituting (10.16b) in (10.16a) yields the equation

$$\partial_\phi^2 f_+^+ + (2\mu_y \cot\phi - 2\mu_x \tan\phi)\partial_\phi f_+^+ + \lambda_+^2 f_+^+ = 0.$$

Since $\lambda_+^2 = 4n(n + \mu_x + \mu_y)$, we directly obtain the result

$$f_+^+ = P_n^{(\mu_x - 1/2, \mu_y - 1/2)}(x), \quad f_+^- = \frac{i}{\lambda_+}\partial_\phi f_+^+,$$

with $x = -\cos 2\phi$, eigenvalues $\lambda_+ = \pm 2\sqrt{n(n+\mu_x+\mu_y)}$ and $n \in \mathbb{N}$. Consequently, for $\epsilon = +$, the eigensolutions of (10.12) are given by

$$F_+(\phi) = P_n^{(\alpha,\beta)}(x) + \frac{i}{\lambda_+}\partial_\phi P_n^{(\alpha,\beta)}(x), \tag{10.17}$$

where the eigenvalues are given by

$$\lambda_+ = \pm 2\sqrt{n(n+\mu_x+\mu_y)}, \quad n \in \mathbb{N},$$

and where the parameters are $\alpha = \mu_x - 1/2$, $\beta = \mu_y - 1/2$ and $x = -\cos 2\phi$. When $\epsilon = +1$, $R_x = R_y$ and the operator $-i\mathscr{J}_2$ can be written as

$$\Lambda_{\mu_x,\mu_y} = \partial_\phi + \frac{A'_{\mu_x,\mu_y}}{A_{\mu_x,\mu_y}}\frac{(\mathbb{I}-R_y)}{2},$$

where

$$A_{\mu_x,\mu_y} = 2^{2(\mu_x+\mu_y)}(\sin|\phi|)^{2\mu_y}(\cos\phi)^{2\mu_x},$$

with $A'(\phi) = \partial_\phi A(\phi)$. This directly establishes that the polynomials defined by (10.17) correspond to the so-called Jacobi-Dunkl polynomials studied in [2].

It is possible to express the eigenfunctions of $\mathscr{J}_2$ in terms of the wavefunctions, which are eigenfunctions of $\mathscr{B}_\phi$. By taking the derivative of equation (10.9) with respect to ϕ for $s_x = s_y = 1$ and adjusting the normalization, one obtains

$$\partial_\phi \Phi_n^{++}(\phi) = 2\sqrt{n(n+\mu_x+\mu_y)}\,\Phi_n^{--}(\phi).$$

Upon substituting this result in (10.17), one finds that for $\epsilon = +1$, the eigenfunctions $F_+(\phi)$ of $\mathscr{J}_2$ and their corresponding eigenvalues are given by

$$F_+(\phi) = \Phi_n^{++}(\phi) \pm i\,\Phi_n^{--}(\phi), \quad \lambda_+ = \pm 2\sqrt{n(n+\mu_x+\mu_y)}. \tag{10.18}$$

The $\epsilon = -1$ case

When $\epsilon = -1$, the equations (10.2.3) and (10.2.3) become

$$\partial_\phi f_-^- + 2\mu_y \cot\phi\, f_-^- = -i\lambda_- f_-^+,$$
$$\partial_\phi f_-^+ - 2\mu_x \tan\phi\, f_-^+ = -i\lambda_- f_-^-.$$

The first equation is easily rewritten as

$$\partial_\phi^2 f_-^+ + (2\mu_y\cot\phi - 2\mu_x\tan\phi)\partial_\phi f_-^+ + (\lambda_-^2 - 4\mu_x\mu_y)f_-^+ - \frac{2\mu_x}{\cos^2\phi}f_-^+ = 0.$$

Given the value of λ_-^2 defined in (10.14), we directly find

$$f_-^+ = \cos\phi\, P_{n-1/2}^{(\mu_x+1/2,\mu_y-1/2)}(x), \quad f_-^- = \frac{i}{\lambda_-}\left(\partial_\phi f_-^+ - 2\mu_x\tan\phi\, f_-^+\right)$$

with $x = -\cos 2\phi$. For $\epsilon = -1$, the eigenfunctions of $\mathscr{J}_2$ and their corresponding eigenvalues thus take the form

$$F_-(\phi) = f_-^+(\phi) \pm f_-^-(\phi), \quad \lambda_- = \pm 2\sqrt{(n+\mu_x)(n+\mu_y)},$$

where $x=-\cos 2\phi$ and where n is a positive half integer. In terms of the wavefunctions, a straightforward computation leads to the expression

$$F_{-}(\phi)=\Phi_n^{-+}(\phi)\mp i\,\Phi_n^{+-}(\phi),\quad \lambda_{-}=\pm 2\sqrt{(n+\mu_x)(n+\mu_y)}. \tag{10.19}$$

Thus we have obtained the eigenfunctions of the operator $\mathscr{J}_2$ in terms of the wavefunctions, which are the eigenfunctions of $\mathscr{B}_\phi$.

10.3 Superintegrability

In this Section we show that the Dunkl oscillator model in the plane is superintegrable. We recover the spectrum of the Hamiltonian algebraically using the parabosonic creation/annihilation operators and obtain the symmetries using the Schwinger construction.

10.3.1 Dynamical algebra and spectrum

We first consider the dynamical algebra of the Dunkl oscillator model. We introduce two commuting sets of parabosonic creation/annihilation operators

$$A_{x_i}=\frac{1}{\sqrt{2}}\left(x_i+\mathscr{D}_{x_i}^{\mu_{x_i}}\right),\qquad A_{x_i}^{\dagger}=\frac{1}{\sqrt{2}}(x_i-\mathscr{D}_{x_i}^{\mu_{x_i}}),$$

where $x_i\in\{x,y\}$. These operators have the non-zero commutation relations

$$[A_x,A_x^{\dagger}]=\mathbb{I}+2\mu_x R_x,\qquad [A_y,A_y^{\dagger}]=\mathbb{I}+2\mu_y R_y.$$

In terms of creation/annihilation operators, the Hamiltonians $\mathscr{H}_x$ and $\mathscr{H}_y$ have the expression

$$\mathscr{H}_x=\frac{1}{2}\{A_x,A_x^{\dagger}\},\qquad \mathscr{H}_y=\frac{1}{2}\{A_y,A_y^{\dagger}\}, \tag{10.20}$$

where $\{x,y\}=xy+yx$ denotes the anti-commutator. Thus the 2-dimensional Hamiltonian of the Dunkl oscillator (10.1) has the simple form

$$\mathscr{H}=\frac{1}{2}\{A_x,A_x^{\dagger}\}+\frac{1}{2}\{A_y,A_y^{\dagger}\}.$$

In the preceding Section, the eigenvalues $\mathscr{E}$ of $\mathscr{H}$ have been obtained analytically by solving the Schrödinger equation. They can also be obtained algebraically. Indeed, we have the additional commutation relations

$$[\mathscr{H}_{x_i}, A_{x_i}] = -A_{x_i}, \qquad [\mathscr{H}_{x_i}, A_{x_i}^{\dagger}] = A_{x_i}^{\dagger} \tag{10.21a}$$

$$\{A_{x_i}, R_{x_i}\} = \{A_{x_i}^{\dagger}, R_{x_i}\} = 0, \qquad [\mathscr{H}_{x_i}, R_{x_i}] = 0, \tag{10.21b}$$

where $x_i \in \{x, y\}$. It is easily seen from the relations (10.20), (10.21a) and (10.21b) that the operators $\mathscr{H}_{x_i}$, A_{x_i}, $A_{x_i}^{\dagger}$ and R_{x_i} realize two independent copies of the parabosonic algebra which we have related to $sl_{-1}(2)$ in [40]. It follows directly from the above commutation relations that

$$\mathscr{E}_x = n_x + \mu_x + 1/2, \quad \mathscr{E}_y = n_y + \mu_y + 1/2, \quad n_x, n_y \in \mathbb{N}.$$

A direct computation shows that the action of the ladder operators A_x, $A_x^{\dagger}$ on the Cartesian eigenbasis $|\, n_x, n_y\,\rangle$ is given by

$$A_x^{\dagger}|\, n_x, n_y\,\rangle = \sqrt{[n_x+1]_{\mu_x}}|\, n_x+1, n_y\,\rangle, A_x|\, n_x, n_y\,\rangle = \sqrt{[n_x]_{\mu_x}}|\, n_x-1, n_y\,\rangle, \tag{10.22}$$

and that of the reflection R_x by

$$R_x|\, n_x, n_y\,\rangle = (-1)^{n_x}|\, n_x, n_y\,\rangle,$$

where $[n]_\mu$ denotes the 'mu-numbers':

$$[n]_\mu = n + \mu(1-(-1)^n).$$

Analogous formulas hold for the action of A_y, $A_y^{\dagger}$ and R_y.

As noted previously, the spectrum of the Hamiltonian $\mathscr{H}$ has the form

$$\mathscr{E}_N = N + \mu_x + \mu_y + 1, \quad N \in \mathbb{N},$$

and exhibits a $N+1$-fold 'accidental' degeneracy at level N. These degeneracies will be explained in terms of the irreducible representations of the symmetry algebra of the Dunkl oscillator.

10.3.2 Superintegrability and the Schwinger-Dunkl algebra

We now exhibit the symmetries of the Hamiltonian (10.1). Let us consider the operator

$$J_3 = \frac{1}{4}\{A_x, A_x^{\dagger}\} - \frac{1}{4}\{A_y, A_y^{\dagger}\} = \frac{1}{2}\left(\mathscr{H}_x - \mathscr{H}_y\right).$$

It is clear that $[\mathscr{H}, J_3] = 0$ and that J_3 is the symmetry corresponding to separation of variables in Cartesian coordinates. Following the Schwinger construction [36], we further introduce

$$J_2 = \frac{1}{2i}\left(A_x^{\dagger}A_y - A_xA_y^{\dagger}\right).$$

A direct computation shows that J_2 is also a symmetry, i.e. $[\mathscr{H}, J_2] = 0$. In addition, expressing the operator J_2 in terms of Dunkl derivatives shows that

$$J_2 = \frac{1}{2i}\left(x\mathscr{D}_y^{\mu_x} - y\mathscr{D}_x^{\mu_x}\right),$$

and hence $J_2 = -\mathscr{J}_2/2$; it is thus seen from (10.12) that J_2 is associated to the separation of variables in polar coordinates. To obtain the complete symmetry algebra, we define a third operator which also commutes with $\mathscr{H}$:

$$J_1 = \frac{1}{2}\left(A_x^{\dagger}A_y + A_xA_y^{\dagger}\right).$$

A direct computation show that the symmetry operators of the Dunkl oscillator in the plane satisfy the following algebra

$$\begin{aligned}
&\{J_1, R_{x_i}\} = 0, \qquad \{J_2, R_{x_i}\} = 0, \qquad [J_3, R_{x_i}] = 0,\\
&[J_2, J_3] = iJ_1, \qquad [J_3, J_1] = iJ_2,\\
&[J_1, J_2] = i\left[J_3 + J_3(\mu_xR_x + \mu_yR_y) - \mathscr{H}(\mu_xR_x - \mu_yR_y)/2\right],
\end{aligned}$$

with $R_x^2 = R_y^2 = \mathbb{I}$, $x_i \in \{x, y\}$ and where the Hamiltonian $\mathscr{H}$ is a central element. We shall refer to the algebra generated by J_1, J_2, J_3, R_x, R_y and $\mathscr{H}$ as the Schwinger-Dunkl algebra $sd(2)$; special cases of it have appeared in other contexts [9, 14]. It is easily seen that $sd(2)$ is a deformation of the Lie algebra $\mathfrak{u}(2)$ by the two involutions R_x, R_y. The Schwinger-Dunkl algebra admits the Casimir operator [9]

$$C = J_1^2 + J_2^2 + J_3^2 + \frac{1}{2}\mu_xR_x + \frac{1}{2}\mu_yR_y + \mu_x\mu_yR_xR_y,$$

which commutes with all the generators. A direct computation shows that in the present realization, the Casimir operator C takes the value

$$C = \frac{1}{4}\mathscr{H}^2 - \frac{1}{4}.$$

Since $\mathscr{H}$ is a central element, we can define

$$\widetilde{C} = C - \mathscr{H}^2/4 + 1/4,$$

and thus $\widetilde{C} = 0$ in this realization.

The irreducible representations of the Schwinger-Dunkl algebra $sd(2)$ can be used to account for the degeneracies of the Hamiltonian (10.1). We shall postpone this study for the second paper of the present series. Note that upon taking $\mu_x = \mu_y = 0$ in the Schwinger-Dunkl algebra, the involutions cease to play an essential role and one recovers the well-known $\mathfrak{su}(2)$ symmetry algebra of the standard quantum harmonic oscillator in the plane.

10.4 Overlap Coefficients

In this section, we obtain the expansion (overlap) coefficients between the Cartesian and polar bases. These expansion coefficients are denoted by $\langle\, k,n;s_x,s_y \mid n_x,n_y \,\rangle$. It is clear that the coefficients will vanish unless the involved states $\mid k,n;s_x,s_y\,\rangle$ and $\mid n_x,n_y\,\rangle$ belong to the same energy eigenspace. The states in the polar basis are the eigenstates of the operator $\mathscr{B}_\phi$ given in (10.7) and satisfy

$$B_\phi \mid k;n;s_x,s_y\,\rangle = \gamma_n \mid k,n;s_x,s_y\,\rangle, \quad \gamma_n = 2n(n+\mu_x+\mu_y),$$

with n a non-negative integer whenever the product $s_x s_y = 1$ and a positive half-integer otherwise. We can consider the relation

$$\gamma_n \langle\, k,n;s_x,s_y \mid n_x,n_y\,\rangle = \langle\, k,n;s_x,s_y \mid \mathscr{B}_\phi \mid n_x,n_y\,\rangle,$$

and expand the action of $\mathscr{B}_\phi$ on the Cartesian basis to obtain a recursion relation for the overlap coefficients. It will prove more convenient to investigate first the overlap coefficients between the Cartesian basis and the eigenbasis of a new operator $\mathscr{Q}$ related to $\mathscr{J}_2$. The eigenstates of this new operator $\mathscr{Q}$ will then be expanded in terms of the polar basis $\mid k,n;s_x,s_y\,\rangle$ to obtain the desired result. For this part, it is convenient to separate the two eigenvalue sectors corresponding to the value of the product $s_x s_y = \pm 1$.

10.4.1 Overlap coefficients for $s_x s_y = +1$

We start by expressing the energy eigenstates in the polar basis in terms of the eigenstates of $\mathscr{J}_2$. As is seen from (10.18), the eigenvectors of the operator $\mathscr{J}_2$ with eigenvalues

$\kappa_n^{\pm}$ that we denote $|\, n,++\rangle_{\mathscr{J}_2}$ and $|\, n,+-\rangle_{\mathscr{J}_2}$ are given by

$$|\, n,++\rangle_{\mathscr{J}_2} = \frac{1}{\sqrt{2}}\Big(|\, k;n;++\rangle + i|\, k,n;--\rangle\Big), \quad \kappa_n^{+} = 2\sqrt{n(n+\mu_x+\mu_y)},$$
$$|\, n,+-\rangle_{\mathscr{J}_2} = \frac{1}{\sqrt{2}}\Big(|\, k;n;++\rangle - i|\, k,n;--\rangle\Big), \quad \kappa_n^{-} = -2\sqrt{n(n+\mu_x+\mu_y)},$$

for $n \neq 0$. For $n = 0$, one has

$$|\, 0,++\rangle_{\mathscr{J}_2} = |\, k,0;++\rangle, \qquad \kappa_0^{+} = 0.$$

We also recall that $R_y|\, n,++\rangle_{\mathscr{J}_2} = |\, n,+-\rangle_{\mathscr{J}_2}$. We now introduce the operator $\mathscr{Q}$ defined by

$$\mathscr{Q} = i\mathscr{J}_2 R_x - \mu_x R_y - \mu_y R_x - (1/2)R_x R_y. \tag{10.23}$$

The relevance of the operator $\mathscr{Q}$ will become clear in Section 5 when the connection between the Schwinger-Dunkl algebra and the Clebsch-Gordan problem of $sl_{-1}(2)$ will be established. In the sector $s_x s_y = +1$, we have $R_x = R_y$ and $\mathscr{Q}$ may be written as

$$\mathscr{Q} = i\mathscr{J}_2 R_y - \mu_x R_y - \mu_y R_y - (1/2)\mathbb{I}.$$

For $n \neq 0$, the eigenvalues $q_n^{\pm}$ and eigenvectors $|\, n,+\pm\rangle_{\mathscr{Q}}$ of $\mathscr{Q}$ are found to be

$$|\, n,++\rangle_{\mathscr{Q}} = \frac{1}{\sqrt{2}}\left(\zeta_n|\, n,++\rangle_{\mathscr{J}_2} + |\, n,+-\rangle_{\mathscr{J}_2}\right), \; q_n^{+} = -2n - \mu_x - \mu_y - 1/2,$$

and

$$|\, n,+-\rangle_{\mathscr{Q}} = \frac{1}{\sqrt{2}}\left(-\zeta_n|\, n,++\rangle_{\mathscr{J}_2} + |\, n,+-\rangle_{\mathscr{J}_2}\right), \; q_n^{-} = 2n + \mu_x + \mu_y - 1/2,$$

where we have defined

$$\zeta_n = \left[\frac{\mu_x + \mu_y - 2i\sqrt{n(n+\mu_x+\mu_y)}}{2n+\mu_x+\mu_y}\right].$$

This amounts to the diagonalization of a 2×2 matrix. We note that $\zeta_n\zeta_n^{*} = 1$. When $n = 0$, one has directly

$$|\, 0,++\rangle_{\mathscr{Q}} = |\, 0,++\rangle_{\mathscr{J}_2}, \quad q_0^{+} = -\mu_x - \mu_y - 1/2.$$

It is possible to regroup the eigenvalues of $\mathscr{Q}$ into a single expression. We have

$$q_\ell = (-1)^{\ell+1}(\ell + \mu_x + \mu_y + 1/2),$$

and the eigenvectors are given by

$$|\, q_{2j}\,\rangle_{\mathscr{Q}} = \frac{1}{\sqrt{2}}\Big(\zeta_j\,|\, j,++\,\rangle_{\mathscr{J}_2} + (1-\delta_{j0})|\, j,+-\,\rangle_{\mathscr{J}_2}\Big),$$
$$|\, q_{2j+1}\,\rangle_{\mathscr{Q}} = \frac{1}{\sqrt{2}}\Big(-\zeta_{j+1}\,|\, j+1,++\,\rangle_{\mathscr{J}_2} + |\, j+1,+-\,\rangle_{\mathscr{J}_2}\Big).$$

In the previous formulas, it should be understood that for the vector $|\, q_0\,\rangle$ the normalization factor $\sqrt{2}$ is not needed.

Having introduced the operator $\mathscr{Q}$, we examine the overlap coefficients between its eigenstates and the eigenstates of $\mathscr{H}$ in the Cartesian basis for a given energy level $\mathscr{E}_N$. In the sector $s_x s_y = +1$, the possible levels take the energy values

$$\mathscr{E}_N = N + \mu_x + \mu_y + 1,$$

where N is an even integer. The eigenspace $\mathscr{E}_N$ is spanned by the vectors

$$|\, 0,N\,\rangle, |\, 1,N-1\,\rangle, \ldots, |\, m,N-m\,\rangle, \ldots, |\, N,0\,\rangle.$$

We shall denote the overlap coefficients by

$$\langle\, q_\ell \,|\, m,N-m\,\rangle = M^{\ell}_{m,N}.$$

To obtain the expression for the expansion coefficients $M^{\ell}_{m,N}$, we start from the relation

$$q_\ell M^{\ell}_{m,N} = \langle\, q_\ell\,|\,\mathscr{Q}\,|\, m,N-m\,\rangle. \tag{10.24}$$

In terms of the parabosonic creation/annihilation operators, the operator $\mathscr{Q}$ reads

$$\mathscr{Q} = (A_x A_y^\dagger - A_x^\dagger A_y)R_x - (\mu_x+\mu_y)R_x - (1/2)\mathbb{I}. \tag{10.25}$$

Upon substituting (10.25) in (10.24) and using the actions (10.22), there comes

$$q_\ell M^{\ell}_{m,N} = A_{m+1} M^{\ell}_{m+1,N} + B_m M^{\ell}_{m,N} + A_m M^{\ell}_{m-1,N},$$

where

$$A_m = (-1)^m\sqrt{[m]_{\mu_x}[N-m+1]_{\mu_y}}, \quad B_m = (-1)^{m+1}(\mu_x+\mu_y) - 1/2.$$

It follows that the overlap coefficients $M^{\ell}_{m,N}$ can be expressed in terms of polynomials $\mathscr{P}_m(q_\ell)$. Indeed, if we define

$$M^{\ell}_{m,N} = M^{\ell}_{0,N}\,\mathscr{P}_m(q_\ell),$$

with $\mathscr{P}_0(q_\ell) = 1$, it transpires that $\mathscr{P}_m(q_\ell)$ are polynomials of degree m in the variable q_ℓ obeying the three-term recurrence relation

$$q_\ell \mathscr{P}_m(q_\ell) = A_{m+1}\mathscr{P}_{m+1}(q_\ell) + B_m \mathscr{P}_m(q_\ell) + A_m \mathscr{P}_{m-1}(q_\ell). \tag{10.26}$$

Upon introducing the monic polynomials $\widehat{\mathscr{P}}_m(q_\ell)$:

$$\mathscr{P}_m(q_\ell) = \frac{\widehat{\mathscr{P}}_m(q_\ell)}{A_1 \cdots A_m},$$

the recurrence relation (10.26) becomes

$$q_\ell \widehat{\mathscr{P}}_m(q_\ell) = \widehat{\mathscr{P}}_{m+1}(q_\ell) + B_m \widehat{\mathscr{P}}_m(q_\ell) + U_m \widehat{\mathscr{P}}_{m-1}(q_\ell), \tag{10.27}$$

where

$$U_n = A_n^2 = [m]_{\mu_x}[N - m + 1]_{\mu_y}. \tag{10.28}$$

Comparing the formulas (10.27) and (10.28) with the formula (10.36) of Appendix A, it is seen that the polynomials $\widehat{\mathscr{P}}_m(q_\ell)$ correspond to the monic dual -1 Hahn polynomials $Q_n(x_\ell; \alpha, \beta; N)$. We thus have

$$\widehat{\mathscr{P}}_m(q_\ell) = 2^{-m} Q_m(x_\ell, \alpha, \beta; N)$$

where the parameter identification is given by

$$\alpha = 2\mu_y + N + 1, \quad \beta = 2\mu_x + N + 1,$$

and the variable x_ℓ takes the values

$$x_\ell = (-1)^{\ell+1}(2\ell + 2\mu_x + 2\mu_y + 1), \quad \ell = 0, \ldots, N.$$

The value of $M_{0,N}^{\ell}$ can be obtained from the requirement that the overlap coefficients provide a unitary transformation between the two bases. Using the orthogonality relation (10.37) of the dual -1 Hahn polynomials, we obtain

$$\langle q_\ell \,|\, m, N - m \rangle = \sqrt{\frac{\omega_{N-\ell}}{U_1 \cdots U_m}}\, Q_m(x_\ell; \alpha, \beta; N),$$

where $\omega_{N-\ell}$ is the weight function (10.38) of the dual -1 Hahn polynomials and N is an even integer. It is seen from the formula (10.38) of Appendix A that if $\mu_x > -1/2$

and $\mu_y > -1/2$, the weight function $\omega_{N-\ell}$ is positive for all $\ell \in \{0,\ldots,N\}$. The overlap coefficients obey the orthonormality relation

$$\sum_{\ell=0}^{N} \langle\, q_\ell \,|\, m, N-m \,\rangle \langle\, n, N-n \,|\, q_\ell \,\rangle = \delta_{nm}.$$

It now possible to obtain the overlap coefficients between the Cartesian and polar wavefunctions of the Dunkl oscillator. We first observe that the eigenstates of $\mathcal{Q}$ have the expansion

$$|\, q_{2n+p} \,\rangle = \left[\tfrac{1+(-1)^p \zeta_{n+p}}{2}\right] |\, k, n+p\,; ++ \,\rangle + \left[\tfrac{1-(-1)^p \zeta_{n+p}}{2i}\right] |\, k, n+p\,; -- \,\rangle,$$

with $p = 0, 1$; the formula is also valid for $n = p = 0$. The inverse relations have the explicit form

$$|\, k, n; ++ \,\rangle = \left[\frac{\zeta_n - 1}{\zeta_n}\right] |\, q_{2n-1} \,\rangle - \left[\frac{\zeta_n + 1}{2i\zeta_n}\right] |\, q_{2n} \,\rangle,$$
$$|\, k, n; -- \,\rangle = \left[\frac{\zeta_n + 1}{\zeta_n}\right] |\, q_{2n-1} \,\rangle + \left[\frac{1 - \zeta_n}{2i\zeta_n}\right] |\, q_{2n} \,\rangle,$$

for $n \neq 0$. These formulas can be used directly to obtain the overlap coefficients

$$\langle\, k, n; ++ \,|\, m, N-m \,\rangle, \quad \langle\, k, n; -- \,|\, m, N-m \,\rangle,$$

as linear combinations of dual -1 Hahn polynomials.

10.4.2 Overlap coefficients for $s_x s_y = -1$

The overlap coefficients in the parity sector $s_x s_y = -1$ are obtained similarly to the case $s_x s_y = 1$. We again start by writing the energy eigenstates in the polar basis in terms of the eigenstates of the operator $\mathscr{J}_2$. It follows from the relation (10.19) that the eigenstates of the operator $\mathscr{J}_2$ with eigenvalues $\sigma_n^{\pm}$ that we denote by $|\, n, -+ \,\rangle_{\mathscr{J}_2}$ and $|\, n, -- \,\rangle_{\mathscr{J}_2}$ are given by

$$|\, n, -+ \,\rangle_{\mathscr{J}_2} = \frac{1}{\sqrt{2}}\Big(|\, k, n; -+ \,\rangle - i\,|\, k, n; +- \,\rangle\Big), \quad \sigma_n^{+} = 2\sqrt{(n+\mu_x)(n+\mu_y)},$$
$$|\, n, -- \,\rangle_{\mathscr{J}_2} = \frac{1}{\sqrt{2}}\Big(|\, k, n; -+ \,\rangle + i\,|\, k, n; +- \,\rangle\Big), \quad \sigma_n^{-} = -2\sqrt{(n+\mu_x)(n+\mu_y)},$$

where $n \in \{1/2, 3/2, \ldots\}$. In this sector, the operator $\mathcal{Q}$ is equivalent to

$$\mathcal{Q} = i\,\mathscr{J}_2 R_x + (\mu_x - \mu_y) R_x + (1/2)\mathbb{I}.$$

The eigenstates $|\, n, -\pm\, \rangle_{\mathscr{Q}}$ and eigenvalues $q_n^{\pm}$ of $\mathscr{Q}$ are easily found to be

$$|\, n, -+\, \rangle_{\mathscr{Q}} = \frac{1}{\sqrt{2}}\left(\xi_n |\, n, -+\, \rangle_{\mathscr{J}_2} + |\, n, --\, \rangle_{\mathscr{J}_2}\right),\ q_n^{+} = 1/2 - 2n - \mu_x - \mu_y,$$

and

$$|\, n, --\, \rangle_{\mathscr{Q}} = \frac{1}{\sqrt{2}}\left(-\xi_n |\, n, -+\, \rangle_{\mathscr{J}_2} + |\, n, --\, \rangle_{\mathscr{J}_2}\right),\ q_n^{-} = 1/2 + 2n + \mu_x + \mu_y,$$

where

$$\xi_n = \left[\frac{\mu_x - \mu_y + 2i\sqrt{(n+\mu_x)(n+\mu_y)}}{2n + \mu_x + \mu_y}\right].$$

It is easily verified that $\xi_n \xi_n^* = 1$. The eigenstates of $\mathscr{Q}$ can be grouped in a single expression. We write

$$q_\ell = (-1)^{\ell+1}(\ell + \mu_x + \mu_y + 1/2).$$

The eigenvectors have the expressions

$$|\, q_{2j+p}\, \rangle_{\mathscr{Q}} = \frac{1}{\sqrt{2}}\Big((-1)^p \xi_{j+1/2} |\, j + 1/2, -+\, \rangle_{\mathscr{J}_2} + |\, j + 1/2, --\, \rangle_{\mathscr{J}_2}\Big).$$

We now compute the overlap coefficients between the eigenstates of $\mathscr{Q}$ and the eigenstates of $\mathscr{H}$ expressed in the Cartesian basis for a given energy level $\mathscr{E}_N$. In the sector $s_x s_y = -1$, the energy takes the values

$$\mathscr{E}_N = N + \mu_x + \mu_y + 1,$$

where N is an odd integer. The eigenspace corresponding to $\mathscr{E}_N$ is spanned by the vectors

$$|\, 0, N\, \rangle, |\, 1, N-1\, \rangle, \cdots |\, m, N-m\, \rangle, \cdots, |\, N, 0\, \rangle.$$

We denote the overlap coefficients by

$$\langle\, q_\ell \,|\, m, N-m\, \rangle = W^{\ell}_{m,N}.$$

The coefficients $W^{\ell}_{m,N}$ can be computed from the relation

$$q_\ell \langle\, q_\ell \,|\, m, N-m\, \rangle = \langle\, q_\ell \,|\, \mathscr{Q} \,|\, m, N-m\, \rangle. \tag{10.29}$$

In terms of the parabosonic operators, the operator $\mathscr{Q}$ acting on the sector $s_x s_y$ reads

$$\mathscr{Q} = (A_x A_y^\dagger - A_x^\dagger A_y) R_x + (\mu_x - \mu_y) R_x + (1/2)\mathbb{I}. \tag{10.30}$$

Upon substituting (10.30) in (10.29) and using the actions (10.22), one finds the recurrence relation

$$q_\ell W^\ell_{m,N} = A_{m+1} W^\ell_{m+1,N} + \widetilde{B}_m W^\ell_{m,N} + A_m W^\ell_{m-1,N},$$

where

$$A_m = (-1)^m \sqrt{[m]_{\mu_x}[N-m+1]_{\mu_y}}, \quad \widetilde{B}_m = (-1)^m(\mu_x - \mu_y) + 1/2.$$

After writing

$$W^\ell_{m,N} = W^\ell_{0,N}\mathscr{P}_m(q_\ell),$$

with $\mathscr{P}_0(q_\ell) = 1$ and introducing the monic polynomials

$$\mathscr{P}_m(q_\ell) = \frac{\widehat{\mathscr{P}}_m(q_\ell)}{A_1 \cdots A_m},$$

one finds that the polynomials $\widehat{\mathscr{P}}_m(q_\ell)$ satisfy the three-term recurrence relation

$$q_\ell \widehat{\mathscr{P}}_m(q_\ell) = \widehat{\mathscr{P}}_{m+1}(q_\ell) + \widetilde{B}_m \widehat{P}_m(q_\ell) + \widetilde{U}_n \widehat{P}_{m-1}(q_\ell), \tag{10.31}$$

with

$$U_m = [m]_{\mu_x}[N-m+1]_{\mu_y}.$$

By comparing the recurrence relation (10.31) with that of the dual −1 Hahn polynomials (10.36), one obtains

$$\mathscr{P}_m(q_\ell) = 2^{-m} Q_m(x_\ell, \alpha, \beta, N),$$

with the parameter identification

$$\alpha = 2\mu_x, \quad \beta = 2\mu_y,$$

and the variable

$$x_\ell = (-1)^\ell(2\ell + 2\mu_x + 2\mu_y + 1).$$

The requirement that the overlap coefficients provide a unitary transformation leads to the relation

$$\langle q_\ell \mid m, N-m \rangle = \sqrt{\frac{w_\ell}{U_1 \cdots U_m}} Q_m(x_\ell, \alpha, \beta, N),$$

where N is an odd integer. The overlap coefficients satisfy the orthogonality relation

$$\sum_{\ell=0}^{N} \langle\, q_\ell \,|\, m, N-m \,\rangle \langle\, q_\ell \,|\, n, N-n \,\rangle = \delta_{nm}.$$

It is again possible to recover the overlap coefficients between the wavefunctions by expressing the eigenvectors of $\mathcal{Q}$ in terms of the eigenstates in the polar basis. One has

$$|\, q_{2j+p} \,\rangle = \left[\tfrac{1+(-1)^p \xi_{j+1/2}}{2}\right] |\, k, j+1/2; -+ \,\rangle + \left[\tfrac{(-1)^p \xi_{j+1/2} - 1}{2i}\right] |\, k, n; +- \,\rangle,$$

with $p \in \{0,1\}$. The inverse relation reads

$$|\, k, j+1/2; -+ \,\rangle = \left[\tfrac{1+\xi_{j+1/2}}{2\xi_{j+1/2}}\right] |\, q_{2j} \,\rangle + \left[\tfrac{\xi_{j+1/2}-1}{2i\xi_{j+1/2}}\right] |\, q_{2j+1} \,\rangle,$$
$$|\, k, j+1/2; +- \,\rangle = \left[\tfrac{\xi_{j+1/2}-1}{2\xi_{j+1/2}}\right] |\, q_{2j} \,\rangle + \left[\tfrac{1+\xi_{j+1/2}}{2i\xi_{j+1/2}}\right] |\, q_{2j+1} \,\rangle.$$

Hence it is seen that the expansion coefficients between the Cartesian and polar bases are given in terms of linear combinations of dual -1 Hahn polynomials. These coefficients can also be expressed in integral form using the separated wavefunctions obtained in Section 2.

10.5 The Schwinger-Dunkl algebra and the Clebsch-Gordan problem

The Schwinger-Dunkl algebra and the dual -1 Hahn polynomials have both appeared in the examination of the Clebsch-Gordan problem for the Hopf algebra $sl_{-1}(2)$ [9, 40]. In this Section, we explain the relationship between the two contexts. This will clarify the introduction of the operator $\mathcal{Q}$ in the previous Section.

10.5.1 $sl_{-1}(2)$ Clebsch–Gordan coefficients and overlap coefficients

The $sl_{-1}(2)$ algebra is generated by the elements A_0, $A_\pm$ and R with the defining relations

$$[A_0, R] = 0, \quad [A_0, A_\pm] = \pm A_\pm, \quad \{A_\pm, R\} = 0, \quad \{A_+, A_-\} = 2A_0,$$

and $R^2 = \mathbb{I}$. It admits the Casimir operator

$$\mathcal{Q} = A_+ A_- R - A_0 R + (1/2) R,$$

which commutes will all the generators. This algebra has infinite-dimensional irreducible modules $V^{(\epsilon,\mu)}$ spanned by the basis vectors $v_n^{(\epsilon,\mu)}$, $n \in \mathbb{N}$. The action of the generators on the basis vectors is

$$A_0 v_n^{(\epsilon,\mu)} = (n+\mu+1/2)v_n^{(\epsilon,\mu)}, \quad R v_n^{(\epsilon,\mu)} = \epsilon(-1)^n v_n^{(\epsilon,\mu)},$$
$$A_+ v_n^{(\epsilon,\mu)} = \sqrt{[n+1]_\mu}\, v_{n+1}^{(\epsilon,\mu)}, \quad A_- v_n^{(\epsilon,\mu)} = \sqrt{[n]_\mu}\, v_{n-1}^{(\epsilon,\mu)}.$$

It is easily seen that $\mathscr{Q}\, v_n^{(\epsilon,\mu)} = -\epsilon\mu\, v_n^{(\epsilon,\mu)}$.

The $sl_{-1}(2)$ algebra is a Hopf algebra and has a non-trivial co-product. Upon taking the tensor product of two irreducible modules $V^{(\epsilon_1,\mu_1)} \otimes V^{(\epsilon_2,\mu_2)}$ spanned by the basis vectors $e_n^{(\epsilon_1,\mu_1)} \otimes e_m^{(\epsilon_2,\mu_2)}$, one obtains a third module $\widetilde{V}$ (in general not irreducible) by adjoining the action

$$\widetilde{A}_0(v\otimes w) = (A_0 v)\otimes w + v\otimes(A_0 w), \quad \widetilde{R}(v\otimes w) = (Rv)\otimes(Rw),$$
$$\widetilde{A}_\pm(v\otimes w) = (A_\pm v)\otimes(Rw) + v\otimes(A_\pm w),$$

where $v \in V^{(\epsilon_1,\mu_1)}$ and $w \in V^{(\epsilon_2,\mu_2)}$. On $\widetilde{V}$, we have the Casimir element

$$\widetilde{\mathscr{Q}} = (A_-^{(1)}A_+^{(2)} - A_+^{(1)}A_-^{(2)})R^{(1)} - (1/2)R^{(1)}R^{(2)} - \epsilon_1\mu_1 R^{(2)} - \epsilon_2\mu_2 R^{(1)},$$

where the superscripts indicate on which module the generators act; e.g. $A_\pm^{(1)} = A_\pm \otimes \mathbb{I}$.

The eigenvalues of $\widetilde{\mathscr{Q}}$ represent the irreducible modules $V^{(\epsilon_i,\mu_i)}$ appearing in the decomposition of $\widetilde{V} = \bigoplus_i V^{(\epsilon_i \mu_i)}$. The Clebsch-Gordan coefficients of $sl_{-1}(2)$ are the expansion coefficients between the direct product basis $e_n^{(\epsilon_1,\mu_1)} \otimes e_m^{(\epsilon_2,\mu_2)}$ and the eigenbasis $f_k^{(\epsilon_1,\mu_i)}$ of the operator $\widetilde{\mathscr{Q}}$; this corresponds to the 'coupled' basis. Given the addition rule of A_0, one has

$$f_N^{(\epsilon_1,\mu_i)} = \sum_{n_1+n_2=N} C_{n_1 n_2 N}^{\mu_1\mu_2\mu_i}\, e_{n_1}^{(\epsilon_1,\mu_1)} \otimes e_{n_2}^{(\epsilon_2,\mu_2)}, \tag{10.32}$$

where $C_{n_1 n_2 N}^{\mu_1\mu_2\mu_i}$ are the Clebsch-Gordan coefficients, which were shown to be given in terms of dual -1 Hahn polynomials in [9, 40].

In our model, it is seen that the operators $\{\mathscr{H}_x, A_x, A_x^\dagger\}$, and $\{\mathscr{H}_y, A_y, A_y^\dagger\}$ realize the two $sl_{-1}(2)$ modules V^{μ_x} and V^{μ_y}, with $\epsilon_x = \epsilon_y = 1$. The Cartesian basis states $|\, n_x, n_y\,\rangle$ correspond to the direct product basis and the operator $\mathscr{Q}$ given in (10.23) corresponds to the Casimir operator $\widetilde{\mathscr{Q}}$. This explains the origin of the operator $\mathscr{Q}$ in our approach to the overlap coefficients.

10.5.2 Occurrence of the Schwinger-Dunkl algebra

In our model, the Schwinger-Dunkl algebra occurs as the symmetry algebra. The algebra $sd(2)$ also appears in the C.G. problem of $sl_{-1}(2)$ as a 'hidden' algebra. We illustrate how this comes about.

In the C.G. problem, it follows from (10.32) that the following operators act as multiple of the identity:

$$A_0^{(1)} + A_0^{(2)}, \quad \mathscr{Q}^{(1)}, \quad \mathscr{Q}^{(2)}, \quad R^{(1)}R^{(2)}. \tag{10.33}$$

In the direct product basis, in addition to the operators (10.33), the operators

$$\widehat{K}_0 = (A_0^{(1)} - A_0^{(2)})/2, \quad \widehat{R} = R^{(1)}$$

and $R^{(2)}$ are also diagonal. In the 'coupled' basis, in addition to (10.33), we have the Casimir operator $\widehat{K}_1 = \widetilde{\mathscr{Q}}$ which is diagonal. Hence, the tensor product basis corresponds to having the operators (10.33) plus $\widehat{K}_0$ and $\widehat{R}$ in diagonal form and the coupled basis corresponds to having the operators (10.33) and $\widehat{K}_1$ in diagonal form. A direct computation shows that the set $\{\widehat{K}_0, \widehat{K}_1, \widehat{R}\}$ generates the Schwinger-Dunkl algebra [9].

We have thus established the connection between our model and the Clebsch-Gordan problem of the algebra $sl_{-1}(2)$.

10.6 Conclusion

We considered the Dunkl oscillator model and showed that it is a superintegrable system. We have exhibited the symmetry algebra that we called the Schwinger-Dunkl algebra and we have obtained the exact solutions of the Schrödinger equation in terms of Jacobi, Laguerre and generalized Hermite polynomials in Cartesian and polar coordinates. The expansion coefficients between the Cartesian and polar bases have been obtained exactly in terms of linear combinations of dual -1 Hahn polynomials and we established the connection between these overlap coefficients and the Clebsch-Gordan problem of the algebra $sl_{-1}(2)$.

The representations of the symmetry algebra of a superintegrable system explain how the degenerate eigenstates of this system are transformed into each other. In the second series of the paper, we shall consider the representations of the Schwinger-Dunkl algebra. As will be seen, these representations exhibit remarkable occurrences of other -1 polynomials.

It would be of interest to consider in a future study the 3D Dunkl oscillator model, which will provide another example of a superintegrable system with reflections. It was shown in [10] that the Bannai–Ito polynomials occur as Racah coefficients of the algebra $sl_{-1}(2)$. Given the connection between the 2D Dunkl oscillator and the Clecbsch-Gordan problem of $sl_{-1}(2)$, one can expect that the Bannai–Ito polynomials will occur in the description of the 3D Dunkl oscillator model.

10.A Appendix A

10.A.1 Formulas for Laguerre polynomials

The Laguerre polynomials $L_n^{(\alpha)}(x)$ are defined by [23]:

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_1F_1\left[\begin{matrix}-n\\ \alpha+1\end{matrix};x\right],$$

where $(a)_n = (a)(a+1)\cdots(a+n-1)$ is the Pochhammer symbol. They obey the orthogonality relation:

$$\int_0^{\infty} e^{-x}x^{\alpha}L_m^{(\alpha)}(x)L_n^{(\alpha)}(x) = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{nm}, \tag{10.34}$$

for $\alpha > -1$.

10.A.2 Formulas for Jacobi polynomials

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are defined by [23]:

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left[\begin{matrix}-n & n+\alpha+\beta+1\\ & \alpha+1\end{matrix};\frac{1-x}{2}\right]$$

They obey the orthogonality relation:

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}P_m^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(x) = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)n!}\delta_{nm}, \tag{10.35}$$

provided that $\alpha > -1$ and $\beta > -1$.

10.A.3 Formulas for dual -1 Hahn polynomials

The monic dual -1 Hahn polynomials $Q_n(x;\alpha,\beta;N)$ have the recurrence relation [41]:

$$xQ_n(x) = Q_{n+1}(x) + b_nQ_n(x) + u_nQ_{n-1}(x).$$

The recurrence coefficients are given by:

$$u_n = 4[n]_\xi[N-n+1]_\zeta, \qquad b_n = \begin{cases} (-1)^{n+1}(2\xi+2\zeta)-1, & N \text{ even}, \\ (-1)^n(2\zeta-2\xi)-1, & N \text{ odd}, \end{cases} \tag{10.36}$$

where

$$\xi = \begin{cases} \frac{\beta-N-1}{2}, & N \text{ even}, \\ \frac{\alpha}{2}, & N \text{ odd}, \end{cases}, \quad \zeta = \begin{cases} \frac{\alpha-N-1}{2} & N \text{ even}, \\ \frac{\beta}{2}, & N \text{ odd}. \end{cases}$$

They obey the orthogonality relation:

$$\sum_{\ell}^{N} \omega_\ell Q_n(x_\ell) Q_m(x_\ell) = v_n \delta_{nm} \tag{10.37}$$

The weight is given by

$$\omega_{2j+q} = \begin{cases} \frac{(-1)^j(-m)_{j+q}}{j!} \frac{(1-\alpha/2)_j(1-\alpha/2-\beta/2)_j}{(1-\beta/2)_j(m+1-\alpha/2-\beta/2)_{j+q}} \frac{(1-\beta/2)_m}{(1-\alpha/2-\beta/2)_m}, & N \text{ even}, \\ \frac{(-1)^j(-m)_j}{j!} \frac{(1/2+\alpha/2)_{j+q}(1/2+\alpha/2+\beta/2)_j}{(1/2+\beta/2)_{j+q}(m+3/2+\alpha/2+\beta/2)_j} \frac{(1/2+\beta/2)_{m+1/2}}{(1+\alpha/2+\beta/2)_{m+1/2}}, & N \text{ odd}. \end{cases} \tag{10.38}$$

where $q \in \{0,1\}$ and with $m = N/2$, $v_n = u_1 \cdots u_n$. The grid points have the expression

$$x_\ell = \begin{cases} (-1)^\ell(2\ell+1-\alpha-\beta), & N \text{ even}, \\ (-1)^\ell(2\ell+1+\alpha+\beta), & N \text{ odd}. \end{cases}$$

10.B Appendix B

We here indicate how it can be simply seen that the Dunkl derivative (10.2) is anti-Hermitian with respect to the scalar product (10.5). This is recorded for completeness and convenience. We need to check that

$$\langle \psi_2 \,|\, \mathscr{D}_x^\mu \psi_1 \rangle = \int_{-\infty}^{\infty} \psi_2^*(x)[\mathscr{D}_x^\mu \psi_1(x)]|x|^{2\mu_x} dx = -\int_{-\infty}^{\infty} [\mathscr{D}_x^\mu \psi_2(x)]^* \psi_1(x)|x|^{2\mu_x} dx = -\langle \mathscr{D}_x^\mu \psi_2 \,|\, \psi_1 \rangle,$$

for all functions $\psi_1(x)$, $\psi_2(x)$ belonging to the L^2 space associated to the scalar produt (10.5). We split the computation in the four possible parity cases for $\psi_1(x)$, $\psi_2(x)$. This is sufficient since any function can be decomposed into its even and odd parts and since the scalar product (10.5) is linear in its arguments.

In the even-even case, one has $\psi_1(x) = \psi_1(-x)$, $\psi_2(x) = \psi_2(-x)$. It follows that

$$\langle \psi_2 \,|\, \mathscr{D}_x^\mu \psi_1 \rangle = \int_{-\infty}^{\infty} \psi_2^*(x)[\mathscr{D}_x^\mu \psi_1(x)]|x|^{2\mu} dx = \int_{-\infty}^{\infty} \psi_2^*(x)[\partial_x \psi_1(x)]|x|^{2\mu} dx = 0,$$

since the integrand in odd. Similarly, we have $\langle \mathscr{D}_x^\mu \psi_2 \,|\, \psi_1 \rangle = 0$.

In the odd-odd case $\psi_1(x) = -\psi_1(-x)$, $\psi_2(x) = -\psi_2(-x)$ and one obtains

$$\int_{-\infty}^{\infty} \psi_2^*(x)[\mathscr{D}_x^\mu \psi_1(x)]|x|^{2\mu} dx = \int_{-\infty}^{\infty} \psi_2^*(x)\left[\partial_x \psi_1(x) + \frac{2\mu}{x}\psi_1(x)\right]|x|^{2\mu} dx = 0,$$

since the integrand is odd. Similarly, we have $\langle \mathscr{D}_x^\mu \psi_2 \,|\, \psi_1 \rangle = 0$.

In the even-odd case, $\psi_1(x) = -\psi_1(-x)$, $\psi_2(x) = \psi_2(-x)$ and it follows that

$$\begin{aligned}
&\int_{-\infty}^{\infty} \psi_2^*(x)[\mathscr{D}_x^\mu \psi_1(x)]|x|^{2\mu} dx = \int_{-\infty}^{\infty} \psi_2^*(x)\left[\partial_x \psi_1(x) + \frac{2\mu}{x}\psi_1(x)\right]|x|^{2\mu} dx \\
&= 2\int_0^{\infty} \psi_2^*(x)\left[\partial_x \psi_1(x) + \frac{2\mu}{x}\psi_1(x)\right] x^{2\mu} dx \\
&= 2\psi_1(x)\psi_2^*(x)x^{2\mu}\Big|_0^{\infty} - 2\int_0^{\infty} \partial_x[\psi_2^*(x)x^{2\mu}]\psi_1(x) dx + 4\mu \int_0^{\infty} \psi_2^*(x)\psi_1(x)x^{2\mu-1} dx \\
&= -2\int_0^{\infty} [\partial_x \psi_2^*(x)]\psi_1(x)|x|^{2\mu} dx = -\int_{-\infty}^{\infty} [\mathscr{D}_x^\mu \psi_2(x)]^* \psi_1(x)|x|^{2\mu} dx,
\end{aligned}$$

where we have used the vanishing conditions on $\psi_1(x)$, $\psi_2(x)$ at infinity. It thus seen that $\langle \psi_2 \,|\, \mathscr{D}_x^\mu \psi_1 \rangle = -\langle \mathscr{D}_x^\mu \psi_2 \,|\, \psi_1 \rangle$.

In the even-odd case, one has $\psi_1(x) = \psi_1(-x)$, $\psi_2(x) = -\psi_2(-x)$ and one obtains

$$\begin{aligned}
&\int_{-\infty}^{\infty} \psi_2^*(x)[\mathscr{D}_x^\mu \psi_1(x)]|x|^{2\mu} dx = \int_{-\infty}^{\infty} \psi_2^*(x)[\partial_x \psi_1(x)]|x|^{2\mu} dx = 2\int_0^{\infty} \psi_2^*(x)[\partial_x \psi_1(x)] x^{2\mu} dx \\
&= 2\psi_1(x)\psi_2^*(x)x^{2\mu}\Big|_0^{\infty} - 2\int_0^{\infty} \left[\partial_x \psi_2^*(x) + \frac{2\mu}{x}\psi_2^*(x)\right]\psi_1(x)x^{2\mu} dx \\
&= -\int_{-\infty}^{\infty} [\mathscr{D}_x^\mu \psi_2(x)]^* \psi_1(x)|x|^{2\mu} dx,
\end{aligned}$$

where we have used the vanishing conditions on $\psi_1(x)$, $\psi_2(x)$ at infinity. Hence we have $\langle \psi_2 \,|\, \mathscr{D}_x^\mu \psi_1 \rangle = -\langle \mathscr{D}_x^\mu \psi_2 \,|\, \psi_1 \rangle$ and the result

$$\langle \psi_2 \,|\, \mathscr{D}_x^\mu \psi_1 \rangle = -\langle \mathscr{D}_x^\mu \psi_2 \,|\, \psi_1 \rangle.$$

is established in all cases.

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Chapitre 11

The Dunkl oscillator in the plane II : representations of the symmetry algebra

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Abstract. The superintegrability, wavefunctions and overlap coefficients of the Dunkl oscillator model in the plane were considered in the first part. Here finite-dimensional representations of the symmetry algebra of the system, called the Schwinger-Dunkl algebra $sd(2)$, are investigated. The algebra $sd(2)$ has six generators, including two involutions and a central element, and can be seen as a deformation of the Lie algebra $\mathfrak{u}(2)$. Two of the symmetry generators, J_3 and J_2, are respectively associated to the separation of variables in Cartesian and polar coordinates. Using the parabosonic creation/annihilation operators, two bases for the representations of $sd(2)$, the Cartesian and circular bases, are constructed. In the Cartesian basis, the operator J_3 is diagonal and the operator J_2 acts in a tridiagonal fashion. In the circular basis, the operator J_2 is block upper-triangular with all blocks 2×2 and the operator J_3 acts in a tridiagonal fashion. The expansion coefficients between the two bases are given by the Krawtchouk polynomials. In the general case, the eigenvectors of J_2 in the circular basis are generated by the Heun polynomials and their components are expressed in terms of the para-Krawtchouk polynomials. In the fully isotropic case, the eigenvectors of J_2 are generated by little -1 Jacobi or ordinary Jacobi polynomials. The basis in which the operator J_2 is diagonal is considered. In this basis, the defining relations of the Schwinger-Dunkl algebra imply that J_3 acts in a block tridiagonal fashion with all blocks 2×2.

The matrix elements of J_3 in this basis are given explicitly.

11.1 Introduction

This is the second part of this series concerned with the analysis of the isotropic Dunkl oscillator model. In part I, the model has been shown to be superintegrable, the wavefunctions have been obtained in Cartesian and polar coordinates and the overlap coefficients have been found [5]. In the present work, the representations of the symmetry algebra of the model, called the Schwinger-Dunkl algebra (see below), are investigated. As shall be seen, this study entails remarkable connections with special functions such as the Heun, little -1 Jacobi and para-Krawtchouk polynomials.

11.1.1 Superintegrability

One recalls that a quantum system defined by a Hamiltonian H in d dimensions is maximally *superintegrable* if it admits $2d-1$ algebraically independent symmetry generators S_i that commute with the Hamiltonian

$$[S_i, H] = 0, \qquad 1 \leqslant i \leqslant 2d-1,$$

where one of the symmetries is the Hamiltonian itself, e.g. $S_1 \equiv H$. Moreover, a superintegrable system is said to be of order ℓ if ℓ is the maximal order of the symmetries S_i in the momentum variables.

11.1.2 The Dunkl oscillator model

The isotropic Dunkl oscillator model [2, 5, 11] in the plane is possibly the simplest two-dimensional system described by a Hamiltonian involving reflections. It is second-order superintegrable and is defined by the Hamiltonian [5]

$$\mathscr{H} = -\frac{1}{2}\left[(\mathscr{D}_x^{\mu_x})^2 + (\mathscr{D}_y^{\mu_y})^2\right] + \frac{1}{2}[x^2+y^2], \tag{11.1}$$

where $\mathscr{D}_{x_i}^{\mu_{x_i}}$ is the Dunkl derivative [4, 14]

$$\mathscr{D}_{x_i}^{\mu_{x_i}} = \partial_{x_i} + \frac{\mu_{x_i}}{x_i}(\mathbb{I} - R_{x_i}), \qquad \partial_{x_i} = \frac{\partial}{\partial x_i},$$

with $\mathbb{I}$ denoting the identity operator and R_{x_i}, $x_i \in \{x, y\}$, standing for the reflection operator with respect to the $x_i = 0$ axis. Hence the reflections R_x, R_y that appear in the Hamiltonian (11.1) have the action

$$R_x f(x,y) = f(-x,y), \qquad R_y f(x,y) = f(x,-y),$$

and thus evidently $R_{x_i}^2 = \mathbb{I}$. In connection with the nomenclature of the standard harmonic oscillator, the model is called isotropic because the quadratic potential is $SO(2)$ invariant. For the full Hamiltonian (11.1) to have this symmetry requires of course that $\mu_x = \mu_y$.

The Schrödinger equation associated to $\mathscr{H}$ is separable in both Cartesian and polar coordinates. The spectrum of energies $\mathscr{E}$ is given by

$$\mathscr{E}_N = N + \mu_x + \mu_y + 1, \qquad N = n_x + n_y, \tag{11.2}$$

where n_x, n_y are non-negative integers. The wavefunctions are well defined for the values $\mu_x, \mu_y \in (-\frac{1}{2}, \infty)$; the case $\mu_x = \mu_y = 0$ corresponds to the standard quantum harmonic oscillator. It is easily seen from (11.2) that the energy level $\mathscr{E}_N$ exhibits a $N+1$-fold degeneracy.

11.1.3 Symmetries of the Dunkl oscillator

The symmetries of the Dunkl oscillator Hamiltonian (11.1) can be obtained by the Schwinger construction using the parabosonic creation/annihilation operators [5, 8, 10]. We consider the operators [14, 15]

$$A_{\pm}^{x_i} = \frac{1}{\sqrt{2}}(x_i \mp \mathscr{D}_{x_i}^{\mu_{x_i}}), \qquad x_i \in \{x, y\}. \tag{11.3}$$

It is verified that the operators $A_{\pm}^{x_i}$ satisfy the following commutation relations:

$$[A_{-}^{x_i}, A_{+}^{x_i}] = \mathbb{I} + 2\mu_{x_i} R_{x_i}, \qquad \{A_{\pm}^{x_i}, R_{x_i}\} = 0, \tag{11.4}$$

where $\{x, y\} = xy + yx$ denotes the anticommutator. In addition to the commutation relations (11.4), one has

$$[A_{\pm}^{x_i}, A_{\pm}^{x_j}] = [A_{\pm}^{x_i}, R_{x_j}] = [R_{x_i}, R_{x_j}] = 0, \qquad i \neq j. \tag{11.5}$$

In terms of the operators (11.3), the Hamiltonian (11.1) takes the form

$$\mathscr{H} = \frac{1}{2}\{A_{+}^{x}, A_{-}^{x}\} + \frac{1}{2}\{A_{+}^{y}, A_{-}^{y}\} = \mathscr{H}_x + \mathscr{H}_y,$$

where

$$\mathscr{H}_{x_i} = \frac{1}{2}\{A_{+}^{x_i}, A_{-}^{x_i}\} = -\frac{1}{2}(\mathscr{D}_{x_i}^{\mu_{x_i}})^2 + \frac{1}{2}x_i^2, \tag{11.6}$$

is the Hamiltonian of the one-dimensional Dunkl oscillator.

The symmetry generators of the Dunkl oscillator model are as follows. Consider the operator

$$J_3 = \frac{1}{4}\{A_-^x, A_+^x\} - \frac{1}{4}\{A_-^y, A_+^y\}. \tag{11.7}$$

It is directly verified that $[\mathscr{H}, J_3] = 0$. Since J_3 can be written as

$$J_3 = \frac{1}{2}(\mathscr{H}_x - \mathscr{H}_y),$$

using (11.6), it is clear that this symmetry corresponds to the separability of the Schrödinger equation in Cartesian coordinates [5]. A second symmetry generator is given by

$$J_2 = \frac{1}{2i}(A_+^x A_-^y - A_-^x A_+^y). \tag{11.8}$$

It is again directly verified that $[J_2, \mathscr{H}] = 0$. In terms of Dunkl derivatives, this operator has the expression

$$J_2 = \frac{1}{2i}\left(x\mathscr{D}_y^{\mu_y} - y\mathscr{D}_x^{\mu_x}\right),$$

and it has been shown [5] that J_2 is the symmetry corresponding to the separation of variables in polar coordinates. A third symmetry J_1, algebraically dependent of J_2, J_3, is obtained by taking $J_1 = -i[J_2, J_3]$. This additional symmetry generator reads

$$J_1 = \frac{1}{2}(A_+^x A_-^y + A_-^x A_+^y). \tag{11.9}$$

In addition to J_i, $i = 1, \ldots, 3$, it is directly checked that the reflections R_x, R_y also commute with $\mathscr{H}$.

11.1.4 The main object: the Schwinger-Dunkl algebra $sd(2)$

The symmetry algebra of the Dunkl oscillator, called the Schwinger-Dunkl algebra, is denoted $sd(2)$ and defined by the commutation relations

$$\{J_1, R_{x_i}\} = 0, \quad \{J_2, R_{x_i}\} = 0, \quad [J_3, R_{x_i}] = 0, \tag{11.10a}$$

$$[J_2, J_3] = iJ_1, \qquad [J_3, J_1] = iJ_2, \tag{11.10b}$$

$$[J_1, J_2] = i\Big(J_3 + J_3(\mu_x R_x + \mu_y R_y) - \mathscr{H}(\mu_x R_x - \mu_y R_y)/2\Big), \tag{11.10c}$$

where $R_{x_i}^2 = \mathbb{I}$, $x_i \in \{x, y\}$, and the Hamiltonian $\mathscr{H}$ is a central element. The algebra $sd(2)$ admits the Casimir operator [7]

$$C = J_1^2 + J_2^2 + J_3^2 + \mu_x R_x/2 + \mu_y R_y/2 + \mu_x \mu_y R_x R_y,$$

which commutes with all the generators. In the present realization, the Casimir operator C takes the value

$$C = \frac{1}{4}(\mathscr{H}^2 - 1).$$

Note that the involution $P = R_x R_y$ also commutes with all the generators and thus can be viewed as a second Casimir operator. Furthermore, it is easily seen that when $\mu_x = \mu_y = 0$, the Schwinger-Dunkl algebra $sd(2)$ reduces to the Lie algebra $\mathfrak{u}(2)$, which is the symmetry algebra of the standard isotropic 2D quantum oscillator in the plane.

The irreducible representations of $sd(2)$ can be used to account for the degeneracies in the spectrum of $\mathscr{H}$. In finite-dimensional representations of degree $N+1$, the action of the symmetry generators J_1, J_2, J_3, R_x and R_y indicate how the degenerate eigenstates of $\mathscr{H}$ corresponding to the energy value $\mathscr{E}_N$ transform into one another under the action of the symmetries. In the following, three bases for the finite-dimensional irreducible representations of $sd(2)$ will be constructed and the explicit formulas for the action of the symmetry generators on each basis will be derived.

11.1.5 Outline

Here is the outline of the paper. In Section 2, we construct the Cartesian basis in which the symmetry generator J_3 is diagonal and J_2 acts in a tridiagonal fashion. In Section 3, we introduce the circular creation/annihilation operators and study the associated circular basis in which J_2 is block upper-triangular and J_3 is tridiagonal. We show that the interbasis expansion coefficients involve the Krawtchouk polynomials and we derive the spectrum of J_2 algebraically. In Section 4, we obtain the eigenvectors of J_2 in the circular basis for odd-dimensional representations and show that these eigenvectors are generated by the Heun polynomials and that their components are para-Krawtchouk polynomials. The fully isotropic case is shown to involve the little -1 Jacobi polynomials. In Section 5, the eigenvectors of J_2 in the circular basis for even-dimensional representations are studied. In Section 6, we examine the basis in which J_2 is diagonal and show that J_3 acts in a six-diagonal fashion on this basis. We conclude with an outlook.

11.2 The Cartesian basis

In this section the Cartesian basis for the finite-dimensional representations of $sd(2)$ is constructed using the realization (11.7), (11.8), (11.9) of the algebra generators in terms of the creation/annihilation operators (11.3). The representation spaces spanned by the Cartesian basis correspond to the spaces of degenerate wavefunctions with energies $\mathscr{E}_N, N \in \mathbb{N}$, separated in Cartesian coordinates, although a different normalization is used for the basis vectors. The action of the $sd(2)$ generators on the wavefunctions were obtained by a direct computation in [5] using the expressions of the symmetries in terms of Dunkl derivatives. Here the actions of the generators and the spectra of the Hamiltonian $\mathscr{H}$ and the symmetry generator J_3 are obtained in a purely algebraic manner.

The Cartesian basis vectors are labeled by two non-negative integers n_x, n_y and are denoted by $|\, n_x, n_y \,\rangle$. These basis vectors are defined by

$$|\, n_x, n_y \,\rangle = (A_+^x)^{n_x}(A_+^y)^{n_y}|\, 0_x, 0_y \,\rangle, \tag{11.11}$$

where $|\, 0_x, 0_y \,\rangle$ is the "vacuum" vector. The vacuum vector has the defining properties

$$A_-^x|\, 0_x, 0_y \,\rangle = 0, \quad A_-^y|\, 0_x, 0_y \,\rangle = 0, \tag{11.12a}$$

$$R_x|\, 0_x, 0_y \,\rangle = |\, 0_x, 0_y \,\rangle, \quad R_y|\, 0_x, 0_y \,\rangle = |\, 0_x, 0_y \,\rangle. \tag{11.12b}$$

The action of the reflection operators and the creation/annihilation operators on the Cartesian basis vectors can be derived from the above definitions and the commutation/anticommutation relations (11.4) and (11.5). From the anticommutation relations

$$\{A_+^x, R_x\} = 0, \quad \{A_+^y, R_y\} = 0,$$

and the vacuum parity conditions (11.12b), it directly follows that

$$R_x|\, n_x, n_y \,\rangle = (-1)^{n_x}|\, n_x, n_y \,\rangle, \quad R_y|\, n_x, n_y \,\rangle = (-1)^{n_y}|\, n_x, n_y \,\rangle. \tag{11.13}$$

By the definition of the basis vectors (11.11), one has also

$$A_+^x|\, n_x, n_y \,\rangle = |\, n_x + 1, n_y \,\rangle, \quad A_+^y|\, n_x, n_y \,\rangle = |\, n_x, n_y + 1 \,\rangle. \tag{11.14}$$

To derive the action of the operators $A_-^{x_i}$ on the Cartesian basis, one needs the commutator identity

$$[A_-^{x_i}, (A_+^{x_i})^n] = (A_+^{x_i})^{n-1}\left[n + \mu_{x_i}(1-(-1)^n)R_{x_i}\right], \tag{11.15}$$

which is easily proven by induction. It is convenient to introduce the μ-numbers [14]

$$[n]_\mu = n + \mu(1-(-1)^n). \tag{11.16}$$

Using the identity (11.15) and the formulas (11.11), (11.12a) and (11.12b), one finds

$$A_-^x|\, n_x, n_y \,\rangle = (A_+^y)^{n_y}[A_-^x, (A_+^x)^{n_x}]|\, 0_x, 0_y \,\rangle = [n_x]_{\mu_x}|\, n_x - 1, n_y \,\rangle, \tag{11.17}$$

and similarly for A_-^y.

Using the results (11.13), (11.14) and (11.17) along with the expressions of the symmetry generators J_i, $i = 1,2,3$, and $\mathscr{H}$, in terms of the operators $A_{\pm}^{x_i}$ given in (11.7), (11.8) and (11.9), one finds that the action of the symmetries on the Cartesian basis is given by

$$J_2 | n_x, n_y \rangle = \frac{1}{2i}\Big([n_y]_{\mu_y} | n_x+1, n_y-1 \rangle - [n_x]_{\mu_x} | n_x-1, n_y+1 \rangle\Big), \tag{11.18a}$$

$$J_3 | n_x, n_y \rangle = \frac{1}{2}\left(n_x - n_y + \mu_x - \mu_y\right) | n_x, n_y \rangle, \tag{11.18b}$$

and the action of J_1 can be obtained directly by commuting J_2 and J_3. The central element $\mathscr{H}$ has the action

$$\mathscr{H} | n_x, n_y \rangle = (n_x + n_y + \mu_x + \mu_y + 1) | n_x, n_y \rangle.$$

Hence the spectra of the symmetry generator J_3 and of the full Hamiltonian $\mathscr{H}$ of the Dunkl oscillator have been recovered in a purely algebraic manner. As is expected, the symmetry operators $J_1, \ldots J_3$ and the involutions R_x, R_y transform the set of vectors $| n_x, n_y \rangle$ with a given value of $N = n_x + n_y$ into one another; these vectors are the degenerate eigenvectors of $\mathscr{H}$ with energy $\mathscr{E}_N = N + \mu_x + \mu_y + 1$.

The preceding results can be used to define an infinite family of $N+1$-dimensional irreducible modules of the Schwinger-Dunkl algebra $sd(2)$ (11.10). Let μ_x, $\mu_y \in \mathbb{R}$ be real numbers such that μ_x, $\mu_y \in (-1/2, \infty)$ and denote by $V_N^{(\mu_x,\mu_y)}$ the $N+1$-dimensional $\mathbb{C}$-vector space spanned by the basis vectors $v_n^{(\mu_x,\mu_y)}$, $n \in \{0, \ldots, N\}$. Consider the vector space $V_N^{(\mu_x,\mu_y)}$ endowed with the following actions of the $sd(2)$ generators:

$$J_1 v_n^{(\mu_x,\mu_y)} = \frac{1}{2}\left([N-n]_{\mu_y} v_{n+1}^{(\mu_x,\mu_y)} + [n]_{\mu_x} v_{n-1}^{(\mu_x,\mu_y)}\right), \tag{11.19a}$$

$$J_2 v_n^{(\mu_x,\mu_y)} = \frac{1}{2i}\left([N-n]_{\mu_y} v_{n+1}^{(\mu_x,\mu_y)} - [n]_{\mu_x} v_{n-1}^{(\mu_x,\mu_y)}\right), \tag{11.19b}$$

$$J_3 v_n^{(\mu_x,\mu_y)} = \left(n + \frac{1}{2}(\mu_x - \mu_y - N)\right) v_n^{(\mu_x,\mu_y)}, \tag{11.19c}$$

$$R_x v_n^{(\mu_x,\mu_y)} = (-1)^n v_n^{(\mu_x,\mu_y)}, \qquad R_y v_n^{(\mu_x,\mu_y)} = (-1)^{N-n} v_n^{(\mu_x,\mu_y)}, \tag{11.19d}$$

where $[n]_\mu$ denotes the μ-numbers (11.16). The central element $\mathscr{H}$ and the Casimir operator have the actions

$$\mathscr{H} v_n^{(\mu_x,\mu_y)} = (N + \mu_x + \mu_y + 1) v_n^{(\mu_x,\mu_y)},$$

and

$$C v_n^{(\mu_x,\mu_y)} = \frac{1}{4}\left\{(N + \mu_x + \mu_y)(N + 2 + \mu_x + \mu_y)\right\} v_n^{(\mu_x,\mu_y)}.$$

It is clear that $V_N^{(\mu_x,\mu_y)}$ is a $sd(2)$-module and its irreducibility follows from the fact that the μ-numbers appearing in the matrix elements of J_1, J_2 are never zero for $\mu_x, \mu_y \in (-1/2, \infty)$. For $\mu_x = \mu_y = 0$, it is directly seen that the $sd(2)$-module $V_N^{(0,0)}$ reduces to the standard $N+1$-dimensional irreducible $\mathfrak{su}(2)$ module.

11.3 The circular basis

In this section, the circular basis for the finite-dimensional representations of $sd(2)$ is constructed using the left/right circular operators. The actions of the symmetries on this basis are obtained and the spectrum of the generator J_2 is derived from these actions. The expansion coefficients between the circular and Cartesian bases, which involve the Krawtchouk polynomials, are also examined.

The left/right circular operators for the 2D Dunkl oscillator are introduced following the analogous construction in the standard 2D harmonic oscillator [1]. We define

$$A^L_{\pm} = \frac{1}{\sqrt{2}}\left(A^x_{\pm} \mp iA^y_{\pm}\right), \quad A^R_{\pm} = \frac{1}{\sqrt{2}}\left(A^x_{\pm} \pm iA^y_{\pm}\right), \tag{11.20}$$

where $A^{x_i}_{\pm}$ are the creation/annihilation operators of the Dunkl oscillator that obey the commutation relations (11.4). The inverse relations are easily seen to be

$$A^x_{\pm} = \frac{1}{\sqrt{2}}\left(A^L_{\pm} + A^R_{\pm}\right), \quad A^y_{\pm} = \frac{\pm i}{\sqrt{2}}\left(A^L_{\pm} - A^R_{\pm}\right).$$

The left/right operators obey the commutation relations

$$\begin{aligned}
&[A^L_-, A^R_-] = 0, \quad [A^L_+, A^R_+] = 0,\\
&[A^R_-, A^L_+] = \mu_x R_x - \mu_y R_y, \quad [A^L_-, A^R_+] = \mu_x R_x - \mu_y R_y,\\
&[A^L_-, A^L_+] = \mathbb{I} + \mu_x R_x + \mu_y R_y, \quad [A^R_-, A^R_+] = \mathbb{I} + \mu_x R_x + \mu_y R_y,
\end{aligned}$$

and the algebraic relations involving the reflections become

$$R_x A^L_{\pm} = -A^R_{\pm} R_x, \quad R_x A^R_{\pm} = -A^L_{\pm} R_x, \quad R_y A^L_{\pm} = A^R_{\pm} R_y, \quad R_y A^R_{\pm} = A^L_{\pm} R_y. \tag{11.21}$$

The circular basis vectors $|\, n_L, n_R\,\rangle$ are labeled by the two non-negative integers n_L, n_R and are defined by

$$|\, n_L, n_R\,\rangle = (A^L_+)^{n_L}(A^R_+)^{n_R}|\, 0_L, 0_R\,\rangle, \tag{11.22}$$

where $|\, 0_L, 0_R\,\rangle$ is the circular vacuum vector with the properties

$$A^L_-|\, 0_L, 0_R\,\rangle = 0, \quad A^R_-|\, 0_L, 0_R\,\rangle = 0, \tag{11.23a}$$

$$R_x|\, 0_L, 0_R\,\rangle = |\, 0_L, 0_R\,\rangle, \quad R_y|\, 0_L, 0_R\,\rangle = |\, 0_L, 0_R\,\rangle. \tag{11.23b}$$

Given the definition (11.22), one has

$$A^L_+|\, n_L, n_R\,\rangle = |\, n_L+1, n_R\,\rangle, \quad A^R_+|\, n_L, n_R\,\rangle = |\, n_L, n_R+1\,\rangle.$$

From the relations (11.21) and the definition (11.22), it follows that

$$R_x | n_L, n_R \rangle = (-1)^{n_L+n_R} | n_R, n_L \rangle, \quad R_y | n_L, n_R \rangle = | n_R, n_L \rangle. \tag{11.24}$$

Consider the commutator identities

$$[A_-^L, (A_+^L)^{n+1}] = (n+1)(A_+^L)^n + \sum_{\alpha=0}^{n} (A_+^L)^{n-\alpha}(A_+^R)^\alpha \left\{(-1)^\alpha \mu_x R_x + \mu_y R_y\right\},$$
$$[A_-^L, (A_+^R)^{n+1}] = \sum_{\beta=0}^{n} (A_+^L)^{n-\beta}(A_+^R)^\beta \left\{(-1)^{n-\beta} \mu_x R_x - \mu_y R_y\right\},$$

which can be proven straightforwardly by induction. From the definition (11.22), the vacuum conditions (11.23a), (11.23b) and the above identities, the action of A_-^L on the circular basis elements $| n_L, n_r \rangle$ can be derived by a direct computation. For $n_L = n_R$, one has

$$A_-^L | n_L, n_R \rangle = n_L | n_L - 1, n_R \rangle,$$

For $n_L > n_R$, one finds

$$A_-^L | n_L, n_R \rangle = n_L | n_L - 1, n_R \rangle + \sum_{j=n_R}^{n_L-1} \{(-1)^{n_R+j} \mu_x + \mu_y\} | n_L + n_R - j - 1, j \rangle.$$

Finally, for $n_L < n_R$, one obtains

$$A_-^L | n_L, n_R \rangle = n_L | n_L - 1, n_R \rangle - \sum_{j=n_L}^{n_R-1} \{(-1)^{n_R+j} \mu_x + \mu_y\} | n_L + n_R - j - 1, j \rangle.$$

To obtain the corresponding formulas for the action of A_-^R, one needs the identities

$$[A_-^R, (A_+^R)^{n+1}] = (n+1)(A_+^R)^n + \sum_{\alpha=0}^{n} (A_+^L)^{n-\alpha}(A_+^R)^\alpha \left\{(-1)^{n-\alpha} \mu_x R_x + \mu_y R_y\right\},$$
$$[A_-^R, (A_+^L)^{n+1}] = \sum_{\beta=0}^{n} (A_+^L)^{n-\beta}(A_+^R)^\beta \left\{(-1)^{\beta} \mu_x R_x - \mu_y R_y\right\}.$$

Using the same procedure as for A_-^L, we obtain the action of A_-^R. For $n_L = n_R$, we have

$$A_-^R | n_L, n_R \rangle = n_R | n_L, n_R - 1 \rangle.$$

When $n_L > n_R$, one finds

$$A_-^R | n_L, n_R \rangle = n_R | n_L, n_R - 1 \rangle + \sum_{j=n_R}^{n_L-1} \{(-1)^{n_R+j} \mu_x - \mu_y\} | n_L + n_R - j - 1, j \rangle,$$

and for $n_R > n_L$, the result is

$$A_-^R | n_L, n_R \rangle = n_R | n_L, n_R - 1 \rangle - \sum_{j=n_L}^{n_R-1} \{(-1)^{n_R+j} \mu_x - \mu_y\} | n_L + n_R - j - 1, j \rangle.$$

As is seen from the formulas, the operators $A_-^{L/R}$ have the effect of sending the circular basis vectors $|\, n_L, n_R\, \rangle$ to *all* circular basis vectors $|\, i_L, j_R\, \rangle$ with $i_L + j_R = n_L + n_R - 1$.

In terms of the circular operators (11.20), the symmetry generators and the central element $\mathscr{H}$ take the rather symmetric form

$$J_1 = \frac{i}{4}\Big(\{A_+^L, A_-^R\} - \{A_-^L, A_+^R\}\Big), \quad J_2 = \frac{1}{4}\Big(\{A_-^R, A_+^R\} - \{A_-^L, A_+^L\}\Big), \tag{11.25a}$$

$$J_3 = \frac{1}{4}\Big(\{A_-^L, A_+^R\} + \{A_+^L, A_-^R\}\Big), \quad \mathscr{H} = \frac{1}{2}\Big(\{A_-^L, A_+^L\} + \{A_-^R, A_+^R\}\Big). \tag{11.25b}$$

Using the above formulas and the actions of the circular operators $A_\pm^{L/R}$, the matrix elements of the $sd(2)$ generators in the circular basis can be computed; they are given below for J_2 and J_3. The action of the Hamiltonian $\mathscr{H}$ is

$$\mathscr{H}|\, n_L, n_R\, \rangle = (n_L + n_R + \mu_x + \mu_y + 1)|\, n_L, n_R\, \rangle.$$

It is clear that the generators preserve the subspace spanned by the basis vectors $\{|\, n_L, n_R\, \rangle \,|\, n_L + n_R = N\}$. As is seen from the action of $\mathscr{H}$, this corresponds to the space of degenerate eigenstates of $\mathscr{H}$ with energy $\mathscr{E}_N$. The properties of representations of the symmetry generators in the circular basis will now be used to derive the transition matrix from the circular basis to the Cartesian basis and to obtain the eigenvalues of J_2 algebraically.

11.3.1 Transition matrix from the circular to the Cartesian basis

We consider the $N+1$-dimensional energy eigenspace spanned by the circular basis vectors denoted by $|\, n_L, n_R\, \rangle$ with $n_L + n_R = N$ and redefine the basis vectors as follows

$$\mathscr{B}_1 := \{f_0 = |\, 0_L, N_R\, \rangle, f_1 = |\, 1_L, (N-1)_R\, \rangle, \ldots, f_N = |\, N_L, 0_R\, \rangle\}.$$

On this basis, a direct computation shows that the generator J_3 has the action

$$J_3 f_n = \frac{1}{2}\Big\{(N-n) f_{n+1} + \xi f_n + n f_{n-1}\Big\}, \tag{11.26}$$

where we have defined

$$\xi = \mu_x - \mu_y.$$

Since J_3 is diagonal in the Cartesian basis and tridiagonal in the circular basis, the two bases are related by a similarity transformation involving orthogonal polynomials.

Let us consider the decomposition of the Cartesian basis vector $v_j^{(\mu_x,\mu_y)}$ of $V_N^{(\mu_x,\mu_y)}$ on the circular basis

$$v_j^{(\mu_x,\mu_y)} = \sum_{n=0}^{N} C_n(j) f_n, \tag{11.27}$$

where $j \in \{0,\ldots,N\}$. Acting on both sides of (11.27) with J_3 and using (11.19c) and (11.26), one arrives at the following recurrence relation satisfied by the expansion coefficients $C_n(j)$:

$$(2j-N)C_n(j) = (n+1)C_{n+1}(j) + (N-n+1)C_{n-1}(j),$$

with $C_{-1} = 0$. Upon factoring out the initial value

$$C_n(j) = C_0(j)P_n(j),$$

we obtain the recurrence relation

$$(2j-N)P_n(j) = (n+1)P_{n+1}(j) + (N-n+1)P_{n-1}(j), \tag{11.28}$$

where $P_0(j) = 1$. It follows from (11.28) that $P_n(x)$ is a polynomial of degree n in x. Upon substituting $P_n(j) = \widehat{P}_n(j)/n!$, we obtain the normalized recurrence relation

$$(j-N/2)\widehat{P}_n(j) = \widehat{P}_{n+1}(j) + \frac{1}{4}n(N-n+1)\widehat{P}_{n-1}(j).$$

It is directly seen that the polynomials $\widehat{P}_n(j)$ are the monic Krawtchouk polynomials $K_n(x;p,N)$ [9] with parameter $p = 1/2$ and variable x evaluated at $x = j$. We thus have

$$C_n(j) = C_0(j)K_n(j;1/2,N),$$

where the constant $C_0(j)$ can be chosen to ensure the unitarity of the transition matrix by using the orthogonality relation of the Krawtchouk polynomials. Despite the differences that the Dunkl and standard harmonic oscillators exhibit, the relations between the circular and Cartesian bases are identical in both cases.

11.3.2 Matrix representation of J_2 and spectrum

The circular representation space can be used to derive the spectrum of the symmetry operator J_2. To exhibit the structure of J_2, we introduce the following notation for the basis vectors:

$$|\, n_L, n_R \,\rangle = |\, \ell, \pm \,\rangle,$$

where

$$\ell = \lfloor |n_L - n_R|/2 \rfloor, \qquad \pm = \mathrm{sign}(n_R - n_L),$$

where $\lfloor x \rfloor$ is the floor function. We adopt the convention that

$$\mathrm{sign}\, 0 = -1$$

for convenience. We denote by $\mathscr{B}_2$ the circular basis such that $n_L + n_R = N$ with the ordering

$$\mathscr{B}_2 = \{|\,0,+\,\rangle, |\,0,-\,\rangle, |\,1,+\,\rangle, |\,1,-\,\rangle, \ldots, \}$$

As an example, consider the case $N = 3$. The basis reads

$$\mathscr{B}_2 = \{|\,0,+\,\rangle, |\,0,-\,\rangle, |\,1,+\,\rangle, |\,1,-\,\rangle\}$$

and corresponds to the following ordering of the standard circular basis vectors $|\,n_L, n_R\,\rangle$:

$$\mathscr{B}_2 = \{|\,1,2\,\rangle, |\,2,1\,\rangle, |\,0,3\,\rangle, |\,3,0\,\rangle\}.$$

For $N = 4$, one has

$$\mathscr{B}_2 = \{|\,0,-\,\rangle, |\,1,+\,\rangle, |\,1,-\,\rangle, |\,2,+\,\rangle, |\,2,-\,\rangle\},$$

which corresponds to

$$\mathscr{B}_2 = \{|\,2,2\,\rangle, |\,1,3\,\rangle, |\,3,1\,\rangle, |\,0,4\,\rangle, |\,4,0\,\rangle\}.$$

Using the action of the operators $A_{\pm}^{L/R}$ and the formulas (11.25a), (11.25b), the matrix representation of J_2 in the circular basis $\mathscr{B}_2$ is derived in a straightforward manner. We find that for N even, the $N+1$-dimensional square matrix representing J_2 in the basis $\mathscr{B}_2$ is block upper-triangular with all blocks 2×2 in addition to a row of 1×2 blocks. The matrix reads

$$[J_2]_{\mathscr{B}_2} = \begin{pmatrix} 0 & \omega_1 & \omega_2 & \cdots & & \omega_m \\ & \Gamma_1 & \Omega_1 & \Omega_2 & \cdots & \Omega_{m-1} \\ & & \Gamma_2 & \Omega_1 & \cdots & \Omega_{m-2} \\ & & & \ddots & & \vdots \\ & & & & \Gamma_{m-1} & \Omega_1 \\ & & & & & \Gamma_m \end{pmatrix}, \tag{11.29}$$

with $m = N/2$ and where we have

$$\Gamma_k = \begin{pmatrix} k + \zeta/2 & -\zeta/2 \\ \zeta/2 & -k - \zeta/2 \end{pmatrix}, \quad \Omega_k = \begin{cases} \begin{pmatrix} -\xi & \xi \\ -\xi & \xi \end{pmatrix} & k \text{ odd}, \\ \begin{pmatrix} \zeta & -\zeta \\ \zeta & -\zeta \end{pmatrix} & k \text{ even}, \end{cases}, \tag{11.30}$$

with ω_k corresponding to the lower part of Ω_k. We have taken

$$\zeta = \mu_x + \mu_y, \qquad \xi = \mu_x - \mu_y.$$

In the N odd case, one obtains

$$[J_2]_{\mathscr{B}_2} = \begin{pmatrix} \widetilde{\Gamma}_0 & \widetilde{\Omega}_1 & \widetilde{\Omega}_2 & \cdots & \widetilde{\Omega}_m \\ & \widetilde{\Gamma}_1 & \widetilde{\Omega}_1 & \cdots & \widetilde{\Omega}_{m-1} \\ & & \ddots & & \\ & & & \widetilde{\Gamma}_{m-1} & \widetilde{\Omega}_1 \\ & & & & \widetilde{\Gamma}_m \end{pmatrix}, \tag{11.31}$$

with $m = (N-1)/2$ and where

$$\widetilde{\Gamma}_k = \begin{pmatrix} (2k+1+\zeta)/2 & \xi/2 \\ -\xi/2 & -(2k+1+\zeta)/2 \end{pmatrix}, \quad \widetilde{\Omega}_k = \begin{cases} \begin{pmatrix} -\xi & -\zeta \\ \zeta & \xi \end{pmatrix} & k \text{ odd}, \\ \begin{pmatrix} \zeta & \xi \\ -\xi & -\zeta \end{pmatrix} & k \text{ even}, \end{cases}. \tag{11.32}$$

Since in both cases the matrices representing J_2 are block upper-triangular, it follows from elementary linear algebra that the set of eigenvalues of J_2 is the union of the sets of eigenvalues of each diagonal block Γ_k or $\widetilde{\Gamma}_k$. By the direct diagonalization of the 2×2 diagonal blocks, we obtain that when N is even, the eigenvalues of J_2 are given by

$$\lambda_k^{\pm} = \pm\sqrt{k(k+\mu_x+\mu_y)}, \qquad k = 0, \ldots, m,$$

where $m = N/2$ and where the eigenvalue $\lambda_0^- = 0$ is non-degenerate. When N is odd, the spectrum of J_2 has the form

$$\lambda_k^{\pm} = \pm\sqrt{(k+\mu_x+1/2)(k+\mu_y+1/2)}, \qquad k = 0, \ldots, m',$$

where $m' = (N-1)/2$. These eigenvalues are indeed the eigenvalues of J_2 that were obtained in the first part [5] by solving the differential equation arising from the realization of J_2 in terms of Dunkl derivatives; here they have been obtained in a purely algebraic manner. It is seen from the matrices (11.29), (11.31) that when $\mu_x = \mu_y = 0$, the matrices representing J_2 are diagonal. This corresponds to the standard result for the harmonic oscillator, where the circular basis is the eigenbasis of the symmetry J_2.

Given that in the case of the Schwinger-Dunkl algebra $sd(2)$, the circular basis does not diagonalize J_2 directly, it is of interest to inquire about the eigenvectors of J_2 in this basis. This is the subject of the next two sections.

11.4 Diagonalization of J_2: the N even case

This section is devoted to the computation of the eigenvectors of J_2 in the circular basis for the N even case. To perform the calculation, we shall make use of an auxiliary operator $\mathscr{Q}$ whose eigenvectors have been related to those of J_2 in the previous paper [5]. The evaluation of the eigenvectors of $\mathscr{Q}$ is somewhat involved and consequently it is instructive to first expose the main steps of the computation.

Firstly, the structure of $\mathscr{Q}$ will be used to reduce the eigenvalue problem to a system of recurrence relations for the components of the eigenvectors. Secondly, using generating functions, the recurrence system will be transformed into a system of differential equations and the solutions will be expressed in terms of Heun polynomials. Thirdly, using well-known properties of Heun functions, the explicit expressions for the components of the eigenvectors will be obtained in terms of a special case of complementary Bannai-Ito polynomials [6] which correspond to para-Krawtchouk polynomials [18]. Lastly, the relation between the eigenvectors of J_2 and $\mathscr{Q}$ obtained in the first paper [5] will be used to write the final expression for the eigenvectors of J_2 in the circular basis.

11.4.1 The operator $\mathscr{Q}$ and its simultaneous eigenvalue equation

The operator $\mathscr{Q}$ has been used in the first part [5] to obtain the overlap coefficients between the wavefunctions in Cartesian and polar coordinates of the 2D Dunkl oscillator [7]. It is defined in terms of J_2 through the relation

$$\mathscr{Q} = -2iJ_2R_x - \mu_x R_y - \mu_y R_x - (1/2)R_xR_y. \tag{11.33}$$

Given the action (11.24) of the reflections operators, it is seen that R_x, R_y have the following matrix representation in the circular basis $\mathscr{B}_2$:

$$R_x = R_y = \text{diag}(1,\sigma_1,\sigma_1,\cdots,\sigma_1), \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{11.34}$$

Using the formula (11.34) for the reflections and the formulas (11.29), (11.30) for the expression of J_2 in the N even case, one obtains from (11.33)

$$[\mathscr{Q}]_{\mathscr{B}_2} = \begin{pmatrix} \phi_0 & \delta_1 & \delta_2 & \cdots & \delta_m \\ & \Phi_1 & \Delta_1 & \cdots & \Delta_{m-1} \\ & & \ddots & & \vdots \\ & & & \Phi_{m-1} & \Delta_1 \\ & & & & \Phi_m, \end{pmatrix},$$

with $m = N/2$, $\phi_0 = -\zeta - 1/2$ and where

$$\Phi_k = \begin{pmatrix} i\zeta - 1/2 & -2ik - (1+i)\zeta \\ 2ik - (1-i)\zeta & -i\zeta - 1/2 \end{pmatrix}, \quad \Delta_m = \begin{cases} \begin{pmatrix} -2i\xi & 2i\xi \\ -2i\xi & 2i\xi \end{pmatrix} & m \text{ odd} \\ \begin{pmatrix} 2i\zeta & -2i\zeta \\ 2i\zeta & -2i\zeta \end{pmatrix} & m \text{ even} \end{cases}. \tag{11.35}$$

The 1×2 blocks δ_i correspond to the lower part of the blocks Δ_i. From the block upper-triangular structure, it follows that the eigenvalues $v_k^{\pm}$ of $\mathscr{Q}$ are given by

$$v_k^+ = 2k + \zeta - 1/2, \quad v_k^- = -(2k + \zeta + 1/2), \quad k = 1, \ldots, m, \tag{11.36}$$

and we also have $v_0^- = -\zeta - 1/2$. Let us denote by $|\, k, \pm \,\rangle_{\mathscr{Q}}$ the eigenvectors of $\mathscr{Q}$ with eigenvalues $v_k^{\pm}$. We wish to evaluate the components of these eigenvectors in the circular basis. We define their expansion in the circular basis by

$$|\, k, + \,\rangle_{\mathscr{Q}} = \sum_{\substack{\ell=0 \\ \sigma=\pm}}^{k} u_\ell^\sigma(k) |\, \ell, \sigma \,\rangle, \qquad |\, k, - \,\rangle_{\mathscr{Q}} = \sum_{\substack{\ell=0 \\ \sigma=\pm}}^{k} v_\ell^\sigma(k) |\, \ell, \sigma \,\rangle, \tag{11.37}$$

for $k = 1, \ldots, m$ and where the vectors $|\, \ell, \pm \,\rangle$ are vectors of the circular basis $\mathscr{B}_2$. It is clear from the matrix representation of $\mathscr{Q}$ that $|\, 0, - \,\rangle_{\mathscr{Q}} = |\, 0, - \,\rangle$ and thus $v_0^-(0) = 1$.

We shall study the simultaneous eigenvalue equation for the operator $\mathscr{Q}$. Since the matrix representing $\mathscr{Q}$ is block upper-triangular, the matrix of eigenvectors will have the same structure. We define the matrix of eigenvectors of $\mathscr{Q}$ as follows:

$$W = \begin{pmatrix} 1 & \widetilde{V}_{01} & \widetilde{V}_{02} & \cdots & \widetilde{V}_{0m} \\ & V_{11} & V_{12} & \cdots & V_{1m} \\ & & V_{22} & & \vdots \\ & & & \ddots & V_{m-1m} \\ & & & & V_{mm} \end{pmatrix},$$

where

$$V_{\ell,k} = \begin{pmatrix} u_\ell^+(k) & v_\ell^+(k) \\ u_\ell^-(k) & v_\ell^-(k) \end{pmatrix},$$

and where $\widetilde{V}_{\ell k}$ is the 1×2 block corresponding to the lower part of $V_{\ell k}$. The simultaneous eigenvalue equation for the matrix $[\mathscr{Q}]_{\mathscr{B}_2}$ can be written as

$$W \cdot L = [\mathscr{Q}]_{\mathscr{B}_2} \cdot W, \tag{11.38}$$

with

$$L = \text{diag}(\nu_0^{(-)}, \Lambda_1, \cdots, \Lambda_m), \quad \Lambda_k = \begin{pmatrix} 2k + \zeta - 1/2 & 0 \\ 0 & -2k - \zeta - 1/2 \end{pmatrix}.$$

As will be seen, the components (11.37) of the eigenvectors of $\mathscr{Q}$ can be derived from the eigenvalue equation (11.38) by solving the associated system of recurrence relations.

11.4.2 Recurrence relations

It will prove convenient to consider the two sectors corresponding to the eigenvalues ν_k^+, ν_k^- separately. In block form, for $\ell = 1, \ldots, m$ and $k = 1, \ldots, m$, the eigenvalue equation (11.38) can be written in the form

$$V_{\ell k} \Lambda_k = \Phi_\ell V_{\ell k} + \sum_{j=1}^{k-\ell} \Delta_j V_{jk}, \tag{11.39}$$

with Φ_ℓ and Δ_j given in (11.35) and where the range of the sum has been determined by the structure of the eigenvector matrix W.

The ν_k^+ eigenvalue sector

We consider the eigenvectors $|\, k, + \rangle_{\mathscr{Q}}$ of $\mathscr{Q}$ with the expansion

$$|\, k, + \rangle_{\mathscr{Q}} = \sum_{\substack{\ell=0 \\ \sigma=\pm}}^{k} u_\ell^\sigma(k) |\, \ell, \sigma \rangle, \tag{11.40}$$

and associated to the eigenvalue $\nu_k^+ = 2k + \zeta - 1/2$. It is understood that $u_0^+(k)$ does not belong to this decomposition. For $\ell = 1, \ldots, m$, it directly seen that the eigenvalue equation (11.39) is equivalent to the following system of recurrence relations:

$$[2k + (1-i)\zeta] u_\ell^+ = [-2i\ell - (1+i)\zeta] u_\ell^- - 2i \sum_{j=\ell+1}^{k} \{(-1)^{j-\ell} \mu_x + \mu_y\} B_j, \tag{11.41a}$$

$$[2k + (1+i)\zeta] u_\ell^- = [2i\ell - (1-i)\zeta] u_\ell^+ - 2i \sum_{j=\ell+1}^{k} \{(-1)^{j-\ell} \mu_x + \mu_y\} B_j, \tag{11.41b}$$

where we have defined

$$B_j = u_j^- - u_j^+$$

and where the explicit dependence of the components $u_\ell^\pm$ on k has been dropped for notational convenience. The case $\ell = 0$ is treated below. The system of recurrence relations (11.41) is "reversed": the values of $u_i^\pm$ are obtained from the values of $u_j^\pm$ with $i < j$ and $j < p$. The terminating conditions are at $\ell = k$. In this case (11.41) reduces to

$$[2k + (1-i)\zeta]u_k^+ = [-(2k)i - (1+i)\zeta]u_k^-, \tag{11.42a}$$

$$[2k + (1+i)\zeta]u_k^- = [(2k)i - (1-i)\zeta]u_k^+. \tag{11.42b}$$

In accordance to the system (11.42), we choose the following terminating conditions

$$u_k^+ = -i, \quad u_k^- = 1.$$

Upon introducing

$$A_j = u_j^+ + u_j^-,$$

the system (11.41) is directly seen to be equivalent to

$$[k+\zeta]A_\ell = -i(\ell+\zeta)B_\ell - 2i \sum_{j=\ell+1}^{k} \{(-1)^{j-\ell}\mu_x + \mu_y\}B_j, \tag{11.43a}$$

$$[k]B_\ell = i\ell A_\ell. \tag{11.43b}$$

The above system accounts for the $\ell = 0$ case. Indeed, it is seen that $B_0 = 0$ and hence $u_0^- = A_0/2$. These equations can be simplified by factoring out the terminating conditions

$$A_\ell = \alpha_0 \widehat{A}_\ell, \qquad B_\ell = \beta_0 \widehat{B}_\ell,$$

where $\alpha_0 = (1-i)$ and $\beta_0 = (1+i)$. It is seen that the normalized components $\widehat{A}_\ell$, $\widehat{B}_\ell$ are real and satisfy the system

$$(k+\zeta)\widehat{A}_\ell = [\ell+\zeta]\widehat{B}_\ell + 2\sum_{j=\ell+1}^{k} \{(-1)^{j-\ell}\mu_x + \mu_y\}\widehat{B}_j, \tag{11.44a}$$

$$k\widehat{B}_\ell = \ell \widehat{A}_\ell, \tag{11.44b}$$

with the terminating conditions $\widehat{A}_k = \widehat{B}_k = 1$. The system (11.44) can be simplified by introducing the reversed components $\widetilde{a}_\ell = \widehat{A}_{k-\ell}$ and $\widetilde{b}_\ell = \widehat{B}_{k-\ell}$. Using the index n, the system takes the usual form

$$(k+\zeta)\widetilde{a}_n = (k-n+\zeta)\widetilde{b}_n + 2\sum_{\alpha=0}^{n-1}\{(-1)^{n+\alpha}\mu_x + \mu_y\}\widetilde{b}_\alpha \tag{11.45a}$$

$$k\,\widetilde{b}_n = (k-n)\widetilde{a}_n, \tag{11.45b}$$

with the initial conditions $\widetilde{a}_0 = \widetilde{b}_0 = 1$. Hence the components $u_\ell^{\pm}(k)$ of the eigenvector $|\,k,\pm\,\rangle_{\mathscr{Q}}$ of the operator $\mathscr{Q}$ have the expression

$$u_\ell^-(k) = \frac{\alpha_0\widetilde{a}_{k-\ell} + \beta_0\widetilde{b}_{k-\ell}}{2}, \quad u_\ell^+(k) = \frac{\alpha_0\widetilde{a}_{k-\ell} - \beta_0\widetilde{b}_{k-\ell}}{2}, \tag{11.46}$$

where $\widetilde{a}_n$ and $\widetilde{b}_n$ are the unique solutions to the system (11.45).

The ν_k^- eigenvalue sector

We consider the eigenvectors $|\,k,-\,\rangle_{\mathscr{Q}}$ corresponding to the eigenvalue ν_k^- of $\mathscr{Q}$ with the circular basis expansion

$$|\,k,-\,\rangle_{\mathscr{Q}} = \sum_{\substack{\ell=0 \\ \sigma=\pm}}^{k} v_\ell^\sigma(k)|\,\ell,\sigma\,\rangle,$$

and associated eigenvalue $\nu_k^- = -2k-\zeta-1/2$. An analysis similar to the preceding one shows that the components $v_\ell^{\pm}(k)$ differ from the components $u_\ell^{\pm}(k)$ only by their terminating conditions. Again choosing $v_k^-(k)=1$, we find

$$v_k^+(k) = \frac{(1+i)k+\zeta}{(1-i)k-\zeta}, \qquad v_k^-(k) = 1.$$

This yields

$$v_\ell^-(k) = \frac{\gamma_0\widetilde{a}_{k-\ell} + \epsilon_0\widetilde{b}_{k-\ell}}{2}, \quad v_\ell^+(k) = \frac{\gamma_0\widetilde{a}_{k-\ell} + \epsilon_0\widetilde{b}_{k-\ell}}{2}, \tag{11.47}$$

where

$$\gamma_0 = \frac{2i(k+\zeta)}{(1+i)k+i\zeta}, \quad \epsilon_0 = \frac{2k}{(1+i)k+i\zeta},$$

and where $v_0^- = \gamma_0\widetilde{a}_k$.

11.4.3 Generating function and Heun polynomials

We have seen that the evaluation of the components of the eigenvectors of $\mathscr{Q}$ in the circular basis depends on the solution of the recurrence system (11.45). As it turns out, an explicit solution for $\widetilde{a}_n(k)$ and $\widetilde{b}_n(k)$ can be obtained using generating functions.

We introduce the ordinary generating functions

$$\widetilde{A}(z) = \sum_{n\geqslant 0} \widetilde{a}_n z^n, \qquad \widetilde{B}(z) = \sum_{n\geqslant 0} \widetilde{b}_n z^n.$$

We shall make use of the elementary identities

$$z\partial_z \widetilde{A}(z) = \sum_{n\geqslant 0} n\widetilde{a}_n z^n, \qquad (1-z)^{-1}\widetilde{A}(z) = \sum_{n\geqslant 0}\left(\sum_{0\leqslant k\leqslant n} \widetilde{a}_k\right) z^n, \tag{11.48a}$$

$$(1+z)^{-1}\widetilde{A}(z) = \sum_{n\geqslant 0}\left(\sum_{0\leqslant k\leqslant n} (-1)^{k+n}\widetilde{a}_k\right) z^n. \tag{11.48b}$$

Using the above identities, it is easily seen that the system of recurrence relations (11.45) for the quantities $\widetilde{a}_n$, $\widetilde{b}_n$ is equivalent to the following system of differential equations for the generating functions $\widetilde{A}(z)$, $\widetilde{B}(z)$:

$$(k+\zeta)\widetilde{A}(z) = (k-\zeta - z\partial_z)\widetilde{B}(z) + \frac{2\mu_x}{1+z}\widetilde{B}(z) + \frac{2\mu_y}{1-z}\widetilde{B}(z), \tag{11.49a}$$

$$k\widetilde{B}(z) = (k - z\partial_z)\widetilde{A}(z). \tag{11.49b}$$

By direct substitution, we find that the generating function $\widetilde{A}(z)$ satisfies the second-order differential equation

$$\widetilde{A}''(z) + \left(\frac{1-2k-\zeta}{z} + \frac{2\mu_y}{z-1} + \frac{2\mu_x}{1+z}\right)\widetilde{A}'(z) + \left(\frac{-2k\zeta z + 2k\xi}{z(z-1)(z+1)}\right)\widetilde{A}(z) = 0. \tag{11.50}$$

This corresponds to Heun's differential equation [3, 13]. The general form of the Heun differential equation is

$$w''(z) + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a}\right)w'(z) + \frac{\alpha\beta z - q}{z(z-1)(z-a)}w(z) = 0, \tag{11.51}$$

with $\alpha + \beta + 1 = \gamma + \delta + \epsilon$. Comparing (11.51) with (11.50), we thus write

$$\widetilde{A}(z) = H\ell(a, q; \alpha, \beta, \gamma, \delta, z) \tag{11.52}$$

with the parameters

$$a = -1, \quad q = 2k(\mu_y - \mu_x), \quad \alpha = -2k, \tag{11.53a}$$

$$\beta = \mu_x + \mu_y, \quad \gamma = 1 - 2k - \mu_x - \mu_y, \quad \delta = 2\mu_y. \tag{11.53b}$$

The function $H\ell(a, q; \alpha, \beta, \gamma, \delta)$ denotes the solution to (11.51) that corresponds to the exponent 0 at $z = 0$ and assumes the value 1 at that point. This is obviously the case of $\widetilde{A}(z)$. It will be seen that $\widetilde{A}(z)$ is in fact a polynomial of degree $2k$, and hence that the Heun function (11.52) is in fact a *Heun polynomial*. Given the system (11.49), we also have

$$\widetilde{B}(z) = k^{-1}(k - z\partial_z)\widetilde{A}(z).$$

11.4.4 Expansion of Heun polynomials in the complementary Bannai-Ito polynomials

The well-studied properties of Heun functions can be used to obtain a closed form formula for the coefficients $\widetilde{a}_n$ and hence for $\widetilde{b}_n$. In what follows, it will be shown that the Heun polynomial in (11.52) can be expanded in terms of a special case of the complementary Bannai-Ito polynomials corresponding to the para-Krawtchouk polynomials.

Consider the solution $H\ell(a,q;\alpha,\beta,\gamma,\delta)$ to the equation (11.51) and its Maclaurin expansion

$$H\ell(a,q;\alpha,\beta,\gamma,\delta) = \sum_{n=0}^{\infty} c_n z^n,$$

where $c_{-1} = 0$, $c_0 = 1$. The coefficients c_n obey the three-term recurrence relation [3, 13]

$$R_n c_{n+1} - (Q_n + q)c_n + P_n c_{n-1} = 0,$$

where

$$R_n = a(n+1)(n+\gamma), \quad Q_n = n\big[(n-1+\gamma)(1+a) + a\delta + \epsilon\big], \tag{11.54a}$$

$$P_n = (n-1+\alpha)(n-1+\beta). \tag{11.54b}$$

The identification of $\widetilde{A}(z)$ as a Heun function enables one to reduce the evaluation of $\widetilde{a}_n$ to the solution of a three-term recurrence relation. It is seen that with the choice of paramaters (11.53), the expansion coefficients c_n of the Heun function $\widetilde{A}(z)$ truncate at degree $k = 2n$. Hence $\widetilde{A}(z)$ is a polynomial of degree $2k$ in z. For convenience, we use the symbol $\mathscr{P}_n = \widetilde{a}_n$ in the computations to follow. Using the parameters (11.53) in the recurrence coefficients (11.54) for the expansion coefficients in

$$\widetilde{A}(z) = \sum_{n\geqslant 0} \mathscr{P}_n z^n,$$

we can obtain the recurrence relation for $\mathscr{P}_n$. Upon setting $N = 2k$ and dividing by $(N-2n)$ we find that $\mathscr{P}_n(\xi)$ is a symmetric polynomial of degree n in the variable ξ obeying the recurrence relation

$$\sigma_{n+1}\mathscr{P}_{n+1}(\xi) - \kappa_n \mathscr{P}_{n-1}(\xi) = \xi\,\mathscr{P}_n(\xi),$$

with $\mathscr{P}_{-1} = 0$, $\mathscr{P}_1 = 1$ and where

$$\sigma_n = \frac{n(N+\zeta-n)}{(2n-N-2)}, \quad \kappa_n = \frac{(N+1-n)(n+\zeta-1)}{(2n-N)}.$$

As k takes integer values, it is seen that when $k=n$, a singularity appears in the recurrence relation. Notwithstanding this, we proceed with the computation; the effect of the pole in $k=n$ on the results is treated below.

Introducing the monic polynomials $\mathscr{P}_n(\xi)=\frac{\widehat{\mathscr{P}}_n(\xi)}{\sigma_1\cdots\sigma_n}$, the recurrence relation becomes

$$\widehat{\mathscr{P}}_{n+1}(\xi)+u_n\widehat{\mathscr{P}}_{n-1}(\xi)=\xi\,\widehat{\mathscr{P}}_n(\xi), \tag{11.55}$$

where

$$u_n=-\frac{n(N+1-n)(N-n+\zeta)(n+\zeta-1)}{(N-2n)(N-2n+2)}.$$

The monic polynomials $\widehat{\mathscr{P}}_n(\xi)$ can be identified with the complementary Bannai-Ito polynomials (CBI).

The monic CBI polynomials [6, 16], denoted $I_n(x;\rho_1,\rho_2,r_1,r_2)$, obey the recurrence relation

$$I_{n+1}(x)+(-1)^n\rho_2 I_n(x)+\tau_n I_{n-1}(x)=xI_n(x), \tag{11.56}$$

where

$$\tau_{2n}=-\frac{n(n+\rho_1-r_1+1/2)(n+\rho_1-r_2+1/2)(n-r_1-r_2)}{(2n+g)(2n+g+1)}, \tag{11.57a}$$

$$\tau_{2n+1}=-\frac{(n+g+1)(n+\rho_1+\rho_2+1)(n+\rho_2-r_1+1/2)(n+\rho_2-r_2+1/2)}{(2n+g+1)(2n+g+2)} \tag{11.57b}$$

and with $g=\rho_1+\rho_2-r_1-r_2$. They have the hypergeometric representation

$$I_{2n}(x)=R_n(x),\quad I_{2n+1}(x)=(x-\rho_2)Q_n(x),$$

where

$$R_n(x)=\eta_n\,{}_4F_3\left[\begin{matrix}-n,n+g+1,\rho_2+x,\rho_2-x\\ \rho_1+\rho_2+1,\rho_2-r_1+1/2,\rho_2-r_2+1/2\end{matrix};1\right], \tag{11.58}$$

$$Q_n(x)=\iota_n\,{}_4F_3\left[\begin{matrix}-n,n+g+2,\rho_2+x+1,\rho_2-x+1\\ \rho_1+\rho_2+2,\rho_2-r_1+3/2,\rho_2-r_2+3/2\end{matrix};1\right], \tag{11.59}$$

with

$$\eta_n=\frac{(\rho_1+\rho_2+1)_n(\rho_2-r_1+1/2)_n(\rho_2-r_2+1/2)}{(n+g+1)_n},$$

$$\iota_n=\frac{(\rho_1+\rho_2+2)_n(\rho_2-r_1+3/2)_n(\rho_2-r_2+3/2)}{(n+g+2)_n},$$

and where $(a)_n=(a)(a+1)\cdots(a+n-1)$ is the Pochhammer symbol.

Comparing the recurrence formulas (11.55) with (11.56) and (11.57), it is seen that the polynomials $\widehat{\mathscr{P}}_n(\xi)$ are monic CBI polynomials

$$\widehat{P}_n(\xi) = I_n\left(\xi/2; \rho_1, \rho_2, r_1, r_2\right), \tag{11.60}$$

with

$$\rho_1 = \frac{\zeta - 2}{2}, \quad \rho_2 = 0, \quad r_1 = \frac{2k+\zeta}{2}, \quad r_2 = 0. \tag{11.61}$$

The parametrization (11.61) is a special case of CBI polynomials. This case corresponds to the para-Krawtchouk polynomials constructed in [18] in the context of perfect state transfer in spin chains.

Since there is a singularity in the recurrence coefficients for the polynomials $\mathscr{P}_n(\xi)$, the correspondence between the polynomials $\mathscr{P}_n(\xi)$ and the CBI polynomials outlined above is valid only for $n = 0, \ldots, k$ and hence the Heun polynomial $\widetilde{A}(z)$ generates only the first k para-Krawtchouk polynomials. As is easily seen by induction, the recurrence relation (11.55) generates center-symmetric polynomials $\mathscr{P}_n$. Hence for $n > k$, we have $\mathscr{P}_n(\xi) = \mathscr{P}_{2k-n}(\xi)$. Putting the preceding results together, we write

$$\widetilde{a}_n = \frac{(-1)^n 4^n}{n!} \frac{(k+1-n)_n}{(2k+\zeta-n)_n} I_n(\xi/2; \rho_1, \rho_2, r_1, r_2) \tag{11.62}$$

for $n \leqslant k$ and

$$\widetilde{a}_n = \widetilde{a}_{2k-n}, \quad n = k+1, \ldots, 2k$$

for $n = k+1 \cdots, 2k$.

The hypergeometric expression of the CBI polynomials (11.58) provides an explicit formula for the coefficients $\widetilde{a}_n$ and the coefficients $\widetilde{b}_n$ are easily evaluated from the recurrence system (11.45). Combining those results with the formulas (11.46) and (11.47) yields the expansion coefficients of the eigenvectors of the operator $\mathscr{Q}$ in the circular basis. Note that these expansion coefficients only involve $\widetilde{a}_j$, $\widetilde{b}_j$ with $j = 0, \ldots, k$ and hence only (11.62) is needed.

11.4.5 Eigenvectors of J_2

To obtain the expansion coefficients of the eigenvectors of J_2 in the circular basis, it is necessary to relate the eigenvectors of J_2 to those of $\mathscr{Q}$. This relation has been obtained in the previous paper [5]. In the present notation, we have

$$|\,k,+\,\rangle_{\mathscr{Q}} = \frac{1}{\sqrt{2}}\left(|\,k,+\,\rangle_{J_2} - \omega_k|\,k,-\,\rangle_{J_2}\right),$$
$$|\,k,-\,\rangle_{\mathscr{Q}} = \frac{1}{\sqrt{2}}\left(|\,k,+\,\rangle_{J_2} + \omega_k|\,k,-\,\rangle_{J_2}\right)$$

where $|\,k,\pm\,\rangle_{J_2}$ are the eigenvectors of J_2 corresponding to the eigenvalues

$$\lambda_{\pm} = \sqrt{k(k+\zeta)},$$

and where the coefficient ω_k is

$$\omega_k = \frac{\zeta - 2i\sqrt{k(k+\zeta)}}{2k+\zeta}.$$

The inverse relation, which allows to express the eigenvectors of J_2 in terms of the known eigenvectors of $\mathscr{Q}$ reads

$$|\,k,+\,\rangle_{J_2} = \frac{1}{\sqrt{2}}(|\,k,+\,\rangle_{\mathscr{Q}} + |\,k,-\,\rangle_{\mathscr{Q}}),$$
$$|\,k,-\,\rangle_{J_2} = \frac{-1}{\omega_k\sqrt{2}}(|\,k,+\,\rangle_{\mathscr{Q}} - |\,k,-\,\rangle_{\mathscr{Q}}).$$

11.4.6 The fully isotropic case

We now consider the case $\mu_x = \mu_y = \mu$. This corresponds to a fully isotropic 2D Dunkl oscillator, where two "identical" parabosonic oscillators are combined. Returning to the system of differential equations (11.49) for the generating functions, one has

$$(k+2\mu)\widetilde{A}(z) = (k-2\mu-z\partial_z)\widetilde{B}(z) + \frac{4\mu}{1-z^2}\widetilde{B}(z),$$
$$k\widetilde{B}(z) = (k-z\partial_z)\widetilde{A}(z).$$

Solving for $\widetilde{A}(z)$, we find

$$z(z^2-1)\widetilde{A}''(z) + \left(z^2(2\mu+1-2k)+2\mu+2k-1\right)\widetilde{A}'(z) - 4k\mu z\widetilde{A}(z) = 0.$$

The solution corresponding to the initial value $\widetilde{a}_0 = 1$ is given by

$$\widetilde{A}(z) = {}_2F_1\left[\begin{matrix} -k,\mu \\ 1-k-\mu \end{matrix}; z^2\right],$$

and we also have

$$\widetilde{B}(z) = \widetilde{A}(z) - \frac{2k\mu z^2}{k+\mu-1}\,{}_2F_1\left[\begin{matrix} 1-k,1+\mu \\ 2-k-\mu \end{matrix}; z^2\right].$$

Hence in the isotropic case, the generating functions are simply the Jacobi polynomials. It follows from the hypergeometric generating function that

$$\widetilde{a}_{2n} = \frac{(-k)_n(\mu)_n}{(1-k-\mu)_n n!}, \qquad \widetilde{a}_{2n+1} = 0,$$

and

$$\widetilde{b}_{2n} = \left(\frac{k-2n}{k}\right)\frac{(-k)_n(\mu)_n}{(1-k-\mu)_n n!}, \qquad \widetilde{b}_{2n+1} = 0.$$

Thus it is seen that the in the fully isotropic case $\mu_x = \mu_y = \mu$, the formulas for the expansion coefficients of the eigenvectors of $\mathscr{Q}$ simplify substantially.

11.5 Diagonalization of J_2: the N odd case

In this section, we obtain the expression for the eigenvectors of J_2 in the circular basis when $N = n_L + n_R$ is an odd integer. In spirit, the computation is similar to the N even case presented in the previous section. We proceed along the same lines.

11.5.1 The operator $\mathscr{Q}$ and its simultaneous eigenvalue equation

The operator $\mathscr{Q}$ is defined by

$$\mathscr{Q} = -2iJ_2R_x - \mu_x R_y - \mu_y R_x - (1/2)R_xR_y.$$

Given the action (11.24) of the reflections operators, it is seen that they have the following matrix representation in the circular basis $\mathscr{B}_2$:

$$R_y = -R_x = \text{diag}(\sigma_1, \ldots, \sigma_1), \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Using the matrix representation of J_2 in the circular basis $\mathscr{B}_2$ given in (11.31), one finds

$$[\mathscr{Q}]_{\mathscr{B}_2} = \begin{pmatrix} \widetilde{\Phi}_0 & \widetilde{\Delta}_1 & \widetilde{\Delta}_2 & \cdots & \widetilde{\Delta}_m \\ & \widetilde{\Phi}_1 & \widetilde{\Delta}_1 & \cdots & \widetilde{\Delta}_{m-1} \\ & & \ddots & & \\ & & & \widetilde{\Phi}_{m-1} & \widetilde{\Delta}_1 \\ & & & & \widetilde{\Phi}_m \end{pmatrix},$$

with $m=(N-1)/2$ and where

$$\widetilde{\Phi}_k=\begin{pmatrix}1/2+i\xi & i(2k+\zeta+1)-\xi\\ -i(2k+\zeta+1)-\xi & 1/2-i\xi\end{pmatrix},\quad \widetilde{\Delta}_m=\begin{cases}\begin{pmatrix}-2i\zeta & -2i\xi\\ 2i\xi & 2i\zeta\end{pmatrix} & m \text{ odd}\\ \begin{pmatrix}2i\xi & 2i\zeta\\ -2i\zeta & -2i\xi\end{pmatrix} & m \text{ even}\end{cases}. \tag{11.63}$$

From the block upper-triangular structure, it follows that the eigenvalues $\nu_k^{\pm}$ of $\mathscr{Q}$ are

$$\nu_k^{+}=2k+\zeta+3/2,\qquad \nu_k^{-}=-(2k+\zeta+1/2),$$

for $k=0,\ldots,m$. Although a different labeling has been used, it is directly checked that the eigenvalues of $\mathscr{Q}$ are the same for the N even and N odd case, except for the additional one. We denote the eigenvectors of $\mathscr{Q}$ corresponding to the eigenvalues $\nu_k^{\pm}$ by $|\,k,\pm\,\rangle_{\mathscr{Q}}$ and define their expansion in the circular basis by

$$|\,k,+\,\rangle_{\mathscr{Q}}=\sum_{\substack{\ell=0\\ \sigma=\pm}}^{k}u_\ell^{\sigma}(k)|\,\ell,\sigma\,\rangle,\qquad |\,k,-\,\rangle_{\mathscr{Q}}=\sum_{\substack{\ell=0\\ \sigma=\pm}}^{k}v_\ell^{\sigma}(k)|\,\ell,\sigma\,\rangle,$$

for $k=0,\ldots,m$ and where the vectors $|\,\ell,\pm\,\rangle$ are the vectors of the circular basis $\mathscr{B}_2$.

We shall once again study the simultaneous eigenvalue equation for the operator $\mathscr{Q}$. We define the matrix of eigenvectors

$$W=\begin{pmatrix}V_{00} & V_{01} & \cdots & V_{0m}\\ & V_{11} & \cdots & V_{1m}\\ & & \ddots & \\ & & & V_{mm}\end{pmatrix},$$

where

$$V_{\ell k}=\begin{pmatrix}v_\ell^{+}(k) & u_\ell^{+}(k)\\ v_\ell^{-}(k) & u_\ell^{-}(k)\end{pmatrix}.$$

The simultaneous eigenvalue equation for the matrix $[\mathscr{Q}]_{\mathscr{B}_2}$ reads

$$W\cdot L=[\mathscr{Q}]_{\mathscr{B}_2}W, \tag{11.64}$$

with

$$L=\operatorname{diag}(\Lambda_0,\cdots,\Lambda_m),\qquad \Lambda_k=\begin{pmatrix}\nu_k^{-} & 0\\ 0 & \nu_k^{+}\end{pmatrix}. \tag{11.65}$$

As in section 4, the simultaneous equation (11.64) will be shown to be equivalent to a system of recurrence relation for the components $u_\ell(k)^{\pm}$, $v_\ell^{\pm}(k)$ of the eigenvectors of $\mathscr{Q}$ in the circular basis.

11.5.2 Recurrence relations

We now construct the recurrence systems for the component of the eigenvectors of the operator $\mathscr{Q}$. For $k = 0, \ldots, m$ and $\ell = 0, \ldots, m$, the simultaneous equation (11.64) takes the form

$$V_{\ell k}\Lambda_k = \widetilde{\Phi}_\ell V_{\ell k} + \sum_{j=1}^{k-\ell} \widetilde{\Delta}_j V_{jk}. \tag{11.66}$$

The v_k^- eigenvalue sector

Let us begin by considering the eigenvectors $|\,k,-\,\rangle_{\mathscr{Q}}$ of $\mathscr{Q}$ corresponding to the eigenvalue v_k^- and their expansion in the circular basis

$$|\,k,-\,\rangle_{\mathscr{Q}} = \sum_{\substack{\ell=0\\ \sigma=\pm}}^{k} v_\ell^\sigma(k)|\,\ell,\sigma\,\rangle.$$

It is directly seen from (11.66),(11.63) and (11.65) that we have

$$[2k+1+\zeta+i\xi]v_\ell^+ = [\xi - i(2\ell+\zeta+1)]v_\ell^- - 2i\mu_x \sum_{j=\ell+1}^{k}(-1)^{j-\ell}A_j - 2i\mu_y \sum_{j=\ell+1}^{k} B_j, \tag{11.67a}$$

$$[2k+1+\zeta-i\xi]v_\ell^- = [\xi + i(2\ell+\zeta+1)]v_\ell^+ + 2i\mu_x \sum_{j=\ell+1}^{k}(-1)^{j-\ell}A_j - 2i\mu_y \sum_{j=\ell+1}^{k} B_j, \tag{11.67b}$$

where we have defined $A_j = v_j^+ + v_j^-$ and $B_j = v_j^- - v_j^+$ and where the explicit dependence on k of the components $v_\ell^\pm(k)$ has been dropped for notational convenience. The recurrence system (11.67) is "reversed". The terminating conditions are at $\ell = k$. In this case, (11.67) becomes

$$[2k+1+\zeta+i\xi]v_k^+ = [\xi - i(2k+\zeta+1)]v_k^-,$$
$$[2k+1+\zeta-i\xi]v_k^- = [\xi + i(2k+\zeta+1)]v_k^+.$$

Choosing $v_k^- = 1$, we obtain the terminating conditions

$$v_k^+ = -i, \qquad v_k^- = 1.$$

Upon taking

$$A_\ell = \alpha_0\widehat{A}_\ell, \quad B_\ell = \beta_0\widehat{B}_\ell,$$

where $\alpha_0 = (1-i)$ and $\beta_0 = (1+i)$, it is easily seen that $\widehat{A}_\ell$ and $\widehat{B}_\ell$ are real and that the system (11.67) is equivalent to

$$[k+\mu_y+1/2]\widehat{A}_\ell = (\ell+\mu_y+1/2)\widehat{B}_\ell + 2\mu_y \sum_{j=\ell+1}^{k} \widehat{B}_j,$$

$$[k+\mu_x+1/2]\widehat{B}_\ell = (\ell+\mu_x+1/2)\widehat{A}_\ell + 2\mu_x \sum_{j=\ell+1}^{k} (-1)^{j-\ell}\widehat{A}_j,$$

with the terminating conditions $\widehat{A}_k = 1$ and $\widehat{B}_k = 1$. Introducing the reversed components $\widetilde{a}_n = \widehat{A}_{k-n}$, $\widetilde{b}_n = \widehat{B}_{k-n}$, we obtain the system

$$[k+\mu_y+1/2]\widetilde{a}_n = [k-n+\mu_y+1/2]\widetilde{b}_n + 2\mu_y \sum_{j=0}^{n-1} \widetilde{b}_j, \tag{11.68a}$$

$$[k+\mu_x+1/2]\widetilde{b}_n = [k-n+\mu_x+1/2]\widetilde{a}_n + 2\mu_x(-1)^n \sum_{j=0}^{n-1} (-1)^j \widetilde{a}_j, \tag{11.68b}$$

with the initial conditions $\widetilde{a}_0 = 1$ and $\widetilde{b}_0 = 1$. Taking into account all the preceding transformations, we have

$$v_\ell^-(k) = \frac{\alpha_0\widetilde{a}_{k-\ell} + \beta_0\widetilde{b}_{k-\ell}}{2}, \quad v_\ell^+(k) = \frac{\alpha_0\widetilde{a}_{k-\ell} - \beta_0\widetilde{b}_{k-\ell}}{2}, \tag{11.69}$$

where $\widetilde{a}_\ell$, $\widetilde{b}_\ell$ are the unique solutions to the recurrence system (11.68).

The v_k^+ eigenvalue sector

Let us now consider the eigenvectors $|\, k,+\,\rangle_{\mathscr{Q}}$ corresponding to the eigenvalue v_k^+ with expansion

$$|\, k,+\,\rangle_{\mathscr{Q}} = \sum_{\substack{\ell=0\\ \sigma=\pm}}^{k} u_\ell^\sigma(k) |\, \ell,\sigma\,\rangle,$$

in the circular basis. Proceeding along the same lines as in the previous computation, we find that the terminating condition are of the form

$$v_k^+ = \frac{i(2k+1+\zeta+i\xi)}{2k+1+\zeta-i\xi}, \qquad v_k^- = 1,$$

and that the components are given by

$$u_\ell^-(k) = \frac{\gamma_0\widetilde{a}_{k-\ell} + \epsilon_0\widetilde{b}_{k-\ell}}{2}, \quad u_\ell^+(k) = \frac{\gamma_0\widetilde{a}_{k-\ell} - \epsilon_0\widetilde{b}_{k-\ell}}{2}, \tag{11.70}$$

where

$$\gamma_0 = \frac{(1+i)(2k+1+2\mu_y)}{2k+1+\zeta-i\xi}, \quad \epsilon_0 = \frac{(1-i)(2k+1+2\mu_x)}{2k+1+\zeta-i\xi}.$$

11.5.3 Generating functions and Heun polynomials

As is seen from the formulas (11.69) and (11.70), the main part of the components of the eigenvectors of $\mathscr{Q}$ in the circular basis is given by the solutions to the recurrence system (11.68). As in section 4 for the N even case, we bring the ordinary generating functions

$$\widetilde{A}(z) = \sum_n \widetilde{a}_n z^n, \qquad \widetilde{B}(z) = \sum_n \widetilde{b}_n z^n.$$

Upon using the identities (11.48), we obtain from (11.68) the associated differential system

$$(k+\mu_y+1/2)\widetilde{A}(z) = (k-\mu_y+1/2-z\partial_z)\widetilde{B}(z) + \frac{2\mu_y}{1-z}\widetilde{B}(z), \tag{11.71a}$$

$$(k+\mu_x+1/2)\widetilde{B}(z) = (k-\mu_x+1/2-z\partial_z)\widetilde{A}(z) + \frac{2\mu_x}{1+z}\widetilde{A}(z). \tag{11.71b}$$

By direct substitution, we find that the generating functions are expressed in terms of the Heun functions

$$\widetilde{A}(z) = (1+z)H\ell(a, q_A; \alpha_A, \beta_A, \gamma_A, \delta_A, z), \tag{11.72a}$$

$$\widetilde{B}(z) = (1-z)H\ell(a, q_B; \alpha_B, \beta_B, \gamma_B, \delta_B, -z), \tag{11.72b}$$

where

$$a = -1, \quad q_A = 2k(\mu_y - \mu_x - 1), \quad \alpha_A = -2k, \tag{11.73a}$$

$$\beta_A = \mu_x + \mu_y + 1, \quad \gamma_A = -2k - \mu_x - \mu_y, \quad \delta_A = 2\mu_y. \tag{11.73b}$$

and where the parameters $q_B, \cdots \delta_B$ are obtained from (11.73) by the transformation $\mu_x \leftrightarrow \mu_y$. The form of the parameters involved in the Heun functions show that once again one has a truncation at degree $2k+1$ and hence the Heun functions appearing in (11.72) are in fact Heun polynomials. The generating functions $\widetilde{A}(z)$ and $\widetilde{B}(z)$ are polynomials of degree $2k+1$.

11.5.4 Expansion of Heun polynomials in complementary Bannai-Ito polynomials

The expansion of the Heun polynomials can be obtained using the associated three-term recurrence relation. Since the expansion coefficients of $\widetilde{B}(z)$ and $\widetilde{A}(z)$ are related by the simple relation $z \leftrightarrow -z$ and $\mu_x \leftrightarrow \mu_y$, we shall focus on the expansion of $\widetilde{A}(z)$.

We examine the expansion of the Heun polynomial appearing in (11.72a)

$$H\ell(a,q_A,\alpha_A,\beta_A,\gamma_A,\delta_A,z)=\sum_r \mathscr{P}_r z^r, \tag{11.74}$$

where the parameters are given in (11.73). Using the recurrence coefficients given in (11.54), one finds that the expansion coefficients $\mathscr{P}_n$ are polynomials of degree r in $\xi = \mu_x - \mu_y$ that obey the recurrence relation

$$\sigma_{r+1}\mathscr{P}_{r+1}(\xi)+\kappa_r\mathscr{P}_{r-1}(\xi)=(1+\xi)\mathscr{P}_r(\xi), \tag{11.75}$$

with $\mathscr{P}_{-1}=0$, $\mathscr{P}_0=1$ and where

$$\sigma_{r+1}=\frac{(r+1)(N+\zeta-r)}{2r-N}, \quad \kappa_r=\frac{(N+1-r)(r+\zeta)}{2r-N},$$

where we have taken $N=2k$. Introducing the monic polynomials $\mathscr{P}_r(\xi)=\frac{\widehat{\mathscr{P}}_r(\xi)}{\sigma_1\cdots\sigma_r}$, we obtain

$$\widehat{\mathscr{P}}_{r+1}(\xi)+u_r\widehat{\mathscr{P}}_{r-1}(\xi)=(1+\xi)\widehat{\mathscr{P}}_r(\xi) \tag{11.76}$$

where

$$u_r=-\frac{r(N+1-r)(N+\zeta-r-1)(r+\zeta)}{(N-2r)(N-2r+2)}.$$

Upon comparing (11.76) with the recurrence coefficients for the CBI polynomials given in (11.57), it is directly seen that the polynomials defined by the recurrence (11.76) correspond to CBI polynomials with the parametrization

$$\rho_1=\frac{\zeta-1}{2}, \quad r_1=\frac{2k+\zeta+1}{2}, \quad \rho=0, \quad r_2=0. \tag{11.77}$$

Hence, when $r\leqslant k$, we have

$$\mathscr{P}_r(\xi)=\frac{(-1)^r4^r}{r!}\frac{(k+1-r)_r}{(2k+\zeta+1-r)_r}I_r((1+\xi)/2;\rho_1,\rho_2,r_1,r_2), \quad r\leqslant k, \tag{11.78}$$

where ρ_1, ρ_2, r_1 and r_2 are given by (11.77) and $I_n(x;\rho_1,\rho_2,r_1,r_2)$ are the complementary Bannai-Ito polynomials. As can be seen by the recurrence relation (11.75), the expansion coefficients of the Heun function (11.74) truncate at order $2k+1$ and are center-symmetric. Hence, for $r>k$, we have

$$\mathscr{P}_r(\xi)=\mathscr{P}_{2k+1-r}(\xi), \qquad r>k. \tag{11.79}$$

Taking into account the relation between $\widetilde{A}(z)$ and $\widetilde{B}(z)$, the expansion coefficients of the Heun function appearing in (11.72b), denoted by $\mathscr{T}_n(\xi)$, are easily seen to be

$$\mathscr{T}_r(\xi) = \frac{4^r}{r!}\frac{(k+1-r)_r}{(2k+\zeta+r-1)_r} I_r((1-\xi)/2;\rho_1,\rho_2,r_1,r_2), \quad r \leqslant 2k, \tag{11.80}$$

$$\mathscr{T}_r(\xi) = \mathscr{T}_{2k+1-r}(\xi), \quad r > 2k. \tag{11.81}$$

Collecting all the previous results, we write

$$\widetilde{a}_n = \mathscr{P}_n(\xi) + \mathscr{P}_{n-1}(\xi), \tag{11.82a}$$

$$\widetilde{b}_n = \mathscr{T}_n(\xi) + \mathscr{T}_{n-1}(\xi), \tag{11.82b}$$

where $\mathscr{P}_{-1} = 0$, $\mathscr{T}_{-1} = 0$ and $\mathscr{P}_n(\xi)$, $\mathscr{T}_n(\xi)$ are given by (11.78), (11.79), (11.80) and (11.81).

11.5.5 Eigenvectors of J_2

To obtain the expansion of the eigenvectors of J_2 in the circular basis, one must relate the eigenvectors of the operator $\mathscr{Q}$ to the eigenvectors of J_2. This relation has been obtained in the previous paper [5]. We have

$$|\,k,\pm\,\rangle_{\mathscr{Q}} = \frac{1}{\sqrt{2}}\left(|\,k,+\,\rangle_{J_2} \mp v_k|\,k,-\,\rangle_{J_2}\right)$$

where $|\,k,\pm\,\rangle_{J_2}$ are the eigenvectors of J_2 corresponding to the eigenvalues

$$\lambda_k^{\pm} = \pm\sqrt{(k+\mu_x+1/2)(k+\mu_y+1/2)}, \qquad k = 0,\ldots,m,$$

and where

$$v_k = \left[\frac{\xi + 2i\sqrt{(k+\mu_x+1/2)(k+\mu_y+1/2)}}{2k+\zeta+1}\right].$$

The inverse relation reads

$$|\,k,+\,\rangle_{J_2} = \frac{1}{\sqrt{2}}(|\,k,+\,\rangle_{\mathscr{Q}} + |\,k,-\,\rangle_{\mathscr{Q}}), \tag{11.83a}$$

$$|\,k,-\,\rangle_{J_2} = \frac{-1}{v_k\sqrt{2}}(|\,k,+\,\rangle_{\mathscr{Q}} - |\,k,-\,\rangle_{\mathscr{Q}}) \tag{11.83b}$$

Using the relations (11.83), the results (11.82), (11.70) and (11.69), one has an explicit expression for the expansion of the eigenvectors of J_2 in the circular basis for the case N odd.

11.5.6 The fully isotropic case : -1 Jacobi polynomials

We consider again the case $\mu_x = \mu_y = \mu$ which corresponds to the fully isotropic Dunkl oscillator, where two independent identical parabosonic oscillators are combined. We return to the system of differential equations for the generating functions of the components of the eigenvectors of $\mathscr{Q}$ given in (11.71). When $\mu_x = \mu_y = \mu$, one has

$$(k+\mu+1/2)\widetilde{A}(z) = (k-\mu+1/2-z\partial_z)\widetilde{B}(z) + \frac{2\mu}{1-z}\widetilde{B}(z),$$
$$(k+\mu+1/2)\widetilde{B}(z) = (k-\mu+1/2-z\partial_z)\widetilde{A}(z) + \frac{2\mu}{1+z}\widetilde{A}(z).$$

It is easily seen from the above formulas that $\widetilde{A}(z) = \widetilde{B}(-z)$. Hence the generating function $\widetilde{A}(z)$ satisfies the differential equation

$$(k+\mu+1/2)\widetilde{A}(z) = (k-\mu+1/2-z\partial_z)\widetilde{A}(-z) + \frac{2\mu}{1-z}\widetilde{A}(-z),$$

which may be cast in the form of an eigenvalue equation

$$L\widetilde{A}(z) = 4\mu\widetilde{A}(z),$$

where

$$L = 2(1-z)\partial_z R + \left[(-2k-1+2\mu) + \frac{2k+1+2\mu}{z}\right](\mathbb{I}-R),$$

where $Rf(z) = f(-z)$. It is recognized that the operator L is a special case of the defining operator of the little -1 Jacobi polynomials [17].

The little -1 Jacobi polynomials, denoted by $P_n^{-1}(x)$, obey the eigenvalue equation

$$\Omega P_n(x) = \lambda_n P_n(x), \qquad \lambda_n = \begin{cases} -2n & n \text{ even} \\ 2(\alpha+\beta+n+1) & n \text{ odd} \end{cases},$$

where

$$\Omega = 2(1-x)\partial_x R + (\alpha+\beta+1-\alpha/x)(\mathbb{I}-R).$$

Comparing the operators Ω and L, it is seen that the generating function $\widetilde{A}(z)$ corresponds to a -1 Jacobi polynomial of degree $n = 2k+1$ with parameters

$$\alpha = -2k-2\mu-1, \qquad \beta = 4\mu-1.$$

Using this identification and the explicit formula for the little −1 Jacobi polynomials derived in [17], we obtain

$$\widetilde{A}(z) = {}_2F_1\left[\begin{matrix}-k,\mu\\ -\mu-k\end{matrix};z^2\right] + \frac{\mu z}{k+\mu}\,{}_2F_1\left[\begin{matrix}-k,\mu+1\\ -k-\mu+1\end{matrix};z^2\right].$$

This directly yields the following result for the recurrence coefficients $\widetilde{a}_n$:

$$\widetilde{a}_{2n} = \frac{(-k)_n(\mu)_n}{n!(-\mu-k)_n}, \quad \widetilde{a}_{2n+1} = \frac{\mu}{k+\mu}\frac{(-k)_n(\mu+1)_n}{n!(-\mu-k+1)_n}.$$

Using the symmetry $\widetilde{B}(z) = \widetilde{A}(-z)$, we also obtain

$$\widetilde{b}_{2n} = \frac{(-k)_n(\mu)_n}{n!(-\mu-k)_n}, \quad \widetilde{b}_{2n+1} = \frac{-\mu}{k+\mu}\frac{(-k)_n(\mu+1)_n}{n!(-\mu-k+1)_n}.$$

Once again, the components of the eigenvectors of $\mathscr{Q}$ drastically simplify in the isotropic case $\mu_x = \mu_y = \mu$. Moreover, the preceding computations entail a relation between a special case of Heun polynomials and the little −1 Jacobi polynomials.

11.6 Representations of $sd(2)$ in the J_2 eigenbasis

We now investigate the representation space in which the operator J_2 is diagonal. The matrix elements of the generators of the Schwinger-Dunkl algebra will be derived using the defining relations of $sd(2)$, which read

$$\{J_1, R_{x_i}\} = 0, \quad \{J_2, R_{x_i}\} = 0, \quad [J_3, R_{x_i}] = 0,$$
$$[J_2, J_3] = iJ_1, \qquad [J_3, J_1] = iJ_2,$$
$$[J_1, J_2] = i[J_3 + J_3(\mu_x R_x + \mu_y R_y) - \mathscr{H}(\mu_x R_x - \mu_y R_y)/2],$$

where $R_x^2 = R_y^2 = \mathbb{I}$. It will prove convenient to treat the even and odd dimensional representations separately.

11.6.1 The N odd case

We first consider the case where N is odd. The representation space $\mathscr{C}$ is spanned in this case by the basis vectors $|\,k,\pm\,\rangle$ with $k \in \{0,\ldots,m\}$ on which the generator J_2 acts in a diagonal fashion

$$J_2|\,k,\pm\,\rangle = \lambda_k^{\pm}|\,k,\pm\,\rangle, \qquad k = 0,\ldots,m,$$

where $m=(N-1)/2$ and where the eigenvalues of J_2, derived in section 3, are given by

$$\lambda_k^{\pm}=\pm\sqrt{(k+\mu_x+1/2)(k+\mu_y+1/2)},\quad k=0,\dots,m.$$

Since R_x, R_y anti-commute with J_2 and given that R_xR_y is central in the algebra $sd(2)$, we can take

$$R_x|\,k,\pm\,\rangle=\epsilon|\,k,\mp\,\rangle,\qquad R_y|\,k,\pm\,\rangle=|\,k,\mp\,\rangle,$$

where $\epsilon=\pm1$. Here we choose $\epsilon=-1$, which corresponds to the representation encountered in the model. In the basis $\{|\,0,+\,\rangle,|\,0,-\,\rangle,\cdots,|\,m,+\,\rangle,|\,m,-\,\rangle\}$, the matrices representing the involutions R_x, R_y have the form

$$R_y=-R_x=\text{diag}(\sigma_1,\dots,\sigma_1),\qquad \sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},$$

which is identical to their action in the circular basis. The Hamiltonian $\mathscr{H}$ has the action

$$\mathscr{H}|\,k,\pm\,\rangle=(N+\mu_x+\mu_y+1)|\,k,\pm\,\rangle.$$

The action of the operator J_3 on this representation space can be derived by imposing the commutation relation (11.6) using (11.6) to define J_1. The action of J_3 on the basis $|\,k,\pm\,\rangle$ is taken to be

$$J_3|\,k,+\,\rangle=\sum_{\substack{j=0\\ \sigma=\pm}}^{m}M_{jk}^{\sigma}|\,j,\sigma\,\rangle,\quad J_3|\,k,-\,\rangle=\sum_{\substack{j=0\\ \sigma=\pm}}^{m}N_{jk}^{\sigma}|\,j,\sigma\,\rangle.$$

With these definitions, it is easily seen that the commutation relation (11.6) is equivalent to the following system of relations

$$[(\lambda_k^+)^2-2(\lambda_j^+\lambda_k^+)+(\lambda_j^+)^2-1]M_{jk}^+=(\mu_y-\mu_x)N_{jk}^+,\tag{11.84a}$$

$$[(\lambda_k^+)^2-2(\lambda_j^-\lambda_k^+)+(\lambda_j^-)^2-1]M_{jk}^-=(\mu_y-\mu_x)N_{jk}^-+\delta_{jk}\frac{\zeta(N+1+\zeta)}{2},\tag{11.84b}$$

$$[(\lambda_k^-)^2-2(\lambda_j^+\lambda_k^-)+(\lambda_j^+)^2-1]N_{jk}^+=(\mu_y-\mu_x)M_{jk}^++\delta_{jk}\frac{\zeta(N+1+\zeta)}{2},\tag{11.84c}$$

$$[(\lambda_k^-)^2-2(\lambda_j^-\lambda_k^-)+(\lambda_j^-)^2-1]N_{jk}^-=(\mu_y-\mu_x)M_{jk}^-,\tag{11.84d}$$

where the relations (11.84a),(11.84b) were obtained by acting on $|\,k,+\,\rangle$ and the relations (11.84c),(11.84d) by acting on $|\,k,-\,\rangle$. It follows directly from the solution of the system (11.84) that J_3 acts in a six-diagonal fashion on the eigenbasis of J_2. For $j=k$, we obtain

$$M_{kk}^+=\frac{\xi\zeta(N+\zeta+1)}{2(2k+\zeta)(2k+\zeta+2)},\quad N_{kk}^+=\frac{\zeta(N+\zeta+1)}{2(2k+\zeta)(2k+\zeta+2)},\tag{11.85a}$$

$$M_{kk}^-=\frac{\zeta(N+\zeta+1)}{2(2k+\zeta)(2k+\zeta+2)},\quad N_{kk}^-=\frac{\xi\zeta(N+\zeta+1)}{2(2k+\zeta)(2k+\zeta+2)},\tag{11.85b}$$

where $\xi = \mu_x - \mu_y$, $\zeta = \mu_x + \mu_y$. For $j = k + \ell$ or $j = k - \ell$ with $\ell > 1$, only the trivial solution occurs, so the matrix representing J_3 in the eigenbasis of J_2 is block tridiagonal with all block $2\times\ 2$. Hence it acts in a six-diagonal fashion on the eigenbasis of J_2. Using the commutation relations and the system (11.84), it is possible to obtain an expression for the matrix elements of J_3 which involves a set of arbitrary non-zero parameters $\{\beta_n\}$ for $n = 0,\ldots,m$. After considerable algebra, one finds that the matrix J_3 has the form

$$J_3 = \begin{pmatrix} C_0 & U_1 & & & \\ D_0 & C_1 & U_2 & & \\ & D_1 & C_2 & \ddots & \\ & & \ddots & \ddots & U_m \\ & & & D_{m-1} & C_m \end{pmatrix},$$

where the blocks are given by

$$U_k = \beta_k \begin{pmatrix} M^+_{k-1k} & 1 \\ 1 & M^+_{k-1k} \end{pmatrix}, \quad C_k = \begin{pmatrix} M^+_{kk} & N^+_{kk} \\ M^-_{kk} & N^-_{kk} \end{pmatrix}, \quad D_k = \beta^{-1}_{k+1} \begin{pmatrix} M^+_{k+1k} & N^+_{k+1k} \\ N^+_{k+1k} & M^+_{k+1k} \end{pmatrix}.$$

The matrix elements of the central blocks are given by (11.85). The components of the upper blocks U_k have the form

$$M^+_{k-1k} = \frac{1}{2\xi}\left[1 - 4(k+\mu_x)(k+\mu_y) - 4\left\{(k+\mu_x-1/2)_2(k+\mu_y-1/2)_2\right\}^{1/2}\right].$$

The matrix elements of the lower blocks have the form

$$M^+_{k+1k} = \frac{\xi(k+1)(2k-N+1)(k+1+\zeta)(2k+2\zeta+N+3)}{4(2k+\zeta+2)(2k+\zeta+1)_3},$$

$$N^+_{k+1k} = E_k M^+_{k+1k}.$$

where

$$E_k = \frac{1}{2\xi}\left[1 - 4(k+\mu_x+1)(k+\mu_y+1) + 4\{(k+\mu_x+1/2)_2(k+\mu_y+1/2)_2\}^{1/2}\right].$$

We note that these matrix elements are valid for $\mu_x \neq \mu_y$. In the latter case, the form of the spectrum of J_2 changes and the computation has to be redone from the start.

11.6.2 The N even case

We now consider the N even case. The representation space $\mathscr{C}$ is spanned in this case by the basis vectors $|\,0,-\,\rangle$ and $|\,k,\pm\,\rangle$ with $k = 1,\ldots,m$ on which the operator J_2 acts in a diagonal fashion

$$J_2|\,k,\pm\,\rangle = \lambda^{\pm}_k|\,k,\pm\,\rangle, \qquad k = 0,\ldots,m,$$

where the eigenvalues of J_2, determined in section 3, are given by the formula

$$\lambda_k^{\pm} = \pm\sqrt{k(k+\zeta)}, \quad k = 0,\ldots,m.$$

Note that the eigenvalue λ_0 is non-degenerate. In this representation, we choose the following action for the involutions R_x, R_y:

$$R_{x_i}|\,0,-\,\rangle = |\,0,-\,\rangle, \quad R_{x_i}|\,k,\pm\,\rangle = |\,k,\mp\,\rangle,$$

and hence the reflections have the matrix representation

$$R_x = R_y = \mathrm{diag}(1,\sigma_1,\ldots,\sigma_1), \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The central element (Hamiltonian) $\mathscr{H}$ has the familiar action

$$\mathscr{H}|\,k,\pm\,\rangle = (N+\mu_x+\mu_y+1)|\,k,\pm\,\rangle.$$

Following the same steps as in (6.1), a direct computation shows that the in this case J_3 has the matrix representation

$$J_3 = \begin{pmatrix} c_0 & u_1 & & & \\ d_0 & C_1 & U_2 & & \\ & D_1 & C_2 & \ddots & \\ & & \ddots & \ddots & U_m \\ & & & D_{m-1} & C_m \end{pmatrix}.$$

The special 2×1 and 1×1 blocks are given by

$$c_0 = \frac{\xi(N+\zeta+1)}{2(1+\zeta)}, \quad u_1 = \begin{pmatrix} \alpha_1 & \alpha_1 \end{pmatrix}, \quad d_0 = \alpha_1^{-1}\begin{pmatrix} w_N & w_N \end{pmatrix}^t,$$

with

$$w_N = \frac{(N/2)(1+2\mu_x)(1+2\mu_y)(N/2+\zeta+1)}{2(1+\zeta)^2(2+\zeta)}.$$

The 2×2 blocks have the form

$$U_k = \alpha_k \begin{pmatrix} 1 & N_{k-1k}^{+} \\ N_{k-1k}^{+} & 1 \end{pmatrix}, \quad C_k = \begin{pmatrix} M_{kk}^{+} & N_{kk}^{+} \\ N_{kk}^{+} & M_{kk}^{+} \end{pmatrix}, \quad D_k = \alpha_{k+1}^{-1}\begin{pmatrix} M_{k+1k}^{+} & N_{k+1k}^{+} \\ N_{k+1k}^{+} & M_{k+1k}^{+} \end{pmatrix}.$$

where

$$M_{kk}^{+} = \frac{\xi\zeta(N+\zeta+1)}{2(2k-1+\zeta)(2k+1+\zeta)}, \quad N_{kk}^{+} = \frac{-\xi(N+\zeta+1)}{2(2k-1+\zeta)(2k+1+\zeta)},$$

$$N_{k-1k}^{+} = \zeta^{-1}\left\{\zeta + 2(k-1)(k+\zeta) - 2[(k-1)_2(k-1+\zeta)_2]^{1/2}\right\},$$

$$M_{k+1k}^{+} = \frac{(N/2-k)(N/2+k+1+\zeta)(2k+1+2\mu_x)(2k+1+2\mu_y)\left\{\zeta+2k(k+\zeta+1)+2[(k)_2(k+\zeta)_2]^{1/2}\right\}}{4(2k+1+\zeta)(2k+\zeta)_3},$$

$$N_{k+1k}^{+} = \frac{\zeta(N/2-k)(N/2+k+\zeta+1)(2k+1+2\mu_x)(2k+1+2\mu_y)}{4(2k+\zeta+1)(2k+\zeta)_3}.$$

The parameters of the sequence $\{\alpha_k\}$ are arbitrary but non-zero; they could be fixed, for example, by examining the action of J_2 on the eigenstates of the 2D Dunkl oscillator in the polar coordinate representation. We have thus obtained the action of the operator J_3 on the eigenstates of J_2. Recall that J_3 is the symmetry operator associated to the separation of variables in Cartesian coordinates and J_2 is the symmetry associated to the separation of variables in polar coordinates.

11.7 Conclusion

We have investigated the finite-dimensional irreducible representations of the Schwinger-Dunkl algebra $sd(2)$, which is the symmetry algebra of the two-dimensional Dunkl oscillator in the plane. The action of the symmetry generators in the representations were obtained in three different bases. In the Cartesian basis, the symmetry generator J_3 associated to separation of variables in Cartesian coordinates is diagonal, and the symmetry J_2 is tridiagonal. In the circular basis, the operator J_3 acts in a three-diagonal fashion and J_2 has a block upper-triangular structure with all blocks 2×2. The eigenvalues of J_2 can be evaluated algebraically in the circular basis and the expansion coefficients for the eigenvectors of J_2 in this basis are generated by Heun polynomials and are expressed in terms of the para-Krawtchouk polynomials. Finally, it was shown that in the eigenbasis of J_2, the operator J_3 acts in a block tridiagonal fashion with all blocks 2×2, that is, that J_3 is six-diagonal.

It has been seen that the Dunkl oscillator model is superintegrable and closely related to the -1 orthogonal polynomials of the Bannai-Ito scheme. In this connection, the study of the 3D Dunkl oscillator model and the singular 2D Dunkl oscillator could also provide additional insight in the physical interpretation of the orthogonal polynomials of the Bannai-Ito scheme.

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Chapitre 12

The singular and the $2:1$ anisotropic Dunkl oscillators in the plane

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Abstract. Two Dunkl oscillator models are considered: one singular and the other with a $2:1$ frequency ratio. These models are defined by Hamiltonians which include the reflection operators in the two variables x and y. The singular or caged Dunkl oscillator is second-order superintegrable and admits separation of variables in both Cartesian and polar coordinates. The spectrum of the Hamiltonian is obtained algebraically and the separated wavefunctions are given in the terms of Jacobi, Laguerre and generalized Hermite polynomials. The symmetry generators are constructed from the $\mathfrak{su}(1,1)$ dynamical operators of the one-dimensional model and generate a cubic symmetry algebra. In terms of the symmetries responsible for the separation of variables, the symmetry algebra of the singular Dunkl oscillator is quadratic and can be identified with a special case of the Askey-Wilson algebra $AW(3)$ with central involutions. The $2:1$ anisotropic Dunkl oscillator model is also second-order superintegrable. The energies of the system are obtained algebraically, the symmetry generators are constructed using the dynamical operators and the resulting symmetry algebra is quadratic. The general system appears to admit separation of variables only in Cartesian coordinates. Special cases where separation occurs in both Cartesian and parabolic coordinates are considered. In the latter case the wavefunctions satisfy the biconfluent Heun equation and depend on a transcendental separation constant.

12.1 Introduction

This paper purports to analyze the singular and the $2:1$ anisotropic Dunkl oscillator models in the plane. These two-dimensional quantum systems are defined by Hamiltonians of Dunkl type which involve the reflection operators in the x and y variables. As shall be seen, these two models exhibit many interesting properties: they are second-order superintegrable, exactly solvable and, in certain cases, they allow separation of variables in more than one coordinate system.

A quantum system with n degrees of freedom described by a Hamiltonian H is (maximally) superintegrable if it possesses $2n-1$ algebraically independent symmetry generators S_i such that

$$[H, S_i] = 0, \qquad i = 1, \dots, 2n-1,$$

where one of the symmetries is the Hamiltonian itself. For such a system, it is impossible for all the symmetry generators to commute with one another and hence the S_i generate a non-Abelian symmetry algebra. If m is the maximal order of the symmetry operators (apart from H) in the momenta, the system is said to be m^{th}- order superintegrable.

First order superintegrability is associated to geometrical symmetries and to Lie algebras [39] whereas second order superintegrability is typically associated to quadratic symmetry algebras [16, 17, 18, 27, 28] and to separation of variables in more than one coordinate system [1, 8, 23, 24, 37]. For example, in the Euclidean plane, all second-order superintegrable systems of the general form

$$H = -\frac{1}{2}\nabla^2 + V(x,y),$$

are known and have been classified [59]. The possible systems are the singular or caged oscillator:

$$V(x,y) = \omega(x^2+y^2) + \frac{\alpha}{x^2} + \frac{\beta}{y^2}, \tag{12.1}$$

which separates in Cartesian and polar coordinates; the anisotropic oscillator with a $2:1$ frequency ratio:

$$V(x,y) = \omega(4x^2+y^2) + \frac{\gamma}{y^2}, \tag{12.2}$$

which separates in Cartesian and parabolic coordinates and the Coulomb problem:

$$V(r,\phi) = \frac{\alpha}{2r} + \frac{1}{4r^2}\left(\frac{\beta_1}{\cos^2(\phi/2)} + \frac{\beta_2}{\sin^2(\phi/2)}\right),$$

which separates in polar and parabolic coordinates. The fourth superintegrable system admits separation in two mutually perpendicular parabolic coordinate systems. We note in passing that only the first two systems (12.1) and (12.2) are genuinely different by virtue of the Levi-Civita mapping [33]; this topic is discussed in the conclusion.

In view of the special properties and applications of superintegrable models, there is considerable interest in enlarging the set of documented systems with this property. Recent advances in this perspective include the study of superintegrable systems with higher order symmetries [25, 26, 34, 50, 51], the construction of new superintegrable models from exceptional polynomials [35, 42], the search for discretized superintegrable systems [36] and the examination of models described by Hamiltonians involving reflection operators [10, 11, 21, 22, 40, 41, 43, 44].

Hamiltonians that include reflection operators have most notably occurred in the study of integrable systems of Calogero-Sutherland type [2, 7, 32, 48] and their generalizations [20, 31]. They also arise in the study of parabosonic oscillators [38, 45, 47]. These models are best described in terms of Dunkl operators [49], which are differential/difference operators that include reflections [5]. These operators are central in the theory of multivariate orthogonal polynomials [6] and are at the heart of Dunkl harmonic analysis [46], which is currently under active development . Furthermore, the recent study of polynomial eigenfunctions of first and second order differential/difference operators of Dunkl type has led to the discovery of several new families of classical orthogonal polynomials of a single variable known as -1 polynomials, also referred to as polynomials of Bannai-Ito type [12, 54, 55, 56, 57, 58]. These new polynomials are related to Jordan algebras [14, 53] and quadratic algebras with reflections [12, 13, 52].

This motivates the study of superintegrable and exactly solvable models that involve reflections. Recently, we introduced the Dunkl oscillator model in the plane [10, 11] described by the Hamiltonian

$$\mathscr{H} = -\frac{1}{2}[(\mathscr{D}_x^{\mu_x})^2 + (\mathscr{D}_y^{\mu_y})^2] + \frac{1}{2}(x^2+y^2), \tag{12.3}$$

where $\mathscr{D}_x^{\mu_x}$ stands for the Dunkl derivative

$$\mathscr{D}_x^{\mu_x} = \partial_x + \frac{\mu_x}{x}(1-R_x), \tag{12.4}$$

where $R_x f(x) = f(-x)$ is the reflection operator. This is possibly the simplest two-dimensional model with reflections and it corresponds to the combination of two independent parabosonic oscillators [45]. The Dunkl oscillator has been shown to be second-order superintegrable and its wavefunctions, overlap coefficients and symmetry algebra have been related to -1 polynomials.

We shall here consider two extensions of the Hamiltonian (12.3). The first one, called the singular Dunkl oscillator, corresponds to the Hamiltonian (12.3) with additional singular terms proportional to x^{-2} and y^{-2}. The second one, called the $2:1$ anisotropic Dunkl oscillator, corresponds to a singular Dunkl oscillator in the y direction combined with a Dunkl oscillator with twice the frequency in the x direction.

The two-dimensional singular Dunkl oscillator will be shown to be second-order superintegrable and to admit separation of variables in Cartesian and polar coordinates. Its separated wavefunctions will be obtained in terms of Jacobi, Laguerre and generalized Hermite polynomials. A cubic symmetry algebra with reflections will be found for this model, as opposed to the linear Lie-type algebra extended with reflections obtained for the ordinary Dunkl oscillator (12.3) in [10]. In terms of the symmetries responsible for the separation of variables, the invariance algebra is quadratic and will be identified to the Hahn algebra with central involutions; the Hahn algebra is a special case of the Askey-Wilson algebra $AW(3)$ [16]. The appearance of the Hahn algebra as symmetry algebra will also establish that the expansion coefficients between the Cartesian and polar bases are given in terms of the dual Hahn polynomials.

The anisotropic Dunkl oscillator will also be shown to be second-order superintegrable and its quadratic symmetry algebra will be constructed with the dynamical (spectrum-generating) operators of the one-dimensional components. It will be seen that for this model the separation of variables is not possible in general. Special cases where separation in parabolic coordinates do occur will be examined; they correspond to the combination of either a singular or an ordinary Dunkl oscillator in one direction with a standard harmonic oscillator with twice the frequency in the other direction. We shall show in one of these special cases that the wavefunctions in parabolic coordinates are expressed in terms of biconfluent Heun functions which depend on a transcendental parameter.

The organization of the remainder of this article is straightforward. Section 2 is dedicated to the analysis of the singular oscillator. Section 3 bears on the 2 : 1 anisotropic Dunkl oscillator. Section 4 concludes the paper with remarks on the Dunkl-Coulomb problem and on the Levi-Civita mapping for models involving Dunkl derivatives.

12.2 The singular Dunkl oscillator

In this section, the singular Dunkl oscillator model in the plane is introduced. The model can be considered both as a generalization of the model (12.1) with the standard derivatives replaced by the Dunkl derivatives or as an extension of the Hamiltonian (12.3) with additional singular terms in the potential.

12.2.1 Hamiltonian, dynamical symmetries and spectrum

The singular Dunkl oscillator in the plane is described by the Hamiltonian

$$\mathscr{H} = -\frac{1}{2}\left[(\mathscr{D}_x^{\mu_x})^2 + (\mathscr{D}_y^{\mu_y})^2\right] + \frac{1}{2}(x^2+y^2) + \frac{(\alpha_x + \beta_x R_x)}{2x^2} + \frac{(\alpha_y + \beta_y R_y)}{2y^2}, \tag{12.5}$$

where $\mathscr{D}_{x_i}^{\mu_{x_i}}$ is the Dunkl derivative (12.4) whose square has the expression

$$(\mathscr{D}_{x_i}^{\mu_{x_i}})^2 = \partial_{x_i}^2 + \frac{2\mu_{x_i}}{x_i}\partial_{x_i} - \frac{\mu_{x_i}}{x_i^2}(1-R_{x_i}), \qquad \partial_{x_i} = \frac{\partial}{\partial x_i},$$

and where R_{x_i} is the reflection operator

$$R_{x_i} f(x_i) = f(-x_i), \qquad i = 1,2,$$

with $x_1 = x$ and $x_2 = y$. The parameters α_{x_i}, β_{x_i} obey the quantization conditions

$$\alpha_{x_i} = 2k_{x_i}^+(k_{x_i}^+ + \mu_{x_i} - 1/2) + 2k_{x_i}^-(k_{x_i}^- + \mu_{x_i} + 1/2), \tag{12.6a}$$

$$\beta_{x_i} = 2k_{x_i}^+(k_{x_i}^+ + \mu_{x_i} - 1/2) - 2k_{x_i}^-(k_{x_i}^- + \mu_{x_i} + 1/2), \tag{12.6b}$$

with $k_{x_i}^\pm \in \mathbb{Z}$. The quantization conditions (12.6) can be seen to arise from the parity requirements (due to the reflections) on the solutions of the Schrödinger equation associated to the Hamiltonian (12.5) (see subsection 2.2.1).

Strikingly, the singular Dunkl oscillator (12.5) exhibits a $\mathfrak{su}(1,1)$ dynamical symmetry similar to that of the ordinary singular oscillator [33]. To see this, one first introduces two commuting sets $(a_x, a_x^\dagger)$, $(a_y, a_y^\dagger)$ of parabosonic creation/annihilation operators [38, 45] :

$$a_{x_i} = \frac{1}{\sqrt{2}}\left(x_i + \mathscr{D}_{x_i}^{\mu_{x_i}}\right), \quad a_{x_i}^\dagger = \frac{1}{\sqrt{2}}\left(x_i - \mathscr{D}_{x_i}^{\mu_{x_i}}\right). \tag{12.7}$$

These operators satisfy the following commutation relations:

$$[a_{x_i}, a_{x_i}^\dagger] = 1 + 2\mu_{x_i} R_{x_i}, \quad \{a_{x_i}, R_{x_i}\} = 0, \quad \{a_{x_i}^\dagger, R_{x_i}\} = 0,$$

where $[a,b] = ab - ba$ and $\{a,b\} = ab + ba$. Upon defining the generators

$$A_{x_i}^\dagger = (a_{x_i}^\dagger)^2 - \frac{(\alpha_{x_i} + \beta_{x_i} R_{x_i})}{2x_i^2}, \quad A_{x_i} = (a_{x_i})^2 - \frac{(\alpha_{x_i} + \beta_{x_i} R_{x_i})}{2x_i^2}, \tag{12.8}$$

a direct computation shows that

$$[\mathscr{H}_{x_i}, A_{x_i}^\dagger] = 2A_{x_i}^\dagger, \quad [\mathscr{H}_{x_i}, A_{x_i}] = -2A_{x_i}, \quad [A_{x_i}^\dagger, A_{x_i}] = -4\mathscr{H}_{x_i}, \tag{12.9}$$

where $\mathscr{H}_{x_i}$ is the Hamiltonian of the one-dimensional singular Dunkl oscillator

$$\mathscr{H}_{x_i} = -\frac{1}{2}(\mathscr{D}_{x_i}^{\mu_{x_i}})^2 + \frac{x_i^2}{2} + \frac{(\alpha_{x_i} + \beta_{x_i} R_{x_i})}{2x_i^2}. \tag{12.10}$$

It is also easily verified that

$$[\mathscr{H}_{x_i}, R_{x_i}] = 0, \quad [A_{x_i}^\dagger, R_{x_i}] = 0, \quad [A_{x_i}, R_{x_i}] = 0. \tag{12.11}$$

The algebra (12.9) is forthwith identified with the Lie algebra $\mathfrak{su}(1,1)$. Indeed, upon taking

$$2J_0 = \mathscr{H}_{x_i}, \quad 2J_+ = A_{x_i}^\dagger, \quad 2J_- = A_{x_i}, \tag{12.12}$$

the defining relations of $\mathfrak{su}(1,1)$ are recovered:

$$[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = -2J_0. \tag{12.13}$$

We also have in this case $J_\pm^\dagger = J_\mp$. The Casimir operator $\mathscr{C}$ of the algebra (12.13) is of the form

$$\mathscr{C} = J_0^2 - J_+ J_- - J_0.$$

In the realization (12.8), (12.9), (12.10) the Casimir operator may be expressed as

$$C_{x_i} = \mathscr{H}_{x_i}^2 - A_{x_i}^\dagger A_{x_i} - 2\mathscr{H}_{x_i},$$

and is seen to have the following action on functions of argument x_i:

$$C_{x_i} f(x_i) = (\mu_{x_i}^2 + \alpha_{x_i} - 3/4) f(x_i) + (\beta_{x_i} - \mu_{x_i}) f(-x_i).$$

Recall that the reflection operator R_{x_i} commutes with all the generators and can thus be simultaneously diagonalized with C_{x_i}. The operator C_{x_i} hence take two possible values depending on the parity of $f(x_i)$. On even functions, one has

$$C_{x_i} f(x_i) = 4\delta_{x_i}(\delta_{x_i} - 1) f(x_i), \qquad \delta_{x_i} = k_{x_i}^+ + \mu_{x_i}/2 + 1/4, \tag{12.14}$$

and on odd functions, one finds

$$C_{x_i} f(x_i) = 4\epsilon_{x_i}(\epsilon_{x_i} - 1) f(x_i), \qquad \epsilon_{x_i} = k_{x_i}^- + \mu_{x_i}/2 + 3/4. \tag{12.15}$$

It is possible to introduce an invariant operator Q_{x_i} given by

$$Q_{x_i} = \mathscr{H}_{x_i}^2 - A_{x_i}^\dagger A_{x_i} - 2\mathscr{H}_{x_i} + (\mu_{x_i} - \beta_{x_i}) R_{x_i}, \tag{12.16}$$

which commutes with all the generators $\mathscr{H}_{x_i}$, A_{x_i}, $A_{x_i}^\dagger$ and acts as a multiple of the identity on the space of functions (even and odd) of argument x_i. The value of the multiple is

$$q_{x_i} = \mu_{x_i}^2 + \alpha_{x_i} - 3/4. \tag{12.17}$$

It follows from the above considerations that the eigenstates of each one-dimensional singular Dunkl oscillator span the space of a direct sum of two irreducible $\mathfrak{su}(1,1)$ representations; one for each parity case. The representation theory of $\mathfrak{su}(1,1)$ can be used to obtain the spectrum of $\mathscr{H}_{x_i}$. In point of fact, it is known [19] that in the positive discrete series of irreducible unitary representations of $\mathfrak{su}(1,1)$ in which the Casimir operator takes the value $\mathscr{C} = \nu(\nu - 1)$, where ν is a positive real number, the spectrum of J_0 is of the form $n+\nu$, where n is a non-negative integer. Given the identification (12.12) and the Casimir values (12.14), (12.15) it follows that the spectrum of $\mathscr{H}_{x_i}$ is

$$E_n^+ = 2n + \nu_{x_i}^+ + 1/2, \qquad E_n^- = 2n + \nu_{x_i}^- + 3/2, \tag{12.18}$$

where

$$\nu_{x_i}^{\pm} = 2k_{x_i}^{\pm} + \mu_{x_i},$$

and where n is a non-negative integer. The $\pm$ sign is associated to the eigenvalues of the reflection R_{x_i}. The following conditions must hold on the values of the parameters:

$$\nu_{x_i}^+ + 1/2 > 0, \quad \text{and} \quad \nu_{x_i}^- + 3/2 > 0. \tag{12.19}$$

It follows from (12.18) that the spectrum of the full Hamiltonian (12.5) splits in four sectors labeled by the eigenvalues (s_x, s_y) of the reflection operators R_x, R_y. The expression for the spectrum is

$$E_{n_x n_y}^{s_x s_y} = 2(n_x + n_y) + \nu_x^{s_x} + \nu_y^{s_y} + \theta_{s_x} + \theta_{s_y} + 1, \tag{12.20}$$

where $s_{x_i} = \pm 1$ and where

$$\theta_{s_x} = \begin{cases} 0 & \text{if } s_x = 1, \\ 1 & \text{if } s_x = -1. \end{cases} \tag{12.21}$$

It is understood that for example when $s_x = -1$, one should read $\nu_x^{s_x}$ as ν_x^-.

12.2.2 Exact solutions and separation of variables

It is possible to obtain explicitly the wavefunctions Ψ satisfying the Schrödinger equation

$$\mathscr{H}\Psi = E\Psi, \tag{12.22}$$

associated to the Hamiltonian (12.5) in both Cartesian and polar coordinates.

Cartesian coordinates

The Hamiltonian (12.5) obviously separates in Cartesian coordinates and in these coordinates the separated wavefunctions $\psi(x_i)$ are those of the one-dimensional singular Dunkl oscillator (12.10). The eigenfunctions $\psi(x)$ of $\mathscr{H}_x$ are easily seen to satisfy the differential equation

$$\psi''(x) + \frac{2\mu_x}{x}\psi'(x) + \left\{2E - x^2 - \frac{\alpha_x + \mu_x}{x^2}\right\}\psi(x) + \left\{\frac{\mu_x - \beta_x}{x^2}\right\}R_x\psi(x) = 0. \tag{12.23}$$

Since the reflection R_x commutes with the one-dimensional Hamiltonian $\mathscr{H}_x$, the eigenfunctions can be chosen to have a definite parity. For the even sector, defined by $R_x\psi^+(x) = \psi^+(x)$, one finds that the normalizable solution to (12.23) is given by

$$\psi^+_{n_x}(x) = (-1)^{n_x}\sqrt{\frac{n_x!}{\Gamma(n_x+\nu_x^+ + 1/2)}}\, e^{-x^2/2} x^{2k_x^+} L_{n_x}^{(\nu_x^+ - 1/2)}(x^2), \tag{12.24}$$

with

$$E^+_{n_x} = 2n_x + \nu_x^+ + 1/2,$$

and where $L_n^{(\alpha)}(x)$ are the Laguerre polynomials [30]. For the odd sector, defined by $R_x\psi^-(x) = -\psi^-(x)$, the wavefunctions are of the form

$$\psi^-_{n_x}(x) = (-1)^{n_x}\sqrt{\frac{n_x!}{\Gamma(n_x+\nu_x^- + 3/2)}}\, e^{-x^2/2} x^{2k_x^- + 1} L_{n_x}^{(\nu_x^- + 1/2)}(x^2), \tag{12.25}$$

with

$$E^-_{n_x} = 2n_x + \nu_x^- + 3/2.$$

Hence, as announced, the wavefunctions of the two-dimensional Hamiltonian (12.5) split in four parity sectors labeled by the eigenvalues of the reflection operators R_x, R_y and are given by

$$\Psi^{s_x s_y}_{n_x n_y}(x,y) = \psi^{s_x}_{n_x}(x)\psi^{s_y}_{n_y}(y), \tag{12.26}$$

with energies $E^{s_x,s_y}_{n_x,n_y}$ as in (12.20) and with $\psi^{s_x}_{n_y}$ given by (12.24), (12.25). Using the orthogonality relation of the Laguerre polynomials, it is directly checked that the wavefunctions (12.26) enjoy the orthogonality relation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Psi^{s_x s_y}_{n_x n_y}(x,y)[\Psi^{s'_x s'_y}_{n'_x n'_y}(x,y)]^* |x|^{2\mu_x}|y|^{2\mu_y}\,\mathrm{d}x\,\mathrm{d}y = \delta_{n_x n'_x}\delta_{n_y n'_y}\delta_{s_x s'_x}\delta_{s_y s'_y}. \tag{12.27}$$

Let us point out that a direct computation shows [10] that the Dunkl derivative (12.4) is anti-Hermitian with respect to the scalar product

$$\langle f \mid g \rangle = \int_{-\infty}^{\infty} g(x) f^*(x)|x|^{2\mu_x}\,\mathrm{d}x.$$

Spacing of energy levels in the singular Dunkl oscillator

It is directly seen from (12.18) that for generic values of k_x^+, k_x^-, the full spectrum of the one-dimensional singular Dunkl oscillator which comprises both the even and odd sectors is not equidistant, in contradistinction with the situation for the ordinary singular oscillator. An equidistant spectrum is obtained by taking $k_x^+ = k_x^- = k_x$. In this case, the energies (12.18) and wavefunctions can both be synthesized in single formulas which are close to the corresponding ones for the ordinary Dunkl oscillator [10]. In this case, one finds for the energies

$$E_{n_x} = n_x + \nu_x + 1/2, \quad n_x = 0, 1, \ldots.$$

The wavefunctions are expressed as

$$\psi_{n_x}(x) = e^{-x^2/2} x^{2k_x} H_{n_x}^{\nu_x}(x),$$

where $H_n^\gamma(x)$ are the generalized Hermite polynomials [3]

$$H_{2m+p}^{\gamma}(x) = (-1)^m \sqrt{\frac{m!}{\Gamma(2m+p+\gamma+1/2)}}\, x^p L_m^{(\gamma+p-1/2)}(x^2), \tag{12.28}$$

with $p \in \{0, 1\}$. The wavefunctions of the full two-dimensional model have then the expression

$$\Psi_{n_x,n_y}(x,y) = e^{-(x^2+y^2)/2} x^{2k_x} y^{2k_y} H_{n_x}^{\nu_x}(x) H_{n_y}^{\nu_y}(y),$$

with $E_{n_x,n_y} = (n_x + n_y) + \nu_x + \nu_y + 1$, where n_x, n_y are non-negative integers, as the corresponding energies. It is directly seen that upon taking $k_x = k_y = 0$ in the above formulas, one recovers the results found in [10] for the Dunkl oscillator model.

Polar coordinates

In polar coordinates

$$x = \rho \cos\phi, \quad y = \rho \sin\phi,$$

the reflection operators have the action

$$R_x \Psi(\rho, \phi) = \Psi(\rho, \pi - \phi), \quad R_y \Psi(\rho, \phi) = \Psi(\rho, -\phi).$$

The Schrödinger equation (12.22) associated to the Hamiltonian (12.5) takes the form

$$\left\{ \mathscr{A}_\rho + \frac{1}{\rho^2} \mathscr{B}_\phi \right\} \Psi(\rho, \phi) = E \Psi(\rho, \phi), \tag{12.29}$$

where $\mathscr{A}_\rho$ has the expression

$$\mathscr{A}_\rho = -\frac{1}{2}\left\{\partial_\rho^2 + \frac{1}{\rho}\partial_\rho\right\} - \frac{1}{\rho}(\mu_x+\mu_y)\partial_\rho + \frac{1}{2}\rho^2,$$

and where B_ϕ is given by

$$\mathscr{B}_\phi = -\frac{1}{2}\partial_\phi^2 + (\mu_x \tan\phi - \mu_y \cot\phi)\partial_\phi + \left\{\frac{\mu_x+\alpha_x}{2\cos^2\phi}\right\} + \left\{\frac{\mu_y+\alpha_y}{2\sin^2\phi}\right\}$$
$$+ \left\{\frac{\beta_x-\mu_x}{2\cos^2\phi}\right\} R_x + \left\{\frac{\beta_y-\mu_y}{2\sin^2\phi}\right\} R_y.$$

It is easy to see that the equation (12.29) admits separation in polar coordinates. Upon taking $\Psi(\rho,\phi) = P(\rho)\Phi(\phi)$, we obtain the pair of ordinary differential equations

$$\mathscr{B}_\phi\Phi(\phi) - \frac{m^2}{2}\Phi(\phi) = 0, \tag{12.30a}$$

$$\mathscr{A}_\rho P(\rho) + \left(\frac{m^2}{2\rho^2} - E\right)P(\rho) = 0, \tag{12.30b}$$

where $m^2/2$ is the separation constant. The solutions to (12.30a) split in four parity sectors labeled by the eigenvalues s_x, s_y of the reflection operators R_x, R_y. The angular wavefunctions are found to be

$$\Phi_n^{s_x s_y}(\phi) = N_n \cos^{2k_x^{s_x}+\theta_{s_x}}\phi\, \sin^{2k_y^{s_y}+\theta_{s_y}}\phi\, P_{n-\theta_{s_x}/2-\theta_{s_y}/2}^{(\nu_y^{s_y}+\theta_{s_y}-1/2,\, \nu_x^{s_x}+\theta_{s_x}-1/2)}(\cos 2\phi),$$

where $P_n^{(\alpha,\beta)}(x)$ are the Jacobi polynomials [30] and where θ_{s_x}, θ_{s_y} are as in (12.21). The admissible values of n are as follows. If either s_x or s_y is negative, n is a positive half-integer. If $s_x = s_y = 1$, n is a non-negative integer and if $s_x = s_y = -1$, n is a positive integer. The normalization constant is

$$N_n = \sqrt{\frac{(2n+\nu_x^{s_x}+\nu_y^{s_y})\Gamma(n+\nu_x^{s_x}+\nu_y^{s_y}+\frac{\theta_{s_x}}{2}+\frac{\theta_{s_y}}{2})(n-\theta_{s_x}/2-\theta_{s_y}/2)!}{2\Gamma(n+\nu_x^{s_x}+\frac{\theta_{s_x}}{2}-\frac{\theta_{s_y}}{2}+1/2)\Gamma(n+\nu_y^{s_y}+\frac{\theta_{s_y}}{2}-\frac{\theta_{s_x}}{2}+1/2)}},$$

where $\Gamma(x)$ is the Gamma function. The separation constant has the expression

$$m^2 = 4(n+k_x^{s_x}+k_y^{s_y})(n+k_x^{s_x}+k_y^{s_y}+\mu_x+\mu_y), \tag{12.31}$$

and the wavefunctions obey the orthogonality relation

$$\int_0^{2\pi} \Phi_n^{s_x s_y}(\phi)\Phi_{n'}^{s'_x s'_y}(\phi)|\cos\phi|^{2\mu_x}|\sin\phi|^{2\mu_y}\,\mathrm{d}\phi = \delta_{nn'}\delta_{s_x,s'_x}\delta_{s_y,s'_y}.$$

The normalizable solution to the radial equation (12.30b) is found to be

$$P_\ell(\rho) = N_\ell\, e^{-\rho^2/2} \rho^{2n+2k_x^{s_x}+2k_y^{s_y}} L_\ell^{(2n+2k_x^{s_x}+2k_y^{s_y})}(\rho^2).$$

and in terms of the quantum numbers associated to the polar basis, the energies of the Hamiltonian (12.5) are

$$E_{n\ell} = 2(n+\ell) + \nu_x^{s_x} + \nu_y^{s_y} + 1,$$

where ℓ is a non-negative integer. The normalization factor is

$$N_\ell = \sqrt{\frac{2\ell!}{\Gamma(2n+2k_x^{s_x}+2k_y^{s_y}+\ell+1)}}.$$

and the radial wavefunctions obey the orthogonality relation

$$\int_0^\infty P_\ell(\rho) P_{\ell'}(\rho) \rho^{2\mu_x+2\mu_y+1}\, \mathrm{d}\rho = \delta_{\ell\ell'}.$$

12.2.3 Integrals of motion and symmetry algebra

The integrals of motion of the two-dimensional Dunkl oscillator are most naturally obtained by combining the $\mathfrak{su}(1,1)$ dynamical operators of the one-dimensional model. We define

$$B_0 = \mathscr{H}_x - \mathscr{H}_y, \quad B_+ = A_x^\dagger A_y, \quad B_- = A_x A_y^\dagger. \tag{12.32}$$

It is directly checked that the operators (12.32) are symmetries of the Hamiltonian (12.5)

$$[\mathscr{H}, B_0] = [\mathscr{H}, B_\pm] = 0.$$

A straightforward computation shows that the following commutation relations hold:

$$[B_0, B_\pm] = \pm 4 B_\pm, \tag{12.33a}$$

$$[B_-, B_+] = B_0^3 + u_1 B_0 + u_2, \tag{12.33b}$$

where

$$u_1 = \xi_x R_x + \xi_y R_y - \mathscr{H}^2 - w_x - w_y, \quad u_2 = \mathscr{H}(\xi_y R_y - \xi_x R_x + w_x - w_y),$$

and with

$$\xi_{x_i} = 2(\mu_{x_i} - \beta_{x_i}), \quad w_{x_i} = 2(\mu_{x_i}^2 + \alpha_{x_i} - 3/4).$$

Since the full Hamiltonian $\mathscr{H}$ given by (12.5) and the reflections R_x, R_y are central elements, they will act as multiples of the identity in any irreducible representation of (12.33) and consequently the operators u_1, u_2 can be treated as "structure constants".

It follows from the above considerations that the singular Dunkl oscillator (12.5) is superintegrable with a cubic symmetry algebra given by (12.33). The energies of the Hamiltonian (12.5) could be derived algebraically from the irreducible representations of the algebra (12.33) [33]. The basis of operators $\{B_0, B_\pm\}$ generating the symmetry algebra (12.33) and defined by (12.32) will be referred to as the "ladder" basis.

12.2.4 Symmetries, separability and the Hahn algebra with involutions

When considering a Hamiltonian that admits separation of variables in more than one coordinate system, an alternative approach to finding the symmetry generators consists in identifying the symmetries responsible for the separation of variables [37]. We shall consider this approach here and relate it to the ladder approach of the preceding subsection.

The symmetry associated to the separation in Cartesian coordinates has already been found and is obviously given by

$$K_1 = B_0 = \mathscr{H}_x - \mathscr{H}_y.$$

When acting on the separated wavefunctions in Cartesian coordinates $\Psi(x,y)$ given by (12.26), this operator is diagonal with eigenvalues

$$\lambda = 2(n_x - n_y) + \nu_x^{s_x} - \nu_y^{s_y} + \theta_{s_x} - \theta_{s_y}, \qquad n_x, n_y \in \{0,1,\dots\}.$$

The symmetry associated to the separation of variables in polar coordinates can be obtained by analogy with the standard singular oscillator case [9]. We consider the operator

$$K_2 = (x\mathscr{D}_y^{\mu_y} - y\mathscr{D}_x^{\mu_x})^2 - \frac{y^2}{x^2}(\alpha_x + \beta_x R_x) - \frac{x^2}{y^2}(\alpha_y + \beta_y R_y) - 1/2.$$

It is directly checked that $[\mathscr{H}, K_2] = 0$ and that K_2 is hence a symmetry. The assertion that K_2 is the symmetry associated to the separation of variables in polar coordinates stems from the following expression for K_2:

$$K_2 = -2\mathscr{B}_\phi + (\alpha_x + \beta_x R_x) + (\alpha_y + \beta_y R_y) - 2\mu_x\mu_y(1 - R_x R_y) - 1/2,$$

which is easily obtained by a direct computation. Thus the operator K_2 acts in a diagonal fashion on the separated wavefunctions $\Psi(\rho,\phi)$ with eigenvalues

$$\lambda = -m^2 + \alpha_x + \alpha_y + \beta_x s_x + \beta_y s_y - 2\mu_x\mu_y(1 - s_x s_y) - 1/2,$$

where m^2 is as given by (12.31). The symmetry K_2 can be expressed in terms of the operators of the ladder basis (12.32). Upon inspection, one finds

$$K_2 = B_+ + B_- + \frac{1}{2}B_0^2 - \frac{1}{2}\mathscr{H}^2 + \mu_x R_x + \mu_y R_y + 2\mu_x\mu_y R_x R_y. \tag{12.34}$$

With this identification, the symmetry algebra (12.33) can be written in terms of the symmetries K_1, K_2 and their commutator $K_3 = [K_1, K_2]$. Using the commutation relations (12.33), the symmetry algebra becomes

$$[K_1, K_2] = K_3, \tag{12.35a}$$

$$[K_2, K_3] = 8\{K_1, K_2\} + \gamma_1 K_1 + \gamma_2, \tag{12.35b}$$

$$[K_3, K_1] = 8K_1^2 - 16K_2 + \gamma_3, \tag{12.35c}$$

where the "structure constants" have the form

$$\gamma_1 = -8(2\beta_x R_x + 2\beta_y R_y + 4\mu_x\mu_y R_x R_y + w_x + w_y),$$

$$\gamma_2 = 8\mathscr{H}(\xi_y R_y - \xi_x R_x + w_x - w_y),$$

$$\gamma_3 = 16(\mu_x R_x + \mu_y R_y + 2\mu_x\mu_y R_x R_y) - 8\mathscr{H}^2.$$

Under the transformation (12.34), the symmetry algebra (12.33) has become quadratic. The algebra (12.35) is a special case of the Askey-Wilson algebra $AW(3)$ known as the Hahn algebra [60] with additional central involutions R_x, R_y; other presentations of the this algebra (without reflections) are found in [29]. The algebra (12.35) has the Casimir operator

$$\mathscr{Q} = 8\{K_1^2, K_2\} + (56 + \gamma_1)K_1^2 - 16K_2^2 + K_3^2 + 2\gamma_2 K_1 + (2\gamma_3 + 16)K_2,$$

which commutes with all generators K_1, K_2 and K_3. In the present realization, one finds that the operator $\mathscr{Q}$ takes the value

$$\mathscr{Q} = \zeta_1\mathscr{H}^2 - \zeta_2 R_x - \zeta_3 R_y + \zeta_4 R_x R_y + \zeta_5,$$

with

$$\zeta_1 = 16\Big\{(\mu_x^2+\mu_y^2+\alpha_x+\alpha_y+2)+(\beta_x-2\mu_x)R_x+(\beta_y-2\mu_y)R_y-2\mu_x\mu_yR_xR_y\Big\},$$
$$\zeta_2 = 64\Big\{(\beta_x-2\mu_x)(\mu_y^2+\alpha_y-3/4)-\mu_x+\mu_x\alpha_y\Big\},$$
$$\zeta_3 = 64\Big\{(\beta_y-2\mu_y)(\mu_x^2+\alpha_x-3/4)-\mu_y+\mu_y\alpha_x\Big\},$$
$$\zeta_4 = 64(\beta_x\mu_y+\beta_y\mu_x-\beta_x\beta_y),$$
$$\zeta_5 = 64\Big\{\mu_x^2+\mu_y^2-\alpha_x(\mu_y^2+\alpha_y-3/4)-\alpha_y(\mu_x^2-3/4)-1/2\Big\}.$$

Because of the direct connection between the irreducible representations of the Askey-Wilson algebra $AW(3)$ and the Askey scheme of orthogonal polynomials [60], the occurrence of the Hahn algebra with reflections (12.35) as a symmetry algebra of the 2D singular Dunkl oscillator model suffices to establish that the dual Hahn polynomials act as overlap coefficients between the polar and Cartesian bases [9]. This result contrasts with the situation in the case of the 2D Dunkl oscillator model, for which the overlap coefficients were found in terms of the dual -1 Hahn polynomials. This difference is explained by the fact that in the ordinary Dunkl oscillator case, the reflections anticommute with the raising/lowering operators and consequently the space of degenerate eigenfunctions of a given energy is labeled by the eigenvalues of the *product* R_xR_y and thus for example the sectors corresponding to $s_x = s_y = 1$ and $s_x = s_y = -1$ are "coupled". In the singular oscillator case the spaces corresponding to different values of s_x, s_y are fully "decoupled".

12.3 The $2:1$ anisotropic Dunkl oscillator

We shall now introduce our second two-dimensional Dunkl oscillator model: the two-dimensional anisotropic Dunkl oscillator with a $2:1$ frequency ratio. The standard 2:1 oscillator is known to be one of the two-dimensional models which is superintegrable of order two and admits separations in both Cartesian and parabolic coordinates [8]; it is correspondingly of interest to consider its Dunkl analogue. It will be shown that this system is also second-order superintegrable, but does not seem to admit separation of variables except in Cartesian coordinates. We shall however present special cases of the general model for which separation in parabolic coordinates occurs.

12.3.1 Hamiltonian, dynamical symmetries and spectrum

The $2:1$ anisotropic Dunkl oscillator is defined by the Hamiltonian

$$\mathcal{H} = -\frac{1}{2}[(\mathcal{D}_x^{\mu_x})^2 + (\mathcal{D}_y^{\mu_y})^2] + \frac{1}{2}(4x^2 + y^2) + \frac{\alpha_y + \beta_y R_y}{2y^2}, \tag{12.36}$$

where

$$\alpha_y = 2k_y^+(k_y^+ + \mu_y - 1/2) + 2k_y^-(k_y^- + \mu_y + 1/2),$$
$$\beta_y = 2k_y^+(k_y^+ + \mu_y - 1/2) - 2k_y^-(k_y^- + \mu_y + 1/2),$$

and with $k_y \in \mathbb{Z}$, $2k_y^+ + \mu_y > -1/2$ and $2k_y^- + \mu_y > -3/2$. It is seen that (12.36) corresponds to the combination of a one-dimensional singular Dunkl oscillator in the y direction and an ordinary one-dimensional Dunkl oscillator with twice the frequency in the x direction. The dynamical symmetries of the y part of the anisotropic oscillator (12.36) described by the Hamiltonian

$$\mathcal{H}_y = -\frac{1}{2}(\mathcal{D}_y^{\mu_y})^2 + \frac{1}{2}y^2 + \frac{\alpha_y + \beta_y R_y}{2y^2}, \tag{12.37}$$

have been studied in the previous section. The dynamical operators $A_y^{\dagger}$, A_y are defined by (12.8) and together with $\mathcal{H}_y$ they generate the $\mathfrak{su}(1,1)$ Lie algebra (12.9) with the invariant operator Q_y defined (12.16) taking the value (12.17). The spectrum of $\mathcal{H}_y$ is known to be of the form

$$E_n^+ = 2n + \nu_y^+ + 1/2, \quad E_n^- = 2n + \nu_y^- + 3/2,$$

where n is a non-negative integer. The dynamical symmetries of the x part of the Hamiltonian (12.36)

$$\mathcal{H}_x = -\frac{1}{2}(\mathcal{D}_x^{\mu_x})^2 + 2x^2, \tag{12.38}$$

are easily obtained. We introduce the operators

$$c_x = \sqrt{2}(x + \frac{1}{2}\mathcal{D}_x^{\mu_x}), \quad c_x^{\dagger} = \sqrt{2}(x - \frac{1}{2}\mathcal{D}_x^{\mu_x}). \tag{12.39}$$

It is directly checked that the following commutation relations hold

$$[\mathcal{H}_x, c_x] = -2c_x, \quad [\mathcal{H}_x, c_x^{\dagger}] = 2c_x^{\dagger}, \quad [c_x, c_x^{\dagger}] = 2 + 4\mu_x R_x, \quad \{c_x, c_x^{\dagger}\} = 2\mathcal{H}_x,$$
$$[\mathcal{H}_x, R_x] = 0, \quad \{c_x, R_x\} = 0, \quad \{c_x^{\dagger}, R_x\} = 0.$$

The dynamical algebra (12.39) is directly identified with the $sl_{-1}(2)$ algebra [52]. The algebra (12.39) admits the Casimir operator

$$Q = c_x^\dagger c_x R_x - \mathscr{H}_x R_x + R_x,$$

which commutes with all the dynamical operators $\mathscr{H}_x$, c_x, $c_x^\dagger$ and acts as a multiple of the identity:

$$Q = q\mathbb{I}, \quad q = -2\mu_x.$$

Using the representation theory of $sl_{-1}(2)$ [52], the expression for the spectrum of $\mathscr{H}_x$ is found to be

$$E_n = 2n + 2\mu_x + 1, \quad n = 0, 1, \ldots$$

It follows that the spectrum of the two-dimensional anisotropic Dunkl oscillator (12.36) is given by

$$E_{n_x,n_y}^{s_y} = 2(n_x + n_y) + 2\mu_x + \nu_y^{s_y} + \theta_{s_y} + 3/2, \tag{12.40}$$

where $\nu_y^{\pm} = 2k_y^{\pm} + \mu_y$.

12.3.2 Exact solutions and separation of variables

It is possible to write down in Cartesian coordinates the exact solutions of the Schrödinger equation corresponding to the Hamiltonian (12.36). The wavefunctions are again of the form $\Psi(x, y) = \varphi(x)\psi(y)$ where $\varphi(x)$ is a wavefunction of the ordinary Dunkl oscillator with frequency 2 and $\psi(y)$ is a wavefunction of the singular Dunkl oscillator.

The solutions to the equation $\mathscr{H}_x \varphi(x) = E\varphi(x)$ have been derived in [10, 45]. They take the form

$$\varphi_{n_x}(x) = 2^{(\mu_x+1/2)/2}\, e^{-x^2} H_{n_x}^{\mu_x}(\sqrt{2}x),$$

where $H_n^{\mu}(x)$ denotes the generalized Hermite polynomials defined in (12.28). The corresponding energies are

$$E_{n_x} = 2n_x + 2\mu_x + 1, \quad n_x = 0, 1, \ldots,$$

The solutions to the equation $\mathscr{H}_y \psi(y) = E\psi(y)$ have been found in the preceding section in terms of Laguerre polynomials and are given by (12.24) and (12.25). It follows that the exact solutions of the Schrödinger equation of the $2:1$ anisotropic Dunkl oscillator are of the form

$$\Psi^{s_y}_{n_x n_y}(x,y) = \sqrt{\frac{2^{\mu_x+1/2}\, n_y!}{\Gamma(n_y+\nu_y^{s_y}+\theta_y+1/2)}}\, e^{-(2x^2+y^2)/2} y^{2k_y^{s_y}+\theta_{s_y}} H^{\mu_x}_{n_x}(\sqrt{2}x) L_{n_y}^{(\nu_y^{s_y}+\theta_y-1/2)}(y^2),$$

with energies given by (12.40) and where $s_y = \pm 1$.

A direct inspection of the Hamiltonian (12.36) shows that this Hamiltonian does not seem to admit separation of variable in any other coordinate system. This situation differs with that of the standard anisotropic oscillator in the plane (12.2) which admits separation of variable in parabolic coordinates.

12.3.3 Integrals of motion and symmetry algebra

The dynamical operators of the anisotropic oscillator (12.36) can again be used to obtain its symmetry generators and establish the superintegrability of the model. Proceeding as in the Section 2, we introduce the operators

$$F_0 = \mathscr{H}_x - \mathscr{H}_y, \quad F_+ = c_x^{\dagger} A_y, \quad F_- = c_x A_y^{\dagger},$$

where $\mathscr{H}_x$ is given by (12.38), $\mathscr{H}_y$ by (12.37), A_y, $A_y^{\dagger}$ by (12.8) and c_x, $c_x^{\dagger}$ by (12.39). A direct examination shows that the operators F_0 and $F_{\pm}$ are symmetries of the anisotropic Dunkl oscillator Hamiltonian (12.36); $[\mathscr{H}_x, F_0] = [\mathscr{H}_x, F_{\pm}] = 0$. A straightforward computation shows that these operators generate the following quadratic algebra:

$$[F_0, F_{\pm}] = \pm 4F_{\pm}, \quad \{F_{\pm}, R_x\} = 0, \quad [F_0, R_x] = 0, \tag{12.41a}$$

$$[F_-, F_+] = \frac{3}{2}F_0^2 + z_1 F_0^2 R_x + z_2 F_0 + z_3 F_0 R_x + z_4 R_x + z_5. \tag{12.41b}$$

with

$$z_1 = \mu_x, \quad z_2 = -\mathscr{H}, \quad z_3 = -2\mu_x \mathscr{H},$$

$$z_4 = \mu_x \mathscr{H}^2 + 2\mu_x(\xi_y R_y - w_y), \quad z_5 = (\xi_y R_y - w_y) - \frac{1}{2}\mathscr{H}^2.$$

The operators $\mathscr{H}$ and R_y are central elements in the algebra (12.41). Introducing the operator

$$F_1 = \frac{1}{\sqrt{2}}(F_+ + F_-),$$

one finds that

$$F_1 = \frac{1}{2}\{\mathscr{D}_y^{\mu_y}, (x\mathscr{D}_y^{\mu_y} - y\mathscr{D}_x^{\mu_x})\} + xy^2 - \frac{x}{y^2}(\alpha_y + \beta_y R_y), \tag{12.42}$$

where $\{x, y\} = xy + yx$. The operator F_1 is thus seen to be a Dunkl analogue of the generalized Runge-Lenz vector [15], which is the symmetry associated to the separation of variables in parabolic coordinates in the standard anisotropic oscillator case. The Dunkl Hamiltonian (12.36) does not however separate in this coordinate system due to the presence of singular terms in both the x and y parts.

We shall now examine two special cases of the $2:1$ anisotropic Dunkl oscillator for which the separation in parabolic coordinates is however possible. These special cases are obtained by removing the reflections in the x-part (12.38) which prevents the separation in parabolic coordinates.

12.3.4 Special case I

The first special case of the $2:1$ anisotropic Dunkl oscillator that we consider is described by the Hamiltonian

$$\mathscr{H}_I = -\frac{1}{2}[\partial_x^2 + (\mathscr{D}_y^{\mu_y})^2] + \frac{1}{2}(4x^2 + y^2) + \frac{\alpha_y + \beta_y R_y}{2y^2}, \tag{12.43}$$

with the usual quantization conditions

$$\alpha_y = 2k_y^+(k_y^+ + \mu_y - 1/2) + 2k_y^-(k_y^- + \mu_y + 1/2),$$
$$\beta_y = 2k_y^+(k_y^+ + \mu_y - 1/2) - 2k_y^-(k_y^- + \mu_y + 1/2),$$

where $k_y \in \mathbb{Z}$. It is easily seen that this Hamiltonian is obtained from (12.36) by taking $\mu_x = 0$ and hence it is superintegrable and its symmetry algebra is obtained directly from (12.41). Moreover, the Hamiltonian (12.43) corresponds to the combination of a standard harmonic oscillator in the x direction with a singular Dunkl oscillator in the y direction. In parabolic coordinates

$$x = \frac{1}{2}(u^2 - v^2), \qquad y = uv,$$

the Schrödinger equation $\mathscr{H}_I\Psi(u,v) = E\Psi(u,v)$, where $\mathscr{H}_I$ is the Hamiltonian given by (12.43), takes the form

$$[\mathscr{C}_u + \mathscr{C}_v]\Psi(u,v) = -2E(u^2 + v^2)\Psi(u,v),$$

where

$$\mathscr{C}_u = \partial_u^2 + \frac{2\mu_y}{u}\partial_u - \frac{\mu_y}{u^2}(1-R_y) - \frac{1}{u^2}(\alpha_y+\beta_y R_y) - u^6.$$

Since the reflection operator R_y commutes with $\mathscr{H}_I$, the wavefunction $\Psi(u,v)$ can be taken to have a definite parity. In the even sector, defined by the relation $R_y\Psi^+(u,v) = \Psi^+(u,v)$, the separation Ansatz $\Psi^+(u,v) = U(u)V(v)$ yields the following pair of ordinary differential equations:

$$U''(u) + \frac{2\mu_y}{u}U'(u) + \left\{2Eu^2 - \frac{\alpha_y+\beta_y}{u^2} - u^6\right\}U(u) = \Lambda^+ U(u), \tag{12.44a}$$

$$V''(v) + \frac{2\mu_y}{v}V'(v) + \left\{2Ev^2 - \frac{\alpha_y+\beta_y}{v^2} - v^6\right\}V(v) = -\Lambda^+ V(v), \tag{12.44b}$$

where Λ^+ is the separation constant. Upon imposing the condition $U(-u) = U(u)$, the solution to the equation (12.44a) is seen to be given by

$$U^+(u) = e^{-u^4/4}u^{2k_y^+}B(\nu_y^+ - 1/2; 0; E; \frac{\Lambda^+}{\sqrt{2}}; \frac{u^2}{\sqrt{2}}),$$

where $B(\alpha;\beta;\gamma;\delta;z)$ is the Heun biconfluent function. The Heun biconfluent function is defined as the solution to the differential equation

$$B''(z) - \frac{(-\alpha+\beta z+2z^2-1)}{z}B'(z) - \frac{1}{2}\frac{(\alpha\beta+\beta+\delta+z(2\alpha+4-2\gamma))}{z}B(z) = 0,$$

with initial conditions $B(0) = 1$ and $B'(0) = (\alpha\beta+\beta+\delta)/(2\alpha+2)$. Similarly, the solution for $V(v)$ is directly given by

$$V^+(v) = e^{-v^4/4}v^{2k_y^+}B(\nu_y^+ - 1/2; 0; E; \frac{-\Lambda^+}{\sqrt{2}}; \frac{v^2}{\sqrt{2}}).$$

In the odd sector, defined by the relation $R_y\Psi(u,v) = -\Psi(u,v)$, we obtain

$$U^-(u) = e^{-u^4/4}u^{2k_y^-+1}B(\nu_y^- + 1/2; 0; E; \frac{\Lambda^-}{\sqrt{2}}; \frac{u^2}{\sqrt{2}}),$$

and

$$V^-(v) = e^{-v^4/4}u^{2k_y^-+1}B(\nu_y^- + 1/2; 0; E; \frac{-\Lambda^-}{\sqrt{2}}; \frac{v^2}{\sqrt{2}}).$$

In parabolic coordinates, the operator F_1 given in (12.42) can readily be shown to be diagonal and its eigenvalues can be related to the separation constants $\Lambda^\pm$. As is the case for the standard anisotropic oscillator (12.2), the parameters Λ obey a transcendental equation and cannot be expressed explicitly [33].

12.3.5 Special case II

Another special case of the $2:1$ anisotropic Dunkl oscillator which admits separation of variables in parabolic coordinates is described by the Hamiltonian

$$\mathcal{H}_{II} = -\frac{1}{2}[\partial_x^2 + (\mathcal{D}_y^{\mu_y})^2] + \frac{1}{2}(4x^2 + y^2). \tag{12.45}$$

The Hamiltonian (12.45) is obtained by taking $\mu_x = 0$, $\alpha_y = 0$ and $\beta_y = 0$ in (12.36). It corresponds to the combination of an ordinary Dunkl oscillator in the y direction and a standard oscillator with twice the frequency in the x direction. For this Hamiltonian, the symmetry algebra is not a special case of the algebra (12.41); this follows from the fact that the dynamical operators of the singular Dunkl oscillator involve the squares of the dynamical symmetries of the ordinary Dunkl oscillator.

The dynamical operators in the present special case are rather the creation and annihilation operators of the standard oscillator

$$g_x = \sqrt{2}(x + \frac{1}{2}\partial_x), \quad g_x^\dagger = \sqrt{2}(x - \frac{1}{2}\partial_x),$$

which are part of the algebra

$$[g_x, g_x^\dagger] = 2, \quad [\mathcal{H}_x, g_x] = -2g_x, \quad [\mathcal{H}_x, g_x^\dagger] = 2g_x^\dagger,$$

and those of the one-dimensional Dunkl (or parabosonic) oscillator

$$h_y = \frac{1}{\sqrt{2}}(y + \mathcal{D}_y^{\mu_y}), \quad h_y^\dagger = \frac{1}{\sqrt{2}}(y - \mathcal{D}_y^{\mu_y}),$$

which obey the $sl_{-1}(2)$ algebra relations

$$[h_y, h_y^\dagger] = 1 + 2\mu_y R_y, \quad [\mathcal{H}_y, h_y] = -h_y, \quad [\mathcal{H}_y, h_y^\dagger] = h_y^\dagger.$$

The symmetries of the Hamiltonian (12.45) are of the form

$$T_0 = \mathcal{H}_x - \mathcal{H}_y, \quad T_+ = g_x^\dagger h_y, \quad T_- = g_x h_y^\dagger,$$

and they generate the following algebra:

$$[T_0, T_\pm] = \pm 3T_\pm, \quad \{T_\pm, R_y\} = 0, \quad [T_0, R_y] = 0,$$

$$[T_+, T_-] = \frac{3}{2}T_0 + \mu_y T_0 R_y + \mathcal{H}(\mu_y R_y - 1/2) + 2\mu_y R_y.$$

Similarly to the preceding special case, the separation of variable in parabolic coordinates can be performed and the wavefunctions in these coordinates satisfy the biconfluent Heun equation. Since the computations are analogous to those already presented here, we omit the details.

12.4 Conclusion

In this paper, we have considered two extensions of the Dunkl oscillator model: one with additional singular terms in the potential and the other with singular terms and a $2:1$ frequency ratio. We showed that the singular Dunkl oscillator is second-order superintegrable and exhibited its symmetry generators. We also identified the symmetry algebra as an extension with central involution operators of a special case of the Askey-Wilson algebra $AW(3)$. We also obtained the exact solutions in both Cartesian and polar coordinates. For the $2:1$ anisotropic Dunkl oscillator, we showed the system to be also second-order superintegrable, exhibited the symmetries and obtained the algebra they generate. Special cases for which separation of variables in parabolic coordinates occurs were also considered and their symmetry algebras found. In one instance, it was shown that the wavefunctions in parabolic coordinates obey the biconfluent Heun equation.

The models investigated here can be considered as generalizations of the standard singular oscillator (12.1) and $2:1$ anisotropic oscillator (12.2) with the derivatives replaced by the Dunkl derivatives. In this context, it is natural to consider the Hamiltonian corresponding to the Dunkl-Coulomb problem

$$H = -\frac{1}{2}\nabla_{\mathscr{D}}^2 + \frac{\alpha}{r},$$

where $\nabla_{\mathscr{D}}^2$ is the Dunkl-Laplacian, which in Cartesian coordinates reads

$$\nabla_{\mathscr{D}}^2 = (\mathscr{D}_x^{\mu_x})^2 + (\mathscr{D}_y^{\mu_y})^2.$$

The spectrum of this Hamiltonian can be evaluated algebraically using the observation that r, $r\nabla_{\mathscr{D}}^2$ and $x\partial_x + y\partial_y$ obey the $\mathfrak{sl}_2$ relations [4]. The expression for the values of the energy then depend on a single quantum number and the spectrum of H exhibits accidental degeneracies; this suggests that the Dunkl-Coulomb Hamiltonian in the plane is also superintegrable. In the standard case, it is known that the singular oscillator model (12.1) can be related to the Coulomb problem via the Levi-Civita mapping. This is not so with the singular Dunkl oscillator and Dunkl-Coulomb problem since the Levi-Civita mapping amounts to a passage from Cartesian to parabolic coordinates, a coordinate system in which the Dunkl Laplacian does not separate. Hence the Dunkl-Coulomb problem is genuinely different from the singular Dunkl oscillator and shall be considered elsewhere.

It would be of interest in a future study to identify and characterize other novel superintegrable systems of Dunkl type. The Dunkl-type models defined on the circle, in the

3-dimensional Euclidean space and on the 2-sphere are of particular interest. Given the relation of Dunkl oscillator models and -1 polynomials, this study could provide further insight into the emerging Bannai-Ito scheme of -1 orthogonal polynomials.

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Chapitre 13

The Dunkl oscillator in three dimensions

V. X. Genest, L. Vinet et A. Zhedanov. The Dunkl oscillator in three dimensions (2014). *Journal of Physics: Conference Series* **512** 012010

Abstract. The isotropic Dunkl oscillator model in three-dimensional Euclidean space is considered. The system is shown to be maximally superintegrable and its symmetries are obtained by the Schwinger construction using the raising/lowering operators of the dynamical $sl_{-1}(2)$ algebra of the one-dimensional Dunkl oscillator. The invariance algebra generated by the constants of motion, an extension of $\mathfrak{u}(3)$ with reflections, is called the Schwinger-Dunkl algebra $sd(3)$. The system is shown to admit separation of variables in Cartesian, polar (cylindrical) and spherical coordinates and the corresponding separated solutions are expressed in terms of generalized Hermite, Laguerre and Jacobi polynomials.

13.1 Introduction

This paper is concerned with the study of the isotropic Dunkl oscillator model in three-dimensional Euclidean space. This model, described by a Hamiltonian involving reflection operators, will be shown to be both maximally superintegrable and exactly solvable. The constants of motion will be obtained by the Schwinger construction using the $sl_{-1}(2)$ dynamical symmetry of the parabose oscillator in one dimension. The invariance algebra generated by the symmetries will be seen to be an extension of $\mathfrak{u}(3)$ by involutions and shall be called the Schwinger-Dunkl algebra $sd(3)$. The Schrödinger equation of the system will be seen to admit separation of variables in Cartesian, polar (cylindrical) and spherical coordinates. The corresponding separated solutions will be found and expressed in terms of generalized Hermite, Laguerre and Jacobi polynomials.

13.1.1 Superintegrability

Let us first recall the notion of superintegrability. A quantum system with d degrees of freedom defined by a Hamiltonian H is *maximally superintegrable* if it admits $2d-1$ algebraically independent symmetry operators S_i that commute with the Hamiltonian, i.e.

$$[H, S_i] = 0, \qquad 1 \leq i \leq 2d-1,$$

where one of the symmetries is the Hamiltonian itself, e.g. $S_1 = H$. A superintegrable system is said to be of order ℓ if ℓ is the maximal degree of the constants of motion S_i, excluding the Hamiltonian, in the momentum variables. The $\ell = 1$ case is associated to symmetries of geometric nature and to Lie algebras whereas the $\ell = 2$ case is typically associated to multiseparability of the Schrödinger equation and to quadratic invariance algebras.

A substantial amount of work has been done on superintegrable systems, motivated in part by their numerous applications, exact solvability and connections with the theory of special functions. In view of these properties, the search for new superintegrable models and their characterization is of significant interest in mathematical physics. For a recent review on superintegrable systems, one can consult [9].

13.1.2 Dunkl oscillator models in the plane

A series of novel superintegrable models in the plane with Hamiltonians involving reflection operators have been introduced recently [4, 5, 7]. The simplest of these systems, called the Dunkl oscillator in the plane, is defined by the following Hamiltonian [4]:

$$H = -\frac{1}{2}\left[\mathscr{D}_1^2 + \mathscr{D}_2^2\right] + \frac{1}{2}\left[x_1^2 + x_2^2\right], \qquad (13.1)$$

where $\mathscr{D}_i$ stands for the Dunkl derivative [3]

$$\mathscr{D}_i = \partial_i + \frac{\mu_i}{x_i}(1 - R_i), \qquad i = 1, 2, \qquad (13.2)$$

with $\partial_i = \frac{\partial}{\partial x_i}$ and where R_i is the reflection operator with respect to the $x_i = 0$ axis, i.e

$$R_1 f(x_1, x_2) = f(-x_1, x_2), \qquad R_2 f(x_1, x_2) = f(x_1, -x_2).$$

The Hamiltonian (13.1) can obviously be written as

$$H = H_1 + H_2$$

where H_i, $i = 1,2$, is the one-dimensional Dunkl oscillator Hamiltonian

$$H_i = -\frac{1}{2}\mathscr{D}_i^2 + \frac{1}{2}x_i^2. \tag{13.3}$$

In [4], the Hamiltonian (13.1) was shown to be maximally superintegrable and its two independent constants of motion, obtained by the Schwinger construction, were seen to generate a $\mathfrak{u}(2)$ algebra extended with involutions. This algebra was called the Schwinger-Dunkl algebra $sd(2)$. It was further shown that the Schrödinger equation associated to (13.1) admits separation of variables in both Cartesian and polar coordinates and the separated wavefunctions were given explicitly. Furthermore, the overlap coefficients between the separated wavefunctions were expressed in terms of the dual -1 Hahn polynomials through a correspondence with the Clebsch-Gordan problem of $sl_{-1}(2)$, a $q \to -1$ limit of the quantum algebra $sl_q(2)$ (see Section II for the definition of $sl_{-1}(2)$ and [11] for background). In [5], the representation theory of the Schwinger-Dunkl algebra $sd(2)$ was examined. The results obtained in this paper further strengthened the idea that Dunkl oscillator models are showcases for -1 orthogonal polynomials (see [6, 12]).

Generalizations of the Dunkl oscillator in the plane (13.1) were considered in [7], where the singular and the $2:1$ anisotropic Dunkl oscillators were investigated. The two systems were shown to be superintegrable, their constants of motion were constructed and their (quadratic) invariance algebra was given. It was also shown that in some cases these models exhibit multiseparability. Here we pursue our investigations on Dunkl oscillator models by considering the three-dimensional version of the system (13.1) in Euclidean space.

13.1.3 The three-dimensional Dunkl oscillator

The Dunkl oscillator in three-dimensional Euclidean space is defined by the Hamiltonian

$$\mathscr{H} = -\frac{1}{2}\left[\mathscr{D}_1^2 + \mathscr{D}_2^2 + \mathscr{D}_3^2\right] + \frac{1}{2}\left[x_1^2 + x_2^2 + x_3^2\right], \tag{13.4}$$

where $\mathscr{D}_i$ is the Dunkl derivative given by (13.2). An elementary calculation shows that the square of the Dunkl derivative has the expression

$$\mathscr{D}_i^2 = \partial_i^2 + \frac{2\mu_i}{x_i}\partial_i - \frac{\mu_i}{x_i^2}(1 - R_i).$$

As is directly seen, the Hamiltonian (13.4) corresponds to the combination of three independent (commuting) one-dimensional Dunkl oscillators with Hamiltonians H_i given by (13.3). It directly follows from the formulas (13.2) and (13.4) that when $\mu_i = 0$, $i = 1,2,3$, the Dunkl oscillator Hamiltonian (13.4) reduces to that of the standard isotropic harmonic oscillator in three dimensions. We note that the term "isotropic" here refers to the fact that the three independent one-dimensional oscillator entering the total Hamiltonian (13.4) have the same frequency. To obtain a 3D Dunkl oscillator isotropic in the sense of being $O(3)$-invariant, one must also take $\mu_1 = \mu_2 = \mu_3$.

13.1.4 Outline

The paper is organized in the following way. In section II, the $sl_{-1}(2)$ dynamical symmetry and the Schwinger construction are used to obtain the symmetries of the total Hamiltonian (13.4), thus establishing its superintegrability. The commutation relations satisfied by the constants of motion, which define the Schwinger-Dunkl algebra $sd(3)$, are also exhibited in two different bases. In section III, the separated solutions of the Schrödinger equation associated to (13.4) are given in Cartesian, polar and spherical coordinates. For each coordinate system, the symmetries responsible for the separation of variables are given. A short conclusion follows.

13.2 Superintegrability

In this section, the $sl_{-1}(2)$ dynamical algebra of the three-dimensional Dunkl oscillator is made explicit and is exploited to obtain the spectrum of the Hamiltonian. Following the Schwinger construction, the constants of motion are constructed using the raising/lowering operators of the $sl_{-1}(2)$ algebra of the one-dimensional constituents. The Schwinger-Dunkl algebra $sd(3)$ formed by the symmetries is seen to correspond to a deformation of the Lie algebra $\mathfrak{u}(3)$ by involutions and the defining relations of $sd(3)$ are given in two different bases.

13.2.1 Dynamical algebra and spectrum

The three-dimensional Dunkl oscillator Hamiltonian (13.4) possesses an $sl_{-1}(2)$ dynamical symmetry inherited from the one of its one-dimensional constituents H_i. This can be seen as follows. Consider the following operators:

$$A_{\pm}^{(i)} = \frac{1}{\sqrt{2}}\left(x_i \mp \mathscr{D}_i\right), \tag{13.5}$$

and define $A_0^{(i)} = H_i$ with H_i given by (13.3). It is easily checked that these operators, together with the reflection operator R_i, satisfy the defining relations of the $sl_{-1}(2)$ algebra which are of the form [11]

$$[A_0^{(i)}, A_{\pm}^{(i)}] = \pm A_{\pm}^{(i)}, \quad [A_0^{(i)}, R_i] = 0, \quad \{A_+^{(i)}, A_-^{(i)}\} = 2A_0^{(i)}, \quad \{A_{\pm}^{(i)}, R_i\} = 0, \tag{13.6}$$

where $[x, y] = xy - yx$ and $\{x, y\} = xy + yx$. As is seen from the above commutation relations, the operators $A_{\pm}^{(i)}$ act as raising/lowering operators for the one-dimensional Hamiltonians $A_0^{(i)} = H_i$. In the realization (13.5), the $sl_{-1}(2)$ Casimir operator

$$Q^{(i)} = A_+^{(i)} A_-^{(i)} R_i - A_0^{(i)} R_i + R_i/2,$$

is a multiple of the identity $Q^{(i)} = -\mu_i$. It thus follows that for $\mu_i > -1/2$, the operators $A_{\pm}^{(i)}$, $A_0^{(i)}$ and R_i realize the positive-discrete series of representations of $sl_{-1}(2)$ [11]. In these representations, the spectrum $\mathscr{E}^{(i)}$ of the operators $A_0^{(i)}$ is given by

$$\mathscr{E}_{n_i}^{(i)} = n_i + \mu_i + 1/2,$$

where n_i is a non-negative integer. Given that the full Hamiltonian (13.4) of the three-dimensional Dunkl oscillator is of the form $\mathscr{H} = H_1 + H_2 + H_3$, it follows that its energy eigenvalues $\mathscr{E}$ have the expression

$$\mathscr{E}_N = N + \mu_1 + \mu_2 + \mu_3 + 3/2, \tag{13.7}$$

where N is a non-negative integer. Since the integer N can be written as $N = n_1 + n_2 + n_3$, the spectrum of $\mathscr{H}$ has degeneracy

$$g_N = \sum_{n_3=0}^{N} (N - n_3 + 1) = (N+1)(N+2)/2,$$

at energy level $\mathscr{E}_N$. Hence the three-dimensional Dunkl oscillator exhibits the same degeneracy in its spectrum as the standard three-dimensional isotropic oscillator.

The coproduct of $sl_{-1}(2)$ (see [11]) can be used to construct the $sl_{-1}(2)$ dynamical algebra of the total Hamiltonian $\mathscr{H}$. Indeed, upon defining the operators

$$\mathscr{A}_{\pm} = A_{\pm}^{(1)} R_2 R_3 + A_{\pm}^{(2)} R_3 + A_{\pm}^{(3)}, \quad \mathscr{R} = R_1 R_2 R_3, \qquad \mathscr{A}_0 = \mathscr{H},$$

it is directly checked that one has

$$[\mathscr{A}_0, \mathscr{A}_{\pm}] = \pm \mathscr{A}_{\pm}, \quad [\mathscr{A}_0, \mathscr{R}] = 0, \quad \{\mathscr{A}_+, \mathscr{A}_-\} = 2\mathscr{A}_0, \quad \{\mathscr{A}_{\pm}, \mathscr{R}\} = 0.$$

Hence the operators $\mathscr{A}_{\pm}$ act as the raising/lowering operators for the full Hamiltonian $\mathscr{H}$ of the three-dimensional Dunkl oscillator.

13.2.2 Constants of motion and the Schwinger-Dunkl algebra $sd(3)$

Given that $\mathscr{H} = H_1 + H_2 + H_3$, it is clear that combining a raising operator for H_i with a lowering operator for H_j when $i \neq j$ will result in an operator preserving the eigenspace associated to a given energy $\mathscr{E}_N$, thus producing a constant of motion of the total Hamiltonian $\mathscr{H}$. Moreover, it is obvious that the one-dimensional components H_i commute with the total Hamiltonian $\mathscr{H}$ and hence these one-dimensional Hamiltonians can be considered as constants of the motion. Furthermore, it is observed that the total Hamiltonian commutes with the reflection operators

$$[\mathscr{H}, R_i] = 0, \qquad i = 1,2,3,$$

so that the reflections can be considered as symmetries. Following the Schwinger construction of $\mathfrak{u}(3)$ with standard creation/annihilation operators, we define

$$J_1 = \frac{1}{i}\big(A_+^{(2)}A_-^{(3)} - A_-^{(2)}A_+^{(3)}\big), \quad J_2 = \frac{1}{i}\big(A_+^{(3)}A_-^{(1)} - A_-^{(3)}A_+^{(1)}\big),$$
$$J_3 = \frac{1}{i}\big(A_+^{(1)}A_-^{(2)} - A_-^{(1)}A_+^{(2)}\big).$$

The operators J_i, $i = 1,2,3$, can be interpreted as Dunkl "rotation" generators since in terms of Dunkl derivatives, they read

$$J_1 = \frac{1}{i}(x_2\mathscr{D}_3 - x_3\mathscr{D}_2), \quad J_2 = \frac{1}{i}(x_3\mathscr{D}_1 - x_1\mathscr{D}_3), \quad J_3 = \frac{1}{i}(x_1\mathscr{D}_2 - x_2\mathscr{D}_1). \tag{13.8}$$

We also introduce the operators

$$K_1 = \big(A_+^{(2)}A_-^{(3)} + A_-^{(2)}A_+^{(3)}\big), K_2 = \big(A_+^{(3)}A_-^{(1)} + A_-^{(3)}A_+^{(1)}\big), K_3 = \big(A_+^{(1)}A_-^{(2)} + A_-^{(1)}A_+^{(2)}\big),$$

which in coordinates have the expression

$$K_1 = (x_2x_3 - \mathscr{D}_2\mathscr{D}_3), \quad K_2 = (x_3x_1 - \mathscr{D}_3\mathscr{D}_1), \quad K_3 = (x_1x_2 - \mathscr{D}_1\mathscr{D}_2). \tag{13.9}$$

It is directly checked that the operators J_i, K_i, $i = 1,2,3$, are symmetries of the total Dunkl oscillator Hamiltonian (13.4), that is

$$[J_i, \mathscr{H}] = [K_i, \mathscr{H}] = 0,$$

for $i = 1,2,3$. Upon writing

$$L_1 = H_1/2, \quad L_2 = H_2/2, \quad L_3 = H_3/2,$$

which satisfy $\mathscr{H} = 2L_1 + 2L_2 + 2L_3$, a direct computation shows that the non-zero commutation relations between the symmetries are given by

$$[J_j, J_k] = i\epsilon_{jk\ell}J_\ell(1 + 2\mu_\ell R_\ell), \qquad [K_j, K_k] = -i\epsilon_{jk\ell}J_\ell(1 + 2\mu_\ell R_\ell), \tag{13.10a}$$

where ϵ_{ijk} stands for the totally antisymmetric tensor with summation over repeated indices understood and

$$[J_j, K_k] = -i\epsilon_{jk\ell}K_\ell(1 + 2\mu_\ell R_\ell), \quad [J_j, K_j] = -i\epsilon_{jk\ell}L_k(1 + 2\mu_\ell R_\ell), \tag{13.10b}$$

for $j \neq k$. One also has

$$[J_j, L_k] = ig_{jk}K_j/2, \quad [K_j, L_k] = -ig_{jk}J_j/2, \tag{13.10c}$$

where g_{ij} are the elements of a 3×3 antisymmetric matrix with $g_{12}=g_{23}=1$, $g_{13}=-1$. The commutation relations involving the reflection operators are of the form

$$[L_i,R_i]=[L_j,R_k]=[J_i,R_i]=[K_i,R_i]=0, \tag{13.10d}$$

$$\{J_j,R_k\}=\{K_j,R_k\}=0, \tag{13.10e}$$

where $j\neq k$. Hence it follows that the three-dimensional Dunkl oscillator model is maximally superintegrable. The invariance algebra is defined by the commutation and anticommutation relations (13.10) and we shall refer to this algebra as the Schwinger-Dunkl algebra $sd(3)$. It is clear from the defining relations (13.10) that the algebra $sd(3)$ corresponds to a deformation of the $\mathfrak{u}(3)$ Lie algebra by involutions; the central element here is of course the total Hamiltonian $\mathcal{H}=H_1+H_2+H_3$. If one takes $\mu_1=\mu_2=\mu_3=0$ in the commutation relations (13.10), one recovers the $\mathfrak{u}(3)$ symmetry algebra of the standard isotropic harmonic oscillator in three dimensions realized by the standard creation/annihilation operators.

13.2.3 An alternative presentation of $sd(3)$

The Schwinger-Dunkl algebra $sd(3)$ obtained here can be seen as a "rank two" version of the Schwinger-Dunkl algebra $sd(2)$ which has appeared in [4] as the symmetry algebra of the Dunkl oscillator in the plane. It is possible to present another basis for the symmetries of the three-dimensional Dunkl oscillator in which the $sd(2)$ algebra explicitly appears as a subalgebra of $sd(3)$. In order to define this basis, it is convenient to introduce the standard 3×3 Gell-Mann matrices [1] denoted by Λ_i, $i=1,\ldots,8$, and obeying the $\mathfrak{su}(3)$ commutation relations

$$[\Lambda_i,\Lambda_j]=if^{ijk}\Lambda_k,$$

with $f^{123}=2$, $f^{458}=f^{678}=\sqrt{3}$ and

$$f^{147}=f^{165}=f^{246}=f^{257}=f^{345}=f^{376}=1.$$

The symmetries of the three-dimensional Dunkl oscillator can be expressed in terms of these matrices as follows. One takes

$$M_j=(A_+^{(1)},A_+^{(2)},A_+^{(3)})\,\Lambda_j\,(A_-^{(1)},A_-^{(2)},A_-^{(3)})^t, \tag{13.11}$$

for $j=1,2,4,5,6,7$ and also

$$M_3=\frac{1}{4}\Big(\{A_+^{(1)},A_-^{(1)}\}-\{A_+^{(2)}A_-^{(2)}\}\Big),$$
$$M_8=\frac{1}{4\sqrt{3}}\Big(\{A_+^{(1)},A_-^{(1)}\}+\{A_+^{(2)},A_-^{(2)}\}-2\{A_+^{(3)},A_-^{(3)}\}\Big).$$

It is straightforward to verify that the operators M_i, $i = 1, \ldots, 8$ commute with the Hamiltonian $\mathscr{H}$ of the three-dimensional Dunkl oscillator. Using the commutation relations (13.6) as well as the extra relation

$$[A_{-}^{(i)}, A_{+}^{(j)}] = \delta_{ij}(1 + 2\mu_i R_i),$$

the $sd(3)$ commutation relations expressed in the basis of the symmetries M_i can easily be obtained. Consider the constants of motion

$$M_1 = \frac{1}{2}\left(A_{+}^{(1)}A_{-}^{(2)} + A_{-}^{(1)}A_{+}^{(2)}\right), \quad M_2 = \frac{1}{2i}\left(A_{+}^{(1)}A_{-}^{(2)} - A_{-}^{(1)}A_{+}^{(2)}\right),$$

as well as M_3. These symmetry operators satisfy the commutation relations of the Schwinger-Dunkl algebra $sd(2)$

$$[M_2, M_3] = iM_1, \quad [M_3, M_1] = iM_2, \tag{13.12a}$$

$$[M_1, M_2] = i\Big(M_3 + M_3(\mu_1 R_1 + \mu_2 R_2) - \frac{1}{3}\left(\mathscr{H} + \sqrt{3}M_8\right)(\mu_1 R_1 - \mu_2 R_2)\Big), \tag{13.12b}$$

$$\{M_1, R_i\} = 0, \quad \{M_2, R_i\} = 0, \quad [M_3, R_i] = 0, \tag{13.12c}$$

for $i = 1, 2$. In the subalgebra (13.12), the operator $\left(\mathscr{H} + \sqrt{3}M_8\right)$ is central and corresponds to the Hamiltonian of the Dunkl oscillator in the plane

$$\frac{2}{3}\left(\mathscr{H} + \sqrt{3}M_8\right) = H_1 + H_2.$$

13.3 Separated Solutions: Cartesian, cylindrical and spherical coordinates

In this section, the exact solutions of the time-independent Schrödinger equation

$$\mathscr{H}\Psi = \mathscr{E}\Psi, \tag{13.13}$$

associated to the three-dimensional Dunkl oscillator Hamiltonian (13.4) are obtained in Cartesian, polar (cylindrical) and spherical coordinates. The operators responsible for the separation of variables in each of these coordinate systems are given explicitly.

13.3.1 Cartesian coordinates

Since $\mathscr{H} = H_1 + H_2 + H_3$, where H_i, $i = 1, 2, 3$, are the one-dimensional Dunkl oscillator Hamiltonians

$$H_i = -\frac{1}{2}\mathscr{D}_i^2 + \frac{1}{2}x_i^2,$$

it is obvious that the Schrödinger equation (13.13) admits separation of variable in Cartesian coordinates $\{x_1, x_2, x_3\}$. In this coordinate system, the separated solutions are of the form

$$\Psi(x_1, x_2, x_3) = \psi(x_1)\psi(x_2)\psi(x_3),$$

where $\psi(x_i)$ are solutions of the one-dimensional Schrödinger equation

$$\left[-\frac{1}{2}\mathscr{D}_i^2 + \frac{1}{2}x_i^2\right]\psi(x_i) = \mathscr{E}^{(i)}\psi(x_i). \tag{13.14}$$

The regular solutions of (13.14) are well known [4, 10]. To obtain these solutions, one uses the fact that the reflection operator R_i commutes with the one-dimensional Hamiltonian H_i, which allows to diagonalize both operators simultaneously. For the one-dimensional problem, the two sectors corresponding to the possible eigenvalues $s_i = \pm 1$ of the reflection operator R_i can be recombined to give the following expression for the wavefunctions:

$$\psi_{n_i}(x_i) = e^{-x_i^2/2} H_{n_i}^{\mu_i}(x_i), \tag{13.15}$$

where n_i is a non-negative integer. The corresponding energy eigenvalues are

$$\mathscr{E}_{n_i}^{(i)} = n_i + \mu_i + 1/2,$$

and $H_n^{\mu}(x)$ stands for the generalized Hermite polynomials [2]

$$H_{2m+p}^{\mu}(x) = (-1)^n \sqrt{\frac{n!}{\Gamma(m+p+\mu+1/2)}}\, x^p L_m^{(\mu-1/2+p)}(x^2), \tag{13.16}$$

with $p \in \{0,1\}$. In (13.16), $L_n^{(\alpha)}(x)$ are the standard Laguerre polynomials [8] and $\Gamma(x)$ is the classical Gamma function [1]. The wavefunctions (13.15) satisfy

$$R_i \psi_{n_i}(x_i) = (-1)^{n_i} \psi_{n_i}(x_i),$$

so that the eigenvalue s_i of the reflection operator R_i is given by the parity of n_i. Using the orthogonality relation of the Laguerre polynomials, one finds that the wavefunctions (13.15) satisfy the orthogonality condition

$$\int_{-\infty}^{\infty} \psi_{n_i}(x_i)\psi_{n_i'}(x_i)|x_i|^{2\mu_i}\,\mathrm{d}x_i = \delta_{n_i,n_i'}.$$

In Cartesian coordinates, the separated solution of the Schrödinger equation associated to the three-dimensional Dunkl oscillator Hamiltonian (13.4) are thus given by

$$\Psi_{n_1,n_2,n_3}(x_1,x_2,x_3) = e^{-(x_1^2+x_2^2+x_3^2)/2} H_{n_1}^{\mu_1}(x_1) H_{n_2}^{\mu_2}(x_2) H_{n_3}^{\mu_3}(x_3), \tag{13.17}$$

and the corresponding energy $\mathscr{E}$ is

$$\mathscr{E} = n_1 + n_2 + n_3 + \mu_1 + \mu_2 + \mu_3 + 3/2, \tag{13.18}$$

where n_i, $i = 1,2,3$, are non-negative integers. It is directly seen from (13.16), (13.17) and (13.18) that if one takes $\mu_i = 0$, the solutions and the spectrum of the isotropic three-dimensional oscillator in Cartesian coordinates are recovered.

13.3.2 Cylindrical coordinates

In cylindrical coordinates

$$x_1 = \rho\cos\varphi, \quad x_2 = \rho\sin\varphi, \quad x_3 = z.$$

the Hamiltonian (13.4) of the three-dimensional Dunkl oscillator reads

$$\mathscr{H} = \mathscr{A}_\rho + \frac{1}{\rho^2}\mathscr{B}_\varphi + \mathscr{C}_z,$$

where

$$\mathscr{A}_\rho = -\frac{1}{2}\left[\partial_\rho^2 + \frac{1}{\rho}\partial_\rho\right] - \frac{(\mu_1+\mu_2)}{\rho}\partial_\rho + \frac{1}{2}\rho^2, \tag{13.19a}$$

$$\mathscr{B}_\varphi = -\frac{1}{2}\partial_\varphi^2 + (\mu_1\tan\varphi - \mu_2\cot\varphi)\partial_\varphi + \frac{\mu_1}{2\cos^2\varphi}(1-R_1) + \frac{\mu_2}{2\sin^2\varphi}(1-R_2), \tag{13.19b}$$

$$\mathscr{C}_z = -\frac{1}{2}\partial_z^2 - \frac{\mu_3}{z}\partial_z + \frac{1}{2}z^2 + \frac{\mu_3}{2z^2}(1-R_3). \tag{13.19c}$$

The reflection operators are easily seen to have the action

$$R_1 f(\rho,\varphi,z) = f(\rho,\pi-\phi,z), \quad R_2 f(\rho,\varphi,z) = f(\rho,-\varphi,z), \quad R_3 f(\rho,\varphi,z) = f(\rho,\varphi,-z).$$

Upon taking $\Psi(\rho,\varphi,z) = P(\rho)\Phi(\varphi)\psi(z)$, one finds that (13.13) is equivalent to the system of ordinary equations

$$\mathscr{A}_\rho P(\rho) - \widetilde{\mathscr{E}}P(\rho) + \frac{k^2}{2\rho^2}P(\rho) = 0, \tag{13.20a}$$

$$\mathscr{B}_\varphi \Phi(\varphi) - \frac{k^2}{2}\Phi(\varphi) = 0, \tag{13.20b}$$

$$\mathscr{C}_z \psi(z) = \mathscr{E}^{(3)}\psi(z), \tag{13.20c}$$

where $\mathscr{E}^{(3)}$, $k^2/2$ are the separation constants and where $\widetilde{\mathscr{E}} = \mathscr{E} - \mathscr{E}^{(3)}$. The solutions to the equation are given by (13.15) and (13.16) with $\mathscr{E}^{(3)} = n_3 + \mu_3 + 1/2$. The solutions to (13.20a) and (13.20b) have been obtained in [4]. For the angular part, the solutions are labeled by the eigenvalues s_1, s_2 with $s_i = \pm 1$ of the reflection operators R_1, R_2 and read

$$\Phi_m^{(s_1,s_2)}(\varphi) = \eta_m \cos^{e_1}\varphi \sin^{e_2}\varphi\, P_{m-e_1/2-e_2/2}^{(\mu_2+e_2-1/2,\mu_1+e_1-1/2)}(\cos 2\varphi), \tag{13.21}$$

where $P_n^{(\alpha,\beta)}(x)$ are the classical Jacobi polynomials [8], η_m is a normalization factor and where e_1, e_2 are the indicator functions for the eigenvalues of the reflections R_1 and R_2, i.e.:

$$e_i = \begin{cases} 0, & \text{if } s_i = 1, \\ 1, & \text{if } s_i = -1, \end{cases}$$

for $i = 1,2$. When $s_1 s_2 = -1$, m is a positive half-integer whereas when $s_1 s_2 = 1$, m is a non-negative integer; note also that for $m = 0$, only the $s_1 = s_2 = 1$ state exists. In all parity cases, the separation constant takes the value

$$k^2 = 4m(m + \mu_1 + \mu_2).$$

If one takes

$$\eta_m = \left[\frac{(2m + \mu_1 + \mu_2)\Gamma(m + \mu_1 + \mu_2 + \frac{e_1}{2} + \frac{e_2}{2})(m - \frac{e_1}{2} - \frac{e_2}{2})!}{2\,\Gamma(m + \mu_1 + \frac{e_1}{2} - \frac{e_2}{2} + 1/2)\Gamma(m + \mu_2 + \frac{e_2}{2} - \frac{e_1}{2} + 1/2)} \right]^{1/2},$$

as the normalization factor, the angular part of the separated wavefunction satisfy the orthogonality relation

$$\int_0^{2\pi} \Phi_m^{(s_1,s_2)} \Phi_{m'}^{(s_1',s_2')} |\cos\phi|^{2\mu_1} |\sin\phi|^{2\mu_2}\, \mathrm{d}\phi = \delta_{m,m'} \delta_{s_1,s_1'} \delta_{s_2,s_2'},$$

which can be deduced from the orthogonality relation satisfied by the Jacobi polynomials [8]. The radial solutions have the expression

$$P_{n_\rho}(\rho) = \left[\frac{2\, n_\rho!}{\Gamma(n_\rho + 2m + \mu_1 + \mu_2 + 1)} \right]^{1/2} e^{-\rho^2/2} \rho^{2m} L_{n_\rho}^{(2m+\mu_1+\mu_2)}(\rho^2), \tag{13.22}$$

where n_ρ is a non-negative integer and where $L_n^{(\alpha)}(x)$ are the Laguerre polynomials. They satisfy the orthogonality relation

$$\int_0^{\infty} P_{n_\rho}(\rho) P_{n_\rho'}(\rho)\, \rho^{2\mu_1 + 2\mu_2 + 1}\, \mathrm{d}\rho = \delta_{n_\rho, n_\rho'}.$$

The separated wavefunctions of the three-dimensional Dunkl oscillator in cylindrical coordinates are thus given by

$$\Psi_{n_\rho,m,n_z}(\rho,\varphi,z) = P_{n_\rho}(\rho) \Phi_m^{(s_1,s_2)}(\varphi) \psi_{n_z}(z)$$

where $P_{n_\rho}(\rho)$, $\Phi_m^{(s_1,s_2)}(\varphi)$ and $\psi_{n_z}(z)$ are given by (13.22), (13.21) and (13.15), respectively. The energy $\mathscr{E}$ is expressed as

$$\mathscr{E} = 2n_\rho + 2m + n_z + \mu_1 + \mu_2 + \mu_3 + 3/2,$$

where n_ρ, n_z are non-negative integers and where m is a non-negative integer when $s_1 s_2 = 1$ or a positive half-integer when $s_1 s_2 = -1$. In the cylindrical basis, the operators $\mathscr{C}_z$ and $\mathscr{B}_\phi$ are diagonal with eigenvalues $\mathscr{E}^{(3)}$ and $k^2/2$. A direct computation shows that one has

$$J_3^2 = 2\mathscr{B}_\varphi + 2\mu_1\mu_2(1 - R_1 R_2),$$

where J_3 is the symmetry given in (13.8). It thus follows that J_3 and H_3 are the symmetries responsible for the separation of variables in cylindrical coordinates.

13.3.3 Spherical coordinates

In spherical coordinates

$$x_1 = r\cos\phi\sin\theta, \quad x_2 = r\sin\phi\sin\theta, \quad x_3 = r\cos\theta,$$

the Hamiltonian (13.4) of the three-dimensional Dunkl oscillator takes the form

$$\mathscr{H} = \mathscr{M}_r + \frac{1}{r^2}\mathscr{N}_\theta + \frac{1}{r^2\sin^2\theta}\mathscr{B}_\phi,$$

where $\mathscr{B}_\phi$ is given by (13.19b) and where

$$\mathscr{M}_r = -\frac{1}{2}\partial_r^2 - \frac{(1+\mu_1+\mu_2+\mu_3)}{r}\partial_r + \frac{1}{2}r^2, \tag{13.23a}$$

$$\mathscr{N}_\theta = -\frac{1}{2}\partial_\theta^2 + (\mu_3\tan\theta - (1/2+\mu_1+\mu_2)\cot\theta)\partial_\theta + \frac{\mu_3}{2\cos^2\theta}(1-R_3). \tag{13.23b}$$

The reflection operators have the action

$$R_1 f(r,\theta,\phi) = f(r,\theta,\pi-\phi), \quad R_2 f(r,\theta,\phi) = f(r,\theta,-\phi), \quad R_3 f(r,\theta,\phi) = f(r,\pi-\theta,\phi).$$

Upon taking $\Psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$ in the Schrödinger equation (13.13), one finds the following system of ordinary differential equations

$$\left[\mathscr{M}_r + \left(\frac{q^2}{2r^2} - \mathscr{E}\right)\right]R(r) = 0, \tag{13.24a}$$

$$\left[\mathscr{N}_\theta + \left(\frac{k^2}{2\sin^2\theta} - \frac{q^2}{2}\right)\right]\Theta(\theta) = 0, \tag{13.24b}$$

$$\left[\mathscr{B}_\phi - \frac{k^2}{2}\right]\Phi(\phi) = 0, \tag{13.24c}$$

where $k^2/2$ and $q^2/2$ are the separation constants. It is directly seen that the azimuthal solution $\Phi(\phi)$ to (13.24c) is given by (13.21) with value of the separation constant $k^2 = 4m(m+\mu_1+\mu_2)$. The zenithal solutions $\Theta(\theta)$ are labeled by the eigenvalue $s_3 = \pm 1$ of the reflection operator R_3. One has

$$\Theta_\ell^{(s_3)}(\theta) = \iota_\ell \cos^{e_3}\theta\sin^{2m}\theta\, P_{\ell-e_3/2}^{(2m+\mu_1+\mu_2,\mu_3+e_3-1/2)}(\cos 2\theta), \tag{13.25}$$

where the value of the separation constant is $q^2 = 4(\ell+m)(\ell+m+\mu_1+\mu_2+\mu_3+1/2)$. When $s_3 = 1$, ℓ is a non-negative integer whereas ℓ is a positive half-integer when $s_3 = -1$. The normalization constant has the expression

$$\iota_\ell = \left[\frac{(2\ell+2m+\mu_1+\mu_2+\mu_3+1/2)\Gamma(\ell+2m+\mu_1+\mu_2+\mu_3+1/2+e_3/2)(\ell-e_3/2)!}{\Gamma(\ell+2m+\mu_1+\mu_2+1-e_3/2)\Gamma(\ell+\mu_3+1/2+e_3/2)}\right]^{1/2}.$$

The radial solutions are given by

$$R_{n_r}(r) = \left[\frac{2n_r!}{\Gamma(n_r+\alpha+1)}\right]^{1/2} e^{-r^2/2} r^{2(\ell+m)} L_{n_r}^{(\alpha)}(r^2), \tag{13.26}$$

with $\alpha = 2(\ell+m)+\mu_1+\mu_2+\mu_3+1/2$. The separated wavefunctions of the three-dimensional Dunkl oscillator in spherical coordinates thus read

$$\Psi_{n_r,\ell,m}^{(s_1,s_2,s_3)}(r,\theta,\phi) = R_{n_r}(r)\Phi_m^{(s_1,s_2)}(\phi)\Theta_\ell^{(s_3)}(\theta), \tag{13.27}$$

where $R(r)$, $\Theta(\theta)$ and $\Phi(\phi)$ are respectively given by (13.26), (13.25) and (13.21) and correspond to the total energy

$$\mathscr{E} = 2(n_r+\ell+m)+\mu_1+\mu_2+\mu_3+3/2. \tag{13.28}$$

These wavefunctions are eigenfunctions of the reflection operators R_i with eigenvalues $s_i = \pm 1$ for $i = 1,2,3$. When $s_3 = 1$, the quantum number ℓ takes non-negative integer values and when $s_3 = -1$, the number ℓ takes positive half-integer values. Similarly, when $s_1 s_2 = 1$, the quantum number m is a non-negative integer and when $s_1 s_2 = -1$, m is a positive half-integer. The wavefunctions satisfy the orthogonality relation

$$\begin{aligned}&\int_0^\infty\int_0^\pi\int_0^{2\pi} r^{2\mu_1+2\mu_2+2\mu_3}|\sin\theta|^{2\mu_1+2\mu_2}|\cos\theta|^{2\mu_3}|\cos\phi|^{2\mu_1}|\sin\phi|^{2\mu_2}\, r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi\\ &R_{n_r'}(r)R_{n_r}(r)\Theta_{\ell'}^{(s_3')}(\theta)\Theta_\ell^{(s_3)}(\theta)\Phi_{m'}^{(s_1',s_2')}(\phi)\Phi_m^{(s_1,s_2)}(\phi) = \delta_{n_r,n_r'}\delta_{\ell,\ell'}\delta_{s_3,s_3'}\delta_{m,m'}\delta_{s_1,s_1'}\delta_{s_2,s_2'}.\end{aligned}$$

In analogy with the standard three-dimensional oscillator, the symmetries responsible for the separation of variables in spherical coordinates are related to the Dunkl "rotation" generators. Indeed, one has that the operator

$$J_3^2 = \left\{\frac{1}{i}\left(x_1\mathscr{D}_2 - x_2\mathscr{D}_1\right)\right\}^2 = 2\mathscr{B}_\phi + 2\mu_1\mu_2(1-R_1R_2), \tag{13.29}$$

is diagonal on the separated wavefunction in spherical coordinates and has eigenvalues

$$J_3^2\,\Psi_{n_r,\ell,m}^{(s_1,s_2,s_3)}(r,\theta,\phi) = \left[4m(m+\mu_1+\mu_2)+2\mu_1\mu_2(1-s_1s_2)\right]\Psi_{n_r,\ell,m}^{(s_1,s_2,s_3)}(r,\theta,\phi).$$

Furthermore, a direct computations shows that the Dunkl total angular momentum operator

$$\mathbf{J}^2 = \left\{\frac{1}{i}\left(x_2\mathscr{D}_3 - x_3\mathscr{D}_2\right)\right\}^2 + \left\{\frac{1}{i}\left(x_3\mathscr{D}_1 - x_1\mathscr{D}_3\right)\right\}^2 + \left\{\frac{1}{i}\left(x_1\mathscr{D}_2 - x_2\mathscr{D}_1\right)\right\}^2$$

has the following expression in spherical coordinates:

$$\begin{aligned}\mathbf{J}^2 = 2\left(\mathscr{N}_\theta + \frac{1}{\sin^2\theta}\mathscr{B}_\phi\right) &+ 2\mu_1\mu_2(1-R_1R_2) + 2\mu_2\mu_3(1-R_2R_3) + 2\mu_1\mu_3(1-R_1R_3)\\ &+ \mu_1(1-R_1) + \mu_2(1-R_2) + \mu_3(1-R_3),\end{aligned}$$

where $\mathcal{N}_\theta$ and $\mathcal{B}_\phi$ are as in (13.23b) and (13.19b). It thus follows that the separated wavefunctions in spherical coordinates $\Psi_{n_r,\ell,m}^{(s_1,s_2,s_3)}(r,\theta,\phi)$ satisfy

$$\mathbf{J}^2\Psi_{n_r,\ell,m}^{(s_1,s_2,s_3)}(r,\theta,\phi)=\lambda_{\ell,m}\Psi_{n_r,\ell,m}^{(s_1,s_2,s_3)}(r,\theta,\phi),$$

with

$$\lambda_{\ell,m}=4(\ell+m)(\ell+m+\mu_1+\mu_2+\mu_3+1/2)+2\mu_1\mu_2(1-s_1s_2)+2\mu_1\mu_3(1-s_1s_3)$$
$$+2\mu_2\mu_3(1-s_2s_2)+\mu_1(1-s_1)+\mu_2(1-s_2)+\mu_3(1-s_3).$$

13.4 Conclusion

In this paper, we have examined the isotropic Dunkl oscillator model in three-dimensional Euclidean space and we have shown that this system is maximally superintegrable. The symmetries of the model were exhibited and the invariance algebra they generate, called the Schwinger-Dunkl algebra $sd(3)$, has been seen to be a deformation of the Lie algebra $\mathfrak{u}(3)$ by involutions. So far, we have examined Dunkl systems with oscillator type potentials. In view of the superintegrability and importance of the Coulomb problem, the examination of the Dunkl-Coulomb problem is of considerable interest. This will be the subject of a future publication.

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Chapitre 14

The Bannai–Ito algebra and a superintegrable system with reflections on the 2-sphere

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Abstract. A quantum superintegrable model with reflections on the 2-sphere is introduced. Its two algebraically independent constants of motion generate a central extension of the Bannai–Ito algebra. The Schrödinger equation separates in spherical coordinates and its exact solutions are presented. It is further observed that the Hamiltonian of the system arises in the addition of three representations of the $sl_{-1}(2)$ algebra (the dynamical algebra of the one-dimensional parabosonic oscillator). The contraction from the two-sphere to the Euclidean plane yields the Dunkl oscillator in two dimensions and its Schwinger-Dunkl symmetry algebra $sd(2)$.

14.1 Introduction

The class of superintegrable quantum systems is of particular interest as a laboratory for the study of symmetries, their algebraic description and their representations. A quantum system in n dimensions with Hamiltonian H is said to be maximally superintegrable if it possesses $2n-1$ algebraically independent constants of motion c_i for $i = 1,\ldots,2n-1$ commuting with H, $[H, c_i] = 0$, where one of these constants is H. A system is of order ℓ if the maximum order in momenta of the constants of motion (other than H) is ℓ. Empirically, superintegrable systems turn out to be exactly solvable.

The category of models that has been most analyzed is that of systems governed by scalar Hamiltonians of the form

$$H = \Delta + V, \tag{14.1a}$$

where Δ is the standard Laplacian

$$\Delta = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \sqrt{\det g}\, g^{ij} \frac{\partial}{\partial x_j}, \tag{14.1b}$$

on spaces with metric g_{ij} in coordinates $\{x_i\}$. In two dimensions, the second order superintegrable models of that type have been identified and classified [25]. As a matter of fact, they can all be obtained from the generic 3-parameter model on the two-sphere by limits (contractions) and specializations (see [20] for details). By observing that the Hamiltonian of the generic model on the two-sphere can be constructed through the addition of three $\mathfrak{su}(1,1)$ algebras, the constants of motion were identified in [9, 14] as the intermediate Casimir operators arising in the stepwise combination process. These were further shown to generate the Racah algebra which is the quadratic algebra with two independent generators that captures the bispectrality of the Racah polynomials sitting atop the discrete side of the Askey tableau of hypergeometric polynomials [22]. This identification of the symmetry algebra hence allows to associate the Racah polynomials to the generic 3-parameter superintegrable system on the 2-sphere. A threefold connection between the polynomials of the Askey scheme, the second-order superintegrable models and their symmetry algebras can further be achieved by performing on the Racah polynomials and the Racah algebras the contractions and specializations that lead from the generic model on the two-sphere to the other second-order superintegrable systems [20].

The exploration of another category of superintegrable models has been undertaken recently: it bears on systems whose Hamiltonians involve reflections. Typically, the involutions arise in Dunkl operators [4] which are defined as follows in the special univariate case:

$$\mathscr{D}_x^{\mu} = \partial_x + \frac{\mu}{x}(1 - R_x), \tag{14.2}$$

where ∂_x stands for the derivative with respect to x and where R_x is the reflection operator such that $R_x f(x) = f(-x)$. One of the simplest dynamical systems with reflections is the parabose oscillator in one dimension with Hamiltonian (see [26])

$$H = \left(\mathscr{D}_x^{\mu}\right)^2 + x^2.$$

Other one-dimensional models have been discussed (see for example [18, 21, 28]). Recall that Dunkl operators are most useful in the study of multivariate orthogonal polynomials associated to reflection groups [5] and that of symmetric functions [24] as well as in the analysis of exactly solvable quantum many-body systems of Calogero-Sutherland type (see for instance [23]).

Interestingly, the theory of univariate orthogonal polynomials that are eigenfunctions of differential or difference operators of Dunkl type has also been the object of attention lately [10, 13, 31, 32, 33, 34, 35]. These are now referred to as -1 polynomials and a scheme similar to the Askey one has emerged for them. Taking a place analogous to that of the Racah polynomials are the Bannai–Ito polynomials, which are sitting at the top of one side of the -1 tableau and which were introduced in a combinatorial context [3]. The characteristic properties of these polynomials are encoded [7, 16, 31] in an algebra bearing the Bannai–Ito name which has 3 generators L_1, L_2 and L_3 verifying the following relations given in terms of anticommutators ($\{A,B\} = AB + BA$):

$$\{L_1,L_2\} = L_3 + \omega_3, \qquad \{L_2,L_3\} = L_1 + \omega_1, \qquad \{L_3,L_1\} = L_2 + \omega_2, \tag{14.3}$$

where ω_1, ω_2, ω_3 are central. Introduced in [31], the algebra (14.3) is the structure behind the bispectrality property of the Bannai–Ito polynomials. It corresponds to a $q \to -1$ limit of the Askey-Wilson algebra [36], which is the algebra behind the bispectrality property of the q-polynomials of the Askey scheme [22]; it has also been used in [30] to study structure relations for -1 polynomials of the Bannai–Ito family. The special case with $\omega_1 = \omega_2 = \omega_3 = 0$ has been studied in [2, 17] as an anticommutator version of the Lie algebra $\mathfrak{su}(2)$.

The examination of superintegrable systems with reflections has mostly focused so far on Dunkl oscillators in the plane [11, 12, 15] and in $\mathbb{R}^3$ [8]. These are formed out of combinations of one-dimensional parabose systems (with the inclusions of possible singular terms). They all are superintegrable and exactly solvable. In the "isotropic" case, the symmetry algebra denoted $sd(n)$ is a deformation of $\mathfrak{su}(n)$ with n the number of dimensions. The Dunkl oscillators have proved to be showcases for -1 polynomials. An infinite family of higher order ($\ell > 2$) superintegrable models with reflections has also been obtained with the help of the little -1 Jacobi polynomials [27].

The purpose of this paper is to introduce and analyze an elegant superintegrable model with reflections on the two-sphere. The symmetry algebra will be seen to be a central extension of the Bannai–Ito algebra, a first physical occurrence as such of this algebra, as far as we know. This model-algebra pairing will present itself as the analog in the presence of reflections of the teaming of the Racah algebra with the so-called generic 3-parameter system on the two-sphere. It entails a relation [7] between Dunkl harmonic analysis on the 2-sphere and the representation theory of $sl_{-1}(2)$, a $q \to -1$ limit of the quantum algebra $\mathcal{U}_q(\mathfrak{sl}_2)$ that can be identified with the dynamical algebra of the parabose oscillator [29].

The outline of the paper is as follows. In Section 2, the model is described, the constants of motion are exhibited and the invariance algebra they generate is identified as a central extension of the Bannai–Ito algebra. In section 3, the separated solutions of the model are given explicitly in two different spherical coordinate systems in terms of Jacobi polynomials and the symmetries responsible for the separation of variables are identified. In section 4, it will be shown how the model can be constructed from the addition of three $sl_{-1}(2)$ realizations and it will be seen that the constants of motion can be interpreted as Casimir operators arising in this Racah problem. The contraction from the two-sphere to the Euclidean plane will be examined in Section 5 and it will be shown how the Dunkl oscillator and its symmetry algebra are recovered in this limit. Some perspectives are offered in the conclusion.

14.2 The model on S^2, superintegrability and symmetry algebra

We shall begin by introducing the system on the 2-sphere that will be studied. Its symmetries will be given explicitly and the algebra they generate, a central extension of the Bannai–Ito algebra, will be presented.

14.2.1 The model on S^2

Let $s_1^2 + s_2^2 + s_3^2 = 1$ be the usual embedding of the unit two-sphere in the three-dimensional Euclidean space with coordinates s_1, s_2, s_3. Consider the model governed by the Hamiltonian

$$\mathscr{H} = J_1^2 + J_2^2 + J_3^2 + \frac{\mu_1}{s_1^2}(\mu_1 - R_1) + \frac{\mu_2}{s_2^2}(\mu_2 - R_2) + \frac{\mu_3}{s_3^2}(\mu_3 - R_3), \tag{14.4}$$

where the μ_i are real parameters such that $\mu_i > -1/2$, a condition required for the normalizability of the wavefunctions (see section 3). The operators J_i appearing in (14.4) are the familiar angular momentum generators

$$J_1 = \frac{1}{i}\left(s_2\partial_{s_3} - s_3\partial_{s_2}\right),\; J_2 = \frac{1}{i}\left(s_3\partial_{s_1} - s_1\partial_{s_3}\right),\; J_3 = \frac{1}{i}\left(s_1\partial_{s_2} - s_2\partial_{s_1}\right),$$

that obey the $\mathfrak{so}(3)$ commutation relations

$$[J_1, J_2] = iJ_3, \quad [J_2, J_3] = iJ_1, \quad [J_3, J_1] = iJ_2.$$

The operators R_i in (14.4) are the reflection operators with respect to the $s_i = 0$ plane, i.e. $R_i f(s_i) = f(-s_i)$. Since these reflections are improper rotations, the Hamiltonian (14.4) has a well defined action on functions defined on the unit sphere. In terms of the standard Laplacian operator Δ_{S^2} on the two-sphere [1], the Hamiltonian (14.4) reads

$$\mathscr{H} = -\Delta_{S^2} + \frac{\mu_1}{s_1^2}(\mu_1 - R_1) + \frac{\mu_2}{s_2^2}(\mu_2 - R_2) + \frac{\mu_3}{s_3^2}(\mu_3 - R_3).$$

14.2.2 Superintegrability

It is possible to exhibit two algebraically independent conserved quantities for the model described by the Hamiltonian (14.4). Let L_1 and L_3 be defined as follows:

$$L_1 = \left(iJ_1 + \mu_2 \frac{s_3}{s_2} R_2 - \mu_3 \frac{s_2}{s_3} R_3\right) R_2 + \mu_2 R_3 + \mu_3 R_2 + \frac{1}{2} R_2 R_3, \tag{14.5a}$$

$$L_3 = \left(iJ_3 + \mu_1 \frac{s_2}{s_1} R_1 - \mu_2 \frac{s_1}{s_2} R_2\right) R_1 + \mu_1 R_2 + \mu_2 R_1 + \frac{1}{2} R_1 R_2. \tag{14.5b}$$

A direct computation shows that one has

$$[\mathscr{H}, L_1] = [\mathscr{H}, L_3] = 0,$$

and hence L_1, L_3 are constants of the motion. Moreover, it can be checked that

$$[\mathscr{H}, R_i] = 0, \qquad i = 1,2,3.$$

and thus the reflection operators are also (discrete) symmetries of the system (14.4).

It is clear from (14.5) that L_1 and L_3 are algebraically independent from one another and hence it follows that the model with Hamiltonian (14.4) on the two-sphere is maximally superintegrable. Since the constants of motion are of first order in the derivatives, the order of superintegrability is $\ell = 1$. While this case is generally associated to geometrical symmetries and Lie invariance algebras for systems of the type (14.1), this is not so in the presence of reflections. In fact, as will be seen next, the invariance algebra is not a Lie algebra.

14.2.3 Symmetry algebra

To examine the algebra generated by the symmetries L_1 and L_3, it is convenient to introduce the operator L_2 defined as

$$L_2 = \left(-iJ_2 + \mu_1 \frac{s_3}{s_1} R_1 - \mu_3 \frac{s_1}{s_3} R_3\right) R_1 R_2 + \mu_1 R_3 + \mu_3 R_1 + \frac{1}{2} R_1 R_3, \tag{14.6}$$

and the operator C given by

$$C = -L_1 R_2 R_3 - L_2 R_1 R_3 - L_3 R_1 R_2 + \mu_1 R_1 + \mu_2 R_2 + \mu_3 R_3 + \frac{1}{2}. \tag{14.7}$$

It is directly verified that both L_2 and C commute with the Hamiltonian (14.4). Moreover, a straightforward calculation shows that C also commutes with the symmetries

$$[C, L_i] = 0, \qquad i = 1,2,3.$$

Furthermore, one can verify that the Hamiltonian of the system (14.4) can be expressed in terms of C as follows:

$$\mathscr{H} = C^2 + C.$$

Upon defining

$$\mathscr{Q} = CR_1R_2R_3,$$

which also commutes with the constants of motion L_i and the Hamiltonian $\mathscr{H}$, it is verified that the following relations hold:

$$\{L_1, L_2\} = L_3 - 2\mu_3\mathscr{Q} + 2\mu_1\mu_2, \tag{14.8a}$$

$$\{L_2, L_3\} = L_1 - 2\mu_1\mathscr{Q} + 2\mu_2\mu_3, \tag{14.8b}$$

$$\{L_3, L_1\} = L_2 - 2\mu_2\mathscr{Q} + 2\mu_1\mu_3. \tag{14.8c}$$

The invariance algebra (14.8) generated by the constants of motion L_i of the system (14.4) corresponds to a central extension of the Bannai–Ito algebra (14.3) where the central operator is $\mathscr{Q}$. In the realization (14.5), (14.6), (14.7), the Casimir operator of the Bannai–Ito algebra, which has the expression [31]

$$\mathbf{L}^2 = L_1^2 + L_2^2 + L_3^2,$$

is related to C in the following way:

$$\mathbf{L}^2 = C^2 + \mu_1^2 + \mu_2^2 + \mu_3^2 - 1/4.$$

Note that one has $C^2 = \mathscr{Q}^2$ since C commutes with $R_1R_2R_3$; further observe that the commutation relations (14.8) are invariant under any cyclic permutation of the pairs (L_i, μ_i), $i = 1,2,3$. Since the reflections R_i are also (discrete) symmetries of the Hamiltonian (14.4), their commutation relations with the other constants of motion L_1, L_2, L_3 can be included as part of the symmetry algebra. One finds that

$$\{L_i, R_j\} = R_k + 2\mu_j R_j R_k + 2\mu_k, \quad [L_i, R_i] = 0,$$

where $i \neq j \neq k$. The commutation relations involving C and the reflections are

$$\{C, R_i\} = -2L_i R_1R_2R_3 - R_i - 2\mu_i,$$

for $i = 1,2,3$.

14.3 Exact solution

In this section, the exact solutions of the Schrödinger equation

$$\mathscr{H}\Psi = \mathscr{E}\Psi, \tag{14.9}$$

associated to the Hamiltonian (14.4) are obtained by separation of variables in two different spherical coordinate systems.

14.3.1 Standard spherical coordinates

In the usual spherical coordinates

$$s_1 = \cos\phi \sin\theta, \quad s_2 = \sin\phi \sin\theta, \quad s_3 = \cos\theta, \tag{14.10}$$

the Hamiltonian (14.4) takes the form

$$\mathscr{H} = \mathscr{F}_\theta + \frac{1}{\sin^2\theta}\mathscr{G}_\phi, \tag{14.11}$$

where

$$\mathscr{F}_\theta = -\partial_\theta^2 - \cot\theta\,\partial_\theta + \frac{\mu_3}{\cos^2\theta}(\mu_3 - R_3), \tag{14.12a}$$

$$\mathscr{G}_\phi = -\partial_\phi^2 + \frac{\mu_1}{\cos^2\phi}(\mu_1 - R_1) + \frac{\mu_2}{\sin^2\phi}(\mu_2 - R_2), \tag{14.12b}$$

and where the reflections have the actions

$$R_1 f(\theta,\phi) = f(\theta, \pi-\phi), \quad R_2 f(\theta,\phi) = f(\theta,-\phi), \quad R_3 f(\theta,\phi) = f(\pi-\theta,\phi).$$

It is clear from the expression (14.11) that the Hamiltonian $\mathscr{H}$ separates in the spherical coordinates (14.10). Moreover, since $\mathscr{H}$ commutes with the three reflections R_i, they can all be diagonalized simultaneously. Upon taking $\Psi(\theta,\phi) = \Theta(\theta)\Phi(\phi)$ in (14.9), one finds the system of ordinary equations

$$\left[\mathscr{F}_\theta + \frac{m^2}{\sin^2\theta} - \mathscr{E}\right]\Theta(\theta) = 0, \tag{14.13a}$$

$$\left[\mathscr{G}_\phi - m^2\right]\Phi(\phi) = 0, \tag{14.13b}$$

where m^2 is the separation constant. The regular solutions to (14.13) can be obtained from the results of [8]. Indeed, up to a gauge transformation with the function $G(s_1,s_2,s_3) = |s_1|^{\mu_1}||s_2|^{\mu_2}|s_3|^{\mu_3}$, the system (14.13) is equivalent to the angular equations arising in the separation of variables in spherical coordinates of the Schrödinger equation for the three-dimensional Dunkl oscillator.

Using this observation and the results of [8], one finds that the azimuthal solutions have the following expression:

$$\Phi_n^{(e_1,e_2)}(\phi) = \left(\frac{(n+\mu_1+\mu_2)\left(\frac{n-e_1-e_2}{2}\right)!\,\Gamma(\frac{n+e_1+e_2}{2}+\mu_1+\mu_2)}{2\,\Gamma(\frac{n+e_1-e_2}{2}+\mu_1+1/2)\Gamma(\frac{n+e_2-e_1}{2}+\mu_2+1/2)}\right)^{1/2}$$
$$|\cos\phi|^{\mu_1}|\sin\phi|^{\mu_2}\cos^{e_1}\phi\sin^{e_2}\phi\, P_{(n-e_1-e_2)/2}^{(\mu_2-1/2+e_2,\mu_1-1/2+e_1)}(\cos 2\phi), \quad (14.14)$$

where $P_n^{(\alpha,\beta)}(x)$ are the Jacobi polynomials [22], $\Gamma(z)$ is the Gamma function [1] and where $e_i \in \{0,1\}$. The azimuthal solutions (14.14) satisfy the eigenvalue equations

$$R_1\Phi_n^{(e_1,e_2)}(\phi) = (1-2e_1)\Phi_n^{(e_1,e_2)}(\phi), \qquad R_2\Phi_n^{(e_1,e_2)}(\phi) = (1-2e_2)\Phi_n^{(e_1,e_2)}(\phi),$$

with respect to the reflections. They obey the orthogonality relation

$$\int_0^{2\pi} \Phi_n^{(e_1,e_2)}(\phi)\Phi_{n'}^{(e_1',e_2')}(\phi)\,\mathrm{d}\phi = \delta_{nn'}\delta_{e_1e_1'}\delta_{e_2e_2'},$$

as can be checked by comparing with the orthogonality relation satisfied by the Jacobi polynomials [22]. The separation constant here takes the value $m^2 = (n+\mu_1+\mu_2)^2$. When $e_1+e_2=1$, n is a positive odd integer, when $e_1+e_2=0$, n is a non-negative even integer and when $e_1+e_2=2$, n is a positive even integer. The regular solutions to the zenithal equation (14.13a) are of the form [8]

$$\Theta_{n;N}^{(e_3)}(\theta) = \left(\frac{(N+\mu_1+\mu_2+\mu_3+1/2)(\frac{N-n-e_3}{2})!\Gamma(\frac{N+n+e_3}{2}+\mu_1+\mu_2+\mu_3+1/2)}{\Gamma(\frac{N+n-e_3}{2}+\mu_1+\mu_2+1)\Gamma(\frac{N-n+e_3}{2}+\mu_3+1/2)}\right)^{1/2}$$
$$|\sin\theta|^{\mu_1+\mu_2}|\cos\theta|^{\mu_3}\sin^n\theta\cos^{e_3}\theta\, P_{(N-n-e_3)/2}^{(n+\mu_1+\mu_2,\mu_3-1/2+e_3)}(\cos 2\theta). \quad (14.15)$$

with $e_3 \in \{0,1\}$. The following eigenvalue equation holds:

$$R_3\Theta_{n;N}^{(e_3)}(\theta) = (1-2e_3)\Theta_{n;N}^{(e_3)}(\theta).$$

The energy $\mathscr{E}$ corresponding to the solution (14.15) is given by

$$\mathscr{E}_N = (N+\mu_1+\mu_2+\mu_3)^2 + (N+\mu_1+\mu_2+\mu_3), \quad (14.16)$$

where N is a non-negative integer. The complete wavefunctions of the Hamiltonian (14.4) on the 2-sphere with energy $\mathscr{E}_N$ given by (14.16) thus have the expression

$$\Psi_{n;N}^{(e_1,e_2,e_3)}(\theta,\phi) = \Theta_{n;N}^{(e_3)}(\theta)\Phi_n^{(e_1,e_2)}(\phi),$$

where the zenithal and azimuthal parts are given by (14.15) and (14.14), respectively. By a direct counting of the admissible states (taking into account the fact that values of the quantum numbers N, n, e_i yielding half-integer or negative values of k in the Jacobi polynomials $P_k^{(\alpha,\beta)}(x)$ are not admissible), it is seen that the $\mathscr{E}_N$ energy eigenspace is $(2N+1)$-fold degenerate. Furthermore, one observes that

$$R_1R_2R_3\Psi_{n;N}^{(e_1,e_2,e_3)}(\theta,\phi) = (-1)^N\Psi_{n;N}^{(e_1,e_2,e_3)}(\theta,\phi),$$

and that the wavefunctions satisfy the orthogonality relation

$$\int_0^{\pi}\int_0^{2\pi}\Psi_{n;N}^{(e_1,e_2,e_3)}(\theta,\phi)\Psi_{n';N'}^{(e_1',e_2',e_3')}(\theta,\phi)\,\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta = \delta_{nn'}\delta_{NN'}\delta_{e_1e_1'}\delta_{e_2e_2'}\delta_{e_3e_3'}.$$

The symmetry operator responsible for the separation of variables in spherical coordinates is L_3. Indeed, a direct computation shows that upon defining $Z = L_3R_1R_2$ one has

$$Z^2 - Z + \frac{1}{4} = \mathscr{G}_{\phi},$$

where $\mathscr{G}_{\phi}$ is given by (14.12b).

14.3.2 Alternative spherical coordinates

The Schrödinger equation (14.9) associated to the Hamiltonian on the 2-sphere (14.4) also separates in the alternative spherical coordinate system where the coordinates s_1, s_2, s_3 of the 2-sphere are parametrized as follows:

$$s_1 = \cos\vartheta, \quad s_2 = \cos\varphi\sin\vartheta, \quad s_3 = \sin\varphi\sin\vartheta. \tag{14.17}$$

In these coordinates the Hamiltonian (14.4) takes the from

$$\mathscr{H} = \widetilde{\mathscr{F}}_{\vartheta} + \frac{1}{\sin^2\vartheta}\widetilde{\mathscr{G}}_{\varphi},$$

where

$$\widetilde{\mathscr{F}}_{\vartheta} = -\partial_{\vartheta}^2 - \cot\vartheta\,\partial_{\vartheta} + \frac{\mu_1}{\cos^2\vartheta}(\mu_1 - R_1), \tag{14.18a}$$

$$\widetilde{\mathscr{G}}_{\varphi} = -\partial_{\varphi}^2 + \frac{\mu_2}{\cos^2\varphi}(\mu_2 - R_2) + \frac{\mu_3}{\sin^2\varphi}(\mu_3 - R_3), \tag{14.18b}$$

and where the reflections have the actions

$$R_1f(\vartheta,\varphi) = f(\pi-\vartheta,\varphi), \quad R_2f(\vartheta,\varphi) = f(\vartheta,\pi-\varphi), \quad R_3f(\vartheta,\varphi) = f(\vartheta,-\varphi). \tag{14.19}$$

Upon comparing (14.18) with (14.12) it is clear that the solutions to the Schrödinger equation (14.9) in the alternative coordinate system (14.17) have the expression

$$\Psi_{n;N}^{(e_1,e_2,e_3)}(\vartheta,\varphi) = \pi\,\Theta_{n;N}^{(e_1)}(\vartheta)\Phi_n^{(e_2,e_3)}(\varphi),$$

where $\pi = (123)$ indicates the permutation applied to the parameters (μ_1, μ_2, μ_3). The symmetry associated to the separation of variables in the alternative coordinate system (14.17) is L_1 since upon taking $Y = L_1 R_2 R_3$, one finds that

$$Y^2 - Y + \frac{1}{4} = \widetilde{\mathscr{G}}_{\varphi}.$$

As illustrated above, the origin of the two independent constants of motion L_3 and L_1 of the model (14.4) on the two-sphere can be traced back to the multiseparability of the Schrödinger equation in the usual and alternative spherical coordinates, respectively. This situation is analogous to the one arising in the analysis of the generic three-parameter system on the 2-sphere (without reflections) for which the symmetries generating the Racah algebra are associated to the separation of variables in different spherical coordinate systems [9]. In this case, the expansion coefficients coefficients between the separated wavefunctions in the coordinate systems (14.10) and (14.17) coincide with the $6j$ symbols of $\mathfrak{su}(1,1)$.

14.4 Connection with $sl_{-1}(2)$

In this section, it is shown how of the Hamiltonian (14.4) of the model on the 2-sphere arises in the combination of three realizations of the $sl_{-1}(2)$ algebra, which we loosely refer to as the Racah problem for $sl_{-1}(2)$.

14.4.1 $sl_{-1}(2)$ algebra

The $sl_{-1}(2)$ algebra was introduced in [29] as a $q \to -1$ limit of the quantum algebra $\mathscr{U}_q(\mathfrak{sl}_2)$. It has three three generators $A_{\pm}$, A_0 and one involution P and is defined by the relations

$$[A_0, A_{\pm}] = \pm A_{\pm},\ [A_0, P] = 0,\ \{A_+, A_-\} = 2A_0,\ \{A_{\pm}, P\} = 0,\ P^2 = 1. \tag{14.20}$$

The Casimir operator of $sl_{-1}(2)$, which commutes with all generators, is given by

$$Q = A_+ A_- P - A_0 P + P/2.$$

Let $A_0^{(i)}$, $A_{\pm}^{(i)}$ and $P^{(i)}$, $i = 1,2,3$, denote three mutually commuting sets of $sl_{-1}(2)$ generators. Using the coproduct of $sl_{-1}(2)$ (see [29]), the three sets can be combined to produce a fourth set of generators satisfying the relations (14.20). The elements of this fourth set $\{\mathscr{A}_0, \mathscr{A}_{\pm}, \mathscr{P}\}$ are defined by

$$\mathscr{A}_0 = A_0^{(1)} + A_0^{(2)} + A_0^{(3)}, \tag{14.21a}$$

$$\mathscr{A}_{\pm} = A_{\pm}^{(1)} P^{(2)} P^{(3)} + A_{\pm}^{(2)} P^{(3)} + A_{\pm}^{(3)}, \tag{14.21b}$$

$$\mathscr{P} = P^{(1)} P^{(2)} P^{(3)}. \tag{14.21c}$$

It is easily verified that the operators (14.21) indeed satisfy the defining relations (14.20) of $sl_{-1}(2)$. In the combining of these independent sets of $sl_{-1}(2)$ generators, three types of Casimir operators should be distinguished. The initial Casimir operators $Q^{(i)}$

$$Q^{(i)} = A_{+}^{(i)}A_{-}^{(i)}P^{(i)} - A_{0}^{(i)}P^{(i)} + P^{(i)}/2, \quad i = 1,2,3,$$

which are attached to each independent set of $sl_{-1}(2)$ generators. The two intermediate Casimir operators $\mathscr{Q}^{(ij)}$

$$Q^{(ij)} = (A_{-}^{(i)}A_{+}^{(j)} - A_{+}^{(i)}A_{-}^{(j)})P^{(i)} + Q^{(i)}P^{(j)} + Q^{(j)}P^{(i)} - P^{(i)}P^{(j)}/2,$$

for $(ij) = (12),(23)$ which are associated to the step-wise combination process. The total Casimir operator $\mathscr{Q}$ associated to the fourth set

$$\mathscr{Q} = \mathscr{A}_{+}\mathscr{A}_{-}\mathscr{P} - \mathscr{A}_{0}\mathscr{P} + \mathscr{P}/2,$$

which has the expression

$$\mathscr{Q} = (A_{-}^{(1)}A_{+}^{(3)} - A_{+}^{(1)}A_{-}^{(3)})P^{(1)} - Q^{(2)}P^{(1)}P^{(3)} + Q^{(12)}P^{(3)} + Q^{(23)}P^{(1)}.$$

The intermediate Casimir operators commute with both the initial and the total Casimir operators and with $\mathscr{P}$, but do not commute amongst themselves. As a matter of fact, it was established in [16] that on representations spaces where the total Casimir operator $\mathscr{Q}$ is diagonal, the intermediate Casimir operators $Q^{(ij)}$ generate the Bannai–Ito algebra.

14.4.2 Differential/Difference realization and the model on the 2-sphere

The connection between the combination of three $sl_{-1}(2)$ algebras and the superintegrable system on the two-sphere defined by (14.4) can now be established. Consider the following differential/difference realization of $sl_{-1}(2)$:

$$A_{\pm}^{(i)} = \frac{1}{\sqrt{2}}\left[s_i \mp \partial_{s_i} \pm \frac{\mu_i}{s_i}R_i\right], \tag{14.22a}$$

$$A_{0}^{(i)} = \frac{1}{2}\left[-\partial_{s_i}^2 + s_i^2 + \frac{\mu_i}{s_i^2}(\mu_i - R_i)\right], \tag{14.22b}$$

$$P^{(i)} = R_i. \tag{14.22c}$$

Note that with respect to the uniform measure on the real line, $A_0^{(i)}$ is Hermitian and $A_{\pm}^{(i)}$ are Hermitian conjugates. Up to a gauge transformation of the generators

$$z \to G(s_i)^{-1}zG(s_i), \qquad z \in \{A_{\pm}^{(i)}, A_0^{(i)}, P^{(i)}\},$$

with gauge function $G(s_i) = |s_i|^{\mu_i}$, the realization (14.22) is equivalent to the realization of $sl_{-1}(2)$ arising in the one-dimensional parabose oscillator [11]. Using (14.22), the initial Casimir operators $Q^{(i)}$ are seen to have the action

$$Q^{(i)} f(s_i) = -\mu_i f(s_i).$$

and the intermediate Casimir operators take the form

$$\begin{aligned} Q^{(12)} &= \left[(s_1\partial_{s_2} - s_2\partial_{s_1}) + \mu_1 \frac{s_2}{s_1} R_1 - \mu_2 \frac{s_1}{s_2} R_2\right] R_1 + \mu_1 R_2 + \mu_2 R_1 + \frac{1}{2} R_1 R_2, \\ Q^{(23)} &= \left[(s_2\partial_{s_3} - s_3\partial_{s_2}) + \mu_2 \frac{s_3}{s_2} R_2 - \mu_3 \frac{s_2}{s_3} R_3\right] R_2 + \mu_2 R_3 + \mu_3 R_2 + \frac{1}{2} R_2 R_3. \end{aligned} \tag{14.23}$$

Furthermore, an explicit computation shows that upon defining $\Omega = \mathscr{Q}\mathscr{P}$, one finds

$$\Omega^2 + \Omega = J_1^2 + J_2^2 + J_3^2 + (s_1^2 + s_2^2 + s_3^2)\left(\frac{\mu_1}{s_1^2}(\mu_1 - R_1) + \frac{\mu_2}{s_2^2}(\mu_2 - R_2) + \frac{\mu_3}{s_3^2}(\mu_3 - R_3)\right). \tag{14.24}$$

Upon comparing the expressions (14.23) for the intermediate Casimir operators with the formulas (14.5) for the constants of motion, it is seen that

$$Q^{(12)} = -L_3, \qquad Q^{(23)} = -L_1,$$

and thus that the intermediate Casimir coincide with the constants of motion. Upon comparing (14.24) with the Hamiltonian (14.4), it is also seen that

$$\Omega^2 + \Omega = \mathscr{H},$$

given the condition $s_1^2 + s_2^2 + s_3^2 = 1$. This condition can be ensured in general. Indeed, one checks that

$$X^2 = \frac{1}{2}(\mathscr{A}_+ + \mathscr{A}_-)^2 = s_1^2 + s_2^2 + s_3^2.$$

Since X^2 commutes with Ω and all the intermediate Casimir operators, it is central in the invariance algebra (14.8) and can thus be treated as a constant. Hence one can take $X^2 = 1$ without loss of generality and complete the identification of the quadratic combination $\Omega^2 + \Omega$ with the Hamiltonian $\mathscr{H}$.

The analysis of the model on the 2-sphere defined by the Hamiltonian (14.4) is thus related to the combination of three independent realizations of the $sl_{-1}(2)$ algebra. The constants of motion of the system correspond to the intermediate Casimir operators arising in this combination and the Hamiltonian is related to a quadratic combination of the total Casimir operator (times $\mathscr{P}$).

It is worth pointing out that in [9, 14] the Hamiltonian of the three-parameter model on the two-sphere (without reflections) was directly identified to the total Casimir operator in the combination of three $\mathfrak{su}(1,1)$ realizations. Here the total Casimir operator $\mathscr{Q}$ (or equivalently Ω) is of *first* order in derivatives and hence a quadratic combination must me taken to recover the Hamiltonian (14.4) of the model. As a remark, let us note that the relation (14.24) is reminiscent of chiral supersymmetry. Indeed, if one defines a new Hamiltonian by $H = \mathscr{H} + 1/4$, then $H = \frac{1}{2}\{Q, Q\}$ where $Q = \Omega + 1/2$. In this picture, $Q = \Omega + 1/2$ can be interpreted as a chiral supercharge for H. See also [6] for related considerations.

14.5 Superintegrable model in the plane from contraction

The results obtained so far are analogous to those of [9, 14] where the analysis of the generic 3-parameter system on the two-sphere was cast in the framework of the Racah problem for the $\mathfrak{su}(1,1)$ algebra. Given that the model (14.4) is the analogue with reflections of the generic 3-parameter system on the 2-sphere and since all second-order superintegrable systems can be obtained from the latter [20], it is natural to ask whether contractions of (14.4) lead to other superintegrable systems with reflections. The answer to that question is in the positive. As an example, we describe in this section how the Dunkl oscillator model in the plane and its conserved quantities can be obtained from a contraction of the system (14.4) on the two-sphere and its symmetries.

14.5.1 Contraction of the Hamiltonian

Consider the Hamiltonian

$$\mathscr{H} = J_1^2 + J_2^2 + J_3^2 + (s_1^2 + s_2^2 + s_3^2)\left(\frac{\mu_1^2 - \mu_1 R_1}{s_1^2} + \frac{\mu_2^2 - \mu_2 R_2}{s_2^2} + \frac{\mu_3^2 - \mu_3 R_3}{s_3^2}\right), \tag{14.25}$$

which is equivalent to (14.4) in view of the results of section 4. The two-sphere $s_1^2 + s_2^2 + s_3^3 = r^2$ of radius r can be contracted to the Euclidean plane with coordinates x_1, x_2 by taking the limit as $r \to \infty$ in

$$x_1 = r\frac{s_1}{s_3}, \quad x_2 = r\frac{s_2}{s_3}, \quad s_3^2 = r^2 - s_1^2 - s_2^2, \tag{14.26}$$

Changing the variables in (14.25) according to the prescription (14.26) and defining $\mu_3 = \widehat{\mu}_3 r^2$, a direct computation shows that

$$\widetilde{\mathscr{H}} \equiv \lim_{r\to\infty} \frac{1}{r^2}\Big(\mathscr{H} - \mu_3^2 + \mu_3 R_3\Big) = \\ -\partial_{x_1}^2 - \partial_{x_2}^2 + \frac{\mu_1}{x_1^2}(\mu_1 - R_1) + \frac{\mu_2}{x_2^2}(\mu_2 - R_2) + \widehat{\mu}_3^2(x_1^2 + x_2^2). \tag{14.27}$$

The Hamiltonian $\widetilde{\mathscr{H}}$ corresponds, up to a gauge transformation, to the Hamiltonian of the Dunkl oscillator model in the plane. Indeed, taking $\widetilde{\mathscr{H}} \to |x_1|^{-\mu_1}|x_2|^{-\mu_2}\widetilde{\mathscr{H}}|x_2|^{\mu_2}|x_1|^{\mu_1}$, one finds that

$$\widetilde{\mathscr{H}} \to -\left[(\mathscr{D}_{x_1}^{\mu_1})^2 + (\mathscr{D}_{x_2}^{\mu_2})^2\right] + \widehat{\mu}_3^2(x_1^2 + x_2^2), \tag{14.28}$$

where $\mathscr{D}_x^{\mu}$ is the Dunkl derivative (14.2). Taking $\widehat{\mu}_3 = 1$, the Hamiltonian (14.28) coincides with that of the model examined in [11].

14.5.2 Contraction of the constants of motion

The conserved quantities of the Dunkl oscillator in the plane can be recovered by a contraction of the symmetry operators of the Hamiltonian (14.25). Since the reflections commute with (14.25), one can consider the following form of the constants of motion:

$$\widetilde{L}_1 = \frac{1}{i}\left[(s_2\partial_{s_3} - s_3\partial_{s_2}) + \mu_2\frac{s_3}{s_2}R_2 - \mu_3\frac{s_2}{s_3}R_3\right],$$
$$\widetilde{L}_2 = \frac{1}{i}\left[(s_3\partial_{s_1} - s_1\partial_{s_3}) + \mu_3\frac{s_1}{s_3}R_3 - \mu_1\frac{s_3}{s_1}R_1\right],$$
$$\widetilde{L}_3 = \frac{1}{i}\left[(s_1\partial_{s_2} - s_2\partial_{s_1}) + \mu_1\frac{s_2}{s_1}R_1 - \mu_2\frac{s_1}{s_2}R_2\right],$$

in lieu of the symmetries L_1, L_2, L_3 in (14.5) and (14.6). The operators $\widetilde{L}_i$ commute with (14.25) and satisfy the relations

$$[\widetilde{L}_1, \widetilde{L}_2] = i\widetilde{L}_3(1 + 2\mu_3 R_3),$$
$$[\widetilde{L}_2, \widetilde{L}_3] = i\widetilde{L}_1(1 + 2\mu_1 R_1),$$
$$[\widetilde{L}_3, \widetilde{L}_1] = i\widetilde{L}_2(1 + 2\mu_2 R_2).$$

The commutation relations with the reflections are given by

$$\{\widetilde{L}_i, R_j\} = [\widetilde{L}_i, R_i] = 0,$$

where $i \neq j$. The contraction of $\widetilde{L}_3$ directly yields a conserved quantity for (14.27). Indeed, one finds using (14.26)

$$\mathscr{J}_2 \equiv \lim_{r\to\infty} \widetilde{L}_3 = \frac{1}{i}\left[(x_1\partial_{x_2} - x_2\partial_{x_1}) + \mu_1\frac{x_2}{x_1}R_1 - \mu_2\frac{x_1}{x_2}R_2\right], \tag{14.29}$$

which commute with $\widetilde{\mathscr{H}}$ given by (14.27). The contraction of the symmetries $\widetilde{L}_1$, $\widetilde{L}_2$ cannot lead to constants of motion for the resulting Hamiltonian (14.27) since these two operators anticommute with the term involving the reflection operator R_3 which is added before the $r \to \infty$ limit is taken (see (14.27)). However, the contraction of $\widetilde{L}_1^2$ and $\widetilde{L}_2^2$, which commute with R_3, will yield symmetries of the contracted Hamiltonian (14.27). Computing the squares of $\widetilde{L}_1$, $\widetilde{L}_2$ and using (14.26), one finds

$$\widetilde{\mathscr{H}}_2 \equiv \lim_{r\to\infty} \frac{1}{r^2}\left(\widetilde{L}_1^2 + \mu_3 R_3 + 2\mu_2\mu_3 R_2 R_3\right) = -\partial_{x_2}^2 + \widehat{\mu}_3^2 x_2^2 + \frac{\mu_2}{x_2^2}(\mu_2 - R_2),$$
$$\widetilde{\mathscr{H}}_1 \equiv \lim_{r\to\infty} \frac{1}{r^2}\left(\widetilde{L}_2^2 + \mu_3 R_3 + 2\mu_1\mu_3 R_1 R_3\right) = -\partial_{x_1}^2 + \widehat{\mu}_3^2 x_1^2 + \frac{\mu_1}{x_1^2}(\mu_1 - R_1).$$

It is clear that $\widetilde{\mathscr{H}} = \widetilde{\mathscr{H}}_1 + \widetilde{\mathscr{H}}_2$ and hence one can define

$$\mathscr{J}_1 = \widetilde{\mathscr{H}}_1 - \widetilde{\mathscr{H}}_2, \tag{14.30}$$

as a second constant of motion. The operators $\mathscr{J}_2$, $\mathscr{J}_1$ respectively given by (14.29), (14.30) correspond (up to a constant and a gauge transformation) to the symmetries of the Dunkl oscillator in the plane obtained in [11] which were found to generate the Schwinger-Dunkl algebra $sd(2)$ (see also [12] for the representation theory of $sd(2)$).

14.6 Conclusion

Recapping, we have shown that the model defined by the Hamiltonian (14.4) on the two-sphere is superintegrable and exactly solvable. The constants of motion were explicitly obtained and were seen to be related to the separability of the Schrödinger equation associated to (14.4) in different spherical coordinate systems. Moreover, it was observed that these symmetries generate a central extension of the Bannai–Ito algebra. The relation between the superintegrable system (14.4) and the Racah problem for the $sl_{-1}(2)$ was also established. Furthermore, the contraction from the two-sphere to the Euclidean plane was examined and it was shown how the Dunkl oscillator in the plane and its symmetry algebra can be recovered in this limit.

The results of this paper are complementary to those presented in [7] where the connection between the combination of three $sl_{-1}(2)$ representations and the Dunkl Laplacian operator is used to study the $6j$ problem for $sl_{-1}(2)$ in the position representation and to provide a further characterization of the Bannai–Ito polynomials as interbasis expansion coefficients. Basis functions for irreducible representations of the Bannai–Ito algebra are also constructed in [7] and expressed in terms of the Dunkl spherical harmonics.

In view of the results obtained in [19] relating the generic model on the three-sphere to the bivariate Wilson polynomials, it would of interest to consider the analogous S^3 model with reflections. This could provide a framework for the study of multivariate -1 orthogonal polynomials.

Also of interest is the study of the Dunkl oscillator models involving more complicated reflection groups.

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Chapitre 15

A Dirac–Dunkl equation on S^2 and the Bannai–Ito algebra

H. De Bie, V. X. Genest et L. Vinet (2015). A Dirac–Dunkl equation on S^2 and the Bannai-Ito algebra. Soumis à *Communications in Mathematical Physics*.

Abstract. The Dirac–Dunkl operator on the 2-sphere associated to the $\mathbb{Z}_2^3$ reflection group is considered. Its symmetries are found and are shown to generate the Bannai–Ito algebra. Representations of the Bannai–Ito algebra are constructed using ladder operators. Eigenfunctions of the spherical Dirac-Dunkl operator are obtained using a Cauchy–Kovalevskaia extension theorem. These eigenfunctions, which correspond to Dunkl monogenics, are seen to support finite-dimensional irreducible representations of the Bannai–Ito algebra.

15.1 Introduction

The purpose of this paper is to study the Dirac–Dunkl operator on the two-sphere for the $\mathbb{Z}_2^3$ reflection group and to investigate its relation with the Bannai–Ito algebra.

The Bannai–Ito algebra is the associative algebra over the field of real numbers with generators I_1, I_2, and I_3 satisfying the relations

$$\{I_1, I_2\} = I_3 + \alpha_3, \quad \{I_2, I_3\} = I_1 + \alpha_1, \quad \{I_3, I_1\} = I_2 + \alpha_2, \tag{15.1}$$

where $\{a, b\} = ab + ba$ is the anticommutator and where α_i, $i = 1, 2, 3$, are real structure constants. The algebra (15.1) was first presented in [27] as the algebraic structure encoding the bispectral properties of the Bannai–Ito polynomials, which together with the Complementary Bannai–Ito polynomials are the parents of the family of -1 polynomials [15, 27]. The Bannai–Ito algebra also arises in representation theoretic problems [17] and in superintegrable systems [16]; see [2] for a recent overview.

Following their introduction in [8, 9, 10], Dunkl operators have appeared in various areas. They enter the study of Calogero–Moser–Sutherland models [28], they play a central role in the theory of multivariate orthogonal polynomials associated to reflection groups [11], they give rise to families of stochastic processes [20, 25], and they can be used to construct quantum superintegrable systems involving reflections [13, 14]. Dunkl operators also find applications in harmonic analysis and integral transforms [7, 24], as they naturally lead to the Laplace-Dunkl operators, which are second-order differential/difference operator that generalize the standard Laplace operator.

In a recent paper [19], the analysis of the Laplace-Dunkl operator on the two-sphere associated to the $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$ Abelian reflection group was cast in the frame of the Racah problem for the Hopf algebra $sl_{-1}(2)$ [26], which is closely related to the Lie superalgebra $\mathfrak{osp}(1|2)$. It was established that the Laplace-Dunkl operator on the two-sphere Δ_{S^2} can be expressed as a quadratic polynomial in the Casimir operator corresponding to the three-fold tensor product of unitary irreducible representations of $sl_{-1}(2)$. A central extension of the Bannai–Ito algebra was seen to emerge as the invariance algebra for Δ_{S^2} and subspaces of the space of Dunkl harmonics that transform according to irreducible representations of the Bannai–Ito algebra were identified.

It is well known that the square root of the standard Laplace operator is the Dirac operator, which is a Clifford-valued first order differential operator. The study of Dirac operators is at the core of Clifford analysis, which can be viewed as a refinement of harmonic analysis [5]. Dirac operators also lend themselves to generalizations involving Dunkl operators [3, 6, 23]. These so-called Dirac-Dunkl operators are the square roots of the corresponding Laplace-Dunkl operators and as such, they exhibit additional structure which makes their analysis both interesting and enlightening.

In this paper, the Dirac-Dunkl operator on the two-sphere associated to the $\mathbb{Z}_2^3$ Abelian reflection group will be examined. We shall begin by discussing the Laplace– and Dirac– Dunkl operators in $\mathbb{R}^3$. The Dirac–Dunkl operator will be defined in terms of the Pauli matrices, which play the role of Dirac's gamma matrices for the three-dimensional Euclidean space. It will be shown that the Laplace– and Dirac– Dunkl operators can be embedded in a realization of $\mathfrak{osp}(1|2)$. The notion of Dunkl monogenics, which are homogeneous polynomial null solutions of the Dirac–Dunkl operator, will be reviewed as well as the corresponding Fischer theorem, which describes the decomposition of the space of homogeneous polynomials in terms of Dunkl monogenics. The Dirac–Dunkl operator on the two-sphere, to be called spherical, will then be defined in terms of generalized "angular momentum" operators written in terms of Dunkl operators and its relation with the spherical Laplace–Dunkl operator will be made explicit. The algebraic interpretation of the Dirac–Dunkl operator will proceed from noting its connection with the sCasimir operator of $\mathfrak{osp}(1|2)$. The symmetries of the spherical Dirac–Dunkl operator will be determined. Remarkably, these symmetries will be seen to satisfy the defining relations of the Bannai–Ito algebra. The relevant finite-dimensional unitary irreducible representations of the Bannai–Ito algebra will be constructed using ladder operators. An explicit basis for the eigenfunctions of the spherical Dirac–Dunkl operator will be obtained. The basis functions, which span the space of Dunkl monogenics, will be constructed systematically using a Cauchy–Kovalevskaia (CK) extension theorem. It will be shown that these spherical wavefunctions, which generalize spherical spinors, transform irreducibly under the action of the Bannai–Ito algebra.

The paper is divided as follows.

- Section 2: Dirac– and Laplace– Dunkl operators in $\mathbb{R}^3$ and S^2, $\mathfrak{osp}(1|2)$ algebra
- Section 3: Symmetries of the Dirac–Dunkl operator on S^2, Bannai–Ito algebra
- Section 4: Ladder operators, Representations of the Bannai–Ito algebra
- Section 5: CK extension, Eigenfunctions of the spherical Dirac–Dunkl operator

15.2 Laplace– and Dirac– Dunkl operators for $\mathbb{Z}_2^3$

In this section, we introduce the Laplace– and Dirac– Dunkl operators associated to the $\mathbb{Z}_2^3$ reflection group. We show that these operators can be embedded in a realization of $\mathfrak{osp}(1|2)$. We define the Dunkl monogenics and the Dunkl harmonics and review the Fischer decomposition theorem. We introduce the spherical Laplace– and Dirac– Dunkl operators and we give their relation.

15.2.1 Laplace– and Dirac– Dunkl operators in $\mathbb{R}^3$

Let $\vec{x} = (x_1, x_2, x_3)$ denote the coordinate vector in $\mathbb{R}^3$ and let μ_i, $i = 1,2,3$, be real numbers such that $\mu_i \geqslant 0$. The Dunkl operators associated to the $\mathbb{Z}_2^3$ reflection group, denoted by T_i, are given by

$$T_i = \partial_{x_i} + \frac{\mu_i}{x_i}(1 - R_i), \qquad i = 1,2,3, \tag{15.2}$$

where

$$R_i f(x_i) = f(-x_i)$$

is the reflection operator. It is obvious that the operators T_i, T_j commute with one another. We define the Dirac–Dunkl operator in $\mathbb{R}^3$, to be denoted by $\underline{D}$, as follows:

$$\underline{D} = \sigma_1 T_1 + \sigma_2 T_2 + \sigma_3 T_3, \tag{15.3}$$

where the σ_i are the familiar Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These matrices satisfy the identities

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k, \qquad \sigma_i\sigma_j = i\epsilon_{ijk}\sigma_k + \delta_{ij},$$

where $[a,b] = ab - ba$ is the commutator, where ϵ_{ijk} is the Levi-Civita symbol and where summation over repeated indices is implied. The Pauli matrices provide a representation of the Euclidean Clifford algebra with three generators on $\mathbb{C}^2$, i.e. on the space of two-spinors. Indeed, one has

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}, \qquad i,j = 1,2,3. \tag{15.4}$$

As a direct consequence of (15.4), one has

$$\underline{D}^2 = \Delta = T_1^2 + T_2^2 + T_3^2, \tag{15.5}$$

where Δ is the Laplace–Dunkl operator in $\mathbb{R}^3$.

The Dirac–Dunkl and the Laplace–Dunkl operators (15.3) and (15.5) can be embedded in a realization of the Lie superalgebra $\mathfrak{osp}(1|2)$. Let $\underline{x}$ and $||\vec{x}||^2$ be the operators defined by

$$\underline{x} = \sigma_1 x_1 + \sigma_2 x_2 + \sigma_3 x_3, \qquad ||\vec{x}||^2 = \underline{x}^2 = x_1^2 + x_2^2 + x_3^2,$$

and let $\mathbb{E}$ stand for the Euler (or dilation) operator

$$\mathbb{E} = x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3}.$$

A direct calculation shows that one has

$$\begin{aligned}
&\{\underline{x},\underline{x}\} = 2\,||\vec{x}||^2, && \{\underline{D},\underline{D}\} = 2\,\Delta, && \{\underline{x},\underline{D}\} = 2\,(\mathbb{E}+\gamma_3), \\
&[\underline{D}, ||\vec{x}||^2] = 2\,\underline{x}, && [\mathbb{E}+\gamma_3, \underline{x}] = \underline{x}, && [\mathbb{E}+\gamma_3, \underline{D}] = -\underline{D}, \qquad [\Delta, \underline{x}] = 2\underline{D}, \\
&[\mathbb{E}+\gamma_3, \Delta] = -2\,\Delta, && [\mathbb{E}+\gamma_3, ||\underline{x}||^2] = 2\,||\vec{x}||^2, && [\Delta, ||\vec{x}||^2] = 4\,(\mathbb{E}+\gamma_3),
\end{aligned} \tag{15.6}$$

where

$$\gamma_3 = \mu_1 + \mu_2 + \mu_3 + 3/2.$$

The commutation relations (15.6) are seen to correspond to those of the $\mathfrak{osp}(1|2)$ Lie superalgebra [12]. In fact, the relations (15.6) hold in any dimension and for any choice of the reflection group with different values of the constant γ [3, 23].

Let $\mathscr{P}_N(\mathbb{R}^3)$ denote the space of homogeneous polynomials of degree N in $\mathbb{R}^3$, where N is a non-negative integer. The space of *Dunkl monogenics* of degree N for the reflection group $\mathbb{Z}_2^3$ shall be denoted by $\mathscr{M}_N(\mathbb{R}^3)$. It is defined as

$$\mathscr{M}_N(\mathbb{R}^3) := \mathrm{Ker}\,\underline{D} \bigcap (\mathscr{P}_N(\mathbb{R}^3) \otimes \mathbb{C}^2). \tag{15.7}$$

Similarly, the space of scalar *Dunkl harmonics* of degree N for the reflection group $\mathbb{Z}_2^3$ is denoted by $\mathscr{H}_N(\mathbb{R}^3)$ and defined as [11]

$$\mathscr{H}_N(\mathbb{R}^3) := \mathrm{Ker}\,\Delta \bigcap \mathscr{P}_N(\mathbb{R}^3).$$

The space of spinor-valued Dunkl harmonics has a direct sum decomposition in terms of the Dunkl monogenics. This decomposition reads

$$\mathscr{H}_N(\mathbb{R}^3) \otimes \mathbb{C}^2 = \mathscr{M}_N(\mathbb{R}^3) \oplus \underline{x}\,\mathscr{M}_{N-1}(\mathbb{R}^3).$$

For $\gamma_3 > 0$, which is automatically satisfied when $\mu_1, \mu_2, \mu_3 \geqslant 0$, the following direct sum decomposition holds [23]:

$$\mathscr{P}_N(\mathbb{R}^3) \otimes \mathbb{C}^2 = \bigoplus_{k=0}^{N} \underline{x}^k \mathscr{M}_{N-k}(\mathbb{R}^3). \tag{15.8}$$

The above is called the *Fischer decomposition* and will play an important role in what follows.

15.2.2 Laplace– and Dirac– operators on S^2

The explicit expression for the Dunkl operators (15.2) allows to write the Laplace–Dunkl operator (15.5) as

$$\Delta = \sum_{i=1}^{3} \partial_{x_i}^2 + \frac{2\mu_i}{x_i}\partial_{x_i} - \frac{\mu_i}{x_i^2}(1 - R_i). \tag{15.9}$$

Since reflections are elements of $O(3)$, the Laplace–Dunkl operator (15.9), like the standard Laplace operator, separates in spherical coordinates. Consequently, it can be restricted to functions defined on the unit sphere. Let Δ_{S^2} denote the restriction of (15.9) to the two-sphere, which shall be referred to as the spherical Laplace–Dunkl operator. It is seen that Δ_{S^2} can be expressed as

$$\Delta_{S^2} = ||\vec{x}||^2 \Delta - \mathbb{E}(\mathbb{E} + 2\mu_1 + 2\mu_2 + 2\mu_3 + 1). \tag{15.10}$$

The spherical Laplace–Dunkl operator can also be written in terms of the Dunkl angular momentum operators. These operators are defined as

$$L_1 = \frac{1}{i}(x_2 T_3 - x_3 T_2), \quad L_2 = \frac{1}{i}(x_3 T_1 - x_1 T_3), \quad L_3 = \frac{1}{i}(x_1 T_2 - x_2 T_1), \tag{15.11}$$

and satisfy the commutation relations

$$[L_i, L_j] = i\epsilon_{ijk} L_k(1+2\mu_k R_k), \qquad [L_i, R_i] = 0, \qquad \{L_i, R_j\} = 0. \tag{15.12}$$

Taking into account the relation (15.10), a direct calculation shows that [18]

$$-\Delta_{S^2} = L_1^2 + L_2^2 + L_3^2 - 2 \sum_{1\leqslant i<j\leqslant 3} \mu_i\mu_j(1-R_iR_j) - \sum_{1\leqslant j\leqslant 3} \mu_j(1-R_j). \tag{15.13}$$

When $\mu_1 = \mu_2 = \mu_3 = 0$, the relation (15.13) reduces to the standard relation between the Laplace operator on the two-sphere and the angular momentum operators.

Let us now introduce the main object of study: the Dirac–Dunkl operator on the two-sphere. This operator, denoted by Γ, is defined as

$$\Gamma = \vec{\sigma}\cdot\vec{L} + \vec{\mu}\cdot\vec{R}, \tag{15.14}$$

where $\vec{\mu} = (\mu_1, \mu_2, \mu_3)$ and $\vec{R} = (R_1, R_2, R_3)$. When $\mu_1 = \mu_2 = \mu_3 = 0$, all reflections disappear and (15.14) reduces to the standard Hamiltonian describing spin-orbit interaction. The operator (15.14) is linked to the spherical Laplace–Dunkl operator by a quadratic relation. Upon using the equations (15.4), (15.12) and (15.13), one finds that

$$\Gamma^2 + \Gamma = -\Delta_{S^2} + (\mu_1+\mu_2+\mu_3)(\mu_1+\mu_2+\mu_3+1). \tag{15.15}$$

A relation akin to (15.15) was derived in [19]; it involved a scalar operator instead of Γ. The Dirac–Dunkl operator on the two-sphere has a natural algebraic interpretation in terms of the realization (15.6) of the $\mathfrak{osp}(1|2)$ algebra. It corresponds, up to an additive constant, to the so-called sCasimir operator. Indeed, it is verified that

$$\{\Gamma+1, \underline{x}\} = 0, \qquad \{\Gamma+1, \underline{D}\} = 0,$$

and that

$$[\Gamma+1, \mathbb{E}] = 0, \quad [\Gamma+1, ||\vec{x}||^2] = 0, \quad [\Gamma+1, \Delta] = 0.$$

Hence $\Gamma+1$ anticommutes with the odd generators and commutes with the even generators of $\mathfrak{osp}(1|2)$, which is the defining property of the sCasimir operator [22]. The spherical Dirac–Dunkl operator is usually written as a commutator (see for example [4]). For (15.14), one has

$$\Gamma + 1 = \frac{1}{2}\big([\underline{D}, \underline{x}] - 1\big). \tag{15.16}$$

The space of Dunkl monogenics $\mathscr{M}_N(\mathbb{R}^3)$ of degree N is an eigenspace for this operator. Indeed, upon using (15.16), the $\mathfrak{osp}(1|2)$ relations (15.6) and the fact that

$$\underline{D}\,\mathscr{M}_N(\mathbb{R}^3) = 0, \qquad \mathbb{E}\,\mathscr{M}_N(\mathbb{R}^3) = N\mathscr{M}_N(\mathbb{R}^3),$$

one can write

$$(\Gamma+1)\mathscr{M}_N(\mathbb{R}^3) = \frac{1}{2}\Big([\underline{D},\underline{x}]-1\Big)\mathscr{M}_N(\mathbb{R}^3) = \frac{1}{2}\Big(\underline{D}\,\underline{x}-1\Big)\mathscr{M}_N(\mathbb{R}^3)$$
$$= \frac{1}{2}\Big(\{\underline{x},\underline{D}\}-1\Big)\mathscr{M}_N(\mathbb{R}^3) = \frac{1}{2}\Big(2(\mathbb{E}+\gamma_3)-1\Big)\mathscr{M}_N(\mathbb{R}^3),$$

which gives

$$(\Gamma+1)\,\mathscr{M}_N(\mathbb{R}^3) = (N+\mu_1+\mu_2+\mu_3+1)\,\mathscr{M}_N(\mathbb{R}^3), \tag{15.17}$$

where $N = 0,1,2,\ldots$ is a non-negative integer.

15.3 Symmetries of the spherical Dirac–Dunkl operator

In this section, the symmetries of the spherical Dirac–Dunkl operator are obtained and are seen to satisfy the defining relations of the Bannai–Ito algebra.

Introduce the operators J_i defined by

$$J_i = L_i + \sigma_i(\mu_j R_j + \mu_k R_k + 1/2), \qquad i = 1,2,3, \tag{15.18}$$

where (ijk) is a cyclic permutation of $\{1,2,3\}$. The operators J_i are symmetries of the spherical Dirac–Dunkl operator, as it is verified that

$$[\Gamma, J_i] = 0, \qquad i = 1,2,3.$$

The operator Γ can be expressed in terms of the symmetries J_i in the following way:

$$\Gamma = \sigma_1 J_1 + \sigma_2 J_2 + \sigma_3 J_3 - \mu_1 R_1 - \mu_2 R_2 - \mu_3 R_3 - 3/2.$$

A direct calculation shows that the operators J_i satisfy the commutation relations

$$[J_i, J_j] = i\epsilon_{ijk}\Big(J_k + 2\mu_k\,(\Gamma+1)\,\sigma_k R_k + 2\mu_i\mu_j\,\sigma_k R_i R_j\Big). \tag{15.19}$$

The operator Γ also admits the three involutions

$$Z_i = \sigma_i R_i, \qquad Z_i^2 = 1, \qquad i = 1,2,3, \tag{15.20}$$

as symmetry operators, i.e

$$[\Gamma, Z_i] = 0, \qquad i = 1,2,3.$$

The commutation relations between Z_i and J_i read

$$[J_i, Z_i] = 0, \qquad \{J_i, Z_j\} = 0, \qquad \{Z_i, Z_j\} = 0, \qquad i \neq j. \tag{15.21}$$

The involutions (15.20) and the relations (15.21) can be exploited to give another presentation of the symmetry algebra of Γ. Let K_i, $i = 1,2,3$, be defined as follows

$$K_i = -i\, J_i\, Z_j\, Z_k, \tag{15.22}$$

where (ijk) is again a cyclic permutation of $\{1,2,3\}$. Since the operators J_i and Z_i both commute with Γ, it follows that the operators K_i also commute with Γ. Upon combining the relations (15.19) and (15.21), one finds that the symmetries K_i satisfy the commutation relations

$$\{K_i, K_j\} = \epsilon_{ijk}\Big(K_k + 2\mu_k\,(\Gamma+1)R_1R_2R_3 + 2\mu_i\mu_j\Big). \tag{15.23}$$

The invariance algebra (15.23) of the Dirac–Dunkl operator thus has the form of the Bannai–Ito algebra. More precisely, (15.23) can be viewed as a central extension of the Bannai–Ito algebra given the presence of the central element $(\Gamma+1)R_1R_2R_3$ on the right hand side. In terms of the symmetries K_i, the Γ operator reads

$$\Gamma = K_1R_2R_3 + K_2R_1R_3 + K_3R_1R_2 - \mu_1R_1 - \mu_2R_2 - \mu_3R_3 - 3/2. \tag{15.24}$$

The commutation relations between the symmetries K_i and the involutions Z_i are

$$[K_i, Z_j] = 0.$$

The Casimir operator of the Bannai–Ito algebra, denoted by Q, has the expression [27]

$$Q = K_1^2 + K_2^2 + K_3^2. \tag{15.25}$$

It is easily verified that Q commutes with K_i and Z_i for $i = 1,2,3$. A direct calculation shows that in the realization (15.22), the Casimir operator (15.25) can be written as

$$Q = (\Gamma+1)^2 + \mu_1^2 + \mu_2^2 + \mu_3^2 - 1/4.$$

It follows from (15.17) and (15.23) that the space of Dunkl monogenics $\mathscr{M}_N(\mathbb{R}^3)$ of degree N carries representations of the Bannai–Ito algebra (15.1). The precise content of $\mathscr{M}_N(\mathbb{R}^3)$ in representations of the Bannai–Ito algebra will be determined in section 5.

15.4 Representations of the Bannai–Ito algebra

In this section, the representations of the Bannai–Ito algebra corresponding to the realization (15.23) are constructed using ladder operators.

On the space of Dunkl monogenics $\mathscr{M}_N(\mathbb{R}^3)$ of degree N, the symmetries K_i of the Dirac–Dunkl operator satisfy the commutation relations

$$\{K_1, K_2\} = K_3 + \omega_3, \quad \{K_2, K_3\} = K_1 + \omega_1, \quad \{K_3, K_1\} = K_2 + \omega_2, \tag{15.26}$$

with structure constants

$$\omega_3 = 2\mu_1\mu_2 + 2\mu_3\mu_N, \quad \omega_1 = 2\mu_2\mu_3 + 2\mu_1\mu_N, \quad \omega_2 = 2\mu_3\mu_1 + 2\mu_2\mu_N, \tag{15.27}$$

and where we have defined

$$\mu_N = (-1)^N (N + \mu_1 + \mu_2 + \mu_3 + 1). \tag{15.28}$$

On $\mathscr{M}_N(\mathbb{R}^3)$, the Casimir operator (15.25) takes the value

$$Q = (N + \mu_1 + \mu_2 + \mu_3 + 1)^2 + \mu_1^2 + \mu_2^2 + \mu_3^2 - 1/4 \equiv q_N. \tag{15.29}$$

We seek to construct the representations of the Bannai–Ito algebra (15.26) on the space spanned by the orthonormal basis vectors $|N, k\rangle$ characterized by the eigenvalue equations

$$K_3 |N, k\rangle = \lambda_k |N, k\rangle, \qquad Q |N, k\rangle = q_N |N, k\rangle. \tag{15.30}$$

Since the operators K_i are potential observables, it is formally assumed that

$$K_i^\dagger = K_i, \qquad i = 1, 2, 3. \tag{15.31}$$

To characterize the representation, one needs to determine the spectrum of K_3 and the action of the operator K_1 on the basis vectors $|N, k\rangle$.

Introduce the operators K_+ and K_- defined by [27]

$$\begin{aligned} K_+ &= (K_1 + K_2)(K_3 - 1/2) - (\omega_1 + \omega_2)/2, \\ K_- &= (K_1 - K_2)(K_3 + 1/2) + (\omega_1 - \omega_2)/2. \end{aligned} \tag{15.32}$$

Using the defining relations (15.26) and the Hermiticity condition (15.31), it is seen that the operators $K_\pm$ are skew-Hermitian, i.e.

$$K_\pm^\dagger = -K_\pm.$$

Moreover, a direct calculation shows that they satisfy the commutation relations

$$\{K_3, K_+\} = K_+, \quad \{K_3, K_-\} = -K_-, \tag{15.33}$$

whence it follows that

$$[K_3, K_+^2] = 0, \qquad [K_3, K_-^2] = 0.$$

Using (15.33), one can write

$$K_3K_+|N,k\rangle = (K_+ - K_+K_3)|N,k\rangle = (1-\lambda_k)K_+|N,k\rangle,$$
$$K_3K_-|N,k\rangle = (-K_- - K_-K_3)|N,k\rangle = (-1-\lambda_k)K_-|N,k\rangle. \tag{15.34}$$

The above relations indicate that $K_+|N,k\rangle$ and $K_-|N,k\rangle$ are eigenvectors of K_3 with eigenvalues $(1-\lambda_k)$ and $-(1+\lambda_k)$, respectively. One has the two inequalities

$$||K_+|N,k\rangle||^2 = \langle N,k|\, K_+^\dagger K_+ \,|N,k\rangle \geqslant 0, \tag{15.35a}$$
$$||K_-|N,k\rangle||^2 = \langle N,k|\, K_-^\dagger K_- \,|N,k\rangle \geqslant 0. \tag{15.35b}$$

Consider the LHS of (15.35a). The operator $K_+^\dagger$ is of the form

$$K_+^\dagger = (K_3 - 1/2)(K_1 + K_2) - (\omega_1 + \omega_2)/2.$$

Upon using the commutation relations (15.26) and (15.29), it is seen that

$$K_+^\dagger K_+ = (K_3 - 1/2)^2(Q - K_3^2 + K_3 + \omega_3) - (\omega_1 + \omega_2)^2/4. \tag{15.36}$$

Using the above expression in (15.35a) with (15.27) and (15.29), one finds a factorized form for the inequality

$$-(\mu_1 + \mu_2 + 1/2 - \lambda_k)(\mu_N + \mu_3 + 1/2 - \lambda_k)(\lambda_k + \mu_1 + \mu_2 - 1/2)(\lambda_k + \mu_N + \mu_3 - 1/2) \geqslant 0. \tag{15.37}$$

which is equivalent to

$$\begin{cases} \mu_1 + \mu_2 \leqslant |\lambda_k - 1/2| \leqslant N + \mu_1 + \mu_2 + 2\mu_3 + 1, & N \text{ even}, \\ \mu_1 + \mu_2 \leqslant |\lambda_k - 1/2| \leqslant N + \mu_1 + \mu_2 + 1, & N \text{ odd}. \end{cases} \tag{15.38}$$

Proceeding similarly for K_-, one can write

$$K_-^\dagger K_- = (K_3 + 1/2)^2(Q - K_3^2 - K_3 - \omega_3) - (\omega_1 - \omega_2)^2/4. \tag{15.39}$$

and one finds that (15.35b) amounts to

$$-(\mu_1 - \mu_2 - 1/2 - \lambda_k)(\mu_3 - \mu_N - 1/2 - \lambda_k)(\lambda_k + \mu_1 - \mu_2 + 1/2)(\lambda_k + \mu_3 - \mu_N + 1/2) \geqslant 0, \tag{15.40}$$

which can be expressed as

$$\begin{cases} |\mu_1 - \mu_2| \leqslant |\lambda_k + 1/2| \leqslant N + \mu_1 + \mu_2 + 1, & N \text{ even}, \\ |\mu_1 - \mu_2| \leqslant |\lambda_k + 1/2| \leqslant N + \mu_1 + \mu_2 + 2\mu_3 + 1, & N \text{ odd}, \end{cases} \tag{15.41}$$

Upon combining (15.38) and (15.41), we choose

$$\lambda_0 = \mu_1 + \mu_2 + 1/2,$$

whence it follows from (15.37) that

$$K_+|\,N,0\,\rangle = 0.$$

Let us mention that the other choices $\lambda_0 = -(\mu_1+\mu_2+1/2)$ and $\widetilde{\lambda}_0 = \pm(-\mu_1-\mu_2+1/2)$ permitted by (15.38) do not lead to admissible representations.

Starting from the vector $|\,N,0\,\rangle$ with eigenvalue λ_0, one can obtain a string of eigenvectors of K_3 with different eigenvalues by successively applying K_+ and K_-. The eigenvalues

$$\lambda_k = (-1)^k(k+\mu_1+\mu_2+1/2), \qquad k=0,1,2,3,\ldots. \tag{15.42}$$

are obtained by applying K_3 on the vectors

$$|\,N,0\,\rangle, \quad K_-|\,N,0\,\rangle, \quad K_+K_-|\,N,0\,\rangle, \quad K_-K_+K_-|\,N,0\,\rangle,\ldots \tag{15.43}$$

One needs to alternate the application of K_+ and K_- since $K_\pm^2$ commute with K_3 and hence their action does not produce an eigenvector with a different eigenvalue. Using (15.42), one can write

$$||K_+|\,N,k\,\rangle||^2 = \begin{cases} \rho_k^{(N)}, & k \text{ even},\\ \rho_{k+1}^{(N)}, & k \text{ odd},\end{cases}$$

where

$$\rho_k^{(N)} = -k(k+2\mu_1+2\mu_2)(k+\mu_1+\mu_2+\mu_3+\mu_N)(k+\mu_1+\mu_2-\mu_3-\mu_N), \tag{15.44}$$

and also

$$||K_-|\,N,k\,\rangle||^2 = \begin{cases} \sigma_{k+1}^{(N)}, & k \text{ even},\\ \sigma_k^{(N)}, & k \text{ odd},\end{cases}$$

with

$$\sigma_k^{(N)} = -(k+2\mu_1)(k+2\mu_2)(k+\mu_1+\mu_2-\mu_3+\mu_N)(k+\mu_1+\mu_2+\mu_3-\mu_N). \tag{15.45}$$

It is verified that the positivity conditions $\rho_k^{(N)}>0$ and $\sigma_k^{(N)}>0$ are satisfied for all $k=0,1,\ldots,N$, provided that $\mu_i \geqslant 0$ for $i=1,2,3$. Following (15.43), (15.44) and (15.45), we define the orthonormal basis vectors $|\,N,k\,\rangle$ from $|\,N,0\,\rangle$ as follows:

$$|\,N,k+1\,\rangle = \begin{cases} \frac{1}{\sqrt{||K_-|\,N,k\,\rangle||^2}}\,K_-|\,N,k\,\rangle, & k \text{ even},\\ \frac{-1}{\sqrt{||K_+|\,N,k\,\rangle||^2}}\,K_+|\,N,k\,\rangle, & k \text{ odd},\end{cases} \tag{15.46}$$

where the phase factor was chosen to ensure the condition $K_{\pm}^{\dagger} = -K_{\pm}$. From (15.36), (15.39), (15.43) and (15.46), the actions of the ladder operators $K_{\pm}$ are seen to have the expressions

$$
\begin{aligned}
K_{+}|N,k\rangle &= \begin{cases} \sqrt{\rho_{k}^{(N)}}\,|N,k-1\rangle, & k \text{ even},\\ -\sqrt{\rho_{k+1}^{(N)}}\,|N,k+1\rangle, & k \text{ odd},\end{cases}\\
K_{-}|N,k\rangle &= \begin{cases} \sqrt{\sigma_{k+1}^{(N)}}\,|N,k+1\rangle, & k \text{ even},\\ -\sqrt{\sigma_{k}^{(N)}}\,|N,k-1\rangle, & k \text{ odd}.\end{cases}
\end{aligned}
\tag{15.47}
$$

As is observed in (15.44) and (15.45), one has $K_{+}|N,N\rangle = 0$ when N is odd and $K_{-}|N,N\rangle = 0$ when N is even. As a result, the representation has dimension $N+1$. Moreover, it immediately follows from the actions (15.47) that the representation is irreducible, as there are no invariant subspaces.

Let us now give the actions of the generators. The eigenvalues of K_3 are of the form

$$
K_3|N,k\rangle = (-1)^{k}(k+\mu_1+\mu_2+1/2)|N,k\rangle, \qquad k = 0,1,\ldots,N.
$$

The action of the operator K_1 in the basis $|N,k\rangle$ can be obtained directly from the definitions (15.32) and the actions (15.47). One finds that K_1 acts in the tridiagonal fashion

$$
K_1|N,k\rangle = U_{k+1}|N,k+1\rangle + V_k|N,k\rangle + U_k|N,k-1\rangle,
$$

with

$$
U_k = \sqrt{A_{k-1}C_k}, \quad V_k = \mu_2+\mu_3+1/2-A_k-C_k,
$$

where the coefficients A_k and C_k read

$$
\begin{aligned}
A_k &= \begin{cases} \frac{(k+2\mu_2+1)(k+\mu_1+\mu_2+\mu_3-\mu_N+1)}{2(k+\mu_1+\mu_2+1)}, & k \text{ even},\\ \frac{(k+2\mu_1+2\mu_2+1)(k+\mu_1+\mu_2+\mu_3+\mu_N+1)}{2(k+\mu_1+\mu_2+1)}, & k \text{ odd},\end{cases}\\
C_k &= \begin{cases} -\frac{k(k+\mu_1+\mu_2-\mu_3-\mu_N)}{2(k+\mu_1+\mu_2)}, & k \text{ even},\\ -\frac{(k+2\mu_1)(k+\mu_1+\mu_2-\mu_3+\mu_N)}{2(k+\mu_1+\mu_2)}, & k \text{ odd}.\end{cases}
\end{aligned}
\tag{15.48}
$$

For $\mu_i \geqslant 0$, $i = 1,2,3$, one has $U_\ell > 0$ for $\ell = 1,\ldots,N$ and $U_0 = U_{N+1} = 0$. Hence in the basis $|N,k\rangle$, the operator K_1 is represented by a symmetric $(N+1)\times(N+1)$ matrix.

It is observed that the commutation relations (15.26) along with the structure constants (15.27) and the Casimir value (15.29) are invariant under any cyclic permutation of the pairs (K_i,μ_i) for $i = 1,2,3$. Consequently, the matrix elements of the generators in other bases, for example bases in which K_1 or K_2 are diagonal, can be obtained directly by applying the corresponding cyclic permutation on the parameters μ_i.

15.5 Eigenfunctions of the spherical Dirac–Dunkl operator

In this section, a basis for the space of Dunkl monogenics $\mathscr{M}_N(\mathbb{R}^3)$ of degree N is constructed using a Cauchy-Kovalevskaia extension theorem. It is shown that the basis functions transform irreducibly under the action of the Bannai–Ito algebra. The wavefunctions are shown to be orthogonal with respect to a scalar product defined as an integral over the 2-sphere.

15.5.1 Cauchy-Kovalevskaia map

Let $\underline{\widetilde{D}}$, $\underline{\widetilde{x}}$ and $\widetilde{\mathbb{E}}$ be defined as follows:

$$\underline{\widetilde{D}} = \sigma_1 T_1 + \sigma_2 T_2, \qquad \underline{\widetilde{x}} = \sigma_1 x_1 + \sigma_2 x_2, \quad \widetilde{\mathbb{E}} = x_1 \partial_{x_1} + x_2 \partial_{x_2}.$$

There is an isomorphism $\mathbf{CK}_{x_3}^{\mu_3} : \mathscr{P}_N(\mathbb{R}^2) \otimes \mathbb{C}^2 \longrightarrow \mathscr{M}_N(\mathbb{R}^3)$, between the space of spinor-valued homogeneous polynomials of degree N in the variables (x_1, x_2) and the space of Dunkl monogenics of degree N in the variables (x_1, x_2, x_3).

Proposition 1. *The isomorphism $\mathbf{CK}_{x_3}^{\mu_3}$ between $\mathscr{P}_N(\mathbb{R}^2) \otimes \mathbb{C}^2$ and $\mathscr{M}_N(\mathbb{R}^3)$ has the explicit expression*

$$\mathbf{CK}_{x_3}^{\mu_3} = {}_0F_1\left(\begin{matrix} - \\ \mu_3 + 1/2 \end{matrix} \;\middle|\; -\left(\frac{x_3 \underline{\widetilde{D}}}{2}\right)^2 \right) - \frac{\sigma_3 x_3 \underline{\widetilde{D}}}{2\mu_3 + 1} \; {}_0F_1\left(\begin{matrix} - \\ \mu_3 + 3/2 \end{matrix} \;\middle|\; -\left(\frac{x_3 \underline{\widetilde{D}}}{2}\right)^2 \right), \tag{15.49}$$

where ${}_pF_q$ is the generalized hypergeometric series [1].

Proof. Let $p(x_1, x_2) \in \mathscr{P}_n(\mathbb{R}^2) \otimes \mathbb{C}^2$. We set

$$\mathbf{CK}_{x_3}^{\mu_3}[p(x_1, x_2)] = \sum_{\alpha=0}^{n} (\sigma_3 x_3)^\alpha p_\alpha(x_1, x_2),$$

with $p_0(x_1, x_2) \equiv p(x_1, x_2)$ and $p_\alpha(x_1, x_2) \in \mathscr{P}_{n-\alpha}(\mathbb{R}^2) \otimes \mathbb{C}^2$ and we determine the $p_\alpha(x_1, x_2)$ such that $\mathbf{CK}_{x_3}^{\mu_3}[p(x_1, x_2)]$ is in the kernel of $\underline{D}$. One has

$$\begin{aligned} \underline{D}\, \mathbf{CK}_{x_3}^{\mu_3}[p(x_1, x_2)] &= \sum_{\alpha=0}^{n} (-\sigma_3 x_3)^\alpha (\sigma_1 T_1 + \sigma_2 T_2) p_\alpha(x_1, x_2) + \sum_{\alpha=1}^{n} \sigma_3^{\alpha+1} (T_3 x_3^\alpha) p_\alpha(x_1, x_2) \\ &= \sum_{\alpha=0}^{n} (-\sigma_3 x_3)^\alpha (\sigma_1 T_1 + \sigma_2 T_2) p_\alpha(x_1, x_2) + \sum_{\alpha=1}^{n} \sigma_3^{\alpha+1} \left[\alpha + \mu_3(1 - (-1)^\alpha)\right] x_3^{\alpha-1} p_\alpha(x_1, x_2). \end{aligned}$$

Imposing the condition $\underline{D}\, \mathbf{CK}_{x_3}^{\mu_3}[p(x_1, x_2)] = 0$ leads to the equations

$$\sum_{\alpha=0}^{n} (-1)^{\alpha+1} (\sigma_3 x_3)^\alpha (\sigma_1 T_1 + \sigma_2 T_2) p_\alpha(x_1, x_2) = \sum_{\alpha=0}^{n-1} (\sigma_3 x_3)^\alpha [\alpha + \mu_3(1 + (-1)^\alpha)] p_{\alpha+1}(x_1, x_2),$$

from which one finds that

$$
\begin{aligned}
p_{2\alpha}(x_1,x_2) &= \left[\frac{(-1)^{\alpha}}{2^{2\alpha}\,\alpha!\,(\mu_3+1/2)_{\alpha}}\right](\sigma_1 T_1+\sigma_2 T_2)^{2\alpha}\,p(x_1,x_2),\\
p_{2\alpha+1}(x_1,x_2) &= \left[\frac{(-1)^{\alpha+1}}{2^{2\alpha+1}\,\alpha!\,(\mu_3+1/2)(\mu_3+3/2)_{\alpha}}\right](\sigma_1 T_1+\sigma_2 T_2)^{2\alpha+1}\,p(x_1,x_2),
\end{aligned}
$$

where $(a)_n$ stands for the Pochhammer symbol. It is seen that the above corresponds to the hypergeometric expression (15.49). □

The inverse of the isomorphism $\mathbf{CK}_{x_3}^{\mu_3}$ is clearly given by I_{x_3} with $I_{x_3}f(x_1,x_2,x_3)=f(x_1,x_2,0)$. When $\mu_3=0$, the operator $\mathbf{CK}_{x_3}^{\mu_3}$ reduces to the well-known Cauchy-Kovalevskaia extension operator for the standard Dirac operator, as determined in [5]. It is manifest that proposition 1 can be extended to any dimension. Thus, in a similar fashion, one has the isomorphism

$$
\mathbf{CK}_{x_2}^{\mu_2}:\mathscr{P}_k(\mathbb{R})\otimes\mathbb{C}^2\longrightarrow\mathscr{M}_k(\mathbb{R}^2),
$$

between the space of spinor-valued homogeneous polynomials in the variable x_1 and the space of Dunkl monogenics of degree k in the variables (x_1,x_2). This isomorphism has the explicit expression

$$
\mathbf{CK}_{x_2}^{\mu_2}={}_0F_1\left(\begin{matrix}-\\ \mu_2+1/2\end{matrix}\;\middle|\;-\left(\frac{x_2\,\sigma_1 T_1}{2}\right)^2\right)-\frac{\sigma_2\,x_2\,(\sigma_1 T_1)}{2\mu_2+1}\,{}_0F_1\left(\begin{matrix}-\\ \mu_2+3/2\end{matrix}\;\middle|\;-\left(\frac{x_2\,\sigma_1 T_1}{2}\right)^2\right). \tag{15.50}
$$

15.5.2 A basis for $\mathscr{M}_N(\mathbb{R}^3)$

Let us now show how a basis for the space of Dunkl monogenics of degree N in $\mathbb{R}^3$ can be constructed using the $\mathbf{CK}_{x_i}^{\mu_i}$ extension operators and the Fischer decomposition theorem (15.8). Let $\chi_+=(1,0)^{\top}$ and $\chi_-=(0,1)^{\top}$ denote the basis spinors; one has $\mathbb{C}^2=\mathrm{Span}\{\chi_\pm\}$. Consider the following tower of **CK** extensions and Fischer decompositions:

$$
\begin{array}{ccccc}
 & & & \mathscr{P}_N(\mathbb{R}^2)\otimes\mathbb{C}^2 \xrightarrow[\mathbf{CK}_{x_3}^{\mu_3}]{} \mathscr{M}_N(\mathbb{R}^3) & \\
 & & & \| & \\
\mathrm{Span}\{x_1^N\,\chi_\pm\} = \mathscr{P}_N(\mathbb{R})\otimes\mathbb{C}^2 & \xrightarrow[\mathbf{CK}_{x_2}^{\mu_2}]{} & \mathscr{M}_N(\mathbb{R}^2) \rightsquigarrow & \mathscr{M}_N(\mathbb{R}^2) & \\
 & & & \oplus & \\
\mathrm{Span}\{x_1^{N-1}\,\chi_\pm\} = \mathscr{P}_{N-1}(\mathbb{R})\otimes\mathbb{C}^2 & \xrightarrow[\mathbf{CK}_{x_2}^{\mu_2}]{} & \mathscr{M}_{N-1}(\mathbb{R}^2) \rightsquigarrow & \underline{\tilde{x}}\mathscr{M}_{N-1}(\mathbb{R}^2) & \\
 & & & \oplus & \\
\vdots & & & \vdots & \\
 & & & \oplus & \\
\mathrm{Span}\{x_1^k\,\chi_\pm\} = \mathscr{P}_k(\mathbb{R})\otimes\mathbb{C}^2 & \xrightarrow[\mathbf{CK}_{x_2}^{\mu_2}]{} & \mathscr{M}_k(\mathbb{R}^2) \rightsquigarrow & \underline{\tilde{x}}^{N-k}\mathscr{M}_k(\mathbb{R}^2) & \sim \psi_{k,\pm}^{(N)} \\
 & & & \oplus & \\
\vdots & & & \vdots & \\
 & & & \oplus & \\
\mathrm{Span}\{x_1\,\chi_\pm\} = \mathscr{P}_1(\mathbb{R})\otimes\mathbb{C}^2 & \xrightarrow[\mathbf{CK}_{x_2}^{\mu_2}]{} & \mathscr{M}_1(\mathbb{R}^2) \rightsquigarrow & \underline{\tilde{x}}^{N-1}\mathscr{M}_1(\mathbb{R}^2) & \\
 & & & \oplus & \\
\mathrm{Span}\{\chi_\pm\} = \mathscr{P}_0(\mathbb{R})\otimes\mathbb{C}^2 & \xrightarrow[\mathbf{CK}_{x_2}^{\mu_2}]{} & \mathscr{M}_0(\mathbb{R}^2) \rightsquigarrow & \underline{\tilde{x}}^{N}\mathscr{M}_0(\mathbb{R}^2) &
\end{array}
$$

Diagram 1. Horizontally, application of the **CK** map and multiplication by $\underline{\tilde{x}}$. Vertically, Fischer decomposition theorem for $\mathscr{P}_N(\mathbb{R}^2)\otimes\mathbb{C}^2$.

As can be seen from the above diagram, the spinors

$$\psi_{k,\pm}^{(N)} = \mathbf{CK}_{x_3}^{\mu_3}\Big[\underline{\tilde{x}}^{N-k}\,\mathbf{CK}_{x_2}^{\mu_2}[x_1^k]\Big]\chi_\pm, \qquad k=0,1,\ldots,N, \tag{15.51}$$

provide a basis for the space of Dunkl monogenics of degree N in (x_1,x_2,x_3). The basis spinors (15.51) can be calculated explicitly. To perform the calculation, one needs the identities

$$
\begin{aligned}
\underline{\tilde{D}}^{2\alpha}\,\underline{\tilde{x}}^{2\beta}\,M_k &= 2^{2\alpha}(-\beta)_\alpha(1-k-\beta-\gamma_2)_\alpha\,\underline{\tilde{x}}^{2\beta-2\alpha}\,M_k,\\
\underline{\tilde{D}}^{2\alpha+1}\,\underline{\tilde{x}}^{2\beta}\,M_k &= \beta\,2^{2\alpha+1}(1-\beta)_\alpha(1-k-\beta-\gamma_2)_\alpha\,\underline{\tilde{x}}^{2\beta-2\alpha-1}M_k,\\
\underline{\tilde{D}}^{2\alpha}\,\underline{\tilde{x}}^{2\beta+1}\,M_k &= 2^{2\alpha}\,(-\beta)_\alpha\,(-k-\beta-\gamma_2)_\alpha\,\underline{\tilde{x}}^{2\beta-2\alpha+1}M_k,\\
\underline{\tilde{D}}^{2\alpha+1}\,\underline{\tilde{x}}^{2\beta+1}\,M_k &= (k+\beta+\gamma_2)\,2^{2\alpha+1}(-\beta)_\alpha(1-k-\beta-\gamma_2)_\alpha\,\underline{\tilde{x}}^{2\beta-2\alpha}\,M_k,
\end{aligned}
\tag{15.52}
$$

where $M_k\in\mathscr{M}_k(\mathbb{R}^2)$ and $\gamma_2=\mu_1+\mu_2+1$. The formulas (15.52), given in [3] for arbitrary dimension, are easily obtained from the relations

$$\underline{\tilde{D}}\,\underline{\tilde{x}}^{2\beta}M_k = 2\beta\,\underline{\tilde{x}}^{2\beta-1}M_k, \qquad \underline{\tilde{D}}\,\underline{\tilde{x}}^{2\beta+1}M_k = 2(\beta+k+\gamma_2)\underline{\tilde{x}}^{2\beta}M_k,$$

which follow from the commutation relations

$$[\underline{\tilde{D}},\underline{\tilde{x}}^2] = 2\underline{\tilde{x}}, \quad \{\underline{\tilde{D}},\underline{\tilde{x}}\} = 2(\tilde{\mathbb{E}}+\gamma_2). \tag{15.53}$$

Similar formulas hold in the one-dimensional case. To present the result, we shall need the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, defined as [21]

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\, {}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix} \,\Big|\, \frac{1-x}{2}\right).$$

The following identity:

$$(x+y)^m P_m^{(\alpha,\beta)}\left(\frac{x-y}{x+y}\right) = \frac{(\alpha+1)_m}{m!}\, x^m\, {}_2F_1\left(\begin{matrix} -m, -m-\beta \\ \alpha+1 \end{matrix} \,\Big|\, -\frac{y}{x}\right),$$

will also be needed.

Computing (15.51) using the definitions (15.49), (15.50), the formulas (15.52) and the above identity, a long but otherwise straightforward calculation shows that the basis spinors have the expression

$$\psi_{k,\pm}^{(N)} = q_{N-k}(x_3, \underset{\sim}{x})\, m_k(x_2,x_1)\, \chi_\pm, \qquad k = 0,\ldots,N, \tag{15.54}$$

where

$$m_k(x_1,x_2) = \mathbf{CK}_{x_2}^{\mu_2}[x_1^k].$$

One has

$$q_{N-k}(x_3, \underset{\sim}{x}) = \frac{\beta!}{(\mu_3+1/2)_\beta}(x_1^2+x_2^2+x_3^2)^\beta \times \begin{cases} P_\beta^{(\mu_3-1/2, k+\mu_1+\mu_2)}\left(\frac{x_1^2+x_2^2-x_3^2}{x_1^2+x_2^2+x_3^2}\right) & N-k = 2\beta, \\ \qquad - \frac{\sigma_3 x_3 \underset{\sim}{x}}{x_1^2+x_2^2+x_3^2} P_{\beta-1}^{(\mu_3+1/2, k+\mu_1+\mu_2+1)}\left(\frac{x_1^2+x_2^2-x_3^2}{x_1^2+x_2^2+x_3^2}\right), & \\ \underset{\sim}{x} P_\beta^{(\mu_3-1/2, k+\mu_1+\mu_2+1)}\left(\frac{x_1^2+x_2^2-x_3^2}{x_1^2+x_2^2+x_3^2}\right), & N-k = 2\beta+1, \\ \qquad -\sigma_3 x_3 \left(\frac{k+\beta+\mu_1+\mu_2+1}{\beta+\mu_3+1/2}\right) P_\beta^{(\mu_3+1/2, k+\mu_1+\mu_2)}\left(\frac{x_1^2+x_2^2-x_3^2}{x_1^2+x_2^2+x_3^2}\right), & \end{cases} \tag{15.55}$$

and

$$m_k(x_2,x_1) = \frac{\beta!}{(\mu_2+1/2)_\beta}(x_1^2+x_2^2)^\beta \times \begin{cases} P_\beta^{(\mu_2-1/2,\mu_1-1/2)}\left(\frac{x_1^2-x_2^2}{x_1^2+x_2^2}\right) - \frac{\sigma_2 x_2 \sigma_1 x_1}{x_1^2+x_2^2} P_{\beta-1}^{(\mu_2+1/2,\mu_1+1/2)}\left(\frac{x_1^2-x_2^2}{x_1^2+x_2^2}\right), & k = 2\beta, \\ x_1 P_\beta^{(\mu_2-1/2,\mu_1+1/2)}\left(\frac{x_1^2-x_2^2}{x_1^2+x_2^2}\right) - \sigma_2\, x_2\, \sigma_1 \left(\frac{\beta+\mu_1+1/2}{\beta+\mu_2+1/2}\right) P_\beta^{(\mu_2+1/2,\mu_1-1/2)}\left(\frac{x_1^2-x_2^2}{x_1^2+x_2^2}\right), & k = 2\beta+1. \end{cases} \tag{15.56}$$

15.5.3 Basis spinors and representations of the Bannai–Ito algebra

The basis vectors $\psi^{(N)}_{k,\pm}$ transform irreducibly under the action of the Bannai–Ito algebra. This can be established as follows. By construction, $\psi^{(N)}_{k,\pm} \in \mathscr{M}_N(\mathbb{R}^3)$, and thus (15.17) gives

$$(\Gamma+1)\,\psi^{(N)}_{k,\pm} = (N+\mu_1+\mu_2+\mu_3+1)\,\psi^{(N)}_{k,\pm}.$$

Hence we have

$$Q\psi^{(N)}_{k,\pm} = \big((\Gamma+1)^2+\mu_1^2+\mu_2^2+\mu_3^2-1/4\big) = q_N\,\psi^{(N)}_{k,\pm}, \tag{15.57}$$

as in (15.30). The spinors (15.54) are also eigenvectors of K_3. To prove this result, one first observes that K_3 can be written as

$$K_3 = -\frac{1}{2}\big([\underline{\widetilde{x}},\underline{\widetilde{D}}]+1\big)R_1R_2.$$

Since K_3 acts only on the variables (x_1,x_2) and since $[K_3,\underline{\widetilde{x}}]=0$, one has

$$\begin{aligned}
K_3\,\psi^{(N)}_{k,\pm} &= K_3\,\mathbf{CK}^{\mu_3}_{x_3}\Big[\underline{\widetilde{x}}^{N-k}\,\mathbf{CK}^{\mu_2}_{x_2}\big[x_1^k\big]\Big]\chi_\pm = \mathbf{CK}^{\mu_3}_{x_3}\Big[\underline{\widetilde{x}}^{N-k}\,K_3\,\mathbf{CK}^{\mu_2}_{x_2}\big[x_1^k\big]\Big]\chi_\pm\\
&= -\frac{(-1)^k}{2}\mathbf{CK}^{\mu_3}_{x_3}\Big[\underline{\widetilde{x}}^{N-k}\,\Big(\underline{\widetilde{x}}\,\underline{\widetilde{D}}-\underline{\widetilde{D}}\,\underline{\widetilde{x}}+1\Big)\mathbf{CK}^{\mu_2}_{x_2}\big[x_1^k\big]\Big]\chi_\pm\\
&= -\frac{(-1)^k}{2}\mathbf{CK}^{\mu_3}_{x_3}\Big[\underline{\widetilde{x}}^{N-k}\,\Big(-2(\widetilde{E}+\gamma_2)+1\Big)\mathbf{CK}^{\mu_2}_{x_2}\big[x_1^k\big]\Big]\chi_\pm,
\end{aligned}$$

where in the last step the commutation relations (15.53) were used. Using the properties

$$\underline{\widetilde{D}}\,\mathbf{CK}^{\mu_2}_{x_2}\big[x_1^k\big]=0,\quad \widetilde{\mathbb{E}}\,\mathbf{CK}^{\mu_2}_{x_2}\big[x_1^k\big]=k\,\mathbf{CK}^{\mu_2}_{x_2}\big[x_1^k\big],\qquad R_1R_2\,\mathbf{CK}^{\mu_2}_{x_2}\big[x_1^k\big]=(-1)^k\,\mathbf{CK}^{\mu_2}_{x_2}\big[x_1^k\big].$$

one finds that

$$K_3\,\psi^{(N)}_{k,\pm} = (-1)^k(k+\mu_1+\mu_2+1/2)\,\psi^{(N)}_{k,\pm}. \tag{15.58}$$

Upon combining (15.57) and (15.58), it is seen that the spinors (15.51) satisfy the defining properties of the basis vectors $|\,N,k\,\rangle$ for the representations of the Bannai–Ito algebra constructed in section 4. The spinors $\psi^{(N)}_{k,\pm}$ however possess an extra label $\pm$ associated to the eigenvalues of the symmetry operator $Z_3=\sigma_3R_3$. Indeed, it is directly verified from the explicit expression (15.55) and (15.56) that one has

$$Z_3\,\psi^{(N)}_{k,\pm} = \pm(-1)^{N-k}\,\psi^{(N)}_{k,\pm}.$$

It follows that each of the two independent sets of basis vectors

$$\{\psi^{(N)}_{k,+}\mid k=0,1,\ldots,N\},\qquad \{\psi^{(N)}_{k,-}\mid k=0,1,\ldots,N\},$$

supports a unitary $(N+1)$-dimensional irreducible representation of the Bannai–Ito algebra as constructed in section 4. As a consequence, the space of Dunkl monogenics $\mathscr{M}_N(\mathbb{R}^3)$ of degree N can be expressed as a direct sum of two such representations. Since $\dim \mathscr{M}_N(\mathbb{R}^3) = 2\times(N+1)$, the dimensions of the spaces match.

15.5.4 Normalized wavefunctions

The wavefunctions (15.54) can be presented in a normalized fashion. We define

$$\Psi_{k,\pm}^{(N)}(x_1,x_2,x_3) = \Theta_{N,k}(x_1,x_2,x_3)\,\Phi_k(x_1,x_2)\,\chi_\pm, \tag{15.59}$$

with

$$\Phi_k(x_1,x_2) = \sqrt{\frac{\beta!\,\Gamma(\beta+\mu_1+\mu_2+1)}{2\Gamma(\beta+\mu_1+1/2)\Gamma(\beta+\mu_2+1/2)}}\,(x_1^2+x_2^2)^\beta$$
$$\times\begin{cases} P_\beta^{(\mu_2-1/2,\mu_1-1/2)}\left(\frac{x_1^2-x_2^2}{x_1^2+x_2^2}\right)\mathbb{1} & k=2\beta,\\ \qquad +\frac{\sigma_1x_1\,\sigma_2x_2}{x_1^2+x_2^2}\,P_{\beta-1}^{(\mu_2+1/2,\mu_1+1/2)}\left(\frac{x_1^2-x_2^2}{x_1^2+x_2^2}\right), & \\ \sqrt{\frac{\beta+\mu_2+1/2}{\beta+\mu_1+1/2}}\,x_1\,P_\beta^{(\mu_2-1/2,\mu_1+1/2)}\left(\frac{x_1^2-x_2^2}{x_1^2+x_2^2}\right)\mathbb{1} & k=2\beta+1,\\ \qquad -\sqrt{\frac{\beta+\mu_1+1/2}{\beta+\mu_2+1/2}}\,\sigma_2\,\sigma_1\,x_2\,P_\beta^{(\mu_2+1/2,\mu_1-1/2)}\left(\frac{x_1^2-x_2^2}{x_1^2+x_2^2}\right), & \end{cases} \tag{15.60}$$

and where

$$\Theta_{N,k}(x_1,x_2,x_3) = \sqrt{\frac{\beta!\,\Gamma(\beta+k+\mu_1+\mu_2+\mu_3+3/2)}{\Gamma(\beta+\mu_3+1/2)\,\Gamma(\beta+k+\mu_1+\mu_2+1)}}$$
$$\times\begin{cases} P_\beta^{(\mu_3-1/2,k+\mu_1+\mu_2+1)}\left(\frac{x_1^2+x_2^2-x_3^2}{x_1^2+x_2^2+x_3^2}\right)\mathbb{1} & N-k=2\beta,\\ \qquad +\frac{(\sigma_1x_1+\sigma_2x_2)\,\sigma_3x_3}{x_1^2+x_2^2+x_3^2}\,P_{\beta-1}^{(\mu_3+1/2,k+\mu_1+\mu_2+1)}\left(\frac{x_1^2+x_2^2-x_3^2}{x_1^2+x_2^2+x_3^2}\right), & \\ \sqrt{\frac{\beta+\mu_3+1/2}{\beta+k+\mu_1+\mu_2+1}}\,(\sigma_1x_1+\sigma_2x_2)\,P_\beta^{(\mu_3-1/2,k+\mu_1+\mu_2+1)}\left(\frac{x_1^2+x_2^2-x_3^2}{x_1^2+x_2^2+x_3^2}\right) & N-k=2\beta+1.\\ \qquad -\sqrt{\frac{k+\beta+\mu_1+\mu_2+1}{\beta+\mu_3+1/2}}\,\sigma_3x_3\,P_\beta^{(\mu_3+1/2,k+\mu_1+\mu_2)}\left(\frac{x_1^2+x_2^2-x_3^2}{x_1^2+x_2^2+x_3^2}\right), & \end{cases} \tag{15.61}$$

In (15.60) and (15.61), the symbol $\mathbb{1}$ stands for the 2×2 identity operator and $\Gamma(x)$ is the standard Gamma function [1]. Introduce the scalar product

$$\langle\Lambda,\Psi\rangle = \int_{S^2}(\Lambda^\dagger\cdot\Psi)\,h(x_1,x_2,x_3)\,\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3, \tag{15.62}$$

where $h(x_1,x_2,x_3)$ is the $\mathbb{Z}_2^3$ invariant weight function [11]

$$h(x_1,x_2,x_3) = |x_1|^{2\mu_1}|x_2|^{2\mu_2}|x_3|^{2\mu_3}.$$

It is directly verified (see for example [13]) that the spherical Dirac-Dunkl operator Γ and its symmetry operators K_i, Z_i are self-adjoint with respect to the scalar product (15.62). Upon writing the wavefunctions (15.59) in the spherical coordinates, it follows from the orthogonality relation of the Jacobi polynomials (see for example [21]) that the wavefunctions (15.59) satisfy the orthogonality relation

$$\langle \Psi^{(N')}_{k',j}, \Psi^{(N)}_{k,j'} \rangle = \delta_{kk'}\delta_{NN'}\delta_{jj'}.$$

15.5.5 Role of the Bannai–Ito polynomials

Let us briefly discuss the role played by the Bannai–Ito polynomials in the present picture. It is known that these polynomials arise as overlap coefficients between the respective eigenbases of any pair of generators of the Bannai–Ito algebra in the representations (15.30) [17, 27]. We introduce the basis $\Upsilon^{(N)}_{s,\pm}$ defined by

$$\Upsilon^{(N)}_{s,\pm} = \widetilde{\Theta}_{N,s}(x_2,x_3,x_1)\,\widetilde{\Phi}_s(x_2,x_3)\,\chi_\pm, \qquad s = 0,\ldots,N, \tag{15.63}$$

where $\widetilde{\Theta}$ and $\widetilde{\Phi}$ are obtained from (15.60) and (15.61) by applying the permutation $(\mu_1,\mu_2,\mu_3) \to (\mu_2,\mu_3,\mu_1)$. It is easily seen from (15.22) and (15.24) that the wavefunctions (15.63) satisfy the eigenvalue equations

$$\begin{aligned}
(\Gamma+1)\,\Upsilon^{(N)}_{s,\pm} &= (N+\mu_1+\mu_2+\mu_3+1)\,\Upsilon^{(N)}_{s,\pm},\\
K_1\,\Upsilon^{(N)}_{s,\pm} &= (-1)^s(s+\mu_2+\mu_3+1/2)\,\Upsilon^{(N)}_{s,\pm},\\
\sigma_3 R_3\,\Upsilon^{(N)}_{s,\pm} &= \pm(-1)^{N-s}\,\Upsilon^{(N)}_{s,\pm}.
\end{aligned}$$

With the scalar product (15.62), the overlap coefficients between the bases $\Psi^{(N)}_{k,\pm}$ and $\Upsilon^{(N)}_{s,\pm}$ are defined as

$$\langle \Upsilon^{(N)}_{s,q}, \Psi^{(N)}_{k,r} \rangle = W^{(N)}_{s,k;q,r}.$$

The coefficients $W^{(N)}_{s,k;q,r}$ can be expressed in terms of the Bannai–Ito polynomials (see [19]).

15.6 Conclusion

In this paper, we considered the Dirac–Dunkl operator on the two-sphere associated to the $\mathbb{Z}_2^3$ Abelian reflection group. We have obtained its symmetries and shown that they generate the Bannai–Ito algebra. We have built the relevant representations of the Bannai–Ito algebra using ladder operators. Finally, using a Cauchy-Kovalevskaia extension theorem, we have constructed the eigenfunctions of the spherical Dirac–Dunkl operator and we have shown that they transform according to irreducible representations of the Bannai–Ito algebra.

As observed in this paper, the formulas (15.1) can be considered as a three-parameter deformation of the algebra sl_2 and as such, it can be considered to have rank one. It would of great interest in the future to generalize the Bannai–Ito algebra to arbitrary rank. In that regard, the study of the Dirac–Dunkl operator in n dimensions associated to the $\mathbb{Z}_2^n$ reflection group is interesting.

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Partie III

Tableau de Bannai–Ito et structures algébriques associées

Introduction

L'une des avancées récentes dans la théorie des polynômes orthogonaux est la découverte de plusieurs nouvelles familles de polynômes orthogonaux hypergéométriques qui correspondent à des limites $q \to -1$ des q-polynômes du tableau de Askey [60, 61, 62, 63]. Tout comme les polynômes du tableau de Askey, les polynômes « -1 » peuvent être organisés au sein d'une hiérarchie appelée *tableau de Bannai–Ito*, dont la construction n'est toujours pas achevée. Au sommet de cette hiérarchie trônent les familles des polynômes de Bannai–Ito et des polynômes de Bannai–Ito complémentaires, qui dépendent chacune de quatre paramètres. Ces deux familles ont plusieurs descendants qui s'obtiennent à partir de limites ou de spécialisations. À l'instar des polynômes du tableau de Askey, les polynômes du tableau de Bannai-Ito sont bispectraux. Une famille de polynômes orthogonaux $\{P_n(x)\}$ est dite bispectrale si en plus d'obéir à la relation de récurrence à trois termes caractéristique de tous les polynômes orthogonaux

$$xP_n(x) = P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x),$$

où $P_{-1}(x) = 0$ et $P_0(x) = 1$ et où $\{b_n\}$, $\{c_n\}$ sont des suites de nombres, elle satisfait aussi à une équation aux valeurs propres

$$\mathscr{L} P_n(x) = \lambda_n P_n(x),$$

où $\mathscr{L}$ est un opérateur différentiel, aux différences ou aux q-différences. La propriété de bispectralité d'une famille de polynômes orthogonaux est importante du point de vue de ses applications physiques potentielles: tous les polynômes orthogonaux qui apparaissent dans un cadre physique sont bispectraux. La caractéristique qui distingue les polynômes -1 de ceux du tableau d'Askey est que les opérateurs $\mathscr{L}$ qu'ils diagonalisent font intervenir des opérateurs de réflexion $RP_n(x) = P_n(-x)$.

Dans cette partie de la thèse, on étudie plusieurs familles de polynômes -1 et on examine les structures algébriques qui leurs sont associées. On démontre d'abord la bispectralité des polynômes de Bannai–Ito complémentaires. On définit ensuite une nouvelle famille de polynômes -1 appelés polynômes de Chihara, que l'on caractérise. Puis, on montre que les polynômes de Bannai–Ito sont les coefficients de Racah de l'algèbre $\mathfrak{osp}(1|2)$. Ceci nous conduit à examiner la structure algébrique qui est associée aux polynômes duaux -1 de Hahn dans le contexte du problème de Clebsch-Gordan de $\mathfrak{osp}(1|2)$. On propose une q-déformation des polynômes de Bannai–Ito en considérant les coefficients de Racah de la superalgèbre quantique $\mathfrak{osp}_q(1|2)$. Finalement, on montre que l'algèbre associée aux q-polynômes de Bannai–Ito, appelée algèbre de q-Bannai–Ito, sert d'algèbre de covariance pour $\mathfrak{osp}_q(1|2)$.

Chapitre 16

Bispectrality of the Complementary Bannai–Ito polynomials

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Abstract. A one-parameter family of operators that have the Complementary Bannai–Ito (CBI) polynomials as eigenfunctions is obtained. The CBI polynomials are the kernel partners of the Bannai–Ito polynomials and also correspond to a $q \to -1$ limit of the Askey-Wilson polynomials. The eigenvalue equations for the CBI polynomials are found to involve second order Dunkl shift operators with reflections and exhibit quadratic spectra. The algebra associated to the CBI polynomials is given and seen to be a deformation of the Askey-Wilson algebra with an involution. The relation between the CBI polynomials and the recently discovered dual -1 Hahn and para-Krawtchouk polynomials, as well as their relation with the symmetric Hahn polynomials, is also discussed.

16.1 Introduction

One of the recent advances in the theory of orthogonal polynomials (OPs) has been the discovery of several new families of ”classical” OPs that correspond to $q \to -1$ limits of q-polynomials of the Askey scheme [20, 22, 25, 26]. The word ”classical” here refers to the fact that in addition to obeying the three-term relation

$$\mathscr{P}_{n+1}(x) + \beta_n \mathscr{P}_n(x) + \gamma_n \mathscr{P}_{n-1}(x) = x\mathscr{P}_n(x),$$

the polynomials $\mathscr{P}_n(x)$ also satisfy an eigenvalue equation of the form

$$\mathscr{L}\mathscr{P}_n(x) = \lambda_n \mathscr{P}_n(x).$$

The novelty of these families of -1 orthogonal polynomials lies in the fact that for each family the operator $\mathscr{L}$ is a differential or difference operator that also contains the reflection operator $Rf(x) = f(-x)$ [24]. Such differential/difference operators are said to be of Dunkl type [4], notwithstanding the fact that the operators $\mathscr{L}$ differ from the standard Dunkl operators in that they preserve the linear space of polynomials of any given maximal degree. In this connection, these -1 OPs have also been referred to as Dunkl orthogonal polynomials.

With the discovery and characterization of these Dunkl polynomials, a -1 scheme of OPs, completing the Askey scheme, is beginning to emerge. At the top of the discrete variable branch of this -1 scheme lie two families of orthogonal polynomials: the Bannai–Ito (BI) polynomials and their kernel partners the Complementary Bannai–Ito polynomials (CBI); both families correspond to different $q \to -1$ limits of the Askey-Wilson polynomials.

The Bannai–Ito polynomials were originally identified by Bannai and Ito themselves in [1] where they recognized that these OPs correspond to the $q \to -1$ limit of the q-Racah polynomials. However, it is only recently [22] that the Dunkl shift operator $\mathscr{L}$ admitting the BI polynomials as eigenfunctions has been constructed. The BI polynomials and their special cases enjoy the Leonard duality property, a property they share with all members of the discrete part of the Askey scheme [1, 14]. This means that in addition to satisfying a three-term recurrence relation, the BI polynomials also obey a three-term difference equation. From the algebraic point of view, this property corresponds to the existence of an associated *Leonard pair* [17].

Amongst the discrete-variable -1 polynomials, there are families that do not possess the Leonard duality property. That is the case of the Complementary Bannai-Ito polynomials and their descendants [20, 22]. This situation is connected to the fact that in these cases the difference operator of the corresponding q-polynomials do not admit a $q \to -1$ limit. In [20], a *five-term* difference equation was nevertheless constructed for the dual -1 Hahn polynomials and the defining Dunkl operator for these polynomials was found.

In this paper, a one-parameter family of Dunkl operators $\mathscr{D}_\alpha$ of which the Complementary Bannai–Ito polynomials are eigenfunctions is derived, thus establishing the bispectrality of the CBI polynomials. The operators of this family involve reflections and are of second order in discrete shifts; they are diagonalized by the CBI polynomials with a quadratic spectrum. The corresponding five-term difference equation satisfied by the CBI polynomials is presented. Moreover, an algebra associated to the CBI polynomials is derived. This quadratic algebra, called the *Complementary Bannai–Ito algebra*, is defined in terms of four generators. It can be seen as a deformation with an involution of the quadratic Hahn algebra $QH(3)$ [8, 30], which is a special case of the Askey-Wilson $AW(3)$ algebra [18, 29].

The paper, which provides a comprehensive description of the CBI polynomials and their properties, is organized in the following way. In Section 1, we present a review of the Bannai–Ito polynomials. In Section 2, we define the Complementary Bannai–Ito polynomials and obtain their recurrence and orthogonality relations. In Section 3, we use a proper $q \to -1$ limit of the Askey-Wilson difference operator to construct an operator $\mathscr{D}$ of which the CBI polynomials are eigenfunctions. We use a "hidden" eigenvalue equation to show that one has in fact a one-parameter family of operators $\mathscr{D}_\alpha$, parametrized by a complex number α, that is diagonalized by the CBI polynomials. In Section 4, we derive the CBI algebra and present some aspects of its irreducible representations. In Section 5, we discuss the relation between the CBI polynomials and three other families of OPs: the dual -1 Hahn, the para-Krawtchouk and the classical Hahn polynomials; these OP families are respectively a limit and two special cases of the CBI polynomials. We conclude with a perspective on the continuum limit and an outlook.

16.2 Bannai–Ito polynomials

The Bannai–Ito polynomials were introduced in 1984 [1] in the complete classification of orthogonal polynomials possessing the Leonard duality property (see Section 4). It was shown that they can be obtained as a $q \to -1$ limit of the q-Racah polynomials and some of their properties were derived. Recently [22], it was observed that the BI polynomials also occur as eigensolutions of a first order Dunkl shift operator. In the following, we review some of the properties of the BI polynomials; we use the presentation of [22].

The monic BI polynomials $B_n(x;\rho_1,\rho_2,r_1,r_2)$, denoted $B_n(x)$ for notational convenience, satisfy the three-term recurrence relation

$$B_{n+1}(x)+(\rho_1-A_n-C_n)B_n(x)+A_{n-1}C_nB_{n-1}(x)=xB_n(x), \tag{16.1}$$

with the initial conditions $B_{-1}(x)=0$ and $B_0(x)=1$. The recurrence coefficients A_n and C_n are given by

$$A_n=\begin{cases}\frac{(n+2\rho_1-2r_1+1)(n+2\rho_1-2r_2+1)}{4(n+g+1)}, & n \text{ even},\\ \frac{(n+2g+1)(n+2\rho_1+2\rho_2+1)}{4(n+g+1)}, & n \text{ odd},\end{cases} \tag{16.2a}$$

$$C_n=\begin{cases}-\frac{n(n-2r_1-2r_2)}{4(n+g)}, & n \text{ even},\\ -\frac{(n+2\rho_2-2r_2)(n+2\rho_2-2r_1)}{4(n+g)}, & n \text{ odd},\end{cases} \tag{16.2b}$$

where

$$g=\rho_1+\rho_2-r_1-r_2.$$

It is seen from the above formulas that the positivity condition $u_n=A_{n-1}C_n>0$ cannot be satisfied for all $n\in\mathbb{N}$ [3]. Hence it follows that the Bannai–Ito polynomials can only form a *finite* set of positive-definite orthogonal polynomials $B_0(x),\ldots,B_N(x)$, which occurs when the "local" positivity condition $u_i > 0$ for $i \in \{1,\ldots,N\}$ and the truncation conditions $u_0 = 0$, $u_{N+1} = 0$ are satisfied. If these conditions are fulfilled, the BI polynomials $B_n(x)$ satisfy the discrete orthogonality relation

$$\sum_{k=0}^{N} w_k B_n(x_k) B_m(x_k) = h_n \delta_{nm},$$

with respect to the positive weight w_k. The spectral points x_k are the simple roots of the polynomial $B_{N+1}(x)$. The explicit formulae for the weight function w_k and the grid points x_k depend on the realization of the truncation condition $u_{N+1} = 0$.

If N is even, it follows from (16.2) that the condition $u_{N+1} = 0$ is tantamount to one of the following requirements:

$$\begin{aligned} &1)\, r_1 - \rho_1 = \frac{N+1}{2}, \quad 2)\, r_2 - \rho_1 = \frac{N+1}{2}, \\ &3)\, r_1 - \rho_2 = \frac{N+1}{2}, \quad 4)\, r_2 - \rho_2 = \frac{N+1}{2}. \end{aligned}$$

For the cases 1) and 2), the grid points have the expression

$$x_k = (-1)^k (k/2 + \rho_1 + 1/4) - 1/4, \tag{16.3}$$

and the weights take the form

$$w_k = \frac{(-1)^\nu}{\ell!} \frac{(\rho_1 - r_1 + 1/2)_{\ell+\nu} (\rho_1 - r_2 + 1/2)_{\ell+\nu} (\rho_1 + \rho_2 + 1)_\ell (2\rho_1 + 1)_\ell}{(\rho_1 + r_1 + 1/2)_{\ell+\nu} (\rho_1 + r_2 + 1/2)_{\ell+\nu} (\rho_1 - \rho_2 + 1)_\ell}, \tag{16.4}$$

where one has $k = 2\ell + \nu$ with $\nu \in \{0,1\}$ and where $(a)_n = a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol. For the cases 3) and 4), the formulae (16.3) and (16.4) hold under the substitution $\rho_1 \leftrightarrow \rho_2$.

If N is odd, it follows from (16.2) that the condition $u_{N+1} = 0$ is equivalent to one of the following restrictions:

$$i)\, \rho_1 + \rho_2 = -\frac{N+1}{2}, \quad ii)\, r_1 + r_2 = \frac{N+1}{2}, \quad iii)\, \rho_1 + \rho_2 - r_1 - r_2 = -\frac{N+1}{2}.$$

The condition $iii)$ leads to a singularity in u_n when $n = (N+1)/2$ and hence only the conditions $i)$ and $ii)$ are admissible. For the case $i)$, the formulae (16.3) and (16.4) hold under the substitution $\rho_1 \leftrightarrow \rho_2$. For the case $ii)$, the spectral points are given by

$$x_k = (-1)^k (r_1 - k/2 - 1/4) - 1/4,$$

and the weight function is given by (16.4) with the substitutions $(\rho_1, \rho_2, r_1, r_2) \to -(r_1, r_2, \rho_1, \rho_2)$.

The Bannai–Ito polynomials can be obtained from a $q \to -1$ limit of the Askey-Wilson polynomials and also have the Bannai–Ito algebra as their characteristic algebra (see [7] and [22]).

16.3 CBI polynomials

In this section we define the *Complementary Bannai–Ito* polynomials through a Christoffel transformation of the Bannai–Ito polynomials. We derive their recurrence relation, hypergeometric representation and orthogonality relations from their kernel properties.

The Complementary Bannai–Ito polynomials $I_n(x;\rho_1,\rho_2,r_1,r_2)$, denoted $I_n(x)$ for convenience, are defined from the BI polynomials $B_n(x)$ by the transformation [22]

$$I_n(x) = \frac{B_{n+1}(x) - A_n B_n(x)}{x-\rho_1}, \tag{16.5}$$

where A_n is as in (16.2). The transformation (16.5) is an example of a Christoffel transformation [16]. It is easily seen from the definition (16.5) that $I_n(x)$ is a monic polynomial of degree n in x. The inverse relation for the CBI polynomials is given by a Geronimus [31] transformation and has the expression

$$B_n(x) = I_n(x) - C_n I_{n-1}(x). \tag{16.6}$$

This formula can be verified by direct substitution of (16.5) in (16.6) which yields back the defining relation (16.1) of the BI polynomials. In the reverse, the substitution of (16.6) in (16.5) yields the three-term recurrence relation [11]

$$I_{n+1}(x) + (\rho_1 - A_n - C_{n+1}) I_n(x) + A_n C_n I_{n-1}(x) = x I_n(x), \tag{16.7}$$

where A_n and C_n are given by (16.2). The recurrence relation (16.7) can be written explicitly as

$$I_{n+1}(x) + (-1)^n \rho_2 I_n(x) + \tau_n I_{n-1}(x) = x I_n(x), \tag{16.8}$$

where τ_n is given by

$$\tau_{2n} = -\frac{n(n+\rho_1-r_1+1/2)(n+\rho_1-r_2+1/2)(n-r_1-r_2)}{(2n+g)(2n+g+1)}, \tag{16.9a}$$

$$\tau_{2n+1} = -\frac{(n+g+1)(n+\rho_1+\rho_2+1)(n+\rho_2-r_1+1/2)(n+\rho_2-r_2+1/2)}{(2n+g+1)(2n+g+2)}, \tag{16.9b}$$

and where $g = \rho_1+\rho_2-r_1-r_2$. One has also the initial conditions $I_0 = 1$ and $I_1 = x - \rho_2$. The CBI polynomials are kernel polynomials of the BI polynomials. Indeed, by noting that

$$A_n = B_{n+1}(\rho_1)/B_n(\rho_1),$$

which follows by induction from (16.1), the transformation (16.5) may be cast in the form

$$I_n(x) = (x-\rho_1)^{-1}\left[B_{n+1}(x) - \frac{B_{n+1}(\rho_1)}{B_n(\rho_1)} B_n(x)\right]. \tag{16.10}$$

It is manifest from (16.10) that $I_n(x)$ are the kernel polynomials associated to $B_n(x)$ with kernel parameter ρ_1 [3]. Since the BI polynomials $B_n(x)$ are orthogonal with respect to a linear functional $\sigma^{(i)}$:

$$\langle \sigma^{(i)}, B_n(x) B_m(x) \rangle = 0, \qquad n \neq m,$$

where the upper index on $\sigma^{(i)}$ designates the possible functionals associated to the various truncation conditions, it follows from (16.10) that we have [3]

$$\langle \sigma^{(i)}, (x-\rho_1) I_n(x) I_m(x) \rangle = 0, \qquad n \neq m. \tag{16.11}$$

Hence the orthogonality and positive-definiteness of the CBI polynomials can be studied using the formulae (16.9) and (16.11) .

It is seen from (16.9) that the condition $\tau_n > 0$ cannot be ensured for all n and hence the Complementary Bannai–Ito polynomials can only form a *finite* system of positive-definite orthogonal polynomials $I_0(x), \ldots, I_N(x)$, provided that the "local" positivity $\tau_n > 0$, $n \in \{1, \ldots, N\}$, and truncation conditions $\tau_0 = 0$ and $\tau_{N+1} = 0$ are satisfied.

When N is even, the truncation conditions $\tau_0 = 0$ and $\tau_{N+1} = 0$ are equivalent to one of the four prescriptions

$$1)\, \rho_2 - r_1 = -\frac{N+1}{2}, \quad 2)\, \rho_2 - r_2 = -\frac{N+1}{2}, \tag{16.12a}$$

$$3)\, \rho_1 + \rho_2 = -\frac{N+2}{2}, \quad 4)\, g = -\frac{N+2}{2}. \tag{16.12b}$$

Since the condition 4) leads to a singularity in τ_n, only the conditions 1), 2) and 3) are admissible. For all three conditions and assuming that the positivity conditions are satisfied, the CBI polynomials enjoy the orthogonality relation

$$\sum_{k=0}^{N} \widetilde{w}_k I_n(x_k) I_m(x_k) = \widetilde{h}_n \delta_{nm}, \tag{16.13}$$

where the spectral points are given by

$$x_k = (-1)^k (k/2 + \rho_2 + 1/4) - 1/4$$

and the positive weights are

$$\widetilde{w}_k = (x_k - \rho_1) w_k,$$

with w_k defined by (16.4) with the substitution $\rho_1 \leftrightarrow \rho_2$.

When N is odd, the truncation conditions $\tau_0 = 0$ and $\tau_{N+1} = 0$ are tantamount to

$$i) r_1 - \rho_1 = \frac{N+2}{2}, \quad ii) r_1 + r_2 = \frac{N+1}{2}, \quad iii) r_2 - \rho_1 = \frac{N+2}{2}. \tag{16.14}$$

If the positivity condition $\tau_n > 0$ is satisfied for $n \in \{1, \ldots, N\}$, the CBI polynomials will enjoy the orthogonality relation (16.13) with respect to the positive definite weight function $\widetilde{w}_k$. When either condition $i)$ or $ii)$ is satisfied, the spectral points are given by

$$x_k = (-1)^k (r_1 - k/2 - 1/4) - 1/4,$$

together with the weight function $\widetilde{w}_k = (x_k - \rho_1) w_k$ where w_k is given by (16.4) with the replacement $(\rho_1, \rho_2, r_1, r_2) = -(r_1, r_2, \rho_1, \rho_2)$. Finally, the orthogonality relation for the truncation condition $iii)$ is obtained from the preceding case under the exchange $r_1 \leftrightarrow r_2$.

Let us now illustrate when positive-definiteness occurs for the CBI polynomials. We first consider the even N case. It is sufficient to take

$$\rho_1 = \left(\frac{\frac{a+b}{2} + c + N}{2}\right), \quad \rho_2 = \left(\frac{\frac{a+b}{2} - 1}{2}\right), \quad r_1 = \left(\frac{\frac{a+b}{2} + N}{2}\right), \quad r_2 = \left(\frac{a-b}{4}\right), \tag{16.15}$$

where a, b and c are arbitrary positive parameters. Assuming (16.15), the recurrence coefficients (16.9) become

$$\tau_n = \begin{cases} \frac{n(N-n+a)(n+c+1)(n+b+c+N+1)}{16(n+g)(n+g+1)}, & n \text{ even}, \\ \frac{(N-n+1)(n+b-1)(n+b+c)(n+a+b+c+N)}{16(n+g)(n+g+1)}, & n \text{ odd}, \end{cases} \tag{16.16}$$

where $g = (b + c - 1)/2$. It is obvious from (16.16) that the positivity and truncation conditions are satisfied for $n \in \{1, \ldots, N\}$; this corresponds to the case 1) of (16.12).

Consider the situation when $N > 1$ is odd. We introduce the parametrization

$$\rho_1 = \left(\frac{\frac{\zeta+\xi}{2} + \chi + N}{2}\right), \quad \rho_2 = \left(\frac{\zeta - \xi}{4}\right), \quad r_1 = \left(\frac{\frac{\zeta+\xi}{2} + N + 1}{2}\right), \quad r_2 = -\left(\frac{\zeta + \xi}{4}\right), \tag{16.17}$$

where ζ, ξ and χ are arbitrary positive parameters. The recurrence coefficients become

$$\tau_n = \begin{cases} \frac{n(N-n+1)(n+\chi)(n+\zeta+\xi+\chi+N+1)}{16(n+g)(n+g+1)}, & n \text{ even}, \\ \frac{(N-n+\xi+1)(n+\zeta)(n+\zeta+\chi)(n+\zeta+\chi+N+1)}{16(n+g)(n+g+1)}, & n \text{ odd}, \end{cases} \tag{16.18}$$

with $g = (\zeta + \chi - 1)/2$. Assuming (16.17), the positivity and truncation conditions are manifestly fulfilled; this corresponds to the condition $ii)$ of (16.14). The other cases can be treated in similar fashion.

It is possible to derive a hypergeometric representation for the CBI polynomials using a method [22, 26] which is analogous to Chihara's construction of symmetric orthogonal polynomials [3] and closely related to the scheme developed in [15] (see also [2]). Given the three-term recurrence relation (16.8), it follows by induction that the polynomials $I_n(x)$ can be written as

$$I_{2n} = R_n(x^2), \qquad I_{2n+1} = (x-\rho_2)Q_n(x^2), \tag{16.19}$$

where $R_n(x^2)$ and $Q_n(x^2)$ are monic polynomials of degree n. It follows directly from (16.19) and (16.8) that the polynomials $R_n(x^2)$ and $Q_n(x^2)$ obey the following system of recurrence relations

$$R_n(z) = Q_n(z) + \tau_{2n}Q_{n-1}(z), \qquad (z-\rho_2^2)Q_n(z) = R_{n+1}(z) + \tau_{2n+1}R_n(z).$$

This system is equivalent to the following pair of equations:

$$R_{n+1}(z) + (\rho_2^2 + \tau_{2n} + \tau_{2n+1})R_n(z) + \tau_{2n-1}\tau_{2n}R_{n-1}(z) = zR_n(z),$$
$$Q_{n+1}(z) + (\rho_2^2 + \tau_{2n+1} + \tau_{2n+2})Q_n(z) + \tau_{2n}\tau_{2n+1}Q_{n-1}(z) = zQ_n(z).$$

These recurrence relations can be identified with those of the Wilson polynomials [13]. From this identification, we obtain

$$R_n(x^2) = \eta_n\, {}_4F_3\left[\begin{matrix} -n, n+g+1, \rho_2+x, \rho_2-x \\ \rho_1+\rho_2+1, \rho_2-r_1+1/2, \rho_2-r_2+1/2 \end{matrix}; 1\right], \tag{16.20a}$$

$$Q_n(x^2) = \iota_n\, {}_4F_3\left[\begin{matrix} -n, n+g+2, \rho_2+1+x, \rho_2+1-x \\ \rho_1+\rho_2+2, \rho_2-r_1+3/2, \rho_2-r_2+3/2 \end{matrix}; 1\right], \tag{16.20b}$$

where ${}_pF_q$ denotes the generalized hypergeometric function [5] and where the normalization coefficients, which ensure that the polynomials are monic, are given by

$$\eta_n = \frac{(\rho_1+\rho_2+1)_n(\rho_2-r_1+1/2)_n(\rho_2-r_2+1/2)_n}{(n+g+1)_n},$$
$$\iota_n = \frac{(\rho_1+\rho_2+2)_n(\rho_2-r_1+3/2)_n(\rho_2-r_2+3/2)_n}{(n+g+2)_n}.$$

Thus the monic CBI polynomials have the hypergeometric representation (16.19). For definiteness and future reference, let us now gather the preceding results in the following proposition.

Proposition 2. *The Complementary Bannai–Ito polynomials $I_n(x;\rho_1,\rho_2,r_1,r_2)$ are the kernel polynomials of the Bannai–Ito polynomials $B_n(x;\rho_1,\rho_2,r_1,r_2)$ with kernel parameter ρ_1. The monic CBI polynomials obey the three-term recurrence relation*

$$I_{n+1}(x) + (-1)^n\rho_2 I_n(x) + \tau_n I_{n-1}(x) = xI_n(x),$$

where τ_n is given by (16.9)*. They have the explicit hypergeometric representation*

$$I_{2n}(x) = R_n(x^2), \quad I_{2n+1}(x) = (x-\rho_2)Q_n(x^2),$$

where $R_n(x^2)$ and $Q_n(x^2)$ are as specified by (16.20)*. If the truncation condition $\tau_{N+1} = 0$ and the positivity condition $\tau_n > 0$, $n \in \{1,\ldots,N\}$, are satisfied, the CBI polynomials obey the orthogonality relation*

$$\sum_{k=0}^{N} \widetilde{w}_k I_n(x_k) I_m(x_k) = \widetilde{h}_n \delta_{nm},$$

with respect to the positive weights $\widetilde{w}_k$. The grid points x_k correspond to the simple roots of the polynomial $I_{N+1}(x)$. The formulas for the weights and grid points depend on the truncation condition. With $w_k(\rho_1,\rho_2,r_1,r_2)$ given as in (16.4)*, one has*

1. *For $r_1 = \frac{N+1}{2} + \rho_2$, $r_2 = \frac{N+1}{2} + \rho_2$ or $\rho_1 = -\frac{N+2}{2} - \rho_2$ with N even:*

$$x_k = (-1)^k(\rho_2 + k/2 + 1/4) - 1/4, \quad \widetilde{w}_k = (x_k - \rho_1) w_k(\rho_2, \rho_1, r_1, r_2).$$

2. *For $r_1 = \frac{N+2}{2} + \rho_1$ or $r_1 = \frac{N+1}{2} - r_2$ with N odd:*

$$x_k = (-1)^k(r_1 - k/2 - 1/4) - 1/4, \quad \widetilde{w}_k = (x_k - \rho_1) w_k(-r_1, -r_2, -\rho_1, -\rho_2).$$

3. *For $r_2 = \frac{N+2}{2} + \rho_1$ with N odd:*

$$x_k = (-1)^k(r_2 - k/2 - 1/4) - 1/4, \quad \widetilde{w}_k = (x_k - \rho_1) w_k(-r_2, -r_1, -\rho_1, -\rho_2).$$

Proof. The proof follows from the above considerations. □

Note that the normalization factor $\widetilde{h}_n$ appearing in (16.13) can easily be evaluated in terms of the product $\tau_1\tau_2\cdots\tau_n$.

The Complementary Bannai–Ito polynomials can be obtained from the Askey-Wilson polynomials upon taking the $q \to -1$ limit [22]. Consider the Askey-Wilson polynomials [13] $p_n(z;a,b,c,d)$

$$p_n(z;a,b,c,d) = a^{-n}(ab,ac,ad;q)_n\, {}_4\phi_3\left(\begin{matrix} q^{-n} & abcdq^{n-1} & az & az^{-1} \\ & ab & ac & ad \end{matrix}\,\middle|\, q\,;q\right), \tag{16.21}$$

where ϕ denotes the basic generalized hypergeometric function [5]. These polynomials depend on the argument $x = (z+z^{-1})/2$ and on four complex parameters a, b, c and d. They obey the recurrence relation [13]

$$\alpha_n p_{n+1}(z) + (a + a^{-1} - \alpha_n - \gamma_n)p_n(z) + \gamma_n p_{n-1}(z) = (z+z^{-1})p_n(z), \tag{16.22}$$

where the coefficients are

$$\alpha_n = \frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{a(1-abcdq^{2n-1})(1-abcdq^{2n})},$$
$$\gamma_n = \frac{a(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n-2})(1-abcdq^{2n-1})}.$$

To recover the CBI polynomials, we consider the parametrization

$$a = ie^{\epsilon(2\rho_1+3/2)}, \quad b = -ie^{\epsilon(2\rho_2+1/2)}, \quad c = ie^{\epsilon(-2r_2+1/2)}, \quad d = ie^{\epsilon(-2r_1+1/2)}, \tag{16.23a}$$
$$q = -e^{\epsilon}, \quad z = ie^{-2\epsilon y}. \tag{16.23b}$$

It can be verified that the limit $q \to -1$ of the Askey-Wilson polynomials

$$\lim_{q\to -1} p_n(z) = p_n^*(y),$$

exists [22] and that $p_n^*(y)$ is a polynomial of degree n in the variable y. Dividing the recurrence relation (16.22) by $1+q$ and taking the limit $\epsilon \to 0$, which amounts to taking $q \to -1$, one finds that the recurrence relation of the limit polynomials $p_n^*(y)$ is

$$\alpha_n^* p_{n+1}^*(y) + (-1)^n \rho_2\, p_n^*(y) + \gamma_n^* p_{n-1}^*(y) = (y-1/4)p_n^*(y)$$

where

$$\alpha_{2n}^* = -\frac{(n+\rho_1+\rho_2+1)(n+g+1)}{(2n+g+1)}, \alpha_{2n+1}^* = -\frac{(n+\rho_1-r_1+3/2)(n+\rho_1-r_2+3/2)}{(2n+g+2)}, \tag{16.24a}$$
$$\gamma_{2n}^* = \frac{n(n-r_1-r_2)}{(2n+g+1)}, \gamma_{2n+1}^* = \frac{(n+\rho_2-r_1+1/2)(n+\rho_2-r_2+1/2)}{(2n+g+2)}. \tag{16.24b}$$

From (16.9) and (16.24), one has the identification

$$\widehat{p_n^*}(y) = I_n(y-1/4), \tag{16.25}$$

where $\widehat{p_n^*}$ are the monic version of the limit polynomials $p_n^*(y)$. Consequently, the CBI polynomial correspond to a $q \to -1$ limit of the Askey-Wilson polynomials, up to a shift in argument. This property will be used in the next section to construct a Dunkl operator that has the CBI polynomials as eigenfunctions.

16.4 Bispectrality of CBI polynomials

In this section, we obtain a family of second order Dunkl shift operators for which the Complementary Bannai–Ito polynomials are eigenfunctions with eigenvalues quadratic in n. This family will be constructed from a limit of a quadratic combination of the Askey-Wilson q-difference operator. We shall refer to these operators as the defining operators of the CBI polynomials.

Consider the Askey-Wilson polynomials $p_n(x)$ defined by (16.21). They obey the q-difference equation [13]

$$\Big(\Omega(z)E_{z,q} + \Omega(z^{-1})E_{z,q^{-1}} - (\Omega(z) + \Omega(z^{-1}))\mathbb{I}\Big)p_n(z) = \Lambda_n p_n(z), \tag{16.26}$$

where $E_{z,q}f(z) = f(qz)$ is the q-shift operator and $\mathbb{I}$ denotes the identity. The lhs of (16.26) is the Askey-Wilson operator. The eigenvalues take the form

$$\Lambda_n = (q^{-n} - 1)(1 - abcdq^{n-1}),$$

and the coefficient $\Omega(z)$ is given by

$$\Omega(z) = \frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)}.$$

We now consider the limiting form of the q-difference equation (16.26) when $q \to -1$. As done previously, we choose the parametrization (16.23), which correspond to the CBI polynomials. We already showed that the Askey-Wilson polynomials $p_n(z)$ become the Complementary Bannai–Ito polynomials $p_n^*(y)$. In the limit $q \to -1$, the q-shift operation $p_n(z) \to p_n(qz)$ becomes $p_n^*(y) \to p_n^*(-y+1/2)$ while $p_n(z) \to p_n(q^{-1}z)$ is reduced to $p_n^*(y) \to p_n^*(-y-1/2)$.

It is natural to expect that in the limit $q \to -1$, the equation (16.26) will yield a defining operator for the CBI polynomials. However a direct computation shows that the limit $\epsilon \to 0$ of the equation (16.26) with the parametrization (16.23) *does not exist*. It is hence impossible to find the desired operator for the CBI polynomials directly from a limiting procedure on equation (16.26). Nevertheless, it is possible to work around this difficulty by choosing an appropriate *quadratic* combination of the Askey-Wilson operator that survives the limit $q \to -1$. A similar procedure was used in [20] to establish the bispectrality of the dual -1 Hahn polynomials.

Let $\mathscr{O}$ denote the Askey-Wilson operator

$$\mathscr{O} = \Omega(z)E_{z,q} + \Omega(z^{-1})E_{z,q^{-1}} - (\Omega(z) + \Omega(z^{-1}))\mathbb{I},$$

which acts on the space of functions $f(z)$ of argument z. We consider the following quadratic combination

$$\mathscr{T} = c^{(2)}\mathscr{O}^2 + c^{(1)}\mathscr{O}, \tag{16.27}$$

with

$$c^{(2)} = \frac{1}{16(1+q)^2}, \qquad c^{(1)} = \frac{1}{4}\left(\frac{1}{(q+1)^2} - \frac{g+1}{q+1}\right),$$

where $g = \rho_1 + \rho_2 - r_1 - r_2$. Since the operator $\mathscr{O}$ acts diagonally on the Askey-Wilson polynomials, we have

$$\mathscr{T} p_n(z) = (c^{(2)}\Lambda_n^2 + c^{(1)}\Lambda_n)p_n(x). \tag{16.28}$$

Upon taking the limit $\epsilon \to 0$ with the parametrization (16.23), the relation (16.28) becomes

$$\begin{aligned}\Phi_1(y)p_n^*(y+1) + \{\Phi_5(y) - \Phi_2(y) - \Phi_3(y)\}p_n^*(1/2-y) + \{\Phi_3(y) - \Phi_4(y) - \Phi_5(y)\}p_n^*(y)\\ + \{\Phi_4(y) - \Phi_1(y)\}p_n^*(-y-1/2) + \Phi_2(y)p_n^*(y-1) = \kappa_n p_n^*(y)\end{aligned} \tag{16.29}$$

where the eigenvalues are

$$\kappa_{2n} = n^2 + (g+1)n, \qquad \kappa_{2n+1} = n^2 + (g+2)n + g^2 + 2g + 5/4.$$

The coefficients $\Phi_i(y)$ are given by

$$\begin{aligned}
\Phi_1(y) &= \frac{(y+\rho_1+3/4)(y+\rho_2+3/4)(y-r_1+1/4)(y-r_2+1/4)}{4(y+1/4)(y+3/4)},\\
\Phi_2(y) &= \frac{(y-\rho_1-5/4)(y-\rho_2-1/4)(y+r_1-3/4)(y+r_2-3/4)}{4(y-1/4)(y-3/4)},\\
\Phi_3(y) &= \frac{(y+\rho_1+3/4)(y-\rho_2-1/4)(y-r_1+1/4)(y-r_2+1/4)}{4(y-1/4)(y+1/4)},\\
\Phi_4(y) &= \frac{(y+\rho_1+3/4)(y+\rho_2-1/4)(y-r_1+1/4)(y-r_2+1/4)}{4(y-1/4)(y+1/4)},\\
\Phi_5(y) &= \frac{(y-\rho_2-1/4)}{4(y-1/4)}\left\{2y^2 - y + v\right\},
\end{aligned}$$

where v takes the form

$$v = r_1 + r_2 + 2r_1r_2 - 2\rho_1 - 2(r_1+r_2)\rho_1 - 4\rho_2 + 1/8 - 2g^2.$$

By the identification (16.25), the relation (16.29) gives the Complementary Bannai–Ito polynomials $I_n(x)$ as eigenfunctions of a second order Dunkl shift operator, hence establishing their bispectrality property. In operator form, the equation (16.29) may be rewritten as

$$\mathscr{H} I_n(y-1/4) = \kappa_n I_n(y-1/4),$$

where $\mathscr{H}$ has the expression

$$\mathscr{H} = \Phi_1 T^1 + (\Phi_4 - \Phi_1)T^{1/2}R + (\Phi_3 - \Phi_4 - \Phi_5)\mathbb{I} + (\Phi_5 - \Phi_2 - \Phi_3)T^{-1/2}R + \Phi_2 T^{-1},$$

where $T^h f(y) = f(y+h)$ and $Rf(y) = f(-y)$. Upon applying the unitary transformation

$$\widetilde{\mathscr{H}} = T^{1/4}\mathscr{H}T^{-1/4},$$

on the operator $\mathscr{H}$ and changing the variable from y to x, the eigenvalue equation (16.29) for the CBI polynomials becomes

$$\widetilde{\mathscr{H}} I_n(x) = \kappa_n I_n(x),$$

where we have

$$\widetilde{\mathscr{H}} = \widetilde{\Phi}_1 T^+ + (\widetilde{\Phi}_4 - \widetilde{\Phi}_1)T^+ R + (\widetilde{\Phi}_3 - \widetilde{\Phi}_4 - \widetilde{\Phi}_5)\mathbb{I} + (\widetilde{\Phi}_5 - \widetilde{\Phi}_2 - \widetilde{\Phi}_3)R + \widetilde{\Phi}_2 T^-, \tag{16.30}$$

with $T^+ = T^1$ and $T^- = T^-$ the usual shift operators in x. The coefficients now have the expression

$$\widetilde{\Phi}_i = \Phi_i(x + 1/4).$$

We now turn to the study of the uniqueness of the operator $\mathscr{H}$ which defines the eigenvalue equation of the Complementary Bannai–Ito polynomials (apart from trivial affine transformations). Quite strikingly, a one-parameter family of such operators can be constructed. This peculiarity is due to the presence of a ”hidden” symmetry in the CBI polynomials. To see this, we recall the relation (16.19) for the CBI polynomials

$$I_{2n} = R_n(x^2), \qquad I_{2n+1} = (x - \rho_2)Q_n(x^2),$$

where $R_n(x^2)$ and $Q_n(x^2)$ are monic polynomials of degree n. From the above relation, it is easily seen that

$$I_{2n}(-x) = I_{2n}(x), \quad \text{and} \quad I_{2n+1}(-x) = \frac{(x+\rho_2)}{(\rho_2 - x)} I_{2n+1}(x).$$

The above equations are equivalent to the following non-trivial "hidden" eigenvalue equation for the CBI polynomials

$$\frac{(\rho_2 - x)}{2x}\big(I_n(-x) - I_n(x)\big) = \mu_n I_n(x),$$

where $\mu_{2n}=0$ and $\mu_{2n+1}=1$. In operator form, we write

$$\frac{(x-\rho_2)}{2x}\Big(\mathbb{I}-R\Big)I_n(x)=\mathscr{U}I_n(x)=\mu_n I_n(x). \tag{16.31}$$

The equation (16.31) indicates that adding $\alpha\mathscr{U}$ to the operator (16.30) will give another eigenvalue equation for the Complementary Bannai–Ito polynomials. The modified operator

$$\widetilde{\mathscr{H}}'=\widetilde{\mathscr{H}}+\alpha\mathscr{U},$$

will have the same spectrum as $\widetilde{\mathscr{H}}$ in the even sector; in the odd sector, the eigenvalues will differ by the constant parameter α.

For definiteness and future reference, let us now collect the preceding results in the following theorem.

Theorem 1. *Let $\mathscr{D}_0$ be the second order Dunkl shift operator acting on the space of functions $f(x)$ of argument x*

$$\mathscr{D}_0=A(x)T^{+}+B(x)T^{-}+C(x)R+D(x)T^{+}R-(A(x)+B(x)+C(x)+D(x))\mathbb{I}, \tag{16.32}$$

where $T^{\pm}f(x)=f(x\pm1)$ and $Rf(x)=f(-x)$, with the coefficients

$$A(x)=\frac{(x+\rho_1+1)(x+\rho_2+1)(2x-2r_1+1)(2x-2r_2+1)}{8(x+1)(2x+1)},$$

$$B(x)=\frac{(x-\rho_2)(x-\rho_1-1)(2x+2r_1-1)(2x+2r_2-1)}{8x(2x-1)},$$

$$C(x)=\frac{(x-\rho_2)(4x^2+\omega)}{8x}-\frac{(x-\rho_2)(x+\rho_1+1)(2x-2r_1+1)(2x-2r_2+1)}{8x(2x+1)}-B(x),$$

$$D(x)=\frac{\rho_2(x+\rho_1+1)(2x-2r_1+1)(2x-2r_2+1)}{8x(x+1)(2x+1)},$$

and with

$$\omega=4\rho_1-4(r_1+r_2)\rho_1+4r_1r_2-6(r_1+r_2)+5.$$

Furthermore, let $\alpha\in\mathbb{C}$ be a complex number and denote the monic Complementary Bannai-Ito polynomials by $I_n(x)$. Then the following eigenvalue equation is satisfied:

$$\mathscr{D}_\alpha I_n(x)=\Lambda_n^{(\alpha)}I_n(x), \tag{16.33}$$

where the eigenvalues are

$$\Lambda_{2n}^{(\alpha)}=n^2+(g+1)n,\qquad \Lambda_{2n+1}^{(\alpha)}=n^2+(g+2)n+\alpha, \tag{16.34}$$

and where we have defined

$$\mathscr{D}_\alpha=\mathscr{D}_0+\alpha\,\frac{(x-\rho_2)}{2x}(\mathbb{I}-R). \tag{16.35}$$

Proof. The result follows from the above considerations. □

We now discuss the CBI polynomials in the context of the Leonard duality. A family of orthogonal polynomials $\mathscr{P}_n(x)$ is said to possess the *Leonard duality property* if it satisfies both a three-term recurrence relation with respect to n and a three-term difference equation of the form

$$\theta(x_k)\mathscr{P}_n(x_{k+1})+\nu(x_k)\mathscr{P}_n(x_k)+\mu(x_k)\mathscr{P}_n(x_{k-1})=\vartheta_n\mathscr{P}_n(x_k),$$

on a discrete set of points x_k, $k\in\mathbb{Z}$. The classification of the polynomials with this property was first accomplished by Leonard in [14]; his theorem was later generalized to include infinite dimensional grids by Bannai and Ito [1]. It turns out that the Complementary Bannai–Ito polynomials lie beyond the scope of the Leonard duality. Indeed, the operators $\mathscr{D}_\alpha$ can be used to show that the CBI polynomials obey a *five-term* difference equation on an infinite-dimensional grid. This result is obtained in the following way.

First consider the grid x_k defined by

$$x_k=(-1)^k(k/2+h+1/4)-1/4,\qquad k\in\mathbb{Z},\tag{16.36}$$

where h is an arbitrary real parameter. It is easily seen that the grid (16.36) is preserved by the operators appearing in (16.32). Explicitly, we have

$$T^+x_k=\begin{cases}x_{k+2} & k\text{ even},\\ x_{k-2} & k\text{ odd},\end{cases}\qquad T^-x_k=\begin{cases}x_{k-2} & k\text{ even},\\ x_{k+2} & k\text{ odd},\end{cases}$$

$$Rx_k=\begin{cases}x_{k-1} & k\text{ even},\\ x_{k+1} & k\text{ odd},\end{cases}\qquad T^+R\,x_k=\begin{cases}x_{k+1} & k\text{ even},\\ x_{k-1} & k\text{ odd}.\end{cases}$$

Referring to $\mathscr{D}_0$, one finds the following five-term difference equation for the CBI polynomials:

$$\begin{aligned}u(x_k)I_n(x_{k+2})+v(x_k)I_n(x_{k+1})+m(x_k)I_n(x_k)&\\ +t(x_k)I_n(x_{k-1})+r(x_k)I_m(x_{k-2})&=\Lambda_n^{(0)}I_n(x_k)\end{aligned}\tag{16.37}$$

where we have

$$u(x_k)=\begin{cases}A(x_k) & k\text{ even},\\ B(x_k) & k\text{ odd},\end{cases}\qquad v(x_k)=\begin{cases}D(x_k) & k\text{ even},\\ C(x_k) & k\text{ odd},\end{cases}$$

$$t(x_k)=\begin{cases}C(x_k) & k\text{ even},\\ D(x_k) & k\text{ odd},\end{cases}\qquad r(x_k)=\begin{cases}B(x_k) & k\text{ even},\\ A(x_k) & k\text{ odd},\end{cases}$$

and $-m(x_k) = u(x_k) + v(x_k) + t(x_k) + r(x_k)$. A similar relation can be found for any value of α. Moreover, it is possible to obtain another 5-term difference equation by considering the alternative grid

$$\widetilde{x}_k = (-1)^k (h - k/2 - 1/4) - 1/4, \qquad k \in \mathbb{Z},$$

and proceeding along the same lines.

16.5 The CBI algebra

The Bannai–Ito polynomials have as an underlying algebraic structure the so-called BI algebra [22], which corresponds to a $q \to -1$ limit of the Askey-Wilson ($AW(3)$) algebra [29]. The algebra $AW(3)$ and the related concept of Leonard pairs [17, 19, 23], describe polynomials which possess the Leonard duality. In this section, we obtain the algebraic structure that encodes the properties of the CBI polynomials.

We begin by a formal definition of the CBI algebra.

Definition 2. The Complementary Bannai–Ito (CBI) algebra is generated by the elements κ_1, κ_2, κ_3 and the involution r satisfying the relations

$$[\kappa_1, r] = 0, \qquad \{\kappa_2, r\} = 2\delta_3, \qquad \{\kappa_3, r\} = 0, \qquad [\kappa_1, \kappa_2] = \kappa_3, \tag{16.38a}$$

$$[\kappa_1, \kappa_3] = \frac{1}{2}\{\kappa_1, \kappa_2\} - \delta_2 \kappa_3 r - \delta_3 \kappa_1 r + \delta_1 \kappa_2 - \delta_1 \delta_3 r, \quad r^2 = \mathbb{I}, \tag{16.38b}$$

$$[\kappa_3, \kappa_2] = \frac{1}{2}\kappa_2^2 + \delta_2 \kappa_2^2 r + 2\delta_3 \kappa_1 r + 2\delta_3 \kappa_3 r + \kappa_1 + \delta_4 r + \delta_5, \tag{16.38c}$$

where $[x, y] = xy - yx$ and $\{x, y\} = xy + yx$. The CBI algebra (16.38) admits the Casimir operator

$$Q = \frac{1}{2}\{\kappa_2^2, \kappa_1\} - \frac{\delta_2}{2}\kappa_2^2 r + \kappa_1^2 - \kappa_3^2 + (\delta_1 - 1/4)\kappa_2^2 + (\delta_3 - \delta_2)\kappa_1 r + 2\delta_5 \kappa_1 + (\delta_1\delta_3 - \delta_2\delta_5) r, \tag{16.39}$$

which commutes with all the generators.

We define the operators

$$K_1 = \mathscr{D}_\alpha, \qquad K_2 = x, \tag{16.40}$$

where K_2 is the operator multiplication by x and where $\mathscr{D}_\alpha$ is as given by (16.35). We introduce the involution [7]

$$P = R + \frac{\rho_2}{x}(\mathbb{I} - R). \tag{16.41}$$

It is easily seen that $P^2 = \mathbb{I}$. Finally, we define a fourth operator K_3 as follows:

$$K_3 = A(x)T^+ - B(x)T^- + [\alpha(x-\rho_2) - 2xC(x)]R - (1+2x)D(x)T^+R. \tag{16.42}$$

A direct computation shows that the operators K_1, K_2 and K_3, together with the involution P, realize the CBI algebra (16.38) under the identifications

$$K_1 = \kappa_1, \quad K_2 = \kappa_2, \quad K_3 = \kappa_3, \quad P = r.$$

The structure constants take the form

$$\delta_1 = \alpha(g - \alpha + 1), \quad \delta_2 = g - 2\alpha + 3/2, \quad \delta_3 = \rho_2, \quad \delta_5 = \alpha(\rho_2 - 1/2) + \omega/8, \tag{16.43a}$$

$$\delta_4 = \alpha(2\rho_2^2 - \rho_2 + 1/2) + \rho_2\omega/4 + (8\rho_1 r_1 r_2 + 4r_1 r_2 - 2\rho_1 + 2r_1 + 2r_2 - 3)/8. \tag{16.43b}$$

It is worth pointing out that even though the BI and CBI polynomials can be obtained from one another by a Christoffel (resp Geronimus) transformation and that they can both be obtained from the Askey-Wilson polynomials by very similar $q \to -1$ limits, their underlying algebraic structure are very dissimilar [22]. In the realization (16.40), (16.41), (16.42) the Casimir operator (16.39) acts a multiple of the identity

$$Q\,f(x) = q\,f(x),$$

where q is a complicated function of the five parameters ρ_1, ρ_2, r_1, r_2 and α.

The realization (16.40), (16.41), (16.42) can be used to obtain irreducible representations of the algebra (16.38) in two "dual" bases. In the first basis $\{v_n, n \in \mathbb{N}\}$, the operator κ_1 is diagonal:

$$\kappa_1 v_n = \Lambda_n^{(\alpha)} v_n,$$

where $\Lambda_n^{(\alpha)}$ is given by (16.34). Since κ_1 and r commute, the operator r can also be taken diagonal in this representation. Since $r^2 = \mathbb{I}$, one finds

$$r v_n = \epsilon(-1)^n v_n,$$

where $\epsilon = \pm 1$ is a representation parameter. Given the fact that the representation parameter ϵ is only a global multiplication factor of r, one can choose $\epsilon = 1$ without loss of generality. Because r is diagonal in the basis v_n, the matrix elements of κ_2 in the basis v_n can be calculated in a way similar to the one employed to obtain the representations of the Hahn algebra [8], with additional parity requirements. It is straightforward to show that in the basis v_n, upon choosing the initial condition $a_0 = 0$, the operator κ_2 is tridiagonal with the action

$$\kappa_2 v_n = a_{n+1} v_{n+1} + b_n v_n + a_n v_{n-1},$$

where we have

$$a_n = \sqrt{\tau_n}, \qquad b_n = (-1)^n \rho_2, \tag{16.44}$$

with τ_n given as in (16.9). We thus have the following result.

Proposition 3. *Let V be the infinite dimensional $\mathbb{C}$-vector space spanned by the basis vectors $\{v_n | n \in \mathbb{N}\}$ endowed with the actions*

$$\kappa_1 v_n = \Lambda_n^{(\alpha)} v_n, \quad r v_n = (-1)^n v_n,$$
$$\kappa_2 v_n = \sqrt{\tau_{n+1}}\, v_{n+1} + (-1)^n \rho_2\, v_n + \sqrt{\tau_n}\, v_{n-1},$$
$$\kappa_3 v_n = (\Lambda_{n+1}^{(\alpha)} - \Lambda_n^{(\alpha)}) \sqrt{\tau_{n+1}} v_{n+1} - (\Lambda_n^{(\alpha)} - \Lambda_{n-1}^{(\alpha)}) \sqrt{\tau_n} v_{n-1},$$

where $\Lambda_n^{(\alpha)}$ and τ_n are given by (16.34) *and* (16.9)*, respectively. Then V is a module for the CBI algebra* (16.38) *with structure constants taking the values* (16.43)*. The module is irreducible if none of the truncation conditions* (16.12) *and* (16.14) *are satisfied.*

Proof. The above considerations show that V is indeed a CBI-module. The irreducibility stems from the fact that if the none of the truncation conditions (16.12) and (16.14) are satisfied, then τ_n is never zero. □

Corollary. *If one of the truncation conditions* (16.12) *or* (16.14) *is satisfied, then V is no longer irreducible. One can restrict to the subspace spanned by the basis vectors $\{v_n | n = 0, \ldots, N\}$ and obtain a $N+1$-dimensional irreducible CBI-module.*

Thus the CBI algebra admits infinite dimensional representations where κ_1, r are diagonal and κ_2 is tridiagonal with matrix elements (16.44). Is is readily checked that

$$P I_n(x) = (-1)^n I_n(x)$$

and hence it is clear that the basis vectors v_n correspond to the CBI polynomials themselves

$$v_n = I_n(x).$$

Alternatively, we can consider the "dual" basis $\{\psi_k, k \in \mathbb{Z}\}$, in which the operator κ_2 is diagonal

$$\kappa_2 \psi_k = \vartheta_k \psi_k,$$

with the Bannai–Ito spectrum

$$\vartheta_k = (-1)^k (k/2 + t + 1/4) - 1/4, \tag{16.45}$$

where t an arbitrary real constant. In this basis, the involution r cannot be diagonal. Let $A_{\ell,k}$ be the matrix elements of r in the basis ψ_k. We have

$$r\psi_k = \sum_{\ell} A_{\ell,k} \psi_\ell.$$

Written in the basis ψ_k, the anticommutation relation $\{\kappa_2, r\} = 2\rho_2$ has the simple form

$$\sum_{\ell} A_{\ell,k} \{\vartheta_\ell + \vartheta_k\} \psi_\ell = 2\rho_2 \psi_k. \tag{16.46}$$

For $\ell = k$, this yields

$$A_{2k,2k} = \frac{\rho_2}{k+t}, \qquad A_{2k+1,2k+1} = -\frac{\rho_2}{k+t+1}.$$

When $\ell \neq k$, the equation (16.46) reduces to

$$A_{k,\ell} \{\vartheta_k + \vartheta_\ell\} = 0.$$

From the definition (16.45) of the eigenvalues ϑ_k, one notes that

$$\vartheta_{2k+1} + \vartheta_{2k+2} = 0. \tag{16.47}$$

It follows from (16.47) that in the basis ψ_k, the operator r is block diagonal with all blocks 2×2. Upon demanding that the other commutation relations of (16.38) be satisfied, it can be shown [7] that in this basis, the operator κ_1 becomes 5-diagonal. This result is expected since the CBI polynomials obey a 5-term difference equation of the form (16.37) on the Bannai–Ito grid.

We have obtained that the CBI polynomials are eigenfunctions of a one-parameter family of operators of the form (16.35) and that two operators $\mathscr{D}_\alpha$, $\mathscr{D}_\beta$ of this family are related by the "hidden" symmetry operator of the CBI polynomials given by (16.31). In the CBI algebra, the transformation $\mathscr{D}_\alpha \to \mathscr{D}_{\alpha+\beta}$ is equivalent to defining

$$\widetilde{K}_1 = K_1 + \frac{\beta}{2}(1 - P), \tag{16.48}$$

while leaving K_2 and P unchanged. The operator K_3 is transformed to

$$\widetilde{K}_3 = K_3 - \beta P K_2 + \beta\delta_3.$$

Upon using $\widetilde{K}_2 = K_2$, one finds that the algebra becomes

$$[\widetilde{K}_1, P] = 0, \qquad \{\widetilde{K}_2, P\} = 2\widetilde{\delta}_3, \qquad \{\widetilde{K}_3, P\} = 0, \qquad [\widetilde{K}_1, \widetilde{K}_2] = \widetilde{K}_3,$$
$$[\widetilde{K}_1, \widetilde{K}_3] = \frac{1}{2}\{\widetilde{K}_1, \widetilde{K}_2\} - \widetilde{\delta}_2\widetilde{K}_3 P - \widetilde{\delta}_3\widetilde{K}_1 P + \widetilde{\delta}_1\widetilde{K}_2 - \widetilde{\widetilde{\delta}_1}\widetilde{\delta}_3 P,$$
$$[\widetilde{K}_3, \widetilde{K}_2] = \frac{1}{2}\widetilde{K}_2^2 + \widetilde{\delta}_2\widetilde{K}_2^2 P + 2\widetilde{\delta}_3\widetilde{K}_1 P + 2\widetilde{\delta}_3\widetilde{K}_3 P + \widetilde{K}_1 + \widetilde{\delta}_4 P + \widetilde{\delta}_5,$$

with the structures constants

$$\widetilde{\delta}_1 = \delta_1 + \beta(\delta_2 - 1/2), \quad \widetilde{\delta}_2 = \delta_2 - 2\beta, \quad \widetilde{\delta_3} = \delta_3,$$
$$\widetilde{\delta}_4 = \delta_4 + \beta(2\delta_3^2 - \delta_3 + 1/2), \quad \widetilde{\delta}_5 = \delta_5 + \beta(\delta_3 - 1/2).$$

It is thus seen that the transformation (16.48) leaves the general form of the CBI algebra (16.38) unaffected and corresponds only to a change in the structure parameters.

16.6 Three OPs families related to the CBI polynomials

In this section, we exhibit the relationship between the Complementary Bannai–Ito polynomials and three other families of orthogonal polynomials : the recently discovered dual -1 Hahn [20, 27] and para-Krawtchouk polynomials [28] and the classical symmetric Hahn polynomials.

16.6.1 Dual -1 Hahn polynomials

The dual -1 Hahn polynomials have been introduced in [20] as $q = -1$ limits of the dual q-Hahn polynomials. They have appeared in the context of perfect state transfer in spin chains [27] and also as the Clebsch–Gordan coefficients of the $sl_{-1}(2)$ algebra in [6, 21]. Moreover, the -1 Hahn polynomials have occurred, in their symmetric form, as wavefunctions for finite parabosonic oscillator models [9, 10]. These polynomials[1], denoted $Q_n(x)$, can be obtained from the CBI polynomials through the limit $\rho_1 \to \infty$.

[1] To recover the formulas found in [20], a re-parametrization is necessary.

Taking the limit $\rho_1 \to \infty$ in the (16.9), one obtains the recurrence relation of the monic dual -1 Hahn polynomials:

$$Q_{n+1}(x) + (-1)^n \rho_2 Q_n(x) + \sigma_n Q_{n-1}(x) = x Q_n(x),$$

where r_n has the expression:

$$\sigma_{2n} = -n(n - r_1 - r_2), \quad \sigma_{2n+1} = -(n + \rho_2 - r_1 + 1/2)(n + \rho_2 - r_2 + 1/2).$$

The polynomials $Q_n(x)$ have the hypergeometric representation:

$$Q_{2n}(x) = \xi_{2n}\,{}_3F_2\Big[\begin{matrix} -n, \rho_2 + x, \rho_2 - x \\ \rho_2 - r_1 + 1/2, \rho_2 - r_2 + 1/2 \end{matrix}; 1\Big],$$
$$Q_{2n+1}(x) = \xi_{2n+1}(x - \rho_2)\,{}_3F_2\Big[\begin{matrix} -n, \rho_2 + x + 1, \rho_2 - x + 1 \\ \rho_2 - r_1 + 3/2, \rho_2 - r_2 + 3/2 \end{matrix}; 1\Big],$$

with normalization coefficients

$$\xi_{2n} = (\rho_2 - r_1 + 1/2)_n(\rho_2 - r_2 + 1/2)_n, \quad \xi_{2n+1} = (\rho_2 - r_1 + 3/2)_n(\rho_2 - r_2 + 3/2)_n.$$

These formulas are obtained from (16.20) in the same limit. Dividing (16.35) by ρ_1 and taking the limit $\rho_1 \to \infty$, one finds that the polynomials $Q_n(x)$ satisfy the eigenvalue equation

$$\mathscr{E}^{(\alpha)} Q_n(x) = \nu_n^{(\alpha)} Q_n(x),$$

with eigenvalues

$$\nu_{2n}^{(\alpha)} = n, \qquad \nu_{2n+1}^{(\alpha)} = n + \alpha.$$

The operator $\mathscr{E}^{(\alpha)}$ is found to be

$$\mathscr{E}^{(\alpha)} = \mathscr{E}^{(0)} + \alpha \frac{(x - \rho_2)}{2x}(\mathbb{I} - R),$$

where

$$\mathscr{E}^{(0)} = I(x)T^{+} + J(x)T^{-} + K(x)R + L(x)T^{+}R - (I(x) + J(x) + K(x) + L(x))\mathbb{I}.$$

The coefficients are given by

$$I(x) = \frac{(x + \rho_2 + 1)(2x - 2r_1 + 1)(2x - 2r_2 + 1)}{8(x+1)(2x+1)}, \qquad J(x) = \frac{(\rho_2 - x)(2x + 2r_1 - 1)(2x + 2r_2 - 1)}{8x(2x-1)},$$
$$K(x) = \frac{(x - \rho_2)(4x^2 + 4r_1 r_2 - 1)}{4x(4x^2 - 1)}, \qquad L(x) = \frac{\rho_2(2x - 2r_1 + 1)(2x - 2r_2 + 1)}{8x(x+1)(2x+1)}.$$

Lastly, it is seen that in the limit $\rho_1 \to \infty$, the CBI algebra becomes

$$[\kappa_1, r] = 0, \quad \{\kappa_2, r\} = 2\gamma_3, \quad \{\kappa_3, r\} = 0, \quad [\kappa_1, \kappa_2] = \kappa_3,$$
$$[\kappa_1, \kappa_3] = \gamma_1 \kappa_2 - \gamma_1 \gamma_3 r - \gamma_2 \kappa_3 r,$$
$$[\kappa_3, \kappa_2] = \gamma_2 \kappa_2^2 r + 2\gamma_3 \kappa_1 r + 2\gamma_3 \kappa_3 r + \kappa_1 + \gamma_4 r + \gamma_5,$$

where we have identified $\kappa_1 = \mathscr{E}^{(\alpha)}$, $\kappa_2 = x$ and $P = r$ with P given by (16.41). The structure parameters have the expression

$$\gamma_1 = \alpha(1-\alpha), \quad \gamma_2 = 1 - 2\alpha, \quad \gamma_3 = \rho_2,$$
$$\gamma_4 = \alpha(2\rho_2^2 - \rho_2 + 1/2) + \rho_2(1 - r_2 - r_1) + r_1 r_2 - 1/4,$$
$$\gamma_5 = (2\alpha\rho_2 - \alpha - r_1 - r_2 + 1)/2.$$

Other properties of the polynomials $Q_n(x)$ can be obtained directly using the limiting procedure.

16.6.2 The symmetric Hahn polynomials

It is possible to relate the CBI polynomials to the symmetric Hahn polynomials [12, 13] through a direct identification of the CBI parameters. This identification can be performed in three different ways by examining the cases for which the defining operator $\mathscr{D}_\alpha$ (16.35) of the CBI polynomials reduces to a classical three-term difference operator involving only the discrete shifts T^+, T^- and the identity operator $\mathbb{I}$.

We consider the operator $\mathscr{D}_\alpha$ in (16.35) with the following parameter identification:

$$\rho_1 = -\frac{1}{2}, \quad \rho_2 = 0, \quad \alpha = \frac{1}{2}(1 - r_1 - r_2). \tag{16.49}$$

With these values of the parameters, the eigenvalue equation (16.33) reduces to

$$B(x) I_n(x+1) - (B(x) + D(x)) I_n(x) + D(x) I_n(x-1) = \lambda_n I_n(x), \tag{16.50}$$

with coefficients

$$B(x) = (x - r_1 + 1/2)(x - r_2 + 1/2), \qquad D(x) = (x + r_1 - 1/2)(x + r_2 - 1/2),$$

and eigenvalues

$$\lambda_n = n(n - 2r_1 - 2r_2 + 1).$$

We consider the parametrization

$$r_1 = \frac{N+1}{2}, \qquad \alpha^* = \beta^* = -r_1 - r_2, \tag{16.51}$$

where N is an even or odd integer. It is seen from (16.12) and (16.14) that (16.51) is an admissible truncation condition for both parities of N. Upon introducing the variable $\widetilde{x} = x + r_1 - 1/2$, the coefficients of the eigenvalue equation (16.50) become

$$B(x) = (\widetilde{x} - N)(\widetilde{x} + \alpha^* + 1), \qquad D(x) = \widetilde{x}(\widetilde{x} - \beta^* - N - 1),$$

and the eigenvalues have the expression

$$\lambda_n = n(n + \alpha^* + \beta^* + 1).$$

This corresponds to the difference equation of the Hahn polynomials [13]. With the parametrization (16.49) and (16.51), the recurrence relation (16.7) of the CBI polynomials becomes

$$I_{n+1}(x) + \omega_n I_{n-1}(x) = x I_n(x),$$

where

$$\omega_n = \frac{n(N-n+1)(n+\alpha^*+\beta^*)(n+\alpha^*+\beta^*+N+1)}{4(2n+\alpha^*+\beta^*-1)(2n+\alpha^*+\beta^*+1)},$$

which is indeed the recurrence relation of symmetric Hahn polynomials. A simple calculation shows that upon taking the parametrization (16.51) in structure parameters (16.43), one has

$$\delta_2 = \delta_3 = \delta_4 = 0,$$

and hence the algebra reduces to

$$[K_1, P] = 0, \qquad \{K_2, P\} = 0, \qquad \{K_3, P\} = 0, \qquad [K_1, K_2] = K_3,$$
$$[K_1, K_3] = \frac{1}{2}\{K_1, K_2\} + \delta_1 K_2,$$
$$[K_3, K_2] = \frac{1}{2}K_2^2 + K_1 + \delta_5,$$

with the remaining structure parameters

$$\delta_1 = \frac{1}{4}(r_1 + r_2), \qquad \delta_5 = \frac{1}{4}(r_1 - 1/2)(r_2 - 1/2).$$

Thus we recover the Hahn algebra since the involution P no longer plays a determining role. For reference, we record the two following alternate choices of CBI parameters which also lead to symmetric Hahn polynomials:

$$\rho_2 = r_1 = 0, \quad \alpha = \frac{1}{4}(2\rho_1 - 2r_2 + 3), \quad \text{or} \quad \rho_2 = r_2 = 0, \quad \alpha = \frac{1}{4}(2\rho_1 - 2r_1 + 3).$$

16.6.3 Para-Krawtchouk polynomials

The para-Krawtchouk polynomials have been found [28] in the design of spin chains effecting perfect quantum state transfer. These polynomials are directly connected to the Complementary Bannai–Ito polynomials through the identification

$$\rho_1 = \frac{\gamma - N - 3}{4}, \qquad \rho_2 = 0, \qquad r_1 = \frac{N + 1 + \gamma}{4}, \qquad r_2 = 0,$$

when N is a positive odd integer. (When N is an even integer, the para-Krawtchouk are directly related to the Bannai–Ito polynomials.) In [28], the eigenvalue equation for the para-Krawtchouk polynomials was found; this operator corresponds to the operator (16.35) with a specific value of the free parameter

$$\alpha = \frac{1 - N}{4}. \tag{16.52}$$

It is interesting to note that in the case $\rho_2 = 0$, the CBI polynomials and their descendants become symmetric and thus the "hidden" eigenvalue equation (16.31) appears trivial. Notwithstanding this, the corresponding symmetric polynomials are still eigenfunctions of a Dunkl operator with a free parameter in the odd sector of the spectrum.

16.7 Conclusion

We have presented a systematic study of the Complementary Bannai–Ito polynomials. We showed that these OPs are eigenfunctions of a one-parameter family of second order Dunkl shift operators and that in consequence they satisfy a one-parameter family of five-term difference equations on grids of the Bannai–Ito type. This result makes explicit the bispectrality of the CBI polynomials and places this OPs family outside the scope of the Leonard duality. Moreover, we have obtained the algebraic structure associated to the CBI polynomials which we named the Complementary Bannai–Ito algebra. It was observed that this quadratic algebra is a deformation of the Askey-Wilson algebra with an involution. Lastly, we identified how the CBI polynomials are related to three other families of OPs.

The investigation of the continuum limit of the BI polynomials has led to connections with other families of -1 orthogonal polynomials which satisfy first order differential/Dunkl equations. It is hence of interest to examine the continuum limit of the CBI polynomials; this question will be treated in a future publication.

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Chapitre 17

A “continuous” limit of the Complementary Bannai–Ito polynomials: Chihara polynomials

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Abstract. A novel family of -1 orthogonal polynomials called the Chihara polynomials is characterized. The polynomials are obtained from a “continuous” limit of the Complementary Bannai–Ito polynomials, which are the kernel partners of the Bannai–Ito polynomials. The three-term recurrence relation and the explicit expression in terms of Gauss hypergeometric functions are obtained through a limit process. A one-parameter family of second-order differential Dunkl operators having these polynomials as eigenfunctions is also exhibited. The quadratic algebra with involution encoding this bispectrality is obtained. The orthogonality measure is derived in two different ways: by using Chihara's method for kernel polynomials and, by obtaining the symmetry factor for the one-parameter family of Dunkl operators. It is shown that the polynomials are related to the Big -1 Jacobi polynomials by a Christoffel transformation and that they can be obtained from the Big q-Jacobi by a $q \to -1$ limit. The generalized Gegenbauer/Hermite polynomials are respectively seen to be special/limiting cases of the Chihara polynomials. A one-parameter extension of the generalized Hermite polynomials is proposed.

17.1 Introduction

One of the recent advances in the theory of orthogonal polynomials is the characterization of -1 orthogonal polynomials [11, 13, 21, 22, 23, 24, 25]. The distinguishing property of these polynomials is that they are eigenfunctions of Dunkl-type operators which involve reflections. They also correspond to $q \to -1$ limits of certain q-polynomials of the Askey tableau. The -1 polynomials should be organized in a tableau complementing the latter. Sitting atop this -1 tableau would be the Bannai–Ito polynomials (BI) and their kernel partners, the Complementary Bannai–Ito polynomials (CBI). Both families depend on four real parameters, satisfy a discrete/finite orthogonality relation and correspond to a (different) $q \to -1$ limit of the Askey-Wilson polynomials [1]. The BI polynomials are eigenfunctions of a *first-order* Dunkl difference operator whereas the CBI polynomials are eigenfunctions of a *second-order* Dunkl difference operator. It should be noted that the polynomials of the -1 scheme do not all have the same type of bispectral properties in distinction with what is observed when $q \to 1$ because the second-order q-difference equations of the basic polynomials of the Askey scheme do not always exist in certain $q \to -1$ limits. In this paper, a novel family of -1 orthogonal polynomials stemming from a "continuous" limit of the Complementary Bannai–Ito polynomials will be studied and characterized. Its members will be called Chihara polynomials.

The Bannai–Ito polynomials, written $B_n(x;\rho_1,\rho_2,r_1,r_2)$ in the notation of [11], were first identified by Bannai and Ito themselves in their classification [2] of orthogonal polynomials satisfying the Leonard duality property [18]; they were also seen to correspond to a $q \to -1$ limit of the q-Racah polynomials [2]. A significant step in the characterization of the BI polynomials was made in [21] where it was recognized that the polynomials $B_n(x)$ are eigenfunctions of the most general (self-adjoint) first-order Dunkl shift operator which stabilizes polynomials of a given degree, i.e.

$$\mathscr{L} = \left[\frac{(x-\rho_1)(x-\rho_2)}{2x}\right](\mathbb{I}-R) + \left[\frac{(x-r_1+1/2)(x-r_2+1/2)}{2x+1}\right](T^+R-\mathbb{I}), \tag{17.1}$$

where $T^+f(x) = f(x+1)$ is the shift operator, $Rf(x) = f(-x)$ is the reflection operator and where $\mathbb{I}$ stands for the identity. In the same paper [21], it was also shown that the BI polynomials correspond to a $q \to -1$ limit of the Askey-Wilson polynomials and that the operator (17.1) can be obtained from the Askey-Wilson operator in this limit. An important limiting case of the BI polynomials is found by considering the "continuous" limit, which is obtained upon writing

$$x \to \frac{x}{h}, \quad \rho_1 = \frac{a_1}{h} + b_1, \quad \rho_2 = \frac{a_2}{h} + b_2, \quad r_1 = \frac{a_1}{h}, \quad r_2 = \frac{a_2}{h}, \tag{17.2}$$

and taking $h \to 0$. In this limit, the operator (17.1) becomes, after a rescaling of the variable x, the most general (self-adjoint) first-order *differential* Dunkl operator which preserves the space of polynomials of a given degree, i.e.

$$\mathscr{M} = \left[\frac{(a+b+1)x^2+(ac-b)x+c}{x^2}\right](R-\mathbb{I}) + \left[\frac{2(x-1)(x+c)}{x}\right]\partial_x R. \tag{17.3}$$

The polynomial eigenfunctions of (17.3) have been identified in [23, 25] as the Big -1 Jacobi polynomials $J_n(x;a,b,c)$ introduced in [25]. Alternatively, one can obtain the polynomials $J_n(x;a,b,c)$ by directly applying the limit (17.2) to the BI polynomials. The Big -1 Jacobi polynomials satisfy a continuous orthogonality relation on the interval $[-1,-c]\cup[1,c]$. They also correspond to a $q\to-1$ limit of the Big q-Jacobi polynomials, an observation which was first used to derive their properties in [25]. It is known moreover (see for example [16]) that the Big q-Jacobi polynomials can be obtained from the Askey-Wilson polynomials using a limiting procedure similar to (17.2). Hence the relationships between the Askey-Wilson, Big q-Jacobi, Bannai–Ito and Big -1 Jacobi polynomials can be expressed diagrammatically as follows:

$$\begin{array}{ccc}
\text{Askey-Wilson} & & \text{Bannai–Ito} \\
p_n(x;a,b,c,d\,|\,q) & \xrightarrow{q\to-1} & B_n(x;\rho_1,\rho_2,r_1,r_2) \\
\Big\downarrow {\scriptstyle x\to x/2a \;|\; a\to 0} & & \Big\downarrow {\scriptstyle x\to x/h \;|\; h\to 0} \\
\text{Big } q\text{-Jacobi} & & \text{Big } -1 \text{ Jacobi} \\
P_n(x;\alpha,\beta,\gamma\,|\,q) & \xrightarrow{q\to-1} & J_n(x;a,b,c)
\end{array} \tag{17.4}$$

where the notation of [16] was used for the Askey-Wilson $p_n(x;a,b,c,d\,|\,q)$ and the Big q-Jacobi polynomials $P_n(x;\alpha,\beta,\gamma\,|\,q)$.

In this paper, we shall be concerned with the continuous limit (17.2) of the Complementary Bannai–Ito polynomials $I_n(x;\rho_1,\rho_2,r_1,r_2)$ [11, 21]. The polynomials $C_n(x;\alpha,\beta,\gamma)$ arising in this limit shall be referred to as Chihara polynomials since they have been introduced by T. Chihara in [5] (up to a parameter redefinition). They depend on three real parameters. Using the limit, the recurrence relation and the explicit expression for the polynomials $C_n(x;\alpha,\beta,\gamma)$ in terms of Gauss hypergeometric functions will be obtained from that of the CBI polynomials. The second-order differential Dunkl operator having the Chihara polynomials as eigenfunctions will also be given. The corresponding bispectrality property will be used to construct the algebraic structure behind the Chihara polynomials: a quadratic Jacobi algebra [15] supplemented with an involution. The weight function for the Chihara polynomials will be constructed in two different ways: on the one hand using Chihara's method for kernel polynomials [5] and on the other hand by solving a Pearson-type equation [16]. This measure will be defined on the union of two disjoint intervals. The Chihara polynomials $C_n(x;\alpha,\beta,\gamma)$ will also be seen to correspond to a $q\to-1$ limit of the Big

q-Jacobi polynomials that is different from the one leading to the Big -1 Jacobi. In analogy with (17.4), the following relationships shall be established:

$$\begin{array}{ccc} \text{Askey-Wilson} & \xrightarrow{q\to -1} & \text{Complementary BI} \\ p_n(x;a,b,c,d\,|\,q) & & I_n(x;\rho_1,\rho_2,r_1,r_2) \\ \Big\downarrow {\scriptstyle x\to x/2a,\ a\to 0} & & \Big\downarrow {\scriptstyle x\to x/h,\ h\to 0} \\ \text{Big } q\text{-Jacobi} & \xrightarrow{q\to -1} & \text{Chihara} \\ P_n(x;\alpha,\beta,\gamma\,|\,q) & & C_n(x;\alpha,\beta,\gamma) \end{array} \tag{17.5}$$

Since the CBI polynomials are obtained from the BI polynomials by the Christoffel transform [6] (and vice-versa using the Geronimus transform [14]), it will be shown that the following relations relating the Chihara to the Big -1 Jacobi polynomials hold:

$$\begin{array}{ccc} \text{Bannai–Ito} & \underset{\text{Geronimus}}{\overset{\text{Christoffel}}{\rightleftarrows}} & \text{Complementary BI} \\ B_n(x;\rho_1,\rho_2,r_1,r_2) & & I_n(x;\rho_1,\rho_2,r_1,r_2) \\ \Big\downarrow {\scriptstyle x\to x/h,\ h\to 0} & & \Big\downarrow {\scriptstyle x\to x/h,\ h\to 0} \\ \text{Big } -1 \text{ Jacobi} & \underset{\text{Geronimus}}{\overset{\text{Christoffel}}{\rightleftarrows}} & \text{Chihara} \\ J_n(x;a,b,c) & & C_n(x;\alpha,\beta,\gamma) \end{array} \tag{17.6}$$

Finally, it will be observed that for $\gamma = 0$, the Chihara polynomials $C_n(x;\alpha,\beta,\gamma)$ reduce to the generalized Gegenbauer polynomials and that upon taking the limit $\beta \to \infty$ with $\gamma = 0$, the polynomials $C_n(x;\alpha,\beta,\gamma)$ go to the generalized Hermite polynomials [6]. A one-parameter extension of the generalized Hermite polynomials will also be presented.

The remainder of the paper is organized straightforwardly. In section 2, the main features of the CBI polynomials are reviewed. In section 3, the "continuous" limit is used to define the Chihara polynomials and establish their basic properties. In section 4, the operator having the Chihara polynomials as eigenfunctions is obtained and the algebraic structure behind their bispectrality is exhibited. In section 5, the weight function is derived and the orthogonality relation is given. In section 6, the polynomials are related to the Big -1 Jacobi and Big q-Jacobi polynomials. In section 7, limits and special cases are examined.

17.2 Complementary Bannai–Ito polynomials

In this section, the main properties of the Complementary Bannai–Ito polynomials, which have been obtained in [11, 21], are reviewed. Let ρ_1, ρ_2, r_1, r_2 be real parameters, the monic CBI polynomial $I_n(x;\rho_1,\rho_2,r_1,r_2)$, denoted $I_n(x)$ for notational ease, are defined by

$$\begin{aligned} I_{2n}(x) &= \eta_{2n}\, {}_4F_3\left[\begin{matrix} -n, n+g+1, \rho_2+x, \rho_2-x \\ \rho_1+\rho_2+1, \rho_2-r_1+1/2, \rho_2-r_2+1/2 \end{matrix}; 1\right], \\ I_{2n+1}(x) &= \eta_{2n+1}\,(x-\rho_2)\, {}_4F_3\left[\begin{matrix} -n, n+g+2, \rho_2+x+1, \rho_2-x+1 \\ \rho_1+\rho_2+2, \rho_2-r_1+3/2, \rho_2-r_2+3/2 \end{matrix}; 1\right], \end{aligned} \tag{17.7}$$

where $g = \rho_1+\rho_2-r_1-r_2$ and where ${}_pF_q$ denotes the generalized hypergeometric series [8]. It is directly seen from (17.7) that $I_n(x)$ is a polynomial of degree n in x and that it is symmetric with respect to the exchange of the two parameters r_1, r_2. The coefficients η_n, which ensure that the polynomials are monic (i.e. $I_n(x) = x^n + \mathcal{O}(x^{n-1})$), are given by

$$\eta_{2n} = \frac{(\rho_1+\rho_2+1)_n(\rho_2-r_1+1/2)_n(\rho_2-r_2+1/2)_n}{(n+g+1)_n},$$
$$\eta_{2n+1} = \frac{(\rho_1+\rho_2+2)_n(\rho_2-r_1+3/2)_n(\rho_2-r_2+3/2)_n}{(n+g+2)_n},$$

where $(a)_n = a(a+1)\cdots(a+n-1)$, $(a)_0 \equiv 1$, stands for the Pochhammer symbol. The CBI polynomials satisfy the three-term recurrence relation

$$x I_n(x) = I_{n+1}(x) + (-1)^n \rho_2 I_n(x) + \tau_n I_{n-1}(x), \tag{17.8}$$

subject to the initial conditions $I_{-1}(x) = 0$, $I_0(x) = 1$ and with the recurrence coefficients

$$\begin{aligned} \tau_{2n} &= -\frac{n(n+\rho_1-r_1+1/2)(n+\rho_1-r_2+1/2)(n-r_1-r_2)}{(2n+g)(2n+g+1)}, \\ \tau_{2n+1} &= -\frac{(n+g+1)(n+\rho_1+\rho_2+1)(n+\rho_2-r_1+1/2)(n+\rho_2-r_2+1/2)}{(2n+g+1)(2n+g+2)}. \end{aligned} \tag{17.9}$$

The CBI polynomials form a finite set $\{I_n(x)\}_{n=0}^N$ of positive-definite orthogonal polynomials provided that the truncation and positivity conditions $\tau_{N+1} = 0$ and $\tau_n > 0$ hold for $n = 1,\ldots,N$, where N is a positive integer. Under these conditions, the CBI polynomials obey the orthogonality relation

$$\sum_{i=0}^{N} \omega_i I_n(x_i) I_m(x_i) = h_n^{(N)} \delta_{nm},$$

where the grid points x_i are of the general form

$$x_i = (-1)^i(a+1/4+i/2)-1/4, \quad \text{or} \quad x_i = (-1)^i(b-1/4-i/2)-1/4.$$

The expressions for the grid points x_i and for the weight function ω_i depend on the truncation condition $\tau_{N+1} = 0$, which can be realized in six different ways (three for each possible parity of N). The explicit formulas for each case shall not be needed here and can be found in [11].

One of the most important properties of the Complementary Bannai–Ito polynomials is their bispectrality. Recall that a family of orthogonal polynomials $\{P_n(x)\}$ is bispectral if one has an eigenvalue equation of the form

$$\mathscr{A} P_n(x) = \lambda_n P_n(x),$$

where $\mathscr{A}$ is an operator acting on the argument x of the polynomials. For the CBI polynomials, there is a one-parameter family of eigenvalue equations [11]

$$\mathscr{K}^{(\alpha)} I_n(x) = \Lambda_n^{(\alpha)} I_n(x), \tag{17.10}$$

with eigenvalues $\Lambda_n^{(\alpha)}$

$$\Lambda_{2n}^{(\alpha)} = n(n+g+1), \qquad \Lambda_{2n+1}^{(\alpha)} = n(n+g+2)+\omega+\alpha, \tag{17.11}$$

where

$$\omega = \rho_1(1-r_1-r_2)+r_1r_2-3(r_1+r_2)/2+5/4, \tag{17.12}$$

and where α is an arbitrary parameter. The operator $\mathscr{K}^{(\alpha)}$ is the second-order Dunkl shift operator[1]

$$\mathscr{K}^{(\alpha)} = A_x(T^+-\mathbb{I})+B_x(T^- - R)+C_x(\mathbb{I}-R)+D_x(T^+R-\mathbb{I}), \tag{17.13}$$

where $T^{\pm}f(x) = f(x\pm 1)$, $Rf(x) = f(-x)$ and where the coefficients read

$$\begin{aligned}
A_x &= \frac{(x+\rho_1+1)(x+\rho_2+1)(x-r_1+1/2)(x-r_2+1/2)}{2(x+1)(2x+1)},\\
B_x &= \frac{(x-\rho_1-1)(x-\rho_2)(x+r_1-1/2)(x+r_2-1/2)}{2x(2x-1)},\\
C_x &= \frac{(x+\rho_1+1)(x-\rho_2)(x-r_1+1/2)(x-r_2+1/2)}{2x(2x+1)}+\frac{(\alpha-x^2)(x-\rho_2)}{2x},\\
D_x &= \frac{\rho_2(x+\rho_1+1)(x-r_1+1/2)(x-r_2+1/2)}{2x(x+1)(2x+1)}.
\end{aligned} \tag{17.14}$$

The Complementary Bannai–Ito correspond to a $q\to -1$ limit of the Askey-Wilson polynomials [21]. Consider the Askey-Wilson polynomials [1]

$$p_n(z;a,b,c,d\,|\,q) = a^{-n}(ab,ac,ad;q)_n\, {}_4\phi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, az, az^{-1} \\ ab, ac, ad \end{matrix}\,\middle|\, q;q\right), \tag{17.15}$$

where ${}_p\phi_q$ is the generalized q-hypergeometric function [8]. Upon considering

$$\begin{aligned}
&a = ie^{\epsilon(2\rho_1+3/2)}, \quad b = -ie^{\epsilon(2\rho_2+1/2)}, \quad c = ie^{\epsilon(-2r_2+1/2)}, \quad d = ie^{\epsilon(-2r_1+1/2)},\\
&q = -e^{\epsilon}, \qquad z = ie^{-2\epsilon(x+1/4)},
\end{aligned}$$

and taking the limit $\epsilon\to 0$, one finds that the polynomials (17.15) converge, up to a normalization factor, to the CBI polynomials $I_n(x;\rho_1,\rho_2,r_1,r_2)$.

[1]One should take $\alpha\to\omega+\alpha$ in the operator obtained in [11] to find the expression (17.13)

17.3 A "continuous" limit to Chihara polynomials

In this section, the "continuous" limit of the Complementary Bannai–Ito polynomials will be used to define the Chihara polynomials and obtain the three-term recurrence relation that they satisfy. Let ρ_1, ρ_2, r_1, r_2 be parametrized as follows:

$$\rho_1 = \frac{a_1}{h} + b_1, \quad \rho_2 = \frac{a_2}{h} + b_2, \quad r_1 = \frac{a_1}{h}, \quad r_2 = \frac{a_2}{h}, \tag{17.16}$$

and denote by

$$F_n^{(h)}(x) = h^n I_n(x/h) \tag{17.17}$$

the monic polynomials obtained by replacing $x \to x/h$ in the CBI polynomials. Upon taking $h \to 0$ in the definition (17.7) of the CBI polynomials, where (17.16) has been used, one finds that the limit exists and that it yields

$$\begin{aligned}
\lim_{h\to 0} F_{2n}^{(h)} &= \frac{(a_2^2 - a_1^2)^n (b_2 + 1/2)_n}{(n + b_1 + b_2 + 1)_n} \, {}_2F_1\left[\begin{matrix} -n, n + b_1 + b_2 + 1 \\ b_2 + 1/2 \end{matrix}; \frac{a_2^2 - x^2}{a_2^2 - a_1^2}\right], \\
\lim_{h\to 0} F_{2n+1}^{(h)} &= \frac{(a_2^2 - a_1^2)^n (b_2 + 3/2)_n}{(n + b_1 + b_2 + 2)_n} \, (x - a_2) \, {}_2F_1\left[\begin{matrix} -n, n + b_1 + b_2 + 2 \\ b_2 + 3/2 \end{matrix}; \frac{a_2^2 - x^2}{a_2^2 - a_1^2}\right].
\end{aligned} \tag{17.18}$$

It is directly seen that the variable x in (17.18) can be rescaled and consequently, that there is only three independent parameters. Assuming that $a_1^2 \neq a_2^2$, we can take

$$x \to x\sqrt{a_1^2 - a_2^2}, \quad \alpha = b_2 - 1/2, \quad \beta = b_1 + 1/2, \quad \gamma = a_2/\sqrt{a_1^2 - a_2^2}, \tag{17.19}$$

to rewrite the polynomials (17.18) in terms of the three parameters α, β and γ. We shall moreover assume that γ is real. This construction motivates the following definition.

Definition 3. Let α, β and γ be real parameters. The Chihara polynomials $C_n(x;\alpha,\beta,\gamma)$, denoted $C_n(x)$ for simplicity, are the monic polynomials of degree n in the variable x defined by

$$\begin{aligned}
C_{2n}(x) &= (-1)^n \frac{(\alpha+1)_n}{(n+\alpha+\beta+1)_n} \, {}_2F_1\left[\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; x^2 - \gamma^2\right], \\
C_{2n+1}(x) &= (-1)^n \frac{(\alpha+2)_n}{(n+\alpha+\beta+2)_n} \, (x-\gamma) \, {}_2F_1\left[\begin{matrix} -n, n+\alpha+\beta+2 \\ \alpha+2 \end{matrix}; x^2 - \gamma^2\right].
\end{aligned} \tag{17.20}$$

The polynomials $C_n(x;\alpha,\beta,\gamma)$ (up to redefinition of the parameters) have been considered by Chihara in [5] in a completely different context (see section 5). We shall henceforth refer to the polynomials $C_n(x;\alpha,\beta,\gamma)$ as the Chihara polynomials. They correspond to the continuous limit (17.16), (17.17) as $h \to 0$ of the CBI polynomials with the scaling and reparametrization (17.19).

Using the same limit on (17.8) and (17.9), the recurrence relation satisfied by the Chihara polynomials (17.20) can readily be obtained.

Proposition 4. *[5] The Chihara polynomials $C_n(x)$ defined by* (17.20) *satisfy the recurrence relation*

$$xC_n(x) = C_{n+1}(x) + (-1)^n \gamma C_n(x) + \sigma_n C_{n-1}(x), \tag{17.21}$$

where

$$\sigma_{2n} = \frac{n(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}, \quad \sigma_{2n+1} = \frac{(n+\alpha+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}. \tag{17.22}$$

Proof. By taking the limit (17.16), (17.17) and reparametrization (17.19) on (17.8), (17.9). □

As is directly checked from the recurrence coefficients (17.22), the positivity condition $\sigma_n > 0$ for $n \geqslant 1$ is satisfied if the parameters α and β are such that

$$\alpha > -1, \qquad \beta > -1.$$

By Favard's theorem [6], it follows that the system of polynomials $\{C_n(x;\alpha,\beta,\gamma)\}_{n=0}^{\infty}$ defined by (17.20) is orthogonal with respect to some positive measure on the real line. This measure shall be constructed in section 5.

17.4 Bispectrality of the Chihara polynomials

In this section, the operator having the Chihara polynomials as eigenfunctions is derived and the algebraic structure behind this bispectrality, a quadratic algebra with an involution, is exhibited.

17.4.1 Bispectrality

Consider the family of eigenvalue equations (17.11) satisfied by the CBI polynomials. Upon changing the variable $x \to x/h$, the action of the operator (17.13) becomes

$$\begin{aligned}\mathcal{K}^{(\alpha)} f(x) = A_{x/h}(f(x+h) - f(x)) + B_{x/h}(f(x-h) - f(-x)) \\ + C_{x/h}(f(x) - f(-x)) + D_{x/h}(f(-x-h) - f(x)).\end{aligned}$$

Using the above expression and the parametrization (17.16), the limit as $h \to 0$ can be taken in (17.11) to obtain the family of eigenvalue equations satisfied by the Chihara polynomials.

Proposition 5. *Let ϵ be an arbitrary parameter. The Chihara polynomials $C_n(x;\alpha,\beta,\gamma)$ satisfy the one-parameter family of eigenvalue equations*

$$\mathscr{D}^{(\epsilon)} C_n(x;\alpha,\beta,\gamma) = \lambda_n^{(\epsilon)} C_n(x;\alpha,\beta,\gamma). \tag{17.23}$$

where the eigenvalues are given by

$$\lambda_{2n}^{(\epsilon)} = n(n+\alpha+\beta+1), \qquad \lambda_{2n+1}^{(\epsilon)} = n(n+\alpha+\beta+2)+\epsilon, \tag{17.24}$$

for $n = 0,1,\ldots$. The second-order differential Dunkl operator $\mathscr{D}^{(\epsilon)}$ having the Chihara polynomials as eigenfunctions has the expression

$$\mathscr{D}^{(\epsilon)} = S_x \partial_x^2 + T_x \partial_x R + U_x \partial_x + V_x(\mathbb{I}-R), \tag{17.25}$$

where the coefficients are

$$\begin{aligned}
S_x &= \frac{(x^2-\gamma^2)(x^2-\gamma^2-1)}{4x^2}, \quad T_x = \frac{\gamma(x-\gamma)(x^2-\gamma^2-1)}{4x^3},\\
U_x &= \frac{\gamma(x^2-\gamma^2-1)(2\gamma-x)}{4x^3} + \frac{(x^2-\gamma^2)(\alpha+\beta+3/2)}{2x} - \frac{\alpha+1/2}{2x},\\
V_x &= \frac{\gamma(x^2-\gamma^2-1)(x-3\gamma/2)}{4x^4} - \frac{(x^2-\gamma^2)(\alpha+\beta+3/2)}{4x^2} + \frac{\alpha+1/2}{4x^2} + \epsilon\,\frac{x-\gamma}{2x}.
\end{aligned} \tag{17.26}$$

Proof. We obtain $\mathscr{D}^{(0)}$ first. Consider the operator $\mathscr{K}^{(-\omega)}$. Upon taking $x \to x/h$, the action of this operator on functions of argument x can be cast in the form

$$\begin{aligned}
\mathscr{K}^{(-\omega)} f(x) = {} & A_{x/h}[f(x+h)-f(x)] + B_{x/h}[f(x-h)-f(x)]\\
&+ [B_{x/h}+C_{x/h}-D_{x/h}]f(x)\\
&+ D_{x/h}f(-x-h) - [B_{x/h}+C_{x/h}]f(-x)
\end{aligned} \tag{17.27}$$

Assuming that $f(x)$ is an analytic function, the first term of (17.27) yields

$$\begin{aligned}
&\lim_{h\to 0}\left(A_{x/h}[f(x+h)-f(x)] + B_{x/h}[f(x-h)-f(x)]\right) =\\
&\quad \left[\frac{(x^2-a_1^2)(x^2-a_2^2)}{4x^2}\right] f''(x)\\
&+ \frac{1}{4}\left[x(2b_1+2b_2+3) + \frac{-a_2x - a_2^2(1+2b_1) - 2a_1^2 b_2}{x} + \frac{a_1^2 a_2}{x^2} - \frac{2a_1^2a_2^2}{x^3}\right] f'(x),
\end{aligned}$$

where (17.16) has been used and where $f'(x)$ stands for the derivative with respect to the argument x. With (17.19) this gives the term $S_x\partial_x^2 + U_x\partial_x$ in $\mathscr{D}^{(0)}$. Similarly, using (17.16), the second term of (17.27) produces

$$\lim_{h\to 0}\left(B_{x/h}+C_{x/h}-D_{x/h}\right)f(x) = \left[\frac{3a_1^2a_2^2}{8x^4} - \frac{a_1^2a_2}{4x^3} + \frac{a_2^2b_1 + a_1^2 b_2}{4x^2} + \frac{a_2}{4x} - \frac{2b_1+2b_2+3}{8}\right] f(x).$$

With the parametrization (17.19), this gives the term $V_x\mathbb{I}$ in $\mathscr{D}^{(0)}$. The third term of (17.27) gives

$$\begin{aligned}
\lim_{h\to 0}\left(D_{x/h}f(-x-h) - [B_{x/h}+C_{x/h}]f(-x)\right) = {} & \frac{a_2(x^2-a_1^2)(a_2-x)}{4x^3} f'(-x)\\
&- \left[\frac{3a_1^2a_2^2}{8x^4} - \frac{a_1^2a_2}{4x^3} + \frac{a_2^2b_1+a_1^2b_2}{4x^2} + \frac{a_2}{4x} - \frac{2b_1+2b_2+3}{8}\right] f(-x).
\end{aligned}$$

Using (17.19), this gives the term $-V_x R + T_x \partial_x R$ in $\mathscr{D}^{(0)}$. The arbitrary parameter ϵ can be added to the odd part of the spectrum since the Chihara polynomials satisfy the eigenvalue equation

$$\frac{(x-\gamma)}{2x}(\mathbb{I}-R)C_n(x) = \rho_n C_n(x), \quad \text{with} \quad \rho_n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd,} \end{cases}$$

as can be seen directly from the explicit expression (17.20). This concludes the proof. □

17.4.2 Algebraic Structure

The bispectrality property of the Chihara polynomials can be encoded algebraically. Let κ_1, κ_2 and P be defined as follows

$$\kappa_1 = \mathscr{D}^{(\epsilon)}, \qquad \kappa_2 = x, \qquad P = R + \frac{\gamma}{x}(\mathbb{I}-R),$$

where $\mathscr{D}^{(\epsilon)}$ is given by (17.25), R is the reflection operator and where κ_2 corresponds to multiplication by x. It is directly checked that P is an involution, which means that

$$P^2 = \mathbb{I}.$$

Upon defining a third generator

$$\kappa_3 = [\kappa_1, \kappa_2],$$

with $[a,b] = ab - ba$, a direct computation shows that one has the commutation relations

$$\begin{aligned} [\kappa_3, \kappa_2] &= \frac{1}{2}\kappa_2^2 + \delta_2 \kappa_2^2 P + 2\delta_3 \kappa_3 P - \delta_5 P - \delta_4, \\ [\kappa_1, \kappa_3] &= \frac{1}{2}\{\kappa_1, \kappa_2\} - \delta_2 \kappa_3 P - \delta_3 \kappa_1 P + \delta_1 \kappa_2 - \delta_1 \delta_3 P, \end{aligned} \tag{17.28}$$

where $\{a,b\} = ab + ba$ stands for the anticommutator. The commutation relations involving the involution P are given by

$$[\kappa_1, P] = 0, \qquad \{\kappa_2, P\} = 2\delta_3, \qquad \{\kappa_3, P\} = 0, \tag{17.29}$$

and the structure constants δ_i, $i = 1, \ldots, 5$ are expressed as follows:

$$\begin{aligned} &\delta_1 = \epsilon(\alpha + \beta + 1 - \epsilon), \quad \delta_2 = (\alpha + \beta + 3/2 - 2\epsilon), \quad \delta_3 = \gamma, \\ &\delta_4 = (\gamma^2 + 1)/2, \qquad \delta_5 = \gamma^2(\alpha + \beta + 3/2 - 2\epsilon) + \alpha + 1/2. \end{aligned}$$

The algebra defined by (17.28) and (17.29) corresponds to a Jacobi algebra [15] supplemented with involutions and can be seen as a contraction of the Complementary Bannai–Ito algebra [11].

17.5 Orthogonality of the Chihara polynomials

In this section, we derive the orthogonality relation satisfied by the Chihara polynomials in two different ways. First, the weight function will be constructed directly, following a method proposed by Chihara in [5]. Second, a Pearson-type equation will be solved for the operator (17.25). It is worth noting here that the weight function cannot be obtained from the limit process (17.16) as $h \to 0$. Indeed, while the Complementary Bannai–Ito polynomials $F_n^{(h)}(x) = h^n I_n(x/h)$ approach this limit, they no longer form a (finite) system of orthogonal polynomials. A similar situation occurs in the standard limit from the q-Racah to the Big q-Jacobi polynomials [16] and is discussed by Koornwinder in [17].

17.5.1 Weight function and Chihara's method

Our first approach to the construction of the weight function is based on the method developed by Chihara in [5] to construct systems of orthogonal polynomials from a given a set of orthogonal polynomials and their kernel partners (see also [19]). Since the present context is rather different, the analysis will be taken from the start. The main observation is that the Chihara polynomials (17.20) can be expressed in terms of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ as follows:

$$\begin{aligned} C_{2n}(x;\alpha,\beta,\gamma) &= \frac{(-1)^n n!}{(n+\alpha+\beta+1)_n} P_n^{(\alpha,\beta)}(y(x)), \\ C_{2n+1}(x;\alpha,\beta,\gamma) &= \frac{(-1)^n n!(x-\gamma)}{(n+\alpha+\beta+2)_n} P_n^{(\alpha+1,\beta)}(y(x)), \end{aligned} \tag{17.30}$$

where

$$y(x) = 1 - 2x^2 + 2\gamma^2.$$

The Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ are known [16] to satisfy the orthogonality relation

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(z) P_m^{(\alpha,\beta)}(z)\, \mathrm{d}\psi^{(\alpha,\beta)}(z) = \chi_n^{(\alpha,\beta)} \delta_{nm}, \tag{17.31}$$

with

$$\chi_n^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)n!}, \tag{17.32}$$

where $\Gamma(z)$ is the gamma function and where

$$\mathrm{d}\psi^{(\alpha,\beta)}(z) = (1-z)^{\alpha}(1+z)^{\beta}\, \mathrm{d}z. \tag{17.33}$$

The relation (17.31) is valid provided that $\alpha > -1$, $\beta > -1$. Since the Chihara polynomials are orthogonal (by proposition 4 and Favard's theorem) and given the relation (17.30) and orthogonality relation (17.31), we consider the integral

$$\mathscr{I}_{MN} = \int_{\mathscr{F}} C_M(x) C_N(x)\, \mathrm{d}\phi(x),$$

where the interval $\mathscr{F} = [-\sqrt{1+\gamma^2}, -|\gamma|] \cup [|\gamma|, \sqrt{1+\gamma^2}]$ corresponds to the inverse mapping of the interval $[-1,1]$ for $y(x)$ and where $\phi(x)$ is a distribution function. Upon taking $M = 2m$ and using (17.30), one directly has (up to normalization)

$$\mathscr{I}_{2m,2n} = \int_{|\gamma|}^{\sqrt{1+\gamma^2}} P_m^{(\alpha,\beta)}\big(y(x)\big) P_n^{(\alpha,\beta)}\big(y(x)\big)\,\big[\,\mathrm{d}\phi(x) - \mathrm{d}\phi(-x)\,\big],$$
$$\mathscr{I}_{2m,2n+1} = \int_{|\gamma|}^{\sqrt{1+\gamma^2}} P_m^{(\alpha,\beta)}\big(y(x)\big) P_n^{(\alpha+1,\beta)}\big(y(x)\big)\,\big[\,(x-\gamma)\,\mathrm{d}\phi(x) + (x+\gamma)\,\mathrm{d}\phi(-x)\,\big].$$

In order that $\mathscr{I}_{2n,2m} = \mathscr{I}_{2n,2m+1} = 0$ for $n \neq m$, one must have for $|\gamma| \leqslant x \leqslant \sqrt{1+\gamma^2}$

$$\begin{aligned} &\mathrm{d}\phi(x) - \mathrm{d}\phi(-x) = \mathrm{d}\psi^{(\alpha,\beta)}\big(y(x)\big),\\ &(x-\gamma)\,\mathrm{d}\phi(x) + (x+\gamma)\,\mathrm{d}\phi(-x) = 0, \end{aligned} \tag{17.34}$$

where $\psi^{(\alpha,\beta)}\big(y(x)\big)$ is the distribution appearing in (17.33) with $z = y(x)$. The common solution to the equations (17.34) is seen to be given by

$$\mathrm{d}\phi(x) = \frac{(x+\gamma)}{2|x|}\,\mathrm{d}\psi^{(\alpha,\beta)}(1 - 2x^2 + 2\gamma^2). \tag{17.35}$$

It is easily verified that the condition $\mathscr{I}_{2n+1,2m+1} = 0$ for $n \neq m$ holds. Indeed, upon using (17.35) one finds (up to normalization)

$$\begin{aligned} \mathscr{I}_{2n+1,2m+1} &= \int_{|\gamma|}^{\sqrt{1+\gamma^2}} P_n^{(\alpha+1,\beta)}\big(y(x)\big) P_m^{(\alpha+1,\beta)}\big(y(x)\big)\,\big[\,(x-\gamma)^2\,\mathrm{d}\phi(x) - (x+\gamma)^2\,\mathrm{d}\phi(-x)\,\big]\\ &= \int_{|\gamma|}^{\sqrt{1+\gamma^2}} P_n^{(\alpha+1,\beta)}\big(y(x)\big) P_m^{(\alpha+1,\beta)}\big(y(x)\big)\,\mathrm{d}\psi^{(\alpha+1,\beta)}\big(y(x)\big) = \chi_n^{(\alpha+1,\beta)}\delta_{nm}, \end{aligned}$$

which follows from (17.31). The following result has thus been established.

Proposition 6. *[5] Let α, β and γ be real parameters such that $\alpha, \beta > -1$. The Chihara polynomials $C_n(x;\alpha,\beta,\gamma)$ satisfy the orthogonality relation*

$$\int_{\mathscr{E}} C_n(x) C_m(x)\,\omega(x)\,\mathrm{d}x = k_n\,\delta_{nm}, \tag{17.36}$$

on the interval $\mathscr{E} = [-\sqrt{1+\gamma^2}, -|\gamma|] \cup [|\gamma|, \sqrt{1+\gamma^2}]$. The weight function has the expression

$$\omega(x) = \theta(x)(x+\gamma)(x^2-\gamma^2)^{\alpha}(1+\gamma^2-x^2)^{\beta}, \tag{17.37}$$

where $\theta(x)$ is the sign function. The normalization factor k_n is given by

$$\begin{aligned} k_{2n} &= \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)}\,\frac{n!}{(2n+\alpha+\beta+1)\big[(n+\alpha+\beta+1)_n\big]^2},\\ k_{2n+1} &= \frac{\Gamma(n+\alpha+2)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+2)}\,\frac{n!}{(2n+\alpha+\beta+2)\big[(n+\alpha+\beta+2)_n\big]^2} \end{aligned} \tag{17.38}$$

Proof. The proof of the orthogonality relation follows from the above considerations. The normalization factor is obtained by comparison with that of the Jacobi polynomials (17.32). □

17.5.2 A Pearson-type equation

The weight function for the Chihara polynomials $C_n(x)$ can also be derived from their bispectral property (17.23) by solving a Pearson-type equation. A similar approach was adopted in [23] and led to the weight function for the Big -1 Jacobi polynomials. In view of the recurrence relation (17.21) satisfied by the Chihara polynomials $C_n(x)$, it follows from Favard's theorem that there exists a linear functional σ such that

$$\langle \sigma, C_n(x) C_m(x) \rangle = h_n \delta_{nm}, \tag{17.39}$$

with non-zero constants h_n. Moreover, it follows from (17.23) and from the completeness of the system of polynomials $\{C_n(x)\}$ that the operator $\mathscr{D}^{(\epsilon)}$ defined by (17.25) and (17.26) is symmetric with respect to the functional σ, which means that

$$\langle \sigma, \{\mathscr{D}^{(\epsilon)} V(x)\} W(x) \rangle = \langle \sigma, V(x) \{\mathscr{D}^{(\epsilon)} W(x)\} \rangle,$$

where $V(x)$ and $W(x)$ are arbitrary polynomials. In the positive-definite case $\alpha > -1$, $\beta > -1$, one has $h_n > 0$ and there is a realization of (17.39) in terms of an integral

$$\langle \sigma, C_n(x) C_m(x) \rangle = \int_a^b C_n(x) C_m(x)\, \mathrm{d}\sigma(x),$$

where $\sigma(x)$ is a distribution function and where a, b can be infinite. Let us consider the case where $\omega(x) = \mathrm{d}\sigma(x)/\mathrm{d}x > 0$ inside the interval $[a, b]$. In this case, the following condition must hold:

$$(\omega(x) \mathscr{D}^{(\epsilon)})^* = \omega(x) \mathscr{D}^{(\epsilon)}, \tag{17.40}$$

where $\mathscr{A}^*$ denotes the Lagrange adjoint operator with respect to $\mathscr{A}$. Recall that for a generic Dunkl differential operator

$$\mathscr{A} = \sum_{k=0}^{N} A_k(x) \partial_x^k + \sum_{\ell=0}^{N} B_k(x) \partial_x^k R,$$

where $A_k(x)$ and $B_k(x)$ are real functions, the Lagrange adjoint operator reads [23]

$$\mathscr{A}^* = \sum_{k=0}^{N} (-1)^k \partial_x^k A_k(x) + \sum_{\ell=0}^{N} \partial_x^k B_k(-x) R.$$

These formulas assume that the interval of orthogonality is necessarily symmetric. Let us now derive directly the expression for the weight function $\omega(x)$ from the condition (17.40). Assuming $\epsilon \in \mathbb{R}$, the Lagrange adjoint of $\mathscr{D}^{(\epsilon)}$ reads

$$[\mathscr{D}^{(\epsilon)}]^* = \partial_x^2 S_x + \partial_x T_{-x} R - \partial_x U_x - V_{-x} R + V_x \mathbb{I},$$

where the coefficients are given by (17.26). Upon imposing the condition (17.40), one finds the following equations for the terms in $\partial_x R$ and ∂_x:

$$\begin{aligned} &(x+\gamma)\omega(-x)+(-x+\gamma)\omega(x)=0,\\ &\omega'(x)=\left[\frac{\alpha}{x-\gamma}+\frac{\alpha+1}{x+\gamma}-\frac{2x\beta}{\gamma^2+1-x^2}\right]\omega(x). \end{aligned} \tag{17.41}$$

It is easily seen that the common solution to (17.41) is given by

$$\omega(x)=\theta(x)(x+\gamma)(x^2-\gamma^2)^{\alpha}(1+\gamma^2-x^2)^{\beta}, \tag{17.42}$$

which corresponds to the weight function (17.37) derived above. It is directly checked that with (17.42), the equations for the terms in ∂_x^2, R and $\mathbb{I}$ arising from the symmetry condition (17.40) are identically satisfied. The orthogonality relation (17.36) can be recovered by the requirements that $\omega(x)>0$ on a symmetric interval.

17.6 Chihara polynomials and Big q and -1 Jacobi polynomials

In this section, the connexion between the Chihara polynomials and the Big q-Jacobi and Big -1 Jacobi polynomials is established. In particular, it is shown that the Chihara polynomials are related to the former by a $q\to-1$ limit and to the latter by a Christoffel transformation.

17.6.1 Chihara polynomials and Big -1 Jacobi polynomials

The Big -1 Jacobi polynomials $J_n(x;a,b,c)$ were introduced in [25] as a $q\to-1$ limit of the Big q-Jacobi polynomials. In [23], they were seen to be the polynomials that diagonalize the most general first order differential Dunkl operator preserving the space of polynomials of a given degree (see (17.3)). The Big -1 Jacobi polynomials can be defined by their recurrence relation

$$x\,J_n(x)=J_{n+1}(x)+(1-A_n-C_n)J_n(x)+A_{n-1}C_nJ_{n-1}, \tag{17.43}$$

subject to the initial conditions $J_{-1}(x)=0$, $J_0(x)=1$ and where the recurrence coefficients read

$$A_n=\begin{cases}\frac{(1+c)(a+n+1)}{2n+a+b+2} & n\,\text{even}\\ \frac{(1-c)(n+a+b+1)}{2n+a+b+2} & n\,\text{odd}\end{cases},\quad C_n=\begin{cases}\frac{(1-c)n}{2n+a+b} & n\,\text{even}\\ \frac{(1+c)(n+b)}{2n+a+b} & n\,\text{odd}\end{cases}, \tag{17.44}$$

for $0<c<1$. Consider the monic polynomials $K_n(x)$ obtained from the Big -1 Jacobi polynomials $J_n(x)$ by the Christoffel transformation [6]

$$K_n(x)=\frac{1}{(x-1)}\left[J_{n+1}(x)-\frac{J_{n+1}(1)}{J_n(1)}J_n(x)\right]=(x-1)^{-1}[J_{n+1}(x)-A_nJ_n(x)], \tag{17.45}$$

where we have used the fact that

$$J_{n+1}(1)/J_n(1) = A_n,$$

which easily follows from (17.43) by induction. As is seen from (17.45), the polynomials $K_n(x)$ are kernel partners of the Big -1 Jacobi polynomials with kernel parameter 1. The inverse transformation, called the Geronimus transformation [14], is here given by

$$J_n(x) = K_n(x) - C_n K_{n-1}(x). \tag{17.46}$$

Indeed, it is directly verified that upon substituting (17.45) in (17.46), one recovers the recurrence relation (17.43) satisfied by the Big -1 Jacobi polynomials. In the reverse, upon substituting (17.46) in (17.45), one finds that the kernel polynomials $K_n(x)$ satisfy the recurrence relation

$$xK_n(x) = K_{n+1}(x) + (1 - A_n - C_{n+1})K_n(x) + A_n C_n K_{n-1}(x).$$

Using the expressions (17.44) for the recurrence coefficients, this recurrence relation can be cast in the form

$$xK_n(x) = K_{n+1}(x) + (-1)^{n+1} c K_n(x) + f_n K_{n-1}(x), \tag{17.47}$$

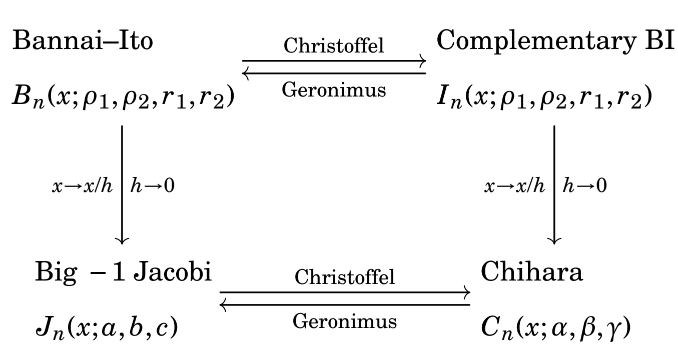
where

$$f_n = \begin{cases} \frac{(1-c^2)n(n+a+1)}{(2n+a+b)(2n+a+b+2)} & n \text{ even} \\ \frac{(1-c^2)(n+b)(n+a+b+1)}{(2n+a+b)(2n+a+b+2)} & n \text{ odd}. \end{cases} \tag{17.48}$$

It follows from the above recurrence relation that the kernel polynomials $K_n(x)$ of the Big -1 Jacobi polynomials correspond to the Chihara polynomials. Indeed, taking $x \to x\sqrt{1-c^2}$ and defining

$$\alpha = b/2 - 1/2, \qquad \beta = a/2 + 1/2, \qquad \frac{c}{\sqrt{1-c^2}} = -\gamma, \tag{17.49}$$

it is directly checked that the recurrence relation (17.47) with coefficients (17.48) corresponds to the recurrence relation (17.21) satisfied by the Chihara polynomials. We have thus established that the Chihara polynomials are the kernel partners of the Big -1 Jacobi polynomials with kernel parameter 1. In view of the fact that the Complementary Bannai–Ito polynomials are the kernel partners of the Bannai–Ito polynomials, we have



The precise limit process from the Bannai–Ito polynomials to the Big q-Jacobi polynomials can be found in [21].

17.6.2 Chihara polynomials and Big q-Jacobi polynomials

The Chihara polynomials also correspond to a $q \to -1$ limit of the Big q-Jacobi polynomials, different from the one leading to the Big -1 Jacobi polynomials. Recall that the monic Big q-Jacobi polynomials $P_n(x;\alpha,\beta,\gamma\,|\,q)$ obey the recurrence relation [16]

$$xP_n(x) = P_{n+1}(x) + (1 - v_n - \nu_n)P_n(x) + v_{n-1}\nu_n P_{n-1}(x), \tag{17.50}$$

with $P_{-1}(x) = 0$, $P_0(x) = 1$ and where

$$\begin{aligned} v_n &= \frac{(1-\alpha q^{n+1})(1-\alpha\beta q^{n+1})(1-\gamma q^{n+1})}{(1-\alpha\beta q^{2n+1})(1-\alpha\beta q^{2n+2})}, \\ \nu_n &= -\alpha\gamma q^{n+1}\,\frac{(1-q)^n(1-\alpha\beta\gamma^{-1}q^n)(1-\beta q^n)}{(1-\alpha\beta q^{2n})(1-\alpha\beta q^{2n+1})}. \end{aligned}$$

Upon writing

$$q = -e^{\epsilon}, \qquad \alpha = e^{2\epsilon\beta}, \qquad \beta = -e^{\epsilon(2\alpha+1)}, \qquad \gamma = -\gamma, \tag{17.51}$$

and taking the limit as $\epsilon \to 0$, the recurrence relation (17.50) is directly seen to converge, up to the redefinition of the variable $x \to x\sqrt{1-\gamma^2}$, to that of the Chihara polynomials (17.21). In view of the well known limit of the Askey-Wilson polynomials to the Big q-Jacobi polynomials, which can be found in [16], we can thus write

$$\begin{array}{ccc} \text{Askey-Wilson} & \xrightarrow{\;q\to-1\;} & \text{Complementary BI} \\ p_n(x;a,b,c,d\,|\,q) & & I_n(x;\rho_1,\rho_2,r_1,r_2) \\ {\scriptstyle x\to x/2a}\Big\downarrow{\scriptstyle a\to 0} & & {\scriptstyle x\to x/h}\Big\downarrow{\scriptstyle h\to 0} \\ \text{Big } q\text{-Jacobi} & \xrightarrow{\;q\to-1\;} & \text{Chihara} \\ P_n(x;\alpha,\beta,\gamma\,|\,q) & & C_n(x;\alpha,\beta,\gamma) \end{array}.$$

It is worth mentioning here that the limit process (17.51) cannot be used to derive the bispectrality property of the Chihara polynomials from the one of the Big q-Jacobi polynomials. Indeed, it can be checked that the q-difference operator diagonalized by the Big q-Jacobi polynomials does not exist in the limit (17.51). A similar situation occurs for the $q \to -1$ limit of the Askey-Wilson polynomials to the Complementary Bannai–Ito polynomials and is discussed in [11].

17.7 Special cases and limits of Chihara polynomials

In this section, three special/limit cases of the Chihara polynomials $C_n(x;\alpha,\beta,\gamma)$ are considered. One special case and one limit case correspond respectively to the generalized Gegenbauer and generalized Hermite polynomials, which are well-known from the theory of symmetric orthogonal polynomials [6]. The third limit case leads to a new bispectral family of -1 orthogonal polynomials which depend on two parameters and which can be seen as a one-parameter extension of the generalized Hermite polynomials.

17.7.1 Generalized Gegenbauer polynomials

It is easy to see from the explicit expression (17.20) that if one takes $\gamma = 0$, the Chihara polynomials $C_n(x;\alpha,\beta,\gamma)$ become symmetric, i.e. $C_n(-x) = (-1)^n C_n(x)$. Denoting by $G_n(x;\alpha,\beta)$ the polynomials obtained by specializing the Chihara polynomials to $\gamma = 0$, one directly has

$$G_{2n}(x) = \frac{(-1)^n(\alpha+1)_n}{(n+\alpha+\beta+1)_n}\, {}_2F_1\left[\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; x^2\right],$$
$$G_{2n+1}(x) = \frac{(-1)^n(\alpha+2)_n}{(n+\alpha+\beta+2)_n}\, x\, {}_2F_1\left[\begin{matrix} -n, n+\alpha+\beta+2 \\ \alpha+2 \end{matrix}; x^2\right].$$

The polynomials $G_n(x)$ are directly identified to the generalized Gegenbauer polynomials (see for example [3, 7]). In view of proposition (4), the polynomials $G_n(x)$ satisfy the recurrence relation

$$x\, G_n(x) = G_{n+1}(x) + \sigma_n\, G_{n-1}(x),$$

with $G_{-1}(x) = 0$, $G_0(x) = 1$ and where σ_n is given by (17.22). It follows from proposition (5) that the polynomials $G_n(x)$ satisfy the family of eigenvalue equations

$$\mathscr{W}^{(\epsilon)} G_n(x) = \lambda_n^{(\epsilon)} G_n(x), \tag{17.52}$$

where the eigenvalues are given by (17.24) and where the operator $\mathscr{W}^{(\epsilon)}$ has the expression

$$\mathscr{W}^{(\epsilon)} = S_x\, \partial_x^2 + U_x\, \partial_x + V_x(\mathbb{I} - R),$$

with the coefficients

$$S_x = \frac{x^2-1}{4}, \quad U_x = \frac{x^2(\alpha+\beta+3/2) - \alpha - 1/2}{2x}, \quad V_x = \frac{\alpha+1/2}{4x^2} - \frac{\alpha+\beta+3/2}{4} + \epsilon/2.$$

Upon taking

$$\epsilon \to (\alpha+1)(\mu+1/2), \quad \beta \to \alpha, \quad \alpha \to \mu - 1/2,$$

the eigenvalue equation (17.52) can be rewritten as

$$\mathcal{Q}G_n(x) = \Upsilon_n G_n(x), \tag{17.53}$$

where

$$\mathcal{Q} = (1-x^2)[D_x^{\mu}]^2 - 2(\alpha+1)xD_x^{\mu}, \tag{17.54}$$

where D_x^{μ} stands for the Dunkl derivative operator

$$D_x^{\mu} = \partial_x + \frac{\mu}{x}(\mathbb{I}-R), \tag{17.55}$$

and where the eigenvalues Υ_n are of the form

$$\Upsilon_{2n} = -2n(2n+2\alpha+2\mu+1), \qquad \Upsilon_{2n+1} = -(2n+2\mu+1)(2n+2\alpha+2). \tag{17.56}$$

The eigenvalue equation (17.53) with the operator (17.54) and eigenvalues (17.56) corresponds to the one obtained by Ben Cheikh and Gaied in their characterization of Dunkl-classical symmetric orthogonal polynomials [4]. The third proposition leads to the orthogonality relation

$$\int_{-1}^{1} G_n(x)G_m(x)\omega(x)\,\mathrm{d}x = k_n\delta_{nm},$$

where the normalization factor k_n is given by (17.38) and where the weight function reads

$$\omega(x) = |x|^{2\alpha+1}(1-x^2)^{\beta}.$$

We have thus established that the generalized Gegenbauer polynomials are -1 orthogonal polynomials which are descendants of the Complementary Bannai–Ito polynomials.

17.7.2 A one-parameter extension of the generalized Hermite polynomials

Another set of bispectral -1 orthogonal polynomials can be obtained upon letting

$$x \to \beta^{-1/2}x, \quad \alpha \to \mu - 1/2, \quad \gamma \to \beta^{-1/2}\gamma,$$

and taking the limit as $\beta \to \infty$. This limit is analogous to the one taking the Jacobi polynomials into the Laguerre polynomials [16]. Let $Y_n(x;\mu,\gamma)$ denote the monic polynomials obtained from the Chihara polynomials in this limit. The following properties of these polynomials can be derived by straightforward computations. The polynomials $Y_n(x;\gamma)$ have the hypergeometric expression

$$\begin{aligned} Y_{2n}(x) &= (-1)^n(\mu+1/2)_n\,{}_1F_1\left[\begin{matrix}-n\\ \mu+1/2\end{matrix};x^2-\gamma^2\right],\\ Y_{2n+1}(x) &= (-1)^n(\mu+3/2)_n\,(x-\gamma)\,{}_1F_1\left[\begin{matrix}-n\\ \mu+3/2\end{matrix};x^2-\gamma^2\right]. \end{aligned}$$

They satisfy the recurrence relation

$$x Y_n(x) = Y_{n+1}(x) + (-1)^n \gamma Y_n(x) + \vartheta_n Y_{n-1}(x),$$

with the coefficients

$$\vartheta_{2n} = n, \qquad \vartheta_{2n+1} = n + \mu + 1/2.$$

The polynomials $Y_n(x;\gamma)$ obey the one-parameter family of eigenvalue equations

$$\mathscr{Z}^{(\epsilon)} Y_n(x) = \lambda_n^{(\epsilon)} Y_n(x),$$

where the spectrum has the form

$$\lambda_{2n}^{(\epsilon)} = n, \qquad \lambda_{2n+1}^{(\epsilon)} = n + \epsilon.$$

The explicit expression for the second-order differential Dunkl operator $\mathscr{Z}^{(\epsilon)}$ is

$$\mathscr{Z}^{(\epsilon)} = S_x \partial_x^2 - T_x \partial_x R + U_x \partial_x + V_x(\mathbb{I} - R),$$

with

$$S_x = \frac{\gamma^2 - x^2}{4x^2}, \qquad T_x = \frac{\gamma(x-\gamma)}{4x^3},$$
$$U_x = \frac{x}{2} + \frac{\gamma}{4x^2} - \frac{\gamma^2}{2x^3} - \frac{\mu+\gamma^2}{2x}, \qquad V_x = \frac{3\gamma^2}{8x^4} - \frac{\gamma}{4x^3} + \frac{\mu+\gamma^2}{4x^2} + \epsilon\frac{x-\gamma}{2x} - \frac{1}{4}.$$

The algebra encoding this bispectrality of the polynomials $Y_n(x)$ is obtained by taking

$$K_1 = \mathscr{Z}^{(\epsilon)}, \quad K_2 = x, \quad P = R + \frac{\gamma}{x}(\mathbb{I} - R),$$

and defining $K_3 = [K_1, K_2]$. One then has the commutation relations

$$[K_1, P] = 0, \quad \{K_2, P\} = 2\gamma, \quad \{K_3, P\} = 0.$$

and

$$[K_2, K_3] = (2\epsilon - 1)K_2^2 P - 2\gamma K_3 P + (\gamma^2 - 2\gamma\epsilon + \mu)P + 1/2,$$
$$[K_3, K_1] = (1 - 2\epsilon)K_3 P + \epsilon(\epsilon - 1)K_2 + \gamma\epsilon(\epsilon - 1)P.$$

The orthogonality relation reads

$$\int_{\mathscr{S}} Y_n(x) Y_m(x) w(x)\, \mathrm{d}x = l_n \delta_{nm}$$

with $\mathscr{S} = (-\infty, -|\gamma|] \cup [|\gamma|, \infty)$ and with the weight function

$$w(x) = \theta(x)(x+\gamma)(x^2 - \gamma^2)^{\mu - 1/2} e^{-x^2}.$$

The normalization factors

$$l_{2n} = n!\, e^{-\gamma^2}\, \Gamma(n + \mu + 1/2), \qquad l_{2n+1} = n!\, e^{-\gamma^2}\, \Gamma(n + \mu + 3/2),$$

are obtained using the observation that the polynomials $Y_n(x;\gamma)$ can be expressed in terms of the standard Laguerre polynomials [16].

17.7.3 Generalized Hermite polynomials

The polynomials $Y_n(x;\mu,\gamma)$ can be can be considered as a one-parameter extension of the generalized Hermite polynomials. Indeed, upon denoting by $H_n^{\mu}(x)$ the polynomials obtained by taking $\gamma=0$ in $Y_n(x;\mu,\gamma)$, one finds that

$$H_{2n}^{\mu}(x)=(-1)^n(\mu+1/2)_n\,{}_1F_1\left[\begin{matrix}-n\\ \mu+1/2\end{matrix};x^2\right],$$
$$H_{2n+1}^{\mu}(x)=(-1)^n(\mu+3/2)_n\,x\,{}_1F_1\left[\begin{matrix}-n\\ \mu+3/2\end{matrix};x^2\right],$$

which corresponds to the generalized Hermite polynomials [6]. It is thus seen that the generalized Hermite polynomials are also -1 orthogonal polynomials that can be obtained from the Complementary Bannai–Ito polynomials. For this special case, the eigenvalue equations can be written (taking $\epsilon\to\epsilon/2$) as

$$\Omega^{(\epsilon)}H_n^{\mu}(x)=\lambda_n^{(\epsilon)}H_n(x),$$

with

$$\lambda_{2n}^{(\epsilon)}=2n,\qquad \lambda_{2n+1}^{(\epsilon)}=2n+\epsilon.$$

and where the operator $\Omega^{(\epsilon)}$ reads

$$\Omega^{(\epsilon)}=-\frac{1}{2}\partial_x^2+\left(x-\frac{\mu}{x}\right)\partial_x+\left(\frac{\mu}{2x^2}+\frac{\epsilon-1}{2}\right)(\mathbb{I}-R).$$

The orthogonality relation then reduces to

$$\int_{-\infty}^{\infty}H_n^{\mu}(x)H_m^{\mu}(x)\,|x|^{\mu}e^{-x^2}\,\mathrm{d}x=l_n\delta_{nm}.$$

Upon taking $\widetilde{\Omega}^{(\epsilon)}=e^{-x^2/2}\Omega^{(\epsilon)}e^{x^2/2}$, the eigenvalue equations can be written as

$$\widetilde{\Omega}^{(\epsilon)}\psi_n(x)=\lambda_n^{(\epsilon)}\psi_n(x),$$

with $\psi_n=e^{-x^2/2}H_n^{\mu}(x)$ and with the eigenvalues

$$\lambda_{2n}^{(\epsilon)}=2n+\mu+1/2,\qquad \lambda_{2n+1}^{(\epsilon)}=2n+\mu+3/2+\epsilon.$$

The operator $\widetilde{\Omega}^{(\epsilon)}$ can be cast in the form

$$\widetilde{\Omega}^{(\epsilon)}=-\frac{1}{2}(D_x^{\mu})^2+\frac{1}{2}x^2+\frac{\epsilon}{2}(\mathbb{I}-R),$$

where D_x^{μ} is the Dunkl derivative (17.55). The operator $\widetilde{\Omega}^{(\epsilon)}$ corresponds to the Hamiltonian of the one-dimensional Dunkl oscillator [20]. Two-dimensional versions of this oscillator models have been considered recently [9, 10, 12].

17.8 Conclusion

In this paper, we have characterized a novel family of -1 orthogonal polynomials in a continuous variable which are obtained from the Complementary Bannai–Ito polynomials by a limit process. These polynomials have been called the Chihara polynomials and it was shown that they diagonalize a second-order differential Dunkl operator with a quadratic spectrum. The orthogonality weight, the recurrence relation and the explicit expression in terms of Gauss hypergeometric function have also been obtained. Moreover, special cases and descendants of these Chihara polynomials have been examined. From these considerations, it was observed that the well-known generalized Gegenbauer/Hermite polynomials are in fact -1 polynomials. In addition, a new class of bispectral -1 orthogonal polynomials which can be interpreted as a one-parameter extension of the generalized Hermite polynomials has been defined.

With the results presented here, the polynomials in the higher portion of the emerging tableau of -1 orthogonal polynomials are now identified and characterized. At the top level of the tableau sit the Bannai–Ito polynomials and their kernel partners, the Complementary Bannai–Ito polynomials. Both sets depend on four parameters. At the next level of this -1 tableau, with three parameters, one has the Big -1 Jacobi polynomials, which are descendants of the BI polynomials, as well as the dual -1 Hahn polynomials (see [22]) and the Chihara polynomials which are descendants of the CBI polynomials. The complete tableau of -1 polynomials with arrows relating them shall be presented in an upcoming review.

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Chapitre 18

The Bannai–Ito polynomials as Racah coefficients of the $sl_{-1}(2)$ algebra

V. X. Genest, L. Vinet et A. Zhedanov (2014). The Bannai–Ito polynomials as Racah coefficients of the $sl_{-1}(2)$ algebra. *Proceedings of the American Mathematical Society* **142** 1545-1560

Abstract. The Bannai-Ito polynomials are shown to arise as Racah coefficients for $sl_{-1}(2)$. This Hopf algebra has four generators including an involution and is defined with both commutation and anticommutation relations. It is also equivalent to the parabosonic oscillator algebra. The coproduct is used to show that the Bannai-Ito algebra acts as the hidden symmetry algebra of the Racah problem for $sl_{-1}(2)$. The Racah coefficients are recovered from a related Leonard pair.

Introduction

The $sl_{-1}(2)$ algebra was introduced recently in [20] as a deformation of the classical $sl(2)$ Lie algebra; it is defined in terms of four generators, including an involution, satisfying both commutation and anticommutation relations. This algebra can also be obtained from the quantum algebra $sl_q(2)$ by taking the limit $q \to -1$ and is furthermore the dynamical algebra of a parabosonic oscillator [10, 13]. We here consider the Racah problem for this algebra.

Recently, a series of orthogonal polynomials corresponding to limits $q \to -1$ of q-polynomials of the Askey scheme were discovered [19, 22, 25, 26]. These polynomials are eigenfunctions of operators of Dunkl type, which involve the reflection operator [7, 24]. Interestingly, these polynomials have also been related to Jordan anticommutator algebras [21]. In most references, so far, these $q = -1$ polynomials have been left buried in the standard classifications. In view of their bispectrality and remarkable properties, a -1 scheme would deserve to be highlighted.

At the top of the discrete variable branch of this $q = -1$ class of polynomials lie the Bannai-Ito (BI) polynomials [2] and their kernel partners, the complementary Bannai-Ito polynomials [22]. Both sets depend on four parameters and are expressible in terms of Wilson polynomials [2, 12, 22]. The BI polynomials possess the Leonard duality property, which in fact led to their initial discovery in [2]. In contradistinction, the complementary BI polynomials and their descendants, the dual $q = -1$ Hahn polynomials [19], are also bispectral but fall beyond the scope of the Leonard duality.

The Clebsch-Gordan problem for $sl_{-1}(2)$ was first solved in [20]; it was shown that the coupling coefficients for two $sl_{-1}(2)$ algebras, also called Clebsch-Gordan or Wigner coefficients, are proportional to the dual $q = -1$ Hahn polynomials [19]. In this paper, we investigate the Racah problem for $sl_{-1}(2)$, which is tantamount to finding the coupling coefficients for three parabosonic oscillators. It is shown that these coefficients are also expressed in terms of $q = -1$ polynomials, in this case the Bannai-Ito polynomials. Our approach consists in constructing the Jordan algebra of the intermediary Casimir operators that appear in the coproduct [6] of three $sl_{-1}(2)$ algebras; this anticommutator algebra coincides with the Bannai-Ito algebra [22], a special case of the Askey-Wilson algebra introduced in [27]. The two Casimir operators are then shown to form a Leonard pair [3, 5, 11, 16, 17, 18], an observation which allows to recover the recurrence relation of the Bannai-Ito polynomials for the overlap (Racah) coefficients.

The outline of the paper is as follows. In section 1, we recall the definition of the $sl_{-1}(2)$ algebra, its irreducible representations and its coproduct structure. We also provide a review of the theory of the Bannai-Ito polynomials and go over the basics of Leonard pairs and the corresponding Askey-Wilson relations [16, 23]. In section 2, we review the Clebsch-Gordan problem for the parabosonic algebra $sl_{-1}(2)$. In section 3, we show that the intermediary Casimir operators (K_1,K_2) of the sum of three $sl_{-1}(2)$ algebras form the Bannai-Ito algebra. In section 4, the operators (K_1,K_2) are re-expressed as a Leonard pair which is used to recover the recurrence relation satisfied by the overlap coefficients (Racah) coefficients. The exact expression for the Racah coefficients is finally obtained up to a phase factor using the orthogonality relation of the BI polynomials. In section 5, we discuss the degenerate case of the Bannai-Ito algebra corresponding to the anticommutator spin algebra [8, 14, 1, 3]. We conclude by explaining that the operators K_1 and K_2, together with their anticommutator K_3, form a Leonard triple. A different

Racah problem, which involves modifying the addition rule of $sl_{-1}(2)$, is considered to that end.

18.1 The $sl_{-1}(2)$ algebra, Bannai-Ito polynomials and Leonard pairs

18.1.1 $sl_{-1}(2)$ essentials

The Hopf algebra $sl_{-1}(2)$ [20] is generated by four operators J_0, J_+, J_- and R satisfying the relations

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_0, R] = 0, \qquad \{J_+, J_-\} = 2J_0, \qquad \{J_\pm, R\} = 0, \tag{18.1}$$

where $[x, y] = xy - yx$ and $\{x, y\} = xy + yx$. The operator R is an involution, which means that it satisfies the property

$$R^2 = \mathfrak{i}\mathfrak{d}, \tag{18.2}$$

where $\mathfrak{i}\mathfrak{d}$ is the identity. The Casimir operator, which commutes with all $sl_{-1}(2)$ elements, is given by

$$\mathcal{Q} = J_+J_-R - J_0R + R/2. \tag{18.3}$$

Let $\epsilon = \pm 1$ and $\mu \geqslant 0$ be two parameters; we denote by (ϵ, μ) the infinite-dimensional vector space spanned by the basis $|\,\epsilon; \mu; n\,\rangle$, $n \in \mathbb{N}$, endowed with the actions

$$\begin{aligned} &J_0|\,\epsilon;\mu;n\,\rangle = (n + \mu + 1/2)|\,\epsilon;\mu;n\,\rangle, \quad R|\,\epsilon;\mu;n\,\rangle = \epsilon(-1)^n|\,\epsilon;\mu;n\,\rangle, \\ &J_+|\,\epsilon;\mu;n\,\rangle = \sqrt{[n+1]_\mu}|\,\epsilon;\mu;n+1\,\rangle, \quad J_-|\,\epsilon;\mu;n\,\rangle = \sqrt{[n]_\mu}|\,\epsilon;\mu;n-1\,\rangle, \end{aligned} \tag{18.4}$$

where $[n]_\mu$ denotes the μ-number

$$[n]_\mu = n + \mu(1 - (-1)^n). \tag{18.5}$$

With the actions (18.4), the vector space (ϵ, μ) forms an irreducible $sl_{-1}(2)$-module. On this module, the Casimir operator is a multiple of the identity

$$\mathcal{Q}|\,\epsilon;\mu;n\,\rangle = -\epsilon\,\mu|\,\epsilon;\mu;n\,\rangle, \tag{18.6}$$

as expected from Schur's lemma. On the space (ϵ,μ), the algebra $sl_{-1}(2)$ is equivalent to the parabosonic oscillator algebra. Indeed, one has

$$[J_-,J_+] = \{J_-,J_+\} - 2J_+J_- = 2J_0 - 2J_+J_-. \tag{18.7}$$

Using the expression (18.3) for the Casimir operator and its action on vectors of (ϵ,μ), we find

$$[J_-,J_+] = 1 + 2\epsilon\mu R. \tag{18.8}$$

The operators $J_\pm$ satisfying the commutation relation (18.8), together with the operator R obeying the relations $R^2 = \mathfrak{i}\mathfrak{d}$ and $\{R,J_\pm\} = 0$, define the parabosonic oscillator algebra [6, 13, 15].

The algebra $sl_{-1}(2)$ admits a non-trivial addition rule, or coproduct. Let (ϵ_1,μ_1) and (ϵ_2,μ_2) be two $sl_{-1}(2)$-modules. A third module can be obtained by taking tensor product $(\epsilon_1,\mu_1)\otimes(\epsilon_2,\mu_2)$ equipped with the transformations

$$\begin{aligned} J_0(v\otimes w) &= (J_0 v)\otimes w + v\otimes(J_0 w),\\ J_\pm(v\otimes w) &= (J_\pm v)\otimes(Rw) + v\otimes(J_\pm w),\\ R(v\otimes w) &= (Rv)\otimes(Rw), \end{aligned} \tag{18.9}$$

where $v\in(\epsilon_1,\mu_1)$ and $w\in(\epsilon_2,\mu_2)$. The addition rule for $sl_{-1}(2)$ can also be presented without referring to any representation. Let $J_0^{(i)}$, $J_\pm^{(i)}$ and $R^{(i)}$ be two mutually commuting sets of $sl_{-1}(2)$ generators. A third algebra, denoted symbolically $3 = 1\oplus 2$, is obtained by defining

$$J_0^{(3)} = J_0^{(1)} + J_0^{(2)}, \qquad J_\pm^{(3)} = J_\pm^{(1)}R^{(2)} + J_\pm^{(2)}, \qquad R^{(3)} = R^{(1)}R^{(2)}. \tag{18.10}$$

It is easily verified that the generators $J_0^{(3)}$, $J_\pm^{(3)}$ and $R^{(3)}$ satisfy the defining relations of $sl_{-1}(2)$ given in (18.1). The Casimir operator for the third algebra, denoted by $\mathcal{Q}_{12}$, is

$$\mathcal{Q}_{12} = J_+^{(3)}J_-^{(3)}R^{(3)} - J_0^{(3)}R^{(3)} + (1/2)R^{(3)} \tag{18.11}$$

18.1.2 Bannai-Ito polynomials

Bannai and Ito discovered their polynomials in 1984 in their complete classification of orthogonal polynomials satisfying the Leonard duality property [2]. These polynomials were shown to be $q=-1$ limits of the q-Racah polynomials and many of their properties (e.g. recurrence relation, weight function, hypergeometric representation) were given in their book [2]. Recently, it was shown in [22] that the Bannai-Ito polynomials also occur naturally as eigensolutions of Dunkl shift operators.In the following, we review some of the properties of the BI polynomials.

The monic BI polynomials satisfy the recurrence relation

$$P_{n+1}(x)+(\rho_1-A_n-C_n)P_n(x)+A_{n-1}C_nP_{n-1}(x)=xP_n(x), \tag{18.12}$$

where

$$A_n=\begin{cases}\frac{(n+1+2\rho_1-2r_1)(n+1+2\rho_1-2r_2)}{4(n+1-r_1-r_2+\rho_1+\rho_2)}, & n \text{ even},\\ \frac{(n+1-2r_1-2r_2+2\rho_1+2\rho_2)(n+1+2\rho_1+2\rho_2)}{4(n+1-r_1-r_2+\rho_1+\rho_2)}, & n \text{ odd},\end{cases} \tag{18.13}$$

$$C_n=\begin{cases}-\frac{n(n-2r_1-2r_2)}{4(n-r_1-r_2+\rho_1+\rho_2)}, & n \text{ even},\\ -\frac{(n-2r_2+2\rho_2)(n-2r_1+2\rho_2)}{4(n-r_1-r_2+\rho_1+\rho_2)}, & n \text{ odd}.\end{cases} \tag{18.14}$$

The polynomials satisfying (18.12) are called positive definite if $U_n=A_{n-1}C_n>0$ for all $n\geqslant 1$. This condition is also equivalent to the existence of a positive orthogonality measure for the polynomials $P_n(x)$. In the case of the BI polynomials, it is seen that this condition cannot be fulfilled for all values of n. However, if $U_i>0$ for $i=1,\dots,N$ and $U_{N+1}=0$, it is known that one has a finite system of orthogonal polynomials $P_0(x),P_1(x),\dots,P_N(x)$ satisfying the discrete orthogonality relation

$$\sum_{s=0}^{N}\omega_s(x_s)P_n(x_s)P_m(x_s)=h_n\delta_{nm}, \quad h_n=u_1,\dots,u_n, \tag{18.15}$$

on the lattice x_s, where $s=0,1,\dots,N$. The discrete points x_s are the simple roots of the polynomial $P_{N+1}(x)$ [4].

When N is an even integer, the truncation condition $U_{N+1}=0$ is equivalent to one of the four possible conditions

$$2(r_i-\rho_k)=N+1, \quad i,k=1,2. \tag{18.16}$$

The case of relevance here is

$$2(r_2-\rho_1)=N+1. \tag{18.17}$$

We introduce the following parametrization:

$$\begin{aligned}&2\rho_1=(b+c), \quad 2\rho_2=(2a+b+c+N+1),\\ &2r_1=(c-b), \quad 2r_2=(b+c+N+1),\end{aligned} \tag{18.18}$$

where a, b and c are arbitrary positive parameters. Assuming (18.18), the coefficient U_n takes the form:

$$U_n = \begin{cases} \frac{n(N+2c+1-n)(n+2a+2b)(n+2a+2b+2c+N+1)}{16(a+b+n)^2}, & n \text{ even}, \\ \frac{(N+1-n)(2a+n)(2b+n)(n+2a+2b+N+1)}{16(a+b+n)^2}, & n \text{ odd}. \end{cases} \tag{18.19}$$

From this expression, it is obvious that $U_{N+1} = 0$ and that the positivity condition $U_n > 0$ is satisfied for $n = 0, \ldots, N$. With this parametrization, the Bannai-Ito polynomials obey the orthogonality relation

$$\textstyle\sum_{\ell=0}^{N} \Omega_\ell P_n(x_\ell) P_m(x_\ell) = \Phi_{N,n} \delta_{nm}. \tag{18.20}$$

The orthogonality grid is given by

$$x_\ell = \tfrac{1}{2}\left[(-1)^\ell(\ell + b + c + 1/2) - 1/2\right]. \tag{18.21}$$

The weight function Ω_ℓ takes the form

$$\Omega_\ell = (-1)^q \frac{(-N/2)_{k+q}(1/2+b)_{k+q}(1+b+c)_k(3/2+a+b+c+N/2)_k}{(1/2+c)_{k+q}(1+b+c+N/2)_{k+q}(1/2-a-N/2)_k k!}, \tag{18.22}$$

where $\ell = 2k+q$ with $q = 0, 1$ and where $(x)_n = (x)(x+1)\cdots(x+n-1)$ stands for the Pochhammer symbol. Furthermore, the normalization factor $\Phi_{N,n}$ is found to be

$$\Phi_{N,n} = \frac{m!k!}{(m-k-q)!}\left[\frac{(1+a+b+k)_{m-k}(1+b+c)_m}{(1/2+a+k+q)_{m-k-q}(1/2+c)_{m-k}}\right] \times \left[\frac{(1/2+b)_{k+q}(m+1+a+b)_{k+q}(m+3/2+a+b+c)_k}{(k+1+a+b)^2_{k+q}}\right], \tag{18.23}$$

where $m = N/2$ and $n = 2k+q$ with $q = 0, 1$. The other truncation conditions in (18.16) can be treated similarly.

When N is an odd integer, the truncation condition $U_{N+1} = 0$ is equivalent to one of the three conditions

$$i)\ \rho_1 + \rho_2 = -\frac{N+1}{2}, \quad ii)\ r_1 + r_2 = \frac{N+1}{2}, \quad iii)\ \rho_1 + \rho_2 - r_1 - r_2 = -\frac{N+1}{2}. \tag{18.24}$$

The condition $iii)$ leads to a singular U_n for $n = (N+1)/2$. Consequently, only the conditions $i)$ and $ii)$ are admissible. The case of relevance here is

$$2(\rho_1 + \rho_2) = -(N+1). \tag{18.25}$$

We introduce the following parametrization:

$$2\rho_1 = (\beta+\gamma), \quad 2\rho_2 = -(\beta+\gamma+N+1),$$
$$2r_1 = (\gamma-\beta), \quad 2r_2 = -(2\alpha+\beta+\gamma+N+1), \tag{18.26}$$

where α, β and γ are arbitrary positive parameters. Assuming (18.26), the coefficient U_n becomes

$$U_n = \begin{cases} \frac{n(N+1-n)(n+2\alpha+2\beta)(n+2\alpha+2\beta+N+1)}{16(\alpha+\beta+n)^2}, & n \text{ even}, \\ \frac{(N+2\gamma+1-n)(2\alpha+n)(2\beta+n)(n+2\alpha+2\beta+2\gamma+N+1)}{16(\alpha+\beta+n)^2}, & n \text{ odd}. \end{cases} \tag{18.27}$$

In this form, the truncation and positivity conditions are manifestly satisfied. With these parameters, the Bannai-Ito polynomials obey the orthogonality relation

$$\sum_{\ell=0}^{N} \Omega_\ell P_n(x_\ell) P_m(x_\ell) = \Phi_{N,n}\delta_{nm}. \tag{18.28}$$

The grid is given by

$$x_\ell = \frac{1}{2}\Big[(-1)^\ell(\ell+\beta+\gamma+1/2)-1/2\Big]. \tag{18.29}$$

The weight function takes the form

$$\Omega_\ell = (-1)^q \frac{(\frac{1-N}{2})_k(\frac{1}{2}+\beta)_{k+q}(1+\beta+\gamma)_k(1+\alpha+\beta+\gamma+\frac{N}{2})_{k+q}}{(\frac{1}{2}+\gamma)_{k+q}(-\alpha-\frac{N}{2})_{k+q}(\frac{3}{2}+\beta+\gamma+\frac{N}{2})_k k!}, \tag{18.30}$$

where $\ell = 2k+q$ with $q = 0,1$ and the normalization factor can be evaluated to

$$\Phi_{N,n} = \frac{(m-1)!k!}{(m-k-1)!}\left[\frac{(1+\alpha+\beta+k)_{m-k}(1+\beta+\gamma)_m}{(1/2+k+q+\alpha)_{m-k-q}(1/2+\gamma)_{m-k-q}}\right] \times \left[\frac{(1/2+\beta)_{k+q}(m+1+\alpha+\beta)_k(m+1/2+\alpha+\beta+\gamma)_{k+q}}{(k+1+\alpha+\beta)^2_{k+q}}\right], \tag{18.31}$$

where $m = (N+1)/2$ and $n = 2k+q$ with $q = 0,1$.

The Bannai-Ito polynomials correspond to the limit $q \to -1$ of the classical Wilson polynomials and admit a hypergeometric representation. The truncated generalized hypergeometric series is defined by

$${}_{p+1}F_q\left(\begin{matrix} -n, a_1, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}; x\right) = \sum_{j=0}^{n} \frac{(-n)_j(a_1)_j\cdots(a_p)_j}{(b_1)_j(b_2)_j\cdots(b_q)_j}\frac{x^j}{j!}. \tag{18.32}$$

We define

$$W_{2n}(x) = \kappa_n^{(1)}\,{}_4F_3\left(\begin{matrix} -n,\, n+g+1,\, \rho_2+x,\, \rho_2-x \\ \rho_1+\rho_2+1,\, \rho_2-r_1+\frac{1}{2},\, \rho_2-r_2+\frac{1}{2} \end{matrix};1\right), \tag{18.33}$$

$$W_{2n+1}(x) = \kappa_n^{(2)}(x-\rho_2)\,{}_4F_3\left(\begin{matrix} -n,\, n+g+2,\, \rho_2+1+x,\, \rho_2+1-x \\ \rho_1+\rho_2+2,\, \rho_2-r_1+\frac{3}{2},\, \rho_2-r_2+\frac{3}{2} \end{matrix};1\right), \tag{18.34}$$

with $g = \rho_1+\rho_2-r_1-r_2$ and where the factors which ensure that the polynomials are monic are given by

$$\kappa_n^{(1)} = \frac{(1+\rho_1+\rho_2)_n(\rho_2-r_1+1/2)_n(\rho_2-r_2+1/2)_n}{(n+g+1)_n}, \tag{18.35}$$

$$\kappa_n^{(2)} = \frac{(2+\rho_1+\rho_2)_n(\rho_2-r_1+3/2)_n(\rho_2-r_2+3/2)_n}{(n+g+2)_n}. \tag{18.36}$$

The monic BI polynomials have the following expression:

$$P_n(x) = W_n(x) - C_n W_{n-1}(x), \tag{18.37}$$

where C_n is given by (18.13).

18.1.3 Leonard pairs and Askey-Wilson relations

Let V be a $\mathbb{C}$-vector space of dimension $N+1$. A square matrix X is said *irreducible tridiagonal* if each of its non-zero entry lies on either the diagonal, sub-diagonal or super-diagonal and if each entry on the super-diagonal and sub-diagonal are non-zero. A Leonard pair on V is an ordered pair of linear transformations $(K_1,K_2) \in \mathrm{End}\,V$ satisfying the following conditions [16]:

- There exists a basis for V with respect to which the matrix representing K_1 is diagonal and the matrix representing K_2 is irreducible tridiagonal.

- There exists a basis for V with respect to which the matrix representing K_2 is diagonal and the matrix representing K_1 is irreducible tridiagonal.

Leonard pairs have deep connections with orthogonal polynomials on finite grids and have also appeared in combinatorics [2, 16]. Given a Leonard pair (K_1,K_2), it is known [18, 23, 27] that K_1, K_2 obey the so-called Askey-Wilson relations

$$\begin{aligned} &K_1^2K_2 - \beta K_1K_2K_1 + K_2K_1^2 - \gamma_1\{K_1,K_2\} - \rho_1K_2 = \gamma_2K_1^2 + \omega K_1 + \eta_1\mathfrak{i}\mathfrak{d}, \\ &K_2^2K_1 - \beta K_2K_1K_2 + K_1K_2^2 - \gamma_2\{K_1K_2\} - \rho_2K_2 = \gamma_1K_1^2 + \omega K_1 + \eta_2\mathfrak{i}\mathfrak{d}, \end{aligned} \tag{18.38}$$

with scalars $\{\beta, \gamma_i, \eta_i, \rho_i\} \in \mathbb{C}$. These scalars are uniquely defined provided that the dimension of the vector space is at least 4. The converse is not always true. Indeed, if one sets $\beta = q + q^{-1}$ and q a root of unity, two linear transformations obeying relations (18.38) do not necessarily form a Leonard pair [17]. In the present work we will nonetheless obtain a Leonard pair satisfying the relations (18.38) with $q = -1$.

We briefly recall how orthogonal polynomials occur in this context. Consider a Leonard pair (K_1, K_2) on a vector space V of dimension $N+1$. By definition, the eigenvalues of K_1 and K_2 are mutually distinct. Denoting the eigenvalues of K_1 by $\lambda_i^{(1)}$ for $i = 0, 1, \ldots, N$, there exists a basis of V in which the matrices representing K_1 and K_2 are of the form

$$K_1 = \begin{pmatrix} \lambda_0^{(1)} & & & \mathbf{0} \\ & \lambda_1^{(1)} & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_N^{(1)} \end{pmatrix}, \qquad K_2 = \begin{pmatrix} a_0 & c_1 & & & \mathbf{0} \\ x_0 & a_1 & c_2 & & \\ & x_1 & a_2 & \ddots & \\ & & \ddots & \ddots & c_N \\ \mathbf{0} & & & x_{N-1} & a_N \end{pmatrix}. \tag{18.39}$$

One can define the sequence of polynomials p_i with $i = 0, \ldots, N$ and initial condition $p_{-1} = 0$ satisfying the recurrence relation

$$y\, p_i(y) = c_{i+1} p_{i+1}(y) + a_i p_i(y) + x_{i-1} p_{i-1}(y). \tag{18.40}$$

The matrix $P_{ij} = p_i(\lambda_j^{(2)})$, where $\lambda_j^{(2)}$, $j \in \{0, 1, \ldots, N\}$, denotes the eigenvalues of K_2, defines the similarity transformation which brings the matrix K_2 to its diagonal form. In physical terms, given a pair (K_1, K_2) of operators expressed in the form (18.39) acting on a state space, the polynomials defined by the recurrence relation (18.40) are the overlap coefficients between the bases in which either K_1 or K_2 is diagonal. For more details, see [16].

18.2 The Clebsch-Gordan problem

The Clebsch-Gordan (CG) problem of $sl_{-1}(2)$ has been solved in [20]. We recall here some of the results concerning this problem which shall prove useful.

The CG problem can be posited in the following way. We consider the $sl_{-1}(2)$-module $(\epsilon_1, \mu_1) \otimes (\epsilon_2, \mu_2)$ or equivalently the addition of two $sl_{-1}(2)$ algebras. It is seen that the operator $J_0^{(3)} = J_0^{(1)} + J_0^{(2)}$ has eigenvalues of the form $\mu_1 + \mu_2 + N + 1$, $N \in \mathbb{N}$. We denote by $| q_{12}, N \rangle$ the state with eigenvalue q_{12} of the Casimir operator $\mathcal{Q}_{12}$ and with a given value N of the total projection. We have

$$\mathcal{Q}_{12} | q_{12}, N \rangle = q_{12} | q_{12}, N \rangle, \qquad J_0^{(3)} | q_{12}, N \rangle = (\mu_1 + \mu_2 + 1 + N) | q_{12}, N \rangle. \tag{18.41}$$

In view of the formula (18.6), the eigenvalues q_{12} of the Casimir operator $\mathcal{Q}_{12}$ can be decomposed as the product

$$q_{12} = -\epsilon_{12}\mu_{12}, \quad \epsilon_{12} = \pm 1, \; \mu_{12} \geqslant 0, \tag{18.42}$$

whence we have $|q_{12}| = \mu_{12}$. The Casimir operator (18.11) of the added algebras can be re-expressed in terms of the local Casimir operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ in the following way:

$$\mathcal{Q}_{12} = \left(J_-^{(1)} J_+^{(2)} - J_+^{(1)} J_-^{(2)} \right) R^{(1)} - (1/2) R^{(1)} R^{(2)} + \mathcal{Q}_1 R^{(2)} + \mathcal{Q}_2 R^{(1)}. \tag{18.43}$$

The state $| q_{12}, N \rangle$ can be decomposed as a linear combination of the tensor product states

$$| q_{12}, N \rangle = \sum_{n_1 + n_2 = N} C_{n_1 n_2 N}^{\mu_1 \mu_2 q_{12}} | \epsilon_1, \mu_1, n_1 \rangle \otimes | \epsilon_2, \mu_2, n_2 \rangle. \tag{18.44}$$

The coefficients $C_{n_1 n_2 N}^{\mu_1 \mu_2 q_{12}}$ are the Clebsch-Gordan coefficients of the $sl_{-1}(2)$ algebra. We note that these coefficients vanish unless $n_1 + n_2 = N$ and that their dependence on ϵ_1 and ϵ_2 is implicit.

The possible values of the eigenvalues q_{12} of the Casimir operator $\mathcal{Q}_{12}$ are given by

$$q_{12} = (-1)^{s+1} \epsilon_1 \epsilon_2 (\mu_1 + \mu_2 + 1/2 + s), \quad s = 0, 1, \ldots, N. \tag{18.45}$$

This result can be derived in the following way. In a given module (ϵ, μ), the eigenvalues λ_{J_0} of J_0 are

$$\lambda_{J_0} = n - \epsilon \mathcal{Q} + 1/2, \quad n \in \mathbb{N}. \tag{18.46}$$

Hence, for a given eigenvalue $\lambda_{J_0} > 0$ of J_0, the eigenvalues q of the Casimir operator $\mathcal{Q}$ which are compatible with λ_{J_0} are, in absolute value,

$$|q| = |\lambda - 1/2|, |\lambda - 3/2|, \ldots \tag{18.47}$$

When considering the coproduct of two $sl_{-1}(2)$ algebras, the eigenvalues of $J_0^{(3)}$ are $\lambda^{(3)} = \mu_1 + \mu_2 + N + 1$. Consequently, for a given value of N, the set of allowed values for the eigenvalues q_{12} of the Casimir $\mathcal{Q}_{12}$, which should be of cardinality $N + 1$, is

$$|q_{12}| = \mu_1 + \mu_2 + N + 1/2, \, \mu_1 + \mu_2 + N - 1/2, \ldots, \mu_1 + \mu_2 + 1/2. \tag{18.48}$$

Thus the admissible values of $\mu_{12} = |q_{12}|$ are given by the above set (18.48).

There remains to evaluate the corresponding values of ϵ_{12}. To that end, we consider the eigenstate $|\, x\,\rangle$ of $\mathscr{Q}_{12}$ corresponding to the maximal value $\mu_{12})_{\max} = \mu_1 + \mu_2 + N + 1/2$. It is seen that this state satisfies the properties

$$J_0^{(3)}|\, x\,\rangle = (\mu_1 + \mu_2 + N + 1)|\, x\,\rangle, \quad J_-^{(3)}|\, x\,\rangle = 0. \tag{18.49}$$

On the one hand, it then follows from (18.49) and (18.4) that

$$R^{(3)}|\, x\,\rangle = \epsilon_{12})_{\max}|\, x\,\rangle, \tag{18.50}$$

where $\epsilon_{12})_{\max}$ is the value of ϵ_{12} corresponding to the maximal value of μ_{12}. On the other hand, it stems from (18.41) and (18.4) that

$$R^{(3)}|\, q_{12}, N\,\rangle = (-1)^N \epsilon_1 \epsilon_2 |\, q_{12}, N\,\rangle. \tag{18.51}$$

We thus have $\epsilon_{12})_{\max} = (-1)^N \epsilon_1 \epsilon_2$. It follows that the eigenvalue q_{12} of the Casimir operator $\mathscr{Q}_{12}$ corresponding to the maximal value of $|q_{12}|$ is given by

$$q_{12} = (-1)^{N+1} \epsilon_1 \epsilon_2 (\mu_1 + \mu_2 + 1/2 + N). \tag{18.52}$$

By induction on N, one is led to the announced form of the eigenvalues (18.45). The Casimir operator $\mathscr{Q}_{12}$ is tridiagonal in the tensor product basis. This allows to obtain a recurrence relation for the Clebsch-Gordan coefficients which, given the spectrum (18.45), is seen to coincide with that of the dual -1 Hahn polynomials [19, 20].

18.3 The Racah problem and Bannai-Ito algebra

The addition rule (18.10) possess an associativity property when three $sl_{-1}(2)$ algebras are added. We consider three mutually commuting sets of $sl_{-1}(2)$ generators $J_0^{(j)}$, $J_\pm^{(j)}$ and $R^{(j)}$ for $j = 1,2,3$. The resulting fourth algebra can be obtained by two different addition sequences. Indeed, one has the two equivalent schemes: $4 = (1 \oplus 2) \oplus 3$ and $4 = 1 \oplus (2 \oplus 3)$. The Racah problem consists in finding the overlap between the respective eigenstates of the intermediary Casimir operators $\mathscr{Q}_{12}$ and $\mathscr{Q}_{23}$ with a fixed eigenvalue q_4 of the total Casimir operator $\mathscr{Q}_4$. Denoting such eigenstates by $|\, q_{12}; q_4, m\,\rangle$ and $|\, q_{23}; q_4, m\,\rangle$, the Racah coefficients are defined as

$$|\, q_{12}; q_4, m\,\rangle = \sum_{q_{23}} R^{\mu_1\mu_2\mu_3}_{q_{12}q_{23}\mu_4} |\, q_{23}; q_4, m\,\rangle, \tag{18.53}$$

where we have by definition

$$\mathcal{Q}_{12}|\, q_{12}, q_4, m\,\rangle = q_{12}|\, q_{12}, q_4, m\,\rangle, \quad \mathcal{Q}_{23}|\, q_{23}, q_4, m\,\rangle = q_{23}|\, q_{23}, q_4, m\,\rangle. \tag{18.54}$$

We note that the Racah coefficients $R^{\mu_1\mu_2\mu_3}_{q_{12}q_{23}\mu_4}$ do not depend on the total projection number m and that their dependence on ϵ_i, $i \in \{1,\ldots,4\}$, is implicit. The problem of finding the overlap coefficients is non-trivial because the operators $\mathcal{Q}_{12}$ and $\mathcal{Q}_{23}$ do not commute, hence they cannot be simultaneously diagonalized. The two intermediary Casimir operators have the following expressions:

$$K_1 = \mathcal{Q}_{12} = \left(J_-^{(1)}J_+^{(2)} - J_+^{(1)}J_-^{(2)}\right)R^{(1)} - R^{(1)}R^{(2)}/2 + \mathcal{Q}_1R^{(2)} + \mathcal{Q}_2R^{(1)}, \tag{18.55}$$

$$K_2 = \mathcal{Q}_{23} = \left(J_-^{(2)}J_+^{(3)} - J_+^{(2)}J_-^{(3)}\right)R^{(2)} - R^{(2)}R^{(3)}/2 + \mathcal{Q}_2R^{(3)} + \mathcal{Q}_3R^{(2)}. \tag{18.56}$$

The full Casimir operator of the fourth algebra $\mathcal{Q}_4$ can also be obtained in a straightforward manner; one finds

$$\mathcal{Q}_4 = \left(J_-^{(1)}J_+^{(3)} - J_+^{(1)}J_-^{(3)}\right)R^{(1)} - \mathcal{Q}_2R^{(1)}R^{(3)} + \mathcal{Q}_{12}R^{(3)} + \mathcal{Q}_{23}R^{(1)}. \tag{18.57}$$

The paramount observation is that the operators K_1, K_2 are closed in frames of a simple algebra with three generators. To see this, one first defines

$$K_3 = (J_+^{(1)}J_-^{(3)} - J_-^{(1)}J_+^{(3)})R^{(1)}R^{(2)} + R^{(1)}R^{(3)}/2 - \mathcal{Q}_1R^{(3)} - \mathcal{Q}_3R^{(1)}. \tag{18.58}$$

Since the operators $\mathcal{Q}_i$ for $i = 1,\ldots,4$ commute with K_1, K_2, K_3 and among themselves, we shall replace them by their corresponding eigenvalues $-\lambda_j$ where $\lambda_j = \epsilon_j\mu_j$. A direct computation shows that the following relations hold:

$$\{K_1, K_2\} = K_3 + \alpha_3, \qquad \{K_2, K_3\} = K_1 + \alpha_1, \qquad \{K_1, K_3\} = K_2 + \alpha_2, \tag{18.59}$$

where the structure constants are given by

$$\alpha_1 = -2(\lambda_1\lambda_2 + \lambda_3\lambda_4), \qquad \alpha_2 = -2(\lambda_1\lambda_4 + \lambda_2\lambda_3), \qquad \alpha_3 = 2(\lambda_1\lambda_3 + \lambda_2\lambda_4). \tag{18.60}$$

Note that the first relation of (18.59) can be considered as a *definition* of K_3. The algebra (18.59) is known as the Bannai-Ito algebra [22], which is, as will be seen below, a special case of the Askey-Wilson algebra (18.38). It admits the Casimir operator

$$\mathcal{Q}_{BI} = K_1^2 + K_2^2 + K_3^2, \tag{18.61}$$

which commutes with all generators. Given the realization (18.60) of this algebra, the Casimir operator takes the value

$$\mathscr{Q}_{BI} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 - 1/4. \tag{18.62}$$

We now look to construct irreducible BI-modules; the degree of these representations is prescribed by the range of possible eigenvalues of the operators $\mathscr{Q}_{12}$ and $\mathscr{Q}_{23}$. For simplicity, we restrict ourselves to the case where ϵ_1, ϵ_2 and ϵ_3 are all equal to 1. The other cases can be treated in similar fashion. It is worth mentioning that ϵ_4 cannot be fixed *a priori* and will in fact depend on the degree of the given module.

In the CG problem, the possible eigenvalues of q_{12} were determined by the value of the total projection operator. For the Racah problem, the overlap coefficients are independent of the total projection and the spectrum of $\mathscr{Q}_{12}$ is restricted only by the value of the total Casimir operator $\mathscr{Q}_4$. From (18.48), one finds that the minimal value of the absolute value of q_{12} is given by

$$|q_{12}|_{\min} = \mu_1 + \mu_2 + 1/2. \tag{18.63}$$

In addition, in view of the addition scheme $4 = (1 \oplus 2) \oplus 3$, we have that the absolute value of the eigenvalues q_4 of the total Casimir operator $\mathscr{Q}_4$ are of the form

$$|q_4| = |q_{12}| + \mu_3 + 1/2 + s_{12,3}, \quad s_{12,3} = 0, 1, \ldots, m \tag{18.64}$$

where m is the total projection of the state $|\, q_{12}; q_4, m\,\rangle$. It is clear that for a given absolute value of $|q_4| = \mu_4$, the maximal value of $|q_{12}|$ corresponds to setting $s_{12,3} = 0$. It then follows that

$$|q_{12}|_{\max} = \mu_4 - \mu_3 - 1/2. \tag{18.65}$$

Considering finite-dimensional representations of degree $N+1$, we find from (18.63) that the eigenvalues q_{12} of the Casimir $\mathscr{Q}_{12}$ are of the form

$$|q_{12}| = \mu_1 + \mu_2 + 1/2, \mu_1 + \mu_2 + 3/2, \ldots, \mu_1 + \mu_2 + 1/2 + N. \tag{18.66}$$

Using (18.65) and (18.66), we obtain

$$N + 1 = \mu_4 - \mu_1 - \mu_2 - \mu_3. \tag{18.67}$$

The spectra of the intermediary Casimir operators $\mathcal{Q}_{12}$ and $\mathcal{Q}_{23}$ are thus given by

$$q_{12} = (-1)^{s_{12}+1}(\mu_1 + \mu_2 + 1/2 + s_{12}), \quad s_{12} = 0, \ldots, N, \tag{18.68}$$

$$q_{23} = (-1)^{s_{23}+1}(\mu_2 + \mu_3 + 1/2 + s_{23}), \quad s_{23} = 0, \ldots, N. \tag{18.69}$$

The parameter ϵ_4 is prescribed by the value of N. Indeed, from the CG problem it is known that the allowed eigenvalues q_4 of the Casimir operator $\mathcal{Q}_4$ are of the form

$$q_4 = (-1)^{k+1}(\mu_{12} + \mu_3 + 1/2 + k), \quad k = 0, \ldots, m \tag{18.70}$$

where m is the total projection. Taking into account the condition (18.67), one finds

$$\epsilon_4 = (-1)^N. \tag{18.71}$$

Having found the explicit expressions for the spectra and dimension in terms of the representation parameters, the matrix representation of the BI algebra can be made explicit.

18.4 Leonard pair and Racah coefficients

Let μ_1, μ_2, μ_3 be fixed (positive) representation parameters and N a positive integer as in (18.67). The operators K_1, K_2 are square matrices of dimension $N+1$ which are easily seen to satisfy the following Askey-Wilson relations:

$$K_1^2 K_2 + 2K_1K_2K_1 + K_2K_1^2 - K_2 = \kappa_3 K_1 + \kappa_2, \tag{18.72}$$

$$K_2^2 K_1 + 2K_2K_1K_2 + K_1K_2^2 - K_1 = \kappa_3 K_2 + \kappa_1, \tag{18.73}$$

where the constants κ_i, $i = 1,2,3$, are given by

$$\begin{aligned} \kappa_1 &= -2(\mu_1\mu_2 + \epsilon_4\mu_3\mu_4), \\ \kappa_2 &= -2(\mu_2\mu_3 + \epsilon_4\mu_1\mu_4), \\ \kappa_3 &= 4(\mu_1\mu_3 + \epsilon_4\mu_2\mu_4), \end{aligned} \tag{18.74}$$

with $\epsilon_4 = (-1)^N$. The matrix representing K_1 can be made diagonal with eigenvalues prescribed by (18.68) and (18.54). In this basis, it is easily seen that the relations (18.72) and (18.73) imply that K_2 must be irreducible tridiagonal. The pair (K_1, K_2) thus forms a Leonard pair. Consequently, there exists a basis in which the matrices K_1 and K_2 can be expressed as

$$K_1 = \mathrm{diag}(\theta_0, \theta_1, \cdots, \theta_N), \qquad K_2 = \begin{pmatrix} b_0 & 1 & & & & \mathbf{0} \\ u_1 & b_1 & 1 & & & \\ & u_2 & b_2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & & b_{N-1} & 1 \\ \mathbf{0} & & & & u_N & b_N \end{pmatrix}, \tag{18.75}$$

where $\theta_i = (-1)^{i+1}(\mu_1 + \mu_2 + 1/2 + i)$ for $i = 0, \ldots, N$ and u_n, b_n are indeterminate. Imposing the relations (18.72) and (18.73) on the two operators and solving for b_n and u_n, one finds

$$-b_n = \begin{cases} (\mu_2 + \mu_3 + 1/2) + \frac{1}{2}\frac{n(n+\mu_1+\mu_2-\mu_3-\epsilon_4\mu_4)}{(n+\mu_1+\mu_2)} \\ -\frac{1}{2}\frac{(n+1+2\mu_2)(n+1+\mu_1+\mu_2+\mu_3-\epsilon_4\mu_4)}{(n+1+\mu_1+\mu_2)}, & \text{for } n \text{ even}, \\ (\mu_2 + \mu_3 + 1/2) + \frac{1}{2}\frac{(n+2\mu_1)(n+\mu_1+\mu_2-\mu_3+\epsilon_4\mu_4)}{(n+\mu_1+\mu_2)} \\ -\frac{1}{2}\frac{(n+1+2\mu_1+2\mu_2)(n+1+\mu_1+\mu_2+\mu_3+\epsilon_4\mu_4)}{(n+1+\mu_1+\mu_2)}, & \text{for } n \text{ odd}, \end{cases} \tag{18.76}$$

$$u_n = \begin{cases} -\frac{1}{4}\frac{n(n+2\mu_1+2\mu_2)(n+\mu_1+\mu_2+\mu_3+\epsilon_4\mu_4)(n+\mu_1+\mu_2-\mu_3-\epsilon_4\mu_4)}{(n+\mu_1+\mu_2)^2}, & \text{for } n \text{ even}, \\ -\frac{1}{4}\frac{(n+2\mu_2)(n+2\mu_1)(n+\mu_1+\mu_2+\mu_3-\epsilon_4\mu_4)(n+\mu_1+\mu_2-\mu_3+\epsilon_4\mu_4)}{(n+\mu_1+\mu_2)^2}, & \text{for } n \text{ odd}. \end{cases} \tag{18.77}$$

The overlap coefficients of the bases in which either $\mathcal{Q}_{12}$ or $\mathcal{Q}_{23}$ is diagonal will thus be proportional to the monic polynomials $\tilde{P}_n(\theta_i^*)$ with $\theta_i^* = (-1)^{i+1}(\mu_2 + \mu_3 + 1/2 + i)$ which obey the recurrence relation

$$\tilde{P}_{n+1}(x) + b_n\tilde{P}_n(x) + u_n\tilde{P}_{n-1}(x) = x\tilde{P}_n(x). \tag{18.78}$$

Defining $P_n(x) = (-2)^{-n}\tilde{P}_n(x)$, we recover the recurrence relation of the monic Bannai-Ito polynomials

$$P_{n+1}(x_s) + (\rho_1 - A_n - C_n)P_n(x_s) + A_{n-1}C_nP_n(x_s) = x_sP_n(x_s), \tag{18.79}$$

with $x_s = -\theta_s^*/2 - 1/4$ and where the identification with the Bannai-Ito parameters is

$$\begin{aligned} \rho_1 &= \frac{1}{2}(\mu_2 + \mu_3), \quad \rho_2 = \frac{1}{2}(\mu_1 + \epsilon_4\mu_4), \\ r_1 &= \frac{1}{2}(\mu_3 - \mu_2), \quad r_2 = \frac{1}{2}(\epsilon_4\mu_4 - \mu_1). \end{aligned} \tag{18.80}$$

The coefficients A_n and C_n are as defined in (18.13). The truncation conditions are the following. On the one hand, if N is even, we have

$$2(r_2-\rho_1)=N+1, \tag{18.81}$$

as well as the identification $a=\mu_1$, $b=\mu_2$ and $c=\mu_3$. On the other hand, if N is odd, we have

$$2(\rho_1+\rho_2)=-(N+1), \tag{18.82}$$

and the identification $\alpha=\mu_1$, $\beta=\mu_2$ and $\gamma=\mu_3$. Is is seen that the Bannai-Ito grids (18.21) and (18.29) coincide, as expected, with the predicted spectrum of the Casimir operator $\mathscr{Q}_{23}$. To determine the normalization constant, we use the unitarity of the transformation which imposes the following orthogonality relation for Racah coefficients:

$$\sum_{q_{23}} R^{\mu_1\mu_2\mu_3}_{qq_{23}\mu_4} R^{\mu_1\mu_2\mu_3}_{q'q_{23}\mu_4} = \delta_{qq'}. \tag{18.83}$$

Using the relations (18.20), (18.28) and (18.83), we obtain

$$R^{\mu_1\mu_2\mu_3}_{q_{12}q_{23}\mu_4} = \sqrt{\frac{\Omega_\ell(x_\ell)}{\Phi_{N,n}}}P_n(\rho_1,\rho_2,r_1,r_2;x_\ell), \tag{18.84}$$

where $P_n(\rho_1,\rho_2,r_1,r_2;x_\ell)$ is the monic Bannai-Ito polynomial. In addition, we have

$$x_\ell=-\frac{1}{2}\left(\theta^*_\ell+1/2\right), \qquad \ell=|q_{23}|-\mu_2-\mu_3-1/2, \qquad n=|q_{12}|-\mu_1-\mu_2-1/2, \tag{18.85}$$

along with the identifications (18.67), (18.80), (18.81) and (18.82). The Racah coefficients (18.84) are thus determined up to a phase factor. Returning the Bannai-Ito algebra (18.59), it is seen that the realization (18.60) is invariant under the permutations $\pi_1=(12)(34)$, $\pi_2=(13)(24)$ and $\pi_3=(14)(23)$ of the representation parameters λ_i, $i=1,\ldots 4$. These transformations generate the Klein four-group. In addition, the operation $\lambda_i\to-\lambda_i$ also leaves (18.59) and (18.60) invariant.

18.5 The Racah problem for the addition of ordinary oscillators

When $\mu=0$, the $sl_{-1}(2)$ algebra reduces to the Heisenberg oscillator algebra endowed however with a non-trivial coproduct. Therefore, the algebra obtained from the Hopf addition rule (18.10) of two $sl_{-1}(2)$ algebras with $\mu_i = 0$ is not as a result a pure oscillator algebra, but a parabosonic algebra. The same assertion holds for the addition of three $sl_{-1}(2)$ algebras with Casimir parameters μ_1, μ_2 and μ_3 all equal to zero. This corresponds to adding three pure oscillator algebras with the addition rule (18.10). Due to the importance of the oscillator algebra, it is worth recording this reduction in some detail. Most algebraic results connected to this skewed addition of three quantum harmonic oscillators have interestingly been obtained previously in [8, 14, 1, 3]. In the case $\mu_1 = \mu_2 = \mu_3 = 0$, the Bannai-Ito (18.59) algebra becomes

$$\{K_1, K_2\} = K_3, \qquad \{K_2, K_3\} = K_1, \qquad \{K_1, K_3\} = K_2. \tag{18.86}$$

This algebra can be seen as an anti-commutator version of the classical $\mathfrak{su}(2)$ Lie algebra. The Askey-Wilson relations simplify to

$$K_1^2 K_2 + 2K_1 K_2 K_1 + K_2 K_1^2 - K_2 = 0, \tag{18.87}$$

$$K_2^2 K_1 + 2K_2 K_1 K_2 + K_1 K_2^2 - K_1 = 0. \tag{18.88}$$

The spectra of the operators K_1 and K_2 are then given by the formula

$$\theta_i = (-1)^{i+1}(i + 1/2). \tag{18.89}$$

Moreover, the degree of the module is $N+1 = \mu_4$ with $\epsilon_4 = (-1)^N$. With these observations, the pair (K_1, K_2) again forms a Leonard pair. The matrices K_1 and K_2 can thus be put in the form

$$K_1 = \mathrm{diag}(\theta_0, \theta_1, \cdots, \theta_N), \qquad K_2 = \begin{pmatrix} b_0 & 1 & & & & \mathbf{0} \\ u_1 & b_1 & 1 & & & \\ & u_2 & b_2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & & b_{N-1} & 1 \\ \mathbf{0} & & & & u_N & b_N \end{pmatrix}. \tag{18.90}$$

In this case, solving for the coefficients b_n and u_n yields on the one hand

$$b_0 = -(N+1)/2, \quad \text{and} \quad b_i = 0 \text{ for } i \neq 0, \tag{18.91}$$

and on the other hand

$$u_n = \frac{(n+N+1)(N+1-n)}{4}. \tag{18.92}$$

The positivity and truncation conditions $u_n > 0$ and u_{N+1} are manifestly satisfied here. As expected, the obtained sequences $\{b_n\}$, $\{u_n\}$ correspond to the specializations $\mu_1 = \mu_2 = \mu_3 = 0$ of the formulas (18.76) and (18.77). Similarly to the Bannai-Ito case, the similarity transformation bringing K_2 into its diagonal form can be constructed with the Bannai-Ito polynomials reduced with the parametrizations $a = 0$, $b = 0$ and $c = 0$ in the N even case and $\alpha = 0$, $\beta = 0$ and $\gamma = 0$ in the N odd case. The explicit hypergeometric representation (18.37) of the corresponding polynomials, the weight functions (18.22), (18.30) as well as the normalization constants can be imported directly without need of a limiting procedure.

Conclusion: the Leonard triple

We considered the Racah problem for the algebra $sl_{-1}(2)$ which acts as the dynamical algebra for a *parabosonic oscillator* and showed that the algebra of the intermediary Casimir operators coincide with the Bannai-Ito algebra. From the knowledge of the Clebsch-Gordan problem, the spectra of the Casimir operators were determined and this allowed to build the relevant finite-dimensional modules for the BI algebra. It was then recognized that the operators $\mathcal{Q}_{12} = K_1$ and $\mathcal{Q}_{23} = K_2$ form a Leonard pair and this observation was used to see that the overlap (Racah) coefficients are given in terms of the Bannai-Ito polynomials.

As is manifest from (18.59), the Bannai-Ito algebra has a Z_3 symmetry with respect to a relabeling of the operators K_i with $i = 1,2,3$. However, the Racah problem considered here provides a specific realization of the BI algebra in terms of the distinct operators $\mathcal{Q}_{12}$, $\mathcal{Q}_{23}$ and K_3, for which this symmetry is not present. In this regard, it is natural to ask whether there exists a situation for which it is the pair (K_2,K_3) or (K_1,K_3) that is realized by intermediate Casimir operators. This question can answered by considering the Racah problem for the addition of three $sl_{-1}(2)$ algebras with different addition rules that lead to a fourth algebra that has nevertheless the same total Casimir $\mathcal{Q}_4$. The first intermediate algebra $\widetilde{(31)} = \widetilde{3} \oplus \widetilde{1}$ is obtained by defining

$$J_0^{(31)} = J_0^{(1)} + J_0^{(3)}, \qquad J_{\pm}^{(31)} = J_{\pm}^{(1)} R^{(3)} + J_{\pm}^{(3)} R^{(2)}, \qquad R^{(31)} = R^{(1)} R^{(3)}, \tag{18.93}$$

which differs from the original coproduct by the presence of $R^{(2)}$. Note that (18.93) implicitly uses (ϵ_2, μ_2) as an auxiliary space. The intermediate Casimir operator $\widetilde{\mathcal{Q}}_{13}$ is then found to coincide with the negative of K_3 as defined in (18.58):

$$\widetilde{\mathscr{Q}}_{31} = -K_3. \tag{18.94}$$

A second intermediate Casimir operator is obtained by using the standard coproduct (18.10) in two ways: one forms the algebra (12), for which $\widetilde{Q}_{12} = K_1$, or one forms the algebra (23) for which $\widetilde{Q}_{23} = K_2$. To ensure consistency, as mentioned before, the full Casimir operator of the fourth algebra $(4) = \widetilde{(31)} \oplus \widetilde{(2)}$ should coincide with (18.57). This is done by defining

$$J_0^{(4)} = J_0^{(31)} + J_0^{(2)}, \qquad J_\pm^{(4)} = J_\pm^{(31)} R^{(2)} + J_\pm^{(2)} R^{(3)}, \qquad R^{(4)} = R^{(31)} R^{(2)}, \tag{18.95}$$

It is readily seen that the generators defined in (18.93) and (18.95) satisfy the defining relations (18.1) of $sl_{-1}(2)$. This fourth algebra is easily seen to admit the same full Casimir operator (18.57). Defining $\widetilde{K_3} = -\widetilde{\mathscr{Q}}_{31}$, $\widetilde{K_1} = K_1$ and $\widetilde{K_2} = K_2$, the algebra (18.59) is recovered with the pair $(\widetilde{K_1}, \widetilde{K_3})$ or $(\widetilde{K_2}, \widetilde{K_3})$ playing the role of the intermediate Casimir operators. The steps of Sections 3, 4 can then be reproduced and this leads one to conclude that K_3 also has a Bannai-Ito type spectrum $\lambda_i^{(3)} = (-1)^i(\mu_1 + \mu_3 + 1/2 + i)$, $i = 0, \ldots, N$ and that (K_2, K_3) and (K_1, K_3) form Leonard pairs. In addition, it follows from this observation that in the realization (18.60) of the Bannai-Ito algebra (18.59) obtained from the operators (18.55), (18.56) and (18.58), the set (K_1, K_2, K_3) constitutes a *Leonard Triple*, which have studied intensively for the q-Racah scheme in [5, 11].

In the case of the algebras $\mathfrak{sl}(2)$ and $sl_q(2)$, it is known that the Clebsch-Gordan coefficients can be obtained from the Racah coefficients in a proper limit. It is not so with the algebra $sl_{-1}(2)$. Indeed, the dual -1 Hahn polynomials are beyond the Leonard duality and do not occur as limits of the Bannai-Ito polynomials. Furthermore, the question of the symmetry algebra underlying the Clebsch-Gordan problem for $sl_{-1}(2)$ remains open. We plan to report on this elsewhere.

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Chapitre 19

The algebra of dual -1 Hahn polynomials and the Clebsch-Gordan problem of $sl_{-1}(2)$

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Abstract. The algebra $\mathscr{H}$ of the dual -1 Hahn polynomials is derived and shown to arise in the Clebsch-Gordan problem of $sl_{-1}(2)$. The dual -1 Hahn polynomials are the bispectral polynomials of a discrete argument obtained from the $q \to -1$ limit of the dual q-Hahn polynomials. The Hopf algebra $sl_{-1}(2)$ has four generators including an involution, it is also a $q \to -1$ limit of the quantum algebra $sl_q(2)$ and furthermore, the dynamical algebra of the parabose oscillator. The algebra $\mathscr{H}$, a two-parameter generalization of $\mathfrak{u}(2)$ with an involution as additional generator, is first derived from the recurrence relation of the -1 Hahn polynomials. It is then shown that $\mathscr{H}$ can be realized in terms of the generators of two added $sl_{-1}(2)$ algebras, so that the Clebsch-Gordan coefficients of $sl_{-1}(2)$ are dual -1 Hahn polynomials. An irreducible representation of $\mathscr{H}$ involving five-diagonal matrices and connected to the difference equation of the dual -1 Hahn polynomials is constructed.

19.1 Introduction

The algebra $sl_{-1}(2)$ has been proposed [16] as a $q \to -1$ limit of the $sl_q(2)$ algebra. It is a Hopf algebra with four generators, including an involution, defined by relations involving both commutators and anti-commutators. This algebra is also the dynamical algebra of a parabosonic oscillator [5, 11, 13, 12].

Recently, a breakthrough in the theory of orthogonal polynomials has been realized with the discovery of a series of classical orthogonal polynomials which are eigenfunctions of continuous or discrete Dunkl operators defined using reflections [4, 15, 18, 19, 20, 21] . These polynomials are referred to as -1 polynomials since they arise as $q \to -1$ limits of q-orthogonal polynomials of the Askey scheme. At the top of the discrete variable branch of these $q = -1$ polynomials lie the Bannai-Ito polynomials and their kernel partners, the complementary Bannai-Ito polynomials. Both sets depend on four parameters and are expressible in terms of Wilson polynomials [1, 8, 18] . The Bannai-Ito polynomials possess the Leonard duality property [14] , which in fact led to their original discovery [1] . Moreover, an algebraic interpretation of these polynomials has been given in terms of the Bannai-Ito algebra, which is a Jordan algebra [17] . In contradistinction to the situation with the Bannai-Ito polynomials, the complementary Bannai-Ito polynomials and their descendants, the dual -1 Hahn polynomials, are bi-spectral (i.e. they obey both a recurrence relation and a difference equation) but they fall outside the scope of the Leonard duality. Moreover, their algebraic interpretation is lacking.

In the present work, we derive the algebra $\mathscr{H}$ of the dual -1 Hahn polynomials and show that it arises as the hidden symmetry algebra of the Clebsch-Gordan problem of $sl_{-1}(2)$. It is already known [16] that the dual -1 Hahn polynomials occur as Clebsch-Gordan coefficients of $sl_{-1}(2)$. Here we recover this result by showing how $\mathscr{H}$ is realized by generators of the coproduct of two $sl_{-1}(2)$ algebras. The algebra $\mathscr{H}$ turns out to be an extension of $\mathfrak{u}(2)$ through the addition of an involution as a generator. We study its finite-dimensional irreducible representations in two bases each diagonalizing a different operator.

The paper is divided as follows. In section 1, we recall basic results on the $sl_{-1}(2)$ algebra and the dual -1 Hahn polynomials. In section 2, we obtain the algebra $\mathscr{H}$ of the dual -1 Hahn polynomials in a specific representation by using the recurrence relation operator and the spectrum of the difference equation. In section 3, we investigate the Clebsch-Gordan problem for $sl_{-1}(2)$ and show that $\mathscr{H}$ appears as the associated algebra. In section 4, an irreducible representation of $\mathscr{H}$ which is "dual" to the one constructed in section 2 is shown to involve five-diagonal matrices. We conclude by discussing another presentation of $\mathscr{H}$ and its relation to the algebras proposed [6, 7] in the context of finite oscillator models.

19.2 $sl_{-1}(2)$ and dual -1 Hahn polynomials

19.2.1 The algebra $sl_{-1}(2)$

The Hopf algebra $sl_{-1}(2)$ is generated by four operators A_0, A_+, A_- and R obeying the relations [16]

$$[A_0, A_\pm] = \pm A_\pm, \quad [A_0, R] = 0, \quad \{A_+, A_-\} = 2A_0, \quad \{A_\pm, R\} = 0, \tag{19.1}$$

where $[a,b] = ab - ba$ and $\{a,b\} = ab + ba$. The operator R is an involution, that is $R^2 = \mathbb{I}$, where $\mathbb{I}$ represents the identity operator. The algebra has the following Casimir operator:

$$Q = A_+A_-R - A_0R + (1/2)R, \tag{19.2}$$

which commutes with all generators. In view of the defining relations (19.1), it is clear that $sl_{-1}(2)$ has a ladder representation. Let μ be a non-negative real number and let ϵ be a parameter taking the values $\epsilon = \pm 1$. Consider the infinite-dimensional vector space (ϵ, μ) spanned by the basis vectors $e_n^{(\epsilon,\mu)}$, $n \in \mathbb{N}$, and endowed with the actions

$$\begin{aligned} A_0\, e_n^{(\epsilon,\mu)} &= (n + \mu + 1/2)\, e_n^{(\epsilon,\mu)}, & R\, e_n^{(\epsilon,\mu)} &= \epsilon(-1)^n\, e_n^{(\epsilon,\mu)}, \\ A_+\, e_n^{(\epsilon,\mu)} &= \sqrt{[n+1]_\mu}\, e_{n+1}^{(\epsilon,\mu)}, & A_-\, e_n^{(\epsilon,\mu)} &= \sqrt{[n]_\mu}\, e_{n-1}^{(\epsilon,\mu)}, \end{aligned} \tag{19.3}$$

where $[n]_\mu$ denotes the μ-number

$$[n]_\mu = n + \mu(1 - (-1)^n). \tag{19.4}$$

It is readily checked that (19.3) defines an irreducible $sl_{-1}(2)$-module. As expected from Schur's lemma, the Casimir operator Q acts on (ϵ, μ) as a multiple of the identity:

$$Q\, e_n^{(\epsilon,\mu)} = -\epsilon\mu\, e_n^{(\epsilon,\mu)}. \tag{19.5}$$

On the space (ϵ, μ), $sl_{-1}(2)$ is equivalent to the dynamical algebra of a parabosonic oscillator. This assertion stems from the following observation. One has

$$[A_-, A_+] = \{A_-, A_+\} - 2A_+A_- = 2A_0 - 2A_+A_-. \tag{19.6}$$

With the use of (19.2) and (19.5), one finds

$$[A_-, A_+] = 1 + 2\epsilon\mu R, \tag{19.7}$$

where the relation is understood to be on the space (ϵ, μ). The operators $A_\pm$ satisfying the commutation relation (19.7), together with the involution R obeying the relations $R^2 = \mathbb{I}$ and $\{R, A_\pm\} = 0$, define the parabosonic oscillator algebra [11, 12, 13, 16] .

The $sl_{-1}(2)$ algebra possesses a non-trivial addition rule, or coproduct [2, 16] . Let (ϵ_a, μ_a) and (ϵ_b, μ_b) be two $sl_{-1}(2)$-modules. A third module is obtained by endowing the tensor product space $(\epsilon_a, \mu_a) \otimes (\epsilon_b, \mu_b)$ with the actions

$$\begin{aligned} A_0(u \otimes v) &= (A_0 u) \otimes v + u \otimes (A_0 v), \\ A_\pm(u \otimes v) &= (A_\pm u) \otimes (Rv) + u \otimes (A_\pm v), \\ R(u \otimes v) &= (Ru) \otimes (Rv), \end{aligned} \tag{19.8}$$

where $u \in (\epsilon_a, \mu_a)$ and $v \in (\epsilon_b, \mu_b)$. This addition rule for $sl_{-1}(2)$ can also be presented in the following way. Let $\{A_0, A_\pm, R_a\}$ and $\{B_0, B_\pm, R_b\}$ be two mutually commuting sets of $sl_{-1}(2)$ generators and denote the corresponding algebras by $\mathscr{A}$ and $\mathscr{B}$. A third algebra, denoted $\mathscr{C} = \mathscr{A} \oplus \mathscr{B}$, is obtained by defining

$$C_0 = A_0 + B_0, \qquad C_\pm = A_\pm R_b + B_\pm, \qquad R_c = R_a R_b. \tag{19.9}$$

It is elementary to verify that the generators C_0, $C_\pm$ and R_c obey the defining relations (19.1) of $sl_{-1}(2)$. The Casimir operator of the resulting algebra

$$Q_{ab} = C_+ C_- R_c - C_0 R_c + (1/2) R_c, \tag{19.10}$$

may be cast in the form

$$Q_{ab} = (A_- B_+ - A_+ B_-) R_a - (1/2) R_a R_b + Q_a R_b + Q_b R_a, \tag{19.11}$$

where Q_i, $i \in \{a, b\}$, are the Casimir operators of the algebras $\mathscr{A}$ and $\mathscr{B}$.

19.2.2 Dual −1 Hahn polynomials

The dual −1 Hahn polynomials $Q_n(x; \alpha, \beta, N)$ have been introduced and investigated [15] as limits of the dual q-Hahn polynomials [8] when $q \to -1$. We here recall their basic properties. The monic dual −1 Hahn polynomials obey the recurrence relation

$$Q_{n+1}(x; \alpha, \beta, N) + b_n Q_n(x; \alpha, \beta, N) + u_n Q_{n-1}(x; \alpha, \beta, N) = x Q_n(x; \alpha, \beta, N). \tag{19.12}$$

The recurrence coefficients are expressed in terms of μ-numbers (19.4) as follows:

$$b_n = \begin{cases} (-1)^{n+1}(2\xi + 2\zeta) - 1, & N \text{ even}, \\ (-1)^{n+1}(2\xi - 2\zeta) - 1, & N \text{ odd}, \end{cases} \qquad u_n = 4[n]_\xi [N - n + 1]_\zeta, \tag{19.13}$$

where the parameters ξ and ζ are given by

$$\xi = \begin{cases} \frac{\beta - N - 1}{2}, & N \text{ even}, \\ \frac{\alpha}{2}, & N \text{ odd}, \end{cases} \qquad \zeta = \begin{cases} \frac{\alpha - N - 1}{2}, & N \text{ even}, \\ \frac{\beta}{2}, & N \text{ odd}. \end{cases} \tag{19.14}$$

It is seen that the truncation conditions $u_0 = 0$ and $u_{N+1} = 0$, necessary for finite orthogonal polynomials, are met. The positivity condition $u_n > 0$ is equivalent to the condition $\alpha > N$ and $\beta > N$ in the case of even N. When N is odd, two conditions are possible to ensure positivity; the relevant situation here is $\alpha > -1$ and $\beta > -1$. The dual -1 Hahn polynomials enjoy the orthogonality relation

$$\sum_{s=0}^{N} \omega_s Q_n(x_s;\alpha,\beta,N) Q_m(x_s;\alpha,\beta,N) = v_n \delta_{nm}. \tag{19.15}$$

The grid and weight function are given by

$$x_s = \begin{cases} (-1)^s(2s+1-\alpha-\beta), & N \text{ even}, \\ (-1)^s(2s+1+\alpha+\beta), & N \text{ odd}, \end{cases} \tag{19.16}$$

$$\omega_{2j+q} = \begin{cases} (-1)^j \frac{(-N/2)_{j+q}}{j!} \frac{(1-\alpha/2)_j (1-\alpha/2-\beta/2)_j}{(1-\beta/2)_j (N/2+1-\alpha/2-\beta/2)_{j+q}}, & N \text{ even}, \\ (-1)^j \frac{(-(N-1)/2)_j}{j!} \frac{(1/2+\alpha/2)_{j+q} (1+\alpha/2+\beta/2)_j}{(1/2+\beta/2)_{j+q} (N/2+3/2+\alpha/2+\beta/2)_j}, & N \text{ odd}, \end{cases} \tag{19.17}$$

where $q \in \{0,1\}$ and where $(a)_n = (a)(a+1)\cdots(a+n-1)$ stands for the Pochhammer symbol. The normalization coefficients v_n take the form

$$v_n = \begin{cases} (-1)^q (16)^{2j+q} j! (1-\alpha/2)_j (-N/2)_{j+q} (\beta/2-N/2)_{j+q} \left(\frac{(1-(\alpha+\beta)/2)_{N/2}}{(1-\beta/2)_{N/2}}\right), & N \text{ even}, \\ (-1)^q (16)^{2j+q} j! (1/2+\alpha/2)_{j+q} (1/2-N/2)_j (-\beta/2-N/2)_{j+q} \left(\frac{(1+(\alpha+\beta)/2)_{\lceil N/2 \rceil}}{((\beta+1)/2)_{\lceil N/2 \rceil}}\right), & N \text{ odd}, \end{cases} \tag{19.18}$$

where $\lceil x \rceil = \lfloor x \rfloor + 1$ and $\lfloor x \rfloor$ denotes the integer part of x. The dual -1 Hahn polynomials admit a hypergeometric representation. Recall that the generalized hypergeometric function ${}_pF_q(z)$ is defined by the infinite series

$${}_pF_q\left[\begin{matrix} a_1 & \cdots & a_p \\ b_1 & \cdots & b_q \end{matrix}; z\right] = \sum_k \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}. \tag{19.19}$$

When N is even, one has [15]

$$Q_{2n}(x) = \gamma_n^{(0)} {}_3F_2\left[\begin{matrix} -n, \delta + \frac{x+1}{4}, \delta - \frac{x+1}{4} \\ -\frac{N}{2}, 1-\frac{\alpha}{2} \end{matrix}; 1\right], \qquad \delta = 1/2 - \frac{\alpha+\beta}{4}, \tag{19.20}$$

$$Q_{2n+1}(x) = \gamma_n^{(1)} (x+1-\tau) {}_3F_2\left[\begin{matrix} -n, \delta + \frac{x+1}{4}, \delta - \frac{x+1}{4} \\ 1-\frac{N}{2}, 1-\frac{\alpha}{2} \end{matrix}; 1\right], \quad \tau = 2N+2-\alpha-\beta, \tag{19.21}$$

where $\gamma_n^{(0)} = 16^n (-N/2)_n (1-\alpha/2)_n$ and $\gamma_n^{(1)} = 16^n (1-N/2)_n (1-\alpha/2)_n$. When N is odd, one rather has

$$Q_{2n}(x) = \phi_n^{(0)} {}_3F_2\left[\begin{matrix} -n, \eta + \frac{x+1}{4}, \eta - \frac{x+1}{4} \\ \frac{1-N}{2}, \frac{\alpha+1}{2} \end{matrix}; 1\right], \qquad \eta = \frac{\alpha+\beta+2}{4}, \tag{19.22}$$

$$Q_{2n+1}(x) = \phi_n^{(1)} (x+1+\alpha-\beta) {}_3F_2\left[\begin{matrix} -n, \eta + \frac{x+1}{4}, \eta - \frac{x+1}{4} \\ \frac{1-N}{2}, \frac{\alpha+3}{2} \end{matrix}; 1\right], \tag{19.23}$$

where $\phi_n^{(0)} = 16^n((1-N)/2)_n((\alpha+1)/2)_n$ and $\phi_n^{(1)} = 16^n((1-N)/2)_n((\alpha+3)/2)_n$.

The dual -1 Hahn polynomials are bispectral but fall outside the scope of the Leonard duality. In point of fact, they satisfy a *five-term* (instead of three-term) difference equation on the grid x_s. This equation is of the form [15]:

$$A(s)Q_n(x_{s+2}) + B(s)Q_n(x_{s+1}) + C(s)Q_n(x_s) + D(s)Q_n(x_{s-1}) + E(s)Q_n(x_{s-2}) = 2nQ_n(x_s). \quad (19.24)$$

It is derived from the $q \to -1$ limit of the operator $L^2 + 2L$, where L is the difference operator of the dual q-Hahn polynomials [8] . The expressions of the coefficients $A(s)$, $B(s)$, $C(s)$, $D(s)$ and $E(s)$ are known explicitly [15] . In relation with this structure of the difference operator, we show in Section 4 that the algebra $\mathscr{H}$ of the dual -1 Hahn polynomials admits an irreducible representation which involves five-diagonal matrices.

19.3 The algebra $\mathscr{H}$ of -1 Hahn polynomials

A specific realization of the algebra of the dual -1 Hahn polynomials is derived by examining the interplay between the recurrence and the difference operators. We consider the finite-dimensional vector space spanned by the basis elements ψ_n, $n \in \{0,\dots,N\}$, and define the following operators:

$$K_1\psi_n = n\psi_n, \qquad 2K_2\psi_n = \psi_{n-1} + b_n\psi_n + u_n\psi_{n+1}, \quad (19.25)$$

where b_n and u_n are as specified by (19.13). Note that ψ_{-1} and ψ_{N+1} do not belong to the vector space so that the action of K_2 on the endpoint vectors ψ_0, ψ_N is given by

$$2K_2\psi_0 = b_0\psi_0 + u_0\psi_1, \qquad 2K_2\psi_N = \psi_{N-1} + b_N\psi_N. \quad (19.26)$$

It is also necessary to introduce the parity operator P, which has the following realization in the basis ψ_n:

$$P\psi_n = (-1)^n\psi_n. \quad (19.27)$$

It is seen that the operators K_1 and K_2, together with the parity operator P, are closed in frames of an algebra which we denote by $\mathscr{H}$. Indeed, a direct computation shows that the following relations hold:

$$[K_1, P] = 0, \qquad \{K_2, P\} = -P - 2\nu, \qquad \{K_3, P\} = 0, \quad (19.28)$$

$$[K_1, K_2] \equiv K_3, \qquad [K_1, K_3] = K_2 + \nu P + 1/2, \tag{19.29}$$

$$[K_3, K_2] = 4K_1 + 4\nu K_1 P - 2\nu K_3 P + \sigma P + \rho, \tag{19.30}$$

where the structure constants are

$$\nu = (\alpha + \beta - 2N - 2)/2, \qquad \sigma = \alpha - \beta + 2N(N + 1 - \beta), \qquad \rho = \beta - \alpha - 2N, \tag{19.31}$$

when N is even and

$$\nu = (\alpha - \beta)/2, \qquad \sigma = -(\alpha + \beta + 2\alpha\beta + 2N\alpha), \qquad \rho = \alpha - \beta - 2N, \tag{19.32}$$

when N is odd. The algebra $\mathscr{H}$, defined by the relations (19.28), (19.29) and (19.30), admits the Casimir operator

$$Q_{\mathscr{H}} = 4K_1^2 + K_2^2 - K_3^2 + K_2 + 2\rho K_1 + 2\nu P, \tag{19.33}$$

which commutes with all generators. The abstract algebra $\mathscr{H}$ is realized by the operators of the dual -1 Hahn polynomials as given by (19.25). In this realization, the Casimir operator (19.33) takes the definite value

$$q_{\mathscr{H}} = \nu^2 + 2\nu - \sigma - \rho - \frac{1}{4}. \tag{19.34}$$

The picture can be summarized as follows. The algebra $\mathscr{H}$ has an irreducible representation of dimension $N+1$ for which the matrix representing K_2 is, up to a multiplicative constant, the Jacobi matrix of the monic dual -1 Hahn polynomials. By construction, one thus has

$$K_1 = \mathrm{diag}(0, 1, \cdots, N), \qquad K_2 = \frac{1}{2}\begin{pmatrix} b_0 & 1 & & & \\ u_1 & b_1 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & 1 \\ & & & u_N & b_N \end{pmatrix}, \qquad P = \mathrm{diag}(1, -1, 1, \cdots, (-1)^N). \tag{19.35}$$

The representation is irreducible since u_n is always non-zero. The transition matrix S with matrix elements

$$S_{ij} = Q_i(x_j; \alpha, \beta, N), \tag{19.36}$$

in the $\{\psi_n\}$ basis provides the similarity transformation diagonalizing K_2. Equivalently, the dual -1 Hahn polynomials are, up to factors, the overlap coefficients of the bases in which either K_1 or K_2 is diagonal. It is clear that in the basis in which K_2 is diagonal, the operators P and K_1 will not be diagonal. In Section 4, the matrix elements of P and K_1 in this basis will be constructed from the commutation relations (19.28), (19.29) and (19.30). Unsurprisingly, the operator K_1 will be shown to be five-diagonal in this basis as expected from the form of the difference equation. We now turn to the Clebsch-Gordan problem.

19.4 The Clebsch-Gordan problem

The Clebsch-Gordan problem for $sl_{-1}(2)$ can be posited in the following way. We consider the $sl_{-1}(2)$-module $(\epsilon_a,\mu_a)\otimes(\epsilon_b,\mu_b)$ that we wish to decompose irreducibly. The basis vectors $e_{n_a}^{(\epsilon_a,\mu_a)}\otimes e_{n_b}^{(\epsilon_b,\mu_b)}$ of the direct product are characterized as eigenvectors of the operators

$$Q_a,\ A_0,\ R_a,\ Q_b,\ B_0,\ R_b, \tag{19.37}$$

with eigenvalues

$$-\epsilon_a\mu_a,\ n_a+\mu_a+1/2,\ (-1)^{n_a}\epsilon_a,\ -\epsilon_b\mu_b,\ n_b+\mu_b+1/2,\ (-1)^{n_b}\epsilon_b, \tag{19.38}$$

respectively. The irreducible modules in the decomposition will be spanned by the elements $e_k^{(\epsilon_{ab},\mu_{ab})}$ refered to as the coupled basis vectors. In each irreducible component, the (total) Casimir operator Q_{ab} of the two added $sl_{-1}(2)$ algebras which reads

$$Q_{ab}=(A_-B_+-A_+B_-)R_a-(1/2)R_aR_b+Q_aR_b+Q_bR_a, \tag{19.39}$$

acts as a multiple of the identity:

$$Q_{ab}=-\epsilon_{ab}\mu_{ab}\,\mathbb{I}. \tag{19.40}$$

The coupled basis elements $e_k^{(\epsilon_{ab},\mu_{ab})}$are the eigenvectors of

$$Q_{ab},\ R_aR_b,\ Q_a,\ Q_b,\ A_0+B_0, \tag{19.41}$$

with eigenvalues

$$-\epsilon_{ab}\mu_{ab},\ \epsilon_{ab},\ -\epsilon_a\mu_a,\ -\epsilon_b\mu_b,\ k+\mu_a+\mu_b+1, \tag{19.42}$$

respectively. The direct product basis is related to the coupled basis by a unitary transformation whose matrix elements are called Clebsch-Gordan coefficients. These overlap coefficients will be zero unless

$$k = n_a + n_b \equiv N. \tag{19.43}$$

We may hence write

$$e_N^{(\epsilon_{ab},\mu_{ab})} = \sum_{n_a+n_b=N} C_{n_a n_b N}^{\mu_a \mu_b \mu_{ab}}\, e_{n_a}^{(\epsilon_a \mu_a)} \otimes e_{n_b}^{(\epsilon_b \mu_b)}, \tag{19.44}$$

where $C_{n_a n_b N}^{\mu_a \mu_b \mu_{ab}}$ are the Clebsch-Gordan coefficients of $sl_{-1}(2)$.

The Clebsch-Gordan problem for $sl_{-1}(2)$ can be solved elegantly by examining the underlying symmetry algebra. It is first observed that for a given N, the following operators act as multiples of the identity operator on both sides of (19.44):

$$\Lambda_1 = 2Q_a, \qquad \Lambda_2 = 2Q_b, \qquad \Lambda_3 = R_a R_b, \qquad \Lambda_4 = A_0 + B_0, \tag{19.45}$$

with multiples

$$\lambda_1 = -2\epsilon_a \mu_a, \qquad \lambda_2 = -2\epsilon_b \mu_b, \qquad \lambda_3 = (-1)^N \epsilon_a \epsilon_b, \qquad \lambda_4 = \mu_a + \mu_b + N + 1. \tag{19.46}$$

Let us now introduce the following operators

$$\kappa_1 = (A_0 - B_0)/2, \qquad \kappa_2 = \Lambda_3 Q_{ab}, \qquad r = R_a. \tag{19.47}$$

The Clebsch-Gordan problem for $sl_{-1}(2)$ is tantamount to finding the overlaps between the eigenvectors of κ_1 and r and the eigenvectors of κ_2. The first set of eigenvectors correspond to the elements of the direct product basis (R.H.S. of (19.44)) since κ_1 and r complement the set (19.45) to give all the labelling operators (19.37). The second set is identified as should be to the coupled basis elements (L.H.S. of (19.44)) since only Q_{ab} needs to be added to (19.45) to recover the complete set of operators (19.41) that are diagonal on the coupled vectors $e_N^{(\epsilon_{ab},\mu_{ab})}$; it will prove convenient to use equivalently $\kappa_2 = \Lambda_3 Q_{ab}$, instead of Q_{ab}.

Let us now consider the algebra which is generated by these operators, i.e. by κ_1, κ_2 and r. Let κ_3 be a fourth generator defined by

$$\kappa_3 \equiv [\kappa_1, \kappa_2]. \tag{19.48}$$

A direct computation shows that:

$$[\kappa_1, r] = 0, \quad \{\kappa_2, r\} = -r + (\lambda_1 + \lambda_2\lambda_3), \quad \{\kappa_3, r\} = 0, \tag{19.49}$$

$$[\kappa_1, \kappa_3] = \kappa_2 - \frac{1}{2}(\lambda_1 + \lambda_2\lambda_3)r + 1/2, \tag{19.50}$$

$$[\kappa_3, \kappa_2] = 4\kappa_1 + (\lambda_1 + \lambda_2\lambda_3)\kappa_3 r - 2(\lambda_1 + \lambda_2\lambda_3)\kappa_1 r + \lambda_4(\lambda_1 - \lambda_2\lambda_3)r. \tag{19.51}$$

In this instance the Casimir operator for the algebra is given by

$$Q_{C.G.} = 4\kappa_1^2 + \kappa_2^2 - \kappa_3^2 + \kappa_2 - (\lambda_1 + \lambda_2\lambda_3)r, \tag{19.52}$$

and acts as a multiple $q_{C.G.}$ of the identity:

$$q_{C.G.} = \frac{1}{4}(\lambda_1 + \lambda_2\lambda_3)^2 + \lambda_4^2 - \frac{5}{4}. \tag{19.53}$$

It is seen that the algebra $\mathscr{H}$ of the dual -1 Hahn polynomials arises as the hidden symmetry algebra of the Clebsch-Gordan problem of $sl_{-1}(2)$. Indeed, redefining $K_1 \to K_1 + \rho/4$ in (19.28), (19.29) and (19.30) yields an algebra of the form (19.49), (19.50) and (19.51).

In order to establish the exact correspondence between the Clebsch-Gordan coefficients of $sl_{-1}(2)$ and the dual -1 Hahn polynomials encompassed by the algebra, it is necessary to determine the spectra of the operators κ_1 and κ_2. In view of the action of A_0 in (19.3), it is clear that $\kappa_1 = (A_0 - B_0)/2$ has a linear spectrum of the form

$$\lambda_{\kappa_1} = n + (\mu_a - \mu_b - N)/2, \quad n \in \{0, \cdots, N\}. \tag{19.54}$$

This spectrum is seen to coincide, up to a translation, with that of operator K_1 in the algebra $\mathscr{H}$ of the dual -1 Hahn polynomials.

The evaluation of the spectrum of Q_{ab} is more delicate. In a given $sl_{-1}(2)$-module (ϵ, μ), it follows from (19.3) and (19.5) that the eigenvalues of A_0 are given by

$$\lambda_{A_0} = n - \epsilon Q + 1/2. \tag{19.55}$$

In reducible representations, it is hence possible from this relation to determine the eigenvalues of the Casimir operator which are compatible with the eigenvalue λ_{A_0} of A_0 that is being considered. So, for a given λ_{A_0}, the absolute value $|q|$ of the possible eigenvalues q of the Casimir operator are

$$|q| = |\lambda_{A_0} - 1/2|, |\lambda_{A_0} - 3/2|, \ldots, \tag{19.56}$$

In the calculation of the Clebsch-Gordan coefficients of $sl_{-1}(2)$, the eigenvalue of $C_0 = A_0 + B_0$ is taken to be $\mu_a + \mu_b + N + 1$. Consequently, the set of the absolute values of the possible eigenvalues of $Q_c = Q_{ab}$ given in (19.39) is of cardinality $N+1$ and is found to be

$$|q_{ab}| = \mu_a + \mu_b + N + 1/2, \mu_a + \mu_b + N - 1/2, \cdots, \mu_a + \mu_b + 1/2. \tag{19.57}$$

Since the eigenvalue of Q_{ab} is $-\epsilon_{ab}\mu_{ab}$, it follows that the admissible values of μ_{ab} are given by the above ensemble (19.57) with $\mu_{ab} = |q_{ab}|$.

There remains to evaluate the associated values of ϵ_{ab}. To that end, consider the eigenvector $\widetilde{e}_0 = e_N^{(\epsilon_{ab}|_{max}, \mu_{ab}|_{\max})}$ of the coupled basis corresponding to the maximal admissible value of μ_{ab}. The state $\widetilde{e}_0$ satisfies the relations

$$C_0 \tilde{e}_0 = (\mu_a + \mu_b + N + 1)\widetilde{e}_0, \qquad C_- \widetilde{e}_0 = 0. \tag{19.58}$$

On the one hand, it then follows from (19.58) and (19.3) that

$$R_c \widetilde{e}_0 = \epsilon_{ab}|_{\max} \widetilde{e}_0. \tag{19.59}$$

On the other hand, the value of $R_c = R_a R_b$ is fixed to be $(-1)^N \epsilon_a \epsilon_b$ on the whole space so that in particular $R_c \widetilde{e}_0 = (-1)^N \epsilon_a \epsilon_b \widetilde{e}_0$. We therefore conclude that for the state with the maximal value $\mu_{ab} = \mu_{ab}|_{\max}$ of μ_{ab}, the corresponding value $\epsilon_{ab}|_{\max}$ of ϵ_{ab} is

$$\epsilon_{ab}|_{max} = (-1)^N \epsilon_a \epsilon_b. \tag{19.60}$$

Since $Q_{ab}\widetilde{e}_0 = Q_{ab} e_N^{(\epsilon_{ab}|_{\max}, \mu_{ab}|_{\max})} = -\epsilon_{ab}|_{\max}\mu_{ab}|_{\max}$, it follows that this eigenvalue q_{ab} of the full Casimir operator Q_{ab} is

$$q_{ab} = (-1)^{N+1} \epsilon_a \epsilon_b (\mu_a + \mu_b + N + 1/2). \tag{19.61}$$

It is easily seen that incrementing the projection from N to $N+1$ adds a new eigenvalue to the set of eigenvalues of Q_{ab} while preserving the admissible values of $(\epsilon_{ab}, \mu_{ab})$ for the original value N of the projection. Thus, by induction, the eigenvalues of the full Casimir operator $Q_c = Q_{ab}$ are given by

$$q_{ab} = -\epsilon_{ab}\mu_{ab} = (-1)^{s+1} \epsilon_a \epsilon_b (\mu_a + \mu_b + s + 1/2), \;\; s = 0, \ldots, N. \tag{19.62}$$

It is thus seen that the spectrum of κ_2 coincide with that of K_2 in the algebra $\mathscr{H}$ and that the Clebsch-Gordan coefficients of $sl_{-1}(2)$ are hence proportional to the dual -1 Hahn polynomials.

For definiteness, let us consider the case $\epsilon_a = 1 = \epsilon_b$; the other cases can be treated similarly. The proportionality constant can be determined by the orthonormality condition of the Clebsch-Gordan coefficients. One has

$$\sum_{\mu_{ab}} C^{\mu_a \mu_b \mu_{ab}}_{n,N-n,N} C^{\mu_a \mu_b \mu_{ab}}_{m,N-m,N} = \delta_{nm}. \tag{19.63}$$

By comparison of the algebras (19.28), (19.29) and (19.30) with (19.49), (19.50) and (19.51), there comes

$$C^{\mu_a,\mu_b,\mu_{ab}}_{n,N-n,N} = \sqrt{\frac{\widetilde{\omega}_k}{v_n}}\, Q_n(z_k;\alpha,\beta,N) \tag{19.64}$$

where

$$\alpha = \begin{cases} 2\mu_b + N + 1, & N \text{ even}, \\ 2\mu_a, & N \text{ odd}, \end{cases} \qquad \beta = \begin{cases} 2\mu_a + N + 1, & N \text{ even}, \\ 2\mu_b, & N \text{ odd}, \end{cases} \tag{19.65}$$

$$z_k = \begin{cases} (-1)^{k+1}(2\mu_a + 2\mu_b + 2k + 1), & N \text{ even}, \\ (-1)^{k}(2\mu_a + 2\mu_b + 2k + 1), & N \text{ odd}, \end{cases} \qquad \widetilde{\omega}_k = \begin{cases} \omega_{N-k}, & N \text{ even}, \\ \omega_k, & N \text{ odd} \end{cases} \tag{19.66}$$

The Clebsch-Gordan coefficients of $sl_{-1}(2)$ have thus been determined up to a phase factor by showing that the algebra underlying this problem coincides with the algebra $\mathscr{H}$ of the dual -1 Hahn polynomials.

19.5 A "dual" representation of $\mathscr{H}$ by pentadiagonal matrices

In section 2, the algebra $\mathscr{H}$ of the dual -1 Hahn polynomials was derived and it was shown that this algebra admits irreducible representations of dimension $N+1$ where K_1, P are diagonal and K_2 is the Jacobi matrix of the dual -1 Hahn polynomials. We now study irreducible representations in the basis where K_2 is diagonal and construct the matrix elements of K_1 and P in that basis. For the reader's convenience, we recall the defining relations of the algebra $\mathscr{H}$

$$[K_1, P] = 0, \qquad \{K_2, P\} = -P - 2\nu, \qquad \{K_3, P\} = 0, \tag{19.67}$$

$$[K_1, K_2] = K_3, \qquad [K_1, K_3] = K_2 + \nu P + 1/2, \tag{19.68}$$

$$[K_3, K_2] = 4K_1 + 4\nu K_1 P - 2\nu K_3 P + \sigma P + \rho, \tag{19.69}$$

with $P^2 = \mathbb{I}$. It is appropriate to separate the N even case from the N odd case. We construct the matrix elements in the N odd case first.

19.5.1 N odd

Consider the basis in which K_2 is diagonal and denote the basis vectors by φ_k, $k \in \{0,\ldots,N\}$. From (19.16), the eigenvalues of K_2 are known and given by

$$\lambda_s = (-1)^s \left(s + 1/2 + \alpha/2 + \beta/2\right). \tag{19.70}$$

One sets

$$K_2\varphi_k = \lambda_k \varphi_k. \tag{19.71}$$

Let P have the matrix elements $M_{\ell k}$ in the basis φ_k so that

$$P\varphi_k = \sum_{\ell} M_{\ell k}\,\varphi_\ell.$$

Consider the vector ϕ_k, with k fixed. Acting with the second relation of (19.67) on φ_k yields

$$\sum_{\ell} M_{\ell k}\,\{\lambda_k + \lambda_\ell + 1\}\,\varphi_\ell = (\beta - \alpha)\varphi_k, \tag{19.72}$$

where we have used the parametrization (19.32). For the term in the sum with $\ell = k$, one gets

$$M_{2p,2p} = \frac{\beta-\alpha}{4p+2+\alpha+\beta}, \qquad M_{2p+1,2p+1} = \frac{\alpha-\beta}{4p+2+\alpha+\beta}. \tag{19.73}$$

The remainder yields

$$\sum_{\ell \neq k} M_{\ell k}\,\{\lambda_k + \lambda_\ell + 1\}\,\varphi_\ell = 0. \tag{19.74}$$

Whence we must have either

$$M_{\ell,k} = 0, \quad \text{or} \quad \{\lambda_k + \lambda_\ell + 1\} = 0. \tag{19.75}$$

Since (19.75) is a linear equation in λ_ℓ (recall that k is fixed), there exists one solution for each possible parity of k. It is seen that

$$\{\lambda_{2p} + \lambda_{2p+1} + 1\} = 0, \tag{19.76}$$

so that (19.74) is ensured for the pairs ($k = 2p, \ell = 2p+1$) and ($k = 2p+1, \ell = 2p$). It follows that $M_{2p+1,2p}$ and $M_{2p,2p+1}$ are arbitrary and that $M_{\ell,k} = 0$ otherwise. Thus P has a block-diagonal structure with 2×2 blocks. By requiring that $P^2 = \mathbb{I}$, one finds that the matrices representing K_2 and P are of the form

$$K_2 = \operatorname{diag}\left(\Lambda_0, \ldots, \Lambda_{\lfloor N/2 \rfloor}\right) \qquad P = \operatorname{diag}\left(\Gamma_0, \ldots, \Gamma_{\lfloor N/2 \rfloor}\right) \tag{19.77}$$

where the 2×2 blocks have the expression

$$\Lambda_p = \begin{pmatrix} \lambda_{2p} & 0 \\ 0 & \lambda_{2p+1} \end{pmatrix}, \qquad \Gamma_p = \begin{pmatrix} \frac{\beta-\alpha}{4p+2+\alpha+\beta} & \frac{2(2p+1+\beta)\gamma_p}{4p+2+\alpha+\beta} \\ \frac{2(2p+1+\alpha)}{(4p+2+\alpha+\beta)\gamma_p} & \frac{\alpha-\beta}{4p+2+\alpha+\beta} \end{pmatrix}, \tag{19.78}$$

and where the real constants γ_p, $p \in \{0, \ldots, \lfloor N/2 \rfloor\}$, define a sequence of non-zero free parameters. These parameters will be treated below. Let K_1 have the matrix elements $N_{\ell k}$ in the basis φ_k so that

$$K_1 \varphi_k = \sum_{\ell} N_{\ell k} \varphi_\ell.$$

It is seen that the commutation relation (19.69) is equivalent to the following linear system of equations:

$$\begin{aligned} &\sum_{\ell} N_{\ell,2p} \left\{ [\lambda_{2p} - \lambda_\ell]^2 + 2\nu \Gamma_p^{(1,1)} \left[(\lambda_{2p} - \lambda_\ell) - 2 \right] - 4 \right\} \varphi_\ell \\ &\quad + \sum_{\ell} N_{\ell,2p+1} \left\{ 2\nu \Gamma_p^{(2,1)} \left[(\lambda_{2p+1} - \lambda_\ell) - 2 \right] \right\} \varphi_\ell = \left\{ \sigma \Gamma_p^{(1,1)} + \rho \right\} \varphi_{2p} + \sigma \Gamma_p^{(2,1)} \varphi_{2p+1}, \end{aligned} \tag{19.79}$$

$$\begin{aligned} &\sum_{\ell} N_{\ell,2p+1} \left\{ \left[\lambda_{2p+1} - \lambda_\ell \right]^2 + 2\nu \Gamma_p^{(2,2)} \left[(\lambda_{2p+1} - \lambda_\ell) - 2 \right] - 4 \right\} \varphi_\ell \\ &\quad + \sum_{\ell} N_{\ell,2p} \left\{ 2\nu \Gamma_p^{(1,2)} \left[(\lambda_{2p} - \lambda_\ell) - 2 \right] \right\} \varphi_\ell = \left\{ \sigma \Gamma_p^{(2,2)} + \rho \right\} \varphi_{2p+1} + \sigma \Gamma_p^{(1,2)} \varphi_{2p}, \end{aligned} \tag{19.80}$$

where $\Gamma_p^{(i,j)}$, $i,j \in \{1,2\}$, denotes the $(i,j)^{\text{th}}$ component of the p^{th} block Γ_p. It follows from the solution of (19.79) and (19.80) that the matrix representing K_1 is block tri-diagonal:

$$K_1 = \begin{pmatrix} C_0 & U_1 & & & \\ D_0 & C_1 & U_2 & & \\ & \ddots & \ddots & \ddots & \\ & & & & U_{\lfloor N/2 \rfloor} \\ & & & D_{\lfloor (N-2)/2 \rfloor} & C_{\lfloor N/2 \rfloor} \end{pmatrix}, \tag{19.81}$$

Using the solution of the linear system (19.79) and (19.80) and requiring that the first relation of (19.67) and the second relation (19.68) are satisfied yields

$$C_p = \begin{pmatrix} 2p - \frac{2p(N+1-2p)(2p+\alpha)}{4p+\alpha+\beta} + \frac{(2p+1)(N-2p)(2p+1+\alpha)}{4p+2+\alpha+\beta} & -\frac{(2p+1+\beta)(2N+2+\alpha+\beta)(\alpha+\beta)\gamma_p}{(4p+\alpha+\beta)(4p+2+\alpha+\beta)(4p+4+\alpha+\beta)} \\ -\frac{(2p+1+\alpha)(2N+2+\alpha+\beta)(\alpha+\beta)}{(4p+\alpha+\beta)(4p+2+\alpha+\beta)(4p+4+\alpha+\beta)\gamma_p} & 2p+1-\frac{(2p+1)(N-2p)(2p+1+\alpha)}{4p+2+\alpha+\beta} + \frac{(2p+2)(N-2p-1)(2p+2+\alpha)}{4p+4+\alpha+\beta} \end{pmatrix}$$

$$U_p = \begin{pmatrix} \frac{(2p-1+\beta)(N+1+2p+\alpha+\beta)\gamma_{p-1}\epsilon_p}{(4p-2+\alpha+\beta)(4p+\alpha+\beta)} & 0 \\ \frac{2(\alpha-\beta)(N+1+2p+\alpha+\beta)\epsilon_p}{(4p-2+\alpha+\beta)(4p+\alpha+\beta)(4p+2+\alpha+\beta)} & \frac{(2p+1+\beta)(N+1+2p+\alpha+\beta)\gamma_p\epsilon_p}{(4p+\alpha+\beta)(4p+2+\alpha+\beta)} \end{pmatrix}$$

$$D_p = \begin{pmatrix} \frac{(2p+2)(N+1-2p)(2p+1+\alpha)(2p+2+\alpha+\beta)}{(4p+2+\alpha+\beta)(4p+4+\alpha+\beta)\gamma_p\epsilon_{p+1}} & \frac{2(2p+2)(N-2p-1)(2p+2+\alpha+\beta)(\alpha-\beta)}{(4p+2+\alpha+\beta)(4p+4+\alpha+\beta)(4p+6+\alpha+\beta)\epsilon_{p+1}} \\ 0 & \frac{(2p+2)(N-2p-1)(2p+3+\alpha)(2p+2+\alpha+\beta)}{(4p+4+\alpha+\beta)(4p+6+\alpha+\beta)\gamma_{p+1}\epsilon_{p+1}} \end{pmatrix}$$

where the constants ϵ_k, $k \in \{1,\ldots,\lfloor N/2\rfloor\}$, define a second set of non-zero free parameters. It can be checked that with their matrix elements so defined, K_1, K_2 and P realize the commutation relations (19.67), (19.68) and (19.69) with the Casimir eigenvalue (19.34). The two sequences of free parameters appearing in the representation can be reduced to one sequence $\{\theta_i\}$ by introducing the following diagonal similarity transformation

$$T_p = \begin{pmatrix} \pi_p\theta_{2p} & 0 \\ 0 & \frac{\pi_p\theta_{2p+1}}{\gamma(p)} \end{pmatrix}, \qquad \pi_p = \prod_{j=0}^{p-1}\frac{1}{\gamma(j)\epsilon(j+1)}, \tag{19.82}$$

where $p \in \{0,\ldots,\lfloor N/2\rfloor\}$, is the block index and $\pi_0 = 1$. It should be noted that $\theta_p \neq 0$. These free parameters correspond to all the possible diagonal similarity transformations that leave the spectrum of K_2 and its ordering invariant. They also correspond to the freedom associated to the substitution

$$S_{ij} = Q_i(x_j;\alpha,\beta,N) \to S'_{ij} = \lambda_j Q_i(x_j;\alpha,\beta,N) \tag{19.83}$$

in the transition matrix (19.36). They could be fixed by unitarity requirements for example. Under the transformation $T^{-1}\mathscr{O}T$, where $\mathscr{O}$ represents any element of the algebra, the matrix elements become

$$\Lambda_p = \begin{pmatrix} \lambda_{2p} & 0 \\ 0 & \lambda_{2p+1} \end{pmatrix}, \qquad \Gamma_p = \begin{pmatrix} \frac{\beta-\alpha}{4p+2+\alpha+\beta} & \frac{2(2p+1+\beta)\theta_{2p+1}}{(4p+2+\alpha+\beta)\theta_{2p}} \\ \frac{2(2p+1+\alpha)\theta_{2p+1}}{(4p+2+\alpha+\beta)\theta_{2p}} & \frac{\alpha-\beta}{4p+2+\alpha+\beta} \end{pmatrix},$$

$$C_p = \begin{pmatrix} 2p - \frac{2p(N+1-2p)(2p+\alpha)}{4p+\alpha+\beta} + \frac{(2p+1)(N-2p)(2p+1+\alpha)}{4p+2+\alpha+\beta} & -\frac{(2p+1+\beta)(2N+2+\alpha+\beta)(\alpha+\beta)\theta_{2p+1}}{(4p+\alpha+\beta)(4p+2+\alpha+\beta)(4p+4+\alpha+\beta)\theta_{2p}} \\ -\frac{(2p+1+\alpha)(2N+2+\alpha+\beta)(\alpha+\beta)\theta_{2p}}{(4p+\alpha+\beta)(4p+2+\alpha+\beta)(4p+4+\alpha+\beta)\theta_{2p+1}} & 2p+1-\frac{(2p+1)(N-2p)(2p+1+\alpha)}{4p+2+\alpha+\beta} + \frac{(2p+2)(N-2p-1)(2p+2+\alpha)}{4p+4+\alpha+\beta} \end{pmatrix}$$

$$U_p = \begin{pmatrix} \frac{(2p-1+\beta)(N+1+2p+\alpha+\beta)\theta_{2p}}{(4p-2+\alpha+\beta)(4p+\alpha+\beta)\theta_{2p-2}} & 0 \\ \frac{2(\alpha-\beta)(N+1+2p+\alpha+\beta)\theta_{2p}}{(4p-2+\alpha+\beta)(4p+\alpha+\beta)(4p+4+\alpha+\beta)\theta_{2p+1}} & \frac{(2p+1+\alpha+\beta)(N+1+2p+\alpha+\beta)\theta_{2p+1}}{(4p+\alpha+\beta)(4p+2+\alpha+\beta)\theta_{2p-1}} \end{pmatrix}$$

$$D_p = \begin{pmatrix} \frac{(2p+2)(N-2p-1)(2p+1+\alpha)(2p+2+\alpha+\beta)\theta_{2p}}{(4p+2+\alpha+\beta)(4p+4+\alpha+\beta)\theta_{2p+2}} & \frac{2(2p+2)(N-2p-1)(\alpha-\beta)(2p+2+\alpha+\beta)\theta_{2p+1}}{(4p+2+\alpha+\beta)(4p+4+\alpha+\beta)(4p+6+\alpha+\beta)\theta_{2p+2}} \\ 0 & \frac{(2p+2)(N-2p-1)(2p+3+\alpha)(2p+2+\alpha+\beta)\theta_{2p+1}}{(4p+4+\alpha+\beta)(4p+6+\alpha+\beta)\theta_{2p+3}} \end{pmatrix},$$

It is thus seen that the algebra $\mathscr{H}$ admits an irreducible representation of dimension $N+1$ where the operator K_2 is diagonal and where K_1 is the five-diagonal matrix with elements as given by the formulas above.

19.5.2 N even

The treatment of the N even case is similar to that of the N odd case. We consider again the basis ϕ_k in which K_2 is diagonal with spectrum

$$\lambda_s = (-1)^s(s + 1/2 - \alpha/2 - \beta/2). \tag{19.84}$$

It follows from the second commutation relation of (19.67) that the matrices K_2 and P are of the form

$$K_2 = \text{diag}(\Lambda_0, \ldots, \Lambda_{(N-2)/2}, \lambda_N), \qquad P = \text{diag}(\Gamma_0, \ldots, \Gamma_{(N-2)/2}, 1), \tag{19.85}$$

where

$$\Lambda_p = \begin{pmatrix} \lambda_{2p} & 0 \\ 0 & \lambda_{2p+1} \end{pmatrix}, \qquad \Gamma_p = \begin{pmatrix} \frac{2N+2-\alpha-\beta}{4p+2-\alpha-\beta} & \frac{2(N+2+2p+-\alpha-\beta)\gamma_p}{4p+2+\alpha+\beta} \\ \frac{2(2p-N)}{(4p+2-\alpha-\beta)\gamma_p} & -\frac{2N+2-\alpha-\beta}{4p+2-\alpha-\beta} \end{pmatrix}. \tag{19.86}$$

Imposing the commutation relations (19.69), (19.67) and the last of (19.68), one obtains the matrix elements of K_1 with two sets of free parameters. The two sets can be reduced to one set $\{\xi_i\}_{i=0}^N$, $\xi_i \neq 0$, corresponding to the possible diagonal transformations preserving the spectrum of K_2 as well as its ordering. One finds

$$\Lambda_p = \begin{pmatrix} \lambda_{2p} & 0 \\ 0 & \lambda_{2p+1} \end{pmatrix}, \qquad \Gamma_p = \begin{pmatrix} \frac{2N+2-\alpha-\beta}{4p+2-\alpha-\beta} & \frac{2(N+2p+2-\alpha-\beta)\xi_{2p+1}}{(4p+2-\alpha-\beta)\xi_{2p}} \\ \frac{2(2p-N)\xi_{2p}}{(4p+2-\alpha-\beta)\xi_{2p+1}} & -\frac{2N+2-\alpha-\beta}{4p+2-\alpha-\beta} \end{pmatrix}.$$

$$C_p = \begin{pmatrix} 2p + \frac{(N-2p)(2p+1)(2p+1-\alpha)}{(4p+2-\alpha-\beta)} - \frac{2p(N+1-2p)(2p-\alpha)}{(4p-\alpha-\beta)} & \frac{(N+2p+2-\alpha-\beta)(\alpha^2-\beta^2)\xi_{2p+1}}{(4p-\alpha-\beta)(4p+2-\alpha-\beta)(4p+4-\alpha-\beta)\xi_{2p}} \\ \frac{(N-2p)(\alpha^2-\beta^2)\xi_{2p}}{(4p-\alpha-\beta)(4p+2-\alpha-\beta)(4p+4-\alpha-\beta)\xi_{2p+1}} & 2p+1-\frac{(N-2p)(2p+1)(2p+1-\alpha)}{(4p+2-\alpha-\beta)} + \frac{(2p+2)(N-2p-1)(2p+2-\alpha)}{(4p+4-\alpha-\beta)} \end{pmatrix}$$

$$U_p = \begin{pmatrix} \frac{2p(N+2-2p)(2p-\alpha-\beta)(2p+N-\alpha-\beta)\xi_{2p}}{(4p-2-\alpha-\beta)(4p-\alpha-\beta)\xi_{2p-2}} & 0 \\ -\frac{4p(N+2-2p)(2p-\alpha-\beta)(2N+2-\alpha-\beta)\xi_{2p}}{(4p-2-\alpha-\beta)(4p-\alpha-\beta)(4p+2-\alpha-\beta)\xi_{2p-1}} & \frac{2p(N+2-2p)(2p-\alpha-\beta)(N+2p+2-\alpha-\beta)\xi_{2p+1}}{(4p-\alpha-\beta)(4p+2-\alpha-\beta)\xi_{2p-1}} \end{pmatrix}$$

$$D_p = \begin{pmatrix} \frac{(2p+2-\alpha)(2p+2-\beta)\xi_{2p}}{(4p+2-\alpha-\beta)(4p+4-\alpha-\beta)\xi_{2p+2}} & \frac{2(2p+2-\alpha)(2p+2-\beta)(2N+2-\alpha-\beta)\xi_{2p+1}}{(N-2p)(4p+2-\alpha-\beta)(4p+4-\alpha-\beta)(4p+6-\alpha-\beta)\xi_{2p+2}} \\ 0 & \frac{(N-2p-2)(2p+2-\alpha)(2p+2-\beta)\xi_{2p+1}}{(N-2p)(4p+4-\alpha-\beta)(4p+6-\alpha-\beta)\xi_{2p+3}} \end{pmatrix}$$

It is straightforward to verify that this reproduces the algebra $\mathscr{H}$ with the fixed value of the Casimir operator (19.34).

19.6 Conclusion

In this paper, we have derived the algebra $\mathscr{H}$ associated to the dual -1 Hahn polynomials and shown that this algebra occurs as the hidden symmetry algebra of the Clebsch-Gordan problem of $sl_{-1}(2)$. We also obtained the irreducible representations of $\mathscr{H}$ which involve five-diagonal matrices and correspond to the difference equation of the dual -1 Hahn polynomials. Although the algebra $\mathscr{H}$ has been derived using a specific realization with a fixed value of the Casimir operator, it can also be considered in an abstract fashion. In concluding we hence wish to offer a different presentation of $\mathscr{H}$ that makes its structure transparent. Upon introducing the following new generators:

$$\widetilde{K_1} = K_1 + \rho/4, \qquad \widetilde{K_2} = \frac{1}{2}(K_2 + \nu P + 1/2), \qquad \widetilde{K_3} = \frac{1}{2}K_3 \tag{19.87}$$

the defining relations now take the form:

$$[\widetilde{K_1}, P] = 0, \qquad \{\widetilde{K_2}, P\} = 0, \qquad \{\widetilde{K_3}, P\} = 0 \tag{19.88}$$

$$[\widetilde{K_1}, \widetilde{K_2}] = \widetilde{K_3}, \quad [\widetilde{K_1}, \widetilde{K_3}] = \widetilde{K_2}, \tag{19.89}$$

$$[\widetilde{K_3}, \widetilde{K_2}] = \widetilde{K_1} + \nu\widetilde{K_1}P + \chi P, \tag{19.90}$$

where $\chi = (\sigma - \nu\rho)/4$. This presentation makes it manifest that $\mathscr{H}$ is a 2-parameter generalization of $\mathfrak{u}(2)$ (allowing for the presence of a central element possibly hidden in χ) with the inclusion of the involution P. The Casimir operator of $\mathscr{H}$ then takes the form

$$Q_{\mathscr{H}} = \widetilde{K}_1^2 + \widetilde{K}_2^2 - \widetilde{K}_3^2 + (\nu/2)P, \tag{19.91}$$

which is clearly a simple one-parameter deformation of the standard $sl(2)$ Casimir operator.

Interestingly, one-parameter versions of this algebra (with either ν or χ equal to zero) have been introduced in studies of finite analogues of the parabosonic oscillator [6, 7] . The wave functions that were found in this context turn out to be symmetrized dual -1 Hahn polynomials. Indeed, it is seen that the substitution

$$\widetilde{K_2} = \frac{1}{2}(K_2 + \nu P + 1/2), \tag{19.92}$$

corresponds to a symmetrization of the recurrence relation of the dual -1 Hahn polynomials (i.e. the suppression of the diagonal term). It is worth pointing out that recently the algebra $\mathscr{H}$ has also been identified as the symmetry algebra of a two-dimensional superintegrable model with reflections [3].

The algebra $\mathscr{H}$ can be considered as an extension of the Askey-Wilson algebra $AW(3)$ [22] . It is known [9, 10] that these algebras are related to double-affine Hecke algebras (DAHA). It would be of interest in the future to explore the possible relation between DAHAs and the algebra $\mathscr{H}$.

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Chapitre 20

The Bannai–Ito algebra and some applications

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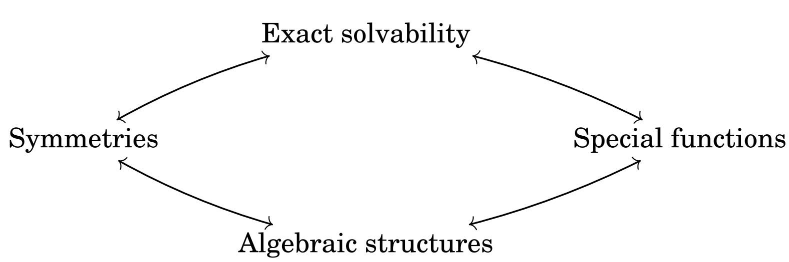
Abstract. The Bannai-Ito algebra is presented together with some of its applications. Its relations with the Bannai-Ito polynomials, the Racah problem for the $sl_{-1}(2)$ algebra, a superintegrable model with reflections and a Dirac-Dunkl equation on the 2-sphere are surveyed.

20.1 Introduction

Exploration through the exact solution of models has a secular tradition in mathematical physics. Empirically, exact solvability is possible in the presence of symmetries, which come in various guises and which are described by a variety of mathematical structures. In many cases, exact solutions are expressed in terms of special functions, whose properties encode the symmetries of the systems in which they arise. This can be represented by the following virtuous circle:



The classical path is the following: start with a model, find its symmetries, determine how these symmetries are mathematically described, work out the representations of that mathematical structure and obtain its relation to special functions to arrive at the solution of the model. However, one can profitably start from any node on this circle. For instance, one can identify and characterize new special functions, determine the algebraic structure they encode, look for models that have this structure as symmetry algebra and proceed to the solution. In this paper, the following path will be taken:

Algebra ⟶ Orthogonal polynomials ⟶ Symmetries ⟶ Exact solutions

The outline of the paper is as follows. In section 2, the Bannai-Ito algebra is introduced and some of its special cases are presented. In section 3, a realization of the Bannai-Ito algebra in terms of discrete shift and reflection operators is exhibited. The Bannai-Ito polynomials and their properties are discussed in section 4. In section 5, the Bannai-Ito algebra is used to derive the recurrence relation satisfied by the Bannai-Ito polynomials. In section 6, the paraboson algebra and the $sl_{-1}(2)$ algebra are introduced. In section 7, the realization of $sl_{-1}(2)$ in terms of Dunkl operators is discussed. In section 8, the Racah problem for $sl_{-1}(2)$ and its relation with the Bannai-Ito algebra is examined. A superintegrable model on the 2-sphere with Bannai-Ito symmetry is studied in section 9. In section 10, a Dunkl-Dirac equation on the 2-sphere with Bannai-Ito symmetry is discussed. A list of open questions is provided in lieu of a conclusion.

20.2 The Bannai-Ito algebra

Throughout the paper, the notation $[A,B] = AB - BA$ and $\{A,B\} = AB + BA$ will be used. Let ω_1, ω_2 and ω_3 be real parameters. The Bannai-Ito algebra is the associative algebra generated by K_1, K_2 and K_3 together with the three relations

$$\{K_1,K_2\} = K_3 + \omega_3, \quad \{K_2,K_3\} = K_1 + \omega_1, \quad \{K_3,K_1\} = K_2 + \omega_2, \tag{20.1}$$

or $\{K_i,K_j\} = K_k + \omega_k$, with (ijk) a cyclic permutation of $(1,2,3)$. The Casimir operator

$$Q = K_1^2 + K_2^2 + K_3^2,$$

commutes with every generator; this property is easily verified with the commutator identity $[AB,C] = A\{B,C\} - \{A,C\}B$. Let us point out two special cases of (20.1) that have been considered previously in the literature.

1. $\omega_1 = \omega_2 = \omega_3 = 0$

The special case with defining relations

$$\{K_1,K_2\} = K_3, \quad \{K_2,K_3\} = K_1, \quad \{K_3,K_1\} = K_2,$$

is sometimes referred to as the *anticommutator spin algebra* [2, 21]; representations of this algebra were examined in [2, 21, 4, 29].

2. $\omega_1 = \omega_2 = 0 \neq \omega_3$

In recent work on the construction of novel finite oscillator models [23, 24], E. Jafarov, N. Stoilova and J. Van der Jeugt introduced the following extension of $\mathfrak{u}(2)$ by an involution R ($R^2 = 1$):

$$[I_3, R] = 0, \quad \{I_1, R\} = 0, \quad \{I_2, R\} = 0,$$
$$[I_3, I_1] = iI_2, \quad [I_2, I_3] = iI_1, \quad [I_1, I_2] = i(I_3 + \omega_3 R).$$

It is easy to check that with

$$K_1 = iI_1 R, \quad K_2 = I_2, \quad K_3 = I_3 R,$$

the above relations are converted into

$$\{K_1, K_3\} = K_2, \quad \{K_2, K_3\} = K_1, \quad \{K_1, K_2\} = K_3 + \omega_3.$$

20.3 A realization of the Bannai-Ito algebra with shift and reflections operators

Let T^+ and R be defined as follows:

$$T^+ f(x) = f(x+1), \quad R f(x) = f(-x).$$

Consider the operator

$$\widehat{K}_1 = F(x)(1-R) + G(x)(T^+ R - 1) + h, \qquad h = \rho_1 + \rho_2 - r_1 - r_2 + 1/2, \tag{20.2}$$

with $F(x)$ and $G(x)$ given by

$$F(x) = \frac{(x-\rho_1)(x-\rho_2)}{x}, \quad G(x) = \frac{(x - r_1 + 1/2)(x - r_2 + 1/2)}{x + 1/2},$$

where ρ_1, ρ_2, r_1, r_2 are four real parameters. It can be shown that $\widehat{K}_1$ is the most general operator of first order in T^+ and R that stabilizes the space of polynomials of a given degree [32]. That is, for any polynomial $Q_n(x)$ of degree n, $[\widehat{K}_1 Q_n(x)]$ is also a polynomial of degree n. Introduce

$$\widehat{K}_2 = 2x + 1/2, \tag{20.3}$$

which is essentially the "multiplication by x" operator and

$$\widehat{K}_3 \equiv \{\widehat{K}_1, \widehat{K}_2\} - 4(\rho_1\rho_2 - r_1 r_2). \tag{20.4}$$

It is directly verified that $\widehat{K}_1$, $\widehat{K}_2$ and $\widehat{K}_3$ satisfy the commutation relations

$$\{\widehat{K}_1, \widehat{K}_2\} = \widehat{K}_3 + \widehat{\omega}_3, \quad \{\widehat{K}_2, \widehat{K}_3\} = \widehat{K}_1 + \widehat{\omega}_1, \quad \{\widehat{K}_3, \widehat{K}_1\} = \widehat{K}_2 + \widehat{\omega}_2, \tag{20.5}$$

where the structure constants $\widehat{\omega}_1$, $\widehat{\omega}_2$ and $\widehat{\omega}_3$ read

$$\widehat{\omega}_1 = 4(\rho_1\rho_2 + r_1 r_2), \quad \widehat{\omega}_2 = 2(\rho_1^2 + \rho_2^2 - r_1^2 - r_2^2), \quad \widehat{\omega}_3 = 4(\rho_1\rho_2 - r_1 r_2). \tag{20.6}$$

The operators $\widehat{K}_1$, $\widehat{K}_2$ and $\widehat{K}_3$ thus realize the Bannai-Ito algebra. In this realization, the Casimir operator acts as a multiple of the identity; one has indeed

$$\widehat{Q} = \widehat{K}_1^2 + \widehat{K}_2^2 + \widehat{K}_3^2 = 2(\rho_1^2 + \rho_2^2 + r_1^2 + r_2^2) - 1/4.$$

20.4 The Bannai-Ito polynomials

Since the operator (20.2) preserves the space of polynomials of a given degree, it is natural to look for its eigenpolynomials, denoted by $B_n(x)$, and their corresponding eigenvalues λ_n. We use the following notation for the generalized hypergeometric series [1]

$${}_rF_s\left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \,\middle|\, z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_r)_k}{(b_1)_k \cdots (b_s)_k} \frac{z^k}{k!},$$

where $(c)_k = c(c+1)\cdots(c+k-1)$, $(c)_0 \equiv 1$ stands for the Pochhammer symbol; note that the above series terminates if one of the a_i is a negative integer. Solving the eigenvalue equation

$$\widehat{K}_1 B_n(x) = \lambda_n B_n(x), \qquad n = 0, 1, 2, \dots \tag{20.7}$$

it is found that the eigenvalues λ_n are given by [32]

$$\lambda_n = (-1)^n (n + h), \tag{20.8}$$

and that the polynomials have the expression

$$\frac{B_n(x)}{c_n} = \begin{cases} {}_4F_3\left(\begin{matrix} -\frac{n}{2}, \frac{n+1}{2}+h, x-r_1+1/2, -x-r_1+1/2 \\ 1-r_1-r_2, \rho_1-r_1+\frac{1}{2}, \rho_2-r_1+\frac{1}{2} \end{matrix} \,\middle|\, 1\right) \\ \quad + \frac{(\frac{n}{2})(x-r_1+\frac{1}{2})}{(\rho_1-r_1+\frac{1}{2})(\rho_2-r_1+\frac{1}{2})} \, {}_4F_3\left(\begin{matrix} 1-\frac{n}{2}, \frac{n+1}{2}+h, x-r_1+3/2, -x-r_1+1/2 \\ 1-r_1-r_2, \rho_1-r_1+\frac{3}{2}, \rho_2-r_1+\frac{3}{2} \end{matrix} \,\middle|\, 1\right) & n \text{ even}, \\ {}_4F_3\left(\begin{matrix} -\frac{n-1}{2}, \frac{n}{2}+h, x-r_1+\frac{1}{2}, -x-r_1+\frac{1}{2} \\ 1-r_1-r_2, \rho_1-r_1+\frac{1}{2}, \rho_2-r_1+\frac{1}{2} \end{matrix} \,\middle|\, 1\right) \\ \quad - \frac{(\frac{n}{2}+h)(x-r_1+\frac{1}{2})}{(\rho_1-r_1+\frac{1}{2})(\rho_2-r_1+\frac{1}{2})} \, {}_4F_3\left(\begin{matrix} -\frac{n-1}{2}, \frac{n+2}{2}+h, x-r_1+\frac{3}{2}, -x-r_1+\frac{1}{2} \\ 1-r_1-r_2, \rho_1-r_1+\frac{3}{2}, \rho_2-r_1+\frac{3}{2} \end{matrix} \,\middle|\, 1\right) & n \text{ odd}, \end{cases} \tag{20.9}$$

where the coefficient

$$c_{2n+p} = (-1)^p \frac{(1-r_1-r_2)_n(\rho_1 - r_1 + 1/2, \rho_2 - r_1 + 1/2)_{n+p}}{(n+h+1/2)_{n+p}}, \qquad p \in \{0,1\},$$

ensures that the polynomials $B_n(x)$ are monic, i.e. $B_n(x) = x^n + \mathscr{O}(x^{n-1})$. The polynomials (20.9) were first written down by Bannai and Ito in their classification of the orthogonal polynomials satisfying the *Leonard duality* property [27, 3], i.e. polynomials $p_n(x)$ satisfying both

- A 3-term recurrence relation with respect to the degree n,
- A 3-term difference equation with respect to a variable index s.

The identification of the defining eigenvalue equation (20.7) of the Bannai-Ito polynomials in [32] has allowed to develop their theory. That they obey a three-term difference equation stems from the fact that there are grids such as

$$x_s = (-1)^s(s/2 + a + 1/4) - 1/4,$$

for which operators of the form

$$H = A(x)R + B(x)T^{+}R + C(x),$$

are tridiagonal in the basis $f(x_s)$

$$Hf(x_s) = \begin{cases} B(x_s)f(x_{s+1}) + A(x_s)f(x_{s-1}) + C(x_s)f(x_s) & s \text{ even}, \\ A(x_s)f(x_{s+1}) + B(x_s)f(x_{s-1}) + C(x_s)f(x_s) & s \text{ odd}. \end{cases}$$

It was observed by Bannai and Ito that the polynomials (20.9) correspond to a $q \to -1$ limit of the q-Racah polynomials (see [26] for the definition of q-Racah polynomials). In this connection, it is worth mentioning that the Bannai-Ito algebra (20.5) generated by the defining operator $\widehat{K}_1$ and the recurrence operator $\widehat{K}_2$ of the Bannai-Ito polynomials can be obtained as a $q \to -1$ limit of the Zhedanov algebra [37], which encodes the bispectral property of the q-Racah polynomials. The Bannai-Ito polynomials $B_n(x)$ have companions

$$I_n(x) = \frac{B_{n+1}(x) - \frac{B_{n+1}(\rho_1)}{B_n(\rho_1)} B_n(x)}{x - \rho_1},$$

called the *complementary* Bannai-Ito polynomials [14]. It has now been understood that the polynomials $B_n(x)$ and $I_n(x)$ are the ancestors of a rich ensemble of polynomials referred to as "−1 orthogonal polynomials" [32, 14, 17, 35, 36, 33, 31]. All polynomials of this scheme are eigenfunctions of first or second order operators of Dunkl type, i.e. which involve reflections.

20.5 The recurrence relation of the BI polynomials from the BI algebra

Let us now show how the Bannai-Ito algebra can be employed to derive the recurrence relation satisfied by the Bannai-Ito polynomials. In order to obtain this relation, one needs to find the action of the operator $\widehat{K}_2$ on the BI polynomials $B_n(x)$. Introduce the operators

$$\widehat{K}_+ = (\widehat{K}_2 + \widehat{K}_3)(\widehat{K}_1 - 1/2) - \frac{\widehat{\omega}_2 + \widehat{\omega}_3}{2}, \quad \widehat{K}_- = (\widehat{K}_2 - \widehat{K}_3)(\widehat{K}_1 + 1/2) + \frac{\widehat{\omega}_2 - \widehat{\omega}_3}{2}, \tag{20.10}$$

where $\widehat{K}_i$ and $\widehat{\omega}_i$ are given by (20.2), (20.3), (20.4) and (20.6). It is readily checked using (20.5) that

$$\{\widehat{K}_1, \widehat{K}_\pm\} = \pm K_\pm.$$

One can directly verify that $\widehat{K}_\pm$ maps polynomials to polynomials. In view of the above, one has

$$\widehat{K}_1\widehat{K}_+B_n(x) = (-\widehat{K}_+\widehat{K}_1 + \widehat{K}_+)B_n(x) = (1-\lambda_n)\widehat{K}_+B_n(x),$$

where λ_n is given by (20.8). It is also seen from (20.8) that

$$1-\lambda_n = \begin{cases} \lambda_{n-1} & n \text{ even}, \\ \lambda_{n+1} & n \text{ odd}. \end{cases}$$

It follows that

$$\widehat{K}_+B_n(x) = \begin{cases} \alpha_n^{(0)}B_{n-1}(x) & n \text{ even}, \\ \alpha_n^{(1)}B_{n+1}(x) & n \text{ odd}. \end{cases}$$

Similarly, one finds

$$\widehat{K}_-B_n(x) = \begin{cases} \beta_n^{(0)}B_{n+1}(x) & n \text{ even}, \\ \beta_n^{(1)}B_{n-1}(x) & n \text{ odd}. \end{cases}$$

The coefficients

$$\alpha_n^{(0)} = \frac{2n(\frac{n}{2}+\rho_1+\rho_2)(r_1+r_2-\frac{n}{2})(\frac{n-1}{2}+h)}{n+h-\frac{1}{2}}, \quad \alpha_n^{(1)} = -4(n+h+1/2),$$

$$\beta_n^{(0)} = 4(n+h+1/2), \quad \beta_n^{(1)} = \frac{4(\rho_1-r_1+\frac{n}{2})(\rho_2-r_1+\frac{n}{2})(\rho_1-r_2+\frac{n}{2})(\rho_2-r_2+\frac{n}{2})}{n+h-1/2},$$

can be obtained from the comparison of the highest order term. Introduce the operator

$$V = \widehat{K}_+(\widehat{K}_1+1/2) + \widehat{K}_-(\widehat{K}_1 - 1/2). \tag{20.11}$$

From the definition (20.10) of $\widehat{K}_{\pm}$, it follows that

$$V = 2\widehat{K}_2(\widehat{K}_1^2 - 1/4) - \widehat{\omega}_3\widehat{K}_1 - \widehat{\omega}_2/2. \tag{20.12}$$

From (20.7), (20.11) and the actions of the operators $\widehat{K}_{\pm}$, we find that V is two-diagonal

$$VB_n(x) = \begin{cases} (\lambda_n + 1/2)\alpha_n^{(0)}B_{n-1}(x) + (\lambda_n - 1/2)\beta_n^{(0)}B_{n+1}(x) & n \text{ even}, \\ (\lambda_n - 1/2)\beta_n^{(1)}B_{n-1}(x) + (\lambda_n + 1/2)\alpha_n^{(1)}B_{n+1}(x) & n \text{ odd}. \end{cases} \tag{20.13}$$

From (20.12) and recalling the definition (20.3) of $\widehat{K}_2$, we have also

$$VB_n(x) = \left[(\lambda_n^2 - 1/4)(4x+1) - \widehat{\omega}_3\lambda_n - \widehat{\omega}_2/2\right]B_n(x). \tag{20.14}$$

Upon combining (20.13) and (20.14), one finds that the Bannai-Ito polynomials satisfy the three-term recurrence relation

$$xB_n(x) = B_{n+1}(x) + (\rho_1 - A_n - C_n)B_n(x) + A_{n-1}C_nB_{n-1}(x),$$

where

$$A_n = \begin{cases} \frac{(n+1+2\rho_1-2r_1)(n+1+2\rho_1-2r_2)}{4(n+\rho_1+\rho_2-r_1-r_2+1)} & n \text{ even}, \\ \frac{(n+1+2\rho_1+2\rho_2-2r_1-2r_2)(n+1+2\rho_1+2\rho_2)}{4(n+\rho_1+\rho_2-r_1-r_2+1)} & n \text{ odd}, \end{cases}$$
$$C_n = \begin{cases} -\frac{n(n-2r_1-2r_2)}{4(n+\rho_1+\rho_2-r_1-r_2)} & n \text{ even}, \\ -\frac{(n+2\rho_2-2r_2)(n+2\rho_2-2r_1)}{4(n+\rho_1+\rho_2-r_1-r_2)} & n \text{ odd}. \end{cases} \tag{20.15}$$

The positivity of the coefficient $A_{n-1}C_n$ restricts the polynomials $B_n(x)$ to being orthogonal on a finite set of points [5].

20.6 The paraboson algebra and $sl_{-1}(2)$

The next realization of the Bannai-Ito algebra will involve $sl_{-1}(2)$; this algebra, introduced in [30], is closely related to the parabosonic oscillator.

20.6.1 The paraboson algebra

Let a and $a^\dagger$ be the generators of the paraboson algebra. These generators satisfy [22]

$$[\{a, a^\dagger\}, a] = -2a, \quad [\{a, a^\dagger\}, a^\dagger] = 2a^\dagger.$$

Setting $H = \frac{1}{2}\{a, a^\dagger\}$, the above relations amount to

$$[H, a] = -a, \quad [H, a^\dagger] = a^\dagger,$$

which correspond to the quantum mechanical equations of an oscillator.

20.6.2 Relation with $\mathfrak{osp}(1|2)$

The paraboson algebra is related to the Lie superalgebra $\mathfrak{osp}(1|2)$ [9]. Indeed, upon setting

$$F_- = a, \quad F_+ = a^\dagger, \quad E_0 = H = \frac{1}{2}\{F_+, F_-\}, \quad E_+ = \frac{1}{2}F_+^2, \quad E_- = \frac{1}{2}F_-^2,$$

and interpreting $F_\pm$ as odd generators, it is directly verified that the generators $F_\pm$, $E_\pm$ and E_0 satisfy the defining relations of $\mathfrak{osp}(1|2)$ [25]:

$$[E_0, F_\pm] = \pm F_\pm, \quad \{F_+, F_-\} = 2E_0, \quad [E_0, E_\pm] = \pm 2E_\pm, \quad [E_-, E_+] = E_0,$$
$$[F_\pm, E_\pm] = 0, \quad [F_\pm, E_\mp] = \mp F_\mp.$$

The $\mathfrak{osp}(1|2)$ Casimir operator reads

$$C_{\mathfrak{osp}(1|2)} = (E_0 - 1/2)^2 - 4E_+E_- - F_+F_-.$$

20.6.3 $sl_q(2)$

Consider now the quantum algebra $sl_q(2)$. It can be presented in terms of the generators A_0 and $A_\pm$ satisfying the commutation relations [34]

$$[A_0, A_\pm] = \pm A_\pm, \quad [A_-, A_+] = 2\frac{q^{A_0} - q^{-A_0}}{q - q^{-1}}.$$

Upon setting

$$B_+ = A_+ q^{(A_0-1)/2}, \quad B_- = q^{(A_0-1)/2} A_-, \quad B_0 = A_0,$$

these relations become

$$[B_0, B_\pm] = \pm B_\pm, \quad B_-B_+ - qB_+B_- = 2\frac{q^{2B_0} - 1}{q^2 - 1}.$$

The $sl_q(2)$ Casimir operator is of the form

$$C_{sl_q(2)} = B_+B_- q^{-B_0} - \frac{2}{(q^2-1)(q-1)}(q^{B_0-1} + q^{-B_0}).$$

Let j be a non-negative integer. The algebra $sl_q(2)$ admits a discrete series representation on the basis $|j,n\rangle$ with the actions

$$q^{B_0}|j,n\rangle = q^{j+n}|j,n\rangle, \qquad n = 0,1,2,\ldots.$$

The algebra has a non-trivial coproduct $\Delta : sl_q(2) \to sl_q(2) \otimes sl_q(2)$ which reads

$$\Delta(B_0) = B_0 \otimes 1 + 1 \otimes B_0, \quad \Delta(B_\pm) = B_\pm \otimes q^{B_0} + 1 \otimes B_\pm.$$

20.6.4 The $sl_{-1}(2)$ algebra as a $q \to -1$ limit of $sl_q(2)$

The $sl_{-1}(2)$ algebra can be obtained as a $q \to -1$ limit of $sl_q(2)$. Let us first introduce the operator R defined as

$$R = \lim_{q\to -1} q^{B_0}.$$

It is easily seen that

$$R|\, j,n\,\rangle = (-1)^{j+n}|\, j,n\,\rangle = \epsilon(-1)^n|\, j,n\,\rangle,$$

where $\epsilon = \pm 1$ depending on the parity of j, thus $R^2 = 1$. When $q \to -1$, one finds that

$$\left.\begin{array}{l} q^{B_0}B_+ = qB_+q^{B_0} \\ B_-q^{B_0} = qq^{B_0}B_- \end{array}\right. \longrightarrow \{R,B_\pm\} = 0,$$

$$B_-B_+ - qB_+B_- = 2\frac{q^{2B_0}-1}{q^2-1} \longrightarrow \{B_+,B_-\} = 2B_0,$$

$$C_{sl_q(2)} \longrightarrow B_+B_-R - B_0R + R/2,$$

$$\Delta(B_\pm) = B_\pm \otimes q^{B_0} + 1 \otimes B_\pm \longrightarrow \Delta(B_\pm) = B_\pm \otimes R + 1 \otimes B_\pm.$$

In summary, $sl_{-1}(2)$ is the algebra generated by J_0, $J_\pm$ and R with the relations [30]

$$[J_0,J_\pm] = \pm J_\pm, \quad [J_0,R] = 0, \quad \{J_\pm,R\} = 0, \quad \{J_+,J_-\} = 2J_0, \quad R^2 = 1. \tag{20.16}$$

The Casimir operator has the expression

$$Q = J_+J_-R - J_0R + R/2, \tag{20.17}$$

and the coproduct is of the form [6]

$$\Delta(J_0) = J_0 \otimes 1 + 1 \otimes J_0, \quad \Delta(J_\pm) = J_\pm \otimes R + 1 \otimes J_\pm, \quad \Delta(R) = R \otimes R. \tag{20.18}$$

The $sl_{-1}(2)$ algebra (20.16) has irreducible and unitary discrete series representations with basis $|\,\epsilon,\mu;n\,\rangle$, where n is a non-negative integer, $\epsilon = \pm 1$ and μ is a real number such that $\mu > -1/2$. These representations are defined by the following actions:

$$J_0|\,\epsilon,\mu;n\,\rangle = (n+\mu+\frac{1}{2})|\,\epsilon,\mu;n\,\rangle, \quad R|\,\epsilon,\mu;n\,\rangle = \epsilon(-1)^n|\,\epsilon,\mu;n\,\rangle,$$

$$J_+|\,\epsilon,\mu;n\,\rangle = \rho_{n+1}|\,\epsilon,\mu;n+1\,\rangle, \quad J_-|\,\epsilon,\mu;n\,\rangle = \rho_n|\,\epsilon,\mu;n-1\,\rangle,$$

where $\rho_n = \sqrt{n+\mu(1-(-1)^n)}$. In these representations, the Casimir operator takes the value

$$Q|\,\epsilon,\mu;n\,\rangle = -\epsilon\mu|\,\epsilon,\mu;n\,\rangle.$$

These modules will be denoted by $V^{(\epsilon,\mu)}$. Let us offer the following remarks.

- The $sl_{-1}(2)$ algebra corresponds to the parabose algebra supplemented by R.
- The $sl_{-1}(2)$ algebra consists of the Cartan generator J_0 and the two odd elements of $\mathfrak{osp}(1|2)$ supplemented by the involution R.
- One has $C_{\mathfrak{osp}(1|2)} = Q^2$, where Q is given by (20.17). Thus the introduction of R allows to take the square-root of $C_{\mathfrak{osp}(1|2)}$.
- In $sl_{-1}(2)$, one has $[J_-, J_+] = 1 - 2QR$. On the module $V^{(\epsilon,\mu)}$, this leads to

  $$[J_-, J_+] = 1 + 2\epsilon\mu R.$$

20.7 Dunkl operators

The irreducible modules $V^{(\epsilon,\mu)}$ of $sl_{-1}(2)$ can be realized by Dunkl operators on the real line. Let R_x be the reflection operator

$$R_x f(x) = f(-x).$$

The $\mathbb{Z}_2$-Dunkl operator on $\mathbb{R}$ is defined by [8]

$$D_x = \frac{\partial}{\partial x} + \frac{\nu}{x}(1 - R_x),$$

where ν is a real number such that $\nu > -1/2$. Upon introducing the operators

$$\widehat{J}_{\pm} = \frac{1}{\sqrt{2}}(x \mp D_x),$$

and defining $\widehat{J}_0 = \frac{1}{2}\{\widehat{J}_-, \widehat{J}_+\}$, it is readily verified that a realization of the $sl_{-1}(2)$-module $V^{(\epsilon,\mu)}$ with $\epsilon = 1$ and $\mu = \nu$ is obtained. In particular, one has

$$[\widehat{J}_-, \widehat{J}_+] = 1 + 2\nu R_x.$$

It can be seen that $\widehat{J}_{\pm}^{\dagger} = \widehat{J}_{\mp}$ with respect to the measure $|x|^{2\nu}\,\mathrm{d}x$ on the real line [10].

20.8 The Racah problem for $sl_{-1}(2)$ and the Bannai-Ito algebra

The Racah problem for $sl_{-1}(2)$ presents itself when the direct product of three irreducible representations is examined. We consider the three-fold tensor product

$$V = V^{(\epsilon_1,\mu_1)} \otimes V^{(\epsilon_2,\mu_2)} \otimes V^{(\epsilon_3,\mu_3)}.$$

It follows from the coproduct formula (20.18) that the generators of $sl_{-1}(2)$ on V are of the form

$$J^{(4)} = J_0^{(1)} + J_0^{(2)} + J_0^{(3)}, \quad J_\pm^{(4)} = J_\pm^{(1)} R^{(2)} R^{(3)} + J_\pm^{(2)} R^{(3)} + J_\pm^{(3)}, \quad R^{(4)} = R^{(1)} R^{(2)} R^{(3)},$$

where the superscripts indicate on which module the generators act. In forming the module V, two sequences are possible: one can first combine (1) and (2) to bring (3) after or one can combine (2) and (3) before adding (1). This is represented by

$$\left(V^{(\epsilon_1,\mu_1)} \otimes V^{(\epsilon_2,\mu_2)}\right) \otimes V^{(\epsilon_3,\mu_3)} \quad \text{or} \quad V^{(\epsilon_1,\mu_1)} \otimes \left(V^{(\epsilon_2,\mu_2)} \otimes V^{(\epsilon_3,\mu_3)}\right). \tag{20.19}$$

These two addition schemes are equivalent and the two corresponding bases are unitarily related. In the following, three types of Casimir operators will be distinguished.

- The initial Casimir operators

$$Q_i = J_+^{(i)} J_-^{(i)} R^{(i)} - (J_0^{(i)} - 1/2) R^{(i)} = -\epsilon_i \mu_i, \qquad i = 1, 2, 3.$$

- The intermediate Casimir operators

$$\begin{aligned} Q_{ij} &= (J_+^{(i)} R^{(j)} + J_+^{(j)})(J_-^{(i)} R^{(j)} + J_-^{(j)}) R^{(i)} R^{(j)} - (J_0^{(i)} + J_0^{(j)} - 1/2) R^{(i)} R^{(j)} \\ &= (J_-^{(i)} J_+^{(j)} - J_+^{(i)} J_-^{(j)}) R^{(i)} - R^{(i)} R^{(j)}/2 + Q_i R^{(j)} + Q_j R^{(i)}, \end{aligned}$$

  where $(ij) = (12), (23)$.

- The total Casimir operator

$$Q_4 = [J_+^{(4)} J_-^{(4)} - (J_0^{(4)} - 1/2)] R^{(4)}.$$

Let $|\, q_{12}, q_4; m\,\rangle$ and $|\, q_{23}, q_4; m\,\rangle$ be the orthonormal bases associated to the two coupling schemes presented in (20.19). These two bases are defined by the relations

$$Q_{12} |\, q_{12}, q_4; m\,\rangle = q_{12} |\, q_{12}, q_4; m\,\rangle, \quad Q_{23} |\, q_{23}, q_4; m\,\rangle = q_{23} |\, q_{23}, q_4; m\,\rangle,$$

and

$$Q_4 |\, -, q_4; m\,\rangle = q_4 |\, -, q_4; m\,\rangle, \quad J_0^{(4)} |\, -, q_4; m\,\rangle = (m + \mu_1 + \mu_2 + \mu_3 + 3/2) |\, -, q_4; m\,\rangle.$$

The Racah problem consists in finding the overlap coefficients

$$\langle\, q_{23}, q_4 \,|\, q_{12}, q_4\,\rangle,$$

between the eigenbases of Q_{12} and Q_{23} with a fixed value q_4 of the total Casimir operator Q_4; as these coefficients do not depend on m, we drop this label. For simplicity, let us now take

$$\epsilon_1 = \epsilon_2 = \epsilon_3 = 1.$$

Upon defining

$$K_1 = -Q_{23}, \qquad K_3 = -Q_{12},$$

one finds that the intermediate Casimir operators of $sl_{-1}(2)$ realize the Bannai-Ito algebra [19]

$$\{K_1, K_3\} = K_2 + \Omega_2, \quad \{K_1, K_2\} = K_3 + \Omega_3, \quad \{K_2, K_3\} = K_1 + \Omega_1, \tag{20.20}$$

with structure constants

$$\Omega_1 = 2(\mu_1\mu + \mu_2\mu_3), \quad \Omega_2 = 2(\mu_1\mu_3 + \mu_2\mu), \quad \Omega_3 = 2(\mu_1\mu_2 + \mu_3\mu), \tag{20.21}$$

where $\mu = \epsilon_4\mu_4 = -q_4$. The first relation in (20.20) can be taken to define K_2 which reads

$$K_2 = (J_+^{(1)}J_-^{(3)} - J_-^{(1)}J_+^{(3)})R^{(1)}R^{(2)} + R^{(1)}R^{(3)}/2 - Q_1R^{(3)} - Q_3R^{(1)}.$$

In the present realization the Casimir operator of the Bannai-Ito algebra becomes

$$Q_{\text{BI}} = \mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2 - 1/4.$$

It has been shown in section 3 that the Bannai-Ito polynomials form a basis for a representation of the BI algebra. It is here relatively easy to construct the representation of the BI algebra on bases of the three-fold tensor product module V with basis vectors defined as eigenvectors of Q_{12} or of Q_{23}. The first step is to obtain the spectra of the intermediate Casimir operators. Simple considerations based on the nature of the $sl_{-1}(2)$ representation show that the eigenvalues q_{12} and q_{23} of Q_{12} and Q_{23} take the form [13, 15, 19, 30]:

$$q_{12} = (-1)^{s_{12}+1}(s_{12} + \mu_1 + \mu_2 + 1/2), \quad q_{23} = (-1)^{s_{23}}(s_{23} + \mu_2 + \mu_3 + 1/2),$$

where $s_{12}, s_{23} = 0, 1, \ldots, N$. The non-negative integer N is specified by

$$N + 1 = \mu_4 - \mu_1 - \mu_2 - \mu_3.$$

Denote the eigenstates of K_3 by $|\, k\,\rangle$ and those of K_1 by $|\, s\,\rangle$; one has

$$K_3|\, k\,\rangle = (-1)^k(k + \mu_1 + \mu_2 + 1/2)|\, k\,\rangle, \quad K_1|\, s\,\rangle = (-1)^s(s + \mu_2 + \mu_3 + 1/2)|\, s\,\rangle.$$

Given the expressions (20.21) for the structure constants Ω_k, one can proceed to determine the $(N+1) \times (N+1)$ matrices that verify the anticommutation relations (20.20). The action of K_1 on $|\, k\,\rangle$ is found to be [19]:

$$K_1|\, k\,\rangle = U_{k+1}|\, k+1\,\rangle + V_k|\, k\,\rangle + U_k|\, k-1\,\rangle,$$

with $V_k = \mu_2 + \mu_3 + 1/2 - B_k - D_k$ and $U_k = \sqrt{B_{k-1}D_k}$ where

$$B_k = \begin{cases} \frac{(k+2\mu_2+1)(k+\mu_1+\mu_2+\mu_3-\mu+1)}{2(k+\mu_1+\mu_2+1)} & k \text{ even}, \\ \frac{(k+2\mu_1+2\mu_2+1)(k+\mu_1+\mu_2+\mu_3+\mu+1)}{2(k+\mu_1+\mu_2+1)} & k \text{ odd}, \end{cases}$$

$$D_k = \begin{cases} -\frac{k(k+\mu_1+\mu_2-\mu_3-\mu)}{2(k+\mu_1+\mu)2)} & n \text{ even}, \\ -\frac{(k+2\mu_1)(k+\mu_1+\mu_2-\mu_3+\mu)}{2(k+\mu_1+\mu_2)} & n \text{ odd}. \end{cases}$$

Under the identifications

$$\rho_1 = \frac{1}{2}(\mu_2 + \mu_3), \quad \rho_2 = \frac{1}{2}(\mu_1 + \mu), \quad r_1 = \frac{1}{2}(\mu_3 - \mu_2), \quad r_2 = \frac{1}{2}(\mu - \mu_1),$$

one has $B_k = 2A_k, D_k = 2C_k$, where A_k and C_k are the recurrence coefficients (20.15) of the Bannai-Ito polynomials. Upon setting

$$\langle s \mid k \rangle = w(s)2^k B_k(x_s), \qquad B_0(x_s) \equiv 1,$$

one has on the one hand

$$\langle s \mid K_1 \mid k \rangle = (-1)^s(s + 2\rho_1 + 1/2)\langle s \mid k \rangle,$$

and on the other hand

$$\langle s \mid K_1 \mid k \rangle = U_{k+1}\langle s \mid k+1 \rangle + V_k\langle s \mid k \rangle + U_{k-1}\langle s \mid k-1 \rangle.$$

Comparing the two RHS yields

$$x_s B_k(x_s) = B_{k+1}(x_s) + (\rho_1 - A_k - C_k)B_k(x_s) + A_{k-1}C_k B_{k-1}(x_s),$$

where x_s are the points of the Bannai-Ito grid

$$x_s = (-1)^s\left(\frac{s}{2} + \rho_1 + 1/4\right) - 1/4, \quad s = 0, \dots, N.$$

Hence the Racah coefficients of $sl_{-1}(2)$ are proportional to the Bannai-Ito polynomials. The algebra (20.20) with structure constants (20.21) is invariant under the cyclic permutations of the pairs (K_i, μ_i). As a result, the representations in the basis where K_1 is diagonal can be obtained directly. In this basis, the operator K_3 is seen to be tridiagonal, which proves again that the Bannai-Ito polynomials possess the Leonard duality property.

20.9 A superintegrable model on S^2 with Bannai-Ito symmetry

We shall now use the analysis of the Racah problem for $sl_{-1}(2)$ and its realization in terms of Dunkl operators to obtain a superintegrable model on the two-sphere. Recall that a quantum system in n

dimensions with Hamiltonian H is maximally superintegrable it it possesses $2n-1$ algebraically independent constants of motion, where one of these constants is H [28]. Let $(s_1, s_2, s_3) \in \mathbb{R}$ and take $s_1^2 + s_2^2 + s_3^2 = 1$. The standard angular momentum operators are

$$L_1 = \frac{1}{i}\left(s_2 \frac{\partial}{\partial s_3} - s_3 \frac{\partial}{\partial s_2}\right), \quad L_2 = \frac{1}{i}\left(s_3 \frac{\partial}{\partial s_1} - s_1 \frac{\partial}{\partial s_3}\right), \quad L_3 = \frac{1}{i}\left(s_1 \frac{\partial}{\partial s_2} - s_2 \frac{\partial}{\partial s_1}\right).$$

The system governed by the Hamiltonian

$$H = L_1^2 + L_2^2 + L_3^2 + \frac{\mu_1}{s_1^2}(\mu_1 - R_1) + \frac{\mu_2}{s_2^2}(\mu_2 - R_2) + \frac{\mu_3}{s_3^2}(\mu_3 - R_3), \tag{20.22}$$

with μ_i, $i = 1,2,3$, real parameters such that $\mu_i > -1/2$ is superintegrable [18].

1. The operators R_i reflect the variable s_i: $R_i f(s_i) = f(-s_i)$.
2. The operators R_i commute with the Hamiltonian: $[H, R_i] = 0$.
3. If one is concerned with the presence of reflection operators in a Hamiltonian, one may replace R_i by $\kappa_i = \pm 1$. This then treats the 8 potential terms

   $$\frac{\mu_1}{s_1^2}(\mu_1 - \kappa_1) + \frac{\mu_2}{s_2^2}(\mu_2 - \kappa_2) + \frac{\mu_3}{s_3^2}(\mu_3 - \kappa_3),$$

   simultaneously much like supersymmetric partners.
4. Rescaling $s_i \to r s_i$ and taking the limit as $r \to \infty$ gives the Hamiltonian of the Dunkl oscillator [10, 11]

   $$\widetilde{H} = -[D_{x_1}^2 + D_{x_2}^2] + \widehat{\mu}_3^2 (x_1^2 + x_2^2),$$

   after appropriate renormalization; see also [12, 16, 20].

It can be checked that the following three quantities commute with the Hamiltonian (20.22) [13, 18]:

$$C_1 = \left(iL_1 + \mu_2 \frac{s_3}{s_2} R_2 - \mu_3 \frac{s_2}{s_3} R_3\right) R_2 + \mu_2 R_3 + \mu_3 R_2 + R_2 R_3/2,$$
$$C_2 = \left(-iL_2 + \mu_1 \frac{s_3}{s_1} R_1 - \mu_3 \frac{s_1}{s_3} R_3\right) R_1 R_2 + \mu_1 R_3 + \mu_3 R_1 + R_1 R_3/2,$$
$$C_3 = \left(iL_3 + \mu_1 \frac{s_2}{s_1} R_1 - \mu_2 \frac{s_1}{s_2} R_2\right) R_1 + \mu_1 R_2 + \mu_2 R_1 + R_1 R_2/2,$$

that is, $[H, C_i] = 0$ for $i = 1,2,3$. To determine the symmetry algebra generated by the above constants of motion, let us return to the Racah problem for $sl_{-1}(2)$. Consider the following (gauge transformed) parabosonic realization of $sl_{-1}(2)$ in the three variables s_i:

$$J_\pm^{(i)} = \frac{1}{\sqrt{2}}\left[s_i \mp \partial_{s_i} \pm \frac{\mu_i}{s_i} R_i\right], \quad J_0^{(i)} = \frac{1,}{2}\left[-\partial_{s_i}^2 + s_i^2 + \frac{\mu_i}{s_i^2}(\mu_i - R_i)\right], \quad R^{(i)} = R_i, \tag{20.23}$$

for $i=1,2,3$. Consider also the addition of these three realizations so that

$$J_0 = J_0^{(1)} + J_0^{(2)} + J_0^{(3)}, \quad J_\pm = J_\pm^{(1)} R^{(2)} R^{(3)} + J_\pm^{(2)} R^{(3)} + J_\pm^{(3)}, \quad R = R^{(1)} R^{(2)} R^{(3)}. \tag{20.24}$$

It is observed that in the realization (20.24), the total Casimir operator can be expressed in terms of the constants of motion as follows:

$$Q = -C_1 R^{(1)} - C_2 R^{(2)} - C_3 R^{(3)} + \mu_1 R^{(2)} R^{(3)} + \mu_2 R^{(1)} R^{(3)} + \mu_3 R^{(1)} R^{(2)} + R/2,$$

Upon taking

$$\Omega = QR,$$

one finds

$$\Omega^2 + \Omega = L_1^2 + L_2^2 + L_3^2 + (s_1^2 + s_2^2 + s_3^2)\left(\frac{\mu_1}{s_1^2}(\mu_1 - R_1) + \frac{\mu_2}{s_2^2}(\mu_2 - R_2) + \frac{\mu_3}{s_3^2}(\mu_3 - R_3)\right), \tag{20.25}$$

so that $H = \Omega^2 + \Omega$ if $s_1^2 + s_2^2 + s_3^2 = 1$. Assuming this constraint can be imposed, H is a quadratic combination of QR. By construction, the intermediate Casimir operators Q_{ij} commute with the total Casimir operator Q and with R and hence with Ω; they thus commute with $H = \Omega^2 + \Omega$ and are the constants of motion. It is indeed found that

$$Q_{12} = -C_3, \qquad Q_{23} = -C_1,$$

in the parabosonic realization (20.23). Let us return to the constraint $s_1^2 + s_2^2 + s_3^2 = 1$. Observe that

$$\frac{1}{2}(J_+ + J_-)^2 = (s_1 R_2 R_3 + s_2 R_3 + s_3)^2 = s_1^2 + s_2^2 + s_3^2.$$

Because $(J_+ + J_-)^2$ commutes with $\Omega = QR$, Q_{12} and Q_{23}, one can impose $s_1^2 + s_2^2 + s_3^2 = 1$. Since it is already known that the intermediate Casimir operators in the addition of three $sl_{-1}(2)$ representations satisfy the Bannai-Ito structure relations, the constants of motion verify

$$\{C_1, C_2\} = C_3 - 2\mu_3 Q + 2\mu_1\mu_2,$$
$$\{C_2, C_3\} = C_1 - 2\mu_1 Q + 2\mu_2\mu_3,$$
$$\{C_3, C_1\} = C_2 - 2\mu_2 Q + 2\mu_3\mu_1,$$

and thus the symmetry algebra of the superintegrable system with Hamiltonian (20.22) is a central extension (with Q begin the central operator) of the Bannai-Ito algebra. Let us note that the relation $H = \Omega^2 + \Omega$ relates to chiral supersymmetry since with $S = \Omega + 1/2$ one has

$$\frac{1}{2}\{S, S\} = H + 1/4.$$

20.10 A Dunkl-Dirac equation on S^2

Consider the $\mathbb{Z}_2$-Dunkl operators

$$D_i = \frac{\partial}{\partial x_i} + \frac{\mu_i}{x_i}(1-R_i), \qquad i = 1,2,\ldots,n,$$

with $\mu_i > -1/2$. The $\mathbb{Z}_2^n$-Dunkl-Laplace operator is

$$\vec{D}^2 = \sum_{i=1}^{n} D_i^2.$$

With γ_n the generators of the Euclidean Clifford algebra

$$\{\gamma_m, \gamma_n\} = 2\delta_{nm},$$

the Dunkl-Dirac operator is

$$\not{D} = \sum_{i=1}^{n} \gamma_i D_i.$$

Clearly, one has $\not{D}^2 = \vec{D}^2$. Let us consider the three-dimensional case. Introduce the Dunkl "angular momentum" operators

$$J_1 = \frac{1}{i}(x_2 D_3 - x_3 D_2), \quad J_2 = \frac{1}{i}(x_3 D_1 - x_1 D_3), \quad J_3 = \frac{1}{i}(x_1 D_2 - x_2 D_1).$$

Their commutation relations are found to be

$$[J_i, J_k] = i\epsilon_{jkl} J_l (1 + 2\mu_l R_l). \tag{20.26}$$

The Dunkl-Laplace equation separates in spherical coordinates; i.e. one can write

$$\vec{D}^2 = D_1^2 + D_2^2 + D_3^2 = \mathcal{M}_r + \frac{1}{r^2}\Delta_{S^2},$$

where Δ_{S^2} is the Dunkl-Laplacian on the 2-sphere. It can be verified that [20]

$$\begin{aligned}\vec{J}^2 &= J_1^2 + J_2^2 + J_3^2 \\ &= -\Delta_{S^2} + 2\mu_1\mu_2(1-R_1R_2) + 2\mu_2\mu_3(1-R_2R_3) + 2\mu_1\mu_3(1-R_1R_3) \\ &\qquad - \mu_1 R_1 - \mu_2 R_2 - \mu_3 R_3 + \mu_1 + \mu_2 + \mu_3.\end{aligned} \tag{20.27}$$

In three dimensions the Euclidean Clifford algebra is realized by the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which satisfy

$$\sigma_i \sigma_j = i\epsilon_{ijk}\sigma_k + \delta_{ij}.$$

Consider the following operator:

$$\Gamma = (\vec{\sigma} \cdot \vec{J}) + \vec{\mu} \cdot \vec{R},$$

with $\vec{\mu}\cdot\vec{R} = \mu_1 R_1 + \mu_2 R_2 + \mu_3 R_3$. Using the commutation relations (20.26) and the expression (20.27) for $\vec{J}^2$, it follows that

$$\Gamma^2 + \Gamma = -\Delta_{S^2} + (\mu_1 + \mu_2 + \mu_3)(\mu_1 + \mu_2 + \mu_3 + 1).$$

This is reminiscent of the expression (20.25) for the superintegrable system with Hamiltonian (20.22) in terms of the $sl_{-1}(2)$ Casimir operator. This justifies calling Γ a Dunkl-Dirac operator on S^2 since a quadratic expression in Γ gives Δ_{S^2}. The symmetries of Γ can be constructed. They are found to have the expression [7]

$$M_i = J_i + \sigma_i(\mu_j R_j + \mu_k R_k + 1/2), \quad (ijk) \text{ cyclic},$$

and one has $[\Gamma, M_i] = 0$. It is seen that the operators

$$X_i = \sigma_i R_i \qquad i = 1,2,3$$

also commute with Γ. Furthermore, one has

$$[M_i, X_i] = 0, \qquad \{M_i, X_j\} = \{M_i, X_k\} = 0.$$

Note that $Y = -iX_1X_2X_3 = R_1R_2R_3$ is central (like Γ). The commutation relations satisfied by the operators M_i are

$$[M_i, M_j] = i\epsilon_{ijk}\left(M_k + 2\mu_k(\Gamma+1)X_k\right) + 2\mu_i\mu_j[X_i, X_j].$$

This is again an extension of $\mathfrak{su}(2)$ with reflections and central elements. Let

$$K_i = M_i X_i Y = M_i \sigma_i R_j R_k.$$

It is readily verified that the operators K_i satisfy

$$\{K_1, K_2\} = K_3 + 2\mu_3(\Gamma+1)Y + 2\mu_1\mu_2,$$
$$\{K_2, K_3\} = K_1 + 2\mu_1(\Gamma+1)Y + 2\mu_2\mu_3,$$
$$\{K_3, K_1\} = K_2 + 2\mu_3(\Gamma+1)Y + 2\mu_3\mu_1,$$

showing that the Bannai-Ito algebra is a symmetry subalgebra of the Dunkl-Dirac equation on S^2. Therefore, the Bannai-Ito algebra is also a symmetry subalgebra of the Dunkl-Laplace equation.

20.11 Conclusion

In this paper, we have presented the Bannai-Ito algebra together with some of its applications. In concluding this overview, we identify some open questions.

1. Representation theory of the Bannai-Ito algebra

   Finite-dimensional representations of the Bannai-Ito algebra associated to certain models were presented. However, the complete characterization of all representations of the Bannai-Ito algebra is not known.

2. Supersymmetry

   The parallel with supersymmetry has been underscored at various points. One may wonder if there is a deeper connection.

3. Dimensional reduction

   It is well known that quantum superintegrable models can be obtained by dimensional reduction. It would be of interest to adapt this framework in the presence of reflections operators. Could the BI algebra can be interpreted as a W-algebra ?

4. Higher ranks

   Of great interest is the extension of the Bannai-Ito algebra to higher ranks, in particular for many-body applications. In this connection, it can be expected that the symmetry analysis of higher dimensional superintegrable models or Dunkl-Dirac equations will be revealing.

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Chapitre 21

The quantum superalgebra $\mathfrak{osp}_q(1|2)$ and a q-generalization of the Bannai–Ito polynomials

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Abstract. The Racah problem for the quantum superalgebra $\mathfrak{osp}_q(1|2)$ is considered. The intermediate Casimir operators are shown to realize a q-deformation of the Bannai–Ito algebra. The Racah coefficients of $\mathfrak{osp}_q(1|2)$ are calculated explicitly in terms of basic orthogonal polynomials that q-generalize the Bannai–Ito polynomials. The relation between these q-deformed Bannai–Ito polynomials and the q-Racah/Askey-Wilson polynomials is discussed.

21.1 Introduction

The goal of this paper is to examine the Racah problem for the quantum superalgebra $\mathfrak{osp}_q(1|2)$ and to present a q-extension of the Bannai–Ito polynomials.

The Bannai–Ito (BI) polynomials were first introduced by Bannai and Ito in their complete classification of the orthogonal polynomials possessing the Leonard duality property [1]. The BI polynomials, denoted by $B_n(x)$, depend on four real parameters ρ_1, ρ_2, r_1, r_2 and can be defined by the three-term recurrence relation

$$xB_n(x) = B_{n+1}(x) + (\rho_1 - A_n - C_n)B_n(x) + A_{n-1}C_n B_{n-1}(x), \tag{21.1}$$

with $B_{-1}(x)=0$, $B_0(x)=1$ and where the recurrence coefficients read

$$A_n = \begin{cases} \frac{(n+2\rho_1-2r_1+1)(n+2\rho_1-2r_2+1)}{4(n+\kappa+1/2)} & n \text{ even} \\ \frac{(n+2\kappa)(n+2\rho_1+2\rho_2+1)}{4(n+\kappa+1/2)} & n \text{ odd} \end{cases},$$
$$C_n = \begin{cases} -\frac{n(n-2r_1-2r_2)}{4(n+\kappa-1/2)} & n \text{ even} \\ -\frac{(n+2\rho_2-2r_1)(n+2\rho_2-2r_2)}{4(n+\kappa-1/2)} & n \text{ odd} \end{cases}, \tag{21.2}$$

with $\kappa = \rho_1+\rho_2-r_1-r_2+1/2$. The polynomials $B_n(x)$ can be obtained as $q\to -1$ limits of the q-Racah [1] or of the Askey-Wilson [22] polynomials, which sit at the top of the Askey scheme of hypergeometric orthogonal polynomials [14]. The Bannai–Ito polynomials are eigenfunctions of the most general self-adjoint first-order shift operator with reflections preserving the space of polynomials of a given degree [22]. Up to affine transformations, this operator has the expression

$$\mathscr{L} = D(x)(1-R)+E(x)(T^{+}R-1)+\kappa, \tag{21.3}$$

with $D(x)$ and $E(x)$ given by

$$D(x)=\frac{(x-\rho_1)(x-\rho_2)}{x}, \quad E(x)=\frac{(x-r_1+1/2)(x-r_2+1/2)}{x+1/2},$$

and where $T^{+}f(x)=f(x+1)$ is the shift operator and $Rf(x)=f(-x)$ is the reflection operator. The BI polynomials satisfy the eigenvalue equation [22]

$$\mathscr{L}B_n(x)=(-1)^n(n+\kappa)B_n(x), \quad n=0,1,2,\ldots \tag{21.4}$$

There is an algebraic structure associated to the BI polynomials which is called the Bannai–Ito algebra [22]. It is defined as the associative algebra over $\mathbb{C}$ with generators A_1, A_2, A_3 obeying the relations

$$\{A_3,A_1\}=A_2+\omega_2, \quad \{A_1,A_2\}=A_3+\omega_3, \quad \{A_2,A_3\}=A_1+\omega_1, \tag{21.5}$$

where $\{x,y\}=xy+yx$ is the anticommutator and where $\omega_1,\omega_2,\omega_3$ are complex structure constants. It is clear that in (21.5) only two of the generators are genuinely independent. The relation between the algebra (21.5) and the polynomials $B_n(x)$ is established by noting that the operators

$$A_1=\mathscr{L}, \quad A_2=2x+1/2,$$

realize the relations (21.5) with values of the structure constants depending on the parameters ρ_1, ρ_1, r_1, r_2. Hence the Bannai–Ito algebra (21.5) encodes, inter alia, the bispectral properties (21.1) and (21.4) of the Bannai–Ito polynomials. Let us mention that since its introduction, the BI algebra has appeared in several instances, notably in connection with generalizations of harmonic [11] and Clifford [3] analysis involving Dunkl operators, and also as a symmetry algebra of superintegrable systems [9]; see [2] for an overview.

It was recently determined that the Bannai–Ito polynomials serve as Racah coefficients in the direct product of three unitary irreducible representations (UIRs) of the algebra $sl_{-1}(2)$ [10]. This algebra, introduced in [21], is closely related to $\mathfrak{osp}(1|2)$ and its UIRs are associated to the one-dimensional para-Bose oscillator [19]. The identification of the Bannai–Ito polynomials as Racah coefficients in [10] followed from the observation that the intermediate Casimir operators entering the Racah problem for $sl_{-1}(2)$ realize the Bannai–Ito algebra.

In this paper we consider the quantum superalgebra $\mathfrak{osp}_q(1|2)$; the Racah coefficients arising in the tensor product of three of its UIRs are calculated explicitly in terms of basic orthogonal polynomials that tend to the Bannai–Ito polynomials in the limit $q \to 1$. We give some of the properties of these q-deformed Bannai–Ito polynomials and discuss their relationship with the q-Racah and Askey-Wilson polynomials. The paper is divided as follows.

In Section two, the definition of the $\mathfrak{osp}_q(1|2)$ algebra is recalled and its extension by the grade involution is defined. UIRs of this extended $\mathfrak{osp}_q(1|2)$ and their Bargmann realizations are presented. In Section 3, the coproduct for $\mathfrak{osp}_q(1|2)$ is used to posit the Racah problem and the intermediate Casimir operators are introduced. It is shown that these operators realize a q-analog of the BI algebra. The finite-dimensional irreducible representations of this q-version of the BI algebra are constructed. These lead to an explicit expression of the Racah coefficients of $\mathfrak{osp}_q(1|2)$ in terms of p-Racah polynomials with base $p = -q$ which tend to the BI polynomials in the $q \to 1$ limit. In the fourth section, the q-analogs of the BI polynomials are defined independently from the Racah problem and their bispectral properties (recurrence relation and eigenvalue equation) are given explicitly; their relation with the Askey-Wilson polynomials is also detailed. In Section 5, the $q \to 1$ limit of the results is discussed. We conclude with an outlook.

21.2 The quantum superalgebra $\mathfrak{osp}_q(1|2)$

In this section, the definition of the quantum superalgebra $\mathfrak{osp}_q(1|2)$ is recalled and its extension by the grade involution is presented. The Hopf structure of this extended $\mathfrak{osp}_q(1|2)$ is described. UIRs of this algebra are constructed and their Bargmann realizations are provided.

21.2.1 Definition and Casimir operator

Let q be a real number with $0 < q < 1$. The quantum superalgebra $\mathfrak{osp}_q(1|2)$ is the algebra presented in terms of one even generator A_0 and two odd generators $A_{\pm}$ obeying the commutation relations [16]

$$[A_0, A_{\pm}] = \pm A_{\pm}, \quad \{A_+, A_-\} = [2A_0]_{q^{1/2}}, \tag{21.6}$$

where $[x, y] = xy - yx$ is the commutator and where $[n]$ is the q-number

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

The abstract $\mathbb{Z}_2$-grading of the algebra (21.6) can be concretely realized by appending the grade involution P to the set of generators and declaring that the even and odd generators respectively commute and anticommute with P. The quantum superalgebra $\mathfrak{osp}_q(1|2)$ can hence be introduced as the algebra with generators A_0, $A_\pm$ and involution P satisfying the commutation relations

$$[A_0, P] = 0, \quad \{A_\pm, P\} = 0, \quad [A_0, A_\pm] = \pm A_\pm, \quad \{A_+, A_-\} = [2A_0]_{q^{1/2}}, \tag{21.7a}$$

with $P^2 = 1$. It is convenient to define the operators

$$K = q^{A_0/2}, \quad K^{-1} = q^{-A_0/2}.$$

In terms of these operators, the relations (21.7a) read

$$\begin{aligned} &KA_+K^{-1} = q^{1/2}A_+, \quad KA_-K^{-1} = q^{-1/2}A_-, \quad KK^{-1} = 1, \\ &[K, P] = 0, \quad [K^{-1}, P] = 0, \quad \{A_\pm, P\} = 0, \quad \{A_+, A_-\} = \frac{K^2 - K^{-2}}{q^{1/2} - q^{-1/2}}. \end{aligned} \tag{21.7b}$$

We shall use both (21.7a) and (21.7b). The Casimir operator of $\mathfrak{osp}_q(1|2)$ reads

$$Q = \left[A_+A_- - \frac{q^{-1/2}K^2 - q^{1/2}K^{-2}}{q - q^{-1}} \right] P. \tag{21.8}$$

It is easily verified that Q commutes with all generators of (21.7). In (21.8), the expression in the square bracket corresponds to the so-called sCasimir operator of $\mathfrak{osp}_q(1|2)$, which commutes with A_0 and anticommutes with $A_\pm$ [17].

21.2.2 Hopf algebraic structure

The algebra (21.7) can be endowed with a Hopf structure. Define the coproduct $\Delta : \mathfrak{osp}_q(1|2) \to \mathfrak{osp}_q(1|2) \otimes \mathfrak{osp}_q(1|2)$ as

$$\Delta(A_\pm) = A_\pm \otimes KP + K^{-1} \otimes A_\pm \quad \Delta(K) = K \otimes K, \quad \Delta(P) = P \otimes P, \tag{21.9}$$

the counit $\epsilon : \mathfrak{osp}_q(1|2) \to \mathbb{C}$ as

$$\epsilon(P) = 1, \quad \epsilon(K) = 1, \quad \epsilon(A_\pm) = 0, \tag{21.10}$$

and the coinverse $\sigma : \mathfrak{osp}_q(1|2) \to \mathfrak{osp}_q(1|2)$ by

$$\sigma(P) = P, \quad \sigma(K) = K^{-1}, \quad \sigma(A_\pm) = q^{\pm 1/2} P A_\pm. \tag{21.11}$$

It is straightforward to verify that with (21.9), (21.10) and (21.11), (21.7) indeed has a Hopf algebraic structure. The conditions on Δ, ϵ and σ are well known; they can be found, for example, in Chap. 4 of [23]. The coproduct given in (21.9) is not cocommutative since $\sigma\Delta \neq \Delta$, where $\sigma(a \otimes b) = b \otimes a$ is the flip automorphism. The alternative coproduct $\widetilde{\Delta} = \sigma\Delta$ and coinverse $\widetilde{S} = S^{-1}$ can be used to define another Hopf algebraic structure for (21.7); we shall not consider it here.

Remark 4. The coproduct (21.9) appears different from the one presented in [16], as it explicitly involves the grade involution P. The two coproducts are however equivalent. For elements in $\mathfrak{osp}_q(1|2)\otimes\mathfrak{osp}_q(1|2)$ a graded product law of the form $(a\otimes b)(c\otimes d) = (-1)^{p(b)}(-1)^{p(c)}(ac\otimes bd)$, where $p(x)$ gives the parity of x, was used in [16] whereas the standard product rule $(a\otimes b)(c\otimes d) = ac\otimes bd$ is used here.

21.2.3 Unitary irreducible $\mathfrak{osp}_q(1|2)$-modules

Let ϵ, μ be real numbers such that $\mu > 0$, $\epsilon = \pm 1$ and let $W^{(\epsilon,\mu)}$ denote the infinite-dimensional vector space spanned by the orthonormal basis vectors $|\,\epsilon,\mu;n\,\rangle$ where n is a non-negative integer. The basis vectors satisfy

$$\langle\,\epsilon,\mu;n'\,|\,\epsilon,\mu;n\,\rangle = \delta_{nn'},$$

where δ is the Kronecker delta. Consider the $\mathfrak{osp}_q(1|2)$ actions

$$\begin{aligned} &A_0\,|\,\epsilon,\mu;n\,\rangle = (n+\mu+1/2)\,|\,\epsilon,\mu;n\,\rangle, && P\,|\,\epsilon,\mu;n\,\rangle = \epsilon(-1)^n\,|\,\epsilon,\mu;n\,\rangle,\\ &A_+|\,\epsilon,\mu;n\,\rangle = \sqrt{\sigma_{n+1}}\,|\,\epsilon,\mu;n+1\,\rangle, && A_-\,|\,\epsilon,\mu;n\,\rangle = \sqrt{\sigma_n}\,|\,\epsilon,\mu;n-1\,\rangle, \end{aligned} \tag{21.12}$$

where σ_n is of the form

$$\sigma_n = [n+\mu]_q - (-1)^n[\mu]_q, \quad n = 0,1,2,\ldots.$$

The vector space $W^{(\epsilon,\mu)}$ endowed with the actions (21.12) forms a unitary irreducible $\mathfrak{osp}_q(1|2)$-module. Indeed, it is verified that the actions (21.12) comply with (21.7). The irreducibility follows from the fact that $\sigma_n > 0$ for $n \geqslant 1$. The module $W^{(\epsilon,\mu)}$ is unitary, as it is realizes the $\star$-conditions

$$A_0^\dagger = A_0, \quad P^\dagger = P, \quad A_\pm^\dagger = A_\mp. \tag{21.13}$$

The representation space $W^{(\epsilon,\mu)}$ can be identified with the state space of the one-dimensional q-deformed parabosonic oscillator [8]. On $W^{(\epsilon,\mu)}$, the Casimir operator (21.8) has the action

$$Q\,|\,\epsilon,\mu;n\,\rangle = -\epsilon\,[\mu]_q\,|\,\epsilon,\mu;n\,\rangle. \tag{21.14}$$

The modules $W^{(\epsilon,\mu)}$ have a Bargmann realization on functions of argument z. In this realization, the basis vectors $|\,\epsilon,\mu;n\,\rangle \equiv e_n^{(\epsilon,\mu)}(z)$ have the expression

$$e_n^{(\epsilon,\mu)}(z) = \frac{z^n}{\sqrt{\sigma_1\sigma_2\cdots\sigma_n}}, \qquad n = 0,1,2,\ldots,$$

and the $\mathfrak{osp}_q(1|2)$ generators take the form

$$\begin{aligned}
&A_0(z) = z\partial_z + \mu + 1/2, \quad K(z) = q^{(\mu+1/2)/2}T_q^{1/2},\\
&P(z) = \epsilon R_z, \qquad A_+(z) = z,\\
&A_-(z) = q^{\mu}\frac{(T_q - R_z)}{(q-q^{-1})z} - q^{-\mu}\frac{(T_q^{-1} - R_z)}{(q-q^{-1})z},
\end{aligned} \tag{21.15}$$

where $T_q^h f(z) = f(q^h z)$ and $R_z f(z) = f(-z)$.

21.3 The Racah problem

In this section, the Racah problem for $\mathfrak{osp}_q(1|2)$ is considered. The intermediate Casimir operators are defined and are seen to generate a q-analog of the Bannai–Ito algebra. The eigenvalues of the intermediate Casimirs are derived and the corresponding representations of the q-extended Bannai–Ito algebra are constructed. The explicit expression of the Racah coefficients for $\mathfrak{osp}_q(1|2)$ in terms of orthogonal polynomials is given.

21.3.1 Outline the problem

The coproduct of $\mathfrak{osp}_q(1|2)$ allows to construct tensor product representations. Consider the $\mathfrak{osp}_q(1|2)$-module defined by

$$W = W^{(\epsilon_1,\mu_1)} \otimes W^{(\epsilon_2,\mu_2)} \otimes W^{(\epsilon_3,\mu_3)}. \tag{21.16}$$

The action of any generator X on W is prescribed by $(1\otimes\Delta)\Delta(X)$ or equivalently by $(\Delta\otimes 1)\Delta(X)$ since the coproduct is coassociative. When considering three-fold tensor product representations, three types of Casimir operators arise. There are three *initial* Casimir operators $Q^{(1)}$, $Q^{(2)}$, $Q^{(3)}$ defined by

$$Q^{(1)} = Q\otimes 1\otimes 1, \quad Q^{(2)} = 1\otimes Q\otimes 1, \quad Q^{(3)} = 1\otimes 1\otimes Q, \tag{21.17}$$

which are associated to each components of the tensor product (21.16). On W, each initial Casimir operator $Q^{(i)}$ acts as a multiple of the identity. In view of (21.14), this multiple denoted by τ_i is given by

$$\tau_i = -\epsilon_i\,[\mu_i]_q, \qquad i = 1,2,3. \tag{21.18}$$

There are two *intermediate* Casimir operators $Q^{(12)}$, $Q^{(23)}$ defined by

$$Q^{(12)} = \Delta(Q) \otimes 1, \quad Q^{(23)} = 1 \otimes \Delta(Q), \tag{21.19}$$

which are associated to $W^{(\epsilon_1,\mu_1)} \otimes W^{(\epsilon_2,\mu_2)}$ and $W^{(\epsilon_2,\mu_2)} \otimes W^{(\epsilon_3,\mu_3)}$, respectively. A direct calculation using (21.8) and (21.9) shows that $\Delta(Q)$ has the expression

$$\Delta(Q) = q^{1/2}\left(A_- K^{-1} P \otimes A_+ K\right) - q^{-1/2}\left(A_+ K^{-1} P \otimes A_- K\right) \\ - [1/2]_q K^{-2} P \otimes K^2 P + Q \otimes K^2 P + K^{-2} P \otimes Q.$$

Finally, there is the *total* Casimir operator $\mathcal{Q}$ defined by

$$\mathcal{Q} = (1 \otimes \Delta)\Delta(Q) = (\Delta \otimes 1)\Delta(Q),$$

which is associated to the whole module W. The total Casimir operator reads

$$\mathcal{Q} = q^{1/2}\left(A_- K^{-1} P \otimes 1 \otimes A_+ K\right) - q^{-1/2}\left(A_+ K^{-1} P \otimes 1 \otimes A_- K\right) \\ - K^{-2} P \otimes Q \otimes K^2 P + \Delta(Q) \otimes K^2 P + K^{-2} P \otimes \Delta(Q). \tag{21.20}$$

The operators $Q^{(12)}$ and $Q^{(23)}$ both commute with $\mathcal{Q}$, but they do not commute with one another. Moreover, the operators $Q^{(12)}$, $Q^{(23)}$ and $\mathcal{Q}$ all commute by construction with the operator E which reads

$$E = (1 \otimes \Delta)\Delta(A_0) = A_0 \otimes 1 \otimes 1 + 1 \otimes A_0 \otimes 1 + 1 \otimes 1 \otimes A_0.$$

Each of $\{Q^{(12)}, \mathcal{Q}, E\}$ and $\{Q^{(23)}, \mathcal{Q}, E\}$ forms a complete set of self-adjoint commuting operators with respect to W.

We introduce two distinct bases for W associated to the two complete sets of commuting operators exhibited above. The first one consists of the orthonormal basis vectors $|\, m;\tau_{12};\tau\,\rangle$ defined by the eigenvalue equations

$$Q^{(12)}|\, m;\tau_{12};\tau\,\rangle = \tau_{12}|\, m;\tau_{12};\tau\,\rangle, \quad \mathcal{Q}|\, m;\tau_{12};\tau\,\rangle = \tau|\, m;\tau_{12};\tau\,\rangle, \\ E|\, m;\tau_{12};\tau\,\rangle = m|\, m;\tau_{12};\tau\,\rangle. \tag{21.21}$$

The second one consists of the orthonormal basis vectors $|\, m;\tau_{23};\tau\,\rangle$ defined by the eigenvalue equations

$$Q^{(23)}|\, m;\tau_{23};\tau\,\rangle = \tau_{23}|\, m;\tau_{23};\tau\,\rangle, \quad \mathcal{Q}|\, m;\tau_{23};\tau\,\rangle = \tau|\, m;\tau_{23};\tau\,\rangle, \\ E|\, m;\tau_{23};\tau\,\rangle = m|\, m;\tau_{23};\tau\,\rangle. \tag{21.22}$$

The Racah problem consist in the determination of the *Racah coefficients*, which are the transition coefficients between these two orthonormal bases. Such coefficients are easily shown to be independent of m [6]. We hence write

$$\begin{bmatrix} \tau_1 & \tau_2 & \tau_3 \\ \tau_{12} & \tau_{23} & \tau \end{bmatrix} = \langle\, m; \tau_{12}; \tau \mid m; \tau_{23}; \tau\,\rangle, \tag{21.23}$$

and refer to the left-hand side of (21.23) as the Racah coefficients for $\mathfrak{osp}_q(1|2)$. For more details on the Racah problem for $\mathfrak{sl}(2)$ and $\mathfrak{sl}_q(2)$, one can consult [13, 24].

21.3.2 Main observation: q-deformation of the Bannai–Ito algebra

The properties of the Racah coefficients are encoded in the algebraic interplay between the intermediate and total Casimir operators. A fruitful approach is therefore to investigate the commutation relations that these operators satisfy [10, 12]. Introduce the operators I_3 and I_1 defined as

$$I_1 = -Q^{(23)}, \qquad I_3 = -Q^{(12)}. \tag{21.24}$$

Let $\{A, B\}_q$ denote the "q-anticommutator"

$$\{A, B\}_q = q^{1/2} AB + q^{-1/2} BA,$$

and introduce the operator I_2 through the relation

$$\{I_3, I_1\}_q \equiv I_2 + (q^{1/2} + q^{-1/2}) \left[Q^{(3)} Q^{(1)} + Q^{(2)} \mathscr{Q} \right].$$

An involved but direct calculation shows that these operators satisfy the relations

$$\{I_i, I_j\}_q = I_k + (q^{1/2} + q^{-1/2}) \left[Q^{(i)} Q^{(j)} + Q^{(k)} \mathscr{Q} \right],$$

where (ijk) is an even permutation of $\{1,2,3\}$. It follows that the bases (21.21) and (21.22) that enter the Racah problem support representations of the algebra

$$\{I_i, I_j\}_q = I_k + \iota_k, \qquad \iota_k = (q^{1/2} + q^{-1/2})(\tau \tau_k + \tau_i \tau_j), \tag{21.25}$$

where (ijk) is an even permutation of $\{1,2,3\}$ and where τ_i is given by (21.18); the values of τ that can occur remain to be evaluated. Since I_3 and I_1 are proportional to $Q^{(12)}$ and $Q^{(23)}$, the Racah coefficients (21.23) coincide with the transition coefficients between the eigenbases of I_3 and I_1 in the appropriate representations of (21.25), which will be studied below. The operator

$$C = (q^{-1/2} - q^{3/2}) I_1 I_2 I_3 + q I_1^2 + q^{-1} I_2^2 + q I_3^2 \\ - (1-q)\iota_1 I_1 - (1-q^{-1})\iota_2 I_2 - (1-q)\iota_3 I_3, \tag{21.26}$$

can be seen to commute with I_1, I_2 and I_3. After considerable algebra, one finds that on the bases (21.21) and (21.22), the operator C takes the value

$$C = -(q - q^{-1})^2 \tau_1 \tau_2 \tau_3 \tau + \tau_1^2 + \tau_2^2 + \tau_3^2 + \tau^2 - q/(1+q)^2. \tag{21.27}$$

The algebra (21.25) stands as a q-deformation of the Bannai–Ito algebra with C as its Casimir operator.

Let us note that the algebra (21.25) can be presented in terms of only two generators. Eliminating I_2 from (21.25), one finds that I_1 and I_3 satisfy

$$I_1^2 I_3 + (q + q^{-1}) I_1 I_3 I_1 + I_3 I_1^2 = I_3 + (q^{1/2} + q^{-1/2}) \iota_2 I_1 + \iota_3, \tag{21.28a}$$

$$I_3^2 I_1 + (q + q^{-1}) I_3 I_1 I_3 + I_1 I_3^2 = I_1 + (q^{1/2} + q^{-1/2}) \iota_2 I_3 + \iota_1. \tag{21.28b}$$

Remark 5. The algebra (21.25) can be obtained from the Zhedanov algebra [26] by the formal substitution $q \to -q$ and scaling of the generators. The Zhedanov algebra was also studied by Koornwinder [15] and Terwilliger [20].

21.3.3 Spectra of the Casimir operators

To investigate the Racah problem, we need to identify which representations of (21.25) arise; this is done by determining the eigenvalues of the intermediate and total Casimir operators of $\mathfrak{osp}_q(1|2)$.

The eigenvalues of the intermediate Casimir operator $Q^{(12)}$, and hence those of I_3, are associated to the decomposition of the two-fold tensor product module $\widetilde{W} = W^{(\epsilon_1,\mu_1)} \otimes W^{(\epsilon_2,\mu_2)}$ in irreducible components. As a vector space, $\widetilde{W}$ has the direct sum decomposition

$$\widetilde{W} = \bigoplus_{n=0}^{\infty} U_n,$$

where each U_n is an eigenspace of $\Delta(A_0)$ with eigenvalue $n + \mu_1 + \mu_2 + 1$. It is seen that U_n is $(N+1)-$dimensional, as it is spanned by vectors $|\, \epsilon_1, \mu_1; n_1 \rangle \otimes |\, \epsilon_2, \mu_2; n_2 \rangle$ such that $n_1 + n_2 = n$. Since $\Delta(A_0)$ and $\Delta(Q)$ commute, U_n is stabilized by $\Delta(Q)$.

Lemma 6. *The eigenvalues of* $\Delta(Q)$ *on* U_n *have the expression*

$$\vartheta_k = (-1)^{k+1} \epsilon_1 \epsilon_2 [k + \mu_1 + \mu_2 + 1/2]_q, \quad k = 0, 1, \ldots, n. \tag{21.29}$$

Proof. By induction on n. The case $n = 0$ is verified directly by applying $\Delta(Q)$ on the single basis vector $|\, \epsilon_1, \mu_1; 0 \rangle \otimes |\, \epsilon_2, \mu_2; 0 \rangle$ of U_0. Suppose that (21.29) holds at level $n-1$ and let $v_k \in U_{n-1}$ for $k = 1, \ldots, n-1$ denote the eigenvectors of $\Delta(Q)$ with eigenvalues (21.29). It is directly seen from the relations (21.7) that the vectors $\Delta(A_+) v_k$ are in U_n and that they are eigenvectors of $\Delta(Q)$ with the same eigenvalues. Consider the vector $w \in U_n$ such that $\Delta(A_-) w = 0$; such a vector is easily constructed in the direct product basis by solving a two-term recurrence relation. It follows from (21.9) that w is an eigenvector of $\Delta(P)$ with eigenvalue $(-1)^n\epsilon_1\epsilon_2$. A calculation shows that w is an eigenvector of $\Delta(Q)$ with eigenvalue ϑ_n. Hence the eigenvalues of $\Delta(Q)$ on U_n are $\{\vartheta_0, \vartheta_1, \dots, \vartheta_{n-1}\} \cup \{\vartheta_n\}$. □

It follows from the above lemma that one has the direct sum decomposition

$$W^{(\epsilon_i,\mu_i)} \otimes W^{(\epsilon_j,\mu_j)} = \bigoplus_{k=0}^{\infty} W^{(\epsilon_{ij}(k),\,\mu_{ij}(k))}, \tag{21.30}$$

where

$$\epsilon_{ij}(k) = (-1)^k \epsilon_1\epsilon_2, \qquad \mu_{ij}(k) = k + \mu_1 + \mu_2 + 1/2. \tag{21.31}$$

Upon using the decomposition (21.30) twice, one finds that the decomposition of the $\mathfrak{osp}_q(1|2)$-module W in irreducible components has the form

$$W = \bigoplus_{N=0}^{\infty} m_N\, W^{(\epsilon_N,\mu_N)},$$

where the multiplicity is $m_N = N+1$ and where

$$\epsilon_N = (-1)^N \epsilon_1\epsilon_2\epsilon_3, \qquad \mu_N = N + \mu_1 + \mu_2 + \mu_3 + 1. \tag{21.32}$$

It follows from the above discussion that the eigenvalues τ of the total Casimir operator $\mathcal{Q}$ are parametrized by the non-negative integer N and read

$$\tau \to \tau_N = -\epsilon_N\, [\mu_N]_q, \qquad N = 0, 1, \dots \tag{21.33}$$

where ϵ_N and μ_N are given by (21.32). The eigenvalues τ_{12} and τ_{23} of the intermediate Casimir operators $Q^{(12)}$, $Q^{(23)}$ are respectively parametrized by the non-negative integers n, s and read

$$\begin{aligned} \tau_{12} &\to \tau_{12}(n) = -\epsilon_{12}(n)[\mu_{12}(n)]_q, \qquad n = 0, 1, \dots, N, \\ \tau_{23} &\to \tau_{23}(s) = -\epsilon_{23}(s)[\mu_{23}(s)]_q, \qquad s = 0, 1, \dots, N, \end{aligned} \tag{21.34}$$

where $\epsilon_{ij}(k)$ and $\mu_{ij}(k)$ are given by (21.31). We can thus write the Racah coefficients for $\mathfrak{osp}_q(1|2)$ as

$$\begin{bmatrix} \tau_1 & \tau_2 & \tau_3 \\ \tau_{12}(n) & \tau_{23}(s) & \tau_N \end{bmatrix}, \qquad n, s \in \{0, 1, \dots, N\}, \qquad N = 0, 1, 2, \dots. \tag{21.35}$$

These coefficients coincide with the interbasis expansion coefficients between the eigenbases of I_1 and I_3 in the $(N+1)$-dimensional representations of the algebra (21.25) with Casimir value (21.27).

21.3.4 Representations

We construct the matrix elements of I_1 in the eigenbasis of I_3. In view of (21.24), (21.31), (21.34), the eigenvectors of I_3 denoted by $|\,N;n\,\rangle$ satisfy

$$I_3|\,N;n\,\rangle = \lambda_n\,|\,N;n\,\rangle, \quad n = 0,1,\ldots,N, \tag{21.36}$$

where $\lambda_n = -\tau_{12}(n)$. The action of the operator I_1 on this basis can be written as

$$I_1|\,N;n\,\rangle = \sum_{k=0}^{N} A_{kn}|\,N;k\,\rangle, \tag{21.37}$$

where A_{kn} are the matrix elements of I_1. In view of (21.36), (21.37) and since the basis vectors are linearly independent, the relation (21.28b) is equivalent to

$$A_{kn}\left[\lambda_n^2 + (q+q^{-1})\lambda_n\lambda_k + \lambda_k^2 - 1\right] = \delta_{kn}[(q^{1/2}+q^{-1/2})\iota_2 + \iota_1]. \tag{21.38}$$

For $k \neq n$, the left-hand side of (21.38) must vanish. It is seen from the expression of the eigenvalues λ_n that A_{kn} can be non-zero only when $k = n \pm 1$ or $k = n$. As a result, the matrix representing I_1 in the I_3 eigenbasis is tridiagonal, i.e.

$$I_1|\,N;n\,\rangle = U_{n+1}|\,N;n+1\,\rangle + V_n|\,N;n\,\rangle + U_{n-1}|\,N;n-1\,\rangle, \tag{21.39}$$

where by definition $U_0 = 0$, $U_{N+1} = 0$ and where we have used the fact that I_1 is self-adjoint. When $n = k$, equation (21.38) gives the following expression for V_n:

$$V_n = \frac{\iota_1 + (q^{1/2}+q^{-1/2})\iota_2\lambda_n}{\lambda_n^2(2+q+q^{-1})-1}. \tag{21.40}$$

If one acts with the relation (21.28a) on $|\,N;n\,\rangle$ and gathers all terms proportional to $|\,N;n\,\rangle$, one finds that U_n^2 satisfies the two-term recurrence relation

$$\begin{aligned}(2\lambda_n + (q+q^{-1})\lambda_{n+1})U_{n+1}^2 + (2\lambda_n + (q+q^{-1})\lambda_n)V_n^2 \\ + (2\lambda_n + (q+q^{-1})\lambda_{n-1})U_n^2 = \lambda_n + (q^{1/2}+q^{-1/2})\iota_2 V_n + \iota_3.\end{aligned} \tag{21.41}$$

The solution to (21.41) can be presented as follows. Let a, b, c and d be defined as

$$\begin{aligned} a &= \epsilon_2\epsilon_3\, q^{\mu_2+\mu_3+1/2}, & b &= -\epsilon_1\epsilon_N\, q^{\mu_1-\mu_N+1/2}, \\ c &= -\epsilon_1\epsilon_N\, q^{\mu_1+\mu_N+1/2}, & d &= \epsilon_2\epsilon_3\, q^{\mu_2-\mu_3+1/2}, \end{aligned} \tag{21.42}$$

and let $p = -q$. Up to an inessential phase factor, one has

$$U_n = (q-q^{-1})^{-1}\sqrt{A_{n-1}C_n}, \tag{21.43}$$

where A_n and C_n read

$$
\begin{aligned}
A_n &= -\frac{(1+abp^n)(1-acp^n)(1-adp^n)(1-abcdp^{n-1})}{a(1-abcdp^{2n-1})(1-abcdp^{2n})},\\
C_n &= \frac{a(1-p^n)(1-bcp^{n-1})(1-bdp^{n-1})(1+cdp^{n-1})}{(1-abcdp^{2n-2})(1-abcdp^{2n-1})}.
\end{aligned}
\tag{21.44}
$$

The coefficients V_n given in (21.40) can be written as

$$
V_n = (q-q^{-1})^{-1}\left[a-a^{-1}-A_n-C_n\right]. \tag{21.45}
$$

With U_n and V_n as in (21.43) and (21.45), the actions (21.36) and (21.39) define $(N+1)$-dimensional representations of (21.28) with value (21.27) of the Casimir operator (21.26). Since $U_n \neq 0$ for $1 \leqslant n \leqslant N$, these representations are irreducible.

The matrix elements of the generators I_1, I_3 in the eigenbasis of I_1 are easily obtained. One observes that the relations (21.25) and Casimir value (21.27) are all invariant under simultaneous cyclic permutations of the generators I_i and representation parameters μ_i and ϵ_i. As a result, the matrix elements of I_1, I_3 in the eigenbasis $\{|\,N;s\,\rangle\}_{s=0}^{N}$ of I_1 are of the form

$$
\begin{aligned}
&I_1|\,N;s\,\rangle = \widetilde{\lambda}_s|\,N;s\,\rangle, \qquad s=0,1,\ldots,N,\\
&I_3|\,N;s\,\rangle = \widetilde{U}_{s+1}|\,N;s+1\,\rangle + \widetilde{V}_s|\,N;s\,\rangle + \widetilde{U}_s|\,N;s-1\,\rangle,
\end{aligned}
\tag{21.46}
$$

where $\widetilde{\lambda}_s$, $\widetilde{U}_s$ and $\widetilde{V}_s$ are obtained from (21.36), (21.43) and (21.45) by applying the permutations $(\mu_1,\mu_2,\mu_3)\to(\mu_2,\mu_3,\mu_1)$ and $(\epsilon_1,\epsilon_2,\epsilon_3)\to(\epsilon_2,\epsilon_3,\epsilon_1)$.

21.3.5 The Racah coefficients of $\mathfrak{osp}_q(1|2)$ as basic orthogonal polynomials

As explained in Subsection 3.3, the Racah coefficients (21.35) of $\mathfrak{osp}_q(1|2)$ coincide with the overlap coefficients $\langle\,N;s\,|\,N;n\,\rangle$. These coefficients can be cast in the form

$$
\langle\,N;s\,|\,N;n\,\rangle = \omega_s G_n(s), \quad \text{where} \quad \omega_s = \langle\,N;s\,|\,N;0\,\rangle \quad \text{and} \quad G_0(s)\equiv 1. \tag{21.47}
$$

Upon considering $\langle\,N;s\,|\,I_1\,|\,N;n\,\rangle$ together with (21.39) and (21.46), one finds that $G_n(s)$ satisfies the three-term recurrence relation

$$
\begin{aligned}
&(-1)^s(aq^s-a^{-1}q^{-s})G_n(s) =\\
&\qquad \sqrt{A_nC_{n+1}}G_{n+1}(s) + [a-a^{-1}-A_n-C_n]G_n(s) + \sqrt{A_{n-1}C_n}G_{n-1}(s),
\end{aligned}
\tag{21.48}
$$

where A_n, C_n are given by (21.44) with the parameterization (21.42).

If one takes

$$\widehat{G}_n(s) = (-1)^n a^n \sqrt{A_0 \cdots A_{n-1}\, C_1 \cdots C_n}\, G_n(s),$$

one finds that $\widehat{G}_n(s)$ satisfies the normalized recurrence relation

$$(p^{-s} + q^{2\mu_2+2\mu_3} p^s)\widehat{G}_n(s) = \widehat{G}_{n+1}(s) + [1 + q^{2\mu_2+2\mu_3}\, p - \check{A}_n - \check{C}_n]\widehat{G}_n(s) + \check{A}_{n-1}\check{C}_n\widehat{G}_{n-1}(s), \quad (21.49)$$

where $p = -q$ and where

$$\check{A}_n = -a\, A_n, \qquad \check{C}_n = -a\, C_n.$$

The recurrence relation (21.49) coincides with the normalized recurrence relation for the p-Racah polynomials $R_n(\mu(s);\alpha,\beta,\gamma,\delta\,|\,p)$ of degree n in the variable $\mu(s) = p^{-s} + \gamma\delta p^{s+1}$ [14]. In consequence, the functions $G_n(s)$ appearing in the coefficients (21.47) are proportional to the p-Racah polynomials

$$R_n(\mu(s);\alpha,\beta,\gamma,\delta\,|\,p) = {}_4\varphi_3\left(\begin{matrix} p^{-n}, \alpha\beta p^{n+1}, p^{-s}, \gamma\delta p^{s+1} \\ \alpha p, \beta\delta p, \gamma p \end{matrix}; p, p\right), \quad (21.50)$$

where ${}_r\varphi_s$ is the generalized basic hypergeometric series [14]

$${}_r\varphi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix}; q, z\right) = \sum_{k=0}^{\infty} \frac{(a_1,\cdots a_r;q)_k}{(b_1,\cdots,b_s;q)_k} (-1)^{(1+s-r)k} q^{(1+s-r)\binom{k}{2}} \frac{z^k}{(q;q)_k}.$$

and where have used the standard notation:

$$(a_1,a_2,\ldots,a_k;q)_s = \prod_{i=1}^{k}(a_i;q)_s, \qquad (a;q)_s = \prod_{k=1}^{s}(1 - q^{k-1}a).$$

Recall that $p = -q$. The relation between the parameters $\alpha,\beta,\gamma,\delta$ of the p-Racah polynomials and those appearing in (21.49) is

$$\begin{aligned} \alpha &= -(-1)^N q^{\mu_1+\mu_2+\mu_3-\mu_N}, & \gamma &= -q^{2\mu_2}, \\ \beta &= -(-1)^N q^{\mu_1+\mu_2-\mu_3+\mu_N}, & \delta &= -q^{2\mu_3}. \end{aligned} \quad (21.51)$$

Using the expression (21.32) for μ_N, it is seen that one has $\alpha\, p = p^{-N}$, which is one of the admissible truncation condition for the p-Racah polynomials.

The vectors $|\, N; n\,\rangle$ being orthonormal, one has the orthogonality relation

$$\sum_{s=0}^{N} \langle\, N; n' \,|\, N; s\,\rangle\langle\, N; s \,|\, N; n\,\rangle = \sum_{s=0}^{N} \omega_s^2 G_n(s) G_{n'}(s) = \delta_{nn'}. \quad (21.52)$$

Since the orthogonality weight for the p-Racah polynomials is unique, one concludes that ω_s in (21.47) is the square root of the p-Racah weight function with parameters (21.51). The weight function Ω_s of the p-Racah polynomials reads [14]

$$\Omega_s(\alpha,\beta,\gamma,\delta;p) = \frac{(\alpha p,\beta\delta p,\gamma p,\gamma\delta p;p)_s}{(p,\alpha^{-1}\gamma\delta p,\beta^{-1}\gamma p,\delta p;p)_s}\frac{1-\gamma\delta p^{2s+1}}{(\alpha\beta p)^s(1-\gamma\delta p)},$$

and the normalization coefficients h_n are

$$h_n(\alpha,\beta,\gamma,\delta;p) = \frac{(\beta^{-1},\gamma\delta p^2;p)_N}{(\beta^{-1}\gamma p,\delta p;p)_N}\frac{(1-\beta p^{-N})(\gamma\delta p)^n}{(1-\beta p^{2n-N})} \times \frac{(p,\beta p,\beta\gamma^{-1}p^{-N},\delta^{-1}p^{-N};p)_n}{(\beta p^{-N},\beta\delta p,\gamma p,p^{-N};p)_n}.$$

The complete and explicit expression for the Racah coefficients of $\mathfrak{osp}_q(1|2)$ arising in the tensor product of three irreducible modules $W^{(\epsilon_i,\mu_i)}$ is thus

$$\begin{bmatrix} \tau_1 & \tau_2 & \tau_3 \\ \tau_{12}(n) & \tau_{23}(s) & \tau_N \end{bmatrix} = (-1)^n\sqrt{\frac{\Omega_s(\alpha,\beta,\gamma,\delta;p)}{h_n(\alpha,\beta,\gamma,\delta;p)}}\,R_n(\mu(s);\alpha,\beta,\gamma,\delta;p),$$

with the parametrization (21.51) and $p=-q$. Let us remark that these Racah coefficients do not depend on the representation parameters $\epsilon_1,\epsilon_2,\epsilon_3$.

Remark 7. The Racah, or $6j$, coefficients of $\mathfrak{osp}_q(1|2)$ were also studied in [18]. The authors considered different representations than the ones considered here. They focused in particular on finite-dimensional representations. The connection with orthogonal polynomials and the algebraic structure (21.25) were not discussed.

Remark 8. The Clebsch-Gordan (CG) problem for $\mathfrak{osp}_q(1|2)$ arising in the tensor product of two irreducible representations was considered in [4] and basic orthogonal polynomials with base $p = -q$ were seen to arise as CG coefficients.

21.4 q-analogs of the Bannai–Ito polynomials and Askey-Wilson polynomials with base $p=-q$

In this section, the basic polynomials with basis $p=-q$ encountered above are presented independently from the Racah problem of $\mathfrak{osp}_q(1|2)$. Their relation with the Askey-Wilson polynomials with base $p=-q$ is discussed.

Consider the recurrence relation (21.48), the recurrence coefficients (21.44) and the parametrization (21.42). Defining $z = a\,p^s$, one is naturally led to introduce the polynomials

$Q_n(x;a,b,c,d;q) \equiv Q_n(x)$ defined by the recurrence relation

$$(z-z^{-1})Q_n(x) = A_n Q_{n+1}(x) + [a-a^{-1}-A_n-C_n]Q_n(x) + C_n Q_{n-1}(x), \tag{21.53}$$

where $x = z - z^{-1}$ and where the recurrence coefficients read

$$A_n = -\frac{(1+abp^n)(1-acp^n)(1-adp^n)(1-abcdp^{n-1})}{a(1-abcdp^{2n-1})(1-abcdp^{2n})},$$
$$C_n = \frac{a(1-p^n)(1-bcp^{n-1})(1-bdp^{n-1})(1+cdp^{n-1})}{(1-abcdp^{2n-2})(1-abcdp^{2n-1})},$$

with $p=-q$. Upon comparing the recurrence relation (21.53) with that of the Askey-Wilson polynomials [14]

$$p_n(y;\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}|q) = {}_4\varphi_3\left(\begin{matrix} q^{-n}, \mathfrak{abcd}q^{n-1}, \mathfrak{a}e^{i\theta}, \mathfrak{a}e^{-i\theta} \\ \mathfrak{ab}, \mathfrak{ac}, \mathfrak{ad} \end{matrix}; q, q\right), \quad y = \cos\theta,$$

it is seen that the polynomials $Q_n(y;a,b,c,d|q)$ can obtained from the Askey-Wilson polynomials by the formal substitutions

$$e^{i\theta} \to iz, \quad \mathfrak{a} \to ia, \quad \mathfrak{b} \to ib, \quad \mathfrak{c} \to -ic, \quad \mathfrak{d} \to -id, \quad q \to -q,$$

where i is the imaginary number. The polynomials $Q_n(x;a,b,c,d|q)$ of degree n in x thus have the hypergeometric expression

$$Q_n(x;a,b,c,d|q) = {}_4\varphi_3\left(\begin{matrix} p^{-n}, abcdp^{n-1}, -az, az^{-1} \\ -ab, ac, ad \end{matrix}; p, p\right), \tag{21.54}$$

where $x = z - z^{-1}$ and where $p = -q$.

The polynomials $Q_n(x;a,b,c,d|q)$ satisfy a difference equation. Introduce the involution $\mathscr{I}_z$ defined by the action

$$\mathscr{I}_z f(z) = f(z^{-1}),$$

and let $\mathscr{D}_z$ be the divided-difference operator

$$\mathscr{D}_z = B(z)(T_q\mathscr{I}_z - 1) + B(-z^{-1})(T_q^{-1}\mathscr{I}_z - 1), \tag{21.55}$$

where $B(z)$ reads

$$B(z) = \frac{(1+az)(1+bz)(1-cz)(1-dz)}{(1+z^2)(1-qz^2)}.$$

The operator (21.55) is very close to the Askey-Wilson operator, the main difference being the presence of the involution $\mathscr{I}_z$. A direct calculation using (21.54) shows that the polynomials $Q_n(x;a,b,c,d|q)$ satisfy the eigenvalue equation

$$\mathscr{D}_z Q_n(x;a,b,c,d|q) = \left[p^{-n}(1-p^n)(1-abcdp^{n-1})\right] Q_n(x;a,b,c,d|q).$$

The operator $\mathscr{D}_z$ can be embedded in a realization of the q-deformed Bannai–Ito algebra (21.28). If one takes

$$\begin{aligned}\mathscr{J}_1 &= \left(\frac{q^{1/2}}{(q-q^{-1})\sqrt{abcd}}\right)\mathscr{D}_z + \left(\frac{q^{1/2}(q-abcd)}{\sqrt{abcd}(q^2-1)}\right),\\ \mathscr{J}_2 &= \frac{z-z^{-1}}{q-q^{-1}},\end{aligned} \tag{21.56}$$

it can be verified that one has

$$\mathscr{J}_2^2\mathscr{J}_1 + (q+q^{-1})\mathscr{J}_2\mathscr{J}_1\mathscr{J}_2 + \mathscr{J}_1\mathscr{J}_2^2 = \mathscr{J}_1 + (q^{1/2}+q^{-1/2})\omega_3\mathscr{J}_2 + \omega_1,$$
$$\mathscr{J}_1^2\mathscr{J}_2 + (q+q^{-1})\mathscr{J}_1\mathscr{J}_2\mathscr{J}_1 + \mathscr{J}_2\mathscr{J}_1^2 = \mathscr{J}_2 + (q^{1/2}+q^{-1/2})\omega_3\mathscr{J}_1 + \omega_2,$$

where the structure constants read

$$\omega_1 = \frac{-q^{-1/2}(abcdq + abq^2 - acq^2 - bcq^2 - adq^2 - bdq^2 + cdq^2 + q^3)}{(1+q)(q-1)^2\sqrt{abcd}},$$
$$\omega_2 = \frac{(a^2bcdq + ab^2cdq - abc^2dq - abcd^2q - abcq^2 - abdq^2 + acdq^2 + bcdq^2)}{(1+q)(q-1)^2abcd},$$
$$\omega_3 = \frac{-abcq - abdq + acdq + bcdq + aq^2 + bq^2 - cq^2 - dq^2}{(1+q)(q-1)^2\sqrt{abcd}}.$$

In the realization (21.56), the Casimir operator (21.26) takes a definite value which is a complicated expression in the parameters a, b, c and d.

21.5 The $q \to 1$ limit

21.5.1 The $q \to 1$ limit of the Racah problem

Consider the defining relations (21.7a) of the $\mathfrak{osp}_q(1|2)$ algebra. In the $q \to 1$ limit, they take the form

$$[\widetilde{A}_0, \widetilde{P}] = 0, \quad \{\widetilde{A}_\pm, \widetilde{P}\} = 0, \quad [\widetilde{A}_0, \widetilde{A}_\pm] = \pm\widetilde{A}_\pm, \quad \{\widetilde{A}_+, \widetilde{A}_-\} = 2\widetilde{A}_0. \tag{21.57}$$

The relations (21.57) define the Lie superalgebra algebra $\mathfrak{osp}(1|2)$ extended by its grade involution, which is also referred to $sl_{-1}(2)$ [21]. In the same limit, the Casimir operator (21.8) reads

$$\widetilde{Q} = [\widetilde{A}_+\widetilde{A}_- - (\widetilde{A}_0 - 1/2)]\widetilde{P}, \tag{21.58}$$

where the expression between the square brackets corresponds to the sCasimir of $\mathfrak{osp}(1|2)$ [17]. The $q \to 1$ limit of the Hopf structure gives the coproduct

$$\begin{aligned}\Delta(\widetilde{A}_0) &= \widetilde{A}_0 \otimes 1 + 1 \otimes \widetilde{A}_0,\\ \Delta(\widetilde{A}_\pm) &= \widetilde{A}_\pm \otimes \widetilde{P} + 1 \otimes \widetilde{A}_\pm, \quad \Delta(\widetilde{P}) = \widetilde{P} \otimes \widetilde{P},\end{aligned} \tag{21.59}$$

as well as the counit and coinverse

$$\begin{aligned} \epsilon(\widetilde{P}) = 1, \quad \epsilon(\widetilde{A}_0) = 0, \quad \epsilon(\widetilde{A}_\pm) = 0, \\ \sigma(\widetilde{P}) = \widetilde{P}, \quad \sigma(\widetilde{A}_0) = -\widetilde{A}_0, \quad \sigma(\widetilde{A}_\pm) = \widetilde{P}\widetilde{A}_\pm, \end{aligned} \tag{21.60}$$

as found in [5]. The unitary $\mathfrak{osp}_q(1|2)$-modules $W^{(\epsilon,\mu)}$ also have a well-defined $q \to 1$ limit to unitary $\mathfrak{osp}(1|2)$-modules $V^{(\epsilon,\mu)}$. The actions (21.12) become

$$\begin{aligned} &\widetilde{A}_0 | \epsilon,\mu;n \rangle = (n+\mu+1/2) | \epsilon,\mu;n \rangle, && \widetilde{P} | \epsilon,\mu;n \rangle = \epsilon(-1)^n | \epsilon,\mu;n \rangle, \\ &\widetilde{A}_+ | \epsilon,\mu;n \rangle = \sqrt{\widetilde{\sigma}_{n+1}} | \epsilon,\mu;n+1 \rangle, && \widetilde{A}_- | \epsilon,\mu;n \rangle = \sqrt{\widetilde{\sigma}_n} | \epsilon,\mu;n-1 \rangle, \end{aligned} \tag{21.61}$$

where $\widetilde{\sigma}_n = n + \mu(1-(-1)^n)$. The modules $V^{(\epsilon,\mu)}$ associated to the actions (21.61) were the $\mathfrak{osp}(1|2)$-modules considered for the Racah problem in [10]. The representations $V^{(\epsilon,\mu)}$ also have Bargmann realization on functions of argument z defined by

$$\begin{aligned} \widetilde{A}_0(z) = z\partial_z + \mu + 1/2, \quad \widetilde{P}(z) = \epsilon R_z, \quad \widetilde{A}_+(z) = z, \\ \widetilde{A}_-(z) = \partial_z + \frac{\mu}{z}(1-R_z). \end{aligned} \tag{21.62}$$

It is seen that in this realization $\widetilde{A}_-(z)$ coincides with the one-dimensional Dunkl derivative [7]. The initial (21.17), intermediate (21.19) and total (21.20) all have well defined limits when $q \to 1$. In this limit, the operators $\widetilde{I}_1 = -\widetilde{Q}^{(23)}$ and $\widetilde{I}_3 = -\widetilde{Q}^{(12)}$ satisfy the Bannai–Ito algebra relations

$$\{\widetilde{I}_i, \widetilde{I}_j\} = \widetilde{I}_k + \omega_k, \quad \omega_k = 2(\mu_i\mu_j + \mu_k\mu), \tag{21.63}$$

where (ijk) is an even permutation of $\{1,2,3\}$. The Casimir (21.26) reduces to

$$\widetilde{C} = \widetilde{I}_1^2 + \widetilde{I}_2^2 + \widetilde{I}_3^2, \tag{21.64}$$

and takes the value

$$\widetilde{C} = \mu_1^2 + \mu_2^2 + \mu_3^2 + \mu^2 - 1/4. \tag{21.65}$$

These results are in accordance with those obtained in [10] and [11]. Adopting the same approach as the one used in this paper, one can obtain the spectra of the intermediate Casimir operators and construct the corresponding representations of (21.63) to find the three-term recurrence relation satisfied by the Racah coefficients of $\mathfrak{osp}(1|2)$ and identify it with that of the Bannai–Ito polynomials.

21.5.2 $q \to 1$ limit of the q-analogs of the Bannai–Ito polynomials

Consider the polynomials defined by the recurrence relation (21.53). Upon taking

$$a = q^{2\rho_1+1/2}, \quad b = -q^{-2r_2+1/2}, \quad c = -q^{2\rho_2+1/2}, \quad d = q^{-2r_1+1/2}, \quad z = q^x,$$

dividing (21.53) by $(q-q^{-1})$ and taking the $q\to 1$ limit, one finds that the the recurrence relation (21.53) becomes, in its normalized form,

$$x\widetilde{Q}_n(x)=\widetilde{Q}_{n+1}(x)+(2\rho_1+1/2-\widetilde{A}_n-\widetilde{C}_n)\widetilde{Q}_n(x)+\widetilde{A}_{n-1}\widetilde{C}_n\widetilde{Q}_{n-1}(x), \tag{21.66}$$

where the coefficients read

$$\widetilde{A}_n=\begin{cases}\frac{(n+2\rho_1-2r_1+1)(n+2\rho_1-2r_2+1)}{2(n+\rho_1+\rho_2-r_1-r_2+1)} & n \text{ even}\\ \frac{(n+2\rho_1+2\rho_2+1)(n+2\rho_1+2\rho_2-2r_1-2r_2+1)}{2(n+\rho_1+\rho_2-r_1-r_2+1)} & n \text{ odd}\end{cases},$$

$$\widetilde{C}_n=\begin{cases}-\frac{n(n-2r_1-2r_2)}{2(n+\rho_1+\rho_2-r_1-r_2)} & n \text{ even}\\ -\frac{(n+2\rho_2-2r_1)(n+2\rho_2-2r_2)}{2(n+\rho_1+\rho_2-r_1-r_2)} & n \text{ odd}\end{cases}.$$

Comparing with the recurrence relation (21.1) satisfied by the Bannai–Ito polynomials, one sees from (21.66) that $\widetilde{Q}_n(x)=2^nB_n(\frac{x-1/2}{2})$. In consequence, the polynomials $Q_n(x;a,b,c,d|q)$ defined by the recurrence relation (21.53) are q-analogs of the Bannai–Ito polynomials. Similarly, upon taking the limit when $q\to 1$ of the divided-difference operator (21.55) with parametrization (21.66), one finds

$$\lim_{q\to 1}\frac{\mathscr{D}_z}{q-q^{-1}}=\left(\frac{(x-2\rho_1-1/2)(x-2\rho_2-1/2)}{2x-1}\right)(T^-R-1)\\ -\left(\frac{(x-2r_1+1/2)(x-2r_2+1/2)}{2x+1}\right)\left(T^+R-1\right),$$

which corresponds to (21.3), up to an affine transformation and a change of variable.

21.6 Conclusion

In this paper, the Racah problem for the quantum superalgebra $\mathfrak{osp}_q(1|2)$ was considered and a family of basic orthogonal polynomials that generalize the Bannai–Ito polynomials was proposed. While these q-analogs of the Bannai–Ito polynomials are formally related to the Askey-Wilson polynomials, the two families of (truncated) polynomials exhibit different algebraic properties, the former arising in the Racah coefficients for the quantum superalgebra $\mathfrak{osp}_q(1|2)$ and the latter arising in the Racah coefficients for the quantum algebra $\mathfrak{sl}_q(2)$.

The results presented here and those of [21] suggest a connection between quantum superalgebras and quantum algebras when $q\to -q$; see also [25]. Also of interest is the investigation of the transformation $q\to -q$ and its consequences for other families of polynomials of the Askey scheme. We plan to report on this in the near future.

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Chapitre 22

The equitable presentation of $\mathfrak{osp}_q(1|2)$ and a q-analog of the Bannai–Ito algebra

V. X. Genest, L. Vinet et A. Zhedanov (2015). The equitable presentation of $\mathfrak{osp}_q(1|2)$ and a q-analog of the Bannai–Ito algebra. Soumis à *Letters in Mathematical Physics*.

Abstract. The equitable presentation of the quantum superalgebra $\mathfrak{osp}_q(1|2)$, in which all generators appear on an equal footing, is exhibited. It is observed that in their equitable presentations, the quantum algebras $\mathfrak{osp}_q(1|2)$ and $\mathfrak{sl}_q(2)$ are related to one another by the formal transformation $q \to -q$. A q-analog of the Bannai–Ito algebra is shown to arise as the covariance algebra of $\mathfrak{osp}_q(1|2)$.

22.1 Introduction

The purpose of this Letter is threefold: to display the equitable, or $\mathbb{Z}_3$-symmetric, presentation of the quantum superalgebra $\mathfrak{osp}_q(1|2)$, to show that the equitable presentations of $\mathfrak{osp}_q(1|2)$ and $sl_q(2)$ are related to one another by the formal transformation $q \to -q$, and to demonstrate that the covariance algebra of $\mathfrak{osp}_q(1|2)$ is a q-analog of the Bannai–Ito algebra.

Our considerations take root in the Racah problem for the $\mathfrak{su}(2)$ algebra, i.e. the coupling of three angular momenta. In this problem, the states are usually described in terms of the quantum numbers j_i associated to the individual angular momenta $\vec{J}_i$ with $i \in \{1,2,3\}$, the quantum number j associated to the total angular momentum $\vec{J} = \vec{J}_1 + \vec{J}_2 + \vec{J}_3$, the quantum number M associated to the projection of the total angular momentum $\vec{J}$ along one axis, and any one of the quantum numbers j_{12}, j_{23}, j_{31} associated to the intermediate angular momenta $\vec{J}_{ij} = \vec{J}_i + \vec{J}_j$ for $(ij) \in \{(12),(23),(31)\}$. These bases are related via Racah coefficients [3]. The main drawback of such bases is their involved behavior under particle permutations. To circumvent this problem, Chakrabarti [2], Lévy-Leblond and Lévy-Nahas [20] devised an "equitable" coupling scheme and showed that there is a "democratic" basis specified by the quantum numbers j_1, j_2, j_3, j and ζ, where ζ is the eigenvalue of the volume operator $\Delta = (\vec{J}_1 \times \vec{J}_2) \cdot \vec{J}_3$. The three angular momenta $\vec{J}_i$ enter symmetrically in this scheme and the states of the democratic basis have definite behaviors under particle permutations.

The Racah–Wilson algebra is the hidden algebraic structure behind the Racah problems of $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$ [11]. The concern for a democratic approach to these Racah problems leads to the equitable presentation of this algebra [9]. In this presentation, the defining relations of the Racah–Wilson algebra are $\mathbb{Z}_3$-symmetric and all the generators appear on an equal footing, whence the epithets "equitable" or "democratic". It was recently shown in [5] that the equitable generators of the Racah/Wilson algebra can also be realized as quadratic expressions in the equitable $\mathfrak{sl}(2)$ generators proposed in [14]. Note that the Racah–Wilson algebra also arises as symmetry algebra for superintegrable systems [16] and encodes the bispectrality of the Racah/Wilson polynomials [6].

The Racah problem can also be posited for the quantum algebra $\mathfrak{sl}_q(2)$. In this case, it is the Askey–Wilson algebra [26], also known as the Zhedanov algebra [17], that arises as the hidden algebraic structure [12]. This algebra encodes the bispectrality of the Askey-Wilson polynomials and, as shown in [13], arises as the covariance algebra for $\mathfrak{sl}_q(2)$. An equitable presentation of the (universal) Askey-Wilson algebra was offered by Terwilliger in [22] who also showed that it can be realized by quadratic combinations of the equitable generators of $\mathfrak{sl}_q(2)$. The equitable presentation of $\mathfrak{sl}_q(2)$ was itself studied in [15]. A democratic presentation for the quantum group $U_q(\mathfrak{g})$ associated with a symmetrizable Kac-Moody algebra $\mathfrak{g}$ was also proposed in [21].

In a recent paper [10], the Racah problem for the quantum superalgebra $\mathfrak{osp}_q(1|2)$ was considered. It was shown that in this case a q-analog of the Bannai–Ito algebra, an algebra proposed in [24], appears as the "hidden" algebraic structure. The algebra obtained in [10] exhibits a $\mathbb{Z}_3$ symmetry and is related to the Askey–Wilson algebra by the formal transformation $q \to -q$.

In this Letter, we display the equitable presentation of the quantum superalgebra $\mathfrak{osp}_q(1|2)$, determine its relation with the equitable presentation of $\mathfrak{sl}_q(2)$ and show that it can be used to realize the q-analog of the Bannai–Ito algebra defined in [10]. The results that we present here enrich the understanding of the quintessential quantum superalgebra $\mathfrak{osp}_q(1|2)$ and shed light on its relationship with other algebraic structures that have appeared recently. The contents of the Letter are as follows.

In Section 2, the definition of $\mathfrak{osp}_q(1|2)$ is reviewed and its extension by the grade involution is presented. A two-parameter family of $\mathfrak{osp}_q(1|2)$-modules is defined. The equitable presentation of $\mathfrak{osp}_q(1|2)$ is introduced and several expressions are given for the Casimir operator. The equitable presentation of $\mathfrak{sl}_q(2)$ is reviewed and compared with the one found for $\mathfrak{osp}_q(1|2)$. In section 3, the realization of the q-deformed Bannai–Ito algebra in terms of the equitable $\mathfrak{osp}_q(1|2)$ generators is presented. A short conclusion follows.

22.2 The $\mathfrak{osp}_q(1|2)$ algebra and its equitable presentation

In this section, we recall the definition of the $\mathfrak{osp}_q(1|2)$ algebra, present its extension by the grade involution, define a family of irreducible representations and display its equitable presentation. We review the equitable presentation of $sl_q(2)$ and compare it with that of $\mathfrak{osp}_q(1|2)$.

22.2.1 Definition of $\mathfrak{osp}_q(1|2)$, the grade involution, and representations

Let q be a complex number which is not a root of unity and let $[n]_q$ denote

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

The quantum superalgebra $\mathfrak{osp}_q(1|2)$ is the $\mathbb{Z}_2$-graded unital associative $\mathbb{C}$-algebra generated by the even element A_0 and the odd elements $A_\pm$ satisfying the relations [18]

$$[A_0, A_\pm] = \pm A_\pm, \qquad \{A_+, A_-\} = [2A_0]_{q^{1/2}},$$

where $[x,y] = xy - yx$ and $\{x,y\} = xy + yx$ respectively stand for the commutator and the anticommutator. The sCasimir operator of $\mathfrak{osp}_q(1|2)$ is defined as [19]

$$S = A_+A_- - [A_0 - 1/2]_q.$$

This operator is readily seen to obey the relations

$$\{S, A_\pm\} = 0, \qquad [S, A_0] = 0.$$

The abstract $\mathbb{Z}_2$-grading of $\mathfrak{osp}_q(1|2)$ can be concretized by adding the grade involution P to the set of generators and by declaring that the even and odd generators respectively commute and anticommute with P. The quantum superalgebra $\mathfrak{osp}_q(1|2)$ can thus be presented as the unital associative ℂ-algebra generated by the elements A_0, $A_\pm$ and the involution P obeying the relations

$$[A_0, A_\pm] = \pm A_\pm, \quad \{A_+, A_-\} = [2A_0]_{q^{1/2}}, \quad \{P, A_\pm\} = 0, \quad [P, A_0] = 0, \quad P^2 = 1. \tag{22.1a}$$

In (22.1a), the parity of the elements no longer needs to be specified. Upon introducing the generators

$$K = q^{A_0}, \qquad K^{-1} = q^{-A_0},$$

one can write the relations (22.1a) in the form

$$\begin{aligned} &KA_+K^{-1} = qA_+, \qquad KA_-K^{-1} = q^{-1}A_-, \qquad KK^{-1} = 1, \quad P^2 = 1, \\ &[K, P] = 0, \qquad [K^{-1}, P] = 0, \qquad \{A_\pm, P\} = 0, \qquad \{A_+, A_-\} = \frac{K - K^{-1}}{q^{1/2} - q^{-1/2}}. \end{aligned} \tag{22.1b}$$

It is directly verified that the Casimir operator

$$Q = \Big(A_+A_- - [A_0 - 1/2]_q\Big)P, \tag{22.2}$$

which is related to the sCasimir operator by $Q = SP$, commutes with all the generators in (22.1). The quantum superalgebra $\mathfrak{osp}_q(1|2)$ can be equipped with a Hopf algebraic structure [18]. Introduce the coproduct map $\Delta : \mathfrak{osp}_q(1|2) \to \mathfrak{osp}_q(1|2) \otimes \mathfrak{osp}_q(1|2)$ with

$$\begin{aligned} &\Delta(A_+) = A_+ \otimes KP + 1 \otimes A_+, \qquad \Delta(A_-) = A_- \otimes P + K^{-1} \otimes A_-, \\ &\Delta(K) = K \otimes K, \quad \Delta(P) = P \otimes P, \end{aligned} \tag{22.3}$$

the counit map $\epsilon : \mathfrak{osp}_q(1|2) \to \mathbb{C}$ with

$$\epsilon(P) = 1, \quad \epsilon(K) = 1, \quad \epsilon(A_\pm) = 0, \tag{22.4}$$

and the coinverse map $\sigma : \mathfrak{osp}_q(1|2) \to \mathfrak{osp}_q(1|2)$ with

$$\sigma(P) = P, \quad \sigma(K) = K^{-1}, \quad \sigma(A_+) = -A_+K^{-1}P, \quad \sigma(A_-) = -KA_-P. \tag{22.5}$$

It is a straightforward exercise to verify that with the coproduct Δ, the counit ϵ and the coinverse σ defined as in (22.3), (22.4) and (22.5), the algebra (22.1) complies with the well-known requirements for a Hopf algebra [25].

Remark 9. Let us note that in computing with the coproduct, one should use the standard product rule $(a \otimes b)(c \otimes d) = (ac \otimes bd)$, as opposed to the usual graded product rule used for superalgebras when the grade involution is not introduced.

Let us now bring a two-parameter family of irreducible representations of $\mathfrak{osp}_q(1|2)$. Let ν be a real parameter and let $e = \pm 1$. Moreover, let $W^{(e,\nu)}$ be the infinite-dimensional vector space spanned by the basis vectors $f_n^{(e,\nu)}$, where n is a non-negative integer. It is verified that the actions

$$\begin{aligned} K\, f_n^{(e,\nu)} &= q^{n+\nu+1/2}\, f_n^{(e,\nu)}, & P\, f_n^{(e,\nu)} &= e(-1)^n\, f_n^{(e,\nu)}, \\ A_+ f_n^{(e,\nu)} &= f_{n+1}^{(e,\nu)}, & A_- f_n^{(e,\nu)} &= \rho_n\, f_{n-1}^{(e,\nu)}, \end{aligned} \tag{22.6}$$

where

$$\rho_n = [n+\nu]_q - (-1)^n [\nu]_q,$$

define representations of $\mathfrak{osp}_q(1|2)$ on $W^{(e,\nu)}$. For generic values of ν, one has $\rho_n \neq 0$ for all $n \geqslant 1$. As a consequence, these representations are irreducible. On $W^{(e,\nu)}$, the Casimir operator (22.2) acts as a multiple of the identity

$$Q\, f_n^{(e,\nu)} = -e\,[\nu]_q\, f_n^{(e,\nu)},$$

as expected from Schur's lemma. Note that the representations $W^{(e,\nu)}$ are associated to the q-analog of the parabosonic oscillator [4].

Remark 10. It is possible to define finite-dimensional representations of (22.1). Indeed, if one takes $\nu = -(N+1)/2$, where N is a even integer, one can use the actions (22.6) to define $(N+1)$-dimensional irreducible representations of $\mathfrak{osp}_q(1|2)$.

The representations $W^{(e,\nu)}$ have a Bargmann realization on functions of the complex argument z. In this realization, the basis vectors $f_n^{(e,\nu)} \equiv f_n^{(e,\nu)}(z)$ read

$$f_n^{(e,\nu)}(z) = z^n, \qquad n = 0,1,2,\ldots,$$

and the generators have the expressions

$$\begin{aligned} &K(z) = q^{\nu+1/2}\, T_q, \quad P(z) = eR_z, \\ &A_+(z) = z, \quad A_-(z) = \frac{q^{\nu}}{q-q^{-1}}\left(\frac{T_q - R_z}{z}\right) - \frac{q^{-\nu}}{q-q^{-1}}\left(\frac{T_q^{-1} - R_z}{z}\right), \end{aligned} \tag{22.7}$$

where $R_z g(z) = g(-z)$ is the reflection operator and where $T_q^{\pm} g(z) = g(q^{\pm 1} z)$ is the q-shift operator.

22.2.2 The equitable presentation of $\mathfrak{osp}_q(1|2)$

Let X, $Y^{\pm}$, Z and ω_y be defined as

$$X = K^{-1}P - (1-q^{-1})A_+K^{-1}P, \quad Y^{\pm} = K^{\pm}P, \quad Z = K^{-1}P + (q^{1/2}+q^{-1/2})A_-P, \quad \omega_y = P. \tag{22.8}$$

Using the relations (22.1), one readily verifies that these operators satisfy

$$\frac{q^{1/2}XY+q^{-1/2}YX}{q^{1/2}+q^{-1/2}}=1, \qquad \frac{q^{1/2}YZ+q^{-1/2}ZY}{q^{1/2}+q^{-1/2}}=1, \qquad \frac{q^{1/2}ZX+q^{-1/2}XZ}{q^{1/2}+q^{-1/2}}=1. \tag{22.9}$$

In addition to $YY^{-1}=1$ and $\omega_y^2=1$, one has also the relations

$$X\omega_y+\omega_y X=2Y^{-1}\omega_y, \qquad Y\omega_y+\omega_y Y=2Y\omega_y, \qquad Z\omega_y+\omega_y Z=2Y^{-1}\omega_y. \tag{22.10}$$

We refer to the relations (22.9) and (22.10) as the *equitable* presentation of $\mathfrak{osp}_q(1|2)$ and to the generators X ,$Y^{\pm}$, Z and ω_y as the equitable generators. It is observed that in this presentation, the generators are more or less on an equal footing; some asymmetry occurs in the relations with the involution ω_y given in (22.10). The standard generators of $\mathfrak{osp}_q(1|2)$ can be expressed as follows in terms of the equitable generators:

$$A_+=\frac{1-XY}{1-q^{-1}}, \quad A_-=\frac{(Z-Y^{-1})\omega_y}{q^{1/2}+q^{-1/2}}, \quad K^{\pm}=Y^{\pm}\omega_y, \quad P=\omega_y. \tag{22.11}$$

In the equitable presentation, the "normalized" Casimir operator

$$\Upsilon=(q-q^{-1})Q, \tag{22.12}$$

can be written in several different ways. One has

$$\begin{aligned}
&\Upsilon=q^{1/2}X-q^{-1/2}Y+q^{1/2}Z-q^{1/2}XYZ, && \Upsilon=q^{1/2}Y-q^{-1/2}Z+q^{1/2}X-q^{1/2}YZX,\\
&\Upsilon=q^{1/2}Z-q^{-1/2}X+q^{1/2}Y-q^{1/2}ZXY, && \Upsilon=q^{1/2}Y-q^{-1/2}Z-q^{-1/2}X+q^{-1/2}ZYX,\\
&\Upsilon=q^{1/2}Z-q^{-1/2}X-q^{-1/2}Y+q^{-1/2}XZY, && \Upsilon=q^{1/2}X-q^{-1/2}Y-q^{-1/2}Z+q^{-1/2}YXZ.
\end{aligned}$$

With respect to the presentation (22.8), the coproduct (22.3) has the expression

$$\begin{aligned}
&\Delta(X)=X\otimes 1+Y^{-1}\otimes(X-1), && \Delta(Z)=Z\otimes 1+Y^{-1}\otimes(Z-1),\\
&\Delta(Y)=Y\otimes Y, && \Delta(\omega_y)=\omega_y\otimes\omega_y.
\end{aligned}$$

On the basis $f_n^{(e,\nu)}$, the equitable generators have the actions

$$\begin{aligned}
X f_n^{(e,\nu)}&=e(-1)^n q^{-(n+\nu+1/2)}\Big(f_n^{(e,\nu)}-(1-q^{-1})f_{n+1}^{(e,\nu)}\Big),\\
Y f_n^{(e,\nu)}&=e(-1)^n q^{n+\nu+1/2} f_n^{(e,\nu)},\\
Z f_n^{(e,\nu)}&=e(-1)^n\Big(q^{-(n+\nu+1/2)} f_n^{(e,\nu)}+(q^{1/2}+q^{-1/2})\rho_n f_{n-1}^{(e,\nu)}\Big).
\end{aligned} \tag{22.13}$$

22.2.3 The equitable presentation of $\mathfrak{sl}_q(2)$

Let us now establish the relation between the equitable presentations of $\mathfrak{osp}_q(1|2)$ and $\mathfrak{sl}_q(2)$. The quantum algebra $\mathfrak{sl}_q(2)$ is defined as the unital $\mathbb{C}$-algebra with generators $\kappa^{\pm}$, J_+, J_- and relations

$$\kappa\kappa^{-1}=\kappa^{-1}\kappa=1, \quad \kappa J_+\kappa^{-1}=qJ_+, \quad \kappa J_-\kappa^{-1}=q^{-1}J_-, \quad [J_+,J_-]=\frac{\kappa-\kappa^{-1}}{q^{1/2}-q^{-1/2}}. \tag{22.14}$$

The equitable generators x, $y^{\pm}$ and z of $\mathfrak{sl}_q(2)$ are given by [15]

$$x = \kappa^{-1} - (q^{1/2} - q^{-1/2})J_+\kappa^{-1}, \qquad y^{\pm} = \kappa^{\pm}, \qquad z = \kappa^{-1} + (1 - q^{-1})J_-, \tag{22.15}$$

and satisfy the relations

$$\frac{q^{1/2}xy - q^{-1/2}yx}{q^{1/2} - q^{-1/2}} = 1, \qquad \frac{q^{1/2}yz - q^{-1/2}zy}{q^{1/2} - q^{-1/2}} = 1, \qquad \frac{q^{1/2}zx - q^{-1/2}xz}{q^{1/2} - q^{-1/2}} = 1. \tag{22.16}$$

It is directly seen that the equitable presentation of $\mathfrak{sl}_q(2)$ given in (22.16) and the equitable presentation of $\mathfrak{osp}_q(1|2)$ given in (22.9) are related to one another by the formal transformation $q \to -q$. This formal relation can also be observed using the standard presentations (22.1) and (22.14). Indeed, upon defining the generators

$$\widetilde{\kappa} = KP, \qquad \widetilde{\kappa}^{-1} = K^{-1}P, \qquad \widetilde{J}_+ = \frac{1}{i}\left(\frac{1 - q^{-1}}{q^{1/2} + q^{-1/2}}\right)A_+, \qquad \widetilde{J}_- = \left(\frac{q^{1/2} + q^{-1/2}}{1 + q^{-1}}\right)A_-P,$$

one finds that they satisfy the relations

$$\widetilde{\kappa}\widetilde{\kappa}^{-1} = \widetilde{\kappa}^{-1}\widetilde{\kappa} = 1, \quad \widetilde{\kappa}\widetilde{J}_+\widetilde{\kappa}^{-1} = -q\widetilde{J}_+, \quad \widetilde{\kappa}\widetilde{J}_-\widetilde{\kappa}^{-1} = -q^{-1}\widetilde{J}_-, \quad [\widetilde{J}_+, \widetilde{J}_-] = \frac{\widetilde{\kappa} - \widetilde{\kappa}^{-1}}{i(q^{1/2} + q^{-1/2})},$$

which indeed corresponds to (22.14) with $q \to -q$.

Remark 11. Note that if one artificially introduces an involution ω_y with $\{J_{\pm}, \omega_y\} = 0$ and $[\kappa, \omega_y] = 0$, relations of the form (22.10) also appear in the equitable presentation of $\mathfrak{sl}_q(2)$.

22.3 A q-generalization of the Bannai–Ito algebra and the covariance algebra of $\mathfrak{osp}_q(1|2)$

In this section, the definitions of the Bannai–Ito algebra and that of its q-extension are reviewed. It is shown that the $\mathbb{Z}_3$-symmetric q-deformed Bannai–Ito algebra can be realized in terms of the equitable $\mathfrak{osp}_q(1|2)$ generators.

22.3.1 The Bannai–Ito algebra and its q-extension

The Bannai–Ito algebra first arose in [24] as the algebraic structure encoding the bispectral properties of the Bannai–Ito polynomials. It also appears as the hidden algebra behind the Racah problem for the Lie superalgebra $\mathfrak{osp}(1|2)$ [8] and as a symmetry algebra for superintegrable systems [1, 7]. The Bannai–Ito algebra is unital associative algebra over $\mathbb{C}$ with generators K_1, K_2, K_3 and relations

$$\{K_1, K_2\} = K_3 + \alpha_3, \quad \{K_2, K_3\} = K_1 + \alpha_1, \quad \{K_3, K_1\} = K_2 + \alpha_2, \tag{22.17}$$

where α_i, $i = 1,2,3$, are structure constants. This algebra admits the Casimir operator

$$L = K_1^2 + K_2^2 + K_3^2, \tag{22.18}$$

which commutes with every generator K_i, $i = 1,2,3$. In [10], a q-deformation of the algebra (22.17) was identified in the study of the Racah problem for $\mathfrak{osp}_q(1|2)$. This q-extension has generators I_1, I_2, I_3 and relations

$$\{I_1, I_2\}_q = I_3 + \iota_3, \quad \{I_2, I_3\}_q = I_1 + \iota_1, \quad \{I_3, I_1\}_q = I_2 + \iota_2, \tag{22.19}$$

where ι_1, ι_2, ι_3 are structure constants and where

$$\{A, B\}_q = q^{1/2}AB + q^{-1/2}BA, \tag{22.20}$$

is the q-anticommutator. The algebra (22.19) is formally related to the Zhedanov algebra by the transformation $q \to -q$ [10]. It has for Casimir operator

$$\Lambda = (q^{-1/2} - q^{3/2})I_1I_2I_3 + qI_1^2 + q^{-1}I_2^2 + qI_3^2 - (1-q)\iota_1 I_1 - (1-q^{-1})\iota_2 I_2 - (1-q)\iota_3 I_3, \tag{22.21}$$

which commutes with all generators I_i, $i = 1,2,3$. It is easily seen that in the limit $q \to 1$, the relations (22.19) and the expression (22.21) tend to the relations (22.17) and to the relation (22.18).

22.3.2 Covariance algebra of $\mathfrak{osp}_q(1|2)$

Let $a^{\pm}$, $b^{\pm}$, $c^{\pm}$ be complex parameters and consider the operators A, B, C defined as follows:

$$\begin{aligned} A &= aX - a^{-1}Y + \frac{bc^{-1}(XY - YX)}{q^{1/2} + q^{-1/2}}, \qquad B = bY - b^{-1}Z + \frac{ca^{-1}(YZ - ZY)}{q^{1/2} + q^{-1/2}}, \\ C &= cZ - c^{-1}X + \frac{ab^{-1}(ZX - XZ)}{q^{1/2} + q^{-1/2}}, \end{aligned} \tag{22.22}$$

where X, Y and Z are the equitable generators of $\mathfrak{osp}_q(1|2)$ defined in (22.8). A direct calculation shows that the elements A, B and C satisfy the relations

$$\begin{aligned} \frac{q^{1/2}AB + q^{-1/2}BA}{q - q^{-1}} &= C + \frac{(a - a^{-1})(b - b^{-1}) - (c - c^{-1})\Upsilon}{q^{1/2} - q^{-1/2}}, \\ \frac{q^{1/2}BC + q^{-1/2}CB}{q - q^{-1}} &= A + \frac{(b - b^{-1})(c - c^{-1}) - (a - a^{-1})\Upsilon}{q^{1/2} - q^{-1/2}}, \\ \frac{q^{1/2}CA + q^{-1/2}AC}{q - q^{-1}} &= B + \frac{(c - c^{-1})(a - a^{-1}) - (b - b^{-1})\Upsilon}{q^{1/2} - q^{-1/2}}, \end{aligned} \tag{22.23}$$

where Υ is the normalized Casimir operator (22.12) of $\mathfrak{osp}_q(1|2)$. The algebra (22.23) is easily seen to be equivalent to the q-deformed Bannai–Ito algebra (22.19). Indeed, upon taking

$$M_1 = \frac{A}{q - q^{-1}}, \quad M_2 = \frac{B}{q - q^{-1}}, \quad M_3 = \frac{C}{q - q^{-1}}, \tag{22.24}$$

one finds that the elements M_1, M_2, M_3 satisfy the defining relations (22.19) of the q-deformed Bannai–Ito algebra

$$\{M_1, M_2\}_q = M_3 + m_3, \quad \{M_2, M_3\}_q = M_1 + m_1, \quad \{M_3, M_1\}_q = M_2 + m_2, \tag{22.25}$$

where the structure constants m_1, m_2 and m_3 read

$$\begin{aligned}
m_1 &= (q^{1/2} + q^{-1/2})\left(\frac{(b-b^{-1})(c-c^{-1}) - (a-a^{-1})\Upsilon}{(q-q^{-1})^2}\right),\\
m_2 &= (q^{1/2} + q^{-1/2})\left(\frac{(c-c^{-1})(a-a^{-1}) - (b-b^{-1})\Upsilon}{(q-q^{-1})^2}\right),\\
m_3 &= (q^{1/2} + q^{-1/2})\left(\frac{(a-a^{-1})(b-b^{-1}) - (c-c^{-1})\Upsilon}{(q-q^{-1})^2}\right).
\end{aligned} \tag{22.26}$$

The presentation (22.25), (22.26) is clearly invariant with respect to the simultaneous cyclic permutation of the generators (M_1, M_2, M_3) and arbitrary parameters (a, b, c); it is thus $\mathbb{Z}_3$-symmetric. In this realization, the Casimir operator (22.21) of the q-deformed Bannai–Ito algebra takes the definite value

$$\begin{aligned}
\Lambda = &\left(\frac{(a-a^{-1})(b-b^{-1})(c-c^{-1})\Upsilon}{(q-q^{-1})^2}\right)\\
&+\left(\frac{a-a^{-1}}{q-q^{-1}}\right)^2 + \left(\frac{b-b^{-1}}{q-q^{-1}}\right)^2 + \left(\frac{c-c^{-1}}{q-q^{-1}}\right)^2 + \left(\frac{\Upsilon}{q-q^{-1}}\right)^2 - \frac{q}{(1+q)^2}.
\end{aligned} \tag{22.27}$$

In view of the above results, one can conclude that the q-deformed Bannai–Ito algebra serves as the covariance algebra for $\mathfrak{osp}_q(1|2)$.

Remark 12. Let us note that if one takes $a = q^{\alpha}$, $b = q^{\beta}$, $c = q^{\gamma}$ and $\Upsilon = -(q^{\delta} - q^{-\delta})$, the structure constants (22.26) and the Casimir value (22.27) are identical to the ones arising in the Racah problem for $\mathfrak{osp}_q(1|2)$ [10].

22.4 Conclusion

In this Letter, the equitable presentation of the quantum superalgebra $\mathfrak{osp}_q(1|2)$ was displayed. It was observed that $\mathfrak{osp}_q(1|2)$ and $\mathfrak{sl}_q(2)$ are related by the formal transformation $q \to -q$ and it was established that the q-deformed Bannai–Ito algebra arises as the covariance algebra of $\mathfrak{osp}_q(1|2)$.

Under the appropriate reparametrization, the q-deformed Bannai–Ito algebra (22.25) with structure constants (22.26) tends to the Bannai–Ito algebra in the $q \to 1$ limit. Similarly in the limit $q \to 1$ the quantum superalgebra $\mathfrak{osp}_q(1|2)$ defined in (22.1) tends to the Lie superalgebra $\mathfrak{osp}(1|2)$ extended by its grade involution, also known as $sl_{-1}(2)$ [23]. However one observes that the operators of the realization (22.8), (22.22), (22.24) do not have a well-defined $q \to 1$ limit. Consequently, one cannot conclude from the above results that the Bannai–Ito algebra is a covariance algebra of $\mathfrak{osp}(1|2)$. The interesting problem of realizing the Bannai–Ito algebra in terms of the Lie superalgebra $\mathfrak{osp}(1|2)$ thus remains.

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Partie IV

Problème de Racah et systèmes superintégrables

Introduction

Le problème de Racah est un problème classique en physique et en théorie de la représentation. Afin d'être concrets, considérons le cas de $\mathfrak{su}(2)$ [64]. Les représentations irréductibles V_j de $\mathfrak{su}(2)$ décrivent les particules élémentaires de spin $j \in \{0, 1/2, 1, \ldots\}$. Le produit tensoriel $V = V_{j_1} \otimes V_{j_2} \otimes V_{j_3}$ correspond au système physique formé de trois particules indépendantes de spins j_1, j_2 et j_3. Du point de vue de la théorie du moment angulaire, il existe deux bases naturelles pour décrire les états de ce système. Dans la première base, les états sont étiquetés par les nombres quantiques j_{12}, j et m, qui correspondent respectivement au moment angulaire du sous-système formé des particules 1 et 2, au moment angulaire total du système et à la valeur de sa projection sur un axe donné. Dans le seconde base, les états sont étiquetés par des nombres quantiques j_{23}, j et m, qui correspondent respectivement au moment angulaire du sous-système formé des particules 2 et 3, au moment angulaire du système complet et à la valeur de sa projection sur un axe donné. Le problème de Racah pour l'algèbre $\mathfrak{su}(2)$ consiste à déterminer les coefficients de transition entre ces deux bases.

De manière plus générale, le problème de Racah se présente lorsqu'on considère le produit tensoriel de trois représentations irréductibles d'une algèbre $\mathscr{A}$ donnée, dont la nature dépend du système physique que l'on souhaite considérer.

Dans cette partie de la thèse, on démontre l'équivalence entre le problème de Racah pour l'algèbre $\mathfrak{su}(1,1)$ et le système superintégrable générique sur la 2-sphère. La découverte inattendue de cette équivalence permet entre autres d'expliquer la nature des symétries de ce système superintégrable important. En outre, on montre l'équivalence entre le problème de Racah pour la superalgèbre de Lie $\mathfrak{osp}(1|2)$ et le système superintégrable générique avec réflexions sur la 2-sphère.

Chapitre 23

Superintegrability in two dimensions and the Racah-Wilson algebra

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Abstract. The analysis of the most general second-order superintegrable system in two dimensions: the generic 3-parameter model on the 2-sphere, is cast in the framework of the Racah problem for the $\mathfrak{su}(1,1)$ algebra. The Hamiltonian of the 3-parameter system and the generators of its quadratic symmetry algebra are seen to correspond to the total and intermediate Casimir operators of the combination of three $\mathfrak{su}(1,1)$ algebras, respectively. The construction makes explicit the isomorphism between the Racah-Wilson algebra, which is the fundamental algebraic structure behind the Racah problem for $\mathfrak{su}(1,1)$, and the invariance algebra of the generic 3-parameter system. It also provides an explanation for the occurrence of the Racah polynomials as overlap coefficients in this context. The irreducible representations of the Racah-Wilson algebra are reviewed as well as their connection with the Askey scheme of classical orthogonal polynomials.

23.1 Introduction

The purpose of this paper is to stress the algebraic equivalence between the analysis of the generic quantum 3-parameter superintegrable system on the 2-sphere and the Racah problem of $\mathfrak{su}(1,1)$. The equivalence will be made explicit by the direct identification of the natural operators of the Racah problem, i.e. the intermediate Casimir operators for the combination of three $\mathfrak{su}(1,1)$ algebras, with the symmetry operators that span the invariance algebra of the system. From this identification will follow the explicit isomorphism between the symmetry algebra of the 3-parameter model and the Racah-Wilson algebra, which is the algebra behind the Racah problem for $\mathfrak{su}(1,1)$. Since the Racah-Wilson algebra is also the algebraic structure that encodes the properties of the Racah and Wilson polynomials, the isomorphism will also provide an elegant explanation for the occurrence of these families of polynomials as overlap coefficients in the 3-parameter system on the 2-sphere.

23.1.1 The Racah-Wilson algebra

The Racah-Wilson algebra is the infinite-dimensional associative algebra generated by the algebraically independent elements K_1, K_2 that satisfy, together with their commutator $K_3 \equiv [K_1, K_2]$, the following commutation relations:

$$
\begin{aligned}
&[K_2, K_3] = a_2 K_2^2 + a_1\{K_1, K_2\} + c_1 K_1 + d K_2 + e_1, \\
&[K_3, K_1] = a_1 K_1^2 + a_2\{K_1, K_2\} + c_2 K_2 + d K_1 + e_2,
\end{aligned}
\tag{23.1}
$$

where the structure constants are assumed to be real (it will be seen that the number of structure constants can be reduced from seven to three by affine transformations of the generators). The algebra (23.1) first appeared in the coupling problem of three angular momenta, i.e. the Racah problem for $\mathfrak{su}(2)$ [8]. It was first observed in [8] that the intermediate Casimir operators in the combination of three angular momenta realize the algebra (23.1) and this observation was exploited to derive the symmetry group of the $6j$-symbol (Racah coefficients). In the same paper, a special case of (23.1) corresponding to $a_1 = 0$ was also seen to occur in the Clebsch-Gordan problem of $\mathfrak{su}(2)$ and the algebraic relations were used to find the symmetries of the $3j$ coefficients. The representations of the algebra (23.1) were presented to some extent in [9] and the link between these representations and the Askey scheme of classical orthogonal polynomials was established. More specifically, it was shown that for certain finite(infinite)-dimensional irreducible representations of (23.1), the Racah (Wilson) polynomials occur as overlap coefficients between the eigenbases of K_1 and K_2. Furthermore, a suitable generalization of (23.1) whose representations encompass the full Askey scheme of basic or q-orthogonal polynomials was also proposed; this generalization is now referred to as the Askey-Wilson algebra [32]. A number of papers followed in which special cases of the algebra (23.1) were seen to occur as symmetry algebras of classical and quantum second-order superintegrable systems in two dimensions [4, 10, 11], some of them based on the combination of two $\mathfrak{su}(1,1)$ algebras [33], i.e. the Clebsch-Gordan problem for $\mathfrak{su}(1,1)$. Quite interestingly, the full Racah algebra was not encountered then. It was only later, with the complete classification of second-order superintegrable systems, that a significant step in this direction was made in the study of the generic 3-parameter system on the 2-sphere [20]. Before discussing this particular model and the results of [20], let us first recall a few points about quantum superintegrability.

23.1.2 Superintegrability

A quantum system described by a Hamiltonian H with d degrees of freedom is *maximally superintegrable* if it admits $2d-1$ algebraically independent symmetry operators S_i satisfying

$$[H, S_i] = 0, \quad 1 \leqslant i \leqslant 2d-1,$$

where one the symmetries is the Hamiltonian itself, e.g. $S_1 \equiv H$. For such a system, it is impossible for all the S_i to commute with one another and hence the symmetries generate a non-Abelian *invariance algebra* for H. In practice, the associated Schrödinger equation

$$H\Psi = E\psi,$$

can be exactly solved both analytically and algebraically. A superintegrable system is said to be order ℓ if ℓ is the maximum order of the symmetries (excluding H) in the momentum variables. First order superintegrable systems ($\ell = 1$) have geometrical symmetries and their invariance algebras are Lie algebras [26]. Second order superintegrable systems ($\ell = 2$) typically admit separation of variables in more than one coordinate system and have quadratic symmetry algebras [12, 13, 25, 27]. In two dimensions, all first and second order superintegrable systems are known and have been classified [2, 3, 18, 19, 31]. The study of superintegrable systems is important in its own right in view of their numerous applications. They also constitute a bedrock for the analysis of symmetries.

23.1.3 The generic 3-parameter system on the 2-sphere

In two dimensions, one of the most important second-order superintegrable systems is the generic 3-parameter model on the 2-sphere [22]. This model is described by the Hamiltonian

$$\mathscr{H} = J_1^2 + J_2^2 + J_3^2 + \frac{k_1^2 - \frac{1}{4}}{x_1^2} + \frac{k_2^2 - \frac{1}{4}}{x_2^2} + \frac{k_3^2 - \frac{1}{4}}{x_3^2}, \tag{23.2}$$

with the constraint $x_1^2 + x_2^2 + x_3^2 = 1$. The operators J_i, $i = 1, 2, 3$, stand for the standard angular momentum generators

$$J_1 = -i(x_2\partial_{x_3} - x_3\partial_{x_2}), \quad J_2 = -i(x_3\partial_{x_1} - x_1\partial_{x_3}), \quad J_3 = -i(x_1\partial_{x_2} - x_2\partial_{x_1}), \tag{23.3}$$

which satisfy the familiar $\mathfrak{so}(3)$ commutation relations

$$[J_i, J_j] = i\sum_{k=1}^{3} \epsilon_{ijk} J_k, \tag{23.4}$$

where ϵ_{ijk} is the totally antisymmetric tensor and ∂_{x_i} denotes differentiation with respect to the variable x_i. It is known (see [22] and references therein) that all second-order superintegrable models in two dimensions are limiting cases of (23.2); hence the generic 3-parameter model on the 2-sphere described by the Hamiltonian (23.2) can be considered as the most general system of such type. The symmetries of $\mathscr{H}$ and the quadratic invariance algebra they span can be found in [17, 19]. Upon taking

$$L_1 = J_1^2 + \frac{a_2 x_3^2}{x_2^2} + \frac{a_3 x_2^2}{x_3^2}, \quad L_2 = J_2^2 + \frac{a_3 x_1^2}{x_3^2} + \frac{a_1 x_3^2}{x_1^2}, \quad L_3 = J_3^2 + \frac{a_1 x_2^2}{x_1^2} + \frac{a_2 x_1^2}{x_2^2}, \tag{23.5}$$

where $a_1 = k_1^2 - 1/4$, $a_2 = k_2^2 - 1/4$ and $a_3 = k_3^2 - 1/4$, it is directly checked that $[\mathscr{H}, L_i] = 0$, i.e. that the operators L_i are constants of motion. Furthermore, upon defining $R = [L_1, L_2]$, the following commutation relations hold:

$$[L_i, R] = 4\{L_i, L_j\} - 4\{L_i, L_k\} - (8 - 16a_j)L_j + (8 - 16a_k)L_k + 8(a_j - a_k), \tag{23.6}$$

where $\{x, y\} = xy + yx$ stands for the anticommutator. The four symmetry operators L_1, L_2, L_3, R and the Hamiltonian $\mathscr{H}$ are not algebraically independent from one another. Indeed, one has

$$\mathscr{H} = L_1 + L_2 + L_3 + a_1 + a_2 + a_3, \tag{23.7}$$

which relates the sum of the three symmetries (23.5) to the Hamiltonian. Moreover, the square of the symmetry operator R can be shown to satisfy [19]

$$\begin{aligned} R^2 = &-\frac{8}{3}\{L_1, L_2, L_3\} - \sum_{i=1}^{3}\left\{(12 - 16a_i)L_i^2 + \frac{1}{3}(16 - 176a_i)L_i + \frac{32}{3}a_i\right\} \\ &+ \frac{52}{3}(\{L_1, L_2\} + \{L_2, L_3\} + \{L_1, L_3\}) + 48(a_1a_2 + a_2a_3 + a_3a_1) - 64a_1a_2a_3, \end{aligned} \tag{23.8}$$

where $\{x_1, x_2, x_3\}$ is the symmetrized sum of six terms of the form $x_i x_j x_k$. In view of the relations (23.7) and (23.8), it is clear that the system described by the Hamiltonian (23.2) possesses three algebraically independent symmetries: $\mathscr{H}$, L_1, L_2, and is hence maximally superintegrable. Moreover, since L_1, L_2 are second order differential operators, the generic 3-parameter system of the sphere is superintegrable of order $\ell = 2$. The solutions of the Schrödinger equation associated to $\mathscr{H}$ have been obtained in [17] by separation of variables in various coordinate systems. In spherical coordinates, these solutions are given in terms of the classical Jacobi polynomials.

23.1.4 The 3-parameter system and Racah polynomials

In a remarkable paper [20], the finite(infinite)-dimensional irreducible representations of the symmetry algebra (23.6), (23.8) have been related to the Racah (Wilson) polynomials. More specifically, it was shown that this symmetry algebra can be realized in terms of difference operators of which the Racah (Wilson) polynomials are eigenfunctions. Given the complexity of the algebraic relations (23.6), (23.8), the result constitutes a *tour de force*. It also implies that the Racah polynomials act as overlap coefficients between the eigenbases of L_1 and L_2 in a given energy eigenspace of the Hamiltonian (23.2). Moreover it allows, through contractions [22], to tie the entire Askey scheme of classical orthogonal polynomials to physical systems.

The occurrence of the Racah (Wilson) polynomials in the representations of the symmetry algebra of the 3-parameter model strongly suggests that the symmetry algebra (23.6), (23.8) can be explicitly written in the form (23.1). More importantly, it indicates the existence of a connection between the 3-parameter system and the Racah problem of either $\mathfrak{su}(1,1)$ or $\mathfrak{su}(2)$, for which the Racah-Wilson algebra is the underlying algebraic structure. As suggested by the work of Kuznetsov [24] and by [33], the connection will be made through the $\mathfrak{su}(1,1)$ algebra. It will be shown explicitly that the analysis of the generic 3-parameter system on the 2-sphere is equivalent to the Racah problem for $\mathfrak{su}(1,1)$. Using a realization of $\mathfrak{su}(1,1)$ in terms of differential operators, the 3-parameter Hamiltonian (23.2) will be seen to correspond to the total Casimir operator of the combination of three $\mathfrak{su}(1,1)$ algebras and the symmetries L_1, L_2, L_3 will be identified with the intermediate Casimir operators. From this identification will follow the explicit isomorphism between the invariance algebra (23.6), (23.8) and the Racah-Wilson algebra which, as we recall, is the fundamental algebraic structure for both the Racah problem for $\mathfrak{su}(1,1)$ and the Askey-Scheme of classical orthogonal polynomials.

23.1.5 Outline

The outline of the paper is as follows. In section 2, the connection between the finite-dimensional irreducible representations of Racah-Wilson algebra and the Racah polynomials is established. This connection is obtained in two equivalent ways. First, the finite-dimensional irreducible representations of the Racah-Wilson algebra are developed in a model-independent fashion and are then related to the Racah polynomials. Second, it is shown that the difference operators corresponding to the difference equation and recurrence relation of the Racah polynomials realize the Racah-Wilson algebra (23.1). The reader who wishes to focus on the connection between the Racah problem and superintegrability could proceeed directly to subsection 2.4. In section 3, the analysis of the Racah problem for $\mathfrak{su}(1,1)$ by means of the Racah algebra is revisited. In section 4, the equivalence between the Racah problem for $\mathfrak{su}(1,1)$ and the generic 3-parameter superintegrable system is established using a differential realization of $\mathfrak{su}(1,1)$. The isomorphism between the invariance algebra of the model and the quadratic Racah algebra is written down explicitly. Some perspectives on future investigations are offered in the conclusion.

23.2 Representations of the Racah-Wilson algebra

In this section, a review of the construction of the irreducible representations of the Racah-Wilson algebra is presented. Another presentation can be found in [9]. Here the emphasis is put on the finite-dimensional representations and on the relation between these representations and the classical Racah polynomials.

23.2.1 The Racah-Wilson algebra and its ladder property

Recall that the defining relations of the Racah-Wilson algebra have the expression

$$[K_1, K_2] = K_3, \tag{23.9a}$$

$$[K_2, K_3] = a_2 K_2^2 + a_1\{K_1, K_2\} + c_1 K_1 + d K_2 + e_1, \tag{23.9b}$$

$$[K_3, K_1] = a_1 K_1^2 + a_2\{K_1, K_2\} + c_2 K_2 + d K_1 + e_2, \tag{23.9c}$$

where it is assumed that $a_1 \cdot a_2 \neq 0$. The algebra admits the Casimir operator [8]

$$\begin{aligned} Q =& a_1\{K_1^2, K_2\} + a_2\{K_1, K_2^2\} + K_3^2 + (a_1^2 + c_1)K_1^2 + (a_2^2 + c_2)K_2^2 \\ & + (d + a_1 a_2)\{K_1, K_2\} + (d a_1 + 2e_1)K_1 + (d a_2 + 2e_2)K_2, \end{aligned} \tag{23.10}$$

which commutes with all generators. It can be seen that the number of parameters in (23.12) can be reduced from seven to three by taking the linear combinations $K_1 \to u_1 K_1 + v_1$, $K_2 \to u_2 K_2 + v_2$, $K_3 \to u_1 u_2 K_3$ and adjusting the coefficients u_i, v_i. A convenient choice for the study of the representations of (23.12) is obtained by taking

$$u_1 = a_2^{-1}, \quad u_2 = a_1^{-1}, \quad v_1 = c_2/2a_2^2, \quad v_2 = c_1/2a_1^2. \tag{23.11}$$

This leads to the following reduced form for the defining relations of the Racah-Wilson algebra:

$$[K_1, K_2] = K_3, \tag{23.12a}$$

$$[K_2, K_3] = K_2^2 + \{K_1, K_2\} + d K_2 + e_1, \tag{23.12b}$$

$$[K_3, K_1] = K_1^2 + \{K_1, K_2\} + d K_1 + e_2, \tag{23.12c}$$

which contains only three parameters d, e_1, e_2. The Casimir (23.10) for the algebra (23.12) is of the form

$$\begin{aligned} Q =& \{K_1^2, K_2\} + \{K_1, K_2^2\} + K_1^2 + K_2^2 + K_3^2 \\ & + (d+1)\{K_1, K_2\} + (2e_1 + d)K_1 + (2e_2 + d)K_2, \end{aligned} \tag{23.13}$$

One of the most important characteristics of the Racah-Wilson algebra is that it possesses a "ladder" property. To exhibit this property, let ω_p be an eigenvector of K_1 with eigenvalue λ_p

$$K_1\omega_p = \lambda_p\omega_p, \tag{23.14}$$

where p is an arbitrary real parameter. One can construct a new eigenvector $\omega_{p'}$ corresponding to a different eigenvalue $\lambda_{p'}$ by taking

$$\omega_{p'} = \left\{\alpha(p)K_1 + \beta(p)K_2 + \gamma(p)K_3\right\}\omega_p, \tag{23.15}$$

where $\alpha(p)$, $\beta(p)$, $\gamma(p)$ are coefficients to be determined. Upon combining the eigenvalue equation for $\omega_{p'}$

$$K_1\omega_{p'} = \lambda_{p'}\omega_{p'} \tag{23.16}$$

with (23.15), using the commutation relations (23.12a), (23.12c) and solving for the coefficients $\alpha(p)$, $\beta(p)$, $\gamma(p)$, it easily seen that the eigenvalues λ_p, $\lambda_{p'}$ must satisfy

$$(\lambda_{p'} - \lambda_p)^2 + (\lambda_{p'} + \lambda_p) = 0. \tag{23.17}$$

For a given value of λ_p, the quadratic equation (23.17) yields two possible values for $\lambda_{p'}$, say λ_+ and λ_-. Without loss of generality, one can define $\lambda_+ = \lambda_{p+1}$ and $\lambda_- = \lambda_{p-1}$. We assume the eigenvalues to be non-degenerate and denote by E_λ the corresponding one-dimensional eigenspaces. It then follows from the analysis above that a generic (p-dependent) algebra elements maps $E_{\lambda_p} \to E_{\lambda_{p-1}} \oplus E_{\lambda_p} \oplus E_{\lambda_{p+1}}$. The element K_2 is thus 3-diagonal and since $K_3 = K_1K_2 - K_2K_1$ with K_1 diagonal, K_3 is 2-diagonal. In the basis with vectors ω_p, one therefore has

$$\begin{aligned}
K_1\,\omega_p &= \lambda_p\,\omega_p,\\
K_2\,\omega_p &= A_{p+1}\,\omega_{p+1} + B_p\,\omega_p + A_p\,\omega_{p-1},\\
K_3\,\omega_p &= g_{p+1}A_{p+1}\,\omega_{p+1} - g_pA_p\,\omega_{p-1},
\end{aligned} \tag{23.18}$$

where $g_p = \lambda_p - \lambda_{p-1}$. Note that for K_2 to be self-adjoint A_p has to be real. The result (23.18) will now be specialized to finite-dimensional irreducible representations. Note that the result (23.18) has been used in [9] to derive infinite-dimensional representations for which the Wilson polynomials act as overlap coefficients between the respective eigenbases of the independent generators K_1, K_2.

23.2.2 Discrete-spectrum and finite-dimensional representations

In finite-dimensional irreducible representations, the spectrum of K_1 is discrete and thus one can denote the eigenvectors of K_1 by ψ_n with n an integer. Then by (23.18) one may write for the actions of the generators

$$K_1\psi_n = \lambda_n\psi_n, \tag{23.19a}$$

$$K_2\psi_n = A_{n+1}\psi_{n+1} + B_n\psi_n + A_n\psi_{n-1}, \tag{23.19b}$$

$$K_3\psi_n = A_{n+1}g_{n+1}\psi_{n+1} - A_n g_n\psi_{n-1}, \tag{23.19c}$$

where $g_n = \lambda_n - \lambda_{n-1}$ and A_n is real. Upon substituting the actions (23.19) in the commutation relation (23.12c) and using (23.12a), one finds that the eigenvalues λ_n satisfy

$$(\lambda_{n+1} - \lambda_n)^2 + (\lambda_n + \lambda_{n+1}) = 0. \tag{23.20}$$

The recurrence relation (23.20) admits two solutions differing only by the sign of the integration constant. Without loss of generality, one can thus write

$$\lambda_n = -(n-\sigma)(n-\sigma+1)/2, \tag{23.21}$$

where σ is a arbitrary real parameter. From (23.12c) and the eigenvalues (23.21), one can evaluate B_n and g_n directly to find

$$B_n = -\frac{\lambda_n^2 + d\,\lambda_n + e_2}{2\lambda_n}, \quad g_n = (\sigma - n). \tag{23.22}$$

At this stage, the actions (23.19) with (23.21) and (23.22) are such that the relations (23.12a) and (23.12c) are satisfied. There remains only to evaluate A_n in (23.19). Upon acting with relation (23.12b) on ψ_n, using (23.19) and (23.22) and then gathering the terms in ψ_n, one obtains the following recurrence relation for A_n^2:

$$2\left\{g_{n+3/2}\,A_{n+1}^2 - g_{n-1/2}\,A_n^2\right\} = B_n^2 + (2\lambda_n + d)B_n + e_1. \tag{23.23}$$

Instead of solving the recurrence relation (23.23), one can use the Casimir operator (23.10) to obtain A_n^2 directly. One first requires that the Casimir operator acts as a multiple of the identity on the basis ψ_n, as is demanded by Schur's lemma for irreducible representations. Hence one takes

$$Q\psi_n = q\,\psi_n. \tag{23.24}$$

Upon substituting (23.10) in (23.24) with the actions (23.19) and then gathering the terms in ψ_n, a straightforward computation yields

$$\begin{aligned}2\left\{g_{n+3/2}g_n A_{n+1}^2 + g_{n+1}g_{n-1/2}A_n^2\right\} &= (2\lambda_n+1)B_n^2 + \lambda_n^2 + (2e_1+d)\lambda_n - q \\ &\quad + (2\lambda_n^2 + 2\lambda_n(d+1) + 2e_2 + d)B_n.\end{aligned} \tag{23.25}$$

Combining (23.23) and (23.25), one obtains

$$4g_{n+1/2}g_{n-1/2}A_n^2 = g_{n-1}g_{n+1}B_nB_{n-1} + e_1(\lambda_n + \lambda_{n-1}) - (e_2 + q). \tag{23.26}$$

It is directly verified that (23.26) indeed satisfies the recurrence relation (23.23). In addition, a straightforward calculation confirms that with (23.19), (23.21), (23.26) and (23.22), the defining relations (23.12) of the Racah-Wilson algebra are satisfied. The formula (23.26) for the matrix elements A_n^2 can be cast in the form

$$A_n^2 = \frac{\mathscr{P}(g_n^2)}{64\, g_n^2\, g_{n-1/2}\, g_{n+1/2}}, \tag{23.27}$$

where $\mathscr{P}(z)$ is the fourth degree polynomial

$$\begin{aligned}\mathscr{P}(z) = z^4 &- (4d+2)z^3 + (4d^2 + 4d + 1 + 8e_2 - 16e_1)z^2 \\ &- 4(d^2 + 2e_2 + 4de_2 + 4q)z + 16e_2^2.\end{aligned} \tag{23.28}$$

The polynomial $\mathscr{P}(g_n^2)$ is referred to as the characteristic polynomial of the algebra, as it determines the representation. Denoting by ξ_j^2, $j = 1, \ldots, 4$, the roots of this polynomial, one may write

$$A_n^2 = \frac{\prod_{j=1}^4 (g_n^2 - \xi_j^2)}{64\, g_n^2\, g_{n-1/2}\, g_{n+1/2}}. \tag{23.29}$$

Using Vieta's formula for the roots of quartic polynomials [29], one finds that the structure parameters d, e_1, e_2 and the Casimir value q are related to the roots ξ_k^2 by

$$\begin{aligned} e_1 &= \frac{1}{64}\{S_1^2 - 4S_2 + 8S_4^{1/2}\}, & e_2 &= \frac{S_4^{1/2}}{4}, \\ d &= \frac{1}{4}\{S_1 - 2\}, & q &= \frac{1}{64}\{4S_1(1 + S_4^{1/2}) + 4S_3 - S_1^2 - 4\}, \end{aligned} \tag{23.30}$$

where $S_1, \cdots S_4$ are the elementary symmetric polynomials

$$S_N = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_N \leqslant N} \xi_{i_1}^2 \cdots \xi_{i_N}^2. \tag{23.31}$$

For real structure parameters, the roots ξ_k^2 are, in general, complex. Furthermore, one sees from (23.30) that the structure parameters remain invariant under any transposition of the roots ξ_k^2 or change of sign of an even number of ξ_k. The explicit formula for A_n^2 therefore reads

$$A_n^2 = \frac{1}{4}\frac{\prod_{j=1}^4 (n - \sigma - \xi_j)(n - \sigma + \xi_j)}{(2n - 2\sigma)^2(2n - 2\sigma + 1)(2n - 2\sigma - 1)}. \tag{23.32}$$

To obtain a finite-dimensional irreducible representation of (23.12), one must have

$$N_1 \leqslant n \leqslant N_2, \tag{23.33}$$

where N_1, N_2 are integers. It is always possible to choose $N_1 = 0$ using the arbitrariness in the parameter σ appearing in (23.21). Thus for $(N+1)$-dimensional irreducible representations to occur, the following conditions must hold:

$$A_0 = 0, \quad A_{N+1} = 0. \tag{23.34}$$

As is seen from (23.29), the truncation conditions (23.34) can be fulfilled if one has

$$g_0 = \pm\xi_i, \quad g_{N+1} = \pm\xi_j, \tag{23.35}$$

where in the generic case $\xi_i \neq \xi_j$. The condition (23.35) implies that for finite-dimensional representations, at least two of the roots are real. The positivity condition $A_n^2 > 0$ induces conditions on the possibles values for the roots ξ_k. As an example, consider the case for which the truncation conditions (23.34) are satisfied through

$$\xi_1 = \sigma, \quad \xi_4 = (\sigma - N - 1). \tag{23.36}$$

Then from (23.32), it follows that $A_n^2 > 0$ for $n = 1, \ldots, N$ if

$$(\sigma < 1/2 \text{ or } \sigma > N + 1/2) \text{ and } \xi_2^2 < (\sigma - 1)^2 \text{ and } \xi_3^2 > (\sigma - N)^2. \tag{23.37}$$

Similar conditions can be found by using the freedom in permuting and changing the signs of the ξ_k. Note that the with the condition on σ, the spectrum (23.21) of K_1 is non-degenerate.

23.2.3 Racah polynomials

The relation between the finite-dimensional irreducible representations of the Racah-Wilson algebra and the classical Racah polynomials is as follows. Let ϕ_s be an eigenvector of K_2:

$$K_2\phi_s = \mu_s\phi_s. \tag{23.38}$$

The duality property of the algebra (23.12) can be used to derive the expression for the spectrum μ_s of K_2. Indeed, it is easily seen that the Racah-Wilson algebra is left invariant if one takes $\widetilde{K}_1 = K_2$, $\widetilde{K}_2 = K_1$, $\widetilde{K}_3 = -K_3$ and performs the replacement $e_1 \leftrightarrow e_2$. From this duality relation and (23.21), it follows that μ_s has the expression

$$\mu_s = -(s - \nu)(s - \nu + 1)/2, \tag{23.39}$$

where ν is an arbitrary parameter. In this basis, it follows from the duality that K_1 is three-diagonal with matrix elements given by the formulas (23.22) and (23.27) with $n \to s$ and $e_1 \leftrightarrow e_2$.

It is appropriate to indicate here the connection with Leonard pairs, which are defined as follows [28]. Two linear transformations (A_1, A_2) form a *Leonard pair* if there exists a basis in which A_1 is diagonal and A_2 is tridiagonal and another basis in which A_1 is tridiagonal and A_2 is diagonal. The preceeding discussion makes clear that K_1, K_2 are hence realizing a Leonard pair.

Since the two sets of basis vectors $\{\phi_s\}$ and $\{\psi_n\}$ for $n, s = 0, \ldots, N$ span the same vector space, the two bases are related to one another by a linear transformation

$$\phi_s = \sum_{n=0}^{N} W_n(s)\psi_n. \tag{23.40}$$

Upon acting with K_2 on each side of (23.40) and writing $W_n(s) = W_0(s)P_n(\mu_s)$ with $P_0(\mu_s) = 1$, one finds that $P_n(\mu_s)$ satisfies

$$\mu_s P_n(\mu_s) = A_{n+1}P_{n+1}(\mu_s) + B_n\psi_n(\mu_s) + A_n P_{n-1}(\mu_s). \tag{23.41}$$

Hence $P_n(\mu_s)$ are polynomials of degree n in the variable μ_s. The above recurrence relation can be put in monic form by taking $P_n(\mu_s) = (A_1 \cdots A_n)^{-1}\widehat{P}_n(x)$. One then has

$$x\widehat{P}_n(x) = \widehat{P}_{n+1}(x) + B_n\widehat{P}_n(x) + A_n^2\widehat{P}_{n-1}(x). \tag{23.42}$$

It follows from (23.42) that the polynomials $\widehat{P}_0(x), \widehat{P}_1(x)\ldots, \widehat{P}_N(x)$ form a finite system of orthogonal polynomials provided that $A_n^2 > 0$ for $n = 1, \ldots, N$ [1]. It can be seen that these polynomials correspond to the classical Racah polynomials. To obtain the identification, one first introduces the monic polynomials $\widehat{H}_n(\widetilde{x})$ in the variable $\widetilde{x} = -2(x+\tau)$ by taking $\widehat{P}_n(x) = (-2)^{-n}\widehat{H}_n(\widetilde{x})$. Upon using the formulas (23.30) for the structure parameters in terms of the roots ξ_k^2 and the explicit expressions (23.21), (23.22) and (23.32) for λ_n, B_n and A_n^2, one finds that the polynomials $\widehat{H}_n(\widetilde{x})$ obey the recurrence relation

$$\widetilde{x}\widehat{H}_n(\widetilde{x}) = \widehat{H}_{n+1}(\widetilde{x}) + \widetilde{B}_n\widehat{H}_n(\widetilde{x}) + \widetilde{A}_n^2\widehat{H}_{n-1}(\widetilde{x}), \tag{23.43}$$

where

$$\begin{aligned} \widetilde{B}_n &= \frac{1}{2}(\sigma-n)(n-\sigma+1) + \frac{\xi_1\xi_2\xi_3\xi_4}{2(\sigma-n)(n-\sigma+1)} + \frac{1}{4}\sum_{j=1}^{4}\xi_j^2 - (2\tau+1/2), \\ \widetilde{A}_n^2 &= \frac{\prod_{j=1}^{4}((\sigma-n)^2-\xi_j^2)}{(2n-2\sigma)^2(2n-2\sigma+1)(2n-2\sigma-1)}. \end{aligned} \tag{23.44}$$

The recurrence relation (23.43) and recurrence coefficients (23.44) can now be compared with those of the monic Racah polynomials $\widehat{R}_n(\lambda(x);\alpha,\beta,\gamma,\delta)$. These polynomials are defined by [23]

$$\widehat{R}_n(\lambda(x);\alpha,\beta,\gamma,\delta) = \frac{(\alpha+1)_n(\beta+\delta+1)_n(\gamma+1)_n}{(n+\alpha+\beta+1)_n}\,{}_4F_3\left[\begin{matrix}-n, n+\alpha+\beta+1, -x, x+\gamma+\delta+1\\ \alpha+1, \beta+\delta+1, \gamma+1\end{matrix};1\right] \tag{23.45}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ denotes the Pochhammer symbol and where ${}_pF_q$ stands for the generalized hypergeometric function [6]. The monic Racah polynomials obey the three-term recurrence relation [23]

$$x\widehat{R}_n(\lambda(x)) = \widehat{R}_{n+1}(\lambda(x)) - (C_n + D_n)\widehat{R}_n(\lambda(x)) + C_{n-1}D_n\widehat{R}_n(\lambda(x)), \tag{23.46}$$

where the recurrence coefficients are given by

$$\begin{aligned} C_n &= \frac{(n+\alpha+1)(n+\alpha+\beta+1)(n+\gamma+1)(n+\beta+\delta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}, \\ D_n &= \frac{n(n+\beta)(n+\alpha-\delta)(n+\alpha+\beta-\gamma)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}, \end{aligned} \tag{23.47}$$

and where it assumed that one of the conditions $\alpha+1=-N$, $\beta+\delta+1=-N$ or $\gamma+1=-N$ holds. Now suppose that the truncation conditions $A_0 = 0$, $A_{N+1} = 0$ are satisfied in (23.44) through

$$g_0 = \sigma = \xi_1, \quad g_{N+1} = (\sigma - N - 1) = \xi_4. \tag{23.48}$$

Then one can adopt the following parametrization for the roots:

$$\xi_1 = -\frac{\alpha+\beta}{2}, \quad \xi_2 = \frac{\beta-\alpha}{2} + \delta, \quad \xi_3 = \frac{\beta-\alpha}{2}, \quad \xi_4 = \gamma - \frac{\alpha+\beta}{2}, \tag{23.49}$$

for which one has $\gamma + 1 = -N$. Then with $\tau = (2+\gamma+\delta)(\gamma+\delta)/8$, the recurrence relation (23.43) is directly seen to coincide with (23.46). It is clear from the above considerations that the parameters ξ_i, $i = 1,\ldots,4$, are much more convenient for the analysis in terms of orthogonal polynomials.

23.2.4 Realization of the Racah algebra

The bispectrality of the Racah polynomials can be used to obtain a realization of the reduced Racah-Wilson algebra (23.12) in terms of difference operators. The bispectral property of the Racah polynomials (23.45) is as follows. On the one hand, the polynomials $R_n(\lambda(x))$ satisfy the three term recurrence relation

$$\lambda(x)R_n(\lambda(x)) = C_nR_{n+1}(\lambda(x)) - (C_n + D_n)R_n(\lambda(x)) + D_nR_{n-1}(\lambda(x)), \tag{23.50}$$

where C_n and D_n are given by (23.46) and where $\lambda(x) = x(x+\gamma+\delta+1)$. On the other hand, the polynomials also satisfy the eigenvalue equation

$$\left[B(x)T^+ - (B(x)+E(x)) + E(x)T^-\right]R_n(\lambda(x)) = \mu_nR_n(\lambda(x)), \tag{23.51}$$

with $\mu_n = n(n+\alpha+\beta+1)$ and where $T^+f(x) = f(x+1)$, $T^-f(x) = f(x-1)$ are the usual shift operators. The coefficients appearing in (23.51) are given by

$$\begin{aligned} B(x) &= \frac{(x+\alpha+1)(x+\beta+\delta+1)(x+\gamma+1)(x+\gamma+\delta+1)}{(2x+\gamma+\delta+1)(2x+\gamma+\delta+2)}, \\ E(x) &= \frac{x(x-\alpha+\gamma+\delta)(x-\beta+\gamma)(x+\delta)}{(2x+\gamma+\delta)(2x+\gamma+\delta+1)}. \end{aligned} \tag{23.52}$$

The recurrence operator (23.50) and the difference operator (23.51) can be used to realize the Racah-Wilson algebra. To this end, the recurrence operator is denoted K_1 and taken to be diagonal, i.e. as in the LHS of (23.50). The difference operator is denoted K_2 and taken to be three-diagonal, i.e. the LHS of (23.51). Upon introducing $K_3 = [K_1, K_2]$, one thus has

$$\begin{aligned} K_1 &= x(x+\gamma+\delta+1), \\ K_2 &= B(x)T^+ + E(x)T^- - (B(x)+E(x)), \\ K_3 &= (2x+\gamma+\delta)E(x)T^- - (2x+\gamma+\delta+2)B(x)T^+. \end{aligned} \tag{23.53}$$

A direct computation shows that the operators K_1, K_2 and K_3 realize the Racah-Wilson algebra (23.9) with structure parameters

$$\begin{aligned} &a_1 = -2, \quad a_2 = -2 \\ &c_1 = -(\alpha+\beta)(2+\alpha+\beta), \quad c_2 = -(\gamma+\delta)(2+\gamma+\delta), \\ &e_1 = -(\alpha+1)(\alpha+\beta)(\beta+\delta+1)(\gamma+1), \quad e_2 = -(\alpha+1)(\beta+\delta+1)(\gamma+1)(\gamma+\delta), \\ &d = \beta(\delta-\gamma-2) - \alpha(2\beta+\gamma+\delta+2) - 2(\gamma+1)(\delta+1). \end{aligned}$$

The canonical form (23.12) can be obtained if one takes

$$K_1 \to a_2^{-1}K_1 + c_2/2a_2^2, \quad K_2 \to a_1^{-1}K_2 + c_1/2a_1^2, \quad K_3 \to a_1^{-1}a_2^{-1}K_3. \tag{23.54}$$

The remaining non-zero structure constants become

$$\begin{aligned} e_1 &\to \frac{1}{4}\left(\frac{\alpha-\beta}{2}\right)\left(\frac{\alpha+\beta}{2}\right)\left(\frac{\alpha+\beta}{2}-\gamma\right)\left(\frac{\alpha-\beta}{2}-\delta\right), \\ e_2 &\to \frac{1}{4}\left(\frac{\gamma-\delta}{2}\right)\left(\frac{\gamma+\delta}{2}\right)\left(\frac{\gamma+\delta}{2}-\alpha\right)\left(\frac{\gamma-\delta}{2}-\beta\right), \\ d &\to \frac{1}{4}\left\{\left(\frac{\gamma-\delta}{2}\right)^2 + \left(\frac{\gamma+\delta}{2}\right)^2 + \left(\frac{\gamma+\delta}{2}-\alpha\right)^2 + \left(\frac{\gamma-\delta}{2}-\beta\right)^2 - 2\right\} \\ &= \frac{1}{4}\left\{\left(\frac{\alpha-\beta}{2}\right)^2 + \left(\frac{\alpha+\beta}{2}\right)^2 + \left(\frac{\alpha+\beta}{2}-\gamma\right)^2 + \left(\frac{\alpha-\beta}{2}-\delta\right)^2 - 2\right\}. \end{aligned} \tag{23.55}$$

It is interesting to note that the duality property of the Racah polynomials is encoded in the Racah-Wilson algebra (23.12) and can be derived directly from the structure constants (23.55). Indeed, the algebra is invariant under the exchange $K_1 \leftrightarrow K_2$, $K_3 \to -K_3$ and $e_1 \leftrightarrow e_2$. This means that if one instead takes K_1 as the diagonal operator in the RHS of (23.51) and K_2 as the tridiagonal operator in the RHS of (23.50), then one finds the same algebra with $e_1 \leftrightarrow e_2$. From (23.55), this is equivalent to the well-known duality property of the Racah polynomials ($n \leftrightarrow x$, $\alpha \leftrightarrow \gamma$, $\beta \leftrightarrow \delta$):

$$R_n(\lambda(x); \alpha, \beta, \gamma, \delta) = R_x(\lambda(n); \gamma, \delta, \alpha, \beta). \tag{23.56}$$

23.3 The Racah problem for $\mathfrak{su}(1,1)$ and the Racah-Wilson algebra

In this section, the Racah problem for $\mathfrak{su}(1,1)$ is revisited using the Racah-Wilson algebra. It shall be shown that the Racah-Wilson algebra is the fundamental algebraic structure behind this problem. Furthermore, it will be seen that our approach encompasses simultaneously the combination of three unitary irreducible $\mathfrak{su}(1,1)$ representations of any series.

23.3.1 Racah problem essentials for $\mathfrak{su}(1,1)$

The $\mathfrak{su}(1,1)$ algebra is generated by the elements J_0, $J_\pm$ with commutation relations

$$[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = -2J_0. \tag{23.57}$$

The Casimir operator, which commutes with all $\mathfrak{su}(1,1)$ elements, is given by

$$\mathscr{C} = J_0^2 - J_+ J_- - J_0. \tag{23.58}$$

Let $J_0^{(i)}$, $J_\pm^{(i)}$, $i = 1,2,3$, denote three mutually commuting sets of generators satisfying the commutation relations (23.57). The three sets can be combined to produce a fourth set of $\mathfrak{su}(1,1)$ generators as follows :

$$J_0^{(4)} = J_0^{(1)} + J_0^{(2)} + J_0^{(3)}, \quad J_\pm^{(4)} = J_\pm^{(1)} + J_\pm^{(2)} + J_\pm^{(3)}. \tag{23.59}$$

The Casimir operator $\mathscr{C}^{(4)} = J_0^{(4)} - J_+^{(4)} J_-^{(4)} - J_0^{(4)}$ for the representation (23.59) of $\mathfrak{su}(1,1)$ is easily seen to have the following expression:

$$\mathscr{C}^{(4)} = \mathscr{C}^{(12)} + \mathscr{C}^{(23)} + \mathscr{C}^{(31)} - \mathscr{C}^{(1)} - \mathscr{C}^{(2)} - \mathscr{C}^{(3)}, \tag{23.60}$$

where $\mathscr{C}^{(i)} = J_0^{(i)} - J_+^{(i)} J_-^{(i)} - J_0^{(i)}$ are the individual Casimir operators and $\mathscr{C}^{(ij)}$ are the intermediate Casimir operators

$$\mathscr{C}^{(ij)} = 2J_0^{(i)} J_0^{(j)} - (J_+^{(i)} J_-^{(j)} + J_+^{(j)} J_-^{(j)}) + \mathscr{C}^{(i)} + \mathscr{C}^{(j)}. \tag{23.61}$$

The full Casimir operator $\mathscr{C}^{(4)}$ commutes with all the intermediate Casimir operators $\mathscr{C}^{(ij)}$ and with all the individual Casimir operators $\mathscr{C}^{(i)}$. The intermediate Casimir operators $\mathscr{C}^{(ij)}$ do not commute with one another but commute with each of the individual Casimir operators $\mathscr{C}^{(i)}$ and with $\mathscr{C}^{(4)}$.

The Racah problem can be posited as follows. Let $V^{(\lambda_i)}$, $i = 1,2,3$, denote a generic unitary irreducible representation space on which the Casimir operator $\mathscr{C}^{(i)}$ has the eigenvalue λ_i. The irreducible unitary representations of $\mathfrak{su}(1,1)$ are known and classified (see for example [30]). Note that $V^{(\lambda_i)}$ may depend on additional parameters other than λ_i and that $V^{(\lambda_i)}$ need not be of the same type as $V^{(\lambda_j)}$. A representation space V for the algebra (23.59) is obtained by taking $V = V^{(\lambda_1)} \otimes V^{(\lambda_2)} \otimes V^{(\lambda_3)}$, where it is understood that each set of $\mathfrak{su}(1,1)$ generators $J_0^{(i)}$, $J_\pm^{(i)}$ acts on the corresponding representation space $V^{(\lambda_i)}$. In general, the representation space V is not irreducible and can be decomposed into irreducible components in two equivalent ways.

- In the first scheme, one decomposes $V^{(\lambda_1)} \otimes V^{(\lambda_2)}$ in irreducible components $V^{(\lambda_{12})}$ and then further decomposes $V^{(\lambda_{12})} \otimes V^{(\lambda_3)}$ for each occurring values of λ_{12}. On the spaces $V^{(\lambda_{12})}$, the intermediate Casimir operator $\mathscr{C}^{(12)}$ acts as $\lambda_{12} \cdot \mathbb{1}$.
- In the second scheme, one first decomposes $V^{(\lambda_2)} \otimes V^{(\lambda_3)}$ in irreducible components $V^{(\lambda_{23})}$ and then further decompose $V^{(\lambda_1)} \otimes V^{(\lambda_{23})}$ for each occurring values of λ_{23}. On the space $V^{(\lambda_{23})}$, the intermediate Casimir operator $\mathscr{C}^{(23)}$ acts as $\lambda_{23} \cdot \mathbb{1}$.

One can define two natural orthonormal bases for the representation space V which correspond to the two different decomposition schemes. For the first scheme, the natural orthonormal basis vectors that span V are denoted $|\,\lambda_{12};\vec{\lambda}\,\rangle$ and are defined by

$$\mathscr{C}^{(12)}|\,\lambda_{12};\vec{\lambda}\,\rangle = \lambda_{12}|\,\lambda_{12};\vec{\lambda}\,\rangle, \quad \mathscr{C}^{(i)}|\,\lambda_{12};\vec{\lambda}\,\rangle = \lambda_i|\,\lambda_{12};\vec{\lambda}\,\rangle, \tag{23.62}$$

where $\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. For the second scheme, the natural orthonormal basis vectors are denoted $|\,\lambda_{23};\vec{\lambda}\,\rangle$ and are defined by

$$\mathscr{C}^{(23)}|\,\lambda_{23};\vec{\lambda}\,\rangle = \lambda_{23}|\,\lambda_{23};\vec{\lambda}\,\rangle, \quad \mathscr{C}^{(i)}|\,\lambda_{23};\vec{\lambda}\,\rangle = \lambda_i|\,\lambda_{23};\vec{\lambda}\,\rangle. \tag{23.63}$$

The three parameters λ_1, λ_2 and λ_3 are given while λ_{12}, λ_{23} and λ_4 vary so that the basis vectors span V. The possible values for these parameters depend on the representations $V^{(\lambda_i)}$ that are involved in the tensor product. For a given value of λ_4, the orthonormal vectors $|\,\lambda_{12};\vec{\lambda}\,\rangle$, $|\,\lambda_{23};\vec{\lambda}\,\rangle$ with admissible values of λ_{12}, λ_{23} span the same space and are thus related by a unitary transformation. One can thus write

$$|\,\lambda_{12};\vec{\lambda}\,\rangle = \sum_{\lambda_{23}} \langle\,\lambda_{23};\vec{\lambda}\,|\,\lambda_{12};\vec{\lambda}\,\rangle\,|\,\lambda_{23};\vec{\lambda}\,\rangle = \sum_{\lambda_{23}} R^{\vec{\lambda}}_{\lambda_{12},\lambda_{23}}|\,\lambda_{23};\vec{\lambda}\,\rangle, \tag{23.64}$$

where the range of the sum depends on the possible values for λ_{23}. Note that these values (or those of λ_{12}) may vary continuously hence the sum in (23.64) can also be an integral. The expansion coefficients $\langle\,\lambda_{23};\vec{\lambda}\,|\,\lambda_{12};\vec{\lambda}\,\rangle = R^{\vec{\lambda}}_{\lambda_{12},\lambda_{23}}$ between the two bases with vectors $|\,\lambda_{12};\vec{\lambda}\,\rangle$ and $|\,\lambda_{23};\vec{\lambda}\,\rangle$ are known as Racah coefficients. These coefficients are usually taken to be real. Since the two bases are orthonormal, the Racah coefficients satisfy the orthogonality relations

$$\sum_{\lambda_{23}} R^{(\lambda)}_{\lambda_{12},\lambda_{23}} R^{(\lambda)}_{\lambda'_{12},\lambda_{23}} = \delta_{\lambda_{12}\lambda'_{12}}, \quad \sum_{\lambda_{12}} R^{(\lambda)}_{\lambda_{12},\lambda_{23}} R^{(\lambda)}_{\lambda_{12},\lambda'_{23}} = \delta_{\lambda_{23}\lambda'_{23}}, \tag{23.65}$$

where each of the summation may become an integral depending on the possible values of λ_{12}, λ_{23}. The evaluation of the coefficients in (23.64) is referred to as the "Racah problem".

23.3.2 The Racah problem and the Racah-Wilson algebra

It will now be shown that the Racah-Wilson algebra is the fundamental structure behind the Racah problem. The idea behind the method is the following. Since the vectors $|\,\lambda_{12};\vec{\lambda}\,\rangle$ and $|\,\lambda_{23};\vec{\lambda}\,\rangle$ are the eigenvectors of the two non-commuting intermediate Casimir operators $\mathscr{C}^{(12)}$ and $\mathscr{C}^{(23)}$, respectively, the information on the structure of the coefficients $\langle\,\lambda_{23};\vec{\lambda}\,|\,\lambda_{12};\vec{\lambda}\,\rangle$ can be obtained by studying their commutation relations. The Casimir operators $\mathscr{C}^{(i)}$, $i=1,\ldots,4$, all act as multiples of the identity on both sets of vectors $|\,\lambda_{12},\vec{\lambda}\,\rangle$, $|\,\lambda_{23},\vec{\lambda}\,\rangle$. Consequently, they can be treated as constants, i.e.:

$$\mathscr{C}^{(i)}=\lambda_i,\quad i=1,\ldots,4. \tag{23.66}$$

Let κ_1 and κ_2 be defined as

$$\kappa_1=-\frac{1}{2}\mathscr{C}^{(12)},\quad \kappa_2=-\frac{1}{2}\mathscr{C}^{(23)}. \tag{23.67}$$

Using the identification (23.66) and the definition (23.61), a direct computation shows that κ_1, κ_2 and their commutator

$$[\kappa_1,\kappa_2]=\kappa_3=\frac{J_0^{(1)}}{2}\left(J_-^{(2)}J_+^{(3)}-J_+^{(2)}J_-^{(3)}\right)+\text{cyclic permutations},$$

satisfy the commutation relations of the Racah-Wilson algebra (23.12)

$$\begin{aligned}
&[\kappa_1,\kappa_2]=\kappa_3,\\
&[\kappa_2,\kappa_3]=\kappa_2^2+\{\kappa_1,\kappa_2\}+d\,\kappa_2+e_1,\\
&[\kappa_3,\kappa_1]=\kappa_1^2+\{\kappa_1,\kappa_2\}+d\,\kappa_1+e_2,
\end{aligned} \tag{23.68}$$

with structure parameters

$$d=\frac{1}{2}(\lambda_1+\lambda_2+\lambda_3+\lambda_4), \tag{23.69a}$$

$$e_1=\frac{1}{4}(\lambda_1-\lambda_4)(\lambda_2-\lambda_3),\quad e_2=\frac{1}{4}(\lambda_1-\lambda_2)(\lambda_4-\lambda_3). \tag{23.69b}$$

It is thus seen that the Racah-Wilson algebra is the fundamental algebraic structure behind the Racah problem for $\mathfrak{su}(1,1)$. One recalls that this result is valid for any choice of representations corresponding to $V^{(\lambda_i)}$ provided that the Casimir operator acts with eigenvalue λ_i. This indicates that it is possible to obtain all types of Racah coefficients corresponding to the combination of the various $\mathfrak{su}(1,1)$ unitary representations through the analysis of the representations of the Racah-Wilson algebra (see for example [14, 15] for possible applications of this scheme). We also note that a similar result holds for the combination of two $\mathfrak{su}(1,1)$ irreducible representations (Clebsch-Gordan problem), where the algebra (23.9) appears with $a_2 \cdot a_1 = 0$. In this case the relevant operators are the intermediate Casimir operator $\kappa_1 = \mathscr{C}^{(12)}$ and the operator $\kappa_2 = J_0^{(1)} - J_0^{(2)}$.

23.3.3 Racah problem for the positive-discrete series

The above results will now be specialized to the positive discrete series of unitary representations of $\mathfrak{su}(1,1)$; these representations will occur in the correspondence between the $\mathfrak{su}(1,1)$ Racah problem and the analysis of the generic 3-parameter superintegrable system on the two sphere. The positive discrete series of unitary irreducible representations of $\mathfrak{su}(1,1)$ are infinite-dimensional and labeled by a positive real number ν. They can be defined by the following actions on a canonical basis $|\,\nu,n\,\rangle$, $n \in \mathbb{N}$:

$$\begin{aligned} J_0|\,\nu,n\,\rangle &= (n+\nu)|\,\nu,n\,\rangle, \\ J_+|\,\nu,n\,\rangle &= \sqrt{(n+1)(n+2\nu)}\,|\,\nu,n+1\,\rangle, \\ J_-|\,\nu,n\,\rangle &= \sqrt{n(n+2\nu-1)}\,|\,\nu,n-1\,\rangle. \end{aligned} \tag{23.70}$$

The action of the Casimir operator $\mathscr{C}$ is given by

$$\mathscr{C}|\,\nu,n\,\rangle = \nu(\nu-1)|\,\nu,n\,\rangle. \tag{23.71}$$

Let us now consider the Racah problem for the combination of three representations of the discrete series, each labeled by a positive number ν_i, $i = 1,2,3$. In this case, the structure constants in the algebra (23.68) have the following expressions:

$$\lambda_i = \nu_i(\nu_i - 1), \quad i = 1,\ldots,4. \tag{23.72}$$

With the eigenvalues of the Casimir operators parametrized as in (23.72), we will replace the notation $|\,\lambda_{ij};\vec{\lambda}\,\rangle$ for instance by $|\,\nu_{ij};\vec{\nu}\,\rangle$. There remains to evaluate the admissible values of ν_{12}, ν_{23} and ν_4. We begin with ν_{12}. In view of the addition rule $J_0^{(12)} = J_0^{(1)} + J_0^{(2)}$ and the actions (23.70), it is not hard to see that the possible values of ν_{12} are of the form

$$\nu_{12} = \nu_1 + \nu_2 + n_{12}, \quad n_{12} \in \mathbb{N}. \tag{23.73}$$

Similarly, one has

$$\nu_{23} = \nu_2 + \nu_3 + n_{23}, \quad n_{23} \in \mathbb{N}. \tag{23.74}$$

Again, the addition rule for $J_0^{(4)} = J_0^{(12)} + J_0^{(3)} = J_0^{(1)} + J_0^{(23)}$ gives for ν_4

$$\nu_4 = \nu_1 + \nu_2 + \nu_3 + N = \nu_{12} + \nu_3 + p_1 = \nu_1 + \nu_{23} + p_2, \tag{23.75}$$

where $N, p_1, p_2 \in \mathbb{N}$. For a given of ν_4, the dimension of the space spanned by the basis vectors $|\,\nu_{12}; \vec{\nu}\,\rangle$ can be evaluated from (23.75) and (23.73) in the following way. If $N = 0$, it is obvious that there is only one possible value for ν_{12}. If $N = 1$, then $\nu_4 = \nu_1 + \nu_2 + \nu_3 + 1$ and hence ν_{12} can take two values corresponding to $n_{12} = 0, 1$. By a direct inductive argument, the dimension of the space spanned by the basis vectors $|\,\nu_{12}; \vec{\nu}\,\rangle$ is $N + 1$.

Let us return to the results of Section 2. Upon comparing the structure constants in (23.68) with (23.30) and using (23.72), it is seen that the $\xi_1, \ldots \xi_4$ of the characteristic polynomial of the Racah-Wilson algebra can be taken to be

$$\xi_1 = (1 - \nu_1 - \nu_2), \quad \xi_2 = (\nu_1 - \nu_2), \quad \xi_3 = (\nu_4 + \nu_3 - 1), \quad \xi_4 = (\nu_3 - \nu_4). \tag{23.76}$$

The free parameter σ in the spectrum of κ_1 given by (23.21) is evaluated to

$$\sigma = 1 - \nu_1 - \nu_2, \tag{23.77}$$

since the minimal value of ν_{12} is $\nu_1 + \nu_2$. From (23.76) and (23.75), it is seen that the truncation conditions

$$\xi_1 = \sigma, \quad \xi_4 = (\sigma - N - 1), \tag{23.78}$$

are satisfied. Thus it follows that the Racah coefficients for the combination of three $\mathfrak{su}(1,1)$ representations of the positive discrete series are expressed in terms of the Racah polynomials with the parameter identification obtained by combining (23.30) and (23.76).

23.4 The 3-parameter superintegrable system on the 2-sphere and the $\mathfrak{su}(1,1)$ Racah problem

The stage has now been set to establish the equivalence between the Racah problem for $\mathfrak{su}(1,1)$ and the analysis of the generic 3-parameter superintegrable system of the two-sphere. To this end, consider the following differential realizations of $\mathfrak{su}(1,1)$

$$\begin{aligned} J_0^{(i)} &= \frac{1}{4}\left(-\partial_{x_i}^2 + x_i^2 + \frac{(k_i^2 - 1/4)}{x_i^2}\right), \\ J_\pm^{(i)} &= \frac{1}{4}\left(\partial_{x_i}^2 \mp 2x_i\partial_{x_i} + (x_i^2 \mp 1) - \frac{(k_i^2 - 1/4)}{x_i^2}\right), \end{aligned} \tag{23.79}$$

where $i = 1, 2, 3$. In these realizations, the Casimir operators $\mathscr{C}^{(i)}$ have actions:

$$\mathscr{C}^{(i)} f(x_i) = \nu_i(\nu_i - 1) f(x_i),$$

where $\nu_i = (k_i+1)/2$. It is thus seen that $\nu_i > 0$ if $k_i > -1$. The operator $J_0^{(i)}$ is the Hamiltonian of the singular oscillator and it has a positive and discrete spectrum. Hence the representation (23.79) realize the positive discrete series of $\mathfrak{su}(1,1)$. Upon using (23.79), it is observed that the full Casimir operator $\mathscr{C}^{(4)}$ and the intermediate Casimir operators $\mathscr{C}^{(ij)}$ have the expressions

$$\begin{aligned}
\mathscr{C}^{(4)} &= \frac{1}{4}\left\{J_1^2+J_2^2+J_3^2+(x_1^2+x_2^2+x_3^2)\left(\frac{a_1}{x_1^2}+\frac{a_2}{x_2^2}+\frac{a_3}{x_3^2}\right)-\frac{3}{4}\right\},\\
\mathscr{C}^{(ij)} &= \frac{1}{4}\left\{J_k^2+\frac{a_i x_j^2}{x_i^2}+\frac{a_j x_i^2}{x_j^2}+a_i+a_j-1\right\},
\end{aligned} \tag{23.80}$$

where the indices i, j, k are such that $\epsilon_{ijk}=1$, $a_i = k_i^2 - 1/4$ and where the operators J_i are the angular momentum generators (23.3). Upon returning to the defining formulas (23.5) for the symmetries of the generic 3-parameter system on the two-sphere, it is directly seen that

$$L_k = 4\mathscr{C}^{(ij)} - a_i - a_j + 1, \tag{23.81}$$

where again the indices are such that $\epsilon_{ijk}=1$. It is also seen that

$$\mathscr{H} = 4\mathscr{C}^{(4)} + 3/4, \tag{23.82}$$

if the condition $x_1^2+x_2^2+x_3^2=1$ is satisfied. This condition can be ensured in general. Indeed, it is verified that

$$S = 2J_0^{(4)} + J_+^{(4)} + J_-^{(4)} = x_1^2+x_2^2+x_3^2. \tag{23.83}$$

Since the operator S commutes with $\mathscr{C}^{(4)}$ and all the intermediate Casimir operators $\mathscr{C}^{(ij)}$, it is central in the algebra (23.68) and thus it can be considered as a constant. Consequently, one can take $S=1$ without loss of generality and this completes the identification of $\mathscr{H}$ with $\mathscr{C}^{(4)}$.

We have thus identified the full Casimir operator $\mathscr{C}^{(4)}$ of the combination of three $\mathfrak{su}(1,1)$ algebras with the Hamiltonian of the generic three-parameter superintegrable system on the two sphere and we have also identified the intermediate Casimir operators $\mathscr{C}^{(12)}$, $\mathscr{C}^{(23)}$ and $\mathscr{C}^{(31)}$ with the symmetries L_3, L_1, L_2, respectively, of this Hamiltonian. In view of the result (23.68), it follows that the symmetry algebra (23.6), (23.8) of the generic 3-parameter system on the two sphere coincides with the Racah-Wilson algebra (23.12) with structure parameters (23.69). We also note that the conditions for the $\nu_i = (k_i+1)/2$ to be positive are the same conditions for the Hamiltonian $\mathscr{H}$ to have normalizable solutions. Moreover, the spectrum found for the full Casimir operator $\mathscr{C}^{(4)}$ yields for the energies (eigenvalues) of the Hamiltonian

$$E_N = 4\nu_4(\nu_4-1)+\frac{3}{4} = \left[2(N+1)+k_1+k_2+k_3\right]^2 - \frac{1}{4}, \tag{23.84}$$

where N is a non-negative integer. This $(N+1)$-fold degenerate spectrum coincides, as should be, with the one obtained in [17] for the spectrum of the Hamiltonian of the generic 3-parameter system.

23.5 Conclusion

In this paper, it has been shown that the analysis of the most general second-order superintegrable system in two dimensions, i.e. the generic 3-parameter system on the 2-sphere, is equivalent to the Racah problem for the positive-discrete series of unitary representations of the Lie algebra $\mathfrak{su}(1,1)$. This correspondence establishes that the symmetry algebra (23.6), (23.8) of the generic 3-parameter system is isomorphic to the reduced Racah-Wilson algebra (23.12). Since the representations of the Racah-Wilson algebra are related to the Racah and Wilson polynomials, this provides an explanation for the connection between the Racah polynomials and the superintegrable 3-parameter system on the 2-sphere.

It has been shown that the Racah-Wilson algebra defining relations can also be realized [5] by taking K_1, K_2 and K_3 as quadratic expressions in the generators (in the equitable presentation) of one $\mathfrak{su}(2)$ algebra. It is relevant to understand how this construction pertains to the relation between the Racah-Wilson algebra and the composition of three $\mathfrak{su}(1,1)$ representations. In differential operator terms, this asks the question of how can one pass from a three- to a one-variable model. This will be explained in a forthcoming publication [7]. Somewhat related would be the algebraic description of the tridiagonalization of ordinary and basic hypergeometric operators [16]. Finally, it is of considerable interest to pursue the analysis of superintegrable models in 3 dimensions along the lines of the present paper. The relation between the generic model on the 3-sphere and Racah/Wilson polynomials in two variables has already been established [21]. It is quite clear that these polynomials should correspond to the $9j$ symbol of $\mathfrak{su}(1,1)$ and that the underlying algebra should describe the symmetries. This analysis should lift the veil on the study and standardization of polynomial algebras of "rank two" associated to bivariate orthogonal polynomials and superintegrable models in three dimensions. We plan to report on these questions in the near future.

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Chapitre 24

The equitable Racah algebra from three $\mathfrak{su}(1,1)$ algebras

V. X. Genest, L. Vinet et A. Zhedanov (2014). The equitable Racah algebra from three $\mathfrak{su}(1,1)$ algebras. *Journal of Physics A: Mathematical and Theoretical* **47** 025203.

Abstract. The Racah algebra, a quadratic algebra with two independent generators, is central in the analysis of superintegrable models and encodes the properties of the Racah polynomials. It is the algebraic structure behind the $\mathfrak{su}(1,1)$ Racah problem as it is realized by the intermediate Casimir operators arising in the addition of three irreducible $\mathfrak{su}(1,1)$ representations. It has been shown that this Racah algebra can also be obtained from quadratic elements in the enveloping algebra of $\mathfrak{su}(2)$. The correspondence between these two realizations is here explained and made explicit.

24.1 Introduction

The Racah algebra, which connects superintegrable models to Racah polynomials [8, 14], is more and more understood to have a universal role [7]. The main objective of this paper is to show that the equitable presentation of the Racah algebra emerges when the addition of three $\mathfrak{su}(1,1)$ representations of the positive-discrete series is considered and furthermore, to establish the relation that this framework has with the one in which the equitable presentation is obtained from quadratic elements in the universal enveloping algebra of $\mathfrak{su}(2)$ [4]. This will be done using Bargmann realizations by reducing the 3-variable model of the 3-summand $\mathfrak{su}(1,1)$ representation to the 1-variable realization of the Racah algebra stemming from the standard representation of $\mathfrak{su}(2)$ on holomorphic functions.

24.1.1 Racah algebra

The Racah algebra is the most general quadratic algebra with two algebraically independent generators, say A and B, which possesses representations with ladder relations [10]. Upon introducing an additional generator C defined by

$$[A,B] = C,$$

where $[x, y] = xy - yx$, the Racah algebra is characterized by the commutation relations

$$[B,C] = B^2 + \{A,B\} + dB + e_1, \tag{24.1a}$$

$$[C,A] = A^2 + \{A,B\} + dA + e_2, \tag{24.1b}$$

where $\{x, y\} = xy + yx$. The Casimir operator, which commutes with all the generators of the Racah algebra, has the expression [9]

$$Q = \{A^2,B\} + \{A,B^2\} + A^2 + B^2 + C^2 + (d+1)\{A,B\} + (2e_1 + d)A + (2e_2 + d)B. \tag{24.2}$$

In the realization that we shall be using, the parameters d, e_1, e_2 are expressed in terms of four real parameters λ_i, $i = 1,\ldots,4$, as follows:

$$d = (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)/2, \quad e_1 = (\lambda_1 - \lambda_4)(\lambda_2 - \lambda_3)/4, \quad e_2 = (\lambda_1 - \lambda_2)(\lambda_4 - \lambda_3)/4.$$

Note that the Racah algebra (24.1) is invariant under the duality transformation $A \leftrightarrow B$, $e_1 \leftrightarrow e_2$.

The Racah algebra is intimately related to the Racah polynomials [7, 9, 10], which sit atop the discrete part of the Askey scheme of hypergeometric orthogonal polynomials [16]. This relation emerges, on the one hand, from the representation theory of the Racah algebra. Indeed, finite-dimensional irreducible representations of (24.1) can be obtained in bases where either A or B is represented by a diagonal matrix. In these representations, the non-diagonal generator is tridiagonal, which means that A and B realize a Leonard pair [19]. In this picture, the Racah polynomials arise as the expansion coefficients between the eigenbases respectively associated to the diagonalization of A and B. On the other hand, one can conversely arrive at the Racah algebra from the bispectrality properties of the Racah polynomials [7]. Indeed, upon identifying A with the recurrence operator (viewed as multiplication by the variable) and taking B as the difference operator of the Racah polynomials, it is checked that the defining relations (24.1) are satisfied with values of the algebra parameters related to those of the Racah polynomials.

Quite significantly, the Racah algebra has been found to be the symmetry algebra of the generic superintegrable 3-parameter system on the 2-sphere [7, 14]. This explains why the overlap coefficients between wavefunctions separated in different spherical coordinate systems are given in terms of Racah polynomials. Moreover, since it has been shown in [15] that all superintegrable systems in two dimensions with constants of motion of degree not higher than two in momenta are limits or specializations of the generic model on the 2-sphere, it follows that the symmetry algebras of these problems can all be obtained as special cases or contractions of the Racah algebra. Note that the Racah polynomials also arise in the interbasis expansion coefficients for the isotropic oscillator in the three-dimensional space of constant positive curvature [11].

Another manifestation of the Racah algebra is in the context of the Racah problem for both the $\mathfrak{su}(2)$ and the $\mathfrak{su}(1,1)$ Lie algebras [9]. The $\mathfrak{su}(1,1)$ case will be reviewed below. In considering the addition of 3 representations of $\mathfrak{su}(1,1)$ from the positive-discrete series, it shall seen that the intermediate Casimir operators associated to pairs of representations do satisfy the defining relations (24.1). In [7], this observation has been related to the determination of the symmetry algebra of the aforementioned superintegrable model on the 2-sphere.

24.1.2 Equitable presentation of the Racah algebra

It is possible to exhibit a $\mathbb{Z}_3$-symmetric or equitable presentation of the Racah algebra [4, 13]. To that end, one first defines X, Y, Ω from A, B, C as follows:

$$X = -2A - \lambda_1, \qquad Y = -2B - \lambda_2, \qquad \Omega = 2C, \tag{24.3}$$

and also introduces a generator Z related to X and Y by the relation

$$X + Y + Z = \lambda_4. \tag{24.4}$$

It follows from (24.3), (24.4) and the definition of C that

$$[X, Y] = [Y, Z] = [Z, X] = 2\Omega.$$

Rewriting the relations (24.1) in terms of X, Y and Z, one easily obtains

$$[X,\Omega] = YX - XZ + (\lambda_1 - \lambda_2 + \lambda_3)Y - (\lambda_1 + \lambda_2 - \lambda_3)Z + f_1, \tag{24.5a}$$

$$[Y,\Omega] = ZY - YX + (\lambda_2 - \lambda_3 + \lambda_1)Z - (\lambda_2 + \lambda_3 - \lambda_1)X + f_2, \tag{24.5b}$$

$$[Z,\Omega] = XZ - ZY + (\lambda_3 - \lambda_1 + \lambda_2)X - (\lambda_3 + \lambda_1 - \lambda_2)Y + f_3, \tag{24.5c}$$

where the structure parameters f_1, f_2, f_3 have the expression

$$f_1 = \left[\lambda_1(\lambda_2 + \lambda_4) + \lambda_3(\lambda_2 - \lambda_4) - 2\lambda_1\lambda_3\right],$$
$$f_2 = \left[\lambda_2(\lambda_3 + \lambda_4) + \lambda_1(\lambda_3 - \lambda_4) - 2\lambda_2\lambda_1\right],$$
$$f_3 = \left[\lambda_3(\lambda_1 + \lambda_4) + \lambda_2(\lambda_1 - \lambda_4) - 2\lambda_3\lambda_2\right].$$

The commutations relations (24.5) are manifestly invariant under cyclic permutations of (X, Y, Z) and $(\lambda_1, \lambda_2, \lambda_3)$ and are referred to as the $\mathbb{Z}_3$-symmetric Racah relations. Given (24.4), they are obviously equivalent to (24.1). Recently, it has been shown that these equitable relations are realized by quadratic elements in $\mathscr{U}(\mathfrak{su}(2))$ [4]. It is the purpose of this paper to establish the correspondence between this last realization of X, Y, Z and the Casimir operators of the $\mathfrak{su}(1,1)$ Racah problem.

24.1.3 Outline

The outline of the paper is as follows. In Section 2, we review the Racah problem for $\mathfrak{su}(1,1)$. We introduce the intermediate Casimir operators, record the relation between them and recall the definition of $6j$-symbols as coefficients between eigenbases corresponding to the diagonalization of two intermediate Casimir operators. In Section 3, we show that the intermediate Casimir operators associated to the $\mathfrak{su}(1,1)$ Racah problem realize the Racah algebra. In Section 4, we consider the Racah problem in the Bargmann picture and show how it reduces to the determination of the overlap coefficients between solutions to pairs of hypergeometric Sturm-Liouville problems. The connection with the realization of the Racah algebra in terms of 3 (linearly related) quadratic elements in $\mathscr{U}(\mathfrak{su}(2))$ is then completed in Section 5 where it is shown that the reduction of the intermediate Casimir operators to hypergeometric Sturm–Liouville operators allows an identification with the quadratic elements in the Bargmann realizations of $\mathfrak{su}(2)$.

24.2 The $\mathfrak{su}(1,1)$ Racah problem

In this section, the Racah problem for the positive-discrete series of irreducible representations of $\mathfrak{su}(1,1)$ is reviewed. The Bargmann realization of these representations is given and the definition of the $6j$-symbols of the algebra in terms of overlap coefficients between eigenbases associated to intermediate Casimir operators is provided.

24.2.1 Positive-discrete series representations and Bargmann realization of $\mathfrak{su}(1,1)$

The $\mathfrak{su}(1,1)$ algebra has three generators $K_{\pm}$, K_0 as its basis elements. These generators obey the commutation relations

$$[K_0, K_{\pm}] = \pm K_{\pm}, \qquad [K_-, K_+] = 2K_0. \tag{24.6}$$

The Casimir operator $\mathscr{Q}$, which commutes with all $\mathfrak{su}(1,1)$ elements, is given by

$$\mathscr{Q} = K_0^2 - K_0 - K_+K_-. \tag{24.7}$$

We shall here be concerned with irreducible representations of (24.6) belonging to the positive-discrete series. These representations are labeled by a positive number ν and can be defined by the following actions of the $\mathfrak{su}(1,1)$ generators on basis vectors e_n, $n \in \mathbb{N}$:

$$K_0 e_n = (n+\nu)e_n, \qquad K_- e_n = n e_{n-1}, \qquad K_+ e_n = (n+2\nu)e_{n+1}. \tag{24.8}$$

One can realize the positive-discrete representations on the space of holomorphic functions of a single variable x. In this realization, the $\mathfrak{su}(1,1)$ generators take the form [17]

$$K_0 = x\partial_x + \nu, \quad K_- = \partial_x, \quad K_+ = x^2\partial_x + 2\nu x, \tag{24.9}$$

and the Casimir operator is a multiple of the identity

$$\mathscr{Q} = \nu(\nu-1).$$

It is easily seen that on the monomial basis

$$e_n(x) = x^n, \tag{24.10}$$

with $n \in \mathbb{N}$, the actions (24.8) are recovered.

24.2.2 Addition schemes for three $\mathfrak{su}(1,1)$ algebras

Consider three mutually commuting sets[1] of $\mathfrak{su}(1,1)$ generators $\mathbf{K}^{(i)} = \{K_0^{(i)}, K_\pm^{(i)}\}$, $i = 1,2,3$, with $[\mathbf{K}^{(i)}, \mathbf{K}^{(j)}] = 0$ for $i \neq j$. These sets of generators can be combined by addition to form four additional ones $\mathbf{K}^{(12)}$, $\mathbf{K}^{(23)}$, $\mathbf{K}^{(31)}$ and $\mathbf{K}^{(4)}$ defined by

$$\mathbf{K}^{(ij)} = \{K_0^{(ij)} \equiv K_0^{(i)} + K_0^{(j)}, K_\pm^{(ij)} \equiv K_\pm^{(i)} + K_\pm^{(j)}\}, \tag{24.11}$$

and

$$\mathbf{K}^{(4)} = \{K_0^{(4)} \equiv K_0^{(1)} + K_0^{(2)} + K_0^{(3)}, K_\pm^{(4)} \equiv K_\pm^{(1)} + K_\pm^{(2)} + K_\pm^{(3)}\}. \tag{24.12}$$

The Casimir operators associated to (24.11) and (24.12) have the expressions:

$$\mathscr{Q}^{(ij)} = [K_0^{(ij)}]^2 - K_0^{(ij)} - K_+^{(ij)}K_-^{(ij)}, \tag{24.13a}$$

$$\mathscr{Q}^{(4)} = [K_0^{(4)}]^2 - K_0^{(4)} - K_+^{(4)}K_-^{(4)}. \tag{24.13b}$$

The Casimir operators (24.13a) will be referred to as the "intermediate Casimirs" whereas the operator (24.13b) will be referred to as the "full Casimir". The intermediate Casimir operators

[1] Here the symbol {} for sets should not be confused with the anticommutator.

$\mathcal{Q}^{(ij)}$ and the full Casimir operator $\mathcal{Q}^{(4)}$ are not independent. Indeed, an elementary calculation shows that

$$\mathcal{Q}^{(4)} = \mathcal{Q}^{(12)} + \mathcal{Q}^{(23)} + \mathcal{Q}^{(31)} - \mathcal{Q}^{(1)} - \mathcal{Q}^{(2)} - \mathcal{Q}^{(3)}, \tag{24.14}$$

where $\mathcal{Q}^{(j)}$, $j = 1,2,3$, are the Casimir operators (24.7) associated to each set $\mathbf{K}^{(i)}$. As is easily verified, the Casimir operators (24.13a) commute with the Casimir operators $\mathcal{Q}^{(i)}$ and with the total Casimir operator $\mathcal{Q}^{(4)}$, but do not commute amongst themselves. The full Casimir operator $\mathcal{Q}^{(4)}$ commutes with both the intermediate Casimirs $\mathcal{Q}^{(ij)}$ and with the individual Casimirs $\mathcal{Q}^{(i)}$.

24.2.3 $6j$-symbols

The $6j$-symbols, also known as the Racah coefficients, arise in the following situation. Consider three irreducible representations of the positive-discrete series labeled by the parameters ν_i, $i = 1,2,3$ associated to the eigenvalues $\nu_i(\nu_i - 1)$ of the individual Casimir operators $\mathcal{Q}^{(i)}$. In this case, the representation parameters ν_{ij} associated to the eigenvalues $\nu_{ij}(\nu_{ij} - 1)$ of the intermediate Casimir operators $\mathcal{Q}^{(ij)}$ have the form $\nu_{ij} = \nu_i + \nu_j + n_{ij}$, where the n_{ij} are non-negative integers. Furthermore, the possible values for the representation parameter ν_4 associated to the eigenvalues $\nu_4(\nu_4 - 1)$ of the full Casimir operator $\mathcal{Q}^{(4)}$ are given by $\nu_4 = \nu_{12} + \nu_3 + \ell = \nu_1 + \nu_{23} + m = \nu_1 + \nu_2 + \nu_3 + k$, where m, ℓ and k are non-negative integers. For details, the reader can consult [3, 18].

For a given value of the total Casimir parameter ν_4, one has a finite-dimensional space on which the pair of (non-commuting) operators $\mathcal{Q}^{(12)}$, $\mathcal{Q}^{(23)}$ act. Each of these operators has a set of eigenvectors $\{\phi_{n_{12}}\}_{n_{12}=0}^{K}$, $\{\chi_{n_{23}}\}_{n_{23}=0}^{K}$ such that

$$\mathcal{Q}^{(12)}\phi_{n_{12}} = \nu_{12}(\nu_{12} - 1)\phi_{n_{12}}, \qquad \mathcal{Q}^{(23)}\chi_{n_{23}} = \nu_{23}(\nu_{23} - 1)\chi_{n_{23}}, \tag{24.15}$$

where $\nu_{12} = \nu_1 + \nu_2 + n_{12}$ and $\nu_{23} = \nu_1 + \nu_2 + n_{23}$. Both sets of basis vectors $\{\phi_{n_{12}}\}$, $\{\chi_{n_{23}}\}$ are eigenvectors of the Casimir operators $\mathcal{Q}^{(i)}$, with $i = 1,\ldots,4$. The $6j$-symbols are the overlap coefficients $W_{n_{12},n_{23}}$ between the two bases

$$\phi_{n_{12}} = \sum_{n_{23}=0}^{K} W_{n_{12},n_{23}}\,\chi_{n_{23}}. \tag{24.16}$$

The dimension $K+1$ of the space can be evaluated straightforwardly in terms of the representation parameters ν_i, $i = 1,\ldots,4$. If $\nu_4 = \nu_1 + \nu_2 + \nu_3 + M$, then it follows from the above considerations that ν_{12} can take the $M+1$ possible values $\nu_{12} \in \{\nu_1 + \nu_2, \nu_1 + \nu_2 + 1, \ldots, \nu_1 + \nu_2 + M\}$ while ν_{23} can take the $M+1$ values $\nu_{23} \in \{\nu_2 + \nu_3, \nu_2 + \nu_3 + 1, \ldots, \nu_2 + \nu_3 + M\}$. Thus, for a fixed value ν_4, the dimension $K+1$ of the space is determined by the value $K = \nu_4 - \nu_1 - \nu_2 - \nu_3$ with $K \in \mathbb{N}$. Note that one can also consider (non-standard) $6j$-symbols for the pairs of operators $\mathcal{Q}^{(23)}$, $\mathcal{Q}^{(31)}$ and $\mathcal{Q}^{(12)}$, $\mathcal{Q}^{(31)}$.

24.3 The Racah algebra and the Racah problem

In this section, it is shown that the Racah algebra is behind the Racah problem for $\mathfrak{su}(1,1)$. This result follows from the determination of the commutation relations satisfied by the intermediate Casimir operators of the Racah problem. The generators satisfying the $\mathbb{Z}_3$-symmetric Racah relations are also exhibited.

24.3.1 Racah Algebra

Let A, B be expressed as follows in terms the intermediate Casimir operators:

$$A = -\mathcal{Q}^{(12)}/2, \qquad B = -\mathcal{Q}^{(23)}/2, \tag{24.17}$$

and define $C = [A,B]$. In the Racah problem, the Casimir operators $\mathcal{Q}^{(i)}$, $i = 1,\ldots,4$ act as multiples of the identity and hence they can be replaced by constants $Q^{(i)} = \lambda_i$, where $\lambda_i = \nu_i(\nu_i - 1)$. A direct computation shows that the operators A, B, together with their commutator C, satisfy the defining relations of the Racah algebra

$$[A,B] = C, \tag{24.18a}$$
$$[B,C] = B^2 + \{A,B\} + \delta B + \epsilon_1, \tag{24.18b}$$
$$[C,A] = A^2 + \{A,B\} + \delta A + \epsilon_2, \tag{24.18c}$$

where

$$\delta = (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)/2, \qquad \epsilon_1 = (\lambda_1 - \lambda_4)(\lambda_2 - \lambda_3)/4, \qquad \epsilon_2 = (\lambda_1 - \lambda_2)(\lambda_4 - \lambda_3)/4. \tag{24.19}$$

The commutation relations (24.18) are most easily verified using the Bargmann realization (24.9) in three variables x, y and z but hold regardless of the representation. It is seen that the relations (24.18) are exactly the defining relations (24.1) of the Racah algebra. In the realization (24.17), it is directly checked that the Casimir operator (24.2) takes the value

$$Q = \frac{1}{4}\left[(\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4)(\lambda_1\lambda_3 - \lambda_2\lambda_4) - \lambda_1\lambda_2 - \lambda_2\lambda_3 - \lambda_3\lambda_4 - \lambda_4\lambda_1\right]. \tag{24.20}$$

It is easy to see that any pair of intermediate Casimir operators $(\mathcal{Q}^{(ij)}, \mathcal{Q}^{(k\ell)})$ will satisfy the relations (24.18). Therefore the intermediate Casimir operators $(\mathcal{Q}^{(12)}, \mathcal{Q}^{(23)}, \mathcal{Q}^{(31)})$ realize a *Leonard Triple* [2].

24.3.2 $\mathbb{Z}_3$-symmetric presentation of the Racah problem

Let X, Y, Z be defined as follows in terms of the intermediate Casimir operators of the $\mathfrak{su}(1,1)$ Racah problem:

$$X = \mathcal{Q}^{(12)} - \mathcal{Q}^{(1)}, \qquad Y = \mathcal{Q}^{(23)} - \mathcal{Q}^{(2)}, \qquad Z = \mathcal{Q}^{(31)} - \mathcal{Q}^{(3)}. \tag{24.21}$$

In view of (24.14), one has

$$X + Y + Z = \mathcal{Q}^{(4)} = \lambda_4.$$

Upon comparing (24.21) with (24.3), it follows that the operators (24.21) obey the $\mathbb{Z}_3$-symmetric Racah relations. It is also seen that the value of the Casimir operator for the Racah algebra (24.20) also possesses this symmetry. It is easy to understand the origin of the $\mathbb{Z}_3$ symmetry in this context: it corresponds to the $\mathbb{Z}_3$ freedom in permuting the three $\mathfrak{su}(1,1)$ representations $\mathbf{K}^{(i)}$. Hence the Racah problem for $\mathfrak{su}(1,1)$ is intrinsically $\mathbb{Z}_3$-symmetric. We shall now consider the Racah problem in the Bargmann picture.

24.4 Sturm–Liouville model for the Racah algebra

In this section, the Racah problem for $\mathfrak{su}(1,1)$ is considered in the Bargmann representation. It is shown that in this picture, the determination of the Racah coefficients is equivalent to obtaining the overlap coefficients between solutions to pairs of hypergeometric Sturm–Liouville problems. This gives a realization of the Racah algebra in terms of differential operators of a single variable.

24.4.1 Racah problem in the Bargmann picture

Consider the problem of determining the Racah coefficients as defined by the set of equations (24.15), (24.16). It is clear from this definition that one can arbitrarily choose the value of the projection operator $K_0^{(4)}$ on the involved basis vectors. Let ψ be an eigenvector of every Casimir operator $\mathcal{Q}^{(i)}$, $i = 1, \ldots, 4$, with the minimal value of this projection. In view of (24.8), this is means that

$$K_0^{(4)}\psi = (\nu_1 + \nu_2 + \nu_3 + M)\psi, \qquad K_-^{(4)}\psi = (K_-^{(1)} + K_-^{(2)} + K_-^{(3)})\psi = 0, \tag{24.22}$$

where M is a non-negative integer. In the Bargmann realization (24.9), $\psi = \psi(x, y, z)$ and it is seen from (24.10) and (24.22) that $\psi(x, y, z)$ can be expressed as a polynomial in the variables x, y, z of total degree M. The conditions (24.22) translate into

$$(x\partial_x + y\partial_y + z\partial_z)\psi(x, y, z) = M\psi(x, y, z), \qquad (\partial_x + \partial_y + \partial_z)\psi(x, y, z) = 0. \tag{24.23}$$

By Euler's homogeneous function theorem, the first condition of (24.23) implies that $\psi(x,y,z)$ is homogeneous of degree M, which means that

$$\psi(\alpha x,\alpha y,\alpha z)=\alpha^M\psi(x,y,z).$$

The second condition of (24.23) implies that $\psi(x,y,z)$ can only depend on the relative variables $(x-y)$ and $(z-y)$. Hence it follows that the most general expression for $\psi(x,y,z)$ is

$$\psi(x,y,z)=(z-y)^M\Phi\left(\frac{x-y}{z-y}\right),$$

where $\Phi(u)$ is a polynomial in u of maximal degree M. It is seen that the action of the intermediate Casimir operators is given by

$$\mathscr{Q}^{(12)}\psi(x,y,z)=(z-y)^M\mathscr{S}_{12}\Phi(u),\qquad \mathscr{Q}^{(23)}\psi(x,y,z)=(z-y)^M\mathscr{S}_{23}\Phi(u), \tag{24.24a}$$

$$\mathscr{Q}^{(31)}\psi(x,y,z)=(z-y)^M\mathscr{S}_{31}\Phi(u), \tag{24.24b}$$

where the one-variable operators $\mathscr{S}_{ij}$ are given by

$$\begin{aligned}
\mathscr{S}_{12}&=u^2(1-u)\partial_u^2+u\left[(M-1-2\nu_1)u+2(\nu_1+\nu_2)\right]\partial_u+2M\nu_1u+(\nu_1+\nu_2)(\nu_1+\nu_2-1),\\
\mathscr{S}_{23}&=u(u-1)\partial_u^2+\left[2(1-M-\nu_2-\nu_3)u+(M-1+2\nu_3)\right]\partial_u+(M+\nu_2+\nu_3)(M+\nu_2+\nu_3-1),\\
\mathscr{S}_{31}&=u(u-1)^2\partial_u^2+(1-u)\left[(M-1-2\nu_1)u+1-M-2\nu_3\right]\partial_u\\
&\qquad+2M\nu_1(1-u)+(\nu_1+\nu_3)(\nu_1+\nu_3-1).
\end{aligned} \tag{24.25}$$

It is elementary to verify that the operators $\mathscr{S}_{ij}$ preserve the space of polynomials of maximal degree M. The operator $\mathscr{S}_{23}$ is a standard hypergeometric operator while $\mathscr{S}_{12}$ and $\mathscr{S}_{13}$ can be reduced to hypergeometric operators by appropriate changes of variables.

Returning to the Racah problem for the intermediate Casimir operators $\mathscr{Q}_{12}$ and $\mathscr{Q}_{23}$, it follows from the above that the equations (24.15) are equivalent to the pair of Sturm–Liouville problems

$$\mathscr{S}_{12}\Phi^{(12)}(u)=\nu_{12}(\nu_{12}-1)\Phi^{(12)}(u),\qquad \mathscr{S}_{23}\Phi^{(23)}(u)=\nu_{23}(\nu_{23}-1)\Phi^{(23)}(u), \tag{24.26}$$

where $\Phi^{(12)}(u)$ and $\Phi^{(23)}(u)$ are required to be polynomials in u of degree not higher than M. In this picture, the Racah decomposition (24.16) becomes

$$\Phi^{(12)}(u)=\sum_{n_{23}=0}^{M}W_{n_{12},n_{23}}\Phi^{(23)}(u),$$

where it is assumed that $\nu_{12} = n_{12} + \nu_1 + \nu_2$ and $\nu_{23} = n_{23} + \nu_2 + \nu_3$. The explicit solutions to the Sturm–Liouville equations (24.26) can be found in terms of Gauss hypergeometric functions. Indeed, consider the eigenvalue equation

$$\mathscr{S}_{12}\Phi^{(12)}(u) = \nu_{12}(\nu_{12}-1)\Phi^{(12)}(u),$$

with $\nu_{12} = n_{12} + \nu_1 + \nu_2$. Then using (24.25), it is directly verified that the polynomial solutions for $\Phi^{(12)}(u)$, up to an inessential constant factor, are given by

$$\Phi^{(12)}(u) = u^{n_{12}}\,{}_2F_1\left[\begin{matrix} n_{12}-M, n_{12}+2\nu_1 \\ 2n_{12}+2\nu_1+2\nu_2 \end{matrix}; u\right],$$

where ${}_2F_1$ is the Gauss hypergeometric function

$${}_2F_1\left[\begin{matrix} a,b \\ c \end{matrix}; z\right] = \sum_{i=0}^{\infty} \frac{(a)_i(b)_i}{(c)_i}\frac{z^i}{i!}, \tag{24.27}$$

and where $(a)_i = (a)(a+1)\cdots(a+i-1)$ stands for the Pochhammer symbol. Using (24.27), the solution for $\Phi^{(12)}(u)$ can also be presented in the form

$$\Phi^{(12)}(u) = u^M\,{}_2F_1\left[\begin{matrix} n_{12}-M, 1-M-n_{12}-2\nu_1-2\nu_2 \\ 1-M-2\nu_1 \end{matrix}; \frac{1}{u}\right].$$

Proceeding similarly for $\Phi^{(23)}(u)$ and $\Phi^{(31)}(u)$, one finds that

$$\Phi^{(23)}(u) = {}_2F_1\left[\begin{matrix} n_{23}-M, 1-M-n_{23}-2\nu_2-2\nu_3 \\ 1-M-2\nu_3 \end{matrix}; u\right],$$
$$\Phi^{(31)}(u) = (1-u)^M\,{}_2F_1\left[\begin{matrix} n_{31}-M, 1-M-n_{31}-2\nu_3-2\nu_1 \\ 1-M-2\nu_1 \end{matrix}; \frac{1}{1-u}\right],$$

where $n_{31} = \nu_{31} - \nu_1 - \nu_3$. Thus, in the Bargmann picture, the Racah coefficients $W_{n_{12},n_{23}}$ occur as overlap coefficients between the solutions of a pair of Sturm–Liouville problems.

24.4.2 One-variable realization of the Racah algebra and equitable presentation

The reduction from a three-variable model to a one-variable model for the Racah problem in the Bargmann picture can be used to exhibit a one-variable realization of the Racah algebra and its equitable presentation. Indeed, it is directly checked that the one-variable operators

$$\kappa_1 = -\mathscr{S}_{12}/2, \qquad \kappa_2 = -\mathscr{S}_{23}/2,$$

together with their commutator $\kappa_3 = [\kappa_1, \kappa_2]$, realize the Racah algebra (24.18) under the identification $\kappa_1 = A$, $\kappa_2 = B$. Furthermore, since $M = \nu_4 - \nu_1 - \nu_2 - \nu_3$, one sees that the relation

$$\mathscr{S}_{12} + \mathscr{S}_{23} + \mathscr{S}_{31} = \nu_4(\nu_4 - 1) + \nu_3(\nu_3 - 1) + \nu_2(\nu_2 - 1) + \nu_1(\nu_1 - 1),$$

holds and one finds that the operators

$$X = \mathscr{S}_{12} - \nu_1(\nu_1 - 1), \qquad Y = \mathscr{S}_{23} - \nu_2(\nu_2 - 1), \qquad Z = \mathscr{S}_{31} - \nu_3(\nu_3 - 1), \tag{24.28}$$

are related by $X + Y + Z = \lambda_4$ and satisfy the $\mathbb{Z}_3$-symmetric Racah relations (24.5) with $\lambda_i = \nu_i(\nu_i - 1)$.

24.5 The Racah algebra and the equitable $\mathfrak{su}(2)$ algebra

In the previous section, the reduction from a three-variable to a one-variable model for the Racah problem was performed and led to a one-variable realization of the Racah algebra. In this section, another interpretation of the operators $\mathscr{S}_{ij}$ in terms of elements in the enveloping algebra of $\mathfrak{su}(2)$ algebra is presented. Using this interpretation, the finite-dimensional irreducible representations of $\mathfrak{su}(2)$ are used to define irreducible representations of the Racah algebra.

24.5.1 Equitable presentation of the $\mathfrak{su}(2)$ algebra

The $\mathfrak{su}(2)$ algebra consists of three generators J_0, $J_\pm$ satisfying the commutation relations

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = 2J_0.$$

The Casimir operator for $\mathfrak{su}(2)$, denoted Δ, is given by

$$\Delta = J_0^2 - J_0 + J_+ J_- \tag{24.29}$$

All unitary irreducible representations of $\mathfrak{su}(2)$ are finite-dimensional. In these representations of dimension $2j+1$, the Casimir operator takes the value $j(j+1)$ with $j \in \{0, 1/2, 1, 3/2, \dots\}$. The $\mathfrak{su}(2)$ algebra has the Bargmann realization

$$J_- = -\partial_u, \qquad J_+ = u^2 \partial_u - 2ju, \qquad J_0 = u\partial_u - j. \tag{24.30}$$

In the realization (24.30), one has $\Delta = j(j+1)$ for the Casimir operator. There exists another presentation of the $\mathfrak{su}(2)$ algebra known as the *equitable* presentation [12]. The equitable basis is defined by

$$E_1 = 2(J_+ - J_0), \qquad E_2 = -2(J_- + J_0), \qquad E_3 = 2J_0. \tag{24.31}$$

in terms of which the commutation relations read

$$[E_i, E_j] = 2(E_i + E_j),$$

where $(ij) \in \{(12),(23),(31)\}$.

24.5.2 Equitable Racah operators from equitable $\mathfrak{su}(2)$ generators

Let us explain how the equitable Racah relations can be realized with quadratic elements in the $\mathfrak{su}(2)$ algebra; this observation has been made in [4]. We have already seen in (24.28) that the operators X, Y, Z of the $\mathbb{Z}_3$-symmetric presentation of the Racah algebra (24.5) can be realized by one-variable differential operators. Let $\mathscr{G}_i$, $i = 1,2,3$, be the following quadratic elements in the equitable $\mathfrak{su}(2)$ generators:

$$\begin{aligned}\mathscr{G}_1 = -\frac{1}{8}\{E_1,E_3\} + \frac{\nu_2-\nu_1}{2}(E_3+E_1) \\ + \frac{1-M-2\nu_1-2\nu_2}{4}(E_1-E_3) + \frac{(M+2\nu_2)(M+4\nu_1+2\nu_2-2)}{4},\end{aligned} \tag{24.32a}$$

$$\begin{aligned}\mathscr{G}_2 = -\frac{1}{8}\{E_2,E_3\} + \frac{\nu_3-\nu_2}{2}(E_2+E_3) \\ + \frac{1-M-2\nu_2-2\nu_3}{4}(E_3-E_2) + \frac{(M+2\nu_3)(M+4\nu_2+2\nu_3-2)}{4},\end{aligned} \tag{24.32b}$$

$$\begin{aligned}\mathscr{G}_3 = -\frac{1}{8}\{E_1,E_2\} + \frac{\nu_1-\nu_3}{2}(E_1+E_2) \\ + \frac{1-M-2\nu_1-2\nu_3}{4}(E_2-E_1) + \frac{(M+2\nu_1)(M+4\nu_3+2\nu_1-2)}{4}.\end{aligned} \tag{24.32c}$$

Then one has $\mathscr{G}_1+\mathscr{G}_2+\mathscr{G}_3 = \lambda_4 = \nu_4(\nu_4-1)$ when $M = \nu_4-\nu_1-\nu_2-\nu_3$. When the Bargmann realization (24.30) is used with $j = M/2$, the operators $\mathscr{G}_i$ are identified with the one-variable realizations $\mathscr{S}_{ij}$ of the intermediate Casimir operators through

$$\mathscr{G}_1 = \mathscr{S}_{12} - \nu_1(\nu_1-1) = X, \tag{24.33}$$

$$\mathscr{G}_2 = \mathscr{S}_{23} - \nu_2(\nu_2-1) = Y, \tag{24.34}$$

$$\mathscr{G}_3 = \mathscr{S}_{31} - \nu_3(\nu_3-1) = Z. \tag{24.35}$$

Hence the quadratic elements $\mathscr{G}_i$ in the $\mathfrak{su}(2)$ generators realize the $\mathbb{Z}_3$-symmetric Racah relations (24.5).

24.5.3 Racah algebra representations from $\mathfrak{su}(2)$ modules

The standard basis for the irreducible representations of $\mathfrak{su}(2)$ and the realization (24.32) of the Racah algebra can be used to construct finite-dimensional representations of the Racah algebra. Let e_n, $n = 0, 1, \ldots, M$, denote the canonical basis vectors for the $M+1$-dimensional irreducible representations of $\mathfrak{su}(2)$. These representations are defined by the actions

$$J_0 e_n = (n - M/2)e_n, \qquad J_+ e_n = \sqrt{(n+1)(M-n)}e_{n+1}, \tag{24.36}$$

$$J_- e_n = \sqrt{n(M-n+1)}e_{n-1}. \tag{24.37}$$

In this basis, the equitable generators (24.31) act in the following way:

$$E_1 e_n = (M-2n)e_n + 2\sqrt{(n+1)(M-n)}e_{n+1}, \tag{24.38a}$$

$$E_2 e_n = -2\sqrt{n(M-n+1)}e_{n-1} + (M-2n)e_n, \tag{24.38b}$$

$$E_3 e_n = (2n-M)e_n. \tag{24.38c}$$

Let A and B be defined as

$$A = -\frac{1}{2}\mathscr{G}_1 - \nu_1(\nu_1 - 1)/2, \qquad B = -\frac{1}{2}\mathscr{G}_2 - \nu_2(\nu_2 - 1)/2, \tag{24.39}$$

where $\mathscr{G}_1$, $\mathscr{G}_2$ are as in (24.32). It follows from (24.33) that the operators A and B realize the Racah algebra (24.18) with $\lambda_i = \nu_i(\nu_i - 1)$ and $\nu_4 = M + \nu_1 + \nu_2 + \nu_3$. A direct computation using (24.38) shows that in the basis e_n, the operators A and B have the actions

$$A\, e_n = \lambda_n^{(A)} e_n + \frac{1}{2}(n + 2\nu_1)\sqrt{(n+1)(M-n)}\, e_{n+1}, \tag{24.40a}$$

$$B\, e_n = \lambda_n^{(B)} e_n + \frac{1}{2}(M - n + 2\nu_3)\sqrt{n(M-n+1)}\, e_{n-1}, \tag{24.40b}$$

where

$$\lambda_n^{(A)} = -(n + \nu_1 + \nu_2)(n + \nu_1 + \nu_2 - 1)/2, \tag{24.41a}$$

$$\lambda_n^{(B)} = -(M - n + \nu_2 + \nu_3)(M - n + \nu_2 + \nu_3 - 1)/2. \tag{24.41b}$$

Since A and B act in a bidiagonal fashion, the expression (24.41) are the eigenvalues of A and B in this representation. In the generic case, the $(M+1)$-dimensional representations of the Racah algebra defined by (24.40) are clearly irreducible. It is convenient at this point to introduce another basis spanned by the basis vectors $\widetilde{e}_n$ which are defined by

$$e_n = \sqrt{\frac{(-1)^n}{n!(-M)_n}\frac{2^n}{(2\nu_1)_n}}\widetilde{e}_n. \tag{24.42}$$

On the basis vectors $\widetilde{e}_n$, the actions (24.40) are

$$A\widetilde{e}_n = \lambda_n^{(A)}\widetilde{e}_n + \widetilde{e}_{n+1}, \tag{24.43a}$$

$$B\widetilde{e}_n = \lambda_n^{(B)}\widetilde{e}_n + \varphi_n\widetilde{e}_{n-1}, \tag{24.43b}$$

where

$$\varphi_n = n(M-n+1)(n+2\nu_1-1)(M-n+2\nu_3)/4. \tag{24.44}$$

From (24.43), it is seen that the basis spanned by the vectors $\widetilde{e}_n$ corresponds to the UD-LD basis for Leonard pairs studied by Terwilliger in [20]. See also [1] for realizations of Leonard pairs using the equitable generators of $\mathfrak{sl}_2$.

24.6 Conclusion

In this paper, we have established the correspondence between two frameworks for the realization of the Racah algebra: the one in which the Racah algebra is realized by the intermediate Casimir operators arising in the combination of three $\mathfrak{su}(1,1)$ representations of the positive-discrete series and the one where the Racah is realized in terms of quadratic elements in the enveloping algbera of $\mathfrak{su}(2)$. We have also exhibited how the $\mathbb{Z}_3$-symmetric, or equitable, presentation of the Racah algebra arises in the context of the Racah problem for $\mathfrak{su}(1,1)$.

In [5, 6], it was shown that the Bannai-Ito (BI) algebra is the algebraic structure behind the Racah problem for the $sl_{-1}(2)$ algebra and a $\mathbb{Z}_3$-symmetric presentation of the BI algebra was offered. In view of the results presented here, it would be of interest to perform the reduction of the number of variables in the $sl_{-1}(2)$ Racah problem to obtain a one-variable realization of the Bannai-Ito algebra and to identify in this case what is the algebraic structure that plays a role analogous to the one played here by $\mathfrak{su}(2)$.

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Chapitre 25

The Racah algebra and superintegrable models

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Abstract. The universal character of the Racah algebra will be illustrated by showing that it is at the center of the relations between the Racah polynomials, the recoupling of three $\mathfrak{su}(1,1)$ representations and the symmetries of the generic second-order superintegrable model on the 2-sphere.

25.1 Introduction

This paper offers a review of the central role that the Racah algebra plays in connection with superintegrable models [2].

25.1.1 Superintegrable models

A quantum system with d degrees of freedom described by a Hamiltonian H is maximally superintegrable (S.I.) if it possesses $2d-1$ algebraically independent constants of motion S_i (also called symmetries) such that:

$$[S_i, H] = 0, \qquad 1 \leq i \leq 2d-1, \tag{25.1}$$

where one of the symmetries is the Hamiltonian. Since the maximal number of symmetries that can be in involution is d, the constants of motion of a superintegrable system generate a non-Abelian algebra whose representations can in general be used to obtain an exact solution to the dynamical equations. A S.I. system is said to be of order ℓ if the maximal order of the symmetries in the momenta (apart from H) is ℓ. We shall be concerned here with second-order ($\ell = 2$) S.I. systems for which the Schrödinger equation is known to admit separation of variables and for which the symmetry algebras are quadratic.

S.I. systems, which include the classical examples of the isotropic harmonic oscillator and of the Coulomb-Kepler problem, are most interesting as models in applications and for pedagogical purposes. In particular, they form the bedrock for the analysis of symmetries and their description. Their study has helped to understand how Lie algebras, superalgebras, quantum algebras, polynomials algebras and algebras with involutions serve that purpose.

25.1.2 Second-order S.I. systems in 2D

The model that we shall focus on is the generic 3-parameter system on the 2-sphere. Its Hamiltonian is

$$H = J_1^2 + J_2^2 + J_3^2 + \frac{a_1}{x_1^2} + \frac{a_2}{x_2^2} + \frac{a_3}{x_3^2}, \qquad a_i = k_i^2 - 1/4, \tag{25.2}$$

where

$$x_1^2 + x_2^2 + x_3^2 = 1, \tag{25.3}$$

and where

$$J_1 = -i(x_2\partial_{x_3} - x_3\partial_{x_2}), \quad J_2 = -i(x_3\partial_{x_1} - x_1\partial_{x_3}), \quad J_3 = -i(x_1\partial_{x_2} - x_2\partial_{x_1}), \tag{25.4}$$

are the familiar angular momentum operators satisfying the commutation relations

$$[J_1, J_2] = iJ_3, \quad [J_2, J_3] = iJ_1, \quad [J_3, J_1] = iJ_2. \tag{25.5}$$

This 3-parameter system is 2^{nd} order superintegrable [9]. Most importantly, all 2^{nd} S.I. systems in 2D can be obtained from this model through specializations, limits and contractions [11]. The superintegrability of H is confirmed by checking that the two operators:

$$L_1 = J_1^2 + \frac{a_2 x_3^2}{x_2^2} + \frac{a_3 x_2^2}{x_3^2}, \quad L_2 = J_2^2 + \frac{a_3 x_1^2}{x_3^2} + \frac{a_1 x_3^2}{x_1^2}, \tag{25.6}$$

commute with H. Kalnins, Miller and Pogosyan [9] have examined the symmetry algebra and presented it as follows. With

$$L_3 = J_3^2 + \frac{a_1 x_2^2}{x_1^2} + \frac{a_2 x_1^2}{x_2^2}, \qquad R = [L_1, L_2], \tag{25.7}$$

one has $[H,L_3]=0$, $H=L_1+L_2+L_3+a_1+a_2+a_3$, and the relations (with (ijk) cyclic)

$$[L_i,R]=4\{L_i,L_j\}-4\{L_i,L_k\}-(8-16a_j)L_j+(8-16a_k)L_k+8(a_j-a_k), \tag{25.8a}$$

$$\begin{aligned} R^2=&-\frac{8}{3}\{L_1,L_2,L_3\}-\sum_{i=1}^{3}\left\{(12-16a_i)L_i^2+\frac{1}{3}(16-176a_i)L_i+\frac{32}{3}a_i\right\}\\ &+\frac{52}{3}(\{L_1,L_2\}+\{L_2,L_3\}+\{L_1,L_3\})+48(a_1a_2+a_2a_3+a_3a_1)\\ &-64a_1a_2a_3, \end{aligned} \tag{25.8b}$$

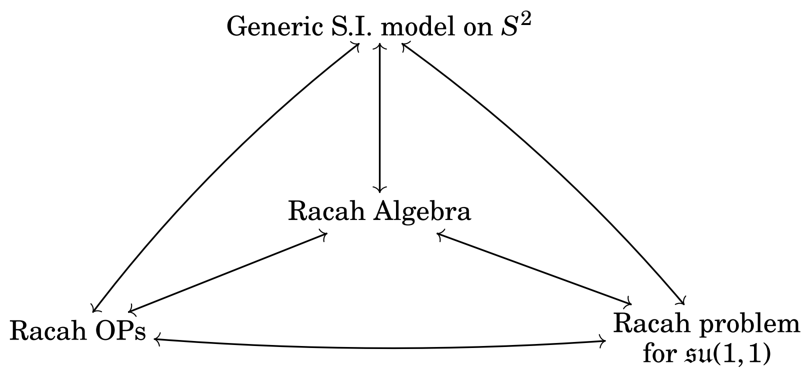
where $\{A,B\}=AB+BA$. Remarkably, Kalnins, Miller and Post [10] have further shown that the quadratic symmmetry algebra can be realized in terms of the difference operators associated to the Racah polynomials, that these same polynomials occur as transition coefficients between bases in which L_1 or L_2 is diagonal and furthermore that contractions of representations of the symmetry algebra lead to the symmetry algebras of the other 2$^{\text{nd}}$ order S.I. systems and other families of orthogonal polynomials [11]. Details about these Racah polynomials [12], denoted by $R_n(\lambda(x);\alpha,\beta,\gamma,\delta)$, will be given later. Suffice it to say for now that they are defined in terms of generalized hypergeometric functions, that they are of degree n in the variable $\lambda(x)=x(x+\gamma+\delta+1)$, that they obey a discrete/finite orthogonality relation and that they sit atop the discrete part of the Askey scheme of hypergeometric orthogonal polynomials [12].

25.1.3 Objectives

In reduced form, the (quadratic) Racah algebra has three generators K_1, K_2, K_3 and the defining relations

$$\begin{aligned} &[K_1,K_2]=K_3,\\ &[K_2,K_3]=K_2^2+\{K_1,K_2\}+dK_2+e_1,\\ &[K_3,K_1]=K_1^2+\{K_1,K_2\}+dK_1+e_2, \end{aligned} \tag{25.9}$$

where d, e_1 and e_2 are real parameters. The objectives of this paper are to show that this algebra has a universal character and intimately connects the generic second-order S.I. system on S^2, the Racah polynomials and the recoupling of three $\mathfrak{su}(1,1)$ representations. Schematically, the goal is to explain the links represented on the following diagram:



25.2 Warming up with a simple model

A key idea in our considerations is that 2[nd] order superintegrable models can be obtained by combining 1D models that are exactly solvable. This can be done in simple cases by straightforward constructions [14] and has also been realized in the R-matrix formalism [5, 13].

Consider the 2D isotropic singular oscillator

$$H = H_{x_1} + H_{x_2}, \tag{25.10}$$

where

$$H_{x_i} = -\frac{1}{2}\partial_{x_i}^2 + \frac{1}{2}\left(x_i^2 + \frac{a_i}{x_i^2}\right), \qquad a_i = k_i^2 - 1/4. \tag{25.11}$$

This is one of the four systems in the classification of second order S.I. systems in Euclidean space [1]. The spectrum of H is

$$E_N = N + (k_1 + k_2 + 1)/2, \qquad N = n_1 + n_2, \qquad n_i \in \mathbb{N}, \tag{25.12}$$

and has a $(N+1)$-fold degeneracy. It is well known that the associated Schrödinger equation separates in Cartesian and polar coordinates. To confirm that this system is maximally superintegrable, one needs to identify two independent constants of motion. This can be done using the $\mathfrak{su}(1,1)$ dynamical algebra of the one-dimensional components. Let

$$B_i^{\pm} = \frac{1}{2}\left[(x_i \mp \partial_{x_i})^2 - \frac{a_i}{x_i^2}\right], \quad i = 1, 2, \tag{25.13}$$

it is readily verified that these operators combine with H_{x_i} to realize the $\mathfrak{su}(1,1)$ algebra since

$$[J_0, J_{\pm}] = \pm J_{\pm}, \qquad [J_+, J_-] = -2J_0, \tag{25.14}$$

with

$$J_0 = H_{x_i}/2 \qquad \text{and} \qquad J_\pm = B_i^\pm/2.$$

In the positive-discrete series of $\mathfrak{su}(1,1)$ representations thus constructed, $B_i^\pm$ act as raising and lowering operators with respect to the eigenvalues of H_{x_i} labeled by n_i. The conserved quantities of the 2D Hamiltonian $H = H_{x_1} + H_{x_2}$ are readily obtained from this observation. Indeed, combining the raising operator for H_{x_1} with the lowering operator for H_{x_2} (or vice-versa) will give an operator that leaves the total energy E_N unchanged and that commutes with H. This is a straightforward generalization of the Schwinger construction for the isotropic harmonic oscillator. The superintegrability of H is thus made manifest by exhibiting the operators

$$C^+ = B_{x_1}^+ B_{x_2}^-, \qquad C^- = B_{x_1}^- B_{x_2}^+, \tag{25.15}$$

to which we conveniently add

$$D = H_{x_1} - H_{x_2}, \tag{25.16}$$

and by noting that

$$[H, C^\pm] = 0, \qquad [H, D] = 0. \tag{25.17}$$

Defining relations for the symmetry algebra formed by the operators $C^\pm$ and D are straightforwardly obtained [14]:

$$\begin{aligned} [D, C^\pm] &= \pm 4 C^\pm, \\ [C^-, C^+] &= D^3 + \alpha_1 D + \alpha_2, \end{aligned} \tag{25.18}$$

where

$$\alpha_1 = -H^2 - 2(k_1^2 + k_2^2 - 2), \qquad \alpha_2 = (2k_1^2 - 2k_2^2)H. \tag{25.19}$$

Since H is central, α_1 and α_2 can be treated as constants on eigenspaces of H. An algebraic solution of the problem is obtained by working out the appropriate representations of this algebra. A special case of (25.18) was found by Higgs in [7] as symmetry algebra of the Coulomb problem on S^2. By taking a different set of operators, it is possible to cast the relations (25.18) in a form that we would say is standard. Let

$$K_1 = \frac{1}{8}\left(H_{x_1} - H_{x_2}\right), \qquad K_2 = \frac{1}{8}\left(C^+ + C^- + \frac{1}{2}(D^2 - H^2)\right). \tag{25.20}$$

It is seen that K_2 can be written as

$$K_2 = \frac{1}{8}\left((x_1\partial_{x_2} - x_2\partial_{x_1})^2 - \frac{a_1 x_2^2}{x_1^2} - \frac{a_2 x_1^2}{x_2^2} - 1/2\right), \tag{25.21}$$

and is purely angular in polar coordinates. If we take

$$K_3 = [K_1, K_2] = \frac{1}{16}(C^+ - C^-), \tag{25.22}$$

the defining relations become

$$\begin{aligned}
&[K_1, K_2] = K_3, \\
&[K_2, K_3] = \{K_1, K_2\} + \delta_1 K_1 + \delta_2, \\
&[K_3, K_1] = K_1^2 - \frac{1}{4}K_2 + \delta_3,
\end{aligned} \tag{25.23}$$

with

$$\delta_1 = -\frac{1}{4}(k_1^2 + k_2^2 - 2), \quad \delta_2 = \frac{1}{32}(k_1 - k_2)(k_1 + k_2)H, \quad \delta_3 = -\frac{1}{64}H^2. \tag{25.24}$$

This presentation allows the identification with the Hahn algebra, a special case of the "generic" Racah algebra (see next section). It is known to appear in connection with the Clebsch-Gordan problem for $\mathfrak{su}(1,1)$ [17]. This suggests a potential link between the isotropic singular oscillator in two dimensions and the Clebsch-Gordan problem for the dynamical algebra of its one-dimensional components. We shall now proceed to discuss the most general second order S.I. model in two dimensions along the lines followed in this section and shall see that the link mentioned above is not fortuitous. However before we do so, we shall introduce thoroughly the Racah algebra, present its finite-dimensional representations and go over the relations these have with Racah polynomials.

25.3 The Racah algebra

The Racah algebra has three generators K_1, K_2 and K_3. In the generic presentation, they obey the relations

$$\begin{aligned}
&[K_1, K_2] = K_3, \\
&[K_2, K_3] = a_2 K_2^2 + a_1\{K_1, K_2\} + c_1 K_1 + d K_2 + e_1, \\
&[K_3, K_1] = a_1 K_1^2 + a_2\{K_1, K_2\} + c_2 K_2 + d K_1 + e_2,
\end{aligned} \tag{25.25}$$

where the parameters a_1, a_2, c_1, c_2, d, e_1 and e_2 are taken to be real. This defines the most general associative quadratic algebra with two independent generators and a ladder property. To see this, let K_1, K_2 be the two independent generators and define $[K_1, K_2] = K_3$. K_1 and K_2 are assumed to be Hermitian, K_3 is thus anti-Hermitian. Consider the most general quadratic relations compatible with the hermiticity conditions:

$$\begin{aligned}
&[K_2, K_3] = a_2 K_2^2 + a_1\{K_1, K_2\} + g_1 K_1^2 + h_1 K_3^2 + c_1 K_1 + d_1 K_2 + e_1, \\
&[K_3, K_1] = a_3 K_1^2 + a_4\{K_1, K_2\} + g_2 K_2^2 + h_2 K_3^2 + c_2 K_2 + d_2 K_1 + e_2.
\end{aligned} \tag{25.26}$$

It follows from the Jacobi identity

$$[K_1,[K_2,K_3]]+[K_3,[K_1,K_2]]+[K_2,[K_3,K_1]]=0,$$

that

$$d_1=d_2, \quad a_3=a_1, \quad a_4=a_2, \quad h_1=h_2=0. \tag{25.27}$$

One then requires $g_1=g_2=0$ to ensure the ladder property (see later) and thus recovers (25.25). This algebra made its appearance in the work of Granovskii and Zhedanov [3] where it was used in the context of the Racah problem of $\mathfrak{su}(2)$ to derive the symmetry group of the $6j$-symbols. It is also known as the Racah-Wilson algebra. When neither a_1 nor a_2 are zero, that is when $a_1 \cdot a_2 \neq 0$, the relations can be put in the following canonical form

$$[K_1,K_2]=K_3, \tag{25.28a}$$

$$[K_2,K_3]=K_2^2+\{K_1,K_2\}+dK_2+e_1, \tag{25.28b}$$

$$[K_3,K_1]=K_1^2+\{K_1,K_2\}+dK_1+e_2, \tag{25.28c}$$

where d, e_1 and e_2 are still real. This presentation thus retains three essential structure parameters and is arrived at by simple affine transformations of the generators $K_i \to u_i K_i + v_i$, $i=1,2,3$. It is verified that this algebra has the following Casimir operator (central element)

$$\begin{aligned} Q =&\{K_1^2,K_2\}+\{K_1,K_2^2\}+K_1^2+K_2^2+K_3^2 \\ &+(d+1)\{K_1,K_2\}+(2e_1+d)K_1+(2e_2+d)K_2, \end{aligned} \tag{25.29}$$

which is cubic in the generators and commutes with each one of them.

25.4 Representations of the Racah algebra and Racah polynomials

We now wish to point out the connection between the Racah algebra and the Racah orthogonal polynomials. This can be done in at least two ways:

1. By constructing the finite-dimensional representations of the algebra.

2. By realizing the algebra in terms of the operators associated to the polynomials.

We shall describe these two approaches in the following. We shall begin this section though by registering the basic definitions and properties of the Racah polynomials that we shall use.

25.4.1 Racah polynomials

The Racah polynomials $R_n(\lambda(x);\alpha,\beta,\gamma,\delta)$ of degree n in $\lambda(x)=x(x+\gamma+\delta+1)$ depend on four real parameters α, β, γ, δ and are defined by the following explicit expression ($n\in\mathbb{N}$):

$$R_n(\lambda(x)) = {}_4F_3\left[\begin{matrix}-n,n+\alpha+\beta+1,-x,x+\gamma+\delta+1\\ \alpha+1,\beta+\delta+1,\gamma+1\end{matrix};1\right], \tag{25.30}$$

where ${}_pF_q$ is the generalized hypergeometric series

$${}_pF_q\left[\begin{matrix}a_1 & \cdots & a_p\\ b_1 & \cdots & b_q\end{matrix};z\right] = \sum_{j=0}^{\infty}\frac{(a_1)_j\cdots(a_p)_j}{(b_1)_j\cdots(b_q)_j}\frac{z^j}{j!}, \tag{25.31}$$

and

$$(a)_j = a(a+1)\cdots(a+j-1).$$

The series in (25.30) truncates since $(-n)_j=0$ for $j\geqslant n+1$. The polynomials thus defined satisfy $R_0(\lambda(x))=1$ and a three-term recurrence relation of the form [12]

$$\lambda(x)R_n(\lambda(x)) = A_nR_{n+1}(\lambda(x))-(A_n+C_n)R_n(\lambda(x))+C_nR_{n-1}(\lambda(x)), \tag{25.32}$$

with

$$\begin{aligned}A_n &= \frac{(n+\alpha+1)(n+\alpha+\beta+1)(n+\beta+\delta+1)(n+\gamma+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)},\\ C_n &= \frac{n(n+\alpha+\beta-\gamma)(n+\alpha-\delta)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}.\end{aligned} \tag{25.33}$$

As usual it is assumed that $R_{-1}(\lambda(x))=0$. Like all polynomials of the Askey scheme, the Racah polynomials are bispectral: in addition to obeying the above recurrence relation, they are aso eigenfunctions of the difference equation

$$\mathcal{L}R_n(\lambda(x)) = n(n+\alpha+\beta+1)R_n(\lambda(x)), \tag{25.34}$$

where

$$\mathcal{L} = B(x)T^{+}+D(x)T^{-}-(B(x)+D(x))\mathbb{I}, \tag{25.35}$$

with

$$T^{\pm}f(x) = f(x\pm 1), \tag{25.36}$$

and

$$\begin{aligned}B(x) &= \frac{(x+\alpha+1)(x+\beta+\delta+1)(x+\gamma+1)(x+\gamma+\delta+1)}{(2x+\gamma+\delta+1)(2x+\gamma+\delta+2)},\\ D(x) &= \frac{x(x-\alpha+\gamma+\delta)(x-\beta+\gamma)(x+\delta)}{(2x+\gamma+\delta)(2x+\gamma+\delta+1)}.\end{aligned} \tag{25.37}$$

Provided one of the following truncation conditions holds:

$$\alpha + 1 = -N, \quad \beta + \delta + 1 = -N, \quad \gamma + 1 = -N,$$

the Racah polynomials $R_n(\lambda(x))$ enjoy a finite orthogonality relation of the form

$$\sum_{x=0}^{N} w_x R_n(\lambda(x)) R_m(\lambda(x)) = h_n \delta_{nm}, \tag{25.38}$$

where w_x and h_n are known explicitly. (For more details on the Racah polynomials see [12] where in particular the limit relations to other OPs of the Askey scheme are provided.)

25.4.2 Finite-dimensional representations

We shall now describe the finite-dimensional unitary representations of the Racah algebra and sketch how they are obtained. We take the defining relations to be in the canonical form (25.28). We begin by taking one generator, say K_1, to be diagnal on the representation space and proceed to show that the Racah algebra has a ladder property.

Let ω_p be a vector of the representation space such that

$$K_1 \omega_p = \lambda_p \omega_p, \qquad p \in \mathbb{R}. \tag{25.39}$$

Suppose we look for another eigenvector $\omega_{p'}$ with eigenvalue $\lambda_{p'}$ that has the form

$$\omega_{p'} = \left\{ \alpha(p) K_1 + \beta(p) K_2 + \gamma(p) K_3 \right\} \omega_p, \tag{25.40}$$

where $\alpha(p)$, $\beta(p)$ and $\gamma(p)$ are coefficients. Imposing the eigenvalue equation

$$K_1 \omega_{p'} = \lambda_{p'} \omega_{p'}, \tag{25.41}$$

using (25.39) and the commutation relations (25.28a) and (25.28c), it is seen that the eigenvalues $\lambda_{p'}$ must satisfy

$$(\lambda_{p'} - \lambda_p)^2 + (\lambda_{p'} + \lambda_p) = 0. \tag{25.42}$$

For a given λ_p, there are two solutions which we can choose to call λ_{p+1} and λ_{p-1}. Assuming that λ_p is non-degenerate and denoting by E_{λ_p} the one-dimensional eigenspace, it follows from the above considerations that a generic element of the algebra will map E_{λ_p} onto $E_{\lambda_{p-1}} \oplus E_{\lambda_p} \oplus E_{\lambda_{p+1}}$. We can thus write

$$\begin{aligned} K_1 \omega_p &= \lambda_p \omega_p, \\ K_2 \omega_p &= U_{p+1} \omega_{p_1} + V_p \omega_p + U_p \omega_{p-1}, \\ K_3 \omega_p &= [K_1, K_2] \omega_p = U_{p+1} g_{p+1} \omega_{p+1} - U_p g_p \omega_{p-1}, \end{aligned} \tag{25.43}$$

with

$$g_p = \lambda_p - \lambda_{p-1}, \tag{25.44}$$

observing that K_2 is tridiagonal and K_3 bidiagonal. Note that K_2 is self-adjoint if U_p is real. Having understood that the representations of the Racah algebra have a ladder structure, we now wish to focus on those representations that are finite-dimensional and for which the spectrum of K_1 is discrete. In the following we shall hence replace the vectors ω_p, $p \in \mathbb{R}$, by the vectors ψ_n, $n \in \mathbb{Z}$, labeled by the discrete index n. The actions (25.43) become

$$\begin{aligned} K_1\psi_n &= \lambda_n \psi_n, \\ K_2\psi_n &= U_{n+1}\psi_{n+1} + V_n\psi_n + U_n\psi_{n-1}, \\ K_3\psi_n &= U_{n+1}g_{n+1}\,\psi_{n+1} - U_n g_n\,\psi_{n-1}. \end{aligned} \tag{25.45}$$

with

$$g_n = \lambda_n - \lambda_{n-1}, \tag{25.46}$$

and there remains to determine λ_n, V_n and U_n. From (25.28c), one finds

$$\lambda_n = (\sigma - n)(n - \sigma + 1)/2, \quad g_n = \sigma - n, \tag{25.47}$$

and

$$V_n = -\frac{\lambda_n^2 + d\lambda_n + e_2}{\lambda_n}, \tag{25.48}$$

where σ is an arbitrary real parameter. We observe that the spectrum is quadratic in n. To find U_n, one uses (25.28b) that yields the following recurrence relation for U_n^2:

$$2(g_{n+3/2}U_{n+1}^2 - g_{n-1/2}U_n^2) = V_n^2 + (2\lambda_n + d)V_n + e_1. \tag{25.49}$$

Instead of solving directly (25.49), it is simpler to use the fact that the Casimir operator Q given in (25.29) is constant on irreducible representation spaces in order to find an expression for

$$2(g_{n+3/2}g_nU_{n+1}^2 + g_{n+1}g_{n-1/2}U_n^2).$$

Upon eliminating U_{n+1} with the help of this result and solving (25.49) for U_n^2 one arrives at

$$U_n^2 = \frac{\mathscr{P}(g_n^2)}{64g_n^2 g_{n-1/2}g_{n+1/2}}, \tag{25.50}$$

where $\mathscr{P}(z)$ is the fourth degree polynomial

$$\begin{aligned} \mathscr{P}(z) =& z^4 - (4d+2)z^3 + (4d^2 + 4d + 1 + 8e_2 - 16e_1)z^2 \\ & - 4(d^2 + 2e_2 + 4de_2 + 4q)z + 16e_2^2. \end{aligned} \tag{25.51}$$

In terms of the roots ξ_k^2 of $\mathscr{P}(g_n^2)$, we can write

$$U_n^2 = \frac{\prod_{k=1}^4 (g_n^2 - \xi_k^2)}{64 g_n^2 g_{n-1/2} g_{n+1/2}}. \tag{25.52}$$

For finite-dimensional representations, the index n is comprised in a finite interval $N_1 \le n \le N_2$, $N_1, N_2 \in \mathbb{Z}$ and the eigenvalues of K_1 are $\lambda_{N_1}, \lambda_{N_1+1}, \ldots, \lambda_{N_2}$. Setting the arbitrary parameter σ in (25.47) equal to $N_1 + \rho$ and calling $N = N_2 - N_1$, we can equivalently restrict the label n to be in the interval

$$0 \le \sigma \le N, \tag{25.53}$$

with the eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_N$ given by

$$\lambda_n = (\rho - n)(n - \rho + 1)/2. \tag{25.54}$$

Clearly, in a $(N+1)$-dimensional representation we must have $U_0 = U_{N+1} = 0$ so that ψ_{-1} and ψ_{N+1} cannot be reached by the actions of K_2 on ψ_0 and ψ_N, respectively. This will be observed if one of the zeros ξ_k^2, $k = 1,2,3,4$, say ξ_i^2 is such that

$$\xi_i^2 = g_0^2 = \rho^2, \tag{25.55a}$$

and another say ξ_j^2, verify

$$\xi_j^2 = g_{N+1}^2 = (\rho - N - 1)^2. \tag{25.55b}$$

25.4.3 Connection with Racah polynomials

Having described the finite-dimensional representations of the Racah algebra in the eigenbasis of K_1, we might wonder what the picture is in the eigenbasis of K_2. It turns out to be very similar. Indeed, it is observed that the defining relations (25.28) are invariant under the exchanges

$$K_1 \leftrightarrow K_2, \qquad e_1 \leftrightarrow e_2. \tag{25.56}$$

It follows that the representation in the bases $\{\phi_s\}$ in which K_2 is diagonal

$$K_2 \phi_s = \mu_s \phi_s, \tag{25.57}$$

can be obtained from those in which K_1 is diagonal using this symmetry property. In this correspondence with the formulas of the last subsection, we make the replacement $n \to s$. It is clear that μ_s will have a from similar to λ_n say

$$\mu_s = (\nu - s)(\nu - s + 1)/2, \tag{25.58}$$

and that K_1 will be tridiagonal in the ϕ_s basis. In other words, K_1 and K_2 realize a Leonard pair [15]. We now claim that the Racah polynomials arise as overlaps between the $\{\psi_n\}$ and the $\{\phi_s\}$ bases; that is, the Racah polynomials appear as expansion coefficients between the basis in which K_1 is diagonal and the one in which K_2 is diagonal.

Let us make this more explicit. Since the bases $\{\phi_s\}$ and $\{\psi_n\}$ span isomorphic spaces we can expand the elements of one in terms of those of the other:

$$\phi_s = \sum_{n=0}^{N} W_n(s)\psi_n. \tag{25.59}$$

Let us write the coefficients $W_n(s)$ in the form

$$W_n(s) = w_0(s)P_n(\mu_s), \tag{25.60}$$

so that $P_0(\mu_s) = 1$. Using (25.57) and (25.45), it is seen upon acting with K_2 on both sides of (25.59) that the quantities $P_n(\mu_s)$ obey

$$\mu_s P_n(\mu_s) = U_{n+1}P_{n+1}(\mu_s) + V_n P_n(\mu_s) + U_n P_{n-1}(\mu_s). \tag{25.61}$$

Given the formulas for U_n and V_n, this recurrence relation is seen to coincide with that of the Racah polynomials. The zeros ξ_k^2 are related to the parameters $\alpha, \beta, \gamma, \delta$ of the polynomials $R_n(\lambda(x); \alpha, \beta, \gamma, \delta)$. If the truncations conditions are satisfied through

$$\xi_1 = \rho, \qquad \xi_4 = (\rho - N - 1), \tag{25.62}$$

(recall (25.55)) the identification is achived by the following parametrization of the roots:

$$\xi_1 = -\frac{\alpha+\beta}{2}, \quad \xi_2 = \frac{\beta-\alpha}{2} + \delta, \quad \xi_3 = \frac{\beta-\alpha}{2}, \quad \xi_4 = \gamma - \frac{\alpha+\beta}{2}. \tag{25.63}$$

25.4.4 The Racah algebra from the Racah polynomials

We have illustrated how the Racah polynomials can be obtained from the Racah algebra by constructing the finite-dimensional representations. Let us indicate now that conversely, the Racah algebra can be identified from the properties of the Racah polynomials. As shall be seen the Racah algebra encodes the bispectrality properties of the polynomials. These properties amount to the fact that in addition to satisfying a three-term recurrence relation (as all orthogonal polynomials must do), the Racah polynomials also obey a difference equation. Let us recall that these relations can be put in the form

$$x(x+\gamma+\delta+1)R_n(\lambda(x)) = \mathscr{M} R_n(\lambda(x)), \tag{25.64a}$$

$$\mathscr{L} R_n(\lambda(x)) = n(n+\alpha+\beta+1)R_n(\lambda(x)), \tag{25.64b}$$

where the *recurrence* operator $\mathscr{M}$ and the *difference* operator $\mathscr{L}$ are given by

$$\mathscr{M} = A_n T_n^+ + C_n T_n^- - (A_n + C_n)\mathbb{I}, \tag{25.65}$$

$$\mathscr{L} = B(x)T_x^+ + D(x)T_x^- - (B(x) + D(x))\mathbb{I}, \tag{25.66}$$

with $T_n^{\pm} f(n) = f(n \pm 1)$, $T_x^{\pm} f(x) = f(x \pm 1)$ and where A_n, C_n are given in (25.33) and $B(x)$, $D(x)$ provided by (25.37). Consider now the realization on functions of x, where $\widetilde{K}_1$ and $\widetilde{K}_2$ are the two operators occurring on the left-hand side of (25.64), i.e. $\widetilde{K}_1$ is the recurrence operator and $\widetilde{K}_2$ is the difference operator:

$$\widetilde{K}_1 = x(x + \gamma + \delta + 1), \qquad \widetilde{K}_2 = \mathscr{L}. \tag{25.67}$$

Performing the affine transformations

$$K_1 = u_1 \widetilde{K}_1 + v_1, \qquad K_2 = u_2 \widetilde{K}_2 + v_2, \tag{25.68}$$

one finds that K_1, K_2 verify the Racah algebra relations (25.28) with

$$\begin{aligned}
e_1 &= \frac{1}{4}\left(\frac{\alpha-\beta}{2}\right)\left(\frac{\alpha+\beta}{2}\right)\left(\frac{\alpha+\beta}{2}-\gamma\right)\left(\frac{\alpha-\beta}{2}-\delta\right),\\
e_2 &= \frac{1}{4}\left(\frac{\gamma-\delta}{2}\right)\left(\frac{\gamma+\delta}{2}\right)\left(\frac{\gamma+\delta}{2}-\alpha\right)\left(\frac{\gamma-\delta}{2}-\beta\right),\\
d &= \frac{1}{4}\left\{\left(\frac{\gamma-\delta}{2}\right)^2+\left(\frac{\gamma+\delta}{2}\right)^2+\left(\frac{\gamma+\delta}{2}-\alpha\right)^2+\left(\frac{\gamma-\delta}{2}-\beta\right)^2-2\right\}.
\end{aligned}$$

We could obviously have taken the realization on functions of n with

$$\widehat{K}_1 = \mathscr{M}, \qquad \widehat{K}_2 = n(n + \alpha + \beta + 1), \tag{25.69}$$

which is bound to lead to the same algebra. The duality property of the Racah polynomials under the exchanges

$$x \leftrightarrow n, \qquad \alpha \leftrightarrow \gamma, \qquad \beta \leftrightarrow \delta, \tag{25.70}$$

which follows from (25.64) is immediately seen to correspond to the symmetry of the Racah algebra under

$$K_1 \leftrightarrow K_2, \qquad K_3 \leftrightarrow -K_3, \qquad e_1 \leftrightarrow e_2, \tag{25.71}$$

that we already observed.

25.5 The generic superintegrable model on S^2 and $\mathfrak{su}(1,1)$

We now return to the generic superintegrable model on S^2 with Hamiltonian (25.2) and constants of motion (25.6) and (25.7). We want to show that it is intimately connected to the $\mathfrak{su}(1,1)$ algebra.

Consider three $\mathfrak{su}(1,1)$ realizations identical to the one introduced in the discussion of the two-dimensional singular oscillator in Section 2:

$$J_0^{(i)} = \frac{1}{4}\left(-\partial_{x_i}^2 + x_2^2 + \frac{a_i}{x_i^2}\right), \quad J_{\pm}^{(i)} = \frac{1}{4}\left((x_i \mp \partial_{x_i})^2 - \frac{a_i}{x_i^2}\right), \tag{25.72}$$

with $a_i = k_i^2 - 1/4$ and $i = 1,2,3$. These provide positive discrete series representations for which the $\mathfrak{su}(1,1)$ Casimir element

$$C^{(i)} = [J_0^{(i)}]^2 - J_+^{(i)} J_-^{(i)} - J_0^{(i)}, \quad i = 1,2,3, \tag{25.73}$$

takes the value

$$C^{(i)} = \nu_i(\nu_i - 1), \qquad \nu_i = (k_i + 1)/2. \tag{25.74}$$

These three sets of $\mathfrak{su}(1,1)$ generators can be added to produce a "fourth" realization:

$$J_0^{(4)} = J_0^{(1)} + J_0^{(2)} + J_0^{(3)}, \quad J_{\pm}^{(4)} = J_{\pm}^{(1)} + J_{\pm}^{(2)} + J_{\pm}^{(3)}. \tag{25.75}$$

Three types of Casimir operators can be distinguished in the process:

1. The "initial" Casimir operators (25.73)
2. The "intermediate" Casimir operators associated to the addition of two representations

   $$C^{(ij)} = [J_0^{(i)} + J_0^{(j)}]^2 - (J_+^{(i)} + J_+^{(j)})(J_-^{(i)} + J_-^{(j)}) - (J_0^{(i)} + J_0^{(j)}), \tag{25.76}$$

   with $(ij) = (12), (23), (31)$.
3. The "full" Casimir operator

   $$C^{(4)} = [J_0^{(4)}]^2 - J_+^{(4)} J_-^{(4)} - J_0^{(4)}. \tag{25.77}$$

By a direct computation one finds that

$$C^{(ij)} = \frac{1}{4}\left\{J_k^2 + \frac{a_i x_j^2}{x_i^2} + \frac{a_j x_i^2}{x_j^2} + a_i + a_j - 1\right\}, \tag{25.78}$$

with (ijk) a cyclic permutation of $(1,2,3)$. Comparing with (25.6) and (25.7), we see that

$$L_1 = 4C^{(23)} - a_2 - a_3 + 1, \quad L_2 = 4C^{(31)} - a_3 - a_1 + 1, \quad L_3 = 4C^{(12)} - a_1 - a_2 + 1, \tag{25.79}$$

and thus observe that the constants of motion of the generic S.I. system on S^2 are basically the intermediate Casimir operators arising in the addition of three $\mathfrak{su}(1,1)$ representations. Similarly one can check that the full Casimir operator $C^{(4)}$ takes the following form when (25.72) and (25.75) are used in (25.77):

$$C^{(4)} = \frac{1}{4}\left\{J_1^2 + J_2^2 + J_3^2 + (x_1^2 + x_2^2 + x_3^2)\left(\frac{a_1}{x_1^2} + \frac{a_2}{x_2^2} + \frac{a_3}{x_3^2}\right) - \frac{3}{4}\right\}. \tag{25.80}$$

As a consequence,

$$H = 4C^{(4)} + \frac{3}{4}, \tag{25.81}$$

if $x_1^2 + x_2^2 + x_3^2 = 1$. At this point we may ask if this restriction to S^2 can generally be ensured in the addition of the three $\mathfrak{su}(1,1)$ representations. That the answer is yes is readily seen. It is noted from (25.72) that

$$2J_0^{(i)} + J_+^{(i)} + J_-^{(i)} = x_i^2, \tag{25.82}$$

and hence that

$$S = 2J_0^{(4)} + J_+^{(4)} + J_-^{(4)} = x_1^2 + x_2^2 + x_3^2. \tag{25.83}$$

Since S commutes with $4C^{(4)} + 3/4$, it is "time-independent" and as a constant can be taken to be 1. We can thus conclude the following. The generic S.I. 3-paramater system is obtained from the addition of three $\mathfrak{su}(1,1)$ realizations. In this identification the restriction $x_1^2 + x_2^2 + x_3^2 = 1$ to S^2 is preserved; the Hamiltonian corresponds to the full Casimir operator $C^{(4)}$ for the addition of the three representations and the constants of motion are obviously the intermediate Casimir operators $C^{(ij)}$ which commute with $C^{(4)}$. The algebra that these intermediate Casimir operators generate is thus the symmetry algebra of the generic $S.I.$ system on S^2. We shall show in the next section that the intermediate Casimir operators in the addition of three $\mathfrak{su}(1,1)$ irreducible representations generate the Racah algebra.

Note that the R-matrix approach has been used in [6] to describe the generic 3-parameter model on S^2 and construct its invariants. One proceeds via dimensional reduction from S^5 to S^2 with a Lax matrix that also involves three $\mathfrak{su}(1,1)$ elements. It is interesting to further point out that the same generic S.I. model on S^2 has been shown to correspond to one of the Krall-Sheffer classes of orthogonal polynomials in two variables [4, 16].

25.6 The Racah problem for $\mathfrak{su}(1,1)$ and the Racah algebra

We loosely refer to the Racah problem as the recoupling of three irreducible representations. Strictly speaking, it is about determining the unitary transformation between two canonical bases corresponding to the steps $(1\oplus 2)\oplus 3$ and $1\oplus(2\oplus 3)$ which are respectively associated to the diagonalization of the intermediate Casimir operators $C^{(12)}$ and $C^{(23)}$. Our goal in this section is to comment on the algebra that $C^{(12)}$ and $C^{(23)}$ generate in the case of $\mathfrak{su}(1,1)$ (and of $\mathfrak{su}(2)$ as a matter of fact).

Consider the addition of three $\mathfrak{su}(1,1)$ representations as in (25.75) and take each initial Casimir operator $C^{(i)}$ to be a multiple of the identity:

$$C^{(i)} = \lambda_i, \qquad i = 1,2,3. \tag{25.84}$$

The intermediate Casimir operators $C^{(12)}$ and $C^{(23)}$ (given by (25.76)) can then be expressed as follows

$$\begin{aligned} C^{(12)} &= 2J_0^{(1)}J_0^{(2)} - (J_+^{(1)}J_-^{(2)} + J_-^{(1)}J_+^{(2)}) + \lambda_1 + \lambda_2, \\ C^{(23)} &= 2J_0^{(2)}J_0^{(3)} - (J_+^{(2)}J_-^{(3)} + J_-^{(2)}J_+^{(3)}) + \lambda_2 + \lambda_3. \end{aligned} \tag{25.85}$$

Further assume that the full Casimir operator $C^{(4)}$ which can be written in the form

$$C^{(4)} = C^{(12)} + C^{(23)} + C^{(31)} - C^{(1)} - C^{(2)} - C^{(3)}, \tag{25.86}$$

is also a multiple of the identity, i.e.

$$C^{(4)} = \lambda_4. \tag{25.87}$$

Denoting by $V^{(\lambda_i)}$ an irreducible $\mathfrak{su}(1,1)$ representation space on which the Casimir operator $C^{(i)}$ is equal to λ_i, we are thus looking at the decomposition of $V^{(\lambda_1)}\otimes V^{(\lambda_2)}\otimes V^{(\lambda_3)}$ in irreducible components $V^{(\lambda_4)}$. Consider now the algebra generated by the intermediate Casimir operators in this Racah problem. Define

$$\kappa_1 = -C^{(12)}/2 \qquad \kappa_2 = -C^{(23)}/2, \tag{25.88}$$

and let

$$\kappa_3 = [\kappa_1, \kappa_2]. \tag{25.89}$$

A direct computation in which $C^{(i)}$ is replaced by λ_i gives, remarkably, the defining relations of the Racah algebra

$$\begin{aligned}
&[\kappa_1,\kappa_2]=\kappa_3,\\
&[\kappa_2,\kappa_3]=\kappa_2^2+\{\kappa_1,\kappa_2\}+d\kappa_2+e_1,\\
&[\kappa_3,\kappa_1]=\kappa_1^2+\{\kappa_1,\kappa_2\}+d\kappa_1+e_2,
\end{aligned} \tag{25.90}$$

where

$$d=\frac{1}{2}(\lambda_1+\lambda_2+\lambda_3+\lambda_4) \quad e_1=\frac{1}{4}(\lambda_1-\lambda_4)(\lambda_2-\lambda_3) \quad e_2=\frac{1}{4}(\lambda_1-\lambda_2)(\lambda_4-\lambda_3) \tag{25.91}$$

The intermediate Casimir operators in the addition of three $\mathfrak{su}(1,1)$ representations form the Racah algebra which is thus the structure behind the Racah problem for $\mathfrak{su}(1,1)$. Combining this with the identification of the intermediate Casimir operators with the constants of motion of the generic S.I. 3-parameter system on S^2, it follows that the (reduced) Racah algebra is the symmetry algebra of this model. In other words, the algebra (25.8) is isomorphic to the Racah algebra (25.28).

We recall from Section (4.3) that the Racah polynomials appear as expansion coefficients between bases for Racah algebra representation spaces in which K_1 is diagonal on the one hand and K_2 is diagonal on the other. Since as we just have seen, K_1 and K_2 can be realized by $C^{(12)}$ and $C^{(23)}$, this naturally relates to the fact that the Racah coefficients (the elements of the matrix relating the bases associated to the 2 step-wise recoupling processes) are Racah polynomials. In the context of the superintegrable model on S^2, these Racah coefficients can be connected to separation of variables. This is seen as follows. The diagonalization of

$$C^{(12)}=\frac{1}{4}(L_3+a_1+a_2-1)=J_3^2+\frac{a_1x_2^2}{x_1^2}+\frac{a_2x_1^2}{x_2^2}, \tag{25.92}$$

brings the separation of variables in the usual spherical coordinates (in which x_1 and x_2 are paired):

$$x_1=r\sin\theta\cos\phi, \quad x_2=r\sin\theta\sin\phi, \quad x_3=r\cos\theta, \tag{25.93}$$

while the diagonalization of

$$C^{(23)}=\frac{1}{4}(L_1+a_2+a_3-1)=J_1^2+\frac{a_2x_3^2}{x_2^2}+\frac{a_3x_2^2}{x_3^2}, \tag{25.94}$$

leads to the separation of variables in another spherical coordinate system (in which x_2 and x_3 are paired) namely,

$$x_1=r\cos\theta, \quad x_2=r\sin\theta\cos\phi, \quad x_3=r\sin\theta\sin\phi. \tag{25.95}$$

Hence the Racah polynomials are in this framework, the overlap coefficients between these two sets of wavefunctions of the generic 3-parameter system on S^2 which are obtained by separating the variables in the considered systems (25.93) and (25.95).

Summing up, the identification of the generic S.I. model on S^2 as the full Casimir operator in the addition of three $\mathfrak{su}(1,1)$ realizations provides a natural way of obtaining the constants of motion (as the intermediate Casimir operators) and of determining the symmetry algebra (as the Racah algebra). This intimately associates the Racah polynomials to the model on S^2. It should be said that representations with continuous spectra are found to bring the Wilson polynomials in the picture. As explained in [11], all the other second order S.I. mdels in two dimensions can be obtained from the generic system on S^2 by contractions and specializations. Correspondingly, when effected on the Racah algebra and the Racah/Wilson polynomials these operations provide the symmetry algebras of all these second order S.I. models and their tagging to orthogonal polynomials of the Askey scheme.

25.7 Conclusion

Let us summarize the main findings and offer perspectives. We presented the Racah algebra and its finite-dimensional representations. We proceeded to explain its universal character and the relations depicted on the diagram presented in Section I. It was shown that the Racah algebra is behind

- The generic 3-parameter superintegrable model on S^2 and hence all second order superintegrable systems in two dimensions
- The Racah problem for $\mathfrak{su}(1,1)$, that is the combination of three irreducible $\mathfrak{su}(1,1)$ representations
- The Racah polynomials that sit atop the discrete part of the Askey scheme of hypergeometric orthogonal polynomials

Looking at three and higher dimensions, the analogous connections between superintegrable models, polynomial algebras and special functions are bound to be illuminating. They are expected to feed the theory of multivariate orthogonal polynomials and their algebraic interpretations.

As an illustration of this, let us mention that Kalkins, Miller and Post have already shown that 2-variable Racah-Wilson polynomials occur in the S^3 model [8]. The construction presented here extends to any dimensions and the three-dimensional model on S^3 arises from the addition of four $\mathfrak{su}(1,1)$ representations and corresponds to the $9j$ or Fano problem for $\mathfrak{su}(1,1)$. The symmetries in this case are expected to lead to a rank 2 version of the Racah algebra which should encode the properties of the general bivariate Racah-Wilson polynomials. We intend to pursue investigations along those lines.

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Chapitre 26

A Laplace-Dunkl equation on S^2 and the Bannai–Ito algebra

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Abstract. The analysis of the $\mathbb{Z}_2^3$ Laplace-Dunkl equation on the 2-sphere is cast in the framework of the Racah problem for the Hopf algebra $sl_{-1}(2)$. The related Dunkl-Laplace operator is shown to correspond to a quadratic expression in the total Casimir operator of the tensor product of three irreducible $sl_{-1}(2)$-modules. The operators commuting with the Dunkl Laplacian are seen to coincide with the intermediate Casimir operators and to realize a central extension of the Bannai–Ito (BI) algebra. Functions on S^2 spanning irreducible modules of the BI algebra are constructed and given explicitly in terms of Jacobi polynomials. The BI polynomials occur as expansion coefficients between two such bases composed of functions separated in different coordinate systems.

26.1 Introduction

The purpose of this paper is to establish a relation between Dunkl harmonic analysis on the 2-sphere and the representation theory of $sl_{-1}(2)$, an algebra obtained as a $q \to -1$ limit of the quantum algebra $\mathscr{U}_q(\mathfrak{sl}_2)$. The Dunkl-Laplace operator on S^2 associated to the Abelian reflection group $\mathbb{Z}_2^3 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ will be expressed as a quadratic polynomial in the total Casimir operator of the tensor product of three irreducible $sl_{-1}(2)$-modules. The operators commuting with the Dunkl Laplacian will be identified with the intermediate Casimir operators arising in the three-fold tensor product. On eigensubspaces of the Dunkl Laplacian, these intermediate Casimir operators will be shown to generate the Bannai–Ito algebra, which is the algebraic structure behind the

Racah problem of $sl_{-1}(2)$. Functions on the 2-sphere providing bases for irreducible modules of the Bannai–Ito algebra will be constructed. It will be shown that the Bannai–Ito polynomials arise here as expansion coefficients between elements of such bases associated to the separation of variables in different spherical coordinate systems.

We first provide background on the entities involved here: the $\mathbb{Z}_2^3$ Dunkl Laplacian and its restriction to the 2-sphere, the $sl_{-1}(2)$ algebra and its Hopf algebra structure and the Bannai–Ito algebra and the associated Bannai–Ito polynomials.

26.1.1 The $\mathbb{Z}_2^3$ Dunkl-Laplacian on S^2

The Dunkl operators and Laplacian were introduced by Dunkl in [4, 5], where a framework for multivariate analysis based on finite reflection groups was developed. These operators have since found a vast number of applications in diverse fields including harmonic analysis and integral transforms [3, 13, 15], orthogonal polynomials and special functions [6], stochastic processes [11] and quantum integrable/superintegrable systems [7, 19]. In the case of the Abelian reflection group $\mathbb{Z}_2^3$, the Dunkl operators $\mathscr{D}_i$, $i = 1,2,3$, associated to each copy of the reflection group $\mathbb{Z}_2$ are defined by

$$\mathscr{D}_i = \partial_{x_i} + \frac{\mu_i}{x_i}(1-R_i), \tag{26.1}$$

with $\mu_i > -1/2$ a real parameter, ∂_{x_i} the partial derivative with respect to the variable x_i and R_i the reflection operator in the $x_i = 0$ plane, i.e. $R_i f(x_i) = f(-x_i)$. The Dunkl Laplacian associated to the $\mathbb{Z}_2^3$ group is defined by

$$\Delta = \mathscr{D}_1^2 + \mathscr{D}_2^2 + \mathscr{D}_3^2, \tag{26.2}$$

and has the following expression:

$$\Delta = \sum_{i=1}^{3} \partial_{x_i}^2 + \frac{2\mu_i}{x_i}\partial_{x_i} - \frac{\mu_i}{x_i^2}(1-R_i).$$

Since the reflections R_i, $i = 1,2,3$, are special rotations in $O(3)$, the Dunkl Laplacian (26.2), like the standard Laplace operator in three variables, separates in the usual spherical coordinates

$$x_1 = r\sin\theta\cos\phi, \quad x_2 = r\sin\theta\sin\phi, \quad x_3 = r\cos\theta, \tag{26.3}$$

with $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$. The operator Δ can thus be restricted to functions defined on the unit sphere. Let Δ_{S^2} denote the angular part of the Dunkl Laplacian (26.2); one has

$$\Delta_{S^2} = L_\theta + \frac{1}{\sin^2\theta} M_\phi, \tag{26.4}$$

where

$$L_\theta = \frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta) + 2\left(\frac{\mu_1+\mu_2}{\tan\theta} - \mu_3\tan\theta\right)\partial_\theta - \frac{\mu_3}{\cos^2\theta}(1-R_3),$$

and

$$M_\phi = \partial_\phi^2 + 2\left(\frac{\mu_2}{\tan\phi} - \mu_1\tan\phi\right)\partial_\phi - \frac{\mu_1}{\cos^2\phi}(1-R_1) - \frac{\mu_2}{\sin^2\phi}(1-R_2),$$

as can be directly checked by expanding (26.2) in spherical coordinates.

26.1.2 The Hopf algebra $\mathbf{sl_{-1}(2)}$

The $sl_{-1}(2)$ algebra was introduced in [16] as the $q \to -1$ limit of the quantum algebra $\mathscr{U}_q(\mathfrak{sl}_2)$ [20]. It is defined as the associative algebra (over $\mathbb{C}$) with generators $A_\pm$, A_0 and P satisfying the relations

$$[A_0, A_\pm] = \pm A_\pm,\ [A_0, P] = 0,\ \{A_+, A_-\} = 2A_0,\ \{A_\pm, P\} = 0,\ P^2 = 1, \tag{26.5}$$

where $[x, y] = xy - yx$ stands for the commutator. This algebra admits the following Casimir operator, which commutes with all generators:

$$C = A_+A_-P - A_0P + P/2. \tag{26.6}$$

The $sl_{-1}(2)$ algebra can be endowed with the structure of a Hopf algebra. One introduces the comultiplication $\Delta : sl_{-1}(2) \to sl_{-1}(2) \otimes sl_{-1}(2)$, the counit $\epsilon : sl_{-1}(2) \to \mathbb{C}$ and the coinverse (antipode) $\sigma : sl_{-1}(2) \to sl_{-1}(2)$ defined by the formulas

$$\begin{aligned}
&\Delta(A_0) = A_0 \otimes 1 + 1 \otimes A_0, \quad \Delta(A_\pm) = A_\pm \otimes P + 1 \otimes A_\pm, \quad \Delta(P) = P \otimes P,\\
&\epsilon(1) = \epsilon(P) = 1, \qquad \epsilon(A_\pm) = \epsilon(A_0) = 0,\\
&\sigma(1) = 1, \quad \sigma(P) = P, \quad \sigma(A_0) = -A_0, \quad \sigma(A_\pm) = PA_\pm.
\end{aligned} \tag{26.7}$$

It is verified that the definitions (26.7) comply with the conditions required for a Hopf algebra [18]. It is worth pointing out that the operators $A_\pm$, A_0 also satisfy the defining relations of the parabosonic algebra for a single paraboson (see [2]).

26.1.3 The Bannai–Ito algebra and polynomials

The Bannai–Ito algebra was introduced in [17] as the algebraic structure encoding the bispectrality property of the Bannai–Ito polynomials. It is defined as the associative algebra (over $\mathbb{C}$) generated by K_1, K_2 and K_3 satisfying the relations

$$\{K_1, K_2\} = K_3 + \alpha_3, \quad \{K_2, K_3\} = K_1 + \alpha_1, \quad \{K_3, K_1\} = K_2 + \alpha_2, \tag{26.8}$$

where $\{x,y\} = xy + yx$ stands for the anticommutator and where α_i, $i = 1,2,3$, are real structure constants. In [17], the algebra was introduced with the structure constants expressed as follows in terms of four real parameters ρ_1, ρ_2, r_1, r_2:

$$\alpha_1 = 4(\rho_1\rho_2 + r_1r_2), \quad \alpha_2 = 2(\rho_1^2 + \rho^2 - r_1^2 - r_2^2), \quad \alpha_3 = 4(\rho_1\rho_2 - r_1r_2),$$

and the generators had the form

$$K_1 = 2\mathscr{L} + (g + 1/2), \quad K_2 = y,$$

with $g = \rho_1 + \rho_2 - r_1 - r_2$ and $\mathscr{L}$ the difference operator

$$\mathscr{L} = \frac{(y-\rho_1)(y-\rho_2)}{2y}(1-R_y) + \frac{(y-r_1+1/2)(y-r_2+1/2)}{2y+1}(T_y^+R_y - 1),$$

where $R_y f(y) = f(-y)$, $T_y^+ f(y) = f(y+1)$. The operator $\mathscr{L}$ is the most general self-adjoint first order difference operator with reflections that stabilizes the space of polynomials of a given degree. As shown in [17], the operator $\mathscr{L}$ admits as eigenfunctions the Bannai–Ito polynomials $B_n(y)$, which were introduced in a combinatorial context by Bannai and Ito in [1]. Their three-term recurrence relation was derived in [17] using the BI algebra (26.8) and reads

$$xB_n(y) = B_{n+1}(y) + (\rho_1 - A_n - C_n)B_n(y) + A_{n-1}C_nB_{n-1}(y), \tag{26.9}$$

where the initial conditions $B_{-1}(x) = 0$, $B_0(x) = 1$ hold and where the recurrence coefficients A_n, C_n are given by

$$A_n = \begin{cases} \frac{(n+2\rho_1-2r_1+1)(n+2\rho_1-2r_2+1)}{4(n+\rho_1+\rho_2-r_1-r_2+1)}, & n \text{ is even,} \\ \frac{(n+2\rho_1+2\rho_2-2r_1-2r_2+1)(n+2\rho_1+2\rho_2+1)}{4(n+\rho_1+\rho_2-r_1-r_2+1)}, & n \text{ is odd,} \end{cases} \tag{26.10a}$$

$$C_n = \begin{cases} -\frac{n(n-2r_1-2r_2)}{4(n+\rho_1+\rho_2-r_1-r_2)}, & n \text{ is even,} \\ -\frac{(n+2\rho_2-2r_2)(n+2\rho_2-2r_1)}{4(n+\rho_1+\rho_2-r_1-r_2)}, & n \text{ is odd.} \end{cases} \tag{26.10b}$$

The polynomials $B_n(y)$ defined by (26.9) are $q \to -1$ limits of either the Askey-Wilson [17] or the q-Racah polynomials [1]. They obey a discrete and finite orthogonality relation of the form

$$\sum_{s=0}^{N} w_s B_n(y_s) B_m(y_s) = h_n \delta_{nm},$$

where the expressions for the grid points y_s, the measure w_s and the normalization constant h_n depend on a set of relations between the parameters. For the complete picture, one may consult the references [8, 17].

26.1.4 Outline

Here is an outline of the paper.

- Section II: Irreducible $sl_{-1}(2)$-modules (positive-discrete series), Realization with Dunkl operators, Racah problem, Intermediate Casimir operators, Relation between the total Casimir and Δ_{S^2}, Spectra of the total and intermediate Casimir operators
- Section III: Commutant of Δ_{S^2}, Bannai–Ito algebra, Finite-dimensional irreducible representations of the BI algebra
- Section IV: Dunkl spherical harmonics for $\mathbb{Z}_2^3$, S^2 basis functions for irreducible modules of the BI algebra, BI polynomials as expansion coefficients between basis functions

26.2 Racah problem of $sl_{-1}(2)$ and Δ_{S^2}

In this section, irreducible $sl_{-1}(2)$-modules of the positive-discrete series and their realizations in terms of the Dunkl operators (26.1) are given. The Racah problem is presented and the intermediate and total Casimir operators are defined. The main result on the relation between the total Casimir operator and the Dunkl Laplacian on S^2 is presented. Moreover, the spectrum of the Dunkl Laplacian is recovered algebraically using this relation.

26.2.1 Representations of the positive-discrete series and their realization in terms of Dunkl operators

Let ϵ and ν be real parameters such that $\epsilon^2 = 1$ and $\nu > -1/2$ and denote by $V^{(\epsilon,\nu)}$ the infinite-dimensional vector space spanned by the orthonormal basis vectors $e_n^{(\epsilon,\nu)}$ with n a non-negative integer. An irreducible $sl_{-1}(2)$-module of the positive-discrete series is obtained by endowing $V^{(\epsilon,\nu)}$ with the actions [16]:

$$A_0\, e_n^{(\epsilon,\nu)} = (n+\nu+1/2)\, e_n^{(\epsilon,\nu)}, \quad P\, e_n^{(\epsilon,\nu)} = \epsilon(-1)^n\, e_n^{(\epsilon,\nu)}, \tag{26.11a}$$

$$A_+\, e_n^{(\epsilon,\nu)} = \sqrt{[n+1]_\nu}\, e_{n+1}^{(\epsilon,\nu)}, \quad A_-\, e_n^{(\epsilon,\nu)} = \sqrt{[n]_\nu}\, e_{n-1}^{(\epsilon,\nu)}, \tag{26.11b}$$

where $[n]_\nu$ is defined by

$$[n]_\nu = n + \nu(1-(-1)^n).$$

It is directly seen that for $\nu > -1/2$, $V^{(\epsilon,\mu)}$ is an irreducible module. Furthermore, it is observed that on this module the spectrum of A_0 is strictly positive and the operators $A_\pm$ are adjoint one of the other. As expected from Schur's lemma, the Casimir operator (26.6) of $sl_{-1}(2)$ acts a multiple of the identity on $V^{(\epsilon,\nu)}$:

$$C\, e_n^{(\epsilon,\nu)} = -\epsilon\, \nu\, e_n^{(\epsilon,\nu)}. \tag{26.12}$$

The $sl_{-1}(2)$-module $V^{(\epsilon,\nu)}$ can be realized using Dunkl operators. Indeed, for each variable x_i, $i = 1,2,3$, one can check that the operators

$$A_0^{(i)} = -\frac{1}{2}\mathscr{D}_i^2 + \frac{1}{2}x_i^2, \quad A_\pm^{(i)} = \frac{1}{\sqrt{2}}(x_i \mp \mathscr{D}_i), \quad P^{(i)} = R_i, \tag{26.13}$$

where $\mathscr{D}_i$ and R_i are as in (26.1), satisfy the defining relations (26.5) of $sl_{-1}(2)$. The Casimir operator $C^{(i)}$ becomes

$$C^{(i)} = A_+^{(i)} A_-^{(i)} P^{(i)} - A_0^{(i)} P^{(i)} + P^{(i)}/2 = -\mu_i, \tag{26.14}$$

and hence the operators (26.13) for $i = 1,2,3$ realize the irreducible module $V^{(\epsilon,\nu)}$ with $\epsilon = \epsilon_i = 1$ and $\nu = \mu_i$. The orthonormal basis vectors $e_n^{(\epsilon_i,\nu_i)}(x_i)$ in this realization are expressed in terms of the generalized Hermite polynomials (see for example [7, 14]) and the space $V^{(\epsilon_i,\nu_i)}$ with $\epsilon_i = 1$ and $\nu_i = \mu_i$ is the L^2 space of square integrable functions of argument x_i with respect to the orthogonality measure of the generalized Hermite polynomials [14]; we shall denote it by $L^2_{\mu_i}$.

26.2.2 The Racah problem, Casimir operators and Δ_{S^2}

The Racah problem for $sl_{-1}(2)$-modules of the positive-discrete series arises when the decomposition in irreducible components of the module $V = V^{(\epsilon_1,\nu_1)} \otimes V^{(\epsilon_2,\nu_2)} \otimes V^{(\epsilon_3,\nu_3)}$ is considered. The action of the $sl_{-1}(2)$ generators on V is prescribed by the coproduct structure (26.7) and one has for $v \in V$

$$A_0 v = (1 \otimes \Delta)\Delta(A_0)v, \quad P v = (1 \otimes \Delta)\Delta(P)v, \quad A_\pm v = (1 \otimes \Delta)\Delta(A_\pm)v. \tag{26.15}$$

Note that $(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta$ since Δ is coassociative. In the realization (26.13), the module V (with $\epsilon_i = 1$ and $\nu_i = \mu_i$) involves functions of the three independent variables x_1, x_2, x_3. The operators satisfying the $sl_{-1}(2)$ relations and acting on functions $f(x_1,x_2,x_3)$ in $L^2_{\mu_1} \otimes L^2_{\mu_2} \otimes L^2_{\mu_3}$ are obtained from (26.13) and (26.15):

$$\begin{aligned} \widetilde{A}_0 &= A_0^{(1)} + A_0^{(2)} + A_0^{(3)}, \quad \widetilde{P} = P^{(1)} P^{(2)} P^{(3)}, \\ \widetilde{A}_\pm &= A_\pm^{(1)} P^{(2)} P^{(3)} + A_\pm^{(2)} P^{(3)} + A_\pm^{(3)}. \end{aligned} \tag{26.16}$$

In combining the modules $V^{(\epsilon_i,\nu_i)}$, $i = 1,2,3$, three types of Casimir operators can be distinguished. The three *initial* Casimir operators are those attached to each components $V^{(\epsilon_i,\mu_i)}$ of V and act as multiplication by $-\epsilon_i \nu_i$ as per (26.12). In the realization (26.13), these are the $C^{(i)}$ given in (26.14). The two *intermediate* Casimir operators are associated to the two equivalent factorizations

$$V = (V^{(\epsilon_1,\nu_1)} \otimes V^{(\epsilon_2,\nu_2)}) \otimes V^{(\epsilon_3,\nu_3)} = V^{(\epsilon_1,\nu_1)} \otimes (V^{(\epsilon_2,\nu_2)} \otimes V^{(\epsilon_3,\nu_3)}), \tag{26.17}$$

and correspond to the operators

$$\Delta(C) \otimes 1 \qquad \text{and} \qquad 1 \otimes \Delta(C), \tag{26.18}$$

where $\Delta(C)$ is obtained from (26.6) and (26.7). In the realization (26.13), these shall be denoted $C^{(ij)}$ with $(ij) = (12), (23)$ and are given by

$$\begin{aligned} C^{(ij)} = &(A_+^{(i)} P^{(i)} + A_+^{(j)} P^{(j)})(A_-^{(i)} P^{(i)} + A_-^{(j)} P^{(j)}) \\ &- (A_0^{(i)} + A_0^{(j)}) P^{(i)} P^{(j)} + P^{(i)} P^{(j)}. \end{aligned} \tag{26.19}$$

The *total* Casimir operator is connected to the whole module V and is of the form $(1 \otimes \Delta)\Delta(C)$. In the realization (26.13), the total Casimir is denoted $\widetilde{C}$ and reads

$$\widetilde{C} = \widetilde{A}_+ \widetilde{A}_- \widetilde{P} - \widetilde{A}_0 \widetilde{P} + \widetilde{P}/2. \tag{26.20}$$

with $\widetilde{A}_0$, $\widetilde{A}_\pm$ and $\widetilde{P}$ given by (26.16). Note that $\widetilde{C}$ does not act as a multiple of the identity on V since in general V is not irreducible.

Remark 13. By construction, the total Casimir operator $\widetilde{C}$ commutes with both the initial and intermediate Casimir operators. Moreover, it is obvious that the two intermediate Casimir operators commute with the initial Casimir operators, but do not commute amongst themselves.

We now relate the total Casimir operator $\widetilde{C}$ to the Dunkl Laplacian operator Δ_{S^2} on the 2-sphere.

Proposition 7. *Let Ω be the following element:*

$$\Omega = \widetilde{C}\widetilde{P}, \tag{26.21}$$

where $\widetilde{C}$ and $\widetilde{P}$ are respectively given by (26.16) *and* (26.20) *in the realization* (26.13)*. One has*

$$-\Delta_{S^2} = \Omega^2 + \Omega - (\mu_1 + \mu_2 + \mu_3)(\mu_1 + \mu_2 + \mu_3 + 1). \tag{26.22}$$

Proof. The relation is obtained by expanding the total Casimir operator (26.20) using (26.13) and by writing the resulting operator in the coordinates (26.3). □

The fact that Ω is a purely angular operator can be understood algebraically as follows. Consider the element $\widetilde{X}$ defined by

$$\widetilde{X} = \frac{1}{\sqrt{2}}\left(\widetilde{A}_+ + \widetilde{A}_-\right).$$

It is directly checked that $\widetilde{X}$ anticommutes with Ω, that is $\{\Omega, \widetilde{X}\} = 0$. It thus follows that $\widetilde{X}^2$ commutes with Ω. Using the expressions (26.16) for the operators $\widetilde{A}_\pm$ in the realization (26.13), it is easily seen that

$$\widetilde{X}^2 = x_1^2 + x_2^2 + x_3^2.$$

Hence Ω commutes with the "radius" operator, which means that it can only be an angular operator.

26.2.3 Spectrum of Δ_{S^2} from the Racah problem

The relation (26.22) can be exploited to algebraically derive the spectrum of Δ_{S^2} from that of Ω using the eigenvalues of the intermediate Casimir operators. In view of (26.18), these eigenvalues can be found from those of $\Delta(C)$ on $V^{(\epsilon_i,\nu_i)} \otimes V^{(\epsilon_j,\nu_j)}$ (see also [10, 9, 16] where this problem was considered). Upon examining the action of $\Delta(A_0)$ on the direct product basis, one obtains using (26.11) the following direct sum decomposition of $V^{(\epsilon_i,\nu_i)} \otimes V^{(\epsilon_j,\nu_j)}$ as a vector space:

$$V^{(\epsilon_i,\nu_i)} \otimes V^{(\epsilon_j,\nu_j)} = \bigoplus_{n=0}^{\infty} U_n,$$

where U_n are the $(n+1)$-dimensional eigenspaces of $\Delta(A_0)$ with eigenvalue $n + \nu_i + \nu_j + 1$. Since $\Delta(C)$ commutes with $\Delta(A_0)$, the action of $\Delta(C)$ stabilizes U_n.

Lemma 14. *The eigenvalues λ_I of $\Delta(C)$ on U_n are given by*

$$\lambda_I(k) = (-1)^{k+1}\epsilon_i\epsilon_j(k + \nu_i + \nu_j + 1/2), \qquad k = 0, \ldots n.$$

Proof. By induction on n. The $n = 0$ case is verified by acting with $\Delta(C)$ on the single basis vector $e_0^{(\epsilon_i,\nu_i)} \otimes e_0^{(\epsilon_j,\nu_j)}$ of U_0. Suppose that the result holds at level $n-1$. Using the fact that $\Delta(C)$ and $\Delta(A_+)$ commute and the induction hypothesis, one obtains from the action of $\Delta(A_+)$ on U_{n-1} eigenvectors of $\Delta(C)$ in U_n with eigenvalues $\lambda_I(k)$ for $k = 0, \ldots, n-1$. Let $v \in U_n$ be such that $\Delta(A_-)v = 0$. Such a vector can explicitly be constructed in the direct product basis by solving the corresponding two-term recurrence relation. It is verified that v is an eigenvector of $\Delta(P)$ with eigenvalue $(-1)^n\epsilon_i\epsilon_j$ and of $\Delta(C)$ with eigenvalue $\lambda_I(n)$. □

As a direct corollary one has the following decomposition of the tensor product module in irreducible components:

$$V^{(\epsilon_i,\nu_i)}\otimes V^{(\epsilon_j,\nu_j)}=\bigoplus_k V^{(\epsilon_{ij}(k),\nu_{ij}(k))}, \tag{26.23}$$

with

$$\epsilon_{ij}(k)=(-1)^k\epsilon_i\epsilon_j,\quad \nu_{ij}(k)=k+\nu_i+\nu_j+1/2,\qquad k\in\mathbb{N}. \tag{26.24}$$

The eigenvalues of the total Casimir operator $(1\otimes\Delta)\Delta(C)$ on V are obtained by using twice the decomposition (26.23) and Lemma 1 on (26.17). It is readily seen performing these decompositions on the LHS of (26.17) that the eigenvalues λ_T of the total Casimir operator are given by

$$\lambda_T=(-1)^{k+1}\epsilon_{12}(\ell)\epsilon_3(k+\nu_{12}(\ell)+\nu_3+1/2),\qquad k,\ell\in\mathbb{N}. \tag{26.25}$$

A similar formula involving ϵ_{23} and ν_{23} is obtained by considering instead the RHS of (26.17). Upon using (26.24), the eigenvalues λ_T can be cast in the form

$$\lambda_T(N)=-\epsilon(N)\nu(N), \tag{26.26}$$

with N a non-negative integer and

$$\epsilon(N)=(-1)^N\epsilon_1\epsilon_2\epsilon_3,\qquad \nu(N)=(N+\nu_1+\nu_2+\nu_3+1). \tag{26.27}$$

The formula (26.26) and (26.27) indicate which irreducible modules appear in the decomposition of V. The multiplicity of $V^{(\epsilon(N),\nu(N))}$ in this decomposition is $N+1$ since for a given value of N there are $N+1$ possible eigenvalues of the intermediate Casimir operators; the decomposition formula for V is thus

$$V=\bigoplus_{N=0}^{\infty} m_N V^{(\epsilon(N),\nu(N))}, \tag{26.28}$$

where $m_N=N+1$ and where $\epsilon(N)$, $\nu(N)$ are given by (26.27).

Returning to the realization (26.16) of the module V with $\epsilon_i=1$ and $\nu_i=\mu_i$, the eigenvalues of $\Omega=\widetilde{C}\widetilde{P}$ are readily obtained. Recalling (26.12), it follows from (26.26) and (26.27) that the eigenvalues ω_N of Ω are

$$\omega_N=-(N+\mu_1+\mu_2+\mu_3+1), \tag{26.29}$$

where N is a non-negative integer. The relation (26.22) then leads to the following.

Proposition 8. *The eigenvalues δ of the Dunkl Laplacian Δ_{S^2} on the 2-sphere are indexed by the non-negative integer N and have the expression*

$$\delta_N=-N(N+2\mu_1+2\mu_2+2\mu_3+1). \tag{26.30}$$

Proof. By proposition 1 and the above considerations. □

The eigenvalues of proposition 2 are in accordance with those obtained in [6]. It is seen that upon specializing (26.30) to $\mu_1 = \mu_2 = \mu_3 = 0$, one recovers the spectrum of the standard Laplacian on the 2-sphere. It is worth mentioning that the formula (26.30) does not provide information on the degeneracy of the eigenvalues. This question will be discussed in the following.

26.3 Commutant of Δ_{S^2} and the Bannai–Ito algebra

In this section, the operators commuting with the Dunkl Laplacian on the 2-sphere are exhibited and are shown to generate a central extension of the Bannai–Ito algebra. The eigensubspaces corresponding to the simultaneous diagonalization of Δ_{S^2} and Ω are seen to support finite-dimensional irreducible representations of the BI algebra and the matrix elements of these representations are constructed.

26.3.1 Commutant of Δ_{S^2} and symmetry algebra

The operators that commute with the Dunkl Laplacian Δ_{S^2} on the 2-sphere, referred to as the *symmetries* of Δ_{S^2}, can be obtained from the relation (26.22) and the framework provided by the Racah problem of $sl_{-1}(2)$. By construction, the intermediate Casimir operators (26.19) commute with the total Casimir (26.20) and with the involution $\widetilde{P}$. As a consequence of (26.22), one thus has

$$[\Delta_{S^2}, C^{(12)}] = [\Delta_{S^2}, C^{(23)}] = 0.$$

Let K_1, K_3 be the following operators:

$$K_1 = -C^{(23)}, \qquad K_3 = -C^{(12)}, \tag{26.31}$$

which obviously commute with the Dunkl Laplacian on S^2. Upon using (26.13) and (26.19), the symmetries K_1, K_3 are seen to have the expressions

$$K_1 = (x_2\mathscr{D}_3 - x_3\mathscr{D}_2)R_2 + \mu_2 R_3 + \mu_3 R_2 + (1/2)R_2R_3, \tag{26.32a}$$

$$K_3 = (x_1\mathscr{D}_2 - x_2\mathscr{D}_1)R_1 + \mu_1 R_2 + \mu_2 R_1 + (1/2)R_1R_2, \tag{26.32b}$$

where $\mathscr{D}_i$ and R_i are given by (26.1). Consider the operator K_2 defined by

$$K_2 = (x_1\mathscr{D}_3 - x_3\mathscr{D}_1)R_1R_2 + \mu_1 R_3 + \mu_3 R_1 + (1/2)R_1R_3. \tag{26.32c}$$

It is verified by an explicit calculation that K_2 is also a symmetry of the Dunkl-Laplacian Δ_{S^2}, i.e. $[\Delta_{S^2}, K_2] = 0$.

Remark 15. Note that K_2 does not correspond to an intermediate Casimir operator since it has a non-trivial action on all three variables x_1, x_2, x_3.

The three operators K_i, $i = 1,2,3$, and the operator Ω given by (26.21) are not independent from one another. As a matter of fact, one has

$$\Omega = -K_1R_2R_3 - K_2R_1R_3 - K_3R_1R_2 + \mu_1R_1 + \mu_2R_2 + \mu_3R_3 + 1/2.$$

We now give the *symmetry algebra* generated by the operators commuting with the Dunkl-Laplace operator Δ_{S^2} on the 2-sphere.

Proposition 9. *Let* Δ_{S^2} *be the Dunkl Laplacian* (26.4) *on the 2-sphere and let* $\widetilde{C}$ *and* K_i*,* $i = 1,2,3$ *be given by* (26.20) *and* (26.32)*, respectively. One has*

$$[\Delta_{S^2}, K_i] = [\Delta_{S^2}, \widetilde{C}] = 0.$$

and the symmetry algebra of Δ_{S^2} *is*

$$\{K_1, K_2\} = K_3 - 2\mu_3\widetilde{C} + 2\mu_1\mu_2, \tag{26.33a}$$

$$\{K_2, K_3\} = K_1 - 2\mu_1\widetilde{C} + 2\mu_2\mu_3, \tag{26.33b}$$

$$\{K_3, K_1\} = K_2 - 2\mu_2\widetilde{C} + 2\mu_1\mu_3. \tag{26.33c}$$

Proof. By an explicit calculation using (26.4) and (26.32). □

The algebra (26.33) corresponds to a central extension of the Bannai–Ito algebra (26.8) by the total Casimir operator $\widetilde{C}$. Since $\widetilde{C}$ (and Ω) commutes with Δ_{S^2}, there is a basis in which they are both diagonal. From (26.27) and (26.29), it follows that the eigenvalues of $\widetilde{C}$ are of the form $-\epsilon\mu$ with

$$\epsilon = (-1)^N, \qquad \mu = (N + \mu_1 + \mu_2 + \mu_3 + 1). \tag{26.34}$$

For a given N, the Δ_{S^2}-eigenspaces arising under the joint diagonalization of Δ_{S^2} and $\widetilde{C}$ (or Ω) are $(N+1)$-dimensional as per the decomposition (26.28) of the tensor product module V in irreducible components. Hence the eigenvalues δ_N of Δ_{S^2} given by (26.30) are at least $(N+1)$-fold degenerate. It can be seen that this degeneracy is in fact higher. Indeed, Δ_{S^2} commutes with every reflection operator R_i, but $\widetilde{C}$ (and Ω) only commute with their product $R_1R_2R_3$. Consequently one can obtain eigenfunctions of Δ_{S^2} with eigenvalue δ_N that are not eigenfunctions of $\widetilde{C}$ by applying any reflection R_i on a given eigenfunction of $\widetilde{C}$. It is known [6] that the eigenspaces corresponding to the eigenvalue δ_N are in fact $(2N+1)$-fold degenerate, as shall be seen in Section 4.

Notwithstanding the degeneracy question, it follows from Proposition 3 and (26.34) that the eigensubspaces of the Laplace-Dunkl operator corresponding to the simultaneous diagonalization of Δ_{S^2} and $\widetilde{C}$ support an $(N+1)$-dimensional module of the Bannai–Ito algebra (26.8) with structure constants taking the values

$$\alpha_1 = 2(\mu_1\mu + \mu_2\mu_3), \quad \alpha_2 = 2(\mu_1\mu_3 + \mu_2\mu), \quad \alpha_3 = 2(\mu_1\mu_2 + \mu_3\mu), \tag{26.35}$$

where $\mu = (-1)^N(N + \mu_1 + \mu_2 + \mu_3 + 1)$. The Casimir operator $\mathbf{K}^2 = K_1^2 + K_2^2 + K_3^2$ of the Bannai–Ito algebra can be expressed in terms of $\widetilde{C}$ as follows:

$$\mathbf{K}^2 = \widetilde{C}^2 + \mu_1^2 + \mu_2^2 + \mu_3^2 - 1/4,$$

and hence using (26.34) one has

$$\mathbf{K}^2 = \mu_1^2 + \mu_2^2 + \mu_3^2 + \mu^2 - 1/4. \tag{26.36}$$

The realization (26.35), (26.36) of the Bannai–Ito algebra corresponds to the one arising in the Racah problem for $sl_{-1}(2)$ studied in [10]. We shall now obtain the matrix elements of the generators in this realization.

26.3.2 Irreducible modules of the Bannai–Ito algebra

We begin by examining the representations of (26.8) with structure constants (26.35) in the eigenbasis $\{\psi_k\}_{k=0}^N$ of K_3. Using the result of Lemma 1 and (26.31), it follows that

$$K_3\psi_k = \omega_k\psi_k, \qquad \omega_k = (-1)^k(k + \mu_1 + \mu_2 + 1/2), \tag{26.37}$$

We define the action of K_1 by

$$K_1\psi_k = \sum_s Z_{s,k}\psi_s. \tag{26.38}$$

From the second relation of (26.8) one finds

$$\sum_s Z_{s,k}\left[(\omega_k + \omega_s)^2 - 1\right]\psi_s = [\alpha_1 + 2\omega_k\alpha_2]\psi_k.$$

When $s = k$, one immediately obtains

$$Z_{k,k} \equiv V_k = \frac{\alpha_1 + 2\omega_k\alpha_2}{4\omega_k^2 - 1}. \tag{26.39}$$

When $s \neq k$, one of the following conditions must hold

$$(\omega_k + \omega_s)^2 - 1 = 0, \quad \text{or} \quad Z_{s,k} = 0.$$

In view of the formula (26.37) for the eigenvalues ω_k, it is directly seen that only $Z_{k+1,k}$, $Z_{k,k}$ and $Z_{k-1,k}$ can be non-vanishing. Thus one can take

$$K_1\psi_k = U_{k+1}\psi_{k+1} + V_k\psi_k + U_k\psi_{k-1}, \tag{26.40}$$

where V_k is given by (26.39) and where U_k remains to be determined. It follows from (26.8) and (26.40) that K_2 has the action

$$K_2\psi_k = (-1)^{k+1}U_{k+1} + W_k\psi_k + (-1)^k U_k\psi_{k-1}, \tag{26.41}$$

where $W_k = 2\omega_k V_k - \alpha_2$. Upon using the actions (26.40), (26.41) in the first relation of (26.8) and comparing the terms in ψ_k, one obtains the recurrence relation for U_k^2

$$2\left\{(-1)^{k+1}U_{k+1}^2 + W_k V_k + (-1)^k U_k^2\right\} = \omega_k + \alpha_3. \tag{26.42}$$

Acting on ψ_k with (26.36) and using the actions (26.40), (26.41), one finds

$$\{\omega_k^2 + W_k^2 + V_k^2 + 2U_k^2 + 2U_{k+1}^2\} = \mu_1^2 + \mu_2^2 + \mu_3^2 + \mu^2 - 1/4. \tag{26.43}$$

The equations (26.42), (26.43) can be used to solve for U_k^2 by eliminating U_{k+1}^2. Straightforward calculations then lead to the following result.

Proposition 10. *Let $\mathscr{W}$ be the $(N+1)$-dimensional vector space spanned by the basis vectors ψ_k, $k = 0,\ldots,N$, and let*

$$\mu = (-1)^N(N+1+\mu_1+\mu_2+\mu_3). \tag{26.44}$$

An irreducible module for the Bannai–Ito algebra (26.8) *with structure constants* (26.35) *is obtained by endowing $\mathscr{W}$ with the actions*

$$K_3\psi_k = \omega_k\psi_k, \tag{26.45a}$$

$$K_2\psi_k = (-1)^{k+1}U_{k+1}\psi_{k+1} + (2\omega_k V_k - \alpha_2)\psi_k + (-1)^k U_k\psi_{k-1}, \tag{26.45b}$$

$$K_1\psi_k = U_{k+1}\psi_{k+1} + V_k\psi_k + U_k\psi_{k-1}, \tag{26.45c}$$

where $\omega_k = (-1)^k(k+\mu_1+\mu_2+1/2)$, $V_k = \mu_2+\mu_3+1/2-B_k-D_k$ and where $U_k = \sqrt{B_{k-1}D_k}$ with

$$B_k = \begin{cases} \frac{(k+2\mu_2+1)(k+\mu_1+\mu_2+\mu_3-\mu+1)}{2(k+\mu_1+\mu_2+1)}, & k \text{ is even},\\ \frac{(k+2\mu_1+2\mu_2+1)(k+\mu_1+\mu_2+\mu_3+\mu+1)}{2(k+\mu_1+\mu_2+1)}, & k \text{ is odd},\end{cases}$$

$$D_k = \begin{cases} \frac{-k(k+\mu_1+\mu_2-\mu_3-\mu)}{2(k+\mu_1+\mu_2)}, & k \text{ is even},\\ \frac{-(k+2\mu_1)(k+\mu_1+\mu_2-\mu_3+\mu)}{2(k+\mu_1+\mu_2)}, & k \text{ is odd}.\end{cases}$$

Proof. One verifies directly that with (26.45) the defining relations (26.8), (26.35) are satisfied. The irreducibility follows from the fact that $U_k \neq 0$ for $\mu_i > -1/2$. □

In view of Proposition 4, it is natural to wonder what the representation matrix elements look like in other bases, say the eigenbases of either K_1 or K_2. These elements are easily obtained from the $\mathbb{Z}_3$ symmetry of the realization (26.35), (26.36). Indeed, it is verified that the algebra (26.8) with (26.35), (26.36) is left invariant by any cyclic transformation of both $\{K_1, K_2, K_3\}$ and $\{\mu_1, \mu_2, \mu_3\}$. As a consequence, the representation matrix elements in the K_1 or K_2 eigenbasis can be obtained directly from Proposition 4 by applying the permutation $\pi = (123)$ or $\pi = (123)^2$ on the generators K_i and the parameters μ_i.

26.4 S^2 basis functions for irreducible Bannai–Ito modules

In this section, a family of orthonormal functions on S^2 that realize bases for the Bannai–Ito modules of Proposition 4 are constructed. It is shown that the Bannai–Ito polynomials arise as the overlap coefficients between two such bases separated in different spherical coordinates.

26.4.1 Harmonics for Δ_{S^2}

It is useful to give here the Dunkl spherical harmonics $Y_N(\theta,\phi)$ which are the regular solutions to the eigenvalue equation

$$\Delta_{S^2} Y_N(\theta,\phi) = \delta_N Y_N(\theta,\phi), \qquad \delta_N = -N(N+2\mu_1+2\mu_2+2\mu_3+1), \tag{26.46}$$

where Δ_{S^2} is given by (26.4). The solutions to (26.46) are well known and are given explicitly in [6] in terms of the generalized Gegenbauer polynomials. We give their expressions here in terms of Jacobi polynomials. In spherical coordinates (26.3), the solutions to (26.46) read

$$Y_{n;N}^{(e_1,e_2,e_3)}(\theta,\phi) = \eta_{n;N}^{(e_1,e_2,e_3)} \cos^{e_3}\theta \sin^n\theta \cos^{e_1}\phi \sin^{e_2}\phi \\ \times P_{(N-n-e_3)/2}^{(n+\mu_1+\mu_2,\mu_3+e_3-1/2)}(\cos 2\theta)\, P_{(n-e_1-e_2)/2}^{(\mu_2+e_2-1/2,\mu_1+e_1-1/2)}(\cos 2\phi), \tag{26.47}$$

where $e_i \in \{0,1\}$, n is a non-negative integer, $\eta_{N,n}^{(e_1,e_2,e_3)}$ is a normalization factor and $P_n^{(\alpha,\beta)}(x)$ are the standard Jacobi polynomials [12]. The harmonics (26.47) satisfy

$$R_i Y_{n;N}^{(e_1,e_2,e_3)}(\theta,\phi) = (1-2e_i) Y_{n;N}^{(e_1,e_2,e_3)}(\theta,\phi).$$

In (26.47), it is understood that half-integer (or negative) indices in $P_n^{(\alpha,\beta)}(x)$ do not provide admissible solutions. Recording the admissible values of n and e_i for a given N, one finds that there are $2N+1$ solutions and

$$R_1R_2R_3 Y_{n;N}^{(e_1,e_2,e_3)}(\theta,\phi) = (-1)^N Y_{n;N}^{(e_1,e_2,e_3)}(\theta,\phi).$$

The normalization factor $\eta_{n;N}^{(e_1,e_2,e_3)}$ is given by

$$\eta_{n;N}^{(e_1,e_2,e_3)} = \left[\frac{(\frac{n-e_1-e_1}{2})!(n+\mu_1+\mu_2)\Gamma(\frac{n+e_1+e_2}{2}+\mu_1+\mu_2)}{2\,\Gamma(\frac{n+e_1-e_2}{2}+\mu_1+1/2)\Gamma(\frac{n+e_2-e_1}{2}+\mu_2+1/2)} \right]^{1/2} \times \left[\frac{(N+\mu_1+\mu_2+\mu_3+1/2)(\frac{N-n-e_3}{2})!\Gamma(\frac{N+n+e_3}{2}+\mu_1+\mu_2+\mu_3+1/2)}{\Gamma(\frac{N+n-e3}{2}+\mu_1+\mu_2+1)\Gamma(\frac{N-n+e_3}{2}+\mu_3+1/2)} \right]^{1/2},$$

where $\Gamma(x)$ stands for the Gamma function and ensures that

$$\int_0^{2\pi}\int_0^{\pi} Y_{n;N}^{(e_1,e_2,e_3)} Y_{n';N'}^{(e_1',e_2',e_3')}\, h(\theta,\phi)\,\sin\theta\,\mathrm{d}\theta\mathrm{d}\phi = \delta_{nn'}\delta_{NN'}\delta_{e_1e_1'}\delta_{e_2e_2'}\delta_{e_3e_3'},$$

where the $\mathbb{Z}_2^3$-invariant weight function $h(\theta,\phi)$ is [6]

$$h(\theta,\phi) = |\cos\theta|^{2\mu_3}|\sin\theta|^{2\mu_1}|\sin\theta|^{2\mu_2}|\cos\phi|^{2\mu_1}|\sin\phi|^{2\mu_2}. \tag{26.48}$$

26.4.2 S^2 basis functions for BI representations

Let $\mathscr{Y}_K^N(\theta,\phi)$, $K=0,\ldots,N$ be the functions on S^2 satisfying

$$\Omega\,\mathscr{Y}_K^N(\theta,\phi) = -(N+\mu_1+\mu_2+\mu_3+1)\mathscr{Y}_K^N(\theta,\phi), \tag{26.49a}$$

$$R_1R_2R_3\,\mathscr{Y}_K^N(\theta,\phi) = (-1)^N\,\mathscr{Y}_K^N(\theta,\phi), \tag{26.49b}$$

$$K_3\,\mathscr{Y}_K^N(\theta,\phi) = (-1)^K(K+\mu_1+\mu_2+1/2)\mathscr{Y}_K^N(\theta,\phi). \tag{26.49c}$$

where Ω is given by (26.21) and where K_3 is given by (26.31). In spherical coordinates (26.3), the operator K_3 has the expression

$$K_3 = \partial_\phi R_1 + \mu_1\tan\phi(1-R_1) + \frac{\mu_2}{\tan\phi}(R_1-R_1R_2) + \mu_1R_2 + \mu_2R_1 + \frac{1}{2}R_1R_2.$$

Since K_3 acts only on ϕ, the functions $\mathscr{Y}_K^N(\theta,\phi)$ can be separated.

The solutions for the azimuthal part are readily obtained from (26.49c) by considering separately the eigenvalue sectors of R_1R_2, which commutes with K_3. For the positive eigenvalue sector, one finds for $K=2k+p$

$$\mathscr{F}_K^{(+)}(\phi) = \zeta_K^{(+)}\left\{ \left[\frac{k+1}{k+\mu_1+\mu_2+1}\right]^{p/2} P_{k+p}^{(\mu_2-1/2,\mu_1-1/2)}(\cos 2\phi) - (-1)^p\left[\frac{k+\mu_1+\mu_2+1}{k+1}\right]^{p/2}\cos\phi\sin\phi\,P_{k+p-1}^{(\mu_2+1/2,\mu_1+1/2)}(\cos 2\phi)\right\}, \tag{26.50a}$$

where $p=0,1$. For the negative eigenvalue sector, the result for $K=2k+p$ is

$$\mathscr{F}_K^{(-)}(\phi) = \zeta_K^{(-)}\left\{ \left[\frac{k+\mu_1+1/2}{k+\mu_2+1/2}\right]^{p/2} \sin\phi P_k^{(\mu_2+1/2,\mu_1-1/2)}(\cos 2\phi) + (-1)^p\left[\frac{k+\mu_2+1/2}{k+\mu_1+1/2}\right]^{p/2}\cos\phi\,P_k^{(\mu_2-1/2,\mu_1+1/2)}(\cos 2\phi)\right\}. \tag{26.50b}$$

The normalization factors are

$$\zeta_K^{(+)} = \sqrt{\frac{(k+p)!\,\Gamma(k+\mu_1+\mu_2+1+p)}{2\,\Gamma(k+\mu_1+1/2+p)\Gamma(k+\mu_2+1/2+p)}},$$
$$\zeta_K^{(-)} = \sqrt{\frac{k!\,\Gamma(k+\mu_1+\mu_2+1)}{2\,\Gamma(k+\mu_1+1/2)\Gamma(k+\mu_2+1/2)}},$$

Using (26.50) the remaining equations (26.49a), (26.49b) can be solved. When $N=2n$ and $K=2k+p$, one finds

$$\mathscr{Y}_K^N(\theta,\phi) = \sqrt{\frac{(n-k-p)!\,\Gamma(n+k+\mu_1+\mu_2+\mu_3+3/2)}{\Gamma(n+k+\mu_1+\mu_2+1)\Gamma(n-k+\mu_3+1/2-p)}} \times$$
$$\Bigg\{\left[\frac{n-k+\mu_3-1/2}{n+k+\mu_1+\mu_2+1}\right]^{p/2} \sin^{2k+2p}\theta\, P_{n-k-p}^{(2k+2p+\mu_1+\mu_2,\mu_3-1/2)}(\cos 2\theta)\mathscr{F}_K^{(+)}(\phi)$$
$$+\left[\frac{n+k+\mu_1+\mu_2+1}{n-k+\mu_3-1/2}\right]^{p/2} \cos\theta\sin^{2k+1}\theta\, P_{n-k-1}^{(2k+1+\mu_1+\mu_2,\mu_3+1/2)}(\cos 2\theta)\mathscr{F}_K^{(-)}(\phi)\Bigg\}. \quad (26.52a)$$

When $N=2n+1$ and $K=2k+p$, the result is

$$\mathscr{Y}_K^N(\theta,\phi) = (-1)^K\sqrt{\frac{(n-k)!\,\Gamma(n+k+\mu_1+\mu_2+\mu_3+3/2+p)}{\Gamma(n-k+\mu_3+1/2)\Gamma(n+k+\mu_1+\mu_2+1+p)}} \times$$
$$\Bigg\{\left[\frac{n+k+\mu_1+\mu_2+1}{n-k+\mu_3+1/2}\right]^{(1-p)/2} \cos\theta\sin^{2k+2p}\theta\, P_{n-k-p}^{(2k+2p+\mu_1+\mu_2,\mu_3+1/2)}(\cos 2\theta)\mathscr{F}_K^{(+)}(\phi)$$
$$-\left[\frac{n-k+\mu_3+1/2}{n+k+\mu_1+\mu_2+1}\right]^{(1-p)/2} \sin^{2k+1}\theta\, P_{n-k}^{(2k+1+\mu_1+\mu_2,\mu_3-1/2)}(\cos 2\theta)\mathscr{F}_K^{(-)}(\phi)\Bigg\}. \quad (26.52b)$$

The solutions to (26.49) can be expressed as linear combinations of the Dunkl spherical harmonics (26.47). For $N=2n$, straightforward calculations lead to the expressions

$$\mathscr{Y}_{2k}^N(\theta,\phi) = \sqrt{\frac{n+k+\mu_1+\mu_2+\mu_3+1/2}{2n+\mu_1+\mu_2+\mu_3+1/2}}\Bigg\{\sqrt{\frac{k+\mu_1+\mu_2}{2k+\mu_1+\mu_2}}Y_{2k;N}^{(0,0,0)}(\theta,\phi)$$
$$-\sqrt{\frac{k}{2k+\mu_1+\mu_2}}Y_{2k;N}^{(1,1,0)}(\theta,\phi)\Bigg\} + \sqrt{\frac{n-k}{2n+\mu_1+\mu_2+\mu_3+1/2}} \times$$
$$\Bigg\{\sqrt{\frac{k+\mu_2+1/2}{2k+\mu_1+\mu_2+1}}Y_{2k+1;N}^{(0,1,1)}(\theta,\phi) + \sqrt{\frac{k+\mu_1+1/2}{2k+\mu_1+\mu_2+1}}Y_{2k+1;N}^{(1,0,1)}(\theta,\phi)\Bigg\},$$

$$\mathscr{Y}_{2k+1}^N(\theta,\phi) = \sqrt{\frac{n-k+\mu_3-1/2}{2n+\mu_1+\mu_2+\mu_3+1/2}}\Bigg\{\sqrt{\frac{k+1}{2k+\mu_1+\mu_2+2}}Y_{2k+2;N}^{(0,0,0)}(\theta,\phi)$$
$$+\sqrt{\frac{k+\mu_1+\mu_2+1}{2k+\mu_1+\mu_2+2}}Y_{2k+2;N}^{(0,1,1)}(\theta,\phi)\Bigg\} + \sqrt{\frac{n+k+\mu_1+\mu_2+1}{2n+\mu_1+\mu_2+\mu_3+1/2}} \times$$
$$\Bigg\{\sqrt{\frac{k+\mu_1+1/2}{2k+\mu_1+\mu_2+1}}Y_{2k+1;N}^{(0,1,1)}(\theta,\phi) - \sqrt{\frac{k+\mu_2+1/2}{2k+\mu_1+\mu_2+1}}Y_{2k+1;N}^{(1,0,1)}(\theta,\phi)\Bigg\},$$

For $N = 2n+1$, one finds

$$\mathscr{Y}_{2k}^{N}(\theta,\phi) = \sqrt{\frac{k+n+\mu_1+\mu_2+1}{2n+\mu_1+\mu_2+\mu_3+3/2}}\left\{\sqrt{\frac{k+\mu_1+\mu_2}{2k+\mu_1+\mu_2}}Y_{2k;N}^{(0,0,1)}(\theta,\phi)\right.$$
$$\left.-\sqrt{\frac{k}{2k+\mu_1+\mu_2}}Y_{2k;N}^{(1,1,1)}(\theta,\phi)\right\} - \sqrt{\frac{n-k+\mu_3+1/2}{2n+\mu_1+\mu_2+\mu_3+3/2}} \times$$
$$\left\{\sqrt{\frac{k+\mu_2+1/2}{2k+\mu_1+\mu_2+1}}Y_{2k+1;N}^{(0,1,0)}(\theta,\phi) + \sqrt{\frac{k+\mu_1+1/2}{2k+\mu_1+\mu_2+1}}Y_{2k+1;N}^{(1,0,0)}(\theta,\phi)\right\},$$

$$\mathscr{Y}_{2k+1}^{N}(\theta,\phi) = \sqrt{\frac{n+k+\mu_1+\mu_2+\mu_3+3/2}{2n+\mu_1+\mu_2+\mu_3+3/2}}\left\{\sqrt{\frac{k+\mu_1+1/2}{2k+\mu_1+\mu_2+1}}Y_{2k+1;N}^{(0,1,0)}(\theta,\phi)\right.$$
$$\left.-\sqrt{\frac{k+\mu_2+1/2}{2k+\mu_1+\mu_2+1}}Y_{2k+1;N}^{(1,0,0)}(\theta,\phi)\right\} - \sqrt{\frac{n-k}{2n+\mu_1+\mu_2+\mu_3+3/2}} \times$$
$$\left\{\sqrt{\frac{k+1}{2k+\mu_1+\mu_2+2}}Y_{2k+2;N}^{(0,0,1)}(\theta,\phi) + \sqrt{\frac{k+\mu_1+\mu_2+1}{2k+\mu_1+\mu_2+2}}Y_{2k+2;N}^{(1,1,1)}(\theta,\phi)\right\}.$$

It follows from the orthogonality relation for the Jacobi polynomials [12] that

$$\int_0^{\pi}\int_0^{2\pi} \mathscr{Y}_K^N(\theta,\phi)\mathscr{Y}_{K'}^{N'}(\theta,\phi)\,h(\theta,\phi)\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta = \delta_{KK'}\delta_{NN'}, \tag{26.53}$$

where $h(\theta,\phi)$ is given by (26.48).

Proposition 11. *The functions $\mathscr{Y}_K^N(\theta,\phi)$ defined by* (26.50), (26.52) *realize the Bannai–Ito modules of Proposition 4. That is, if one takes $\psi_K = \mathscr{Y}_K^N(\theta,\phi)$, the generators* (26.31) *expressed in spherical coordinates have the actions* (26.45).

Proof. The result follows from the fact that the $\mathscr{Y}_K^N(\theta,\phi)$ are solutions to (26.49). One needs only to check for possible phase factors. A check on the highest order term occurring in $K_1\mathscr{Y}_K^N(\theta,\phi)$ confirms the phase factors in (26.52). □

26.4.3 Bannai–Ito polynomials as overlap coefficients

As is seen from (26.49), the simultaneous diagonalization of Ω, $R_1R_2R_3$ and K_3 is associated to the separation of variables of the basis functions $\mathscr{Y}_K^N(\theta,\phi)$ in the usual spherical coordinates

$$x_1 = \sin\theta\cos\phi, \quad x_2 = \sin\theta\sin\phi, \quad x_3 = \cos\theta. \tag{26.54}$$

Consider the basis functions $\mathscr{Z}_S^N(\vartheta,\varphi)$, $S = 0,\ldots,N$, associated to the simultaneous diagonalization of Ω, $R_1R_2R_3$ and K_1. The relations (26.49a), (26.49b) hold and one has

$$K_1\mathscr{Z}_S^N(\vartheta,\varphi) = (-1)^S(S+\mu_2+\mu_3+1/2)\mathscr{Z}_S^N(\vartheta,\varphi). \tag{26.55}$$

The functions $\mathscr{Z}_S^N(\vartheta,\varphi)$ separate in the alternative spherical coordinates

$$x_1 = \cos\vartheta, \quad x_2 = \sin\vartheta\cos\varphi, \quad x_3 = \sin\vartheta\sin\varphi, \tag{26.56}$$

as can be seen from the expression of K_1 obtained using (26.56). Writing Ω in the coordinates (26.56) and comparing the expression with the one obtained using the coordinates (26.54), it is seen that the basis functions $\mathscr{Z}_S^N(\vartheta,\varphi)$ have the expression

$$\mathscr{Z}_S^N(\vartheta,\varphi) = \begin{cases} \pi\mathscr{Y}_S^N(\pi-\vartheta,\varphi), & N \text{ is even,} \\ \pi\mathscr{Y}_S^N(\vartheta,\varphi), & N \text{ is odd,} \end{cases}$$

where $\pi = (123)$ is the permutation applied to the parameters (μ_1,μ_2,μ_3). Since $\{\mathscr{Y}_K^N(\theta,\phi)\}_{K=0}^N$ and $\{\mathscr{Z}_S^N(\vartheta,\varphi)\}_{S=0}^N$ form orthonormal bases for the same space, they are related (at a given point) by a unitary transformation. One hence writes

$$\mathscr{Z}_S^N(\vartheta,\varphi) = \sum_{K=0}^N R_{S,K;N}^{\mu_1,\mu_2,\mu_3}\,\mathscr{Y}_K^N(\theta,\phi). \tag{26.57}$$

Since the coefficients $R_{S,K;N}^{\mu_1,\mu_2,\mu_3}$ are real, their unitarity implies

$$\sum_{S=0}^N R_{S,K;N}^{\mu_1\mu_2\mu_3} R_{S,K';N}^{\mu_1\mu_2\mu_3} = \delta_{KK'}, \quad \sum_{K=0}^N R_{S,K;N}^{\mu_1\mu_2\mu_3} R_{S',K;N}^{\mu_1\mu_2\mu_3} = \delta_{SS'}, \tag{26.58}$$

These transition coefficients can be expressed in terms of the Bannai–Ito polynomials (26.9) as follows. Acting with K_1 on both sides of (26.57), using (26.55) and Proposition 5 and furthermore defining $R_{S,K;N}^{\mu_1,\mu_2,\mu_3} = 2^K[w_{S;N}]^{1/2}B_K(x_S)$ such that $B_0(x_S) = 1$, it seen that $B_K(x_S)$ satisfy the three-term recurrence relation (26.9) of the Bannai–Ito polynomials $B_K(x_S;\rho_1,\rho_2,r_1,r_2)$ with parameters

$$\rho_1 = \frac{\mu_2+\mu_3}{2}, \; \rho_2 = \frac{\mu_1+\mu}{2}, \; r_1 = \frac{\mu_3-\mu_2}{2}, \; r_2 = \frac{\mu-\mu_1}{2}. \tag{26.59}$$

with μ given by (26.44) and where the variable x_S is given by

$$x_S = \frac{1}{2}\left[(-1)^S(S+\mu_2+\mu_3+1/2)-1/2\right]. \tag{26.60}$$

The coefficients $R_{S,K;N}^{\mu_1\mu_2\mu_3}$ coincide with the Racah coefficients of $sl_{-1}(2)$ [10]. Combining (26.58) with the orthogonality relation of the BI polynomial [17], one finds

$$R_{S,K;N}^{\mu_1\mu_2\mu_3} = \sqrt{\frac{w_{S;N}}{u_1u_2\cdots u_K}}B_K(x_S;\rho_1,\rho_2,r_1,r_2). \tag{26.61}$$

with (26.59), (26.60), where $u_n = A_{n-1}C_n$ with A_n, C_n as in (26.10), and where $w_{S;N}$ is of the form

$$w_{S;N} = \frac{1}{h_N}\frac{(-1)^\nu(\rho_1-r_1+1/2;\rho_1-r_2+1/2)_{\ell+\nu}(\rho_1+\rho_2+1;2\rho_1+1)_\ell}{(\rho_1+r_1+1/2;\rho_1+r_2+1/2)_{\ell+\nu}(1;\rho_1-\rho_2+1)_\ell},$$

with $S = 2\ell + \nu$, $\nu = \{0,1\}$ and

$$(a_1; a_2; \ldots; a_k)_n = (a_1)_n (a_2)_n \cdots (a_k)_n, \tag{26.62}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$. The normalization factor h_N is given by

$$h_N = \begin{cases} \frac{(2\rho_1+1; r_1-\rho_2+1/2)_{N/2}}{(\rho_1-\rho_2+1; \rho_1+r_1+1/2)_{N/2}}, & N \text{ even}, \\ \frac{(2\rho_1+1; r_1+r_2)_{(N+1)/2}}{(\rho_1+r_1+1/2; \rho_1+r_2+1/2)_{(N+1)/2}}, & N \text{ odd}. \end{cases}$$

Using the orthogonality relation (26.53) satisfied by the basis functions $\mathscr{Y}_K^N(\theta,\phi)$ on the decomposition formula (26.57), one finds that

$$R_{S,K;N}^{\mu_1\mu_2\mu_3} = \int_0^{\pi} \int_0^{2\pi} \mathscr{Y}_K^N(\theta,\phi)\, \mathscr{Z}_S^N(\vartheta,\varphi)\, h(\theta,\phi) \sin\theta \, \mathrm{d}\phi \, \mathrm{d}\theta,$$

which in light of (26.61) gives an integral formula for the Bannai–Ito polynomials.

26.5 Conclusion

We have established in this paper the algebraic basis for the harmonic analysis on S^2 associated to a $\mathbb{Z}_2^3$ Dunkl Laplacian Δ_S^2. The commutant of Δ_{S^2} was determined in the framework of the Racah problem for $sl_{-1}(2)$ and identified with a central extension of the Bannai–Ito algebra. Two bases for the unitary irreducible representations of this algebra on $L^2(S^2)$ were explicitly constructed in terms of the Dunkl spherical harmonics with the Bannai–Ito orthogonal polynomials arising in their overlaps.

Since the Dunkl operators and Laplacian can be defined for an arbitrary number of variables, it would be natural to look for the extension of the results presented here to spheres in higher dimensions. It would be also of interest to examine the situation on hyperboloids. We plan to pursue the study of these questions in the future.

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Partie V

Polynômes multi-orthogonaux matriciels et applications

Introduction

Le théorème de Favard [36] stipule que toute famille de polynômes orthogonaux $\{P_n(x)\}_{n=0}^{\infty}$ obéit à une relation de récurrence à trois termes de la forme

$$xP_n(x) = P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x),$$

avec $P_{-1}(x) = 0$, $P_0(x) = 1$ et où b_n et c_n sont des coefficients numériques. Réciproquement, il stipule aussi que toute famille $\{P_n(x)\}_{n=0}^{\infty}$ qui obéit à une relation de récurrence à trois termes de la forme ci-haut forme nécessairement une famille orthogonale.

Les polynômes multi-orthogonaux matriciels sont une généralisation des polynômes orthogonaux: ils sont caractérisés par des relations de récurrence d'ordre plus élevé [65]. Par exemple, supposons que l'on ait affaire à la relation de récurrence

$$xP_n(x) = a_n P_{n+2}(x) + b_n P_{n+1}(x) + c_n P_n(x) + d_n P_{n-1}(x) + e_n P_{n-2}(x),$$

avec certaines conditions initiales. Le polynôme vectoriel

$$Q_n(x) = (P_{2n}(x), P_{2n+1}(x))^{\top},$$

obéit alors à la relation de récurrence

$$xQ_n(x) = A_n Q_{n+1}(x) + B_n Q_n(x) + C_n Q_{n-1}(x),$$

où A_n, B_n et C_n sont maintenant des matrices 2×2. En général, les polynômes multi-orthogonaux obéissent à de multiples relations d'orthogonalité matricielles.

Dans cette partie de la thèse, on étudie deux nouvelles familles de polynômes vectoriels multi-orthogonaux qui interviennent dans les éléments de matrices d'exponentielles quadratiques dans les générateurs de $\mathfrak{su}(2)$. On utilise ensuite ces résultats dans l'étude des états cohérents et comprimés de l'oscillateur fini, ce qui mène aussi à la caractérisation d'une famille de polynômes matriciels multi-orthogonaux.

Chapitre 27

d-Orthogonal polynomials and $\mathfrak{su}(2)$

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Abstract. Two families of d-orthogonal polynomials related to $\mathfrak{su}(2)$ are identified and studied. The algebraic setting allows for their full characterization (explicit expressions, recurrence relations, difference equations, generating functions, etc.). In the limit where $\mathfrak{su}(2)$ contracts to the Heisenberg-Weyl algebra $\mathfrak{h}_1$, the polynomials tend to the standard Meixner polynomials and d-Charlier polynomials, respectively.

27.1 Introduction

We identify in this paper two families of d-orthogonal polynomials that are associated to $\mathfrak{su}(2)$. When available, algebraic models for orthogonal polynomials provide a cogent framework for the characterization of these special functions. They also point to the likelihood of seeing the corresponding polynomials occur in the description of physical systems whose symmetry generators form the algebra in question.

d-orthogonal polynomials generalize the standard orthogonal polynomials in that they obey higher recurrence relations. They will be defined below and have been seen to possess various applications [6, 7, 13]. Recently, two of us have uncovered the connection between d-Charlier polynomials and the Heisenberg algebra. Here we pursue this exploration of d-orthogonal polynomials related to Lie algebras by considering the case of $\mathfrak{su}(2)$. Remarkably, two hypergeometric families of such polynomials will be identified and characterized.

27.1.1 d-Orthogonal polynomials

The monic d-orthogonal polynomials $\widehat{P}_n(k)$ of degree n ($\widehat{P}_n(k) = k^n + \ldots$) can be defined by the recurrence relation [10]

$$\widehat{P}_{n+1}(k) = k\widehat{P}_n(k) - \sum_{\mu=0}^{d} a_{n,n-\mu}\widehat{P}_{n-\mu}(k), \tag{27.1}$$

of order $d+1$ with complex coefficients $a_{n,m}$; the initial conditions are $\widehat{P}_n = 0$ if $n < 0$ and $\widehat{P}_0 = 1$. It is assumed that $a_{n,n-d} \neq 0$ (non-degeneracy condition).

When $d = 1$, it is known that under the condition $c_n \neq 0$, the polynomials satisfying three-term recurrence relations

$$\widehat{P}_{n+1}(k) = k\widehat{P}_n(k) - b_n\widehat{P}_n(k) - c_n\widehat{P}_{n-1}(k),$$

are orthogonal with respect to a linear functional σ

$$\langle \sigma, \widehat{P}_n(k)\widehat{P}_m(k)\rangle = \delta_{nm},$$

defined on the space of all polynomials.

When $d > 1$, the polynomials $\widehat{P}_n(k)$ obey *vector orthogonality relations*. This means that there exists a set of d linear functionals σ_i for $i = 0, \ldots, d-1$ such that the following relations hold:

$$\begin{aligned} \langle \sigma_i, \widehat{P}_n\widehat{P}_m\rangle &= 0, \text{ if } m > dn + i, \\ \langle \sigma_i, \widehat{P}_n\widehat{P}_{dn+i}\rangle &\neq 0, \text{ if } n \geqslant 0. \end{aligned}$$

27.1.2 d-Orthogonal polynomials as generalized hypergeometric functions

Of particular interest are d-orthogonal polynomials that can be expressed in terms of generalized hypergeometric functions (see for instance [1, 2, 5]). These functions are denoted ${}_pF_q$ and are defined by

$${}_pF_q\left[\begin{matrix}\{a_p\} \\ \{b_q\}\end{matrix}; \frac{1}{c}\right] := \sum_{\mu=0}^{\infty} \frac{(a_1)_\mu \cdots (a_p)_\mu}{(b_1)_\mu \cdots (b_q)_\mu} \frac{c^{-\mu}}{\mu!},$$

where $(m)_0 = 1$ and $(m)_k = (m)(m+1)\cdots(m+k-1)$ stands for the Pochhammer symbol. In the case where one of the a_i's is a negative integer, say $a_1 = -n$ for $n \in \mathbb{N}$, the series truncates at $\mu = n$ and we can write

$${}_{1+s}F_q\left[\begin{matrix}-n, \{a_s\} \\ \{b_q\}\end{matrix}; \frac{1}{c}\right] = \sum_{\mu=0}^{n} \frac{(-n)_\mu (a_1)_\mu \cdots (a_s)_\mu}{(b_1)_\mu \cdots (b_q)_\mu} \frac{c^{-\mu}}{\mu!}. \tag{27.2}$$

If one of the b_i's is also a negative integer, the corresponding sequence of polynomials is finite.

The classification of d-orthogonal polynomials that have a hypergeometric representation of the form (27.2) has been studied recently in [3]. The results are as follows.

Let $s \geqslant 1$ and let $\{a_s\} = \{a_1, \ldots, a_s\}$ be a set of s polynomials of degree one in the variable k. This set is called s-separable if there is a polynomial $\pi(y)$ such that $\prod_{i=1}^{s}(a_i(k)+y) = [\prod_{i=1}^{s} a_i(k)] + \pi(y)$; an example of such s-separable set is $\{k\, e^{\frac{2\pi i x}{s}}, x = 0\ldots, s-1\}$. If the set $\{a_s\}$ is s-separable, then there exists only $2(d+1)$ classes of d-orthogonal polynomials of type (27.2) corresponding to the cases:

1. $s = 0, \ldots, d-1$ and $q = d$;
2. $s = d$ and $q = 0, \ldots, d-1$;
3. $s = q = d$ and $c \neq 1$;
4. $s = q = d+1$ and $c = 1$ and $\sum_{i=0}^{d+1} a_i(0) - \sum_{i=1}^{d+1} b_i \notin \mathbb{N}$.

Examples of d-orthogonal polynomials belonging to this classification have been found in [4, 14]. We shall here provide more examples that are of particular interest as well as cases that fall outside the scope of the classification given above.

27.1.3 Purpose and outline

We investigate in this paper d-orthogonal polynomials associated to $\mathfrak{su}(2)$. We shall consider two operators S and Q, each defined as the product of exponentials in the Lie algebra elements. We shall hence determine the action of these operators on the canonical $(N+1)$-dimensional irreducible representation spaces of $\mathfrak{su}(2)$. In both cases, the corresponding matrix elements will be found to be expressible in terms of d-orthogonal polynomials, some of them belonging to the above-mentioned classification. The connection with the Lie algebra $\mathfrak{su}(2)$ will be used to fully characterize the two families of d-orthogonal polynomials. The limit as $N \to \infty$, where $\mathfrak{su}(2)$ contracts to $\mathfrak{h}_1$, shall also be studied. In this limit, the polynomials are shown to tend on the one hand to the standard Meixner polynomials and on the other hand to d-Charlier polynomials.

The outline of the paper is as follows. In section 2, we recall, for reference, basic facts about the $\mathfrak{su}(2)$ algebra and its representations. We also define a set of $\mathfrak{su}(2)$-coherent states and summarize how $\mathfrak{su}(2)$ contracts to the Heisenberg-Weyl algebra in the limit as $N \to \infty$. We then define the operators S and Q that shall be studied along with their matrix elements. The biorthogonality and recurrence relations of these matrix elements are made explicit from algebraic considerations. Results obtained in [14] and [15] concerning the Meixner and d-Charlier polynomials shall also be recalled. In section 3, the polynomials arising from the operator S are completely characterized. The result involve two families of polynomials for which the generating functions, difference equations, ladder operators, etc. are explicitly provided. The contraction limit is examined in all these instances and shown to correspond systematically to the characterization of the Meixner polynomials. In section 4, the same program is carried out for the operator Q; the contraction in this case leads to d-Charlier polynomials. We conclude with an outlook.

27.2 The algebra $\mathfrak{su}(2)$, matrix elements and orthogonal polynomials

In this section, we establish notations and definitions that shall be needed throughout the paper.

27.2.1 $\mathfrak{su}(2)$ essentials

The $\mathfrak{su}(2)$ algebra and its irreducible representations

The Lie algebra $\mathfrak{su}(2)$ is generated by three operators J_0, J_+ and J_- that obey the commutation relations

$$[J_+, J_-] = 2J_0, \qquad [J_0, J_\pm] = \pm J_\pm. \tag{27.3}$$

The irreducible unitary representations of $\mathfrak{su}(2)$ are of degree $N+1$, with $N \in \mathbb{N}$. In these representations, $J_0^\dagger = J_0$ and $J_\pm^\dagger = J_\mp$, where † refers to the hermitian conjugate. We shall denote the orthonormal basis vectors by $|N,n\rangle$, for $n = 0,\ldots,N$. The action of the generators on those basis vectors is

$$J_+|N,n\rangle = \sqrt{(n+1)(N-n)}|N,n+1\rangle, \qquad J_-|N,n\rangle = \sqrt{n(N-n+1)}|N,n-1\rangle, \tag{27.4}$$

$$J_0|N,n\rangle = (n-N/2)|N,n\rangle. \tag{27.5}$$

The operators $J_\pm$ will often be referred to as "ladder operators". Note that the action of J_- and J_+ on the end point vectors is $J_-|N,0\rangle = 0$ and $J_+|N,N\rangle = 0$.

It is convenient to introduce the number operator $\mathcal{N}$, which is such that $\mathcal{N}|N,n\rangle = n|N,n\rangle$; it is easily seen that this operator is related to J_0 by the formula $\mathcal{N} = J_0 + N/2$. The most general action of any powers of the ladder operators $J_\pm$ on the basis vectors is expressible in terms of the

Pochhammer symbol. Indeed, one finds

$$J_+^k|\,N,n\,\rangle = \sqrt{\frac{(n+k)!(N-n)!}{n!(N-n-k)!}}|\,N,n+k\,\rangle = \sqrt{(-1)^k(n+1)_k(n-N)_k}|\,N,n+k\,\rangle, \tag{27.6}$$

$$J_-^k|\,N,n\,\rangle = \sqrt{\frac{n!(N-n+k)!}{(n-k)!(N-n)!}}|\,N,n-k\,\rangle = \sqrt{(-1)^k(-n)_k(N-n+1)_k}|\,N,n-k\,\rangle. \tag{27.7}$$

These formulas are obtained by applying (27.4) on the basis vector and by noting that $\frac{(n+k)!}{n!} = (n+1)_k$ and that $\frac{n!}{(n-k)!} = (-1)^k(-n)_k$ [9].

Coherent states

Let us introduce the $\mathfrak{su}(2)$-coherent states $|\,N,\eta\,\rangle$ defined as follows [11]

$$|\,N,\eta\,\rangle := \sqrt{\frac{1}{(1+|\eta|^2)^N}}\sum_{n=0}^{N}\binom{N}{n}^{1/2}\eta^n|\,N,n\,\rangle, \tag{27.8}$$

where η is a complex number. The action of the ladder operators on a specific coherent state $|\,N,\eta\,\rangle$ can be computed directly to find:

$$J_+|\,N,\eta\,\rangle = \eta^{-1}\mathcal{N}|\,N,\eta\,\rangle, \qquad J_-|\,N,\eta\,\rangle = \eta(N-\mathcal{N})|\,N,\eta\,\rangle. \tag{27.9}$$

These relations can be generalized to arbitrary powers of the ladder operators; one writes $\mathcal{N}$ in terms of J_0 and uses commutation relations (27.3) to obtain:

$$\begin{aligned} J_+^k|\,N,\eta\,\rangle &= (-1)^k\eta^{-k}(-\mathcal{N})_k|\,N,\eta\,\rangle,\\ J_-^k|\,N,\eta\,\rangle &= (-1)^k\eta^{k}(\mathcal{N}-N)_k|\,N,\eta\,\rangle. \end{aligned} \tag{27.10}$$

Note that the formulas involving the Pochhammer symbols are to be treated formally.

Contraction of $\mathfrak{su}(2)$ to $\mathfrak{h}_1$

The contraction of $\mathfrak{su}(2)$ is the limiting procedure by which $\mathfrak{su}(2)$ reduces to the Heisenberg–Weyl algebra. The Heisenberg algebra is generated by the creation-annihilation operators $a^\dagger$, a and the identity operator. This algebra is denoted $\mathfrak{h}_1$ and defined by the commutation relations

$$[a,a^\dagger] = \mathfrak{i}\mathfrak{d}, \text{ and } [a,\mathfrak{i}\mathfrak{d}] = [a^\dagger,\mathfrak{i}\mathfrak{d}] = 0, \tag{27.11}$$

where $\mathfrak{i}\mathfrak{d}$ stands for the identity operator. The contraction corresponds to taking the limit as $N\to\infty$; in order for this limit to be well defined, the ladder operators must be rescaled by a factor $\sqrt{N}$. The contracted ladder operators are denoted by

$$J_\pm^{(\infty)} = \lim_{N\to\infty}\frac{J_\pm}{\sqrt{N}}.$$

In this limit, the action of the ladder operators on the basis vectors $|N,n\rangle$ tends to the action of the creation-annihilation operators in the irreducible representation of the $\mathfrak{h}_1$ algebra, which is infinite-dimensional. Indeed, the following formulas are easily derived:

$$\lim_{N\to\infty} \frac{J_+}{\sqrt{N}} |N,n\rangle = \sqrt{n+1}|n+1\rangle,$$
$$\lim_{N\to\infty} \frac{J_-}{\sqrt{N}} |N,n\rangle = \sqrt{n}|n-1\rangle.$$

This limit shall be used to establish the correspondence with studies associated to the Heisenberg-Weyl algebra; see [14].

27.2.2 Operators and their matrix elements

The operators S, Q and their matrix elements, which will be the central objects of study, are now defined.

S, Q, their matrix elements and biorthogonality

Let a and b be complex parameters; we define

$$S := e^{aJ_+^2} e^{bJ_-^2},$$
$$Q := e^{aJ_+} e^{bJ_-^M},$$

with $M \in \mathbb{N}$. These operators can be represented by $(N+1)\times(N+1)$ matrices; their matrix elements are defined as

$$\psi_{k,n} := \langle k,N|S|N,n\rangle,$$
$$\varphi_{k,n} := \langle k,N|Q|N,n\rangle.$$

The two operators S, Q are obviously invertible, with inverses given by $S^{-1} = e^{-bJ_-^2} e^{-aJ_+^2}$ and $Q^{-1} = e^{-bJ_-^M} e^{-aJ_+}$. We denote by $\chi_{n,k} = \langle n,N|S^{-1}|N,k\rangle$ the matrix elements of S^{-1}; this leads to the following biorthogonality relation:

$$\sum_{k=0}^{N} \chi_{m,k}\psi_{k,n} = \sum_{k=0}^{N} \langle m,N|S^{-1}|N,k\rangle\langle k,N|S|N,n\rangle = \langle m,N|S^{-1}S|N,n\rangle = \delta_{nm},$$

where we have used the identity $\sum_{k=0}^{N}|N,k\rangle\langle k,N|| = \mathfrak{i}\mathfrak{d}$, which follows directly from the orthonormality of the basis $\{|N,k\rangle\}_{k=0}^{N}$. A similar relation can be written for the matrix elements of Q^{-1}. If one defines $\varsigma_{n,k} = \langle n,N|Q^{-1}|N,k\rangle$, then

$$\sum_{k=0}^{N} \varsigma_{m,k}\varphi_{k,n} = \delta_{nm}.$$

Recurrence relations and polynomial solutions

The algebraic nature of the operators S and Q leads to recurrence relations satisfied by the matrix elements $\psi_{k,n}$ and $\varphi_{k,n}$. Let us start by showing how the recurrence relation for the $\psi_{k,n}$'s arises. The first step is to observe that

$$(k-N/2)\psi_{k,n} = \langle k,N \mid J_0 S \mid N,n \rangle = \langle k,N \mid S S^{-1} J_0 S \mid N,n \rangle. \tag{27.12}$$

The quantity $S^{-1}J_0S$ is computed using the Baker-Campbell-Hausdorff formula:

$$e^X Y e^{-X} = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \frac{1}{3!}[X,[X,[X,Y]]] + \ldots.$$

Applying this formula and the relations of appendix A, one readily finds

$$S^{-1}J_0S = J_0 - 2bJ_-^2 + 2a[J_+ + 2b(1+2J_0)J_- - 4b^2J_-^3]^2.$$

Substituting this result into (27.12) yields for $\psi_{k,n}$ the recurrence relation

$$\begin{aligned} 2a\sqrt{(n+1)_2(n-N)_2}\,\psi_{k,n+2} &= (k-n)\psi_{k,n} + 2b\sqrt{(-n)_2(N-n+1)_2}\,\psi_{k,n-2} \\ &\quad + 2ab\sum_{t=0}^{3}(-b)^t\xi_t(n,N)\sqrt{(-n)_{2t}(N-n+1)_{2t}}\,\psi_{k,n-2t}, \end{aligned}$$

where the coefficients $\xi_t(n,N)$ are given by:

$$\begin{aligned} \xi_0(n,N) &:= 2[2n-N][2n^2-2nN-N+1], \\ \xi_1(n,N) &:= 4[6n^2-6n(N+2)+N^2+5N+9], \\ \xi_2(n,N) &:= 16(2n-N-4), \\ \xi_3(n,N) &:= 16. \end{aligned} \tag{27.13}$$

The recurrence relation for the matrix elements $\psi_{k,n}$ is not of the form (27.1). However, the jumps on the index n are all even and it is clear from (27.5) that any $\psi_{k,n}$ with different index parity will be zero. Thus, setting $n = 2j+q$, $k = 2\ell+q$ with $q = 0,1$ and introducing the monic polynomial $\psi_{k,n} = \psi_{k,q}\left(a^{-j}\sqrt{\frac{(N-n)!}{(N-q)!n!}}\right)\widehat{A}_j^{(q)}(\ell)$, the recurrence relation becomes

$$\begin{aligned} \widehat{A}_{j+1}^{(q)}(\ell) &= (\ell-j)\widehat{A}_j^{(q)}(\ell) + c[(-n)_2(N-n+1)_2]\widehat{A}_{j-1}^{(q)}(\ell) \\ &\quad + c\sum_{t=0}^{3}(-c)^t\xi_t(n,N)[(-n)_{2t}(N-n+1)_{2t}]\,\widehat{A}_{j-t}^{(q)}(\ell), \end{aligned} \tag{27.14}$$

with $c = ab$. This relation is precisely of the form (27.1); we thus conclude that the matrix elements $\psi_{k,n}$ are given in terms of two families of d-orthogonal polynomials with $d = 3$ corresponding to $q = 0$ and $q = 1$. These two families are fully characterized in the next section.

The recurrence relation for the matrix elements $\varphi_{k,n}$ can be obtained in the same fashion. It is first noted that

$$(k-N/2)\varphi_{k,n} = \langle k,N \mid J_0 Q \mid N,n \rangle = \langle k,N \mid QQ^{-1}J_0 Q \mid N,n \rangle.$$

The computation of $Q^{-1}J_0Q$ proceeds along the same lines. The result is

$$Q^{-1}J_0Q = J_0 + aJ_+ - MbJ_-^M + abM\big(M-1+2J_0\big)J_-^{M-1} - ab^2M^2J_-^{2M-1}. \tag{27.15}$$

Once again, introducing the monic polynomial $\varphi_{k,n} = \varphi_{k,0}\left(a^{-n}\sqrt{\frac{(N-n)!}{N!n!}}\right)\widehat{B}_n(k)$, this becomes

$$\begin{aligned}\widehat{B}_{n+1}(k) &= (k-n)\widehat{B}_n(k) + f\,\zeta_M(n,N)\widehat{B}_{n-M}(k)\\ &\quad + f\,\zeta_{M-1}(n,N)\widehat{B}_{n+1-M} + f^2\zeta_{2M-1}(n,N)\widehat{B}_{n+1-2M},\end{aligned} \tag{27.16}$$

with $f = a^M b$ and where the coefficients $\zeta(n,N)$ are:

$$\begin{aligned}\zeta_M(n,N) &:= (-1)^M M(-n)_M(N-n+1)_M,\\ \zeta_{M-1}(n,N) &:= (-1)^M M(-n)_{M-1}(N-n+1)_{M-1}(2n-M-N+1),\\ \zeta_{2M-1}(n,N) &:= (-1)^{2M-1}M^2(-n)_{2M-1}(N-n+1)_{2M-1}.\end{aligned}$$

Therefore, the polynomials $\widehat{B}_n(k)$ are d-orthogonal with $d = 2M-1$.

Contractions and the $\mathfrak{h}_1$ algebra

It is relevant at this point to recall related results obtained in connection with the Heisenberg algebra $\mathfrak{h}_1$. These are to be compared with the contractions of the polynomials $\widehat{A}_j^{(q)}(\ell)$ and $\widehat{B}_n(k)$ obtained in the next sections.

In [15], two of us investigated the matrix elements

$$\psi_{k,n}^{(\infty)} = \langle k \mid e^{b(a^\dagger)^2} e^{ca^2} \mid n \rangle,$$

where a and $a^\dagger$ are the generators of $\mathfrak{h}_1$ and the vectors $\mid n \rangle$ are the basis vectors of its irreducible representation. It was shown that these matrix elements are given in terms of two series of Meixner polynomials with different parameters. Indeed, with $n = 2j+q$ and $k = 2\ell+q$, one finds[1]

$$\psi_{k,n}^{(\infty)} \propto M_j(\ell; 1/2+q, z).$$

It is clear that this operator corresponds to the contraction of the operator S previously defined. Thus, the polynomials $\widehat{A}_j^{(q)}(k)$ are expected to tend to the Meixner polynomials in the limit as $N\to\infty$ and can be interpreted as a d-orthogonal finite "deformation" of Meixner polynomials.

[1]See appendix B for definition and properties of Meixner polynomials $M_n(x;\beta,z)$.

In [14], we investigated the matrix elements

$$\varphi_{k,n}^{(\infty)} = \langle\, k \mid e^{\beta a^\dagger} e^{\sigma a^M} \mid n\,\rangle.$$

These matrix elements were found to be d-Charlier polynomials with $d = M$. These matrix elements correspond to the contraction of the matrix elements of the operator Q just defined. Consequently, it is expected that the contraction of the operator Q will lead to these d-Charlier. Moreover, the matrix elements $\varphi_{k,n}$ are expected to yield the standard Krawtchouk polynomials $K_n(x;p,N)$ when[2] $M = 1$; the cases $M \neq 1$ correspond therefore to some d-Krawtchouk polynomials.

27.3 Characterization of the $\widehat{A}_j^{(q)}(\ell)$ family

We shall now completely characterize the family of d-orthogonal polynomials arising from the matrix elements of the operator $S = e^{aJ_+^2} e^{bJ_-^2}$; these matrix elements have already been shown to satisfy the recurrence relation (27.14). The general properties are computed first and contractions are studied thereafter.

27.3.1 Properties

Explicit matrix elements

We first look for the explicit expression of the matrix elements $\psi_{k,n} = \langle\, k,N \mid S \mid N,n\,\rangle$. This expression is obtained by setting $n = 2j+q$ and $k = 2\ell+q$, expanding the exponentials in series, using the actions (27.6) and (27.7) and recalling the identity $(a)_{2n} = 2^{2n}\left(\frac{a}{2}\right)_n\left(\frac{a+1}{2}\right)_n$ as well as $(2n+q)! = 2^{2n}\, q!\, n!\, (q+1/2)_n$. Extracting the factor

$$\psi_{k,q} = \frac{a^\ell}{\ell!}\sqrt{\frac{(N-q)!k!}{(N-k)!}},$$

and pulling out the normalization factor $(a)^{-j}\sqrt{\frac{(N-n)!}{(N-q)!n!}}$ ensuring that the polynomials are monic yields

$$\psi_{k,n} = \left((a)^{-j}\sqrt{\frac{(N-n)!}{(N-q)!n!}}\right)\widehat{A}_j^{(q)}(\ell;c,N)\,\psi_{k,q}, \tag{27.17}$$

where we have set

$$\widehat{A}_j^{(q)}(\ell;c,N) := \frac{c^j}{j!}\frac{(N-q)!n!}{(N-n)!}\,{}_2F_3\left[\begin{matrix} -j & -\ell & \\ q+1/2 & \frac{q-N}{2} & \frac{q-N+1}{2} \end{matrix};\frac{1}{16c}\right], \tag{27.18}$$

[2] See appendix B for definition and properties of Krawtchouk polynomials $K_n(x;p,N)$.

with $c = ab$. It is understood that if n and k have different parities, the matrix element $\psi_{k,n}$ is zero.

We thus have an explicit representation of the polynomials $\widehat{A}_j^{(q)}(k;c,N)$ in terms of generalized hypergeometric functions. These polynomials satisfy the recurrence relation (27.12). Moreover, the fact that $\widehat{A}_j^{(q)}(k;c,N)$ is expressed as a ${}_2F_3$ indicates that these polynomials belong to the classification proposed in [3]. Indeed, it is clear that the singleton $\{-\ell\}$ is 1-separable; consequently, the polynomials are of the form (27.2) with $s = 1$. Thus, the polynomials $\widehat{A}_j^{(q)}(k;c,N)$ are examples of d-orthogonal polynomials corresponding to the case 1 of the classification with $s = 1$ and $q = d = 3$.

Explicit inverse matrix elements

The matrix elements $\chi_{n,k}$ of the inverse operator $S^{-1} = e^{-bJ_-^2}e^{-aJ_+^2}$ can also be evaluated explicitly; they can be computed either by directly expanding the exponentials or by inspection. Indeed, one finds the matrix elements of the inverse to be

$$\chi_{n,k} = \psi^{\star}_{N-k,N-n}, \tag{27.19}$$

with $\star$ denoting the substitutions $a \to -a$ and $b \to -b$. If N is an even number of the form $N = 2p + 2q$, the inverse is given by:

$$\chi_{n,k} = (-1)^{j-\ell}\frac{(a)^{j-\ell}}{(p-\ell)!}\sqrt{\frac{(N-k)!n!}{k!(N-n)!}}\widehat{A}_{p-j}^{(q)}(p-\ell;c,N), \tag{27.20}$$

with $n = 2j + q$ and $k = 2\ell + q$. If N is an odd integer, the cases $q = 0$ and $q = 1$ have to be treated separately; since no further difficulty arise and $\chi_{n,k}$ is expressed directly in terms of $\psi_{n,k}$, we omit the details.

Biorthogonality relation

As pointed out in section 2.2.1, the matrix elements of the inverse S^{-1} provide the polynomials entering in the biorthogonality relations of the $\widehat{A}_j^{(q)}(\ell;c,N)$ family. In view of (27.19), this biorthogonality relation reads

$$\sum_{k=0}^{N}\psi_{k,n}\psi^{\star}_{N-k,N-m} = \delta_{nm}. \tag{27.21}$$

If $N = 2p + 2q$, the biorthogonality relation involves two $\widehat{A}_j^{(q)}(\ell;c,N)$ with the same q; in the case where $N = 2p + 1$, the relation will involve two polynomials with different values of q.

Firstly, set $N = 2p + 2q$ and $m = 2j' + q$; the biorthogonality relation (27.21) becomes

$$\sum_{\ell=0}^{p} w_\ell(p)\widehat{A}_j^{(q)}(\ell;c,N)\widehat{A}_{p-j'}^{(q)}(p-\ell;c,N) = (-1)^j\delta_{jj'}$$

with the weight $w_\ell(p) = \frac{(-1)^\ell}{\ell!(p-\ell)!}$.

Secondly, set $N = 2p+1$ and $m = 2j' + q$; the biorthogonality relation is then of the form

$$\sum_{\ell=0}^{p} w_\ell(p) \widehat{A}_j^{(0)}(\ell;c,N)\widehat{A}_{p-j'}^{(1)}(p-\ell;c,N) = (-1)^j \delta_{jj'}.$$

A similar relation can be obtained with the $q=0$ and $q=1$ polynomials in a different order. Thus, the two classes of polynomials corresponding to the values $q=0$ and $q=1$ of the $\widehat{A}_j^{(q)}(\ell;c,N)$ family are interlaced when the degree of the representation is odd and independent if the degree is even.

Generating function

We now derive the generating function for the $\widehat{A}_j^{(q)}(\ell;c,N)$ polynomials. This derivation is based on the action of the operator S on coherent states $|\,N,\eta\,\rangle$. Consider the following function:

$$G(k;\eta) := \frac{1}{\psi_{k,q}}\sqrt{\frac{a^{-q}(N-q)!}{N!}}\sum_{n=0}^{N}\binom{N}{n}^{1/2}\psi_{k,n}\,\eta^n. \tag{27.22}$$

Upon substitution of the expression for the matrix elements, one finds

$$G(k;\eta) = \sum_{n=0}^{N}\widehat{A}_j^{(q)}(\ell;c,N)\frac{(\eta/\sqrt{a})^n}{n!},$$

setting $G(k;\eta)$ as a generating function for the polynomials $\widehat{A}_j^{(q)}(\ell;c,N)$. This generating function $G(k;\eta)$ is in fact the matrix element of S with respect to the coherent state $|\,N,\eta\,\rangle$. Indeed, with definition (27.8), it follows that

$$\begin{aligned} G(k;\eta) &= \frac{1}{\psi_{k,q}}\sqrt{\frac{a^{-q}(N-q)!}{N!}}\sum_{n=0}^{N}\langle\, k,N\,|\,S\,|\,N,n\,\rangle\langle\, n,N\,|\,N,\eta\,\rangle \\ &= \frac{1}{\psi_{k,q}}\sqrt{\frac{a^{-q}(N-q)!}{N!}}\,\langle\, k,N\,|\,S\,|\,N,\eta\,\rangle, \end{aligned}$$

where the normalization factor was omitted. The matrix element $\langle\, k,N\,|\,S\,|\,N,\eta\,\rangle$ can be evaluated by using the spectral decomposition. One first decomposes the matrix element as:

$$\langle\, k,N\,|\,S\,|\,N,\eta\,\rangle = \sum_{\mu=0}^{N}\langle\, k,N\,|\,e^{aJ_+^2}\,|\,N,\mu\,\rangle\langle\,\mu,N\,|\,e^{bJ_-^2}\,|\,N,\eta\,\rangle.$$

The action of the ladder operators $J_\pm$ on the coherent states can then be used to obtain:

$$G(\ell;q;\eta) = (e^{i\pi/2})^{N+q}(-\eta\sqrt{b})^N\sum_{m=0}^{\lfloor (N-q)/2\rfloor}\frac{(1/\sqrt{c})^{2m+q}}{(2m+q)!}(-\ell)_m\,H_{N-2m-q}\left(\frac{e^{i\pi/2}}{2\eta\sqrt{b}}\right), \tag{27.23}$$

where H_n is the standard Hermite polynomial[3].

[3]See appendix B for definition of the Hermite polynomials.

Difference equation

In a fashion dual to the way the recurrence relation was obtained, the difference equation satisfied by the matrix elements $\psi_{k,n}$ can be derived with the help of the Baker-Campbell-Hausdorff formula and the formulas from appendix A. First, observe that

$$(n-N/2)\psi_{k,n} = \langle\, k,N \mid SJ_0 \mid N,n\,\rangle = \langle\, k,N \mid SJ_0S^{-1}S \mid N,n\,\rangle.$$

The operator SJ_0S^{-1} must be evaluated here. Using the formulas given in the appendix, one obtains

$$SJ_0S^{-1} = J_0 - 2aJ_+^2 + 2b[J_- + 2aJ_+(1+2J_0) - 4a^2J_+^3]^2.$$

Substituting the result, one finds

$$\begin{aligned}(n-k)\psi_{k,n} &= 2b\sqrt{(k+1)_2(k-N)_2}\,\psi_{k+2,n} - 2a\sqrt{(-k)_2(N-k+1)_2}\,\psi_{k-2,n}\\ &\quad - 2ab\sum_{t=0}^{3}(-a)^t\xi_t(k,N)\sqrt{(-k)_{2t}(N-k+1)_{2t}}\,\psi_{k-2t,n},\end{aligned}$$

where the coefficients $\xi_t(k,N)$ are those found in (27.13). The difference equation for the matrix elements $\psi_{n,k}$ can straightforwardly be turned into a difference equation for the family of polynomials $\widehat{A}_j^{(q)}(k;c,N)$. Indeed, we have

$$\begin{aligned}(j-\ell)\widehat{A}_j^{(q)}(\ell;c,N) &= c\,\Omega_\ell(q;N)\,\widehat{A}_j^{(q)}(\ell+1;c,N) - \ell\,\widehat{A}_j^{(q)}(\ell-1;c,N)\\ &\quad - c\sum_{t=0}^{3}(-\ell)_t\xi_t(2\ell+q,N)\,\widehat{A}_j^{(q)}(\ell-t;c,N),\end{aligned} \tag{27.24}$$

with

$$\Omega_\ell(N) = \frac{1}{\ell+1}(2\ell+q+1)_2(2\ell+q-N)_2.$$

This difference equation can be written as an eigenvalue problem; denoting $\nabla f(x) = f(x) - f(x-1)$, $\Delta f(x) = f(x+1) - f(x)$ and using the well-known identities

$$f(x+t) = \sum_{w=0}^{t}\binom{t}{w}\Delta^w f(x),$$
$$f(x-t) = \sum_{w=0}^{t}(-1)^w\binom{t}{w}\nabla^w f(x),$$

the recurrence relation can be written as

$$\mathscr{H}^{(q)}\widehat{A}_j^{(q)}(\ell;c,N) = j\widehat{A}_j^{(q)}(\ell;c,N),$$

with

$$\mathscr{H}^{(q)} = \ell\,\nabla + c\,\Omega_\ell(q;N)\sum_{w=0}^{1}\Delta^w - c\sum_{t=0}^{3}(-\ell)_t\xi_t(2\ell+q,N)\sum_{w=0}^{t}(-1)^w\binom{t}{w}\nabla^w.$$

Shift operators

The shift operators can also be derived from the Baker–Campbell–Hausdorff formula and the formulas from appendix A. We have

$$\sqrt{(-n)_2(N-n+1)_2}\,\psi_{k,n-2} = \langle\, k,N \mid S J_-^2 \mid N,n\,\rangle = \langle\, k,N \mid S J_-^2 S^{-1} S \mid N,n\,\rangle.$$

One then easily obtains

$$S J_-^2 S^{-1} = [J_- + 2aJ_+(1+2J_0) - 4a^2 J_+^3]^2.$$

Upon substitution in the previous equation, one gets

$$\sqrt{(-n)_2(N-n+1)_2}\,\psi_{k,n-2} = \sqrt{(k+1)_2(k-N)_2}\,\psi_{k+2,n} - a\sum_{t=0}^{3}(-a)^t \xi_t(k,N)\sqrt{(-k)_{2t}(N-k+1)_{2t}}\,\psi_{k-2t,n}, \tag{27.25}$$

where the coefficients $\xi_t(k,N)$ are those given in (27.13). Using the identities involving the difference operators introduced above, we obtain

$$F^{(q)}\widehat{A}_j^{(q)}(\ell;c,N) = (-2j-q)_2(N-2j-q+1)_2\,\widehat{A}_{j-1}^{(q)}(\ell;c,N),$$

with

$$F^{(q)} = \Omega_\ell(q;N)\sum_{w=0}^{1}\Delta^w - \sum_{t=0}^{3}(-\ell)_t\xi_t(k,N)\sum_{w=0}^{t}(-1)^w\binom{t}{w}\nabla^w.$$

The computation of the backward shift operator for the $\widehat{A}_j^{(q)}(\ell;c,N)$ is much more involved; the operator contains terms up to order 18. The explicit expression will not be given.

d-Orthogonality functionals

From the recurrence relation (27.12) computed in section 2, we concluded that the polynomials $\widehat{A}_j(\ell;c,N)$ are d-orthogonal with $d=3$. We now provide the functionals with respect to which the $\widehat{A}_j(\ell;c,N)$ polynomials are orthogonal.

The initial step is to compute the recurrence relation satisfied by the matrix elements $\chi_{n,k}$ of the inverse operator. First, we note that

$$(k-N/2)\,\chi_{n,k} = \langle\, n,N \mid S^{-1}J_0 \mid N,k\,\rangle = \langle\, n,N \mid S^{-1}J_0 S S^{-1} \mid N,k\,\rangle.$$

Proceeding along similar lines as before, we get

$$S^{-1}J_0 S = J_0 - 2bJ_-^2 + 2a[J_+ + 2b(1+2J_0)J_- - 4b^2 J_-^3]^2.$$

We then find the recurrence relation to be

$$
\begin{aligned}
(k-n)\,\chi_{n,k} &= 2a\sqrt{(-n)_2(N-n+1)_2}\,\chi_{n-2,k} - 2b\sqrt{(n+1)_2(n-N)_2}\,\chi_{n+2,k} \\
&\quad - 2ab\sum_{t=0}^{3}(-b)^t\sigma_t(n,N)\sqrt{(n+1)_{2t}(n-N)_{2t}}\,\chi_{n+2t,k},
\end{aligned}
\tag{27.26}
$$

where the coefficients are:

$$
\begin{aligned}
\sigma_0(n,N) &:= 2(2n-N)(1+2n(n-N)-N),\\
\sigma_1(n,N) &:= 4(6n^2-6n(N+2)+N^2-7N+9),\\
\sigma_2(n,N) &:= 16(2n-N+4),\\
\sigma_3(n,N) &:= 16.
\end{aligned}
$$

From this recurrence relation, the following proposition can be stated.

Proposition 12. *We have*

$$
\chi_{2j+q,2\ell+q} = \sum_{i=0}^{2} Y_i^{(j)}\,\Xi_i(\ell),
$$

where $\Xi_i(\ell) = \chi_{2i+q,2\ell+q}$ *and* Y_i *is a polynomial in the variable* ℓ. *The degree of the polynomial is determined as follows. Suppose* $j = 3\gamma+\delta$ *with* $\delta = 0,1,2$*; then, if* $i \leq \delta$, *the degree of the polynomial is equal to* γ, *otherwise it is equal to* $\gamma-1$.

Proof. This statement is obtained directly from the recurrence relation. Setting $n = 2j+q = 0$ yields the element $\chi_{2(3)+q,k}$ in terms of the elements $\chi_{2(2)+q,k}$, $\chi_{2(1)+q,k}$, $\chi_{q,k}$ with degree 1 in ℓ in the case of $\chi_{0+q,k}$ and 0 for the others. The proposition then follows by induction. □

With this statement, we define the following linear functionals

$$
\mathscr{L}_i^{(q)}[f(x)] = \sum_{x=0}^{\lfloor (N-q)/2\rfloor} \frac{a^x}{x!}\sqrt{\frac{(2x+q)!}{(N-2x-q)!}}\,\Xi_i(x)f(x),
$$

for $i = 0,1,2$. From the biorthogonality relation, it follows that

$$
\mathscr{L}_i^{(q)}[\ell^\gamma \widehat{A}_j^{(q)}(\ell;c,N)]
\begin{cases}
= 0, & \text{if } j \geqslant 3\gamma+i+1;\\
\neq 0, & \text{if } j = 3\gamma+i.
\end{cases}
\tag{27.27}
$$

With these relations, the d-orthogonality of the family of polynomials $\widehat{A}_j^{(q)}(\ell;c,N)$ is manifest.

The algebraic setting at hand has thus allowed to completely characterize the family of polynomials $\widehat{A}_j^{(q)}(\ell;c,N)$. The Baker-Campbell-Hausdorff relation and the formulas from appendix A were used to derive the recurrence relation, the difference equations as well as the forward shift operator. Moreover, the action of the operators $J_\pm$ on the basis vectors $\mid N,n\,\rangle$ of the $(N+1)$-dimensional irreducible representation of $\mathfrak{su}(2)$ was used to compute explicitly the expression for the polynomials $\widehat{A}_j^{(q)}(\ell;c,N)$. As it turns out, these polynomials are d-orthogonal with $d=3$. They have a simple representation as ${}_2F_3$ generalized hypergeometric function and fall into the classification of Ben Cheikh *et al.* due to the 1-separability of the singleton set $\{\ell\}$. The inverse elements were found using the symmetry of the vectors $\mid N,n\,\rangle$ with respect to the action of the operators $J_\pm$; those inverse elements proved useful in obtaining the three orthogonality functionals for the $\widehat{A}_j^{(q)}(\ell;c,N)$'s. In the next subsection, we will look at the contractions of the $\widehat{A}_j^{(q)}(\ell;c,N)$ and their structural formulas.

27.3.2 Contractions

We now turn to the evaluation of the contraction of the $\widehat{A}_j^{(q)}(\ell;c,N)$. We recall that the procedure of contraction, explained in section 2, corresponds to taking the limit as $N\to\infty$ after renormalizing the operators $J_\pm$ by a factor of $\sqrt{N}$.

Contraction of the recurrence relation

Let us first apply the contraction to the recurrence relation. The renormalization of the $\mathfrak{su}(2)$ generators is implemented by substituting c by c/N^2. We denote

$$\lim_{N\to\infty}\widehat{A}_j^{(q)}(\ell;c,N)=\widehat{M}_j(\ell),$$

provided that this limit exists. After straightforward manipulations, one finds that after the renormalization, the terms in $\widehat{A}_{j-3}^{(q)}(\ell;c,N)$ and $\widehat{A}_{j-2}^{(q)}(\ell;c,N)$ are of order $\mathcal{O}(N^{-2})$ and $\mathcal{O}(N^{-1})$, respectively. Consequently, they tend to zero in the limit as $N\to\infty$ and the recurrence relation becomes

$$\begin{aligned}\ell\,\widehat{M}_j(\ell)&=\widehat{M}_{j+1}(\ell)+\Big\{j-c\,[2+4(2j+q)]\Big\}\widehat{M}_j(\ell)\\&\quad-\Big\{c(2j+q)(2j+q-1)[1-4c]\Big\}\widehat{M}_{j-1}(\ell).\end{aligned}$$

The initial 5-term recurrence relation contracts to a 3-term recurrence relation. To identify to which orthogonal polynomial this relation corresponds, we set $-4c=\frac{d}{1-d}$. The recurrence becomes

$$\begin{aligned}\ell\,\widehat{M}_j(\ell)&=\widehat{M}_{j+1}(\ell)+\frac{1}{1-d}\Big\{j+d(j+q+1/2)\Big\}\widehat{M}_j(\ell)\\&\quad+\frac{d}{(1-d)^2}\Big\{(j+q/2)(j+q/2-1/2)\Big\}\widehat{M}_{j-1}(\ell).\end{aligned}\tag{27.28}$$

We recognize in (27.28) the normalized recurrence relation of the Meixner polynomials $M_j(\ell;\beta,d)$ with $\beta=q+1/2$, for $q=0,1$. Thus, as expected, we have

$$\lim_{N\to\infty}\widehat{A}_j^{(q)}(\ell;c,N)=M_j\left(\ell,q+1/2,\frac{c}{c-4}\right).$$

Contraction of the matrix elements

It is relevant to examine directly the contraction of the explicit formulas obtained for the matrix elements $\psi_{n,k}$. To take this limit, one must first expand $\psi_{n,k}$ by writing the generalized hypergeometric function as a truncated sum. Then, using the same renormalization as in the previous contraction, one straightforwardly obtains

$$\lim_{N\to\infty} \psi_{n,k} = \frac{a^{\ell}\, b^{j}}{\ell!\, j!}\sqrt{k!\,n!}\,{}_2F_1\left[\begin{matrix} -j-\ell \\ (q+1/2)\end{matrix};\frac{1}{4c}\right]. \tag{27.29}$$

Here, the explicit expression of the Meixner polynomials in terms of Gauss hypergeometric functions is recovered. The contraction at this level exhibits clearly the relationship between the parameters of the polynomial family $\widehat{A}_j^{(q)}(\ell;c,N)$ and the Meixner polynomials. We have $\beta = q+1/2$ and $1-\frac{1}{d} = \frac{1}{4c}$. This is the result found in [15].[4]

Contraction of the generating function and coherent states

The contraction limit of the generating function $G(k;\eta)$ can also be taken. However, it is easy to see that in the limit $N\to\infty$, the coherent states $|\,N,\eta\,\rangle$ as given by (27.8) are ill-defined. Indeed, under the contraction of the $\mathfrak{su}(2)$ algebra, the radius of the Bloch sphere, on which the coherent states are defined, must also be taken to infinity. Therefore, a well-defined contraction of the coherent states requires to take the limit $N\to\infty$ with the renormalization $\eta\to\eta/\sqrt{N}$ [11]. Note that with this renormalization, the coherent states become eigenstates of the operator $J_-/\sqrt{N}$ in the limit $N\to\infty$.

Performing the same renormalization, the contraction of the generating function yields that of the normalized Meixner polynomials

$$\lim_{N\to\infty} G(k;\eta) = \sum_{n=0}^{\infty} \widehat{M}_j(\ell,q+1/2,c)\frac{(\eta/\sqrt{a})^n}{n!} = e^{b\eta^2}\,\eta^q\;{}_1F_1\left[\begin{matrix} -\ell \\ q+1/2\end{matrix};-\frac{\eta^2}{4a}\right].$$

Contraction of inverse matrix elements and biorthogonality

In section 3.1, the symmetry of the irreducible representation of $\mathfrak{su}(2)$ was used to obtain that $\chi_{n,k} = \psi^{\star}_{N-k,N-n}$. This result is due to the fact that in this representation, there exists a vector $|\,N,0\,\rangle$ which is annihilated by J_- and a vector $|\,N,N\,\rangle$ which is annihilated by the operator J_+. In the contraction limit, the symmetry of the representation is not preserved. Indeed, the vectors of the $\mathfrak{h}_1$ irreducible representation are also labeled by a positive integer n, but this integer is

[4] To recover the full result, one has to take $a\to-\overline{\omega}$, $b\to\omega$ and eliminate the diagonal term in expansion of [15].

unbounded from above; more precisely, there exits no basis vector such that $a^\dagger|\, n\,\rangle = 0$, but the relation $a|\,0\,\rangle = 0$ still holds. Consequently, all relations involving the inverse matrix elements $\chi_{n,k}$ must be recalculated with the contraction applied directly to the operators. We stress that this loss of symmetry under the limit $N \to \infty$ is the *reason* for the drastic change of behavior of the polynomials $\widehat{A}_j^{(q)}(\ell;c,N)$ and for the fact that, in particular, the d-orthogonality reduces to the standard orthogonality.

To find the expression for the inverse matrix elements in the contraction limit, one can simply calculate, directly from the operators, the matrix elements of $S^{-1} = e^{-ba^2}e^{-a(a^\dagger)^2}$ written as $\chi_{n,k} = \langle\, n\,|\,S^{-1}\,|\,k\,\rangle$ or take the contraction of the recurrence relation. Meixner polynomials with changes in the arguments are found.

Other limits involving objects such as the difference equation, the forward shift operators can be taken in the same way, yielding their counterparts for the Meixner polynomial.

27.4 Characterization of the $\widehat{B}_n(k)$ family

In this section, we fully characterize the d-orthogonal polynomials, with $d = 2M-1$, in terms of which the matrix elements of the operator $Q = e^{aJ_+}e^{bJ_-^M}$ are expressed; these polynomials have already been shown to obey the recurrence relation (27.16).

27.4.1 Properties

Explicit expression for the matrix elements of Q

The matrix elements $\varphi_{k,n} = \langle\, k,N\,|\,Q\,|\,N,n\,\rangle$ can be computed explicitly by expanding the exponentials in series and using the actions (27.6) and (27.7). To express the matrix elements in terms of generalized hypergeometric functions, one needs the identities

$$(a)_{Mn} = M^{Mn}\prod_{\beta=0}^{M-1}\left(\frac{a+\beta}{M}\right)_n \text{ and } (Mn+q)! = M^{Mn}\,q!\prod_{\beta=0}^{M-1}\left(\frac{q+\beta+1}{M}\right)_n.$$

Setting $n = Mj+q$ with $q \in \{0,\ldots,M-1\}$, one finds

$$\varphi_{k,n} = \frac{a^{k-q}b^j}{j!(k-q)!q!}\sqrt{k!n!}\sqrt{\frac{(N-q)!(N-q)!}{(N-k)!(N-n)!}}\;{}_{1+M}F_{2M-1}\left[\begin{matrix}-j & \{\alpha_m\} \\ \{\beta_m\} & \{\gamma_m\}\end{matrix};\frac{-1}{(Ma)^M b}\right],$$

with $\{\alpha_m\} = (q-k+m)/M$, $\{\beta_m\} = (q+m+1)/M$ with $q+m+1 = M$ excluded from the sequence and $\{\gamma_m\} = (q-N+m)/M$, with m running from 0 to $M-1$.

To obtain the exact expression for the $\widehat{B}_n(k)$ polynomials, one must pull out the "ground state"

$$\varphi_{k,0} = a^k\binom{N}{k}^{1/2},$$

and the normalization factor $a^{-n}\sqrt{\frac{(N-n)!}{N!n!}}$ from the expression of the matrix elements. The final expression reads, with $f = a^M b$:

$$\varphi_{k,n} = \widehat{B}_n(k;f,N)\left[a^{-n}\sqrt{\frac{(N-n)!}{N!n!}}\right]\varphi_{k,0},$$

with

$$\widehat{B}_n(k;f,N) = \frac{(-1)^q(f)^j(-k)_q}{j!q!}\frac{(N-q)!n!}{(N-n)!}\,{}_{1+M}F_{2M-1}\left[\begin{matrix}-j & \{\alpha_m\} \\ \{\beta_m\} & \{\gamma_m\}\end{matrix};\frac{-1}{M^M f}\right]. \tag{27.30}$$

The recurrence relation (27.16) indicates that these polynomials are d-orthogonal with $d = 2M-1$, but it is clear that the set $\{\alpha_m\}$ is s-separable only for $M = 1$, it is 1-separable in this case. However, with this value of M, the polynomials are expressed as Gauss hypergeometric functions and are simply related to the Krawtchouk polynomials. For $M = 1$, the matrix elements are

$$\varphi_{k,n} = a^k b^n \binom{N}{n}^{1/2}\binom{N}{k}^{1/2} K_n(k;p,N),$$

with $p = -ab$. For any other value M, the polynomials are d-orthogonal extensions of the Krawtchouk ones, but they fall outside the classification of [3].

Matrix elements of Q^{-1}

The matrix elements of the inverse operator $\varsigma_{n,k} = \langle\, n,N \mid Q^{-1} \mid N,k\,\rangle$ with $Q^{-1} = e^{-bJ_-^M}e^{-aJ_+}$ can also be computed directly. Just as for the matrix elements $\psi_{n,k}$ of the S operator, the matrix elements $\varsigma_{n,k}$ of the inverse operator Q^{-1} possess a reflection symmetry; we indeed find

$$\varsigma_{n,k} = \varphi^\star_{N-k,N-n}, \tag{27.31}$$

where $\star$ denotes the replacements $a \to -a$ and $b \to -b$. In terms of the matrix elements $\varphi_{k,n}$, this biorthogonality relation reads

$$\sum_{k=0}^{N}\varphi_{k,n}\varphi^\star_{N-k,N-m} = \delta_{nm}.$$

Because of the asymmetric form of the operator, the explicit expression for the matrix elements $\varsigma_{n,k}$ heavily depends on the behavior of k and N modulo M. An exact expression would thus comprise M^2 cases of residues.

For definiteness, we shall set $M = 2$ whenever general expressions cannot be found in closed-form. Note that this case is relevant in the contraction limit because such powers of the creation-annihilation operators appear in the oscillator algebra $\mathfrak{sch}_1$ [15].

Biorthogonality relations

The biorthogonality of the $\widehat{B}_n(k;f,N)$ polynomials can be written for any value of M. Indeed, one has

$$\sum_{k=0}^{N} w_k \widehat{B}_n(k;f,N)\widehat{B}_{N-m}(N-k;f',N) = (-1)^n \delta_{nm}, \tag{27.32}$$

with $w_k = \frac{(-1)^k}{k!(N-k)!}$ and $f' = (-1)^{M+1} f$. Note that when $M = 1$, this is not exactly the orthogonality relation of the Krawtchouk polynomials. To obtain the orthogonality of the Krawtchouk polynomials from this equation, one must remove the normalization factor and use Pfaff's transformation. This transformation is only available for ${}_2F_1$ hypergeometric functions, thus, $M = 1$ is the only case for which the biorthogonality relation degenerates into the standard orthogonality.

Generating function

The generating function for the $\widehat{B}_n(k;f,N)$ polynomials is obtained as in the previous section; it can be derived explicitly for any value of M. Define

$$G(k;\eta) = \frac{1}{\varphi_{k,0}} \sum_{n=0}^{N} \binom{N}{n}^{1/2} \varphi_{k,n} \eta^n. \tag{27.33}$$

Substituting the expression for the matrix elements, one gets

$$G(k;\eta) = \sum_{n=0}^{N} \widehat{B}_n(k;f,N) \frac{(\eta/a)^n}{n!},$$

yielding a generating function for the polynomials $\widehat{B}_n(k;f,N)$. Once again, this generating function is expressed as the overlap between the vector $|\,N,k\,\rangle$ and $Q|\,N,\eta\,\rangle$. Indeed, it follows from the definition of the coherent states (27.8) that

$$G(k;\eta) = \frac{1}{\varphi_{k,0}} \langle\, k,N\,|\,Q\,|\,N,\eta\,\rangle.$$

Using the action of the ladder operators (27.9) and (27.10), one easily finds

$$G(k;\eta) = \sum_{\mu=0}^{N} \frac{(-\eta/a)^\mu}{\mu!} (-k)_\mu \, {}_MF_0\left[\{\delta_m\};(-1)^M (M\eta)^M b\right], \tag{27.34}$$

with $\{\delta_m\} = \frac{\mu-N+m}{M}$ with $m = 0,\ldots,M-1$. After a slight adjustment in the parameters and use of the identity ${}_1F_0(-a;b) = (1-b)^a$, the generating function for the Krawtchouk (see Appendix B) polynomials can be recovered for $M = 1$.

Difference equation

The algebraic setting can be used to derive the difference equation satisfied by the matrix elements $\varphi_{n,k}$. Again, one writes

$$(n-N/2)\varphi_{k,n} = \langle k,N \mid QJ_0 \mid N,n \rangle = \langle k,N \mid QJ_0Q^{-1}Q \mid N,n \rangle.$$

From the Baker–Campbell–Hausdorff relation, it follows that

$$QJ_0Q^{-1} = J_0 - aJ_+ + Mb[J_- + 2aJ_0 - a^2J_+]^M.$$

Setting $M = 2$, the difference equation for the matrix elements is found to be

$$\begin{aligned}(n-k)\varphi_{k,n} &= 2b\sqrt{(k+1)_2(k-N)_2}\,\varphi_{k+2,n} + 4ab\sqrt{(k+1)(N-k)}\,\zeta_1\,\varphi_{k+1,n}\\ &\quad + 2a^2b\,\zeta_0\,\varphi_{k,n} - 4a^3b\sqrt{k(N-k+1)}\,\zeta_1\,\varphi_{k-1,n}\\ &\quad - a\sqrt{k(N-k+1)}\,\varphi_{k-1,n} + 2a^4b\sqrt{(-k)_2(N-k+1)_2}\,\varphi_{k-2,n},\end{aligned} \tag{27.35}$$

with the coefficients ζ_i:

$$\begin{aligned}\zeta_0 &= 6k^2 - 6kN + N(N+1),\\ \zeta_1 &= 2k - N - 1.\end{aligned}$$

For the $\widehat{B}_n(k)$ polynomials, this equation becomes

$$\begin{aligned}n\widehat{B}_n(k;f,N) &= (2f)(k-N)_2\widehat{B}_n(k+2;f,N) + (4f\zeta_1)(N-k)\widehat{B}_n(k+1;f,N)\\ &\quad + (k+2f\zeta_0)\widehat{B}_n(k;f,N) - k(4f\zeta_1+1)\widehat{B}_n(k-1;f,N)\\ &\quad + (2f)(-k)_2\widehat{B}_n(k-2;f,N).\end{aligned} \tag{27.36}$$

Using the same identities as before, this difference equation can be written as an eigenvalue equation.

Orthogonality functionals

The d-orthogonality functionals can be computed for the polynomials $\widehat{B}_n(k;c,N)$. First we have:

$$(k-N/2)\varsigma_{n,k} = \langle n,N \mid QJ_0 \mid N,k \rangle = \langle n,N \mid QJ_0Q^{-1}Q \mid N,k \rangle.$$

For $M = 2$, the formula (27.15) yields

$$Q^{-1}J_0Q = J_0 - 2bJ_-^2 + aJ_+ + 2ab(1+2J_0)J_- - 4ab^2J_-^3.$$

Substitution in the first equation gives a recurrence relation for the matrix elements of the inverse operator Q^{-1}; this relation is

$$(k-n)\,\varsigma_{n,k} = a\sqrt{n(N-n+1)}\,\varsigma_{n-1,k} + 2ab(2n+1-N)\sqrt{(n+1)(N-n)}\,\varsigma_{n+1,k} \\ -2b\sqrt{(n+1)_2(n-N)_2}\,\varsigma_{n+2,k} - 4ab^2\sqrt{-(n+1)_3(n-N)_3}\,\varsigma_{n+3,k}. \tag{27.37}$$

The form of this recurrence relation suggests the following proposition:

Proposition 13. *We have*

$$\varsigma_{n,k} = \sum_{i=0}^{2} Y_i^{(n)}(k)\Xi_i(k), \tag{27.38}$$

with $\Xi_i(k) = \varsigma_{i,k}$ *and* $Y_i^{(n)}$ *a polynomial in* k*. If* $n = 3\gamma+\delta$ *with* $\delta = 0,1,2$*; then the degree of the polynomial is* γ *when* $i \leqslant \delta$ *and* $\gamma-1$ *otherwise.*

Proof. The proof follows from the recurrence relation and by induction. □

This suggests the definition of the linear functional

$$\mathscr{M}_i f(x) = \sum_{x=0}^{N} a^x \binom{N}{x}^{1/2} \Xi_i(x) f(x). \tag{27.39}$$

It thus follows that

$$\mathscr{M}_i(k^\gamma \widehat{B}_n(k;c,N)) \begin{cases} = 0 & \text{if} n \geqslant 3\gamma+i+1 \\ \neq 0 & \text{if} n = 3\gamma+i \end{cases} \tag{27.40}$$

General orthogonality functionals could be defined in full generality for the operator Q^{-1}. Indeed, one could emulate the procedure to build $2M-1$ orthogonality functionals.

27.4.2 Contractions

The contractions of the polynomials $\widehat{B}_n(k)$ are expected to yield the d-Charlier polynomials considered in [14]. The loss of symmetry also occurs in the contractions of the polynomials $\widehat{B}_n(k)$; consequently, contractions of the orthogonality functionals and ladder operators cannot be taken directly. As those cases were treated in full generality in [14], we simply show how the matrix elements $\varphi_{k,n}$ contract to the particular case of d-Charlier polynomials that was considered.

Contraction of $\varphi_{k,n}$

The required renormalization is $a \to a/\sqrt{N}$ and $b \to b/\sqrt{N^M}$. Once again, the limit is taken by first expanding the generalized hypergeometric function as a truncated sum and performing the indicated substitution. Upon simple transformations, the result is found to be

$$\lim_{N\to\infty} \varphi_{k,n} = \frac{a^{k-q} b^j}{j!(k-q)!q!} \sqrt{k!n!}\, {}_{1+M}F_{M-1} \left[\begin{matrix} -j & \{\alpha_m\} \\ \{\beta_m\} & - \end{matrix} ; \frac{(-1)^{M+1}}{a^M b} \right], \tag{27.41}$$

which gives precisely the matrix elements obtained in [14].

27.5 Conclusion

We studied the d-orthogonal polynomials related to the classical Lie algebra $\mathfrak{su}(2)$. We showed the matrix elements of the operators $S = e^{aJ_+^2} e^{bJ_-^2}$ and $Q = e^{aJ_+} e^{bJ_-^M}$ were given in terms of two families of polynomials $\widehat{A}_j^{(q)}(\ell;c,N)$ and $\widehat{B}_n(k;c,N)$. Using the algebraic setting, we characterized these polynomials; their explicit expressions in terms of hypergeometric functions allowed to identify those that belong to the classification given in [3].

We also studied the contraction limit in which $\mathfrak{su}(2)$ is sent to the Heisenberg algebra $\mathfrak{h}_1$. We showed that the $\widehat{A}_j^{(q)}(\ell;c,N)$ tend to standard Meixner polynomials when $N \to \infty$. The $\widehat{A}_j^{(q)}(\ell;c,N)$ can therefore be seen as discrete d-orthogonal versions of the Meixner polynomials. In addition, it was shown that the polynomials $\widehat{B}_n(k;c,N)$ are some d-orthogonal Krawtchouk polynomials and that they converge to the d-Charlier polynomials in the contraction limit.

In [15], the exponentials of linear and quadratic polynomials in the generators of the $\mathfrak{h}_1$ algebra were unified to yield matrix orthogonal polynomials; these considerations were motivated by the link with the quantum harmonic oscillator. Similarly, considerations regarding the discrete finite quantum oscillator would suggest to convolute the corresponding matrix elements. While providing physically relevant amplitudes for the quantum finite oscillator, this study could lead to d-orthogonal matrix polynomials, such as considered in [12]. We plan to report elsewhere on this.

Appendix A–Useful formulas for $\mathfrak{su}(2)$

The relation

$$J_0 J_\pm^n = J_\pm^n (J_0 \pm n),$$

holds and can be proven straightforwardly by induction on n. Using this identity and the relations $(J_\pm)^\dagger = J_\mp$ as well as $J_0^\dagger = J_0$, it follows that for $Q(J_\pm)$ denoting a polynomial in $J_\pm$, one has

$$[Q(J_\pm), J_0] = \mp J_\pm Q'(J_\pm),$$

where $Q'(x)$ denotes the derivative with respect to x. The preceding formula and the Baker-Campbell-Hausdorff relation lead to the identity

$$e^{Q(J_\pm)}J_0e^{-Q(J_\pm)} = J_0 \mp J_\pm Q'(J_\pm).$$

In addition, we have the relations

$$[J_+, J_-^n] = 2nJ_0J_-^{n-1} + n(n-1)J_-^{n-1},$$
$$[J_-, J_+^n] = -2nJ_+^{n-1}J_0 - n(n-1)J_+^{n-1},$$

which can also be proved by induction on n. With the help of the previous identities, one obtains

$$[J_+, Q(J_-)] = 2J_0Q'(J_-) + J_-Q''(J_-),$$
$$[J_-, Q(J_+)] = -2Q'(J_+)J_0 - J_+Q''(J_+),$$

From these formulas it follows that

$$e^{Q(J_-)}J_+e^{-Q(J_-)} = J_+ - 2J_0Q'(J_-) - J_-[Q''(J_-) + Q'(J_-)^2],$$
$$e^{Q(J_+)}J_-e^{-Q(J_+)} = J_- + 2Q'(J_+)J_0 + J_+[Q''(J_+) - Q'(J_+)^2].$$

Appendix B–Meixner, Hermite and Krawtchouk polynomials

Meixner polynomials

The Meixner polynomials have the hypergeometric representation

$$M_n(x;\beta,d) = {}_2F_1\left[\begin{matrix}-n,-x \\ \beta\end{matrix};1-\frac{1}{d}\right].$$

They satisfy the normalized recurrence relation

$$x\widehat{B}_n(x) = \widehat{B}_{n+1}(x) + \frac{n+(n+\beta)d}{1-d}\widehat{B}_n(x) + \frac{n(n+\beta-1)d}{(1-d)^2}\widehat{B}_{n-1}(x),$$

with

$$M_n(x;\beta,d) = \frac{1}{(\beta)_n}\left(\frac{d-1}{d}\right)^n \widehat{B}_n(x).$$

Hermite polynomials

The Hermite polynomials have the hypergeometric representation

$$H_n(x) = (2x)^n\, {}_2F_0\left[\begin{matrix}-\frac{n}{2},\frac{1-n}{2} \\ -\end{matrix};-\frac{1}{x^2}\right].$$

Krawtchouk polynomials

The Krawtchouk polynomials have the hypergeometric representation

$$K_n(x;p,N) = {}_2F_1\left[\begin{matrix}-n,-x\\ -N\end{matrix};\frac{1}{p}\right].$$

Their orthogonality relation is

$$\sum_{x=0}^{N}\binom{N}{x}p^x(1-p)^{N-x}K_m(x;p,N)K_n(x;p,N) = \frac{(-1)^n n!}{(-N)_n}\left(\frac{1-p}{p}\right)^n\delta_{nm}.$$

They have the generating function

$$(1+t)^{N-x}\left(1-\frac{1-p}{p}t\right)^x = \sum_{n=0}^{N}\binom{N}{n}K_n(x;p,N)t^n.$$

For further details, see [8].

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Chapitre 28

Generalized squeezed-coherent states of the finite one-dimensional oscillator and matrix multi-orthogonality

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Abstract. A set of generalized squeezed-coherent states for the finite $\mathfrak{u}(2)$ oscillator is obtained. These states are given as linear combinations of the mode eigenstates with amplitudes determined by matrix elements of exponentials in the $\mathfrak{su}(2)$ generators. These matrix elements are given in the $(N+1)$-dimensional basis of the finite oscillator eigenstates and are seen to involve 3×3 matrix multi-orthogonal polynomials $Q_n(k)$ in a discrete variable k which have the Krawtchouk and vector-orthogonal polynomials as their building blocks. The algebraic setting allows for the characterization of these polynomials and the computation of mean values in the squeezed-coherent states. In the limit where N goes to infinity and the discrete oscillator approaches the standard harmonic oscillator, the polynomials tend to 2×2 matrix orthogonal polynomials and the squeezed-coherent states tend to those of the standard oscillator.

28.1 Introduction

Discretizations of the standard quantum harmonic oscillator are provided by finite oscillator models (see for instance [1, 7]). We here consider the one based on the Lie algebra $\mathfrak{u}(2) = \mathfrak{u}(1) \oplus \mathfrak{su}(2)$ which has been interpreted as a quantum optical system consisting of $N+1$ equally spaced sensor points [1]. In this connection, we investigate here the matrix elements of exponentials of linear and quadratic expressions in the $\mathfrak{su}(2)$ generators; these operators represent discrete analogues of the squeeze-coherent states operators for the standard quantum oscillator. As shall be seen, these matrix elements are given in terms of matrix multi-orthogonal polynomials.

These polynomials (defined below), generalize the standard orthogonal polynomials by being orthogonal with respect to a *matrix* of functionals [11]. Very few explicit examples have been encountered in the literature; remarkably, our study entails a family of such polynomials and the algebraic setting allows for their characterization.

28.1.1 Finite oscillator and $\mathfrak{u}(2)$ algebra

The standard one-dimensional quantum oscillator is described by the Heisenberg algebra $\mathfrak{h}_1$, with generators a, $a^\dagger$ and $\mathfrak{i}\mathfrak{d}$ obeying

$$[a, a^\dagger] = \mathfrak{i}\mathfrak{d} \text{ and } [a, \mathfrak{i}\mathfrak{d}] = [a^\dagger, \mathfrak{i}\mathfrak{d}] = 0. \tag{28.1}$$

The Hamiltonian is given by $H = a^\dagger a + 1/2$ and with the position operator Q and the momentum operator P defined as follows:

$$Q = \frac{1}{2}(a + a^\dagger), \quad P = -\frac{i}{2}(a - a^\dagger), \tag{28.2}$$

the equations of motion

$$[H, Q] = -iP, \tag{28.3}$$

$$[H, P] = iQ, \tag{28.4}$$

are recovered.

The finite oscillator model is obtained by replacing the Heisenberg algebra by the algebra $\mathfrak{u}(2) = \mathfrak{u}(1) \oplus \mathfrak{su}(2)$. The $\mathfrak{su}(2)$ generators are denoted by J_1, J_2 and J_3 and verify

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \tag{28.5}$$

with ϵ_{ijk} the Levi-Civita symbol. The $\mathfrak{u}(1)$ generator is later to be $\frac{N}{2}\mathfrak{i}\mathfrak{d}$. For the finite oscillator, the correspondence with the physical "observables" is as follows:

$$\text{Position operator: } Q = J_1, \tag{28.6}$$

$$\text{Momentum operator: } P = -J_2, \tag{28.7}$$

$$\text{Hamiltonian: } H = J_3 + \frac{(N+1)}{2}\,\mathrm{i}\mathfrak{d}. \tag{28.8}$$

While this relaxes the functional dependence of the Hamiltonian, it is readily seen that this identification reproduces the Hamilton-Lie equations (28.3) and (28.4).

In quantum optics, such a system can be identified with signals coming from an array of $N+1$ sensor points [1]. The states of this system can be expanded in the eigenbasis of the Hamiltonian $H = J_3 + N/2 + 1/2$, which spans the vector space of the $(N+1)$-dimensional unitary irreducible representation of the $\mathfrak{su}(2)$ algebra. The eigenstates of H are denoted $|\,N,n\,\rangle$ and one has

$$H|\,N,n\,\rangle = (n+1/2)|\,N,n\,\rangle, \tag{28.9}$$

with $n = 0,\ldots,N$. The number n will often be referred to as the mode number and the states $|\,N,n\,\rangle$ as the mode eigenstates. This oscillator model thus only has a finite number of excitations, as opposed to an infinite number for the standard oscillator. Moreover, in this representation, the spectrum of the momentum and position operators P and Q consists of equally-spaced discrete values ranging from $-N/2$ to $N/2$. The position and momentum eigenbases can be obtained from the mode eigenbasis by simple rotations and their overlaps are $\mathfrak{su}(2)$ Wigner functions [1].

It is convenient to introduce the usual shift operators $J_\pm$ and the number operator $\widehat{N}$. These operators are defined by

$$J_\pm = (J_1 \pm \mathrm{i}J_2), \tag{28.10}$$

$$\widehat{N} = J_3 + N/2. \tag{28.11}$$

The action of these operators on the mode eigenstates is given by

$$J_+|\,N,n\,\rangle = \sqrt{(n+1)(N-n)}|\,N,n+1\,\rangle, \tag{28.12}$$

$$J_-|\,N,n\,\rangle = \sqrt{n(N-n+1)}|\,N,n-1\,\rangle, \tag{28.13}$$

$$\widehat{N}|\,N,n\,\rangle = n|\,N,n\,\rangle. \tag{28.14}$$

For the shift operators $J_\pm$, the action of any of their positive powers has the form

$$J_+^\alpha|\,N,n\,\rangle = \sqrt{\frac{(n+\alpha)!(N-n)!}{n!(N-n-\alpha)!}}|\,N,n+\alpha\,\rangle = \sqrt{(-1)^\alpha(n+1)_\alpha(n-N)_\alpha}\,|\,N,n+\alpha\,\rangle, \tag{28.15}$$

$$J_-^\beta|\,N,n\,\rangle = \sqrt{\frac{n!(N-n+\beta)!}{(n-\beta)!(N-n)!}}|\,N,n-\beta\,\rangle = \sqrt{(-1)^\beta(-n)_\beta(N-n+1)_\beta}\,|\,N,n-\beta\,\rangle, \tag{28.16}$$

where $(n)_0 = 1$ and $(n)_\alpha = n(n+1)\cdots(n-\alpha+1)$ stands for the Pochhammer symbol. It is worth noting that in contradistinction with the standard quantum harmonic oscillator, the finite oscillator possesses both a ground state and an anti-ground state. Indeed, one has $J_+|\,N,N\,\rangle = 0$ and $J_-|\,N,0\,\rangle = 0$. This symmetry will play a role in what follows.

28.1.2 Contraction to the standard oscillator

In the limit $N \to \infty$, the finite oscillator tends to the standard quantum harmonic oscillator through the contraction of $\mathfrak{u}(2)$ to $\mathfrak{h}_1$ [2, 15]. In this limit, after an appropriate rescaling, the shift operators $J_\pm$ tend to the operators $a^\dagger$ and a. Precisely, with

$$\lim_{N\to\infty}\frac{J_+}{\sqrt{N}} = a^\dagger, \qquad \lim_{N\to\infty}\frac{J_-}{\sqrt{N}} = a, \tag{28.17}$$

the commutation relation $[a,a^\dagger] = \mathfrak{i}\mathfrak{d}$ of the Heisenberg-Weyl algebra $\mathfrak{h}_1$ is recovered. Moreover, the contraction of the Hamiltonian H leads to the standard quantum oscillator Hamiltonian $H_{\text{osc}} = \frac{1}{2}(P^2+Q^2)$. This limit shall be used to establish the correspondence with studies associated with the standard harmonic oscillator [13].

28.1.3 Exponential operator and generalized coherent states

It is known that the standard one-dimensional harmonic oscillator admits the Schrödinger algebra $\mathfrak{sh}_1$ as dynamical algebra [8, 9]. This algebra is generated by the linears and bilinears in a and $a^\dagger$, that is a, $a^\dagger$, $\mathfrak{i}\mathfrak{d}$, a^2, $(a^\dagger)^2$ and $a^\dagger a$. The representation of the group Sch_1 has been recently constructed and analyzed in the oscillator state basis by two of us [13]. It involved determining the matrix elements of the exponentials of linear and quadratic expressions in a and $a^\dagger$. The study hence had a direct relation to the generalized squeezed-coherent states of the ordinary quantum oscillator [10] .

We pursue here a similar analysis for the finite oscillator. Notwithstanding the fact that the linears and bilinears in J_+ and J_- no longer form a Lie algebra, our purpose is to determine analogously the matrix elements of the fully disentangled exponential operator

$$R(\eta,\xi) = D(\eta)\cdot S(\xi) = e^{\eta J_+}e^{\mu J_3}e^{-\overline{\eta}J_-}\cdot e^{\xi J_+^2/2}e^{-\overline{\xi}J_-^2/2}, \tag{28.18}$$

in the basis of the finite oscillator's states. The parameters η and ξ are complex-valued and $\mu = \log(1+\eta\overline{\eta})$. The matrix elements in the $(N+1)$-dimensional eigenmode basis shall be denoted

$$R_{k,n} = \langle\, k,N\,|\,R(\eta,\xi)\,|\,N,n\,\rangle. \tag{28.19}$$

In parallel with the definition of the standard harmonic oscillator squeezed-coherent states, we introduce the following normalized set of states

$$|\,\eta,\xi\,\rangle := \frac{1}{|\langle\,\eta,\xi\,|\,\eta,\xi\,\rangle|^2} R(\eta,\xi)|\,N,0\,\rangle, \tag{28.20}$$

which are a special case of the generalized coherent states

$$|\,\eta,\xi\,\rangle_n := \frac{1}{A}\sum_k R_{k,n}|\,N,k\,\rangle, \tag{28.21}$$

where A is a normalization factor.

Contrary to the case of the harmonic oscillator, the operator R considered here is not unitary. While the operator $D(\eta)$ can be shown to be unitary [12], such is not the case for the operator $S(\xi)$. Nonetheless, we shall observe that the superpositions of states in (28.20) show spin squeezing and entanglement according to the criteria found [6] and [14]. In addition, the consideration of the fully disentangled form (28.18) allows for the explicit calculation of the matrix elements in terms of known polynomials, which is not possible with other choices of the squeezing operator for the finite oscillator [16].

As previously mentioned, the matrix elements (28.19) will be naturally expressed in terms of a finite family of 3×3 multi-orthogonal matrix polynomials $Q_n(k)$ in the discrete variable k. In the contraction limit, these polynomials tend to the 2×2 matrix orthogonal polynomials encountered in [13].

28.1.4 Matrix multi-orthogonality

Matrix multi-orthogonality has been first studied in the context of Padé-type approximation [5]. The algebraic aspects of matrix multi-orthogonality (recurrence relation, Shohat-Favard theorem, Darboux transformation, etc.) are discussed by Sorokin and Van Iseghem in [11]. Their study is based on matrix orthogonality for vector polynomials. We shall here recall the basic results to be used in what follows.

We first introduce the canonical basis for the vector space of vector polynomials of size q:

$$e_0 = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, \ldots, e_{q-1} = \begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}, e_q = \begin{pmatrix}x\\0\\\vdots\\0\end{pmatrix}, \ldots, e_{2q-1} = \begin{pmatrix}0\\0\\\vdots\\x\end{pmatrix}, e_{2q} = \begin{pmatrix}x^2\\0\\\vdots\\0\end{pmatrix}, \ldots \tag{28.22}$$

For $i = \lambda q + s$ with $i \geqslant 0$ and $s = 0,\ldots,q-1$, the basis vector e_i has the component x^λ in the $s+1^{\text{th}}$ position and zeros everywhere else. A vector polynomial of the form $\alpha_0 e_0 + \cdots + \alpha_n e_n$ with $\alpha_n \neq 0$ will be said of order n. If $q = 1$, this corresponds to a standard polynomial of degree n in x. Let

q and p be positive integers and $H_n(x) = (h_{n,1}(x), h_{n,2}(x), \ldots, h_{n,q}(x))^t$ be a q-vector polynomial of order n, where t denotes the transpose [11]. The vector polynomial $H_n(x)$ is multi-orthogonal if there exists a $p \times q$ matrix of functionals $\Theta_{i,j}$ with $i = 0, \ldots, p$ and $j = 0, \ldots, q$, defined by their moments, such that the following relations hold:

$$\Theta_{1,1}(h_{n,1}(x)x^\nu) + \cdots + \Theta_{1,q}(h_{n,q}(x)x^\nu) = 0, \qquad \nu = 0, \ldots, n_1 - 1, \tag{28.23}$$

$$\cdots$$

$$\Theta_{p,1}(h_{n,1}(x)x^\nu) + \cdots + \Theta_{p,q}(h_{n,q}(x)x^\nu) = 0, \qquad \nu = 0, \ldots, n_p - 1, \tag{28.24}$$

The numbers $(n_1, \ldots, n_p)$ are defined as follows: set $n = \mu p + \delta$, with $\delta = 0, \ldots, p-1$, then $n_1 = \cdots = n_\delta = \mu + 1$ and $n_{\delta+1} = \cdots = n_p = \mu$.

It was shown [11] that such polynomials obey a recurrence relation of the form

$$\begin{aligned} &c_n^{(q)} H_{n+q}(x) + \cdots + c_n^{(1)} H_{n+1}(x) + c_n^{(0)} H_n(x) \\ &\qquad + c_n^{(-1)} H_{n-1}(x) + \cdots + c_n^{(-p)} H_{n-p}(x) = x H_n(x), \end{aligned} \tag{28.25}$$

along with the initial conditions $H_{-p} = \cdots = H_{-1} = 0$; it was also proven [11] that a recurrence of the type (28.25) implies the orthogonality conditions (28.23) and (28.24).

These relations are more easily handled by introducing matrix polynomials, which are matrices of polynomials. Suppose that $p \geqslant q$, the matrix polynomials are obtained by first writing the recurrence relation (28.25) for k consecutive indices. One has

$$x(H_n(x), \ldots, H_{n+k-1}(x)) = (H_{n-p}(x), \ldots, H_{n+k-1+q}(x)) \begin{pmatrix} c_n^{(-p)} & & \\ \vdots & \ddots & c_{n+k-1}^{(-p)} \\ c_n^{(0)} & \cdots & c_{n+k-1}^{(-q)} \\ \vdots & \ddots & c_{n+k-1}^{(0)} \\ c_n^{(q)} & \ddots & \vdots \\ & & c_{n+k-1}^{(q)} \end{pmatrix}. \tag{28.26}$$

One can choose k to be the greatest common divisor of p and q; in this case, we can set $p = \sigma k$, $q = \rho k$ and the matrix on the right hand side can be put in blocks of size $k \times k$. We define the $q \times k$ matrix polynomial by $Q_n(x) = (H_{nk}(x), \ldots, H_{nk+k-1}(x))$. The recurrence relation (28.25) thus becomes

$$x Q_n(x) = \sum_{\ell=-\sigma}^{\rho} \Gamma_n^{(\ell)} Q_{n+\ell}(x). \tag{28.27}$$

At the end points $-\sigma$ and ρ in the sum, the matrix coefficient $\Gamma_n^{-\sigma}$ is an upper triangular invertible matrix and Γ_n^{ρ} is a lower triangular invertible matrix. For $q = 1$, this recurrence relation characterizes vector orthogonality of order p, also called p-orthogonality. If $p = q$ and $\Gamma_n^{-\sigma} = (\Gamma_n^{\rho})^*$, matrix orthogonality is recovered. In this paper, the special case corresponding to $q = 3$ and $p = 9$ will be encountered.

28.1.5 Outline

The outline of the paper is as follows. In section 2, we obtain the recurrence relation satisfied by the matrix elements $R_{k,n}$ and show that they involve 3×3 matrix multi-orthogonal polynomials. In section 3, we express these matrix elements as a finite convolution involving the Krawtchouk polynomials and a family of 3-orthogonal polynomials recently studied in [3]. In section 4, we obtain a biorthogonality relation for the matrix polynomials $Q_n(x)$. In section 5, we calculate the matrix orthogonality functionals $\Theta_{i,j}$ for the polynomials. In section 6, we obtain a difference equation for the polynomials and discuss the dual picture. In section 7, we derive the generating functions and ladder relations. In section 8, we discuss the properties of the states $|\,\eta,\xi\,\rangle$ and study spin squeezing in this system. In section 9, we briefly review the contraction limit $N\to\infty$ and relate our results with those of [13]. We close with concluding remarks in section 10. Appendices containing $\mathfrak{su}(2)$ structure formulas and properties of the Krawtchouk polynomials are included.

28.2 Recurrence relation

We shall begin the analysis by obtaining the recurrence relation satisfied by the matrix elements $R_{k,n}$. We first observe that

$$(k-N/2)R_{k,n}=\langle\, k,N\,|\,J_3R\,|\,N,n\,\rangle=\langle\, k,N\,|\,RR^{-1}J_3R\,|\,N,n\,\rangle, \tag{28.28}$$

where the inverse operator is given by

$$R^{-1}=e^{\frac{\bar{\xi}}{2}J_-^2}e^{-\frac{\xi}{2}J_+^2}e^{\bar{\eta}J_-}e^{-\mu J_3}e^{-\eta J_+}. \tag{28.29}$$

The recurrence relation is obtained by expanding the expression $R^{-1}J_3R$ and acting on the eigenstates $|\,N,n\,\rangle$. This is done using the Baker–Campbell–Hausdorff relation

$$e^ABe^{-A}=B+[A,B]+\frac{1}{2!}[A,[A,B]]+\frac{1}{3!}[A,[A,[A,B]]]+\dots, \tag{28.30}$$

and the formulas for $\mathfrak{su}(2)$ found in appendix A. Using the polar parametrizations $\eta=\rho e^{i\delta}$ and $\xi=re^{i\gamma}$, we obtain

$$\begin{aligned}R^{-1}J_3R&=(1-2p)\mathscr{A}_0+\rho e^{-i\delta}(1-p)\mathscr{A}_-+\rho(1-p)[e^{i\delta}-re^{-i(\delta-\gamma)}]\mathscr{A}_+\\&\quad+(1-2p)re^{i\gamma}\mathscr{A}_+^2-\rho r^2e^{-i(\delta-2\gamma)}(1-p)\mathscr{A}_+^3-2\rho re^{-i(\delta-\gamma)}(1-p)\mathscr{A}_+\mathscr{A}_0\end{aligned} \tag{28.31}$$

where we have defined

$$p=\frac{\rho^2}{1+\rho^2},\qquad \mathscr{A}_0=J_3+\bar{\xi}J_-^2, \tag{28.32}$$

$$\mathscr{A}_-=J_-,\qquad \mathscr{A}_+=J_+-\bar{\xi}(1+2J_3)J_--\bar{\xi}^2J_-^3. \tag{28.33}$$

Introducing this result into (28.28), we find that the matrix elements $R_{k,n}$ obey the recurrence relation

$$c_n^{(3)} R_{k,n+3} + c_n^{(2)} R_{k,n+2} + c_n^{(1)} R_{k,n+1} + c_n^{(0)} R_{n,k} \\ + c_n^{(-1)} R_{k,n-1} + \cdots + c_n^{(-9)} R_{k,n-9} = k\, R_{k,n}.$$

The coefficients $c_n^{(j)}$ can be obtained straightforwardly with the help of a symbolic computation software, their explicit expressions are cumbersome and will thus be omitted here. This recurrence relation is of the form (28.25) with $q = 3$ and $p = 9$; consequently, we look for an expression of the matrix elements $R_{k,n}$ as vector polynomials. From the shape of the recurrence relation, it is natural to define the 3-vector matrix elements

$$\Psi_{k,n} = (R_{k,3n}, R_{k,3n+1}, R_{k,3n+2})^t, \tag{28.34}$$

generated by the 3×3 matrix polynomial $Q_n(k)$,

$$\Psi_{k,n} = Q_n(k) \Psi_{k,0}. \tag{28.35}$$

With these definitions, the recurrence relation (28.34) for the matrix elements $R_{k,n}$ can be expressed as a recurrence relation for the matrix polynomial $Q_n(k)$. We have

$$k\, Q_n(k) = \sum_{j=-3}^{1} \Gamma_n^{(j)}\, Q_{n+j}(k), \tag{28.36}$$

where the matrices $\Gamma_n^{(j)}$ are expressed in terms of the coefficients $c_n^{(j)}$ in the following manner

$$\Gamma_n^{(1)} = \begin{pmatrix} c_{3n}^{(3)} & 0 & 0 \\ c_{3n+1}^{(2)} & c_{3n+1}^{(3)} & 0 \\ c_{3n+2}^{(1)} & c_{3n+2}^{(2)} & c_{3n+2}^{(3)} \end{pmatrix}, \qquad \Gamma_n^{(0)} = \begin{pmatrix} c_{3n}^{(0)} & c_{3n}^{(1)} & c_{3n}^{(2)} \\ c_{3n+1}^{(-1)} & c_{3n+1}^{(0)} & c_{3n+1}^{(1)} \\ c_{3n+2}^{(-2)} & c_{3n+2}^{(-1)} & c_{3n+2}^{(0)} \end{pmatrix}, \tag{28.37}$$

$$\Gamma_n^{(-1)} = \begin{pmatrix} c_{3n}^{(-3)} & c_{3n}^{(-2)} & c_{3n}^{(-1)} \\ c_{3n+1}^{(-4)} & c_{3n+1}^{(-3)} & c_{3n+1}^{(-2)} \\ c_{3n+2}^{(-5)} & c_{3n+2}^{(-4)} & c_{3n+2}^{(-3)} \end{pmatrix}, \qquad \Gamma_n^{(-2)} = \begin{pmatrix} c_{3n}^{(-6)} & c_{3n}^{(-5)} & c_{3n}^{(-4)} \\ c_{3n+1}^{(-7)} & c_{3n+1}^{(-6)} & c_{3n+1}^{(-5)} \\ c_{3n+2}^{(-8)} & c_{3n+2}^{(-7)} & c_{3n+2}^{(-6)} \end{pmatrix}, \tag{28.38}$$

$$\Gamma_n^{(-3)} = \begin{pmatrix} c_{3n}^{(-9)} & c_{3n}^{(-8)} & c_{3n}^{(-7)} \\ 0 & c_{3n+1}^{(-9)} & c_{3n+1}^{(-8)} \\ 0 & 0 & c_{3n+2}^{(-9)} \end{pmatrix}. \tag{28.39}$$

The structure of the matrix polynomial $Q_n(k)$ implies the existence of orthogonality relations of the form (28.23) and (28.24). Equivalently, we will explicitly obtain a set of three 3×3 matrices of functionals, denoted by $\mathscr{F}_i$, with orthogonality conditions

$$\mathscr{F}_i[Q_n(k) k^\nu] = 0 \ \text{ for } \ \nu = 0, \ldots, \lfloor \frac{n-i}{3} \rfloor, \tag{28.40}$$

where $\lfloor x \rfloor$ denotes the integer part of x and where the index i runs from 1 to 3. These orthogonality functionals will be computed from the matrix elements of the inverse operator R^{-1} and will make explicit the multi-orthogonal nature of the matrix polynomials $Q_n(k)$.

28.3 Decomposition of matrix elements

The matrix elements $R_{k,n}$ can be expressed as a finite convolution involving the classical Krawtchouk polynomials and a family of vector orthogonal polynomials studied recently in [3]. Indeed, one writes

$$\begin{aligned} R_{k,n} &= \langle k,N \,|\, R \,|\, N,n \rangle, \\ &= \langle k,N \,|\, e^{\eta J_+} e^{\mu J_3} e^{-\overline{\eta} J_-} \cdot e^{\xi J_+^2/2} e^{-\overline{\xi} J_-^2/2} \,|\, N,n \rangle, \\ &= \sum_{m=0}^{N} \langle k,N \,|\, e^{\eta J_+} e^{\mu J_3} e^{-\overline{\eta} J_-} \,|\, N,m \rangle \langle m,N \,|\, e^{\xi J_+^2/2} e^{-\overline{\xi} J_-^2/2} \,|\, N,n \rangle, \\ &= \sum_{m=0}^{N} \lambda_{k,m} \phi_{m,n}, \end{aligned} \tag{28.41}$$

where we have defined the auxiliary matrix elements

$$\lambda_{k,m} = \langle k,N \,|\, e^{\eta J_+} e^{\mu J_3} e^{-\overline{\eta} J_-} \,|\, N,m \rangle, \tag{28.42}$$

$$\phi_{m,n} = \langle m,N \,|\, e^{\xi J_+^2/2} e^{-\overline{\xi} J_-^2/2} \,|\, N,n \rangle. \tag{28.43}$$

The properties of these intermediary object shall prove useful to further characterize the matrix polynomials $Q_n(k)$ and will also yield the explicit expansion of the states $|\,\eta,\xi\,\rangle$ in the mode eigenbasis.

28.3.1 The matrix elements $\lambda_{k,m}$ and Krawtchouk polynomials

We first study the matrix elements $\lambda_{k,m}$ of the coherent state operator $D(\eta) = e^{\eta J_+} e^{\mu J_3} e^{-\overline{\eta} J_-}$. These matrix elements can be computed directly by expanding the exponentials in series and using the actions (28.15) and (28.16) on the state vector $|\,N,m\,\rangle$. With the identity $\frac{n!}{(n-k)!} = (-1)^k (-n)_k$, we readily obtain

$$\lambda_{k,m} = (-1)^m \frac{\rho^{m+k} e^{i\delta(k-m)}}{\sqrt{(1+\rho^2)^N}} \binom{N}{k}^{1/2} \binom{N}{m}^{1/2} K_m(k;p,N), \tag{28.44}$$

where $p = \frac{\rho^2}{1+\rho^2}$ and $K_m(k;p,N)$ is the Krawtchouk polynomial of degree m, which has the hypergeometric representation

$$K_m(k;p,N) = {}_2F_1\left(\begin{matrix} -m,-k \\ -N \end{matrix} ; \frac{1}{p} \right), \tag{28.45}$$

with $m=0,\dots,N$. This result also follows from the recurrence relation satisfied by the matrix elements $\lambda_{k,m}$. This relation can be obtained from (28.34) or more simply by writing

$$(k-N/2)\,\lambda_{k,m}=\langle\, k,N\mid J_3D(\eta)\mid N,m\,\rangle=\langle\, k,N\mid D(\eta)D^{-1}(\eta)J_3D(\eta)\mid N,m\,\rangle. \tag{28.46}$$

Using the B.–C.–H. relation, the matrix elements $\lambda_{k,m}$ are found to obey

$$\begin{aligned}k\lambda_{k,m}=m\left(\frac{1-\rho^2}{1+\rho^2}\right)\lambda_{k,m}+N\left(\frac{\rho^2}{1+\rho^2}\right)\lambda_{k,m}+\frac{\rho\, e^{i\delta}}{1+\rho^2}\sqrt{(m+1)(N-m)}\,\lambda_{k,m+1}\\ +\frac{\rho\, e^{-i\delta}}{1+\rho^2}\sqrt{m(N-m+1)}\,\lambda_{k,m-1}.\end{aligned} \tag{28.47}$$

Introducing the polynomials $\lambda_{k,m}=(-\overline{\eta})^m\binom{N}{m}^{1/2}K_m(k;p,N)\lambda_{k,0}$ with $p=\frac{\rho^2}{1+\rho^2}$, we recover the three-term recurrence relation of the Krawtchouk polynomials

$$\begin{aligned}-kK_m(k;p,N)=-[p(N-m)+m(1-p)]K_m(k;p,N)\\ +p(N-m)K_{m+1}(k;p,N)+m(1-p)K_{m-1}(k;p,N).\end{aligned} \tag{28.48}$$

The coherent state operator $D(\eta)$ is unitary (see [12]); consequently, we have the following orthogonality relation between the matrix elements $\lambda_{m,k}$:

$$\sum_{k=0}^{N}\lambda_{k,m}\lambda_{k,n}^{*}=\delta_{mn}, \tag{28.49}$$

where x^* is the complex conjugate of x. In $\lambda_{k,n}$, this amounts to the replacement $\delta\to-\delta$. In the following, it shall be useful to write the orthogonality relation (28.49) as the biorthogonality relation

$$\sum_{k=0}^{N}\lambda_{k,m}\lambda_{N-n,N-k}^{*}=\delta_{nm}, \tag{28.50}$$

whose equivalence to (28.49) is shown straightforwardly using the properties of the Krawtchouk polynomials. Expressing the matrix elements as in (28.44) yields the well-known orthogonality relation of the Krawtchouk polynomials

$$\sum_{k=0}^{N}\binom{N}{k}p^k(1-p)^{N-k}K_n(k;p,N)K_m(k;p,N)=\frac{(-1)^n n!}{(-N)_n}\left(\frac{1-p}{p}\right)^n\delta_{nm}. \tag{28.51}$$

28.3.2 The matrix elements $\phi_{m,n}$ and vector orthogonal polynomials

We now turn to the characterization of the matrix elements $\phi_{m,n}$ of the squeezing operator $S(\xi)$. As in the case of the coherent state operator $D(\eta)$, the matrix elements $\phi_{m,n}$ of $S(\xi)=e^{\xi J_+^2/2}e^{-\overline{\xi}J_-^2/2}$

can be computed directly by expanding the exponentials in series and applying the actions (28.15) and (28.16) on the state vector $|\,N,n\,\rangle$. Obviously, any matrix element $\phi_{m',n'}$ with m' and n' of different parities will be zero; consequently, we set $n = 2a + c$ and $m = 2b + c$ with $c = 0,1$ and obtain

$$\phi_{m,n} = (-1)^a \frac{(r/2)^{a+b} e^{i\gamma(b-a)}}{a!b!} \sqrt{\frac{(N-c)!m!}{(N-m)!}} \sqrt{\frac{(N-c)!n!}{(N-n)!}} A_a^{(c)}(b;d,N), \tag{28.52}$$

where the identities $(a)_{2n} = 2^{2n}\left(\frac{a}{2}\right)_n \left(\frac{a+1}{2}\right)_n$ and $(2\sigma+s)! = 2^{2\sigma} s!\sigma!(s+1/2)_\sigma$ were used and where we have defined

$$A_a^{(c)}(b;d,N) = {}_2F_3\left(\begin{matrix} -a & -b & \\ c+1/2 & \frac{c-N}{2} & \frac{c-N+1}{2} \end{matrix}; \frac{1}{d}\right), \tag{28.53}$$

with $d = -4r^2$. The polynomials $A_a^{(c)}$ have been studied in [3]; we review here some basic results. The polynomials $A_a^{(c)}$ are vector orthogonal polynomials of dimension 3, which corresponds to the case $q = 1$ and $p = 3$ of the general setting presented in the introduction. This can be seen by computing the recurrence relation satisfied by the matrix elements $\phi_{m,n}$. Once again, we start with

$$(m - N/2)\phi_{m,n} = \langle\, m,N \,|\, J_3 S(\xi) \,|\, N,n\,\rangle = \langle\, m,N \,|\, S(\xi)S^{-1}(\xi) J_3 S(\xi) \,|\, N,n\,\rangle. \tag{28.54}$$

Using the B.–C.–H. relation and the formulas from Appendix A, we obtain

$$S^{-1}(\xi)J_3 S(\xi) = (J_3 + \bar{\xi} J_-^2) + \xi[J_+ - \bar{\xi}(1+2J_3)J_- - \bar{\xi}^2 J_-^3]^2. \tag{28.55}$$

Substituting this result into (28.54) yields

$$\begin{aligned}(m-n)\phi_{m,n} &= \xi\sqrt{(n+1)_2(n-N)_2}\,\phi_{m,n+2} + \bar{\xi}\sqrt{(-n)_2(N-n+1)_2}\,\phi_{m,n-2} \\ &\quad + \xi\bar{\xi}\sum_{j=0}^{3} \bar{\xi}^j \sqrt{(-n)_{2j}(N-n+1)_{2j}} f_n^{(j)}\,\phi_{m,n-2j},\end{aligned} \tag{28.56}$$

with coefficients

$$f_n^{(0)} = (N-2n)(-1+N+2Nn-2n^2), \tag{28.57}$$

$$f_n^{(1)} = (6n^2 - 12n + N(5-6n) + N^2 + 9), \tag{28.58}$$

$$f_n^{(2)} = (4n - 2N - 8), \tag{28.59}$$

$$f_n^{(3)} = 1. \tag{28.60}$$

Setting $\phi_{2b+c,2a+c} = \frac{(-\bar{\xi})^a}{a!}\sqrt{\frac{(N-c)!n!}{(N-n)!}} A_a^{(c)}(b;d,N)\phi_{2b+c,c}$, the polynomials $A_a^{(c)}(b;d,N)$ are seen to obey the recurrence relation

$$\begin{aligned}(b-a)A_a^{(c)}(b;d,N) &= \frac{-\xi\bar{\xi}}{a+1}(n+1)_2(n-N)_2 A_{a+1}^{(c)}(b;d,N) - aA_{a-1}^{(c)}(b;d,N) \\ &\quad + \xi\bar{\xi}\sum_{j=0}^{3}(-a)_j f_n^{(j)} A_{a-j}^{(c)}(b;d,N).\end{aligned} \tag{28.61}$$

The matrix elements $\phi_{m,n}$ are thus given by two families of polynomials $A_a^{(c)}$ for $c = 0, 1$ of vector orthogonal polynomials of order 3. The matrix elements of the inverse operator $S^{-1}(\xi)$ can also be found by direct computation or by inspection. One readily sees that the matrix elements $\phi_{m,n}$ obey the biorthogonality relation

$$\sum_{m=0}^{N} \phi_{m,n}\phi^*_{N-n',N-m} = \delta_{nn'}, \tag{28.62}$$

In contrast to the situation with the matrix elements $\lambda_{k,m}$ of the displacement operator $D(\eta)$, the biorthogonality relation for the matrix elements $\phi_{m,n}$ is not equivalent to a standard orthogonality relation; this is a consequence of the non-unitarity of $S(\xi)$ and explains the vector-orthogonal nature of the polynomials $A_a^{(c)}(b;d,N)$. From the biorthogonality relation of the matrix elements $\phi_{m,n}$ follow two biorthogonality relations for the polynomials $A_a^{(c)}(b;d,N)$; we have, for $N = 2u+2c$,

$$\sum_{b=0}^{u} (-1)^b \binom{u}{b} A_a^{(c)}(b;d,N) A_{u-a'}^{(c)}(u-b;d,N) = \frac{a!}{(-u)_a[(c+1/2)_u]^2}\left(\frac{1}{d}\right)^u \delta_{aa'}. \tag{28.63}$$

For $N = 2u+1$, we find a biorthogonality relation interlacing the two families $c = 0$ and $c = 1$:

$$\sum_{b=0}^{u} (-1)^b \binom{u}{b} A_a^{(1)}(b;d,N) A_{u-a'}^{(0)}(u-b;d,N) = \frac{a!}{(-u)_a[(1/2)_u(3/2)_u]}\left(\frac{1}{d}\right)^u \delta_{aa'}. \tag{28.64}$$

28.3.3 Full matrix elements and squeezed-coherent states

The results of the two preceding subsections allow to write explicitly the matrix elements $R_{k,n}$; noting that $\phi_{m,n}$ is automatically zero when m and n have different parities, we have, for $n = 2a+c$, the following expression for the full matrix elements:

$$R_{k,n} = \Phi \sum_{b=0}^{\lfloor \frac{N-c}{2} \rfloor} \Theta_b K_{2b+c}(k;p,N) A_a^{(c)}(b;d,N), \tag{28.65}$$

where we have defined

$$\Phi = \frac{1}{\sqrt{(1+\rho^2)^N}} \frac{\eta^k(-\overline{\xi}/2)^a}{a!} \binom{N}{k}^{1/2} \sqrt{\frac{(N-c)!n!}{(N-n)!}}, \tag{28.66}$$

$$\Theta_b = \frac{(-\overline{\eta})^{2b+c}(\xi/2)^b}{b!} \binom{N}{2b+c}^{1/2} \sqrt{\frac{(N-c)!(2b+c)!}{(N-2b-c)!}}. \tag{28.67}$$

The generalized squeezed-coherent states are therefore expressed as the linear combination

$$|\,\eta,\xi\,\rangle_n = \frac{1}{\text{Norm}} \sum_{k=0}^{N} R_{k,n} |\,N,k\,\rangle, \tag{28.68}$$

where Norm is a normalization constant. The expression for the amplitudes simplifies significantly if one considers the standard squeezed-coherent states in which the operator $R(\eta,\xi)$ acts on the vacuum. Indeed, we have

$$|\,\eta,\xi\,\rangle = \frac{1}{\text{Norm}}\sum_{k=0}^{N}\sqrt{\frac{1}{(1+\rho^2)^N}}\binom{N}{k}^{1/2}\eta^k\left(\sum_{b=0}^{\lfloor\frac{N}{2}\rfloor}\frac{(\overline{\eta}^2\xi/2)^b}{b!}(-N)_{2b}K_{2b}(k;p,N)\right)|\,k\,\rangle. \tag{28.69}$$

If the squeezing parameter $\xi = re^{i\gamma}$ is set to zero, we recover the standard normalized $\mathfrak{su}(2)$ coherent states

$$|\,\eta\,\rangle = \sqrt{\frac{1}{(1+\rho^2)^N}}\sum_{k=0}^{N}\binom{N}{k}^{1/2}\eta^k|\,k\,\rangle. \tag{28.70}$$

In section 8, the properties of the states $|\,\eta,\xi\,\rangle$ will be further investigated; in particular, it will be shown that they exhibit spin squeezing when N is even.

28.4 Biorthogonality relation

Given the symmetry of the matrix elements entering the finite convolution yielding $R_{k,n}$, the matrix elements of the inverse operator $S^{-1}(\xi)D^{-1}(\eta)$ are expected to have a similar behavior. Indeed, one finds that

$$\sum_{k=0}^{N} R_{k,n}\tilde{R}_{N-k,N-n'} = \delta_{nn'}, \tag{28.71}$$

where $\sim$ denotes the replacements $\rho \to -\rho$ and $r \to -r$. In terms of the vector polynomials $\Psi_{k,n} = (R_{k,3n}, R_{k,3n+1}, R_{k,3n+2})^t$, this biorthogonality relation takes the form

$$\sum_{k=0}^{N}\Psi_{k,n}(\tilde{\Psi}_{N-k,N-n'})^t = \delta_{nn'}\text{Id}_{3\times 3}. \tag{28.72}$$

This equation can be transformed into a biorthogonality relation for the matrix polynomials $Q_n(k)$. Indeed, one has

$$\sum_{k=0}^{N} Q_n(k)\Psi_{k,0}(\tilde{Q}_{N-n'}(N-k)\tilde{\Psi}_{N-k,0})^t = \delta_{nn'}\text{Id}_{3\times 3}, \tag{28.73}$$

which can be written as

$$\sum_{k=0}^{N} Q_n(k)W(k)(\tilde{Q}_{N-n'}(N-k))^t = \delta_{nn'}\text{Id}_{3\times 3}, \tag{28.74}$$

with the biorthogonality weight matrix

$$W(k) = \begin{pmatrix} R_{k,0}\tilde{R}_{N-k,0} & R_{k,0}\tilde{R}_{N-k,1} & R_{k,0}\tilde{R}_{N-k,2} \\ R_{k,1}\tilde{R}_{N-k,0} & R_{k,1}\tilde{R}_{N-k,1} & R_{k,1}\tilde{R}_{N-k,2} \\ R_{k,2}\tilde{R}_{N-k,0} & R_{k,2}\tilde{R}_{N-k,1} & R_{k,2}\tilde{R}_{N-k,2} \end{pmatrix}. \tag{28.75}$$

Each element of the weight matrix can be computed exactly as a finite sum with the help of the equation (28.65).

28.5 Matrix Orthogonality functionals

From the recurrence relation (28.34) and the results in [11], it is known that there exists a set of three 3×3 matrix of functionals with respect to which the matrix polynomials $Q_n(k)$ are orthogonal. To construct these functionals, we consider the recurrence relation satisfied by the matrix elements of the inverse operator $R^{-1}(\eta,\xi)$. These matrix elements are defined as

$$R^{-1}_{n,k} = \langle n,N \mid R^{-1} \mid N,k \rangle, \tag{28.76}$$

and their recurrence relation is obtained by noting that

$$(k-N/2)R^{-1}_{n,k} = \langle n,N \mid R^{-1}J_0 \mid N,k \rangle = \langle n,N \mid R^{-1}J_0RR^{-1} \mid N,k \rangle. \tag{28.77}$$

The quantity $R^{-1}J_3R$ has been computed previously; recalling that $J_{\pm}^{\dagger} = J_{\mp}$ and that $J_3^{\dagger} = J_3$, the recurrence relation for the inverse 3-vector $\Psi^{-1}_{n,k}$ is given by

$$k\,\Psi^{-1}_{n,k} = \sum_{j=-1}^{3} e_n^{(j)}\Psi^{-1}_{n+j,k}, \tag{28.78}$$

where the coefficients $e_n^{(j)}$ are 3×3 matrices and $e_n^{(-1)}$ as well as $e_n^{(3)}$ are respectively lower and upper triangular invertible matrices. Since

$$\sum_{k=0}^{N} R^{-1}_{m,k}R_{k,n} = \sum_{k=0}^{N} \langle m,N \mid R^{-1} \mid N,k \rangle \langle k,N \mid R \mid N,n \rangle = \delta_{nm}, \tag{28.79}$$

it follows that

$$\sum_{k=0}^{N} \Psi_{k,n}(\Psi^{-1}_{m,k})^t = \delta_{nm}\mathrm{Id}_{3\times 3}. \tag{28.80}$$

We may now state the following proposition.

Proposition 14. *The 3-vector $\Psi^{-1}_{n,k}$ can be expressed as*

$$\Psi^{-1}_{n,k} = \sum_{i=0}^{2} p_i^{(n)}(k)\kappa_i\Xi_i, \tag{28.81}$$

where $\Xi_i = \Psi^{-1}_{i,k}$. The κ_i are 3×3 matrices which depend only on i and $p_i^{(n)}(k)$ are polynomials in the variable k. Let $n = 3\nu + \ell$, for $\ell = 0,1,2$; if $i \leqslant \ell$, the degree of the polynomial $p_i^{(n)}(k)$ is ν, otherwise it is $\nu - 1$.

Proof. We set $n = 0$ in the recurrence relation (28.78), which leads to

$$e_n^{(3)}\Psi^{-1}_{3,k} = (k\,\mathrm{Id}_{3\times 3} - e_0^{(0)})\Psi^{-1}_{0,k} - e^{(1)}\Psi^{-1}_{1,k} - e^{(2)}\Psi^{-1}_{2,k}. \tag{28.82}$$

Since $e_n^{(3)}$ is upper triangular and invertible, it follows that

$$\Psi_{3,k}^{-1} = \sum_{i=0}^{2} p_i^{(3)}(k)\kappa_i \Psi_{i,k}^{-1}, \tag{28.83}$$

where the degree of $p_i^{(3)}(k)$ is 1 for $i = 0$ and 0 for all other indices. This establishes (28.81) for $n = 0$. The proof is then completed by induction. □

From this proposition, it is natural to define the following matrices of functionals.

Definition 16. Let $\mathscr{F}_i$ for $i = 1,2,3$ be the matrix functionals defined by

$$\mathscr{F}_i[\cdot] = \sum_{k=0}^{N} [\cdot]\Psi_{k,0}\Xi_{i-1}^{t}. \tag{28.84}$$

With this definition, the relation (28.80) can be written as

$$\mathscr{F}_i[k^{\nu} Q_n(k)] = 0_{3\times 3} \text{ for } \nu = 0,\ldots,\lfloor\frac{n-i}{3}\rfloor, \tag{28.85}$$

for $i = 1,2,3$. The multi-orthogonality of the matrix polynomials $Q_n(k)$ has thus been made explicit by the direct construction of the orthogonality functionals.

28.6 Difference equation

The matrix polynomials $Q_n(k)$ are bi-spectral; not only do they satisfy a recurrence relation, they also obey a difference equation. In a fashion dual to the approach followed to find the recurrence relation, we observe that

$$(n - N/2)R_{k,n} = \langle k,N \mid RJ_3 \mid N,n \rangle = \langle k,N \mid RJ_3R^{-1}R \mid N,n \rangle. \tag{28.86}$$

Using once again the B.–C.–H relation and formulas from appendix A, we obtain

$$RJ_3R^{-1} = (\mathscr{B}_0 - \xi\mathscr{B}_+^2) - \overline{\xi}[\mathscr{B}_- + \xi\mathscr{B}_+(1+2\mathscr{B}_0) - \xi^2\mathscr{B}_+^3]^2, \tag{28.87}$$

with

$$\mathscr{B}_0 = (1-2p)J_3 - \rho e^{i\delta}(1-p)J_+ - \rho e^{-i\delta}(1-p)J_-, \tag{28.88}$$

$$\mathscr{B}_+ = (1-p)J_+ + 2\rho e^{-i\delta}(1-p)J_3 - pe^{-2i\delta}J_-, \tag{28.89}$$

$$\mathscr{B}_- = -pe^{2i\delta}J_+ + 2\rho e^{i\delta}(1-p)J_3 + (1-p)J_-. \tag{28.90}$$

Expanding these expressions leads to a difference equation of the form

$$nR_{k,n} = \sum_{j=-6}^{6} m_k^{(j)} R_{k+j,n}, \tag{28.91}$$

which can be turned into a difference equation for the matrix polynomials $Q_n(k)$ which is quite involved and not provided here.

It is interesting to observe that this difference equation contains the same number of terms as the recurrence relation, but has a different symmetry. Indeed, the indices run from −6 to 6 in the difference equation. It indicates that the order in which the coherent state operator and the squeezing operator are presented has an impact on the structure of the associated polynomials. If one defines $R' = S(\xi)D(\eta)$ instead of $R = D(\eta)S(\xi)$, one gets 6×6 matrix multi-orthogonal polynomials satisfying a three-term matrix recurrence relation; these polynomials are not, however, orthogonal, because the condition $(\Gamma_n^{(-6)})^* = \Gamma_n^{(6)}$ is not fulfilled.

28.7 Generating functions and ladder relations

The ordinary $\mathfrak{su}(2)$ coherent states $|\,\eta\,\rangle$ can be used to obtain a generating function for the matrix elements $R_{k,n}$. We consider the two-variable function defined by

$$G(x,y) = \frac{1}{(1+\rho^2)^N}\sum_{k,n}\binom{N}{k}^{1/2}\binom{N}{n}^{1/2}\overline{x}^k y^n R_{k,n}. \tag{28.92}$$

Clearly, the function $G(x,y)$ can be viewed as the matrix element of R between the coherent states $|\,x\,\rangle$ and $|\,y\,\rangle$. Introducing a resolution of the identity, we obtain

$$G(x,y) = \langle\, x\,|\,R\,|\,y\,\rangle = \sum_{m=0}^{N}\langle\, x\,|\,D(\eta)\,|\,N,m\,\rangle\langle\, m,N\,|\,S(\xi)\,|\,y\,\rangle. \tag{28.93}$$

The first part of this convolution can be evaluated directly using the action of the generators $J_\pm$ on the states; not surprisingly, we recover, up to a multiplicative factor, the generating function of the Krawtchouk polynomials; the result is

$$\langle\, x\,|\,D(\eta)\,|\,N,m\,\rangle = \frac{\overline{\eta}^m}{(1+\rho^2)^N}\binom{N}{m}^{1/2}(1+\eta\overline{x})^{N-m}\left(\frac{(1-p)}{p}\eta\overline{x}-1\right)^m, \tag{28.94}$$

with $p = \frac{\rho^2}{1+\rho^2}$. The second part in the R.H.S of (28.93) is also readily determined. Setting $m = 2t+s$, one finds

$$\langle\, m,N\,|\,S(\xi)\,|\,y\,\rangle = \frac{\xi^t y^s}{t!}\sqrt{\frac{N!m!}{(N-m)!}}\sum_{k=0}^{\lfloor\frac{N-s}{2}\rfloor}\frac{(-y^2/\xi)^k}{(2k+s)!}(-t)_k\, {}_2F_0\left(\frac{-z(k)}{2},\frac{1-z(k)}{2};-4y^2\overline{\xi}\right), \tag{28.95}$$

where $z(k) = N-2k-s$. The convolution of the two functions given in (28.94) and (28.95) thus yields the generating function for the matrix elements $R_{k,n}$. The ladder relations for the matrix polynomials $Q_n(k)$ can also be constructed explicitly from the observations

$$\begin{aligned}\sqrt{-(n+1)_3(n-N)_3}\,R_{k,n+3} &= \langle\, k,N\,|\,RJ_+^3\,|\,N,n\,\rangle,\\ \sqrt{-(-n)_3(N-n+1)_3}\,R_{k,n-3} &= \langle\, k,N\,|\,RJ_-^3\,|\,N,n\,\rangle.\end{aligned} \tag{28.96}$$

The conjugation of the operator $J_{\pm}^3$ by the operator R leads to complicated expressions which are best evaluated with the assistance of a computer.

28.8 Observables in the squeezed-coherent states

We now further investigate the properties of the states $|\,\eta,\xi\,\rangle$ resulting from the application of the squeezed-coherent operator $D(\eta)S(\xi)$ on the vacuum $|\,N,0\,\rangle$. For systems which possess the $\mathfrak{su}(2)$ symmetry, there exist many different parameters to determine whether a state is squeezed or not (for a review of the parameters that can be used see [6]). In the following, we will adopt [14]

$$Z_{\vec{n}_i}^2 = N\frac{(\Delta J_{\vec{n}_i})^2}{\langle J_{\vec{n}_{i+1}}\rangle^2+\langle J_{\vec{n}_{i+2}}\rangle^2}, \tag{28.97}$$

where the indices are to be understood cyclically. If $Z_{\vec{n}_i}^2<1$, the system is squeezed in the direction $\vec{n}_i$, where the $\vec{n}_i$, $i=1,2,3$ are orthogonal unit vectors. This choice of the squeezing criterion is relevant because of its relation with entanglement [6].

The mean value of any observable $\mathscr{O}$ in the state $|\,\eta,\xi\,\rangle$ can be expressed by

$$\langle\mathscr{O}\rangle = \frac{1}{\kappa(r)}\sum_{i,j}\frac{(\xi/2)^i(\bar{\xi}/2)^j}{i!j!}\sqrt{(2i)!(2j)!(-N)_{2i}(-N)_{2j}}\,\langle\,2j\,|\,D^{-1}(\eta)\mathscr{O}D(\eta)\,|\,2i\,\rangle, \tag{28.98}$$

where κ is the normalization constant, which is given by the hypergeometric function

$$\kappa(r) = |\langle\,\xi,\eta\,|\,\eta,\xi\,\rangle|^2 = {}_3F_0\left(1/2,\frac{-N}{2},\frac{1-N}{2};(2r)^2\right). \tag{28.99}$$

For simplicity, we choose to study squeezing along the axis of the Hamiltonian J_3. We begin by setting $r=0$ to study the behavior of the parameter Z_3^2 in the coherent states. We readily find

$$\langle J_1\rangle = \langle Q\rangle = \frac{N\rho}{1+\rho^2}\cos(\delta) \qquad \langle J_2\rangle = -\langle P\rangle = -\frac{N\rho}{1+\rho^2}\sin(\delta), \tag{28.100}$$

$$\langle J_3\rangle = \langle (H-1/2)\rangle = -\frac{N}{2}\frac{1-\rho^2}{1+\rho^2}, \qquad (\Delta J_3)^2 = \langle J_3^2\rangle - \langle J_3\rangle^2 = \frac{N\rho^2}{(1+\rho^2)^2}. \tag{28.101}$$

From these results, it is seen that for pure coherent states $|\,\eta\,\rangle$, we always have $Z_3^2=1$, which ensures, according to the definition (28.97), that purely coherent states are never squeezed; this can be proved in a straightforward manner for any choice of normalized basis $\{\vec{n}_1,\vec{n}_2,\vec{n}_3\}$[14].

We now investigate squeezing in the $\vec{n}_3=(0,0,1)$ direction for the states $|\,\eta,\xi\,\rangle$. We have that

$$\langle J_1\rangle = \left[\frac{\rho\cos(\delta)}{1+\rho^2}\right]G_n(r), \qquad \langle J_2\rangle = \left[\frac{-\rho\sin(\delta)}{1+\rho^2}\right]G_n(r), \qquad \langle J_3\rangle = \left[\frac{1-\rho^2}{1+\rho^2}\right]H_n(r) \tag{28.102}$$

$$\langle J_3^2\rangle = \frac{1}{(1+\rho^2)^2}\left(\left[1+\rho^4\right]J_n(r)+\rho^2L_n(r)+2\rho^2[\cos(2\delta-\gamma)]M_n(r)\right) \tag{28.103}$$

where we have defined the polynomials

$$G_n(r) = \frac{1}{\kappa(r)} \sum_{i=0}^{\lfloor \frac{N}{2} \rfloor} \frac{(r^2)^i}{i!} (1/2)_i (-N)_{2i} (N-4i), \tag{28.104}$$

$$H_n(r) = \frac{1}{\kappa(r)} \sum_{i=0}^{\lfloor \frac{N}{2} \rfloor} \frac{(r^2)^i}{i!} (1/2)_i (-N)_{2i} (2i - N/2), \tag{28.105}$$

$$J_n(r) = \frac{1}{\kappa(r)} \sum_{i=0}^{\lfloor \frac{N}{2} \rfloor} \frac{(r^2)^i}{i!} (1/2)_i (-N)_{2i} (4i^2 - 2iN + N^2/4), \tag{28.106}$$

$$L_n(r) = \frac{1}{\kappa(r)} \sum_{i=0}^{\lfloor \frac{N}{2} \rfloor} \frac{(r^2)^i}{i!} (1/2)_i (-N)_{2i} (-16i^2 + 8iN - N^2/2 + N) \tag{28.107}$$

$$M_n(r) = \frac{r}{\kappa(r)} \sum_{i=0}^{\lfloor \frac{N}{2} \rfloor} \frac{(r^2)^i}{i!} (1/2)_{i+1} (-N)_{2i+2} \tag{28.108}$$

The exact expression for the squeezing parameter $Z^2_{\vec{n}_z}$ cannot be obtained in closed form; nevertheless, it can easily be computed numerically. We observed that squeezing ($Z^2_{\vec{n}_z} < 1$) along the Hamiltonian axis $\vec{n}_z$ occurs only in the case where N is even, which corresponds to an oscillator having an odd number of points. It is worth noting that a distinction between the N odd and the N even cases also arises in the Fourier-Krawtchouk transform [1], which transforms the finite oscillator wave functions into themselves. The squeezing parameter is plotted against θ in figure 1.

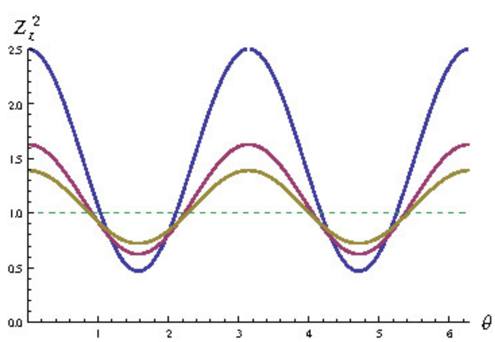

Figure 28.1: Squeezing parameter $Z^2_{\vec{n}_z}$ for $r = 2, 4, 6$ (decreasing amplitudes) with $\rho = 0.8$ and $N = 40$.

28.9 Contraction to the standard oscillator

It is of interest to study the behavior of the polynomials $Q_n(k)$ and of the squeezed-states in the contraction limit $N \to \infty$ where the finite $\mathfrak{u}(2)$ oscillator tends to the standard quantum harmonic oscillator; this limit was studied in detail in [2]. To obtain the proper limit, the parameters of the squeeze-coherent operator must be renormalized according to

$$\rho \to \frac{\rho}{\sqrt{N}}, \quad r \to \frac{r}{N}. \tag{28.109}$$

Upon taking the limit, one finds that the 13-term recurrence relation of the $R_{n,k}$ tends to a 5-term symmetric recurrence relation of the form

$$kR_{k,n} = \sum_{j=-2}^{2} c_n^{(j)} R_{k,n+j}. \tag{28.110}$$

This is indeed the type of recurrence relation obtained in [13]. In the case of the standard oscillator, it is possible to choose a unitary version of the corresponding S operator and to express its matrix elements in terms of 2×2 matrix orthogonal polynomials.

28.10 Conclusion

The matrix elements of the exponential operators corresponding to the squeezed-coherent operators of the finite oscillator have been determined in the energy eigenbasis of this model. They were seen to be given in terms of matrix multi-orthogonal polynomials which have the Krawtchouk and vector orthogonal polynomials as building blocks. The algebraic setting allowed to characterize these polynomials and to explicitly compute their matrix orthogonality functionals. The results have been used to show that the squeezed coherent states of the finite oscillator exhibit squeezing when the dimension of the oscillator $N+1$ is odd.

Appendix A–Useful formulas involving the $\mathfrak{su}(2)$ generators

The relation

$$J_3 J_{\pm}^n = J_{\pm}^n (J_3 \pm n),$$

holds and can be proven straightforwardly by induction on n. Using this identity and the relations $(J_{\pm})^{\dagger} = J_{\mp}$ as well as $J_3^{\dagger} = J_3$, it follows that for $P(J_{\pm})$ denoting a polynomial in $J_{\pm}$, one has

$$[P(J_{\pm}), J_3] = \mp J_{\pm} P'(J_{\pm}),$$

where $P'(x)$ is the derivative of $P(x)$ with respect to x. The preceding formula and the Baker-Campbell-Hausdorff relation lead to the identity

$$e^{P(J_{\pm})} J_3 e^{-P(J_{\pm})} = J_3 \mp J_{\pm} P'(J_{\pm}).$$

In addition, we have the relations

$$[J_+, J_-^n] = 2nJ_3J_-^{n-1} + n(n-1)J_-^{n-1},$$
$$[J_-, J_+^n] = -2nJ_+^{n-1}J_3 - n(n-1)J_+^{n-1},$$

which can also be proved by induction on n. With the help of the previous identities, one obtains

$$[J_+, P(J_-)] = 2J_3P'(J_-) + J_-P''(J_-),$$
$$[J_-, P(J_+)] = -2P'(J_+)J_3 - J_+P''(J_+),$$

From these formulas it follows that

$$e^{P(J_-)}J_+e^{-P(J_-)} = J_+ - 2J_3P'(J_-) - J_-[P''(J_-) + P'(J_-)^2],$$
$$e^{P(J_+)}J_-e^{-P(J_+)} = J_- + 2P'(J_+)J_3 + J_+[P''(J_+) - P'(J_+)^2].$$

Appendix B–Krawtchouk polynomials

The Krawtchouk polynomials have the hypergeometric representation

$$K_n(x;p,N) = {}_2F_1\left[\begin{matrix} -n, -x \\ -N \end{matrix}; \frac{1}{p}\right].$$

Their orthogonality relation is

$$\sum_{x=0}^{N} \binom{N}{x} p^x(1-p)^{N-x} K_m(x;p,N)K_n(x;p,N) = \frac{(-1)^n n!}{(-N)_n}\left(\frac{1-p}{p}\right)^n \delta_{nm}.$$

They have the generating function

$$(1+t)^{N-x}\left(1 - \frac{1-p}{p}t\right)^x = \sum_{n=0}^{N} \binom{N}{n} K_n(x;p,N)t^n.$$

For further details, see [4].

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Conclusion

En guise de conclusion, je propose ici quelques questions de recherche liées émanant directement de la présente thèse et qui mériteraient sans doute d'être étudiées.

- La dualité de Schur-Weyl classique établit un lien entre les n-uplets de produits tensoriels de représentations de $\mathfrak{sl}_2$ et l'algèbre du groupe symétrique $\mathbb{C}S_n$. Des versions de cette dualité existent entre les algèbres quantiques $U_q(\mathfrak{sl}_2)$, $\mathfrak{osp}_q(1|2)$ et la superalgèbre $\mathfrak{osp}(1|2)$ et les algèbres de Hecke, de Birman–Murakami–Wenzl et Brauer. Il serait intéressant de déterminer le lien entre ces algèbres et les algèbres de Racah, Bannai–Ito, Askey–Wilson et q-Bannai–Ito.

- Les algèbres de Racah, de Bannai–Ito, de Askey–Wilson et q-Bannai–Ito peuvent être vues comme des algèbres quadratiques de rang 1. Il serait naturel d'en obtenir des généralisations au rang n et de déterminer leurs liens avec les polynômes orthogonaux multivariés.

- Nous avons étudié un certain nombre de systèmes superintégrables en n dimension construits à l'aide des opérateurs de Dunkl associés au système de racine $A_1^{\times n}$. Il serait pertinent d'examiner des systèmes définis en termes des opérateurs de Dunkl associés à d'autres systèmes de racines.

- Il serait utile d'obtenir des généralisations à multivariées des polynômes du tableau de Bannai–Ito et de comprendre leurs liens avec les systèmes superintégrables avec réflexions en plus hautes dimensions.

- Une q-généralisation des relations entre les groupes $SO(d+1)$, $SO(d,1)$, $E(d)$ et les familles de polynômes multivariés de Krawtchouk, Meixner et Charlier serait désirable. Elle permetterait de caractériser davantage certaines familles de q-polynômes multivariés proposés par Gasper et Rahman.

Contribution aux articles

Cette annexe vise à détailler la contribution de VXG aux articles compris dans cette thèse, conformément aux exigences de la FESP. La description est sommaire. Les contributions sous la forme de commentaires, discussions, suggestions, etc., ne sont pas mentionnées.

- Chapitre 1. Idée originale et première version par AZ et LV. Réécriture complète et ajouts subtantiels par LV et VXG. Calculs par AZ, VXG et LV.
- Chapitre 2. Idée originale de LV et VXG. Rédaction et calculs par VXG.
- Chapitre 3. Idée originale de HM, LV et VXG. Rédaction et calculs par VXG.
- Chapitre 4. Idée originale de LV. Rédaction et calculs par VXG.
- Chapitre 5. Idée originale, rédaction et calculs par VXG.
- Chapitre 6. Idée originale, rédaction et calculs par VXG.
- Chapitre 7. Idée originale de LV et VXG. Calculs par VXG et LV. Rédaction par VXG.
- Chapitre 8. Idée originale de LV. Calculs par VXG et LV. Rédaction par VXG.
- Chapitre 9. Idée originale de LV et VXG. Calculs par JG, JL et VXG. Rédaction par VXG.
- Chapitre 10. Idée originale de LV et MI. Calculs par LV et VXG. Rédaction par VXG et LV.
- Chapitre 11. Idée originale de LV. Calculs et rédaction par VXG.
- Chapitre 12. Idée originale de LV et VXG. Calculs et rédaction par VXG.
- Chapitre 13. Idée originale par VXG. Calculs et rédaction par VXG.
- Chapitre 14. Idée originale par LV, AZ et VXG. Calculs et rédaction par VXG.
- Chapitre 15. Idée originale par LV. Calculs par VXG, HDB et LV. Rédaction par VXG.
- Chapitre 16. Idée originale par AZ et LV. Calculs par AZ et VXG. Rédaction par VXG.

- Chapitre 17. Idée originale par AZ, LV et VXG. Calculs et rédaction par VXG.
- Chapitre 18. Idée originale par LV. Calculs et rédaction par VXG.
- Chapitre 19. Idée originale, calculs et rédaction par VXG.
- Chapitre 20. Idée originale par LV. Rédaction par VXG.
- Chapitre 21. Idée originale par LV et VXG. Calculs et rédaction par VXG.
- Chapitre 22. Idée originale, calculs et rédaction par VXG.
- Chapitre 23. Idée originale par AZ. Calculs par AZ, LV et VXG. Rédaction par VXG et AZ.
- Chapitre 24. Idée originale par LV, AZ et VXG. Calculs par AZ. Rédaction par VXG et LV.
- Chapitre 25. Idée originale et calculs par LV et VXG. Rédaction par LV et VXG.
- Chapitre 26. Idée originale par VXG, AZ et LV. Calculs et rédaction par VXG.
- Chapitre 27. Idée originale par LV et AZ. Calculs et rédaction par VXG.
- Chapitre 28. Idée originale par LV et AZ. Calculs et rédaction par VXG.

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