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Trois études en micro-économie de l'incertain

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SOMMAIRE

L'incertitude en micro-économie constitue le thème central de cette thèse. Les trois études qui constituent la thèse portent principalement sur des questions de statique comparée. En d'autres termes, il s'agit d'étudier comment les agents économiques modifient leurs décisions optimales suite à une variation d'un paramètre exogène du modèle qui décrit leurs comportements. Cette analyse est faite dans le cadre de modèles qui prennent en compte l'incertitude de l'environnement économique des agents.

La première étude examine le choix du mode d'habitation (propriétaire/locataire) en conjonction avec la distribution des revenus à travers le temps. L'analyse repose sur un modèle à deux périodes avec un rendement incertain de l'actif immobilier. Dans une littérature récente, on a fait l'hypothèse que la distribution des revenus à travers le temps peut affecter le choix du mode d'habitation pour les agents ayant un accès limité aux sources d'emprunt. L'objectif principal de la première étude consiste a vérifier cette hypothèse sur la base du modèle théorique à deux périodes.

Le modèle théorique adopté dans la première étude comporte plusieurs variables de décisions et deux paramètres aléatoires. L'analyse des accroissements de risque dans ce genre de modèle pose un défi de taille. Pour résoudre les questions liées aux accroissements de risque dans de tels modèles sans avoir recours à des hypothèses trop restrictives, une étude approfondie des accroissements de risque s'avère nécessaire. Tel est l'objectif poursuivi dans les deux dernières études de la thèse.

Plusieurs définitions d'accroissement de risque existent dans la littérature. La deuxième étude de la thèse démontre que certaines définitions préservent le signe algébrique de la corrélation entre deux paramètres aléatoires. Dans l'étude des choix optimaux des agents, l'utilisation de telles définitions nous permet d'éliminer l'effet de changements de corrélation entre les paramètres aléatoires qui pourraient être induits par un accroissement de risque. L'importance du maintien des signes des coefficients de corrélation vient du fait que la détention d'actifs négativement corrélés permet aux agents de constituer des portefeuilles moins risqués.

La dernière étude de la thèse porte sur les effets des accroissements de risque dans le cadre de modèles de choix de portefeuille d'actifs financiers. L'étude contient huit propositions dont la plus générale provient de l'étude d'un modèle à deux variables de décision et deux paramètres aléatoires. Cette proposition constitue un premier résultat dans l'étude des effets d'un accroissement de risque sur les choix optimaux en présence de plusieurs paramètres aléatoires et plusieurs variables de décision. Un résultat qualitatif commun découle de cette étude: lorsque le rendement d'un actif devient plus risqué, la quantité optimale de l'actif détenue par un agent économique décroît.

RÉSUMÉ

L'incertitude en micro-économie constitue le thème central de cette thèse. Dans la littérature économique, l'analyse de l'incertitude a pris plusieurs formes, mais de nombreuses questions liées au comportement des agents demeurent sans réponse. Les trois études de cette thèse examinent principalement les effets de l'incertitude sur les choix économiques des agents. Ces effets peuvent être analysés de plusieurs façons. L'approche retenue dans cette thèse est celle de la statique comparée.

La méthode de statique comparée consiste d'abord à supposer que les agents ont pris des décisions optimales, c'est-à-dire celles qui maximisent leur utilité tout en respectant leurs contraintes. Les contraintes sont généralement de type budgétaire. Il s'agit ensuite de faire varier un paramètre exogène du modèle de comportement des agents afin d'examiner les effets de cette variation sur les décisions optimales. A titre d'exemple, on peut supposer que le revenu d'un consommateur est exogène. On peut alors se demander comment un agent modifiera ses décisions de consommation à la suite d'une réduction exogène de son revenu.

Les modèles de comportement qui permettent d'inférer les décisions optimales des agents peuvent être définis dans un contexte de certitude. En ce cas, on suppose que les agents connaissent avec certitude l'ensemble des paramètres exogènes qui affectent leur situation économique. Cependant, les agents évoluent rarement dans de telles situations. Il convient alors de modifier les modèles de comportement pour prendre en compte l'incertitude. L'approche retenue dans cette thèse consiste à introduire des actifs financiers ou immobiliers dont les rendements sont *a priori* incertains.

La première étude examine le choix du mode d'habitation (propriétaire/locataire) dans le cadre d'un modèle à deux périodes. La caractéristique fondamentale de ce modèle repose sur la distinction entre la part d'investissement et la part de consommation associée à un actif immobilier. De plus, afin de prendre en compte l'incertitude, on suppose que la part d'investissement de l'actif immobilier rapporte un rendement aléatoire.

Dans un article récent (Fu, 1991), on a fait l'hypothèse que le profil du revenu peut affecter le choix du mode d'habitation pour les agents ayant un accès restreint aux sources d'emprunt. Cependant, la définition du profil de revenu proposée dans le modèle de base du comportement des agents (Henderson and Ioannides, 1983) est inadéquate pour valider ou infirmer cette hypothèse. L'objectif principal de cette première étude consiste à vérifier l'hypothèse de Fu (1991) à l'aide d'une nouvelle définition du profil de revenu. Le choix du mode d'habitation est donc analysé conjointement avec le profil de revenu des agents.

Dans un modèle à deux périodes, le profil de revenu reflète la distribution des revenus totaux (i.e. la somme des revenus des périodes 1 et 2) à travers le temps. Lorsque le revenu de la période 2 est proportionnellement plus important que le revenu de la période 1, le profil de revenu est biaisé vers la période 2, et *vice versa*. Si l'agent économique peut accéder à une source de crédit, il peut alors modifier son profil de revenu à sa guise. Considérons un agent dont le profil serait biaisé vers la période 2. Au début de la période 1, il dispose donc d'un revenu proportionnellement moins important qu'à la période 2. En empruntant au début de la période 1 et en remboursant le principal et les intérêts en début de période 2, le biais du profil de revenu tend à s'amenuiser.

La première étude de la thèse atteint un double objectif. En premier lieu, une définition plus précise du profil de revenu est élaborée. Puis, à partir de cette nouvelle définition et toujours dans le cadre du modèle de base à deux périodes, l'hypothèse de Fu (1991) est confirmée. La validation de cette hypothèse permet de conclure que les agents dont le profil de revenu est biaisé vers la première période accèdent plus facilement au statut de propriétaire. Il s'agit là d'un résultat qui cadre bien avec l'intuition. En effet, même si un agent économique dispose de sources de crédit limitées, cela n'entrave pas l'accès à la propriété si le revenu disponible en début de période est suffisant pour compenser les contraintes associées au crédit.

Dans le cadre du modèle théorique adopté dans la première étude de la thèse, d'autres questions liées à l'incertitude se posent. Quel est l'effet de la spéculation immobilière sur les choix optimaux des agents? Comment une incertitude accrue des taux hypothécaires affecte le choix du mode d'habitation? Ces questions relèvent de l'analyse des accroissements de risque et de leurs effets sur les décisions optimales.

L'analyse des accroissements de risque dans le cadre de modèles à plusieurs variables de décision pose un défi de taille. La littérature économique s'est principalement concentrée sur des modèles à une variable de décision et à un paramètre aléatoire. Depuis les dix dernières années, des études ont été entreprises afin d'obtenir des résultats de statique comparée non ambigus avec des modèles à deux variables de décision et deux paramètres aléatoires. Cependant, les résultats les plus récents reposent sur des modèles restrictifs. Les problèmes découlant des nombreuses interactions entre les variables de décision sont contournés par de fortes hypothèses sur les préférences ou sur les distributions des paramètres aléatoires. Les difficultés posées par un modèle théorique du choix de mode d'habitation comportant quatre variables de décision et deux paramètres aléatoire sont nécessairement importantes. On comprend donc la nécessité d'approfondir l'analyse des accroissements de risque avec plusieurs variables de décision et plusieurs paramètres aléatoires. Ceci constitue le principal objectif poursuivi dans les deux dernières études de la thèse.

La deuxième étude de la thèse porte sur les propriétés des accroissements de risque. Les modèles à deux paramètres aléatoires comportent des difficultés qui n'apparaissent pas dans le cadre de modèles à un seul paramètre aléatoire. Considérons un modèle avec deux paramètres aléatoires, x et y. Lorsque les paramètres sont liés de façon stochastique, la covariance entre x et y peut servir de mesure descriptive pour caractériser la relation stochastique. Plus précisément, le signe algébrique de la covariance indique si x et y sont négativement corrélés (covariance négative) ou positivement corrélés (covariance positive). La corrélation a une importance particulière dans les modèles de détention d'actifs financiers dont les rendements sont incertains. En effet, les actifs financiers qui sont négativement corrélés réduisent la variance des portefeuille dans lesquels ils sont détenus. Il apparaît donc clairement qu'un accroissement de risque du paramètre aléatoire x doit être analysé en tenant compte de la corrélation entre x et y. Il s'agit là d'une difficulté supplémentaire qui est propre aux modèles à deux paramètres aléatoires.

Des contributions récentes ont démontré que la corrélation qui apparaît dans les modèles à deux paramètres aléatoires ne peut pas être ignorée. En particulier, il est possible de construire des exemples d'accroissement de risque d'une variable aléatoire x qui entraîne un changement du signe algébrique de la covariance entre x et y. Ce renversement de signe constitue une modification de la corrélation entre x et y qui induit des changements dans les choix optimaux des agents. Il s'agit donc d'un accroissement de risque *doublé* d'un changement de corrélation. La tâche consiste alors à définir des accroissements de risque qui préservent la corrélation entre les variables stochastiques afin d'obtenir des résultats qui portent *strictement* sur les accroissements de risque.

Plusieurs définitions d'accroissement de risque existent dans la littérature. La deuxième étude de la thèse démontre que certaines définitions préservent le signe algébrique et la valeur numérique de la covariance entre deux paramètres aléatoires. Dans l'étude des choix optimaux des agents, l'utilisation de telles définitions s'avèrent indispensable puisqu'elles permettent d'éliminer l'effet de changements de corrélation entre les paramètres aléatoires qui pourraient être induits par un accroissement de risque.

La dernière étude de la thèse porte sur les effets des accroissements de risque dans le cadre de modèles de choix de portefeuille d'actifs financiers. Afin de dégager des résultats non ambigus, trois types de restrictions sont appliquées. Les hypothèses portent sur les préférences, les distributions des paramètres aléatoires et/ou sur les définitions d'accroissement de risque. L'étude contient huit propositions suivant différents modèles (une ou plusieurs variables de décision / un ou plusieurs paramètres aléatoires) qui sont assortis de différentes restrictions. En dépit des différences qui distinguent ces huit propositions, elles conduisent toutes au même résultat qualitatif: lorsque le rendement d'un actif devient plus risqué, la quantité optimale de l'actif détenue par un agent économique décroît.

La proposition la plus générale provient de l'étude d'un modèle à deux variables de décision et deux paramètres aléatoires. La restriction sur les paramètres aléatoires consiste a postuler une loi normale bivariée (avec dépendance stochastique) tandis que l'accroissement de risque est conditionné par une variable aléatoire. Cette proposition constitue un premier résultat dans l'étude des effets d'un accroissement de risque sur les choix optimaux en présence de plusieurs paramètres aléatoires et plusieurs variables de décision.

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ESSAY 1: THE PROFILE OF INCOME, HOUSING TENURE CHOICE AND HOUSING CONSUMPTION

Introduction

The permanent income hypothesis/life-cycle (PIH/LC) model of consumption is based on the assumption of perfect financial markets. Since borrowing and lending are unrestricted in any fashion, the lifetime profile of income is irrelevant in explaining the optimal consumption path. This is not the case when capital market imperfections are assumed (hereafter designated as the CMI hypothesis). Moreover, one might reasonably think that distortions will be induced in portfolio choices. The economic significance of the profile of income is then closely related to the CMI hypothesis, and desired consumption can be constrained by current resources at some point in time. Thus, consumption might be more sensitive to current resources than what is predicted by the permanent income hypothesis. This has come to be labelled as the excess sensitivity of consumption to current income.

In this paper, the impact of the profile of income upon housing tenure choice and housing consumption is investigated under a CMI hypothesis. Capital market imperfections can be implemented in two ways: *endogenously*, by introducing adverse selection and/or moral hazard (see, e.g., Stiglitz and Weiss 1981), or *exogenously*, by setting a price or quantity ceiling on a particular asset. The CMI hypothesis considered here is an exogenous quantity ceiling on borrowing. Endogenous modelling is beyond the scope of this paper.

The motivation to choose the housing market is twofold. First, the impact of the profile of income[1] upon tenure choice has recently arisen in the literature, but it is still an open issue. Henderson and Ioannides (1983) (hereafter H-I) were among the very first to address the question. They presented a theoretical model capable of explaining why some people choose to rent rather than to own their dwelling unit. Among other things, this choice might depend on the tilt of income.

However Fu (1991) subsequently showed that the tilt of income does not affect the tenure choice "because a perfect financial market, in which personal saving and borrowing are unrestricted, is assumed in the model [H-I]" (Fu, p. 383). Fu also conjectured that under the CMI hypothesis, the

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1 The three expressions profile of income, tilt of income and path of income are synonymous.

tenure choice could be affected by the path of income. Fu's conjecture is intuitively appealing but its validity remains to be proven within the framework of the H-I model. Furthermore, even if the CMI hypothesis is accepted, we still do not know *how* the tenure choice could be affected by the income path. For instance, is it more likely that a typical consumer under financial constraint will prefer to own rather than rent when faced with a tilt of income towards the present period?

In this paper I show *how* the tenure choice could be affected by the income path under the CMI hypothesis and within the framework of the two period H-I model. It is also shown that the definition of income path proposed by H-I is not sufficient to assess the impact upon tenure choice under the CMI hypothesis. A new definition of income path is proposed and shown to be sufficient to obtain the desired results. Intuitively, our definition is related to the one introduced in H-I. Income is tilted if the variation of period 1 income is different from the variation in present value of period 2 income, that is if $dy_1 - dy_2/(1+r) \neq 0$. Wealth changes if the sum of the variation of period 1 income and the variation in present value of period 2 income is not null, in other words if $dy_1 + dy_2/(1+r) \neq 0$. However, in order to obtain a pure tilt of income and a pure wealth change, we define these two concepts in such a way that they have no common components. In H-I model, nothing ensures us that this is the case. This is the most important difference between H-I's definition and ours and it turns out to be the key for assessing accurately the impact of the profile of income upon tenure choice.

A substantial part of the discussion focuses on the variations of optimal values of housing investment[2] (h_i) and housing consumption (h_c) when income is tilted. So let us introduce the following notation:

$\partial h_c/\partial T_j > 0$ (<0) Housing consumption is increasing (decreasing) with respect to a tilt of income towards period j, where ∂T_j (j=1,2) means that income is tilted towards period j or equivalently that income of period j becomes relatively larger than the income of the other period.

[2] Though housing investment is a flow, the two expressions housing investment and housing stock are used interchangeably throughout this paper.

$\partial h_i/\partial T_j > 0$ (<0) Housing investment is increasing (decreasing) with respect to a tilt of income towards period j (j=1,2), where ∂T_j is defined as previously.

It is demonstrated that under the CMI hypothesis, housing investment always increases (decreases) when income is tilted towards period 1 (period 2). This result holds under various assumptions: risk aversion, risk neutrality and nullity of utilization costs of housing capacity incurred by tenants. The proposition is fairly intuitive. When the capital market is perfect, it is analytically shown that saving and borrowing act as a buffer to smooth out the profile of income so that housing investment and housing consumption are unaffected. However, when the CMI hypothesis is assumed, saving and borrowing no longer smooth out perfectly the profile of income and thus housing consumption and investment are modified. Furthermore, when the CMI hypothesis takes the form of an exogenous borrowing ceiling, housing investment can be viewed as the next best instrument to cope with the profile of income when the ceiling constraint is binding because reducing investment provides additional resources in the first period. Thus housing investment does react to the tilt of income when borrowing is bound.

As for tenure choice, two results are established. Firstly, $\partial h_c/\partial T_1 < 0$ ($\partial h_c/\partial T_2 > 0$) provides a sufficient condition to increase (decrease) the probability of owning when income is tilted towards the present (future) period. Secondly, risk neutrality provides a sufficient condition to increase (decrease) the probability of owner-occupancy when income is tilted towards period 1 (period 2). Without one of these two previous sufficient conditions, it is still proven that the effect of a tilt of income upon tenure choice under the CMI hypothesis depends on the magnitude of income elasticity of housing consumption demand relative to that of housing investment. To provide more insight into these results, we must remember that renters (in the H-I model) are those for whom optimal housing consumption (h_c) is greater than optimal housing investment (h_i). Since owners are those for whom $h_i \geq h_c$, potential owners have to bridge the gap between housing investment and housing consumption. The previous results establish the conditions under which this gap increases or decreases. Finally, the role of risk aversion is defined more precisely. We are able to give an analytical expression for the marginal effect of risk aversion upon housing investment and housing consumption when income is tilted. It is also shown that risk aversion limits the efficiency of housing investment as an alternative buffer against the profile of income when saving is constrained.

The second reason why this paper focuses upon the housing market and the profile of income under the CMI hypothesis has to deal with the PIH/LC model of consumption. There is substantial evidence in the literature that the PIH/LC model is empirically rejected because aggregate consumption shows excess sensitivity to current income fluctuations. Capital market imperfections are consistently reported to be at the root of the excess sensitivity of consumption (Marjorie Flavin, 1981, 1985; Fumio Hayashi, 1985; Tullio Jappelli and Marco Pagano, 1989; Stephen Zeldes, 1989). If excess sensitivity is found in aggregate consumption, it should also be found in its components and more specifically in housing consumption. Some findings support this view. Jones (1990) concludes that current wealth does provide both greater explanatory power and higher elasticities than permanent income to assess the housing demand of young owners. Jappelli and Pagano (1989) argue that if the excess sensitivity of aggregate consumption is caused by capital market imperfections, the excess sensitivity should be greater in countries with imperfect capital markets than in countries where these markets are well developed and competitive. They use, among other things, *the proportion of homeowners in young cohorts* as an indicator of imperfections of capital markets. For the countries studied (United States, UK, Japan and Italy), the figures show an inverse relationship between the degree of excess sensitivity of consumption to current income and the proportion of homeowners in young cohorts. This paper provides a theoretical approach that explain these findings in the housing market.

The theoretical explanation builds on the methodology of Jones (1990). As in the H-I model, owners are those for whom housing investment (H*) is equal to housing consumption (H(h*)). The total consumption must equal the permanent income. Since housing investment is an asset, it should be determined in a portfolio choice optimization framework. The portfolio choice is constrained by the gross investible wealth, that is current wealth (NW) plus borrowing (D). It is argued that capital market constraints restrict the ability of young households to borrow on human capital collateral so that NW is the critical budget component for portfolio choices. Furthermore it is hypothesized that, for young households, H(h*) > H* is more likely. It implies that first-time owners equalize housing investment and housing consumption by adjusting their housing investment. Current wealth (NW) will then be the key element determining observed demand. This explains why housing consumption for young owners is more sensitive to current resources than to permanent income or equivalently, why housing consumption shows excess sensitivity. However, this result relies on the critical assumption that H(h*) > H* is more likely for young households.

This paper offers a theoretical answer to explain why it is more likely that optimal housing consumption will be greater than optimal housing investment for young households. First, the new definition of income path is used to show that young households are more likely to face a binding borrowing constraint because their profile of income is tilted towards the second period. The finding of Jappelli (1990) support this result. Intuitively, young households borrow to increase the available resources in the first period to smooth out their profile of income because they seek to avoid distortions in their optimal consumption path. Thereby they increase their chances to be credit constrained. Second, the results relating the tenure choice and the path of income demonstrate that young households under financial constraint are more likely to choose their housing investment and consumption in such a way that H(h*) > H*. They do so precisely because their income path is tilted towards the second period.

Thus, by using a new set of results, more insight is provided to understand why capital market imperfections can lead to excess sensitivity of housing consumption to current resources. The profile of income is identified as the channel through which capital market imperfections are transmitted to modify saving patterns to affect the probability of being credit constrained and to modify consumption-portfolio decisions. This new insight also shows why young households are more likely to choose a consumption path that departs from the PIH/LC predicted path.

The analysis of the excess sensitivity of housing consumption is possible because a conceptual distinction is made between housing investment and housing consumption. In this regard, Henderson and Ioannides (1983) offered a theoretical model; to my knowledge, it is the only tractable model capable of explaining housing tenure choice in combination with a distinction between housing investment and housing consumption.

The rest of the article is divided into five sections. Section 1 outlines the framework used to assess the relationship between tenure choice and income path. Section 2 provides a new definition of tilted income path and Section 3 uses this new definition to analyse comparative-statics results. Section 4 focuses on particular cases of comparative-statics results outlined in Section 3. Finally, Section 5 examines the role of the profile of income in explaining the excess sensitivity of housing consumption.

Section 1: Framework

To examine how tenure choice is affected by a tilt of income, we adopt the H-I model. In this model, housing is used to produce housing services and as an investment good. The housing services are a function of the rate of utilization chosen by occupants and of housing capacity. Defining h_c as capacity and u as the rate of utilization, total services are:

(0.1) $h = h_c f(u)$, $f' > 0$, $f'' < 0$

For a given level u, housing services are directly proportional to capacity. For this reason, h_c will be referred to as housing consumption. Because f'' is negative (i.e. f(u) is concave in u), doubling the rate of utilization will less than double housing services, *ceteris paribus*.

The costs associated with a given rate of utilization differ according to the tenure choice. Tenants are assumed to pay lower costs of utilization, at all rate of utilization, because rental contracts cannot provide for all possible contingencies. In other words, if the costs of a tenant exceed the rental payment, the landlord can only recover a part of the excess costs. Owners face a different situation because they pay the full cost of utilization rates. The rental externality arise from the fact that tenants do not face the social marginal costs of their utilization rates. Defining T(u) and $\tau(u)$ as the cost function of owners and tenants respectively, total costs are:

(0.2) $h_c T(u)$, $T'(u) > 0$ $T''(u) > 0$ Owners

(0.3) $h_c \tau(u)$, $\tau'(u) > 0$ $\tau''(u) > 0$ Tenants

(0.4) $\tau(u) < T(u)$ for all u $\tau'(u) < T'(u)$ for all u

Owners and tenants both have a convex cost function. For a given capacity, doubling the rate of utilization will more than double total costs. Equation **(0.4)** is the formal definition of the rental externality.

Henderson and Ioannides (1983) show that when housing is considered only as a consumption good, owning always dominates renting because a owner utility is always greater than a tenant utility. The proof of this result is based on the concavity of f(u), on the convexity of the cost function and on the rental externality. For renting to be a rational choice, and therefore for housing tenure choice to be an issue, a conceptual distinction between housing consumption (h_c) and housing investment (h_i) must be introduced in conjunction with the rental externality. When housing is also considered as a risky investment, Henderson and Ioannides (1983) demonstrate that owning does not always dominate renting because housing tenure choice is the result of a tradeoff. The tradeoff depend on the rental externality and on the premium required as a compensation for incurring the increased risk associated with the housing investment.

The distinction between housing consumption (h_c) and housing investment (h_i) combined with the rental externality also serves another purpose H-I model because it determines the housing tenure choice. If housing consumption (h_c) is less than housing investment (h_i), it can be shown (see H-I, Section II, A, 1 and Section II, B,3) that it is efficient for the consumer to owner-occupy his housing investment up to h_c and *rent out* the rest (h_i - h_c) because the rental externality is avoided. However, when $h_c > h_i$, it is assumed that the consumer cannot owner-occupy a fraction of his housing consumption and rent the rest (h_c - h_i). Thus, renters are defined by the inequality $h_c > h_i$ and owners are those for whom $h_i \geq h_c$[3]. It should also be clear that owners and renters do not face the same maximization problem because owners rent out a fraction of their housing investment whereas renters rent out the *totality* of it.

To assess the impact of the profile of income upon housing tenure choice, the renters utility maximization problem is considered. The basic idea is then to examine how the optimal choices of h_i and h_c change with respect to the profile of income. Let us denote $h_c - h_i = \Delta$, $\Delta > 0$ for renters, and $\Delta \leq 0$ for owners. Whenever Δ increases after an exogenous variation of the income path, the renter will not modify his tenure choice. However, if Δ decreases the renter is more likely to become

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[3] When $h_c > h_i$, but h_c is near h_i, the consumer could "distort" his investment and consumption choices to increase h_i up to h_c. Thus the consumer owner-occupies his entire housing investment and avoids the rental externality. See H-I, Section II, A, 1 and Section II, B, 3 for details.

owner-occupier because the inequality defining the renting status is less likely to hold. The renters utility maximization problem corresponds to equation (10)[4] in H-I, with an additional constraint on personal saving since we consider the CMI hypothesis. In order to do so, we will say that personal saving (S) must be non-negative[5]. The relevant utility maximization problem of renters is then:

$$\textbf{(1)}\quad MAX_{h_c,\, h_i,\, S,\, u} \quad \begin{aligned} &U\,(y_1 - S - (P - L - R)h_i - Rh_c\,,\; h_c f(u)) \\ &+ E\,\{V\,(y_2 + S(1+r) + (P(1+\theta) - L(1+r) \\ &- (T(\overline{u}) - \tau(\overline{u}))h_i - \tau(u)h_c)\} \end{aligned}$$

subject to $S \geq 0$.

In problem **(1)**, Henderson and Ioannides (1983) make the following important assumptions:

i) *The consumer maximizes a multi period utility function*

V (.) is the indirect utility function of wealth remaining after period 1. To analyse the optimal decisions in period 1, all subsequent periods are compressed into the indirect utility function V (.). This assumes that the optimal decisions are made in future periods. It also assumes that future prices and incomes are held constant for a comparative statics;

ii) *Utility of period 1 and indirect utility V (.) are separable*

U (.) is the utility derived from the period 1 consumption. U (.) and V (.) are assumed to be additively separable. Furthermore, both U (.) and V (.) are assumed to be increasing and strictly quasi concave. U (.) is assumed to be additively separable in its arguments;

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4 Equation numbers in bold character refers to this paper whereas equation numbers in normal character refers to a specifically mentioned article.

5 Henderson and Ioannides (1983) used this methodology to assess the effect of imperfect financial markets hypothesis on owner optimal choices. Another formulation of this constraint is $S \geq B$ where B is an exogenous ceiling that could be set to be positive, nil or negative. Since B is exogenous, the essence of the model is unaffected. Also, in an empirical investigation of consumption and liquidity constraints, Zeldes (1989) used this formulation.

iii) *Absence of personal discount factor*

Utility derived from consumption in period 1 differ from consumption utility in period 2 by a personal discount factor. This factor accounts for the fact that consumption in period 2 occurs later in time. Since consumption decisions for period 2 and all subsequent periods are embedded in the indirect utility function V (.), the personal discount factor is also embedded in V (.);

iv) *For housing consumed in period 1, costs of utilization are incurred in the second period only*

The interpretation of problem **(1)** is as follows. A consumption bundle of two items provides utility in period 1. The items are total housing services h_c f(u) and the numeraire x. Problem **(1)** contains the following implicit budget constraints: $y_1 = x + S + (P\text{-}L\text{-}R)h_i + Rh_c$. It simply means that the income of period 1 (y_1) is allocated between the numeraire x, saving S, housing investment (P - L - R)h_i and housing capacity (consumption) R h_c.

S is the period 1 saving which earns the non-stochastic real rate of interest r. h_i is the housing investment (accumulated units of housing stock) which is rented out to others in period 1 at price R. The constant market purchase price per unit of housing stock is P. A mortgage loan L (per unit of housing stock) at the fixed market rate of interest r is available to the consumer. Thus, (P - L - R) is the net price of housing investment per unit of stock in period 1. R h_c is the housing capacity rented for oneself for consumption purposes at price R.

At the end of period 1, the remaining wealth w provides utility equal to V (w). The wealth w has the following components: the income of period 2 (y_2), the accumulated saving S (1+r) and the accumulated housing investment. The housing investment earns a stochastic return θ and the interest on the mortgage loan is the rate r. When the consumer rents out the housing investment h_i to others, he incurs the uncollectible maintenance costs $(T(\bar{u})\text{-}\tau(\bar{u}))h_i$. These costs reflect the rental externality and are uncertain given that the tenants choose the rate of utilization $\bar{u}$. Thus the net value of housing investment at the end of period 1 is $(P(1+\theta) - L(1+r) - (T(\bar{u})\text{-}\tau(\bar{u}))h_i$. Finally, the wealth is reduced by $\tau(u)h_c$. It represents a payment for damages not covered in the rental contract when the consumer chooses a rate of utilization u in his rented housing capacity. If no damage occurs, the payment is zero.

To facilitate the presentation the following definitions and assumptions are introduced by Henderson and Ioannides (1983):

(1.1) $(P - L - R)(1+r) \equiv \xi > 0$

(1.2) $P(1+\theta) - L(1+r) - (T(\bar{u})-\tau(\bar{u})) \equiv \beta + \gamma > 0$

Equation **(1.1)** simply means that the net price of housing investment per unit of housing stock in period 1 (multiplied by (1+r)) is positive. Equation **(1.2)** assumes that the net value of housing investment a the end of period 1 is positive. This assumption excludes the possibility of costs of defaults. For the convenience of the reader, every term described in this model is also defined in Appendix A.

Problem **(1)** is a Kuhn-Tucker maximization problem. To simplify the derivation of comparative-statics results, we will express **(1)** differently, but equivalently[6] as:

$$\textbf{(2)}\quad MAX_{h_c,\, h_i,\, S,\, u,\, \alpha,\, \lambda} \quad \begin{aligned} L = {} & U\,(y_1 - S - (P - L - R)h_i - Rh_c\,,\ h_c f(u)) \\ & + E\,\{V\,(y_2 + S(1+r) + (P(1+\theta) - L(1+r) \\ & - (T(\overline{u}) - \tau(\overline{u})))h_i - \tau(u)h_c)\} + \lambda\,[S - \alpha^2] \end{aligned}$$

to obtain a standard Lagrangian maximization problem where λ is the Lagrange multiplier associated with the constraint on S and α is a slack variable.

For a consumer to solve problem **(1)**, the Hessian matrix (defined in Appendix B) must be negative definite and a necessary condition for this is that D < 0. D is the determinant of the Hessian matrix under the specifications of the H-I model, D is negative. See Appendix C for all the details.

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[6] See Silberberg 1978, Chapter XII.

The maximization of problem **(2)** generates the following first order conditions with respect to { h_c, h_i, S, u, α, λ }:

$$(3) \qquad L_{h_c} = -Ru_1 + u_2 f(u) - \tau(u)E[v'] = 0$$

$$(4) \qquad L_{h_i} = -(P - L - R)u_1 + E[v'(\beta+\gamma)] = 0$$

$$(5) \qquad L_S = -u_1 + E[v'](1+r) + \lambda = 0$$

$$(6) \qquad L_u = h_c u_2 f'(u) - h_c E[v']\tau'(u) = 0$$

$$(7) \qquad L_\alpha = -2\lambda\alpha = 0$$

$$(8) \qquad L_\lambda = S - \alpha^2 \equiv g^S(S,\alpha) = 0$$

By using equation **(3)** and **(6)** we have:

$$(9) \qquad \frac{f'(u)}{f(u)} \left\{ Ru_1 + \tau(u)E[v'] \right\} = E[v']\tau'(u)$$

Let α and λ be the optimal values associated with **(2)**. When $\alpha \neq 0$, we have an interior solution, and from **(7)** $\lambda = 0$. Knowing that $\lambda = 0$ and from **(5)**, it follows that:

$$(10) \qquad E[v'](1+r) = u_1$$

Substituting **(10)** in **(9)**, we obtain equation (8) of H-I:

$$(11) \qquad \frac{f'(u)}{f(u)} \left\{ R(1+r) + \tau(u) \right\} = \tau'(u)$$

Equations **(10)** and **(11)** are associated with an interior solution for which the constraint on saving S is not binding. **(10)** is simply the first order condition of saving. **(11)** determines the equilibrium rate of utilization u. The left hand side of **(11)** represents the marginal benefit of the rate of utilization and the right hand side is the marginal cost associated with u. Also note that **(11)** is independent of the income levels y_1 and y_2.

When $\alpha = 0$, we have a corner solution, and $\lambda \geq 0$. If $\lambda > 0$, then **(10)** is no longer valid and **(9)** cannot be expressed differently. So we will say that **(9)** is the corner solution version of **(11)**, or of (8) in H-I. It should be noted that **(9)**, the marginal condition for u, is no longer independent of (y_1, y_2) as it was the case for **(11)** or for (8) in H-I. Equation **(6)** must then be included in the differential equation system.

To derive comparative-statics results, we set up the differential equation system:

$$\Omega \begin{bmatrix} dh_c \\ dh_i \\ dS \\ du \\ d\alpha \\ d\lambda \end{bmatrix} = \begin{bmatrix} (Ru_{11})dy_1 + (\tau(u)E[v''])dy_2 \\ ((P-R-L)u_{11})dy_1 - (E[v''(\beta+\gamma)])dy_2 \\ u_{11}dy_1 - ((1+r)E[v''])dy_2 \\ (h_c E[v'']\tau'(u))dy_2 \\ 0 \\ 0 \end{bmatrix}$$

where terms of the (6x6) bordered Hessian matrix Ω are defined in Appendix B.

We will now focus our interest on the case $\alpha = 0$ (corner solution, binding financial constraint) since the case $\alpha \neq 0$ (interior solution) has been extensively examined by H-I and Fu. We therefore set $\alpha = 0$ in the bordered Hessian matrix.

By applying Cramer's rule upon the differential equation system we obtain:[7]

$$\textbf{(12)}\quad dh_i =$$

$$\left[\left(E[v''(\beta+\gamma-\xi)](1+r) - \frac{\xi\, E[v'']\lambda}{E[v']}\right)\left(\frac{h_c^2 u_{11}u_{22}f'R}{(1+r)D}\right) \left(-dy_1 f'R(1+r) - \frac{dy_1 f'R\lambda}{E[v']} - dy_2 f'R\right)\right]$$

$$+\left[(\xi E[v'']\tau^2 + E[v''(\beta+\gamma)](1+r)R\tau)\left(dy_1\frac{h_c u_{11}(f''u_2 - E[v']\tau'')}{(1+r)D}\right) - (\xi E[v'']R\tau + E[v''(\beta+\gamma)](1+r)R^2)\left(dy_2\frac{h_c u_{11}(f''u_2 - E[v']\tau'')}{(1+r)D}\right)\right]$$

$$+\left[dy_1\frac{h_c u_{11}u_{22}\xi f^2(f''u_2 - E[v']\tau'')}{(1+r)D} - dy_2\frac{h_c u_{22}E[v''(\beta+\gamma)]f^2(1+r)(f''u_2 - E[v']\tau'')}{(1+r)D}\right]$$

---

[7] Every comparative-statics results in this paper has been checked with the computer software Macsyma.

(13) $dh_c =$

$$
\begin{aligned}
&\Bigg[\left(\xi E[v''(\beta+\gamma)]\tau + E[v''(\beta+\gamma)^2](1+r)R\right)\left(dy_1 \frac{h_c u_{11}(f''u_2 - E[v']\tau'')}{(1+r)D}\right) \\
&\quad + \left(\xi E[v'']\tau + E[v''(\beta+\gamma)](1+r)R\right)\left(dy_2 \frac{h_c u_{11}\xi(f''u_2 - E[v']\tau'')}{(1+r)^2 D}\right)\Bigg] \\
&+ \Bigg[\left((E[v''(\beta+\gamma)^2] - \xi E[v''(\beta+\gamma)])(1+r) - \frac{\xi E[v''(\beta+\gamma)]\lambda}{E[v']}\right)\left(dy_1 \frac{h_c^2 u_{11} u_{22}(f')^2 R}{(1+r)D}\right) \\
&\quad + \left((E[v''(\beta+\gamma)] - \xi E[v''])\xi(1+r) - \frac{\xi^2 E[v'']\lambda}{E[v']}\right)\left(dy_2 \frac{h_c^2 u_{11} u_{22}(f')^2 R}{(1+r)^2 D}\right)\Bigg] \\
&+ \Bigg[dy_1 \frac{h_c^2 (E[v'']E[v''(\beta+\gamma)^2] - (E[v''(\beta+\gamma)])^2) u_{11} R(\tau')^2}{D} \\
&\quad + dy_2 \frac{h_c (E[v'']E[v''(\beta+\gamma)^2] - (E[v''(\beta+\gamma)])^2)\tau(f''u_2 - E[v']\tau'')}{D} \\
&\quad - dy_2 \frac{h_c^2 (E[v'']E[v''(\beta+\gamma)^2] - (E[v''(\beta+\gamma)])^2)(u_{22}(f')^2 R u_1)}{D}\Bigg]
\end{aligned}
$$

where D in **(12)** and **(13)** is defined in Appendix C. It is also shown in Appendix C that the specifications of the H-I model ensure that $D < 0$.

In H-I and Fu, the time path of income is tilted if $dy_1 - dy_2 / (1+r) \neq 0$ where dy_1 and dy_2 are the exogenous variations of period 1 income and period 2 income respectively. If $dy_1 - dy_2 / (1+r) > 0$ (< 0), the time path is tilted towards period 1 (period 2). If we use this definition in **(12)** and **(13),** it is impossible to express dh_i or dh_c as a function of $dy_1 - dy_2 / (1+r)$. In other words, the H-I definition of wealth and income path variation is not useful to assess the theoretical relationship between the path of income and tenure choice. We propose a more complete definition in the next section.

Section 2: Extended Definition of Wealth and Income Path Variations

Let us suppose that an agent has an initial vector of income $E = (y_1, y_2/(1+r))$, where r is the non-stochastic real rate of interest. y_1 and $y_2/(1+r)$ are first period income and present value of second period income respectively. Let us now consider the vector $(dy_1, dy_2/(1+r))$ as an arbitrary variation of $(y_1, y_2/(1+r))$. The variation $(dy_1, dy_2/(1+r))$ induces two changes: a tilt of income path and a change in wealth. We will say that these variations are the two basic components of any arbitrary variation $(dy_1, dy_2/(1+r))$.

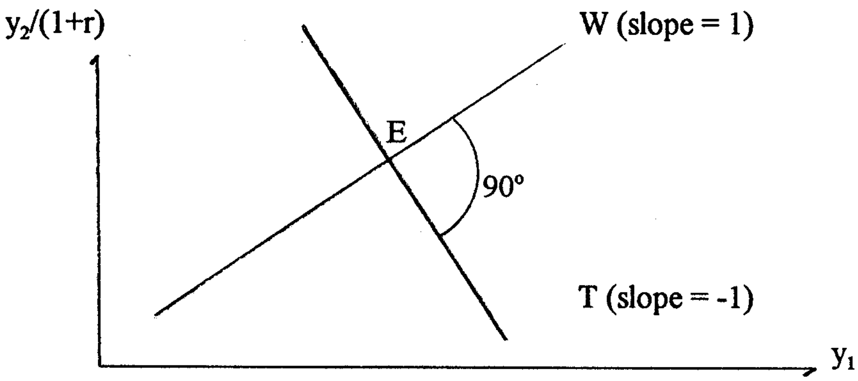

FIGURE 1

In figure 1, any variation $(dy_1, dy_2/(1+r))$ can be expressed as a linear combination of two vectors: one along W and one along T, denoted respectively as $(dy_1^W, dy_2^W/(1+r))$ and $(dy_1^T, dy_2^T/(1+r))$. Since along W the slope is 1, we have:

$$\textbf{(14)}\quad \frac{dy_2^W/(1+r)}{dy_1^W} = 1 \text{ or } (dy_1 - dy_2/(1+r))\Big|_{along\ W} \equiv dy_1^W - dy_2^W/(1+r) = 0$$
$$\text{and} \quad dy_1^W + dy_2^W/(1+r) \neq 0$$

Along T we have:

$$\textbf{(15)}\quad \frac{dy_2^T/(1+r)}{dy_1^T} = -1 \text{ or } (dy_1 + dy_2/(1+r))\Big|_{along\ T} \equiv dy_1^T + dy_2^T/(1+r) = 0$$
$$\text{and} \quad dy_1^T - dy_2^T/(1+r) \neq 0$$

We define $(dy_1^W, dy_2^W/(1+r))$ as the wealth component of $(dy_1, dy_2/(1+r))$ and we will say that wealth is increased (decreased, unchanged) if and only if $dy_1^W+dy_2^W/(1+r)>0$ ($<0, = 0$). Similarly, $(dy_1^T, dy_2^T/(1+r))$ is the tilt component and income is tilted towards period 1 (period 2) if and only if $dy_1^T - dy_2^T/(1+r) > 0$ (< 0); there is no tilt of income if and only if $dy_1^T - dy_2^T/(1+r) = 0$.

From these definitions, it should be clear that a change in wealth or in the time path of income implies specific signs for $(dy_1^T, dy_2^T/(1+r))$ and for $(dy_1^W, dy_2^W/(1+r))$:

<u>Tilt of income towards</u>:	<u>Period 1</u>	<u>Period 2</u>
	$dy_1^T > 0$	$dy_1^T < 0$
	$dy_2^T/(1+r) < 0$	$dy_2^T/(1+r) > 0$
<u>Change in wealth</u>:	<u>Increase</u>	<u>Decrease</u>
	$dy_1^W > 0$	$dy_1^W < 0$
	$dy_2^W/(1+r) > 0$	$dy_2^W/(1+r) < 0$

We can now state that a wealth component and a tilt component are defined by the following properties:

(i) Orthogonality

$$\begin{bmatrix} dy_1^W & dy_2^W/(1+r) \end{bmatrix} \begin{bmatrix} dy_1^T \\ dy_2^T/(1+r) \end{bmatrix} = 0$$

(ii) Decomposition

$$\begin{bmatrix} dy_1^W \\ dy_2^W/(1+r) \end{bmatrix} + \begin{bmatrix} dy_1^T \\ dy_2^T/(1+r) \end{bmatrix} = \begin{bmatrix} dy_1 \\ dy_2/(1+r) \end{bmatrix}$$

$$dy_1^W + dy_2^W/(1+r) \neq 0$$
$$dy_1^W - dy_2^W/(1+r) = 0$$

(iv) Pure tilt of income

$$dy_1^T - dy_2^T/(1+r) \neq 0$$
$$dy_1^T + dy_2^T/(1+r) = 0$$

We note that this definition is perfectly consistent with Fu's argument: within the H-I model where there is no constraint on personal saving and borrowing, the tilt of income should not have any impact upon optimal choices of h_i and h_c. To verify this consistency of the definition, we consider equations 2a and 2b in Fu (reproduced in Appendix D) and we note that both equations are a function of $(dy_1 + dy_2/(1+r))$. We apply Property (ii) to obtain[8]:

$$\textbf{(16)} \quad (dy_1^W + dy_1^T) + dy_2^W/(1+r) + dy_2^T/(1+r) = dy_1 + dy_2/(1+r)$$

and we apply Property (iv) in **(16)** to get:

$$\textbf{(17)} \quad (dy_1^W + dy_2^W/(1+r)) = dy_1 + dy_2/(1+r)$$

Equation **(17)** shows that $(dy_1 + dy_2/(1+r))$ has only a wealth component. Therefore there is no ground for a tilt of income to have an impact upon h_i or h_c. Furthermore, by using **(17)**, the two basic functions of saving can be enlightened. In Fu, equation 2c) (reproduced in Appendix D) presents the correct comparative-statics result for saving when y_1 and y_2 change. To investigate the effect of income path upon saving, dy_1 and dy_2 are replaced by dy_1^T and dy_2^T respectively in accordance with Property (ii). Since equation **(17)** says that $dy_1 + dy_2/(1+r)$ does not contain any income path effect, 2c) in Fu reduces to:

$$\textbf{(17-S1)} \quad dS = 1/2\ (dy_1^T - dy_2^T/(1+r))$$

Equation **(17-S1)** indicates that saving increases (decreases) when the profile of income is tilted

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[8] Transposing both sides of the equality of Property (ii) and postmultiplying the resulting equality by a 2X1 vector of 1 (unity) leads to equation **(16)**.

towards the first period (second period). Thus, when the capital market is perfect (or equivalently when there is no constraint on the amount of saving), saving plays its usual role of carrying resources through time but also serves as a buffer to smooth out the profile of income. The smoothing process ensures that the optimal housing consumption path and the portfolio choice of housing investment are unaffected by the profile of income[9]. Finally, Proposition (iv) is applied in **(17-S1)** to obtain **(17-S2)**, an equivalent version that will be used later:

$$(17-S2) \qquad dS = dy_1^T$$

where $-dy_2^T/(1+r)$ is substituted for dy_1^T

The interpretation of **(17-S2)** is straightforward. Each variation in the income path (dy_1^T) is matched by an equal variation in the *value of saving*. Since saving is expressed in monetary units, dS can be regarded as a variation in the value of saving.

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9 If the optimal choices of housing consumption (h_c) and housing investment (h_i) are left unchanged after a variation of the income path, saving is defined as a *perfectly efficient buffer*. If h_c and/or h_i change after a variation of the income path, saving would be defined as a ***less efficient buffer***. Under the CMI hypothesis, the efficiency of housing investment as a buffer depends wether or not h_c is changed after a variation of the income path.

Section 3: Comparative-Statics Results

With the definitions of the previous section, we now want to assess the impact of a tilted income towards period 1 upon h_i and h_c. In order to do so, we consider equations **(12)** and **(13)**. To implement a tilt of income path, we replace every dy_1 and $dy_2/(1+r)$ by dy_1^T and $dy_2^T/(1+r)$ respectively (by Property (ii)) and we use Property (iv) to substitute $-dy_2^T$ for $(1+r)dy_1^T$. We obtain:

(18)
$$dh_i = + \left[\left(C(1+r) - \frac{\xi E[v'']\lambda}{E[v']} \right) \left(\frac{h_c^2 u_{11} u_{22} f' R}{(1+r)D} \right) \left(\frac{-dy_1^T f' R\lambda}{E[v']} \right) \right]$$
$$+ \left[\left(\xi E[v''](\tau^2 + (1+r)R\tau) + E[v''(\beta+\gamma)]((1+r)R\tau + ((1+r)R)^2) \right) \left(dy_1^T \frac{h_c u_{11} H}{(1+r)D} \right) \right]$$
$$+ (u_{11}\xi + E[v''\ (\beta+\gamma)](1+r)^2) \left(dy_1^T \frac{h_c u_{22} f^2 H}{(1+r)D} \right)$$

(19)
$$dh_c = \left[\left(E[v''(\beta+\gamma-\xi)^2] - \frac{\xi\lambda C}{E[v']} \right) \left(dy_1^T \frac{h_c^2 u_{11} u_{22} (f')^2 R}{(1+r)D} \right) \right]$$
$$+ \left[(\tau\xi C + (1+r)RG) \left(dy_1^T \frac{h_c u_{11} H}{(1+r)D} \right) \right]$$
$$+ \frac{B}{D} h_c \left(h_c R \ [U_{11}(\tau')^2 + (1+r) U_{22} (f')^2 \ U_1] - (1+r)\tau H \right) dy_1^T$$

where $D < 0$. D is the same as the one defined in **(12)** and **(13)**, and is given in Appendix C. B, C, G and H are also defined in Appendix C.

In order to sign **(18)** and **(19)**, we will keep the H-I specifications (see Appendix A for a reminder of signs). This means in particular that we still assume decreasing absolute risk aversion and non-decreasing relative risk aversion. Since every coefficient of dy_1^T in **(18)** is positive, it follows that h_i increases (decreases) when the income path is tilted towards period 1 (period 2). As for equation **(19)**, the resulting effect of a tilted income path is ambiguous because the coefficients of dy_1^T in the second and third squared brackets cannot be signed. Assuming constant absolute risk aversion does not remove this ambiguity. Despite this indefinite result we are able to formulate three propositions:

Proposition 1: For a tilt of income towards period 1 (period 2), housing investment always increases (decreases), when the constraint $S \geq 0$ is binding.

Proof: See Appendix E

Now remembering that owner-occupancy is defined by $h_i \geq h_c$, so that potential owners have to bridge the gap between housing investment and housing consumption, we can establish the two following propositions:

Proposition 2: Under the binding constraint $S \geq 0$, if income is tilted towards period 1 (period 2), then $\partial h_c/\partial T_1 < 0$ ($\partial h_c/\partial T_2 > 0$) is a sufficient condition to increase (decrease) the probability of owner-occupancy.

Proof: See Appendix E

Proposition 3: Let the constraint $S \geq 0$ be binding. If income is tilted towards period 1 (period 2) and $\partial h_c/\partial T_1 > 0$ ($\partial h_c/\partial T_2 < 0$), then the probability of owner-occupancy increases (decreases) if the income elasticity of housing investment is greater than that of housing consumption. The probability of owner-occupancy decreases (increases) if the income elasticity of housing investment is smaller than that of housing consumption.

Proof: See Appendix E

Proposition 1, 2 and 3 are established under the CMI hypothesis, more specifically under the assumption of a binding borrowing constraint. Proposition 1 is interpreted as follows. Since the financial constraint is binding, saving (borrowing) can no longer be used as a buffer against changes of the income path. Housing investment constitutes the next best alternative. As an alternative, housing investment also reacts in the same way as saving with respect to income path; the directions of the variations induced by the tilt of income are identical. Despite its appealing intuition, this interpretation leaves an unanswered question: is housing investment a perfect substitute for saving? Obviously not for a risk averse agent, because in the H-I model, housing investment is a risky asset whereas saving is a safe asset. But if this difference is set aside, how do these two assets compare? In order to answer, **(18)** and **(19)** must be investigated under the risk neutrality assumption. This will be undertaken in the following section.

As for Proposition 2, it states a sufficient condition under which the gap between housing investment and housing consumption increases and decreases. Proposition 3 deals with the case of housing investment and housing consumption moving in the same direction with respect to a change in the path of income. The gap to bridge increases or decreases depending upon the speed of adjustment of investment and consumption with respect to changes in income y_1 and y_2.

Section 4: Two Particular Cases of Comparative-Statics Results

In this section, we will consider separately two alternative hypothesis in order to further investigate equations **(18)** and **(19)**. First, instead of assuming risk aversion we postulate risk neutrality, that is $v''(w) = 0 \ \forall \ w$[10]. This assumption provides additional insights to understand the characteristics of housing investment as a substitute for saving. Setting $v'' = 0$ in **(18)** and **(19)** gives respectively:

$$\textbf{(20.a)} \qquad dh_i = \frac{dy_1^T}{(P-L-R)} \qquad or \qquad \textbf{(20.b)} \qquad (P-L-R)\ dh_i = dy_1^T$$

$$\textbf{(21)} \qquad dh_c = 0$$

Equation **(20)** shows that housing investment can be used as a perfect substitute for saving ***when the maximizing agent is risk neutral*** and faces a binding borrowing constraint. Housing investment is now the alternative buffer to smooth out the profile of income in such a way that housing consumption is unaffected, as shown by equation **(21)**. In this regard, housing investment constitutes a perfect substitute for saving because the smoothing process ensures that optimal housing consumption is left unchanged. Housing investment plays the same two roles that saving does under perfect financial markets: carrying resources through time and smoothing the profile of income to avoid distortions in the optimal housing consumption path. Also, equation **(20.b)** sheds light on the comparison between saving and housing investment. It shows that each variation in the income path (dy_1^T) is matched by an equal variation in the *value of housing investment*. This result is perfectly symmetric with the interpretation of equation **(17-S2)**.

Results in equations **(20)** and **(21)** can also be used to define more precisely the role of risk aversion. Since **(18)** is derived under the assumption of risk aversion and **(20)** is based upon risk-neutrality, the difference between **(18)** and **(20)** can be interpreted as the marginal effect of risk aversion upon h_i

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10 Assuming risk neutrality in the renters utility maximization problem without any constraint on saving, that is assuming risk neutrality in H-I equation (10), requires special attention. The Cramer's rule is not the appropriate methodology to obtain comparative-statics results. Rather, a detailed analysis of the first-order condition of the housing investment is needed. Details are available from the author upon request.

when income is tilted. The same interpretation is valid for **(19)** and **(21)** with respect to h_c. But since $dh_c = 0$ in **(21)** it implies that the comparative-statics result of dh_c in **(19)** is per se the marginal effect of risk aversion upon h_c when income is tilted. The result of equation **(19)** also suggests that under the risk aversion assumption, housing investment does not smooth out perfectly the profile of income since $dh_c \neq 0$. This is the case because risk averse agents do not use risky housing investment as they use saving (the safe asset) to smooth out the profile of income. Under the risk aversion assumption, housing investment still serves as a buffer against variations of the income path but it is no longer a perfect smoothing instrument. Thus the efficiency of the housing investment as a buffer against changes in the profile of income critically depends upon the attitude of the maximizing agent towards risk.

The previous remarks suggest additional intuition. If housing investment does not smooth out perfectly the profile of income, it implies that more (less) income resources are available in the first period if the path of income is tilted towards the first (second) period. Since housing consumption is positively related with the first period income (see H-I and Fu), it should increase (decrease) when the profile of income is tilted towards the first (second) period. In order to prove the validity of this intuition, equation (19) must be signed. It is impossible to do so unless a minor simplification of the H-I model is made. This will be done under the second alternative hypothesis investigation.

Finally, by using the result of equation **(20)**, it is obvious that under the risk neutrality hypothesis housing investment increases (decreases) when income is tilted towards period 1 (period 2). We can state formally the following proposition:

Proposition 4: Under the risk neutrality hypothesis, housing investment always increases (decreases) when income is tilted towards period 1 (period 2) and when $S \geq 0$ is a binding constraint.

Proof: See Appendix E

Also, since $dh_c = 0$ in **(20)** and remembering that owner-occupancy is defined by $h_i \geq h_c$, we use

Proposition 4 to establish the following result:

Proposition 5: Risk neutrality is a sufficient condition to increase (decrease) the probability of owner-occupancy when income is tilted towards period 1 (period 2) and when $S \geq 0$ is a binding constraint.

Proof: See Appendix E

We now turn to a second alternative hypothesis. Let us assume that the total utilization costs per unit of housing capacity incurred by a tenant is zero, that is $\tau(u) = 0 \ \forall \ u$. This also implies that $\tau'(u) = \tau''(u) = 0 \ \forall \ u$. This hypothesis is a minor simplification of the H-I model. We set $\tau = \tau' = \tau''$ = 0 in **(18)** and **(19)** to obtain:

$$\textbf{(22)} \quad dh_i = \left[\left(C(1+r) - \frac{\xi E[v'']\lambda}{E[v']}\right)\left(\frac{h_c^2 u_{11} u_{22} f' R}{(1+r)D}\right)\left(\frac{-dy_1^T f' R\lambda}{E[v']}\right)\right] + \left[(E[v''(\beta+\gamma)]((1+r)R)^2)\left(\frac{dy_1^T h_c u_{11} f'' u_2}{(1+r)D}\right)\right] + \left[(u_{11}\xi + E[v''(\beta+\gamma)](1+r)^2)\left(\frac{dy_1^T h_c u_{22} f^2 f'' u_2}{(1+r)D}\right)\right]$$

$$\textbf{(23)} \quad dh_c = \left[\left(E[v''(\beta+\gamma-\xi)^2] - \frac{\xi\lambda C}{E[v']}\right)\left(\frac{dy_1^T h_c^2 u_{11} u_{22} (f')^2 R}{(1+r)D}\right)\right] + \left[(1+r)RG\left(\frac{dy_1^T h_c u_{11} f'' u_2}{(1+r)D}\right)\right] + \left[\frac{dy_1^T (1+r) h_c^2 B \ u_{22} (f')^2 R u_1}{D}\right]$$

where in **(22)** and **(23)** $D < 0$ under the H-I model specification. B, C, D and G are defined in Appendix C.

By carefully examining every term of dh_i **(22)** and dh_c **(23)** and knowing the signs of the terms listed in Appendix A, it appears that every coefficient of dy_1^T is positive. The most important thing to note is that we can now sign dh_c; when income is tilted towards the first (second) period, dh_c is positive (negative). This result shows that when housing investment does not smooth out perfectly the profile of income because agents are risk averse, housing consumption is affected. Since housing consumption is positively related to first period income, it increases (decreases) when the path of income is tilted towards the first (second) period. This proves the validity of our previous intuition. Equation **(22)** and **(23)** can also be used to formulate the following proposition:

Proposition 6: Under the hypothesis that the utilization costs incurred by tenants are nil ($\tau = \tau' = \tau'' = 0$) and under the binding constraint $S \geq 0$, both housing consumption and housing investment increase (decrease) if income is tilted towards period 1 (period 2). Thus when the profile of income is tilted towards period 1 (period 2), the probability of owner-occupancy increases (decreases) if the income elasticity of housing investment is greater than that of housing consumption. The probability of owner-occupancy decreases (increases) if the income elasticity of housing investment is smaller than that of housing consumption.

Proof: See Appendix E

Despite the fact that the hypothesis $\tau = \tau' = \tau'' = 0$ leads to Proposition 6 and proves the validity of an intuitive result, we have to verify that two important features of the H-I model are preserved. First we must ensure that there is still an externality problem in the rental market. Secondly, to justify the existence of the rental market, we must prove that owning does not always dominate renting.

The externality in the rental market arises from the fact that it is impossible to explicitly provide in rental contracts for all possible contingencies (costs of utilization, excluding the rental price). Therefore, tenants pay less than owners at all rates of utilization, and since tenants do not face their true costs, they over utilize their housing capacity[11]. Assuming $\tau = \tau' = \tau'' = 0$ reinforces the externality in the rental market because it means that it is impossible to explicitly provide in rental

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11 However, tenants must indirectly pay for their over utilization rate in terms of higher contract rents. See equation (6) of H-I.

contracts for *any* contingencies. In that case, the rental contract only specifies the rental price R per unit of housing capacity. For a complete discussion of the externality problem, see Section 1 of H-I.

In order to prove that owning does not always dominate renting, we first consider in the H-I model the tenant problem (10) and the owner problem (13). To these two problems we add a non-negativity constraint on saving since we consider also the CMI hypothesis and we modify 11c) and 14c) of H-I accordingly. Let us associate the symbols * with optimal values for owners and ~ with optimal values for tenants. We then compare maximal utility from owning (v*) with maximal utility from renting (v~) when $h_c^* = h_i^* = h$[12]. We do a Taylor-series expansion of v~ about v* and substitute in for x~, x*, w~, w* from problem (10) and (13) of H-I and from the respective first order conditions of these two problems. To substitute, we also use the Taylor-series expansion of f~ and T~ found in (9) of H-I model (details are available upon request). Rearranging the result gives:

$$\textbf{(24)}\quad \frac{v^{\sim}-v^{*}}{u_1^{*}} < \begin{array}{l} \left[\dfrac{u_2^{*}\delta}{u_1^{*}} - \dfrac{E[v'^{*}]d}{u_1^{*}}\right] h_c^{\sim} \\ -h_c\left[E\left[\left(\dfrac{v'^{*}}{u_1^{*}}\right)\left(\left(T(\bar{u})-\tau(\bar{u})\right)-\left(T(u)-\tau(u)\right)\right)\right]\right] \\ +(h_c^{\sim}-h_i^{\sim})\left[-cov\left(\dfrac{v'^{*}}{u_1},\ P\theta-\left(T(\bar{u})-\tau(\bar{u})\right)\right)-Prem\right] \\ +(h_c^{\sim}-h_i^{\sim})\left[\dfrac{\lambda^{*}}{u_1^{*}(1+r)}E\left[P(1+\theta)-L(1+r)-\left(T(\bar{u})-\tau(\bar{u})\right)\right]\right] \end{array}$$

where:

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12 In the H-I model, when h_c is near h_i the agent can distort his investment and consumption choices to equalize h_i and h_c and then own his dwelling unit. For this agent (now characterized by $h_i^* = h_c^*$), it is rational to do so because the rental externality is avoided. The comparison with a home owner for whom $h_i^* > h_c^*$ is a particular case of equation **(24)**.

$$u_1^* = E[v'^*](1+r) + \lambda^*$$

$$cov\left(\frac{v'^*}{u_1^*},\ P\theta - (T(\overline{u}) - \tau(\overline{u}))\right) = E\left[\left(\frac{v'^*}{u_1^*}\right)(P\theta - (T(\overline{u}) - \tau(\overline{u}))\right] + \frac{\lambda^*}{u_1^*(1+r)} E[P\theta - (T(\overline{u}) - \tau(\overline{u}))] - \frac{E[P\theta - (T(\overline{u}) - \tau(\overline{u}))]}{(1+r)}$$

$$Prem = \frac{E[P\theta - (T(\overline{u}) - \tau(\overline{u}))]}{(1+r)} + \frac{R(1+r)}{(1+r)} - \frac{rP}{(1+r)}$$

$$\frac{E[P(1+\theta) - L(1+r) - (T(\overline{u}) - \tau(\overline{u}))]}{(1+r)}$$: *Expected present value of a unit of housing stock at the beginning of period 2*

We first note that if there is no constraint on saving $\lambda^* = 0$. Setting $\lambda^* = 0$ in **(24)** leads to the same expression found in footnote 4 of H-I page 109, which supposes no constraint on saving. H-I provide an interpretation of each term in squared brackets in **(24)**, except for the last one. From H-I we also know that the first pair of squared brackets has always a negative sign and that the second pair of brackets could be either positive or negative (see H-I page 109) even if we set $\tau = 0\ \forall\ u$. Therefore owning does not always dominate renting even under the joint hypothesis of CMI and $\tau(u) = 0\ \forall\ u$.

Finally, we propose an interpretation of the last term in squared brackets:

$$\frac{\lambda^*}{u_1^*(1+r)} E[P(1+\theta) - L(1+r) - (T(\overline{u}) - \tau(\overline{u}))]$$

in **(24)**. When choosing to own, and hence bring h_c and h_i at least into equality, the agent faces a cost arising form the fact that saving is constrained ($\lambda^* \neq 0$). To fully compensate the agent for incurring this cost, he must receive the expected present value of a unit of housing stock multiplied by λ^*/u_1^*. Thus the required compensation is proportional to λ^*.

Section 5: The Excess Sensitivity of Housing Consumption

In this section, the relationship between excess sensitivity of consumption to current resources and liquidity constraints is analysed. More specifically, if capital market imperfections are reported to be the cause of the excess sensitivity of consumption, it should also be the case for housing consumption. The empirical findings of Jones (1990) support this view. The results show that *current wealth* does provide greater explanatory power and higher elasticities than permanent income to assess the housing demand of young owners. The purpose of this section is to combine the results of the previous section with the methodology of Jones (1990) to bring a more complete theoretical explanation for these empirical findings.

Like Henderson and Ioannides (1983), Jones argues that optimal housing investment (H*) should be distinguished from optimal housing consumption (H(h*)) and that owner occupancy is defined by the condition H* = H(h*). Since in general, optimal housing investment is not equal to optimal housing consumption, potential owners have to adjust either H* or H(h*)). For *potential young owners*, it is hypothesized that H(h*) > H* is more likely to be the observed pattern. It means that a young household who wishes to consume that particular level of housing consumption H(h*) under owner occupancy has to distort his optimal housing investment H* to increase it up to H(h*).

In Jones, housing consumption and housing investment are determined by two different frameworks. Housing consumption and non-housing consumption are determined by the permanent income derived from the PIH\LC model. Housing investment is explained by a portfolio choice subject to a budget constraint:

$$(25) \qquad \sum_{i=1}^{n} A_i = NW + D$$

where A_i is the household's demand for the i^{th} asset, NW is household current wealth (net worth), and D is borrowing (current level of household debt). The right hand side of **(25)** is the predetermined gross investible wealth. Jones argues that the CMI hypothesis restricts the ability of young households to borrow on human capital collateral. Thus NW is the key element of constraint **(25)**.

Since potential young owners have to distort their housing investment H* to become owners-occupiers, it implies that current wealth (NW), not permanent income, is also the key element to explain the housing consumption demand of young households. Thus, the excess sensitivity of housing consumption is more likely to be observed for young households because their borrowing is constrained and their optimal choices are such that $H(h^*) > H^*$.

However, the previous explanation is based upon the critical hypothesis that, for young households, $H(h^*) > H^*$ is more likely to be the observed pattern. By using the results of the previous sections, it can be shown that this pattern is more prevalent because young households under financial constraint are characterized by a profile of income that leads to such a pattern of optimal choices.

We first note that, in the H-I model without financial constraint, when the profile of income is tilted towards the second period (the future), saving decreases. Equation **(17-S1)** relates saving to the profile of income when the capital market is perfect. When the path of income is tilted towards the future, $dy_1^T < 0$ and $dy_2^T > 0$. It is then obvious that saving decreases. For young households, the profile of income is typically tilted towards the future; earnings of the present period are less than earnings of the next periods. Jappelli and Pagano (1989) present evidence that support this argument. They estimated the profile of earnings for four countries (Japan, U.K., U.S.A. and Italy). The figures show, for young households, a path of income tilted towards the future. Thus, by using equation **(17-S1)**, we can state that young households are more likely to be credit constrained because their income path is tilted towards the future which in turn causes an increase in borrowing. Therefore, their chance of being credit constrained increases. This result is also in line with the empirical evidence of Jappelli (1990). For the U.S. economy, it is shown that the probability of being credit constrained is a decreasing function of current income and age.

When young households face a binding financial constraint, the profile of income does have a meaningful economic role because it can alter the optimal choices of consumption and investment, as shown by Proposition 1 to 6. It was also shown that the efficiency of housing investment as a buffer depends upon the attitude towards risk. Assuming that risk aversion is the most prevalent behaviour throughout the economy, it implies that housing investment does not smooth out perfectly the profile of income. This partial failure of the smoothing process modifies the profile of available income resources in the first and the second period, which in turn modifies the consumption choices. For a profile of income tilted towards period 2 (the future), housing consumption decreases as demonstrated by equation **(23)**. Under risk aversion, housing investment also decreases when the path of income is tilted towards the future (see equation **(22)**). Thus, for young households, the gap between housing investment and housing consumption depends upon the income elasticities of housing investment and housing consumption.

Henderson and Ioannides (1987) found that, for those with the profile of income tilted towards the future, the probability of owning is decreasing. If risk aversion is assumed, it implies an income elasticity of housing investment greater than that of housing consumption, which also leads to an increasing gap between H* and H(h*) such that H(h*) > H*. For young potential homeowners under financial constraint, it means further distorting H* to achieve owner-occupancy. The superior magnitude of the income elasticity of housing investment is also confirmed by the increasing probability of owning with respect to wealth found by H-I (1987). The corrections of the comparative-statics results made by Fu (1990) show that the probability of owning increases with wealth if the income elasticity of housing investment is greater than that of housing consumption.

By using equation **(22)** and **(23)** or their associated Proposition 6 in combination with the empirical findings of H-I (1987), it is demonstrated that young potential homeowners under financial constraint are characterized by an increased gap between housing investment and housing consumption such that H(h*) > H*. This pattern of optimal choices stems from the profile of income. To achieve owner-occupancy, housing investment H* must be distorted. Since H* is constrained by equation **(25)** associated with the portfolio choices, housing consumption of first-time young homeowners depends critically upon current wealth (NW), not the permanent income associated with the PIH/LC model of consumption. This explains the excess sensitivity of housing consumption to current resources for first-time young homeowners. This result is also in line with the findings of Hayashi (1985). These findings indicate that the optimal consumption path of young households characterized by low levels of saving (and more likely to be liquidity constrained) departs more markedly from the PIH/LC predicted path than it does for older households with high level of savings.

The profile of income constitutes the channel through which capital market imperfections leads to excess sensitivity of housing consumption. Under perfect financial markets, the path of income modify the saving choices to increase the probability of being credit constrained. With financial constraint, the path of income alters the optimal choices to increase the gap between housing investment and housing consumption. To achieve owner-occupancy now requires to distort the optimal housing investment which depends upon current wealth.

Conclusion

To establish the relationship between tenure choice and income path, we used the H-I model with an additional constraint on personal saving. This additional constraint was included in order to implement the capital market imperfections (CMI) hypothesis. According to Fu's conjecture, income path can have an impact upon tenure choice only under this hypothesis. This paper has proven the validity of Fu's conjecture. We also showed that the definition of income path proposed by H-I was insufficient to assess the relationship between tenure choice and the tilt of income. We provided an extended definition of income path. It should be emphasized that our definition is a key element of this paper. Without it, it would have been impossible to obtain any results.

Three sets of results were established. First, it has been proven that under the CMI hypothesis, housing investment always increases (decreases) when income is tilted towards the present (future) period. This result is robust since it holds under various assumptions: risk aversion, risk neutrality and nullity of utilization costs per unit of housing capacity incurred by tenants. It also shows that housing investment constitutes the next best instrument to smooth out the profile of income when saving is constrained. However, the efficiency of housing investment as a buffer against the path of income depends critically upon the attitude of the maximizing agent towards risk.

Secondly, we have established two distinct sufficient conditions (under the CMI hypothesis) to increase the probability of owner-occupancy when income is tilted. Without these sufficient conditions, we have been able to formulate some propositions by using the magnitude of income elasticity of housing investment and housing consumption in the same fashion as in Fu.

Finally, by using the previous results, the profile of income has been identified as the channel through which capital market imperfections are transmitted to induce excess sensitivity of housing consumption to current resources. The profile of income modifies the saving pattern. This modification increases the chances of being credit constrained. When borrowing is constrained, the profile of income changes the optimal choices in such a way that owner-occupancy is achieved by distorting the optimal housing investment, which in turn depends upon the current wealth, not permanent income.

Appendix A

y_1 and y_2:	income in period 1 and period 2 respectively
S:	period 1 saving which earns the non-stochastic real rate of interest r
P:	constant market purchase price per unit of housing stock
L:	mortgage loan (per unit of housing stock) at the fixed market rate of interest r
R:	rental price per unit of housing capacity
θ:	stochastic return of asset h_i
h_i:	investment in housing (unit of housing stock) which is rented out to others in period 1 at price R
h_c:	housing capacity rented for oneself for consumption purposes at price R
u:	rate of utilization of housing capacity
$h_c\, f(u)$:	total services derived from housing capacity, given the rate of utilization u
T(u):	utilization costs per unit of housing capacity <u>incurred by a owner-occupier or a landlord</u>
$h_c\, T(u)$:	total utilization costs incurred by an owner-occupier
$\tau(u)$:	utilization costs per unit of housing capacity <u>incurred by a tenant</u>
$h_c\, \tau(u)$:	total utilization costs incurred by a tenant
$(T(\bar{u})-\tau(\bar{u}))h_i$:	total uncollectible maintenance costs (<u>incurred by a landlord</u>) that are uncertain given that the tenants choose $\bar{u}$
y_1-S-(P-L-R)h_i-Rh_c = x:	period 1 consumption of the numeraire
u(.):	utility derived from period 1 consumption bundle
V(w):	indirect utility function of wealth remaining after period 1
w:	remaining wealth after period 1
$\bar{u}$:	the uncertain rate of utilization incurred by a landlord. The landlord faces uncertainty because u is chosen by tenants.

<u>According to the H-I model specification, we have the following signs:</u>

$$f(u) > 0, \quad f'(u) > 0, \quad f''(u) < 0$$

$$\tau > 0, \quad \tau'(u) > 0, \quad \tau''(u) > 0$$

$$T > 0, \quad T'(u) > 0, \quad T''(u) > 0$$

$$\tau(u) < T(u) \quad \forall u \qquad and \qquad \tau'(u) < T'(u) \quad \forall u$$

$$\frac{\partial u(.)}{\partial x} \equiv u_1 > 0, \qquad \frac{\partial u(.)}{\partial(h_c f(u))} \equiv u_2 > 0$$

$$\frac{\partial^2 u(.)}{\partial x \partial(h_c f(u))} \equiv u_{12} = u_{21} = 0$$

$$\xi \equiv (P-L-R)(1+r) > 0$$

$$\frac{\partial^2 u(.)}{\partial x^2} \equiv u_{11} < 0, \qquad \frac{\partial^2 u(.)}{\partial(h_c f(u))^2} \equiv u_{22} < 0$$

$$\beta+\gamma \equiv P(1+\theta) - L(1+r) - (T(\bar{u}) - \tau(\bar{u})) > 0 \quad \forall(\theta, \bar{u}) \quad \textit{where } \theta \textit{ and } \bar{u} \textit{ are stochastic}$$

$$(1+r) > 0, \qquad R > 0$$

$$E\left[\frac{\partial^2 v(.)}{\partial w^2}\right] \equiv E[v''(.)] \equiv E[v''] < 0$$

$$E[v''(.)(\beta+\gamma)] \equiv E[v''(\beta+\gamma)] < 0$$

$$E\left[\frac{\partial^2 v(.)}{\partial w}\right] \equiv E[v'(.)] \equiv E[v'] > 0$$

We also define the following terms:

α: a slack variable added in **(1)** to obtain a standard Lagrangian maximisation problem

λ: Lagrange multiplier associated with the constraint $S = \alpha^2$ in problem **(2)**.

$$A \equiv \frac{-v''(.)}{v'(.)} \equiv \frac{-v''}{v'}: \text{ degree of absolute risk aversion}$$

$$F \equiv \frac{-wv''(.)}{v'(.)} \equiv \frac{-wv''}{v'}: \text{ relative risk aversion coefficient}$$

Under the assumptions $dA/dw < 0$ and $dF/dw \geq 0$, the following signs are proven in the Appendix of H-I:

$$E[v'']\, E[v''(\beta+\gamma)^2] - (E[v''(\beta+\gamma)])^2 > 0$$

$$E[v''(\beta+\gamma)(\beta+\gamma-\xi)] < 0$$

$$E[v''(\beta+\gamma-\xi)] > 0$$

Appendix B

$$\Omega = \begin{bmatrix} L_{h_c h_c} & L_{h_c h_i} & L_{h_c S} & L_{h_c u} & L_{h_c \alpha} & g^S_{h_c} \\ L_{h_i h_c} & L_{h_i h_i} & L_{h_i S} & L_{h_i u} & L_{h_i \alpha} & g^S_{h_i} \\ L_{S h_c} & L_{S h_i} & L_{SS} & L_{Su} & L_{S\alpha} & g^S_S \\ L_{u h_c} & L_{u h_i} & L_{uS} & L_{uu} & L_{u\alpha} & g^S_u \\ L_{\alpha h_c} & L_{\alpha h_i} & L_{\alpha S} & L_{\alpha u} & L_{\alpha\alpha} & g^S_\alpha \\ g^S_{h_c} & g^S_{h_i} & g^S_S & g^S_u & g^S_\alpha & 0 \end{bmatrix}$$

$$L_{h_c h_c} = R^2 u_{11} + f^2(u)\, u_{22} + \tau(u)^2 E[v'']$$

$$L_{h_c h_i} = \frac{R\xi u_{11}}{(1+r)} - \tau(u) E[v''(\beta+\gamma)] = L_{h_i h_c}$$

$$L_{h_c S} = R u_{11} - \tau(u) E[v''](1+r) = L_{S h_c}$$

$$L_{h_c u} = u_{22} h_c f'(u) f(u) + E[v''] h_c \tau'(u) \tau(u) = L_{u h_c}$$

$$L_{h_c \alpha} = 0 = L_{\alpha h_c}, \qquad g^S_{h_c} = 0$$

$$L_{h_i h_i} = \left[\frac{\xi}{(1+r)}\right]^2 u_{11} + E[v''(\beta+\gamma)^2]$$

$$L_{h_i S} = \frac{\xi}{(1+r)} u_{11} + E[v''(\beta+\gamma)](1+r) = L_{S h_i}$$

$$L_{h_i u} = -E[v''(\beta+\gamma)]\tau'(u) h_c = L_{u h_i}$$

$$L_{h_i\alpha} = 0 = L_{\alpha h_i}, \qquad g^S_{h_i} = 0 \ .$$

$$L_{SS} = u_{11} + (1+r)^2 E[v'']$$

$$L_{Su} = -E[v'']\tau'(u)h_c(1+r) = L_{uS}$$

$$L_{S\alpha} = 0 = L_{\alpha S}, \qquad g^S_S = 1$$

$$L_{uu} = u_{22}(h_c f'(u))^2 + h_c U_2 f''(u) + E[v''](h_c\tau'(u))^2 - E[v']\tau''(u)h_c$$

$$L_{u\alpha} = 0 = L_{\alpha u}, \qquad g^S_u = 0$$

$$L_{\alpha\alpha} = -2\lambda, \qquad g^S_\alpha = -2\alpha$$

$$\beta \equiv P - L(1+r), \qquad \beta+\gamma > 0$$

$$\gamma \equiv P\theta - (T(\overline{u}) - \tau(\overline{u}))$$

$$\xi \equiv (P - L - R)(1+r) > 0$$

Appendix C

The expression of D in equations **(12)**, **(13)**, **(18)** and **(19)**.

$$
\begin{aligned}
D = & \\
& \left[\frac{h_c}{(1+r)^2}\right]\{\xi^2 f^2 u_{11} + (\xi E[v''(\beta+\gamma)] + G) f^2 (1+r)^2\} u_{22} H \\
& + \{\xi^2 E[v''] \tau^2 + (2(1+r)R\tau + (1+r)^2 R^2) \xi E[v''(\beta+\gamma)]\} u_{11} H \\
& + \{GR^2 u_{11} + B\tau^2\} (1+r)^2 H \\
& + E[v''(\beta+\gamma-\xi)^2]((f')^2 (1+r)^2 h_c R^2 U_{11} u_{22}) \\
& - 2C(f')^2 (1+r) h_c R^2 \frac{\lambda}{E[v']} u_{11} u_{22} \xi \\
& + \xi^2 E[v''] h_c (f')^2 R^2 \left[\frac{\lambda}{E[v']}\right]^2 u_{11} u_{22} \\
& + B(1+r)^2 h_c ((f'\tau - f\tau')^2 u_{22} + R^2 (\tau')^2 u_{11})
\end{aligned}
$$

Henderson and Ioannides (1983) show the following:

$$B \equiv (E[v'']E[v''(\beta+\gamma)^2] - (E[v''(\beta+\gamma)])^2) > 0$$

B > 0 under decreasing absolute risk aversion and non-decreasing relative risk aversion.

$$C \equiv E[v''(\beta+\gamma-\xi)] > 0$$

C > o under decreasing absolute risk aversion.

$$G = E[v''(\beta+\gamma)(\beta+\gamma-\xi)] < 0$$

G < 0 under decreasing absolute risk aversion and non-decreasing relative risk aversion.

$$H = f''u_2 - E[v']\tau'' < 0$$

H < 0 because U(.) and V (.) are increasing. Furthermore, f(u) is concave in u and τ(u) is convex in u. Knowing these results, one can carefully review the expression of D to verify that we have indeed D < 0.

The expression of D in equations **(22)** and **(23)**.

$$
\begin{aligned}
D &= \\
&\left[\frac{h_c}{(1+r)^2}\right] (\xi^2 f^2 u_{11} + (\xi E[v''(\beta+\gamma)] + G) f^2 (1+r)^2)(u_{22} f'' u_2) \\
&+ ((1+r)^2 R^2 \xi E[v''(\beta+\gamma)] u_{11} f'' u_2) \\
&+ (G R^2 u_{11} (1+r)^2 f'' u_2) \\
&+ (E[v''(\beta+\gamma-\xi)^2](f')^2 (1+r)^2 h_c R^2 u_{11} u_{22}) \\
&-2\left(C (f')^2 (1+r) h_c R^2 \frac{\lambda}{E[v']} u_{11} u_{22} \xi \right) \\
&+\left(\xi^2 E[v''] h_c (f')^2 R^2 \left(\frac{\lambda}{E[v']}\right)^2 u_{11} u_{22} \right)
\end{aligned}
$$

Again, knowing that C > 0 and G < 0, one can carefully examine the expression of D to verify that D < 0.

Appendix D

(2a) $d\tilde{h}_c$

$$= \frac{U_{11}}{D}\left\{E[V''']E\left[V'''(P_2-(1+r)P_1)^2\right] - E^2\left[V'''(P_2-(1+r)P_1)\right]\right\} \times [R(1+r)+\tau](1+r)\left(dy_1+\frac{dy_2}{1+r}\right)$$

(2b) $d\tilde{h}_I$

$$= -\frac{U_{11}U_{22}}{D}f^2E[V'''(P_2-(1+r)P_1)] \times (1+r)\left(dy_1+\frac{dy_2}{1+r}\right)$$

(2c) $$d\tilde{S} = \frac{1}{2}\left(dy_1-\frac{dy_2}{1+r}\right)+\frac{1}{2D_{-I}} \times\left\{U_{11}U_{22}f^2-U_{11}E[V'''] \times\left[R^2(1+r)^2-\tau^2\right] - U_{22}f^2E[V'''](1+r)^2\right\} \times\left(dy_1+\frac{dy_2}{1+r}\right) - \left\{P_1+\frac{1}{D_{-I}}(U_{11}[R(1+r)+\tau]R + U_{22}f^2(1+r)) \times E[V'''(P_2-(1+r)P_1)]\right\}d\tilde{h}_I$$

$$D = U_{11}U_{22}f^2E\left[V'''(P_2-(1+r)P_1)^2\right] + \left\{U_{11}[R(1+r)+\tau]^2+U_{22}(1+r)^2f^2\right\} \times\left\{E[V''']E\left[V'''(P_2-(1+r)P_1)^2\right] - E^2\left[V'''(P_2-(1+r)P_1)\right]\right\}$$

$$D_{-I} = U_{11}U_{22}f^2+U_{11}E[V'''][R(1+r)+\tau]^2 + U_{22}f^2E[V'''](1+r)^2$$

Appendix E

Proof of Proposition 1:

A tilt of income towards period 1 implies $dy_1^T > 0$. If the constraint $S \geq 0$ is binding, then $\lambda > 0$. In Appendix C, it is shown that $C > 0$ under decreasing absolute risk aversion. It is also shown that $H < 0$ because U (.) and V (.) are increasing, f(u) is concave in u and $\tau(u)$ is convex in u. Appendix C also demonstrates why $D < 0$ in the H-I model.

$E[V'']$, u_{11} and u_{22} are all negative by assumption of strict quasi concavity. $E[V'] > 0$ because V (.) is assumed to be increasing. $(\beta + \gamma)$ is positive by assumption to exclude costs of defaults. ξ is also assumed to be positive. The sign of all remaining terms are obvious.

Knowing all these signs, a careful examination of **(18)** show that every coefficient of dy_1^T is positive. Therefore, $dh_i > 0$.

Proof of Proposition 2:

Let us denote $h_c - h_i = \Delta$. Owners and renters are defined by $\Delta < 0$ and $\Delta > 0$ respectively. Thus, renters must reduce the gap Δ to achieve ownership. From Proposition 1, we already know that h_i increases when income is tilted towards period 1. The increase of h_i reduces the gap Δ for a given level of h_c. Obviously, if housing consumption *also decreases* when income is tilted towards period 1 $(\partial h_c / \partial T_1 < 0)$, then the gap Δ is reduced and the probability of owner occupancy increases.

Proof of Proposition 3:

h_i increases when income is tilted towards period 1 and therefore reduced the gap Δ for a given level of h_c (see Proposition 1). However, housing consumption increases when income is tilted towards period 1 $(\partial h_c / \partial T_1 > 0)$ and thus increases the gap Δ for a given level of h_i. But if income elasticity of housing investment is greater than (smaller than) that of housing consumption, it implies that h_i changes at a faster (slower) rate than h_c and that the gap Δ is decreasing (increasing). Thus the probability of owner occupancy increases.

Proof of Proposition 4:

Consider equation **(20)**. When income is tilted towards period 1 it implies that $dy_1^T > 0$. (P - L - R) is assumed to be positive because it is the net price of housing investment per unit of stock in period 1. Therefore, $dh_i > 0$.

Proof of Proposition 5:

Let us denote $h_c - h_i = \Delta$. Owners and renters are defined by $\Delta < 0$ and $\Delta > 0$ respectively. Thus, renters must reduce the gap Δ to achieve ownership. From Proposition 4, we already know that h_i increases when income is tilted towards period 1 under risk neutrality. Thus the gap Δ is reduced for a given level of h_c. Equation **(21)** shows that h_c does not change when income is tilted towards period 1 under risk neutrality. Thus the gap Δ is unchanged for a given level of h_i. Combining these results leads to the obvious conclusion that the gap Δ is reduced. Therefore, the probability of owner occupancy increases.

Proof of Proposition 6:

Consider Equation **(22)** to sign dh_i. Since we set $\tau = \tau' = \tau'' = 0$ in **(18)** to obtain **(22)**, it follows that equation **(22)** is a particular case of equation **(18)** and its associated Proposition 1. Thus, from Proposition 1, we know that $dh_i > 0$.

Consider now equation **(23)** to sign dh_c. A tilt of income towards period 1 implies that $dy_1^T > 0$. If the constraint $S \geq 0$ is binding, then $\lambda > 0$. In Appendix C, it is shown that $C > 0$ under decreasing absolute risk aversion. It is also shown that $B > 0$ and $G < 0$ under decreasing absolute risk aversion and non-decreasing relative risk aversion. Appendix C also demonstrates why $D < 0$ in the H-I model.

E [V''], u_{11} and u_{22} are all negative by assumption of strict quasi concavity. E [V'], u_1 and u_2 are all positive because V (.) and U (.) are assumed to be increasing; f'' is negative by the assumed concavity of f(u). The sign of all remaining terms are obvious. Knowing all these signs, a careful examination of **(23)** shows that every coefficient of dy_1^T is positive. Therefore, $dh_c > 0$.

Define $h_c - h_i = \Delta$. Owners and renters are defined by $\Delta < 0$ and $\Delta > 0$ respectively. Thus, renters must reduce the gap Δ to achieve ownership. The gap Δ increases or decreases depending upon the speed of adjustment of h_c and h_i with respect to a change of magnitude dy_1^T. Thus, the rest of the proof is identical to the proof of Proposition 3.

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ESSAY 2: INCREASES IN RISK WITH MULTIPLE DEPENDANT STOCHASTIC PARAMETERS: SOME NEW CONSIDERATIONS

Section 1: Introduction

The analysis of a change in risk with only one source of randomness has been studied extensively. In a context of two stochastically dependent parameters, a new set of questions arises. This short comment will focus on two issues identified by Meyer [1992]. When a model allows for two risky parameters (say x and y, both defined on the support (0,B)), an assumption on the marginal cumulative density function (CDF) of y must be made as x undergoes a change in risk. This is the *ceteris paribus* assumption[1]. The second issue pertains to the stochastic dependance between x and y, i.e. the correlation. In an example on optimal insurance coverage, Meyer [1992] shows that a change in risk with no effect on the marginal CDF of x and y can modify the optimal choice because the correlation is reversed. See Doherty and Schlesinger [1983] for a first analysis of optimal insurance coverage with two random parameters.

In a related paper, Dionne and Gollier [1992] present an alternate definition of a change in risk in a model with two sources of randomness[2]. However, the issues raised by Meyer [1992] are not addressed in their article, neither are the *ceteris paribus* assumption nor the stochastic dependence. Furthermore, it is difficult to see how this definition of a change in risk could be illustrated by a practical example. In Section 2, it is shown that the Dionne and Gollier [1992] definition always preserves the sign of the correlation, provided that a *ceteris paribus* assumption is made. Section 3 discusses the comparative statics results obtained with the Dionne and Gollier [1992] definition of a decrease in risk and also compares these results with the ones derived by Meyer [1992]. Section 4 concludes and illustrates changes in risk with marketplace examples.

Section 2: Deterministic Transformation and Simple Decrease in Risk

To deal with these issues, Meyer [1992] uses a deterministic transformation of the random parameter x, denoted t(x) (see Meyer and Ormiston [1989] for more details). This transformation is applied in the context of an insurance purchase decision model to derive comparative statics results. t(x) transforms x in such a way that t: $(0,B) \rightarrow (0,B)$. If t(x) is further restricted to be non decreasing, x and t(x) respectively have a marginal CDF $F^1(.)$ and $F^0(.)$ satisfying the relation

$F^1(t(x)) = SUP\ \{F^0(w) : t(w) \leq t(x)\}$. A strictly increasing transformation is characterized by $F^1(t(x)) = F^0(x)$. A deterministic transformation does not alter the marginal CDF of x. Because $G(y) = \int_0^B G(y|x)\, dF(x)$ it follows that neither $G(y|x)$ nor $G(y)$ is changed by the transformation.

Meyer [1992] also argues that a non decreasing transformation cannot reverse the correlation sign.

Dionne and Gollier [1992] propose an alternate definition of a change in risk to derive comparative statics results with two random variables. They investigate the same insurance purchase decision model. The two preceding definitions are reproduced for convenience.

Definition 1: (Meyer [1992])

We say that t(x) represents a simple risk reducing deterministic transformation across P of x if and only if it is a deterministic transformation satisfying the following properties:

2.1) $E[t(x)] = E[x]$;

2.2) t(x) is non-decreasing and $t(x) \leq x$ whenever $x \geq P$ and $t(x) \geq x$ whenever $x \leq P$.

Definition 2: (Dionne and Gollier [1991], [1992])

We say that $F_2(x)$ is a simple decrease in risk (sDR) across P of $F_1(x)$ if and only if:

1.1) the mean of x is preserved;

1.2) $F_2(x)$ is greater than $F_1(x)$ whenever x is larger than P and $F_2(x)$ is less than $F_1(x)$ whenever x is less than P.

To obtain unambiguous comparative statics results under the assumption of stochastic dependence, Dionne and Gollier [1992] apply a sDR on the conditional distribution of x, for all possible realizations of the background risk y. The sDR needs not be the same for all $y \in (0,B)$. Furthermore, for every possible realization of y, the conditional mean of x is required to be unchanged (Dionne and Gollier [1991], Definition 6(a)). This can be described as "conditional mean preserving". At this stage, the two issues raised by Meyer [1992] come into play. More precisely, does a sDR on the conditional distribution of x (for all y ∈ (0,B)) change G(y)? Does such a decrease in risk reverse the correlation between x and y?

From the relation $F(x) = \int_0^B F(x|y)\,dG(y)$ it is obvious that F(x) and/or G(y) are affected by a sDR on the conditional distribution of x, at least for some (x,y) pairs. Thus a sDR requires an assumption on G(y). Specifically, the marginal CDF of y is assumed to be unchanged. This assumption is crucial to prove the following proposition.

Proposition 1:

A sDR across P on the conditional distribution of x for all y ∈ (0,B) cannot reverse the correlation sign between x and y if the marginal CDF of y is assumed to be unchanged.

Proof:

Let the correlation between x and y be denoted ρ(x,y). By definition, $\rho \equiv cov(x,y)/\sigma_x\sigma_y$. To prove Proposition 1, we only have to show that COV(x,y) is unaffected by a sDR across P on the conditional distribution of x for all y ∈ (0,B). Using the definition of covariance, we can write:

$$cov(x,y) = \int_0^B\int_0^B xy\,dF(x,r|y)\,dG(y) - \{\int_0^B(\int_0^B x\,dF(x,r|y))\,dG(y)\}\{\int_0^B y\,dG(y)\} \tag{1}$$

where F(x,r|y) stands for the conditional CDF of x for a given level of risk r.

Observe that G(y) is not a function of r because, by assumption, a sDR across P on the conditional distribution of x for all y ∈ (0,B) does not affect G(y). Let dr be a change in risk. Then:

$$\frac{dcov(x,y)}{dr} = \int_0^B\int_0^B xy\,dF_r'(x,r|y)\,dG(y) - \{\int_0^B(\int_0^B x\,dF_r'(x,r|y))\,dG(y)\}\{\int_0^B y\,dG(y)\} \tag{2}$$

From definition 6a of Dionne and Gollier [1991], we have a "conditional mean preserving" sDR i.e. $\Psi(y) \equiv \int_0^B x dF_r^{'}(x,r|y) = 0$ for all y $\in$ (0,B). This implies that the second term on the RHS of equation (2) is zero. Thus, we can write equation (2) as:

$$\frac{dcov(x,y)}{dr} = \int_0^B y \, \Psi(y) \, dG(y) = 0 \tag{3}$$

Q.E.D.

Section 3: Discussion

At this point, it would be interesting to contrast the definition of Dionne and Gollier [1992] with Meyer's [1992] definition. The contrast should be expressed in terms of hypotheses needed to obtain a given comparative statics result. To compare these two definitions, the discussion is recast in a model of portfolio management. Two reasons support this choice. First, the Dionne and Gollier [1992] definition was originally applied in such a model (Dionne and Gollier [1991]), so all the results are already known. Secondly, Meyer [1992] offers an interpretation of his results in a portfolio choice context. However, the interpretation boils down to a particular case because the two assets are perfectly negatively correlated.

Let's assume that the capital can be allocated between two risky assets with stochastic returns x and y defined on the support (0,B). Capital is normalized to unity to give a terminal wealth of Z, where Z = bx+(1-b)y and b stands for the proportion of funds invested in asset x. b* maximizes the expected utility:

$$b^* \in arg \max_b EU(b,H) = \int_0^B \int_0^B U(Z) \; dH(x,y) \tag{4}$$

In this context, a version of Meyer's Theorem 3 still applies and is stated in Theorem 3' below. For convenience, we assume b*>0.

Theorem 3':

Suppose x and y are independent and x undergoes a Rothschild and Stiglitz [1970] decrease in risk that maintains independence between x and y. Then the decision makers who are increasingly relative and decreasingly absolute risk averse with $R_R \leq 1$ will increase b*.

Proof:

It is sufficient to show that U'(Z)(x-y) is concave for all values of $y \in (0,B)$. The first order derivative can be expressed as $U'(Z)\{1 - R_R + R_A y\}$ where R_A and R_R are the absolute risk aversion and the relative risk aversion respectively. Straightforward manipulations yield a second order derivative in terms of R_A and R_R:

$$b[U''(Z)\{1-R_R+R_A y\}+U'(Z)\{-R'_R+R'_A y\}] \tag{5}$$

This expression is positive under the hypotheses of Theorem 3'. QED.

Dionne and Gollier [1991] obtain the same comparative statics result (see their Proposition 4 and Corollaries 2 and 3) with only one assumption on the preferences, namely risk aversion[3]. This suggests that the Dionne and Gollier [1992] definition of a change in risk is more restrictive than Meyer's [1992] definition.

Theorem 4 of Meyer [1992] addresses the case of stochastic dependence. In a portfolio management context, the theorem can be restated as:

Theorem 4':

Let t(x) be increasing with $t(x) \geq x$ for $x \leq y$ and $t(x) \leq x$ for $x \geq y$. If expected utility does not decrease when x is transformed by t(x), then the transformation causes the risk averse decision makers to increase b*.

Proof:

Following the proof of Meyer's [1992] Theorem 2, we have to determine the sign of the derivative of the first order condition with respect to θ, that is the sign of:

$$\int_0^B\int_0^B U'(Z(\theta))\,[x+\theta k(x)-y]\,d^2H(x,y) \quad \text{for all } \theta \text{ in } [0,B] \tag{6}$$

The last expression can be written as:

$$\int_0^B\int_0^B U'(Z(\theta))\,k(x)\,d^2H(x,y) \;+\; \int_0^B\int_0^B U''(Z(\theta))\,[x+\theta k(x)-y]\,b^*k(x)\,d^2H(x,y) \tag{7}$$

The second portion of expression (7) is positive for all $\theta \in (0,B)$ because k(x) and [x+θk(x)-y] have opposite signs for all $\theta \in (0,B)$ and $U''(Z(\theta)) < 0$, $b^* > 0$.

The first portion of expression (7) has the sign of dEU/dθ at θ = 0. Because t(x) is a beneficial change in risk by assumption, it must be that dEU/dθ > 0. The first portion is positive also, and one can conclude that the optimal b* is increased. Q.E.D.

Assuming only the risk aversion hypothesis, Dionne and Gollier [1991] find the same result (see their Proposition 4 and Corollaries 2 and 3). However, the intuition suggests that an additional assumption is needed to obtain unambiguous comparative statics results in a context of stochastic dependance. Both Dionne and Gollier [1992] and Meyer [1992] resort to an additional assumption. Dionne and Gollier [1992] impose a decrease (increase) in risk on the conditional cumulative density function F(x|y) and this change in risk is applied for every potential value of y. As for Meyer [1992], the deterministic transformation is required not to decrease the expected utility.

Section 4: Conclusion and Applications

The Dionne and Gollier [1992] definition of a change in risk shares a very important feature with

Meyer's [1992] definition. They both preserve the sign of the correlation between the two stochastic variables. When combined with appropriate hypotheses, these definitions also lead to the same comparative statics results. However, it is difficult to draw a definitive conclusion on their relative generality because these definitions are not directly comparable. Dionne and Gollier [1992] use the cumulative density function approach whereas Meyer's [1992] definition modifies the realizations of the stochastic variable.

Despite that these definitions of a change in risk are appealing from an analytical point of view, their illustration is treated as a marginal issue in the two articles discussed in this paper. The following examples attempt to provide more detailed illustrations. In the case of a single source of randomness, some examples have already been proposed. For instance, a long position on a European call option on a stock provides a limit to potential losses (i.e. the value of the option) and unlimited potential gains. This position, when compared to a long position on the same stock, reduces the risk because the cumulative distribution of returns is modified. Some weight is shifted from the left to the right of the distribution. Alternatively, we can say that the stochastic returns x are transformed by t(x) in such a way that t(x)=c for all $x \leq c$, where c is a constant. Eeckhoudt and Hansen (1980) offer another example with minimum and/or maximum prices in the context of a competitive firm.

To illustrate the case of two sources of randomness, assume that the stochastic profits (y) of a company are defined by $y=\pi(a,b)$. a and b represent the unregulated stochastic price of electricity and aluminium respectively. The statistical dependance between a and b is summarized by H(a,b)=F(a|b)G(b). H, F and G refer to cumulative density functions. Let's assume now that a regulation specifying a minimum and maximum price of electricity is implemented. For each potential aluminium price, the cumulative density function of electricity prices is modified and the modification needs not to be identical for every aluminium price. In other words, F(a|b) undergoes a decrease in risk for each potential value of b. This change in risk on the conditional cumulative density function illustrates the definition proposed by Dionne and Gollier [1992]. We could also say that, for each potential value of b, a is transformed by t(a|b). Thus this example also illustrates what Dionne and

Gollier [1991] call a "semi-deterministic transformation". As a final note, it can be reasonably argued that such a regulation may change the correlation between a and b but cannot alter the sign of it.

Notes

1 As pointed out by Meyer [1992], this is not the only assumption one might consider:

> "Stochastic dependance, like deterministic dependance, requires that when one of two variables is changed, the other must change also. Unlike deterministic dependence, however, there are a large number of possible changes which can occur. Hence, a researcher still can impose any one of a variety of *ceteris paribus* assumptions concerning the other random parameters".

Meyer [1992], p.11. See also Meyer [1992], note 4.

2 This definition was originally presented in a context of portfolio management. See Dionne and Gollier [1991].

3 Proposition 4 of Dionne and Gollier [1991] considers only the case of stochastic dependence between x and y. However, it is trivial to show the validity of Proposition 4 in a context of stochastic independence.

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ESSAY 3: INCREASES IN RISK AND OPTIMAL PORTFOLIO

Section 1: Introduction

Since the contributions of Rothschild and Stiglitz (1970-1971) there has been a proliferation of articles on the effect of increases in risk on the optimal decision variables of economic problems under uncertainty (see the recent papers by Gollier (1994); Dionne, Eeckhoudt and Gollier (1993); Meyer (1992); Hadar and Seo (1990); Meyer and Ormiston (1994) and Eeckhoudt and Kimball (1992)). Recently, some papers have extended this literature by considering problems with two random parameters but were restricted to applications with only one decision variable which implies that this literature cannot yet study the effect of a general increase in risk on an optimal portfolio along with debt, insurance or even saving under uncertainty. Moreover, as discussed by Levy (1992) in his recent survey, the main drawback of the standard one decision - one random variable model is in the area of finance since the model cannot be used for the study of efficient diversification strategies. The object of our research is to extend significantly this literature by proposing a model with two decision variables and two dependent random variables.

In the literature on optimal portfolio analysis, restrictions are often imposed on the distribution of the rates of return and/or the utility function of the decision makers. Any form of comparative statics analysis becomes very complicated when more than one risky asset is in the portfolio. Ross (1981) showed, for example, that we must restrict the Arrow-Pratt measure of risk aversion in the presence of two risky assets, if one wants to obtain the intuitive result that a decision maker, with decreasing absolute risk aversion, will increase his investment in the risky asset following an increase in his initial wealth. But, as demonstrated by Machina (1982) and Epstein (1985), even the Ross' definition of risk aversion is not strong enough to make the comparative statics analysis if the increment in wealth is random instead of being non-stochastic. Machina needs that the two base wealth distributions being comparable by using the criteria of first-order stochastic dominance. Epstein proposes another set of restrictions to the analysis and shows that his analysis implies mean-variance utility even if his application is restricted to one decision variable or to a two fund separation problem. We know from Meyer (1987) and Epstein (1985) that a mean variance framework does not necessarily imply quadratic utility functions or normal distributions. However, these results do not indicate which utility functions or which distributions of the rates of return are more likely to yield unambiguous comparative statics results. Thus, one of the objectives of this essay is to investigate other utility functions than the quadratic.

Although this form of comparative static analysis is not directly related to our problematic, it is not without any link. It is well known that decreasing absolute risk aversion is a sufficient condition to sign the effect of an increase in initial wealth on the optimal portfolio (one random variable-one decision variable model). Decreasing risk aversion in also part of a set of sufficient conditions (although it is not necessary) to sign the effect of increases in risk of the risky asset on the portfolio composition of risk averse individuals. In general, however, one needs more restrictive assumptions on the utility function to sign the effect of a Rothschild-Stiglitz mean preserving spread on optimal decision variables than for an increase in base wealth. Since the Rothschild-Stiglitz mean preserving spread does increase the variance, the need of additional restrictions suggests that even the mean-variance analysis assumption may not be sufficient to obtain intuitive comparative statics results for portfolio with more than one random asset.

One way that was adopted in the finance literature to simplify the analysis was to propose that risk averse individuals act as if they held the same portfolio of risky assets and only modify the composition between that portfolio and the riskless asset. This approach has been intensively used over the recent years for the comparative statics analysis of mean preserving spreads on the composition of individuals' portfolio (Hadar and Seo (1990); Meyer and Ormiston (1994) and Dionne and Gollier (1992)). This methodology is not free of profound criticism since it cannot explain how the increase in the riskiness of some risky assets affect the composition of the risky fund.

In this paper we propose a detailed analysis of a three assets portfolio and show how the increase in risk of one risky asset affect the composition of risk averse individuals' portfolios. In the next section we revise the main results associated to the comparative statics of increases in risk. We analyse in detail the main results of models with one and two random variables with one decision variable. In Section 3 we propose a model of two random and two decision variables and present its comparative statics in terms of increases in risk of one risky asset. Four examples are studied in detail. The last section summarizes the main results and concludes on the implications for portfolio choices.

Section 2: Increases in Risk and Optimal Portfolio Choice - Literature Review

2.1 One risky asset and one decision variable

Let us consider the standard portfolio problem. A strictly risk averse individual must allocate his initial wealth W_0 between a risky asset Z_1 with a rate of return x_1 and asset Z_0 with a riskless rate of return x_0. His initial wealth can be written as

$$W_0 = z_1 + z_0 \tag{1}$$

where z_1 and z_0 are the initial monetary investments in assets Z_1 and Z_0. The final wealth W depends on the amount invested in Z_1 (Z_0 is automatically determined by the constraint (1)) and is then equal to

$$W(z_1) = z_0 x_0 + z_1 x_1 \tag{2}$$

or, by substituting (1) in (2),

$$W(z_1) = W_0 x_0 + z_1(x_1 - x_0).$$

We assume that x_1 is a continuous random variable that belongs to the interval $[\underline{x}_1, \overline{x}_1]$, where $\underline{x}_1 < x_0 < \overline{x}^1$ to obtain a meaningful portfolio problem. $E_F(x_1)$ is the expected value of x_1 under the distribution $F(x_1)$ with a probability density function $f(x_1)$.

Therefore, the individual's portfolio choice is

$$\underset{z_1}{\text{Max}} \int_{\underline{x}_1}^{\overline{x}_1} U(W_0 x_0 + z_1(x_1 - x_0)) f(x_1) dx_1 \tag{3}$$

where U is a von Neuman-Morgenstern strictly increasing and strictly concave utility function of wealth with $U'(\cdot) > 0$ and $U''(\cdot) < 0$ for all W.

Since W_0, x_0 and x_1 are considered as given, **(3)** can also be written in terms of the single decision variable Z_1

$$\underset{z_1}{\text{Max}}\ E\left[U\left(W\left(z_1\right)\right)\right]. \tag{3}$$

The first-order condition that determine the optimal value of Z_1 is equal to :

$$\int_{\underline{x}_1}^{\overline{x}_1} U'\left(W_0 x_0 + z_{IF}^*\left(x_1 - x_0\right)\right)\left(x_1 - x_0\right) dF\left(x_1\right) = 0 \tag{4}$$

where $dF(x_1)$ is written for $f(x_1)\, dx_1$.

Under strict risk aversion, (4) is necessary and sufficient for a global optimum corresponding to z_{IF}^*. Risk aversion is defined as $U''(.) \le 0$ and strict risk aversion is defined as $U''(.) < 0$. Strict equality is obtained since $(x_1 - x_0)$ changes sign once in the interval $\left[\underline{x}_1, \overline{x}_1\right]$. As shown by Mossin (1973), $E(x_1) > x_0$ implies that z_{IF}^* is strictly positive while $E(x_1) < x_0$ implies $z_{IF}^* < 0$. For the reminder of this section, we will assume that $E(x_1) > x_0$ and that the parameters (the first two moments of x_1) are such that an interior solution exists (see Dionne, Eeckhoudt and Gollier (1993) for more details).

To analyse the effect of a mean preserving spread on the optimal value z_{IF}^*, let us introduce a new distribution $G(x_1)$ with a density function $g(x_1)$. $G(x_1)$ differs from the original distribution $F(x_1)$ because it represents a riskier distribution. If $F(x_1)$ undergoes a mean preserving increase in risk, the resulting distribution is $G(x_1)$. From Rothschild and Stiglitz (1971), we know that we must restrict either the set of utility functions or the set of mean preserving spreads to sign the effects of increases in risk in accordance with the economic intuition. Let us first consider restrictions on the set of utility functions.

Without loss of generality we assume that $G(x_1)$ has the same support $\left[\underline{x}_1, \overline{x}_1\right]$ than $F(x_1)$. Moreover, since we analyse the effect of a mean preserving spread on z_1*, we assume that $E_F(x_1) = E_G(x_1)$. Therefore, using the integral definition of a mean preserving spread : $G(x_1)$ is more risky than $F(x_1)$ for all risk averse individuals $[E_F U(W(z_1) \geq E_G U(W(z_1))]$ if and only if

$$\int_{\underline{x}_1}^{\overline{x}_1} \left(F(x_1) - G(x_1)\right) dx_1 = 0$$

and

$$\int_{\underline{x}_1}^{x_1} \left(F(x_1) - G(x_1)\right) dx_1 \leq 0, \forall x_1 \in \left[\underline{x}_1, \overline{x}_1\right].$$

A risk averse individual will invest less in the risky asset under the more risky distribution $G(x_1)$ than under the less risky distribution $F(x_1)(z_{1G}^* \leq z_{1F}^*)$, if and only if the first order condition (4) evaluated at

$$\int_{\underline{x}_1}^{\overline{x}_1} U'\left(W_0 x_0 + z_{1F}^*(x_1 - x_0)\right)(x_1 - x_0) dG(x_1) \tag{5}$$

is non-positive or, by subtracting (4) to (5) if and only if :

$$\int_{\underline{x}_1}^{\overline{x}_1} U'\left(W_0 x_0 + z_{1F}^*(x_1 - x_0)\right)(x_1 - x_0) dS(x_1) \leq 0, \tag{6}$$

where $S(x_1) \equiv G(x_1) - F(x_1)$ for all x_1.

<u>Proposition 1</u> : Suppose that z_{1F}^* et z_{1G}^* maximize $E[U(W(z_1))]$ under $F(x_1)$ and $G(x_1)$ respectively. Suppose also that $G(x_1)$ represents a mean preserving spread with respect to $F(x_1)$. Then two

sufficient conditions for $z_{IF}^* \geq z_{IG}^*$, for all distribution functions is that $V(z_1, x_1) \equiv U'(W(z_{IF}^*)) \cdot (x_1 - x_0)$ is a concave function of x_1 and that partial risk aversion $P(W_1, W_2)$ is non decreasing and less than one.

Proof: Differentiating twice $V(z_1,x_1)$ with respect to x_1 yields :

$$V'(z_1,x_1) = U'(W(z_1^*))\,(1-P(W_1,W_2))$$

where

$$P(W_1, W_2) \equiv -\frac{U''(W(z_1(w_1,w_2)))z_2}{U'(W(z_1(w_1,w_2)))} \quad \textbf{(8)}$$

$$W_1 \equiv W_0 x_0 \text{ and } W_2 \equiv z_1(x_1 - x_0)$$

and

$$V''(z_1,x_1) = U''(W(z_1^*))(1-P(W_1,W_2)) - U'(W(z_1^*))\frac{\partial P(W_1,W_2)}{\partial W_2} \quad \textbf{(9)}$$

which is weakly negative under the two sufficient conditions of the proposition.

By the definition of $P(w_1, w_2)$, the sufficient conditions of Proposition 1 can be reinterpreted in terms of both the (Arrow-Pratt) measures of absolute risk aversion (A) and relative risk aversion (R). In fact it can be shown that

$$\frac{dP}{dW_2} = \frac{dR}{dW_2} - W_1\frac{dA}{dW_2}. \quad (10)$$

When R is non-decreasing and A is non-increasing, P is non decreasing. However, the converse is not true which reinforces the idea that using the partial measure of risk aversion yields more general results (see Dionne and Gollier (1992) for more details). In fact, Proposition 1 includes the quadratic utility function as a particular case while it is often claimed in many finance books (see Huang and Litzenberger (1988), for example) that a set of sufficient conditions for $V(z_1,x_1)$ to be concave is that the relative risk aversion measure is less than one and increasing and that the measure of absolute risk aversion is decreasing. The last condition rules out the quadratic utility function since $U'''(\cdot) > 0$ is necessary to obtain decreasing absolute risk aversion. Finally, it is clear that we can find easily sufficient conditions to obtain that $V(z_1,x_1)$ is convex in x_1 which means that risk averse individuals may increase their investment in the risky asset when the latter is more risky!

Turning now to conditions on the definitions of mean preserving spread that yield intuitive results for all strictly risk averse individuals (for all strictly concave utility functions), the more general definition for linear payoffs is that of Dionne, Eeckhoudt and Gollier (1993) : "relativity weak increase in risk". However, since their definition is not strong enough for the comparative statics of problems with two random variables we will discuss two more general definitions that will be useful in the next section.

Meyer and Ormiston (1983, 1985) have proposed the definition of "a strong increase in risk" which is a particular case of a mean preserving spread. In a mean preserving spread as defined by Rothschild and Stiglitz (1970), some mass in the center of the probability density function is moved towards the tails, but *not necessarily* outside the original support of the distribution. In a strong increase in risk, the mass taken from the center of the probability density function must be transferred towards the tails, but outside the support of the original distribution. In this sense, a strong increase in risk is a particular case of the mean preserving spread defined by Rothschild and Stiglitz (1970).

<u>Strong increase in risk</u> : Suppose that the support is $[x_b,x_c]$ under distribution $F(x_1)$ and is (x_a,x_d) under distribution $G(x_1)$ with $x_a \le x_b \le x_c \le x_d$. $G(x_1)$ is a strong increase (SIR) in risk with respect to $F(x_1)$ if and only if :

a) $$\int_{x_b}^{x_c} F(x_1)dx_1 = \int_{x_a}^{x_d} G(x_1)dx_1;$$

b) $$\int_{x_a}^{x_1} \left(G(x_1)-F(x_1)\right)dx_1 \ge 0 \qquad \forall x \in [x_a, x_d];$$

c) $G(x_1) - F(x_1)$ is non-increasing in (x_b,x_c), where support of $F(x_1)$ is contained in $[x_b,x_c]$ and support of $G(x_1)$ is contained in $[x_a,x_d]$.

Condition c) illustrates the additional requirement to define the more specific increase in risk. This requirement implies that the weight taken out from the initial distribution is transferred outside the initial support or exactly at its boundaries.

Using that definition we can prove the following proposition:

Proposition 2 : Suppose that z_{IF}* and z_{IG}* maximize $EU(W(z_1))$ under $F(x_1)$ and $G(x_1)$ respectively. Then, sufficient conditions for $z_{IG}^* \leq z_{IF}^*$ for all strictly risk-averse decisions makers are :

a) $G(x_1)$ represents a strong increase in risk in relation to $F(x_1)$;

b) $W(z_1)$ is a linear payoff.

Proof: See Appendix

This proposition cannot be found in the literature. Thus, it represents a contribution, although a minor one. The proof follows closely the general outline of a proof presented by Alarie, Dionne, Eeckhoudt (1992) for linear payoffs. We should mention here that Meyer and Ormiston (1985) did not consider linear payoffs and were not able to sign the effect of an increase in risk on the standard portfolio problem (see Dionne, Eeckhoudt and Gollier (1993) for more details).

Another definition that will be useful for our analysis is that of Simple Increases in Risk proposed by Dionne and Gollier (1992). This definition is also a particular case of a Rothschild-Stiglitz mean preserving spread and is not directly comparable with a Strong Increase in Risk.

Simple Increases in Risk : $G(x_1)$ is a Simple Increase in Risk (sIR) across x_0 of $F(x_1)$ if and only if:

a) it preserves the mean;

b) $G(x_1)$ is larger then $F(x_1)$ whenever x_1 is less than x_0 and $G(x_1)$ is less than $F(x_1)$ whenever x_1 is larger than x_0 : $(G(x_1)-F(x_1))(x_1-x_0) \leq 0 \quad \forall x_1 \in [\underline{x}_1, \overline{x}_1]$.

The above condition implies that a simple increase in risk across x_0 satisfies first degree stochastic dominance on both sides of x_0. The definition of a simple increase in risk also implies that the cumulative distributions $G(x_1)$ and $F(x_1)$ cross only once at x_0. This property is known as the *single crossing property*. A similar result to that of Proposition 2 can be obtained by applying a sIR to the initial distribution $F(x_1)$. The proposition is not reported here since it suffices to replace "strong increase in risk" by "simple increase in risk" in condition a). The proof however is very different. It will be presented when necessary in the next section.

2.2 Two risky assets and one decision variable

Let us now introduce a second random variable x_2 with z_2 being the initial monetary investment in z_2. To simplify the notation, we assume that $z_0 \equiv 0$ and $z_1 + z_2 = 1$ which implies that the end of period wealth is equal to $W(z_1) = z_1x_1 + (1-z_1)x_2$.

With this specification of the two risky asset problem found in the literature, the problem in (3) becomes :

$$\max_{z_1} \int_{\underline{x}_1}^{\overline{x}_1} \int_{\underline{x}_2}^{\overline{x}_2} U(z_1x_1+(1-z_1)x_2)dH(x_1,x_2) \tag{14}$$

where $H(x_1,x_2)$ is the initial joint distribution of $\tilde{x}_1$ and $\tilde{x}_2$. We assume again that there exists an interior solution equal to z_1^* (for more details about the conditions related to the existence of an interior solution, see Ross (1981)). We now address the problem of determining the effect of an increase in risk in the distribution of x_1 on z_1^*.

Hadar and Seo (1990) first extended the literature by showing that the Rothschild and Stiglitz (1971) condition remains necessary and sufficient to obtain that a positive z_1^* will decrease following a mean preserving spread on x_1. However, they considered only the case where x_1 and x_2 are independent random variables which implies that $H(x_1,x_2) = F_1(x_1)\, G(x_2)$ where $F_1(\tilde{x}_1)$ and $G(\tilde{x}_2)$ are the initial marginal distribution of x_1 and x_2 respectively. $H(x_1,x_2)$ is the initial joint distribution of $\tilde{x}_1$ and $\tilde{x}_2$. To conduct the analysis, $F_1(x_1)$ is replaced by $F_2(x_1)$. $F_2(x_1)$ is the resulting distribution when $F_1(x_1)$ undergoes a mean preserving increase in risk.

Extending the above results to dependent random variables is not free of additional assumptions. The first two contributions were those of J. Meyer (1992) and Dionne and Gollier (1992). Since they were discussed in detail in the previous essay, we will only summarize the results that will be useful for our extension.

Meyer (1992) was the first to emphasize that the stochastic dependence between the random variables must not be altered in order to obtain meaningful comparative statics analysis. Meyer and Ormiston (1994) extended the analysis of Hadar and Seo (1990) by defining $H(x_1,x_2) = F_1(x_1|x_2)\, G(x_2)$ and by supposing that the conditional distribution of x_1 is altered in the following way : "as x_1 is changed, the marginal cumulative distribution of x_2 is assumed to be unchanged" (p. 606, with appropriate modifications of notation) which is the definition proposed by Dionne and Gollier (1992). However, Dionne and Gollier (1992) did not consider restrictions on the utility function of the risk averse decision makers but considered increases in risk that permitted intuitive comparative statics results for all risk averse individuals. An example where the conditions imposed on the change of x_1 are met is the following : $x_1^1 = x_1^0 + w$ where w is a random variable which satisfies $E(w|x_1^0,x_2) = 0$ (Meyer and Ormiston, 1994). The example indicates that it is possible to extend directly the results of Hadar and Seo (1990) and consequently those of Rothschild and Stiglitz (1971) even when the random variables are not independent. A sufficient condition is that the noise (w) added to the initial random variable x_1^0 be independent of both x_1^0 and x_2 whatever the dependence between x_1^0 and x_2. More precisely, we must have that $E(w|\, x_1^0, x_2) = E(w) = 0$. Therefore we can summarize the preceding discussion as follows :

<u>Proposition 3</u> : Assume that $U'''(\cdot) \geq 0$ and that $F_2(x_1|x_2)$ is a mean preserving spread of $F_1(x_1|x_2)$. Assume also that $G(x_2)$ is not changed. Then $z_{12}^* \leq z_{11}^*$ if and only if $U'(z)\cdot z$ is concave in z where $z \equiv z_1x_1+(1-z_1)x_2$ and where z_{12}^* and z_{11}^* maximizes $EU(W_1(z))$ over z under $F_2(\cdot)$ and $F_1(\cdot)$ respectively.

<u>Proof</u> : See Meyer and Ormiston (1994).

It is clear that the above definition of mean preserving spread is automatically met when x_1 and x_2 are independent random variables. When compared to propositions 1 and 2, proposition 3 contains an additional assumption on the preferences, that is $U'''(.) \geq 0$. Proposition 3 deals with two risky assets whereas proposition 1 and 2 deal with one risky asset. Therefore, the need of an additional assumption is not surprising.

Dionne and Gollier (1992) show that if a conditional strong increases in risk or if a conditional simple increase in risk (or if a combination of both) is imposed to x_1 no conditions of U are necessary to obtain intuitive results. A conditional strong increase in risk is defined as follows :

Conditional Strong Increase in Risk : $F_2(x_1|x_2)$ is more risky than $F_1(x_1|x_2)$ if and only if

a) $\int_{x_a}^{x_d} x_1 dF_2(x_1|x_2) = \int_{x_b}^{x_c} x_1 dF_1(x_1|x_2)$ where $[x_a,x_d]$ is the support of x_1 after the strong increase in risk and $[x_b,x_c]$ is the support of x_1 before.

b) the distribution of x_1, conditional upon the realization of x_2, undergoes a strong increase in risk (Meyer and Ormiston, 1985) for some $x_2 \in [\underline{x}_2, \overline{x}_2]$.

Conditional Simple Increase in Risk : Let the interval $[\underline{x}_1, \overline{x}_1]$ be the support of x_1. $F_2(x_1|x_2)$ is more risky than $F_1(x_1|x_2)$ if and only if

a) both distributions have the same conditional mean for all x_2;

b) the distribution of x_1 conditional on the realisation of x_2, under gives a simple increase in risk for same $x_2 \in [\underline{x}_2, \overline{x}_2]$.

As shown in the second essay of this thesis, these definitions do not change the sign nor the numerical value of the covariance between x_1 and x_2, provided that the marginal cumulative density function of x_2 remains unchanged. This assumption is known as the *ceteris paribus assumption* (Meyer (1992)). A detailed example is proposed in the next section. Using the above definition, Dionne and Gollier showed that.

Proposition 4 : If the distribution of asset x_1 conditional to x_2 undergoes a strong increase in risk for all possible realizations of x_2 or a simple increase in risk for all possible realization of x or a combination of both definitions for all possible realisations of x_2, then all risk averse decision makers weakly reduce their position on x_1, that is $z_{12}^* < z_{11}^*$ if z_{12}^* is the optimal level of z_1 under the more risky distribution and z_{11}^* is the optimal solution under the less risky distribution.

It is important to note here that the same results cannot be obtained with the less restrictive definitions of Relatively Strong Increase in Risk (Black and Bulkley, 1989) and Relatively Weak Increase in Risk (Dionne, Eeckhoudt and Gollier, 1993) since the two definitions are too general (see Dionne and Gollier (1995) for a counter example).

Up to now we have limited the analysis to problems with one decision variable which are quite restrictive since they limit the portfolio to two assets. Extension of the above results to two decisions variables is difficult since it introduces more than one first order condition. A first attempt to such an extension is proposed in the next section (see however the articles of Dionne and Eeckhoudt (1984) and Eeckhoudt, Meyer and Ormiston (1995) for problems with two decision variables and one risk parameter).

Section 3: A Portfolio with Two Random Variables and Two Decision variables

3.1 The maximization problem

The basic model of the preceding section now becomes the following. Initial wealth is equal to

$$W_0 = z_0 + z_1 + z_2$$

while end of period random wealth is

$$W(z_1,z_2) = W_0x_0 + z_1(x_1-x_0) + z_2(x_2-x_0).$$

Since W_0x_0 is a constant, it can be dropped without any loss of generality in order to simplify the notation. z_1^* and z_2^* solve the following maximization problem :

$$\underset{z_1, z_2}{\text{Max}}\ E\big(U\big(W(z_1, z_2)\big)\big) = \underset{z_1, z_2}{\text{Max}} \int_{\underline{x}_2}^{\overline{x}_2} \int_{\underline{x}_1}^{\overline{x}_1} U\big(z_1(x_1 - x_0) + z_2(x_2 - x_0)\big)\, dH(x_1, x_2) \tag{15}$$

where $[\underline{x}_1, \overline{x}_1]$ and $[\underline{x}_2, \overline{x}_2]$ are respectively the support of x_1 and x_2 and $H(x_1, x_2)$ is the joint distribution of the two random variables. The first order conditions of the above problem are :

$$\int_{\underline{x}_2}^{\overline{x}_2} \int_{\underline{x}_1}^{\overline{x}_1} U'\big(z_1(x_1 - x_0) + z_2(x_2 - x_0)\big)(x_1 - x_0)\, dH(x_1, x_2) = 0, \tag{16}$$

$$\int_{\underline{x}_2}^{\overline{x}_2} \int_{\underline{x}_1}^{\overline{x}_1} U'\big(z_1(x_1 - x_0) + z_2(x_2 - x_0)\big)(x_2 - x_0)\, dH(x_1, x_2) = 0. \tag{17}$$

The above conditions are necessary and sufficient under strict risk aversion or when U is strictly concave.

Let us define the risk premium $m_i = E(x_i - x_0)$, $i = 1,2$. The risk premium is the excess return of asset i above the return of the safe asset. To simplify both the presentation and the interpretation of the results we will assume, without loss of generality, that $E(x_2 - x_0) = 0$ and that $E(x_1 - x_0) > 0$. Even if $E(x_2 - x_0) = 0$, the asset z_2 can still be detained for hedging purposes.

By an application of the definition of the covariance, the two first order conditions can be written as :

$$E\big[U'\big(z_1 (x_1 - x_0) + z_2(x_2 - x_0)\big)\big]\, E(x_1 - x_0) + \text{cov}\big(U'(W), x_1 - x_0\big) = 0 \tag{18}$$

$$\text{cov}(U'(W), x_2 - x_0) = 0. \tag{19}$$

Let us first consider the case where $\text{cov}(x_1, x_2) > 0$. To satisfy (18), the $\text{cov}(U'(W), x_1 - x_0)$ must be negative since $m_1 > 0$. (19) indicates clearly that z_1 and z_2 must have opposite signs. Therefore the solution $z_1^* > 0$ and $z_2^* < 0$ does not preclude (18) and (19) while $z_1^* < 0$ $z_2^* > 0$ is in contradiction with (18). When $m_1 < 0$, the same analysis indicates that $z_1^* < 0$ and $z_2^* > 0$ when the covariance is positive.

Now consider the case where $cov(x_1,x_2) = 0$. (19) indicates clearly that $z_2^* = 0$ and (18) shows that $z_1^* > 0$.

Finally when $cov(x_1,x_2) < 0$, (19) now indicates that z_1^* and z_2^* must have identical signs that strictly differ from zero. From (18), when $m_1 > 0$, $cov(U'(w),x_1-x_0)$ must be negative. Then we obtain that z_1^* and z_2^* are both strictly positive. When $m_1 < 0$, the same analysis indicates that both values must be strictly negative.

In order to find explicit solutions of the above system of equations three examples are presented :

1) U is a quadratic utility function, which means that $U'''(W) = 0$. The two first order conditions become

$$z_1(\sigma_{11}+m_1^2) + z_2\sigma_{12} = m_1 \tag{20}$$

$$z_1\sigma_{12} + z_2\sigma_{22} = 0 \tag{21}$$

where $m_1 \equiv E(x_1-x_0) > 0$ by assumption (see Mossin (1973), for details).

Solving the system of two equations yields the following explicit values for z_1^* and z_2^* :

$$z_1^* = \frac{m_1\sigma_{22}}{\left(\sigma_{11}\sigma_{22} - \sigma_{12}^2 + m_1^2\sigma_{22}\right)} \tag{22}$$

$$z_2^* = \frac{-m_1\sigma_{12}}{\left(\sigma_{11}\sigma_{22} - \sigma_{23}^2 + m_1^2\sigma 22\right)} \tag{23}$$

where $\sigma_{11}\sigma_{22} - \sigma_{12}^2 \equiv D > 0$ since it is the determinant of the variance-covariance matrix. We verify that $z_1^* > 0$ and $z_2^* < 0$ when $\sigma_{12} > 0$ and $z_1^* > 0$ and $z_2^* > 0$ when $\sigma_{12} < 0$. Many other cases of interest are possible. For example, when $m_1 < 0$, $z_1^* < 0$ and $z_2^* > 0$ when

$\sigma_{12} > 0$ and $z_1^* < 0$, $z_2^* < 0$ when $\sigma_{12} < 0$. It is important to repeat here that since the utility function is quadratic, only the first two moments of the distribution do matter. However, as pointed out by Meyer (1987), using the quadratic utility function does not necessary mean that we cover all the possibilities for mean-variance analysis. Our second example is the mean-standard-deviation utility case.

2) We now suppose that the welfare of the risk averse agent is represented by $V(\mu,\sigma)$ where μ is the mean of the portfolio and σ is its standard deviation. To be more precise

$$\mu \equiv E(W) = m_1 z_1 + m_2 z_2 \tag{24}$$

$$\sigma \equiv \left(z_1^2 \sigma_{11} + z_2^2 \sigma_{22} + 2\sigma_{12} z_1 z_2\right)^{1/2} \tag{25}$$

We use the standard deviation in accordance to Meyer's comment that mean-variance analysis correspond to a less broader class of utility function having the appropriate convexity properties. (We shall return to this comment, however.) Maximizing $V(\mu,\sigma)$ over z_1 and z_2 yields as first order conditions (when $m_2 = 0$) :

$$V_1 m_1 + \frac{V_2}{\sqrt{\sigma^2}} \left(z_1 \sigma_{11} + z_2 \sigma_{12}\right) = 0 \tag{26}$$

$$\frac{V_2}{\sqrt{\sigma^2}} \left(z_1 \sigma_{12} + z_2 \sigma_{22}\right) = 0 \tag{27}$$

where V_1 and V_2 are for $dV/d\mu$ and $dV/d\sigma$ respectively, which implies that

$$z_1 = \frac{-V_1 \sigma}{V_2} \left(\frac{m_1 \sigma_{22}}{\sigma_{22} \sigma_{11} - \sigma_{12}^2} \right) \tag{28}$$

$$z_2 = \frac{V_1 \sigma}{V_2} \left(\frac{m_1 \sigma_{22}}{\sigma_{22} \sigma_{11} - \sigma_{12}^2} \right) \tag{29}$$

We observe that the results are similar to those obtained in the preceding case, but z_1 and z_2 are not the explicit solutions. Indeed, $V_1(.)$ and $V_2(.)$ contain z_1 and z_2 as their arguments. Thus, equations **(28)** and **(29)** are not an explicit solution of z_1 and z_2. Moreover, we must take into account that the inverse of the marginal rate of substitution between μ and σ ($-V_1/V_2$) is a function of both μ and σ and σ is itself function of σ_{11}, σ_{22} and σ_{12}. Finally, when $V(\mu,\sigma) = \mu - a\sigma^2$, z_1^* and z_2^* can be derived explicitly. It can be shown that this case corresponds to $EU(W) = -e^{-\gamma w}$ or to constant absolute risk aversion (Epstein, 1985). However, the normality of returns must be assumed.

3) We now assume that x_1 and x_2 are random variables that are bivariate normally distributed, which implies that W is normally distributed. Therefore, applying the Stein's lemma, $\text{cov}(U'(W),x_2-x_0) \equiv EU''(W)\,\text{cov}(z_1(x_1-x_0) + z_2(x_2-x_0), (x_2-x_0)) = 0$ which is equivalent to

$$EU''(W)(z_1\sigma_{12}+z_2\sigma_{22}) = 0 \tag{30}$$

implying that $z_2 = -z_1\frac{\sigma_{12}}{\sigma_{22}}$ from the first order condition for z_2.

Again, by applying the Stein's lemma, the first order condition for z_1 can be rewritten as

$$\begin{aligned} EU'(W)\,E(x_1-x_0) &= -\text{cov}(U'(W), x_1-x_0) \\ &= -E(U''(W))\,\text{cov}(W, x_1-x_0) \\ &= -E(U''(W))(z_1\sigma_{11}+z_2\sigma_{12}). \end{aligned} \tag{31}$$

By substituting the value of z_2 in (31) we obtain

$$EU'(W)\,E(x_1-x_0) = -E(U''(W))\left(z_1\sigma_{11}-z_1\frac{\sigma_{12}^2}{\sigma_{22}}\right) \tag{32}$$

or

$$EU'(W)m_1 = \frac{-E(U''(W))}{\sigma_{22}}\left(z_1\sigma_{11}\sigma_{22}-z_1\sigma_{12}^2\right) \tag{33}$$

which implies that

$$z_1 = -\frac{EU'(W)}{EU''(W)} \frac{m_1\sigma_{22}}{\sigma_{11}\sigma_{22} - \sigma_{12}^2} \tag{34}$$

while

$$z_2 = +\frac{EU'(W)}{EU''(W)} \frac{m_1\sigma_{12}}{\sigma_{11}\sigma_{22} - \sigma_{12}^2} \tag{35}$$

where

$$-\frac{EU'(W)}{EU''(W)} \equiv \frac{1}{A(W)} \tag{36}$$

is the inverse of the coefficient of global absolute risk aversion (Huang and Letzenberger, 1988). When the utility function $U(W) = -e^{-\gamma w}$, since W is normally distributed, it can be shown that $\frac{1}{A(W)} = -\frac{EU'(W)}{EU''(W)} = \frac{1}{\gamma}$, a constant that is the inverse of the Arrow-Pratt measure of risk aversion. In that case, the values of z_1* and z_2* can be derived explicitly :

$$z_1^* = \frac{1}{\gamma} \frac{m_1\sigma_{22}}{\sigma_{11}\sigma_{22} - \sigma_{12}^2} \tag{34}$$

$$z_2^* = -\frac{1}{\gamma} \frac{m_1\sigma_{12}}{\sigma_{11}\sigma_{22} - \sigma_{12}^2} \tag{35}$$

3.2 The comparative static analysis

Let us consider the following comparative static problem : how a mean preserving spread of z_1 affect the composition of the optimal portfolio? This question is very difficult since it implies that we must consider simultaneously the effect of the mean preserving spread on the two decision variables. Even in the case of two independent random variables, Hadar and Seo (1990) indicate that they were not able to solve this comparative static problem. Meyer and Ormiston (1994) state : "Extension of these comparative results to portfolios with more then two assets is difficult. This is because more than one decision variable and first order condition must be analysed. Hadar and Seo made limited progress in this area". (p. 611)

Suppose that we use the following notation. An increase in risk is designed by a partial derivative of the joint distribution function with respect to a parameter r, for risk. The parameter r represents the level of risk. For instance, we say that $F(x_1|x_2,r^1)$ represents an increase in risk with respect to $F(x_1|x_2,r^0)$ if $r^1 > r^0$. $H(x_1,x_2|r)$ is the joint cumulative distribution of x_1 and x_2 for a given risk r. Now in order to take into account of the ceteris paribus assumption we will define $H(x_1,x_2|r) \equiv F(x_1|x_2,r)G(x_2)$ with $d^2H(x_1,x_2|r) = d^2F(x_1,x_2|r)dG(x_2)$.

Differentiating the first order conditions with respect to z_1,z_2 and r yields :

$$\left\{\int_{\underline{x_1}}^{\overline{x_1}}\int_{\underline{x_2}}^{\overline{x_2}} U''(x_1-x_0)^2 dF(x_1|x_2,r)dG(x_2)\right\} dz_1 + \left\{\int_{\underline{x_1}}^{\overline{x_1}}\int_{\underline{x_2}}^{\overline{x_2}} U''(x_1-x_0)(x_2-x_0)dF(x_1|x_2,r)dG(x_2)\right\} dz_2$$
$$+ \int_{\underline{x_1}}^{\overline{x_1}}\int_{\underline{x_2}}^{\overline{x_2}} U'(x_1-x_0) F''_{x_1,r}\, dx_1 dr\, dG(x_2) = 0 \tag{37}$$

$$\left\{\int_{\underline{x_1}}^{\overline{x_1}}\int_{\underline{x_2}}^{\overline{x_2}} U''(x_1-x_0)(x_2-x_0)dF(x_1|x_2,r)dG(x_2)\right\}dz_1 + \left\{\int_{\underline{x_1}}^{\overline{x_1}}\int_{\underline{x_2}}^{\overline{x_2}} U''(x_2-x_0)dF(x_1|x_2,r)dG(x_2)\right\}dz_2$$
$$+ \int_{\underline{x_1}}^{\overline{x_1}}\int_{\underline{x_2}}^{\overline{x_2}} U'(x_2-x_0) F''_{x_1,r}\, dx_1 dr\, dG(x_2) = 0. \tag{38}$$

Rearranging the two above relations in matrix form

$$\begin{bmatrix} \int_{\underline{x}_1}^{\bar{x}_1}\int_{\underline{x}_2}^{\bar{x}_2} U''(x_1-x_0)^2 dF(z_1|x_2,r)dG(x_2) & \int_{\underline{x}_1}^{\bar{x}_1}\int_{\underline{x}_2}^{\bar{x}_2} U''(x_1-x_0)(x_2-x_0)dF(x_1|x_2,r)dG(x_2) \\ \int_{\underline{x}_1}^{\bar{x}_2}\int_{\underline{x}_2}^{\bar{x}_2} U''(x_1-x_0)(x_2-x_0)dF(x_1|x_2,r) & \int_{\underline{x}_1}^{\bar{x}_1}\int_{\underline{x}_2}^{\bar{x}_2} U''(x_2-x_0)^2 dF(x_1|x_2,r)dG(x_2) \end{bmatrix}_{2x2} \begin{bmatrix} dz_1^* \\ dz_2^* \end{bmatrix}_{2x1} \tag{39}$$

$$= \begin{bmatrix} -\int_{\underline{x}_1}^{\bar{x}_1}\int_{\underline{x}_2}^{\bar{x}_2} U'(x_1-x_0)\, F''_{x_1,r}\, dx_1 dG(x_2)dr \\ -\int_{\underline{x}_1}^{\bar{x}_1}\int_{\underline{x}_2}^{\bar{x}_2} U'(x_2-x_0)\, F''_{x_1,r}\, dx_1 dG(x_2)dr \end{bmatrix}_{2x1}$$

and applying the Cramer's rule we obtain :

$$|H|\frac{dz_1^*}{dr} = \left[\int_{\underline{x}_1}^{\bar{x}_1}\int_{\underline{x}_2}^{\bar{x}_2} U'(x_2-x_0)F''_{x_1,r}dx_1\,dG(x_2)\cdot\int_{\underline{x}_1}^{\bar{x}_1}\int_{\underline{x}_2}^{\bar{x}_2} U''(x_1-x_0)(x_2-x_0)dF(x_1|x_2,r)dG(x_2)\right] - \left[\int_{\underline{x}_1}^{\bar{x}_1}\int_{\underline{x}_2}^{\bar{x}_2} U'(x_1-x_0)F''_{x_1,r}dx_1\,dG(x_2)\cdot\int_{\underline{x}_1}^{\bar{x}_1}\int_{\underline{x}_2}^{\bar{x}_2} U''(x_2-x_0)^2 dF(x_1|x_2,r)dG(x_2)\right] \tag{40}$$

$$|H|\frac{dz_2^*}{dr} = \left[\int_{\underline{x}_1}^{\bar{x}_1}\int_{\underline{x}_2}^{\bar{x}_2} U'(x_1-x_0)(x_2-x_0)dF(x_1|x_2,r)dG(x_2)\cdot\int_{\underline{x}_1}^{\bar{x}_1}\int_{\underline{x}_2}^{\bar{x}_2} U'(x_1-x_0)F''_{x_1,r}dx_1\,dG(x_2)\right] - \left[\int_{\underline{x}_1}^{\bar{x}_1}\int_{\underline{x}_2}^{\bar{x}_2} U''(x_1-x_0)^2 dF(x_1|x_2,r)dG(x_2)\cdot\int_{\underline{x}_1}^{\bar{x}_1}\int_{\underline{x}_2}^{\bar{x}_2} U'(x_2-x_0)F''_{x_1,r}dG(x_2)\right] \tag{41}$$

where the determinant of the Hessian Matrix $\underset{2x2}{|H|} \geq 0$ for a maximum and where $F^{''}_{x_1,r}$ is written for an increase in risk. We have that $F^{''}_{x_1,r} = d[F(x_1|x_2,r^1) - F(x_1|x_2,r^0)]$. The parameter r is the level of risk associated with a distribution. We say that r^1 represents an increase in risk with respect to r^0. Since both conditions are symmetric, we will focus the attention on (40). We will first analyse in detail each of the four terms. To simplify the notation, let us rewrite (40) as

$$|H| \frac{dz_1^*}{dr} = \Delta_1 \cdot \Delta_2 - \Delta_3 \cdot \Delta_4 \tag{42}$$

where

$$\Delta_1 \equiv \int_{\underline{x}_1}^{\overline{x}_1} \int_{\underline{x}_2}^{\overline{x}_2} U'(x_2 - x_0) F^{''}_{x_1,r}(x_1 | x_2, r) dx_1 dG(x_2) \tag{43}$$

$$\Delta_2 \equiv \int_{\underline{x}_1}^{\overline{x}_1} \int_{\underline{x}_2}^{\overline{x}_2} U''(x_1 - x_0)(x_2 - x_0) dF(x_1 | x_2, r) dG(x_2) \tag{44}$$

$$\Delta_3 \equiv \int_{\underline{x}_1}^{\overline{x}_1} \int_{\underline{x}_2}^{\overline{x}_2} U'(x_1 - x_0) F^{''}_{x_1,r}(x_1 | x_2, r) dx_1 dG(x_2) \tag{45}$$

$$\Delta_4 \equiv \int_{\underline{x}_1}^{\overline{x}_1} \int_{\underline{x}_2}^{\overline{x}_2} U''(x_2 - x_0)^2 dF(x_1 | x_2, r) dG(x_2) \tag{46}$$

Δ_3 is named the Direct Increase in Risk Effect. It can be rewritten as

$$\Delta_3 = \int_{\underline{x}_1}^{\overline{x}_1} \int_{\underline{x}_2}^{\overline{x}_2} U'(W)(x_1 - x_0) dS(x_1 | x_2) dG(x_2) \tag{47}$$

where $S(x_1|x_2) = F(x_1|x_2,r_2) - F(x_1|x_2,r_1)$ where r_2 is more risky than r_1 by definition.

This term is similar to those analysed in models with two random parameters and one decision variable. It can be shown, by using the technical methods reviewed in the previous section, that Δ_3 is negative when $z_1^* > 0$ if and only if $U'(W)\,W$ is concave in W (Meyer and Ormiston (1994)) when

$dS(x_1|x_2)$ represents a Rothschild-Stiglitz increase in risk. This means that the sufficient conditions on U(W) discussed in the previous section are sufficient to get intuitive comparative statics results. However, this result can be obtained only if $x_1 \in [0, \bar{x}_1]$ as discussed in Meyer and Ormiston (1994) and Hadar and Seo (1990). This last restriction is very restrictive for real portfolio analysis because it implies that the return of the risky asset cannot be negative. Another possibility, is to follow Dionne and Gollier (1992) and show that a conditional Strong increase in risk or a conditional Simple increase in risk on x_1 yields an effect opposite to the sign of z_1^* for all risk averse individuals that is : Sign (Direct effect) = - Sign (z_1^*) which is similar to the results of Dionne, Eeckhoudt, Gollier (1993) with a one decision variable - one random variable model but with a more general definition of increases in risk. Therefore, this first term Δ_3 does not introduce any new difficulty in the analysis for the moment.

Δ_4 is from the second order condition and is always strictly negative under strictly risk aversion. Consequently the sign of product $-\Delta_3\Delta_4$ is equal to that of sign (Δ_3) as in models with one decision variable.

We now analyse the two other terms by starting with Δ_2, the Interaction Effect. This effects links z_1^* and z_2^* via the interaction between the two random parameters. This terms is very difficult to sign because it links three random variables. Moreover, an increase in the product of $(x_1-x_0)(x_2-x_0)$ does not mean a particular variation of $W \equiv z_1^*(x_1-x_0) + z_2^*(x_2-x_0)$ and therefore does not mean a particular variation of $U''(W)$. For the moment we are able to sign this term under two assumptions : 1) U is quadratic; 2) x_1 and x_2 are two random variables which are distributed according to a bivariate normal distribution. In both cases, the third moment of the distribution has no weight.

When the utility function is quadratic, $U'''(W) = 0$ which implies that $U''(W)$ is constant. Therefore

$$\int_{\underline{x}_1}^{\bar{x}_1}\int_{\underline{x}_2}^{\bar{x}_2} U''(x_1-x_0)(x_2-x_0)dF(x_1|x_2,r)dG(x_2) = U''(W)\int_{\underline{x}_1}^{\bar{x}_1}\int_{\underline{x}_2}^{\bar{x}_2}(x_1-x_0)(x_2-x_0)dF(x_1|x_2,r)dG(x_2) \quad (48)$$

Using the definition of the covariance, the right hand side of the above equation can be written as

$$U''(W)m_1m_2 + U''(W)\ \mathrm{cov}(x_1\text{-}x_0,x_2\text{-}x_0)$$

which, under the assumption that $m_2 = 0$, is equal to $U''(W)\ \mathrm{cov}(x_2,x_1)$ since x_0 is a constant. Consequently, with a quadratic utility function, the Interaction Effect term has a sign that is equal to (- sign $\mathrm{cov}(x_1,x_2)$).

We may also assume that x_1 and x_2 follow a bivariate normal distribution and obtain the same result. To obtain the result we will use as a starting point the first order condition for z_2*. But, by symmetry the same result can be obtained from the other first order condition. From (19) this first order condition can be rewritten by using the Stein Lemma as :

$$E(U''(W))\left(z_1\sigma_{12}+ z_2\sigma_{22}\right) \tag{49}$$

which is (30). Differentiating this expression with respect to z_1 yields

$$E(U''(W))\sigma_{12} + E\left[(U'''(W))(x_1-x_0)\right]\left[z_1\sigma_{12}+z_2\sigma_{22}\right] \tag{50}$$

which is reduced to $E(U''(w))\sigma_{12})$ since $[z_1\sigma_{12}+z_2\sigma_{22}] = 0$ from the first order condition (30).

We now analyse Δ_1 which we name Pseudo Increase in Risk Effect. This term measures the effect of an increase in risk of random variable x_1 on z_1, via the fact that z_1* is determined simultaneously with z_2*. In other words, when the risk of x_1 increases, it changes the distribution of x_1 that also affect z_2* which in turn affect z_1* (since both are determined simultaneously).

Let us rewrite Δ_1 :

$$\Delta_1 = \int_{\underline{x}_2}^{\overline{x}_2}\int_{\underline{x}_1}^{\overline{x}_1} U'(x_2-x_0)F''_{x_1,r}\,dx_1\,dG(x_2). \tag{51}$$

By letting $\theta(x_2) \equiv \int_{\underline{x}_1}^{\overline{x}_1} U' F''_{x_1,r} dx_1$, (51) becomes :

$$\int_{\underline{x}_2}^{\overline{x}_2} \theta(x_2)(x_2 - x_0) dG(x_2). \tag{52}$$

By definition

$$\theta(x_2) = \int_{\underline{x}_1}^{\overline{x}_1} U' F''_{x_1,r} dx_1 = \int_{\underline{x}_1}^{\overline{x}_1} U' dS(x_1 | x_2).$$

Integrating by parts and applying the Leibnitz rule

$$\theta(x_2) = U' S(x_1 | x_2) \Big|_{\underline{x}_1}^{\overline{x}_1} - \int_{\underline{x}_1}^{\overline{x}_1} U'' z_1 S(x_1 | x_2) dx_1$$

$$\theta(x_2) = -\int_{\underline{x}_1}^{\overline{x}_1} U'' z_1 \left(d \int_{\underline{x}_1}^{x_1} S(u | x_2) du \right)$$

since

$$S(\underline{x}_1 | x_2) = S(\overline{x}_1 | x_2) = 0.$$

Integrating by parts again

$$\theta(x_2) = -\left\{ U'' z_1 \int_{\underline{x}_1}^{x_1} S(u | x_2) du \Big|_{\underline{x}_1}^{\overline{x}_1} - \int_{\underline{x}_1}^{\overline{x}_1} U''' z_1^2 \left(\int_{\underline{x}_1}^{x_1} S(u | x_2) du \right) dx_1 \right\}$$

or

$$\theta(x_2) = \int_{\underline{x}_1}^{\overline{x}_1} U''' z_1^2 T(x_1 | x_2) dx_1$$

where

$$T(x_1 \mid x_2) = \int_{\underline{x}_1}^{x_1} S(u \mid x_2) du > 0 \tag{53}$$

by Rothschild-Stiglitz definition.

When U(W) is quadratic, $\theta(x_2) = 0$ and the Pseudo increase in risk in nil. When $U'''(w) > 0$, it is necessary to introduce an assumption on $T_{x_2}(x_1 \mid x_2)$ to sign $\theta'(x_2)$ and to solve (52). Since for the moment we cannot give a sign to the Interaction Effect for other distributions than the normal distribution or utility functions different to the quadratic, now we analyse the Pseudo Effect under the normality assumption. (51) can be rewritten as

$$\int_{\underline{x}_1}^{\bar{x}_1} \int_{\underline{x}_2}^{\bar{x}_2} U'(w) (x_2 - x_0)(f_2(x_1 \mid x_2) - f_1(x_1 \mid x_2)) dx_1 dG(x_2). \tag{54}$$

By applying the Stein Lemma, (54) becomes, under the assumption that $m_2 = 0$,

$$EU''(z_1\sigma_{12} + z_2\sigma_{22})\big|_{z_1^*, z_2^*, F_2} - EU''(z_1\sigma_{12} + z_2\sigma_{22})\big|_{z_1^*, z_2^*, F_1} \tag{55}$$

Two remarks must be made about the last expression. First, EU'' is evaluated at z_1^*, z_2^* since the whole expression comes from a first order condition. By the same token, $E(U'')$ is resulting from choices made under F_1 in the first place and F_2 in the second one.

Second, we must consider $(\sigma_{12}, \sigma_{22})_{F_2}$ under F_2 and $(\sigma_{12}, \sigma_{22})_{F_1}$ under F_1 respectively. The ceteris paribus assumption ensures that the marginal CDF, $G(x_2)$, does not change after a conditional MPS on x_1. Thus σ_{22} is the same under F_1 and F_2. Furthermore, it is shown in the second essay of this thesis that this ceteris paribus assumption guarantees that σ_{12} remains unchanged (numerical value and sign) after a conditional MPS on x_1.

These two remarks allow us to conclude about the sign of Pseudo Effect. Note that in the last expression

$$\left(z_1\sigma_{12}+z_2\sigma_{22}\right)\Big|_{z_1^*,z_2^*,F_1} = 0$$

by the first order condition of z_1. We also know that

$$\left(z_1\sigma_{12}+z_2\sigma_{22}\right)\Big|_{z_1^*,z_2^*,F_2} = \left(z_1\sigma_{12}+z_2\sigma_{22}\right)\Big|_{z_1^*,z_2^*,F_1}$$

and we conclude that the Pseudo Effect is zero when x_1 and x_2 are bivariate normally distributed.

We are now in a position to summarize the analysis of the four terms. Since, up to now, it is not possible to sign generally the Interaction Effect Term, we will present four propositions corresponding to the four examples discussed in the previous section.

We first start with the case where the two random variables are bivariate normal distributions. In the Appendix it is shown that the sign of the variation of z_1^* is - sign(z_1^*) when the utility function is exponential or when we apply a conditional simple increase in risk. The sign of the variation of z_2^* is equal to -sign(z_2^*). We can summarize these results with the following proposition :

Proposition 5 : Assume that x_1 and x_2 are two dependent random variables following a bivariate normal distribution with $E(x_1-x_0) > 0$ and $E(x_2-x_0) = 0$. Therefore $z_1^* > 0$ and $z_2^* < 0$ when $\sigma_{12} > 0$ and $z_1^* > 0$ and $z_2^* > 0$ when $\sigma_{12} < 0$ under the initial distribution. Sufficient conditions to obtain that sign(dz_1^*/dr) = - sign(z_1^*) and sign(dz_2^*/dr) = - sign(z_2^*) are that U(W) is exponential in W or that a conditional Simple increase in risk is imposed to x_1.

Proof : See Appendix.

When U(W) is exponential, it is immediate to verify the result of Proposition 5 by differentiating (34′) and (35′) with respect to σ_{11} since we have explicit solutions for z_1^* and z_2^*. This means that we need restriction on U even when random parameters are normally distributed. When a simple increase in risk is imposed to x_1, since the two distributions are normal, the Pseudo Effect is nil which implies that only the direct effect does matter. We must point out that we cannot apply a Strong Increase in Risk since this definition is useful only for distribution with finite supports and we cannot apply the model of Meyer and Ormiston (1994) and Hadar and Seo (1990) since both are restricted to random variables and decision parameters strictly positive.

We now analyse the case of the quadratic utility function. Here again the analysis is easy since we have explicit values of z_1* and z_2* at the optimum. Differentiating (22) and (23) with respect to σ_{11} yields :

Proposition 6 : When U(W) is quadratic sign(dz_1*/dr) = -sign(z_1*) and sign(dz_2*/dr) = -sign(z_2*).

$$\frac{dz_2^*}{d\sigma_{11}} = \frac{-m_1\sigma_{22}^2}{\left(\sigma_{11}\sigma_{22} - \sigma_{12}^2 + m_1^2\sigma_{22}\right)^2}$$

$$\frac{dz_2^*}{d\sigma_{11}} = \frac{m_1\sigma_{21}\sigma_{22}}{\left(\sigma_{11}\sigma_{12} - \sigma_{12}^2 + m_1^2\sigma_{22}\right)^2}$$

Under the assumptions that $\sigma_{12} > 0$, $\frac{dz_1^*}{d\sigma_{11}} < 0$, $\frac{dz_2^*}{d\sigma_{11}} > 0$ which is intuitively acceptable since $z_1^* > 0$ and $z_2^* < 0$. This result can be also obtained from the general model with a quadratic utility function since under the assumption :

$$\text{sign}(\text{Interaction Effect}) = -\text{sign}(\text{cov}(x_1, x_a))$$

$$\text{sign}(\text{Pseudo Effect}) = 0$$

$$\text{sign}(\text{Direct Effect}) = -\text{sign}(z_1^*)$$

which yield $\frac{dz_1^*}{dr} < 0$ and $\frac{dz_2^*}{dr} > 0$.

We must now discuss the ceteris paribus assumption. From Meyer (1992) we know that such an assumption is necessary to isolate the effect of a mean preserving spread on the optimal decision variables. The second essay of this thesis showed that this assumption implies, in Dionne and Gollier model, that the covariance between the random variables is maintained. If we look at the Meyer and Ormiston (1994) contribution, we can easily show that their application to a general mean preserving spread also maintains fixed the covariance between the two random variables.

Here we provide an example where an increase in the variance of x_1 maintains the covariance between the two random variables. The discussion of Meyer (1992) and Meyer and Orminston (1994) provide detailed examples in which one or more hypotheses are violated, leading to a reversal of the covariance sign. To illustrate our case, suppose that the random variables x_1 and x_2 (X_1 and X_2 stand for realized values) have the following realizations in a situation with two states of the world :

Initial Situation			
x_{2i}	x_{1i}	S_1 20	S_2 40
S_1	10	0.3	0.1
S_2	30	0.2	0.4
Final Situation			
x_{2i}	x_{1i}	S_1 15	S_2 45
S_1	10	0.2666	0.1333
S_2	30	0.2333	0.3666

Each entry is a joint probability. The marginal probability $f(x_{1i})$ is the sum of each entry in the column S_i and the marginal probability $f(x_{2i})$ is the sum of each entry in the row S_i. The conditional probability is illustrated as follows:

$$f(X_{1i} = 20 | X_{2i} = 10) = \frac{f(X_{1i} = 20,\ X_{2i} = 10)}{f(X_{2i} = 10)} = \frac{0.3}{0.4} = 0.75$$

We can verify that the means of x_{1i} and x_{2i}, and their conditional means, are identical under both situations. The covariance remains unchanged with a value of 40. Furthermore, the ceteris paribus assumption is respected. Thus, only the variance of x_{1i} increases from 50 to 112.50. Finally, one can easily verify that for every value of x_{2i}, the random variable x_{1i} undergoes an increase in risk as defined by Rothschild and Stiglitz (1970).

We now study the mean variance approach. As shown by Epstein (1985), when U(W) is exponential, the corresponding $V(\mu,\sigma^2) = \mu - a\sigma^2$ which implies positive linear indifference curves in the (μ,σ^2) space. In such case, it is easily obtained that $\frac{dz_1^*}{d\sigma_{11}} = -(V_2)^2 z_1 \sigma_{22}$ which has again the opposite sign of z_1* where V_2 is the partial derivative with respect to σ^2. When U(W) is not exponential, the marginal rate of substitution between μ and σ^2 is no longer a constant along the indifference curves. It is easily verified however that the same results can be obtained when U(W) is quadratic. The detailed discussion in the next paragraphs shows indirectly why this is the case.

<u>Proposition 7</u> : In the mean-variance model sign(dz_1*/dr) = -sign(z_1*) and sign(dz_2*/dr) = -sign(z_2*) when U(W) is quadratic or exponential.

Turning now to the mean-standard derivation space, matters are much more complicated. But a pointed out by Meyer (1987), the correspondence between EU(W) and $V(\mu,\sigma)$ are more directly implementable and permit to obtain a wider range of utilities in the (μ,σ) space that have the appropriate properties of concavity.

Following Meyer (1987) we can write in a model with one random variable

$$EU(W) = \int_{\underline{x}}^{\overline{x}} U(\mu + \sigma_x)\, dF(x) \equiv V(\mu,\sigma)$$

where $x = \frac{W-\mu}{\sigma}$, $E(x) = 0$ and $\sigma^2(x) = 1$.

We can verify that

$$V_{12} = \int_{\underline{x}}^{\overline{x}} U''x\,dF(x) = E(U''x) = cov(U'',x).$$

The last inequality is true since $E(x) = 0$. When $U''' > 0$, $V_{12} > 0$ and this assumption generally yields ambiguous comparative states results. When $U''' = 0$, $V_{12} = 0$.

We now show the following result :

<u>Proposition 8</u> : When the agent utility function is $V(\mu,\sigma)$, if $V_{12} = V_{22} = 0$, then $sign(dz_1{*}/d\sigma_{11}) = -sign(z_1{*})$ if $\left(1 - \frac{z_1^2 D}{\sigma_{22}\sigma_P^2}\right) > 0$ where D is the determinant of the variance-covariance matrix and σ_p^2 is the variance of the portfolio. Since z_1D/σ_p^2 is the elasticity of σ_p with respect of z_1, the variations of $z_1{*}$ and $z_2{*}$ with respect to σ_{11} in the mean-standard deviation model are of the same sign to those of the mean-variance model when this elasticity is low enough to satisfy the condition $\left(1 - \frac{z_1^2 D}{\sigma_{22}\sigma_P^2}\right) > 0$ <u>Proof</u> : See the Appendix.

In the expected utility framework, Proposition 5 (with conditional Simple Increase in Risk) is the most general proposition and extends the existing literature. To our knowledge, this is the very first result obtained with two random variables and two decision variables. In Proposition 5 (with Simple Increase in Risk), no assumption is made on preferences and the two funds separation theorem is not used implicitly as in Hadar and Seo (1990). As for the stochastic parameters, we need the assumption of a bivariate normal distribution but we allow stochastic dependence.

Finally, despite their many differences, Propositions 5 to 8 share one common feature: they all lead to the same comparative statics results ($sign(dz_1*/dr) = -sign(z_1*)$). This suggests a relationship between the expected utility, the mean variance and the mean standard deviation framework. As pointed out earlier, Meyer (1987) has already formalized the relationship between the expected utility and the mean standard deviation framework. However, the relationship was not extended to comparative statics results with two decision variables and two random parameters.

Appendix

<u>Proof of Proposition 2</u> : The proof follows the outline of a similar proof in Alarie, Dionne and Eeckhoudt (1992). Since $U'' < 0$ one has to show that

$$\int_{x_a}^{x_d} U'(x_1 - x_0)\, d[G(x_1) - F(x_1)] = \int_{x_1}^{x_d} U'(x_1 - x_0) S(x_1)\, dx_1 < 0 \tag{A.1}$$

where (A.1) is evaluated at z_1*. To establish the proof, two cases must be considered. These cases correspond to the location of x_0 on supp F. We can have $x_b < x_0 < x_e$ or $x_e < x_0 < E(x_1)$.

Case 1 : $x_b < x_0 < x_e$

$$\int_{x_a}^{x_d} U'(x_1 - x_0) S(x)\, dx = \int_{x_a}^{x_b} U'(x_1 - x_0) S(x_1)\, dx_1 + \int_{x_b}^{x_d} U'(x_1 - x_0) S(x_1)\, dx_1 \tag{A.2}$$

From the definition of a strong increase in risk, the integrands in the RHS of (A.2) alternate in signs starting with a negative one.

Note that U' is decreasing in x_1 :

$$\frac{\partial U'}{\partial x_1} = U'' z_1 < 0. \tag{A.3}$$

Since U' is decreasing in x_1, we have the following inequality.

$$\int_{x_a}^{x_d} U'(x_1 - x_0) S(x_1)\, dx_1 < U'(x_b) \int_{x_a}^{x_b} (x_1 - x_0) s(x_1)\, dx_2 + U'(x_b) \int_{x_b}^{x_d} (x_1 - x_0) s(x_1)\, dx_1 \tag{A.4}$$

where $U'(x_b) = U'(x_0)W_0 + z_1*(x_b - x^0))$. (A.4) is also equal to

$$\int_{x_a}^{x_d} U'(x_1-x_0)s(x_1)dx_1 < U'(x_b)\int_{x_a}^{x_d}(x_1-x_0)s(x_1)dx_1 \tag{A.5}$$

The RHS of (A.5) is zero by the definition of a strong increase in risk :

$$\int_{x_a}^{x_d}(x-x_0)s(x)\,dx < \int_{x_a}^{x_d}(x_1)s(x_1)dx_1 - x_0\int_{x_1}^{x_4} s(x_1)dx_1 = 0 \tag{A.6}$$

It follows immediately that the LHS of (A.5) is negative.

Case 2 : $x_e < x_0 < E(x_1)$

$$\begin{aligned}\int_{x_a}^{x_d} U'(x_1-x_0)s(x_1)dx_1 = &\int_{x_a}^{x_e} U'(x_1-x_0)s(x_1)dx_1 + \int_{x_e}^{x_0} U'(x_1-x_0)s(x_1)dx_1 \\ &+ \int_{x_0}^{x_f} U'(x_1-x_0)s(x_1)dx_1 + \int_{x_f}^{x_f} U'(x_1-x_0)s(x_1)dx_1\end{aligned} \tag{A.7}$$

Again, from the definition of a strong increase in risk, the integrands in (A.7) alternate in signs starting with a negative one.

We already know by (A.3) that U′ is decreasing in x_1 so we have the following inequality :

$$\int_{x_a}^{x_d}(x_1-x_0)s(x_1)dx_1 < U'(x_e)\left|\int_{x_a}^{x_0}(x_1-x_0)s(x_1)dx_1\right| + U'(x_f)\left|\int_{x_0}^{x_d}(x_1-x_0)s(x_1)dx_1\right| \tag{A.8}$$

where $U'(x_1) = U'((x_0)W_0+z_i^*(x_i-x_0))$, i=e,f.

From (A.6) we already know that the sum of the two brackets in the RHS of (A.8) is zero. Since by

(A.3) $U'(x_c) > U'(x_f)$, we simply have to show that the first bracket is negative to complete the proof of Case 2.

$$\int_{x_a}^{x_0} (x_1 - x_0) s(x_1) dx_1 = (x_1 - x_0) S(x_1) \Big|_{x_a}^{x_0} - \int_{x_a}^{x_0} S(x_1) dx = -\int_{x_1}^{x_0} S(x_1) dx_1 < 0 \tag{A.9}$$

The second equality follows from $S(x_1) = 0$ and the third inequality follows from the definition of a Strong increase in risk.

<u>Proof of Proposition 5</u> : We have to show that

$$\text{Sign}\left\{\int_{\underline{x}_2}^{\overline{x}_2}\int_{\underline{x}_1}^{\overline{x}_1} U'(x_1 - x_0) F''_{x_1,r} dx_1 dG(x_1)\right\} = -\text{Sign}(z_1^*)$$

with a simple increase in risk (sIR) across x_0. We say that F_2 is a conditional sIR across x_0 of F_1 if and only if

a) $$\int_{\underline{x}_1}^{\overline{x}_1} x_1 dF_2(x_1 | x_2) = \int_{\underline{x}_1}^{\overline{x}_1} x_1 dF_1(x_1 | x_2) \qquad \forall x_2 \in [\underline{x}_2, \overline{x}_2] \tag{A.10}$$

b) $$s(x_1 | x_2)(x_1 - x_0) = (F_2(x_1 | x_2) - F_1(x_1 | x_2))(x_1 - x_0) \leq 0 \qquad \forall (x_1, x_2) \in [\underline{x}_1, \overline{x}_1] \times [\underline{x}_2, \overline{x}_2] \tag{A.11}$$

We can write

$$\int_{\underline{x}_1}^{\overline{x}_1} U'(x_1 - x_0) F''_{x_1,r} dx = \int_{\underline{x}_1}^{\overline{x}_1} U'(x_1 - x_2) s(x_1 | x_2) dx_1 . \tag{A.12}$$

Integrating by parts, we obtain :

$$\int_{\underline{x}_1}^{\overline{x}_1} U'(x_1-x_0)s(x_1|x_2)dz_1 = U'(x_1-x_0)S(x_1|x_2)\Big|_{\underline{x}_1}^{\overline{x}_1} - \int_{\underline{x}_1}^{\overline{x}_1} z_1U''(x_1-x_0)S(x_1|x_2)dx_1$$

$$- \int_{\underline{x}_1}^{\overline{x}_1} U'S(x_1|x_2)dz_1 = 0 \tag{A.13}$$

From Dionne and Gollier (1992) we know that

$$(x_1-x_0)S(x_1|x_2) = \int_{\underline{x}_1}^{\overline{x}_1} S(t|x_2)dt + t(x_1|x_2).$$

Using the above result we can rewrite the equation (A.13) as :

$$\int_{x_a}^{x_d} U'(x_1-x_0)s(x_1|x_2)dz_1 = -\int_{\underline{x}_1}^{\overline{x}_1} z_1U''T(x_1|x_2)dx_1 - \int_{\underline{x}_1}^{\overline{x}_1} z_1U''\left(\int_{\underline{x}_1}^{x_1} S(t|x_2)dt\right) dx_1$$

$$- \int_{\underline{x}_1}^{\overline{x}_1} U'S(x_1|x_2)dx_1 \tag{A.14}$$

where

$$\int_{\underline{x}_1}^{\overline{x}_1} z_1U''(\cdot)\left(\int_{\underline{x}_1}^{x_1} S(t|x_2)dt\right) dx_1 = -\int U'S(x_1|x_2)dx_1$$

by integration and by using the Leibnitz's rule.

Therefore, (A.14) becomes

$$\int_{\underline{x}_1}^{\overline{x}_1} U'(x_1 - x_0) s(x_1 | x_2) dz_1 = -z_1 \int_{x_1}^{x_1} U'' T(x_1 | x_2) dx_1 \qquad (A.15)$$

where

$$T(x_1 | x_2) = \int_{\underline{x}_1}^{x_1} (t - x_0) d(F_2(t | x_2) - F_1(t | x_2)).$$

We now proceed to show that the sign of $T(x_1 | x_2)$ is negative for a sIR across x_0. Integrating by parts, we obtain :

$$T(x_1 | x_2) = (t - x_0)(F_2(t | x_2) - F_1(t | x_2))\Big|_{\underline{x}_1}^{x_1} - \int_{\underline{x}_1}^{x_1} - F_2(t | x_2) 1 - F_1(t | x_2) dt \qquad (A.16)$$

The first term of the equation (A.16) is negative by the definition of a sIR across x_0. The second term is always positive, since a sIR across x_0 satisfies second degree stochastic dominance. Thus $T(x_1|x_2)$ is always negative $\forall (x_1 x_2) \in [\underline{x}_1, \overline{x}_1] \times [\underline{x}_2, \overline{x}_2]$. We conclude that :

$$\int_{\underline{x}_1}^{\overline{x}_1} U'(x_1 - x_0) s(x_1 | x_2) dx_1 = -z_1 \int_{\underline{x}_1}^{\overline{x}_1} U'' T(x_1 | x_2) dx_1 \qquad (A.17)$$

has the opposite sign of z_1 under risk aversion.

Thus we obtain that :

$$\int_{\underline{x}_2}^{\overline{x}_2} \int_{\underline{x}_1}^{\overline{x}_1} U'(x_1 - x_0) F''_{x_1, r} dx_1 dG(x_2) = \int_{\underline{x}_2}^{\overline{x}_2} \int_{\underline{x}_1}^{\overline{x}_1} U'(x_1 - x_0) s(x_1 | x_2) dx_1 dG(x_2) \qquad (A.18)$$

has the opposite sign of z_1 under risk aversion. <u>Q.E.D.</u>

Proof of Proposition 8 : In the (μ,σ) space, the maximization problem yields as first order conditions :

$$z_1^* : V_1 m_1 + V_2 \frac{\left(z_1\sigma_{11}+z_2\sigma_{12}\right)}{\sigma_p} = 0$$

$$z_2^* : V_2 \frac{\left(z_2\sigma_{22}+z_1\sigma_{12}\right)}{\sigma_p} = 0$$

which imply that

$$V_1 m_1 + V_2 \frac{z_1}{\sigma_{22}} \frac{\left(\sigma_{11}\sigma_{22}-\sigma_{12}^2\right)}{\sigma_p} = 0$$

or that

$$V_1 m_1 + V_2 \frac{z_1}{\sigma_{22}} \frac{D}{\sigma_p} = 0.$$

Total differentiation of F.O.C. for z_1 with respect to z_1, z_2 and σ_{11} gives :

$$\left\{V_{11}m_1^2 + V_{12}\left(\frac{D}{\sigma_p}\right)m_1 + V_{12}\frac{z_1}{\sigma_{22}}\frac{D}{\sigma_p}m_1 + V_{22}\frac{z_1}{\sigma_{22}}\left(\frac{D}{\sigma_p}\right)^2 + \frac{V_{12}}{\sigma_{22}\sigma_p} - \frac{V_2 D}{\sigma_{22}\sigma_p}\left(\frac{D}{\sigma_p}\right)^2\right\}dz_1$$

$$+ \left\{0\ dz_2\right\}$$

$$+ \left\{V_{12}\frac{z_1^2}{\sigma_p}m_1 + V_{22}\frac{z_1}{\sigma_{22}}\frac{D}{\sigma_{22}}\frac{z_1^2}{\sigma_p} + V_2\frac{z_1}{\sigma_p} - V_2\frac{z_1}{\sigma_p}\frac{z_1^2}{\sigma_{22}}\frac{D}{\sigma_p^2}\right\}d\sigma_{11} = 0$$

Total differentiation of F.O.C. for z_2 with respect to z_1, z_2 and σ_{11} gives :

$$\left\{V_2\frac{\sigma_{12}}{\sigma_p}\right\}dz_1 + \left\{V_2\frac{\sigma_{22}}{\sigma_p}\right\}dz_2 + \left\{0\,d\sigma_{11}\right\} = 0$$

We can express these equations in matrix form to obtain :

$$\begin{bmatrix} \left\{V_{11}m_1^2 + V_{12}\frac{D}{\sigma_p}m_1\left\{1+\frac{z_1}{\sigma_{22}}\right\} + V_{22}\frac{z_1}{\sigma_{22}}\left(\frac{D}{\sigma_p}\right)^2 + \frac{V_2}{\sigma_{22}}\frac{D}{\sigma_p}\left\{1-\frac{z_1 D}{\sigma_p^2}\right\}\right\} & 0 \\ V_2\frac{\sigma_{12}}{\sigma_p} & V_2\frac{\sigma_{22}}{\sigma_p} \end{bmatrix}_{2x2} \begin{bmatrix} dz_1 \\ dz_2 \end{bmatrix}_{2x1}$$

$$= \begin{bmatrix} -\left\{\frac{z_1^2}{\sigma_p}\left\{V_{12}m_1 + V_{11}\frac{z_1}{\sigma_{22}}\frac{D}{\sigma_p}\right\} + V_2\frac{z_1}{\sigma_p}\left\{1-\frac{z_1^2}{\sigma_{22}}\frac{D}{\sigma_p^2}\right\}\right\}d\sigma_{11} \\ 0 \end{bmatrix}_{2x1}$$

Applying Cramer's rule :

$$|H|\frac{dz_1^*}{d\sigma_{11}} = -\left\{\frac{z_1^2}{\sigma_p}\left\{V_{12}m_1 + V_{22}\frac{z_1}{\sigma_{22}}\frac{D}{\sigma_p}\right\} + V_2\frac{z_1}{\sigma_{22}}\left\{1-\frac{z_1^2}{\sigma_{22}}\frac{D}{\sigma_p^2}\right\}V_2\frac{\sigma_{22}}{\sigma_p}\right\}$$

where $|H| \geq 0$ for a maximum determinant of the Hessian matrix.

Collecting the terms and rearranging yields :

$$|H|\frac{dz_1^*}{d\sigma_{11}} = -\left\{\frac{z_1^2}{\sigma_p}\frac{z_1}{\sigma_p}\frac{D}{\sigma_{22}}V_2\left\{\frac{V_{22}}{V_2}-\frac{1}{\sigma_p}\right\} + \frac{z_1}{\sigma_p}\{z_1V_{12}m_1+V_2\}V_2\frac{\sigma_{22}}{\sigma_p}\right\}$$

Similarly, one obtains

$$|H|\frac{dz_2^*}{d\sigma_{11}} = -\left\{\frac{z_1^2}{\sigma_p}\frac{z_1}{\sigma_p}\frac{D}{\sigma_{22}}V_2\left\{\frac{V_{22}}{V_2}-\frac{1}{\sigma_p}\right\} + \frac{z_1}{\sigma_p}\{z_1V_{12}m_1+V_2\}V_2\frac{\sigma_{12}}{\sigma_p}\right\}$$

When $V_{22} = V_{12} = 0$ or when $V(\mu,\sigma) = \mu-\alpha\sigma$

$$|H|\frac{dz_1^*}{d\sigma_{11}} = -(V_2)^2\frac{z_1}{\sigma_p^2}\sigma_{22}\left(1-\frac{z_1^2D}{\sigma_{22}\sigma_p^2}\right)$$

and

$$|H|\frac{dz_2^*}{d\sigma_{11}} = -(V_2)^2\frac{\sigma_{12}}{\sigma_p^2}\sigma_{22}\left(1-\frac{z_1^2D}{\sigma_{22}\sigma_p^2}\right)$$

In the mean-variance space, when

$$V(\mu,\sigma^2) = \mu - a\sigma^2$$

we obtain

$$|H|\frac{dz_1}{d\sigma_{11}} = -(V_2)^2 z_1\sigma_{22}.$$

Both measures have the same sign when

$$\frac{d\sigma_p}{dz_1}\frac{z_1}{\sigma_p} = \frac{z_1D}{\sigma_p^2}$$

is low enough.

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CONCLUSION

L'objectif principal de cette thèse consistait à analyser le comportement des agents économiques en présence d'incertitude. La méthode de statique comparée est la technique d'analyse qui a été retenue pour cet examen du comportement des agents.

La première étude portait sur un modèle théorique du choix du mode d'habitation à deux périodes. L'incertitude a été modélisée en introduisant un actif immobilier avec rendement incertain. L'objectif principal consistait à vérifier l'hypothèse selon laquelle la distribution du revenu à travers le temps influence le choix du mode d'habitation si l'accès aux sources de crédit est limité (Fu, 1991). Afin de vérifier cette hypothèse, une nouvelle définition de la distribution du revenu à travers le temps a été élaborée. Selon cette définition, tout changement de la somme des revenus totaux à travers le temps peut être décomposée en deux effets indépendants: un effet lié au profil du revenu et un effet lié à l'accroissement de la richesse. Cette décomposition est en soit un résultat puisque les définitions existantes dans la littérature ne permettaient pas d'isoler ces deux effets. Sans cette nouvelle formulation, il serait impossible d'évaluer l'impact du profil de revenu sur le choix d'habitation. La première étude a permis de valider l'hypothèse de Fu (1991). Cette validation a permis de conclure que les agents dont le profil de revenu était biaisé vers la période 1 avaient de meilleures chances d'accéder au statut de propriétaire.

Les deux dernières études portaient sur les effets d'accroissements de risque dans les modèles à plusieurs variables de décision et plusieurs paramètres aléatoires. La deuxième étude a démontré que certaines définitions d'accroissement de risque préservent la corrélation entre deux paramètres aléatoires puisque le signe algébrique et la valeur numérique de la covariance demeurent inchangés. Les résultats obtenus à partir de ces définitions portent donc strictement sur les accroissements de risque et ne contiennent pas de composantes liées au changement de la corrélation. Le maintien du signe du coefficient de corrélation est important puisqu'il permet d'éviter des changements dans les choix optimaux qui ne résulteraient pas d'un accroissement de risque.

L'effet d'un accroissement de risque sur la composition optimale d'un portefeuille constituait le thème central de la dernière étude. Tous les résultats de cette étude démontrent qu'un actif qui devient plus risqué sera détenu dans une proportion moins grande. Ce résultat demeure valide dans le modèle le plus général à deux variables de décision et deux variables aléatoires. En particulier, la cinquième proposition ne postule aucune hypothèse sur les préférences des agents. Seul une distribution bi-normale (avec dépendence stochastique) est supposée. Il d'agit ici d'un premier résultat concernant les effets d'un accroissement de risque sur les choix optimaux dans un modèle aussi général.

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