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Three essays on self-enforcing risk-sharing contracts

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Cette thèse est composée de trois essais sur les contrats de partage des risques avec contraintes d'engagement des parties. Le premier essai étudie ce type de contrat lorsqu'il est possible pour les parties d'effectuer un transfer financier avant la réalisation de l'état de la nature. Ce paiement ex ante permet de relâcher les contraintes d'engagement ex post et ainsi d'améliorer la relation contractuelle au sens de Pareto. Certaines propriétés dynamiques des trajets optimaux expliquées jusqu'ici dans la littérature à l'aide de modèles avec asymmétrie d'information, sont répliquées par notre modèle avec information symétrique.

Le deuxième essai développe une méthode d'approximation polynômiale de la fonction de valeur qui caractérise le type de contrat étudié. Les polynômes de Chebyshev sont utilisés pour leurs propriétés d'orthogonalité qui permettent d'appliquer la méthode de collocation orthogonale. Cette méthode converge vers la solution en quelques minutes. Un résultat théorique du premier essai qui n'a pu être démontré que dans un environnement à deux états est vérifié et généralisé numériquement dans un environnement à trois états.

Le troisième essai étudie le contrat optimal entre un agent riscophobe, qui peut investir pour augmenter son bien-être la période suivante mais ne peut s'engager à respecter son contrat, et un agent neutre au risque qui s'engage à respecter les termes de son contrat. Une analogie est faite avec la relation financière entre un pays en voie de développement que l'on ne peut forcer à respecter ses engagements, et un financier. Le résultat principal est qu'une diminution de l'endettement du pays ne serait pas nécessairement suivie d'une hausse de l'investissement.

RÉSUMÉ

Cette thèse est composée de trois essais sur les contrats de partage des risques avec contraintes d'engagement des parties. Le premier essai étudie ce type de contrat lorsqu'il est possible pour les parties d'effectuer un transfer financier avant la réalisation de l'état de la nature. Ce paiement ex ante permet de relâcher les contraintes d'engagement ex post et ainsi d'améliorer la relation contractuelle au sens de Parẽto. Le résultat principal est à l'effet que les paiements ex ante ne sont pas stationnaires. Ils dépendent de façon optimale du surplus espéré par chaque agent dans la relation contractuelle. Ce surplus est fonction de l'histoire de réalisation des états de la nature. Lorsqu'un agent prévoit recevoir une faible part du surplus de la relation, ses contraintes ex post sont relativement serrées et par conséquent, on ne peut lui demander d'effectuer un gros paiement ex post. Dans ce cas, il est optimal pour l'agent d'effectuer un paiement ex ante de manière à relâcher ses contraintes ex post. En général, cependant, ces contraintes ne peuvent être complètement éliminées puisqu'un paiement élevé ex ante de la part d'un agent augmente les incitations de l'autre agent à briser son contrat et partir avec ce paiement. Nous montrons que la grosseur du paiement ex ante est inversement reliée au surplus espéré de la relation. Nous montrons également l'émergence de propriétés dynamiques intéressantes même si les chocs subis par le revenu sont indépendants et identiquement distribués d'une période à l'autre. Par exemple, dans un exemple d'assurance à deux états, nous montrons que la dynamique des primes du contrat optimal dépend de l'expérience de l'assuré même s'il n'y a pas d'information privée ni d'apprentissage dans notre modèle.

Le deuxième essai développe une méthode d'approximation polynômiale de la

fonction de valeur qui caractérise le type de contrat étudié. Les polynômes de Chebyshev sont utilisés pour leurs propriétés d'orthogonalité qui permettent d'appliquer la méthode de collocation orthogonale. La collocation orthogonale consiste à évaluer le problème statique à $n+1$ points pour obtenir une approximation polynômiale d'ordre n. L'approximation de la fonction de valeur sera obtenue de façon itérative par l'interpolation à ces $n+1$ points. Nous pouvons ainsi obtenir un approximation polynômiale satisfaisante d'ordre 2, en quelques minutes seulement, sur un microordinateur avec processeur 486.

Nous avons ensuite utilisé cette approximation de la fonction de valeur pour effectuer des simulations du contrat optimal à partir de 100 séries de revenus sur 100 périodes. Nous avons ainsi pu vérifier numériquement dans un environnement à trois états de la nature, qu'une hausse (baisse) du revenu de l'agent riscophobe d'une période à l'autre, diminuera (augmentera) le surplus espéré par l'agent neutre au risque. Grâce aux simulations, ce résultat théorique du premier essai qui n'a pu être démontré que dans un environnement à deux états est vérifié et généralisé numériquement dans un environnement à trois états.

Dans un autre ordre d'idées, une littérature abondante existe sur la pertinence ou non d'effacer une partie de la dette de certains pays très endettés, notamment en Amérique Latine, de manière à stimuler l'investissement du pays et ainsi augmenter l'espérance de remboursements des financiers. Dans la plupart de ces papiers, on étudie le problème en supposant que le pays est déjà dans une situation d'endettement telle qu'il lui soit impossible de rembourser un jour la totalité de sa créance.

Le troisième essai de cette thèse étudie le contrat optimal entre un agent riscophobe, qui peut investir pour augmenter son bien-être la période suivante mais ne peut s'engager à respecter son contrat, et un agent neutre au risque qui s'engage à respecter les termes de son contrat. Une analogie est faite avec la relation financière entre un pays en voie de développement que l'on ne peut forcer a respecter ses engagements, et un financier. Notre modélisation est tout à fait différente de celle généralement rencontrée dans la littérature sur cette question. Dans notre papier, le contrat optimal est auto-exécutoire, c'est-à-dire qu'en tout temps et en toute circonstance, il est dans l'intéret du pays de respecter sa part du contrat. Puisque la banque (ou le consortium bancaire) s'engage à respecter sa part du contrat, le contrat optimal sera tel que les financiers feront des profits positifs au début de la relation de crédit et feront des pertes par la suite.

Le résultat principal du papier est qu'une diminution de l'endettement du pays ne serait pas nécessairement suivie d'une hausse de l'investissement. En fait, le signe de la corrélation entre le niveau d'endettement et l'investissement est fonction des paramètres du modèle.

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À mes parents.

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INTRODUCTION

Cette thèse est composée de trois essais sur les contrats de long terme de partage de risques avec contraintes de banqueroute. La problématique générale est la suivante. Le revenu d'un agent riscophobe est aléatoire. Il cherche donc à diversifier le plus possible son risque à chaque période en transigeant sur les marchés financiers. Il veut également lisser autant que possible sa consommation dans le temps. Nous étudions dans cette thèse le contrat financier d'un agent riscophobe dans un contexte où les pouvoir légaux sont limités, c'est-à-dire un environnement économique dans lequel on ne peut forçer le respect des termes du contrat (ou dans lequel les coûts nécessaires à l'imposition du respect du contrat sont excessivement élevés). Par exemple, dans le cas de la dette des pays en voie de développement (PVD), il n'existe pas d'instance supranationale qui puisse imposer au PVD le remboursement de sa créance. En l'absence d'une telle instance, si le remboursement exigé au PVD devient très élevé, il peut décider de ne plus rembourser sans qu'on puisse l'y forçer. Le contrat optimal dans un tel environnement doit faire un compromis entre le partage des risques et le lissage de la consommation d'une part, et donner les incitations à ne pas déclarer faillite d'autre part. Le but de la thèse est précisément d'étudier l'effet des contraintes de banqueroute sur les contrats optimaux de partage de risques, et ce dans différents environnements économiques.

Nous étudierons donc deux agents en relation de long terme pour partager leurs risques, et lisser leur consommation. La relation est déterminée par un contrat qui prescrit des transferts financiers contingents à la réalisation de l'état de la nature (et possiblement à toute l'histoire de réalisation des états de la nature depuis le début de la relation). Si les deux agents pouvaient s'engager à respecter leur part du contrat, on atteindrait le niveau de premier rang de partage de risque, puisque le contrat de long terme serait une suite de contrat d'une période dans lequel les agents s'engagent à effectuer le transfert permettant le partage optimal des risques. Par exemple, si l'agent 1 est neutre au risque et reçoit un revenu constant, il pourra s'engager à chaque période à verser à l'agent 2 la différence entre son revenu moyen et le revenu réalisé à cette période. L'agent 1 acceptera un telle entente puisque son revenu moyen est inchangé et qu'il est donc indifférent entre signer ou non un tel contrat, et l'agent 2 étant pleinement assuré se verra ravi. Par contre, lorsqu'un agent ne peut s'engager à ne pas faire défaut une fois l'état de la nature réalisé, le partage de risque optimal peut être difficile ou impossible à obtenir puisqu'il est possible qu'un agent décide de ne pas effectuer le transfert prescrit par le contrat, si celui-ci est trop élevé. Dans l'exemple précédent, l'agent 1 pourrait décider de ne pas verser le montant prescrit par le contrat lorsque l'agent 2 reçoit son revenu le plus bas, si les bénéfices escomptés qu'il espère recevoir dans la suite de sa relation financière ne compensent pas pour le gros déboursement qu'il doit faire maintenant. Supposons que l'agent 1 est un consortium bancaire qui prête à l'agent 2, un PVD. Supposons que ces deux agents signent un contrat de partage des risques. En l'absence d'instance légale internationale, si le consortium, par exemple, décide de ne pays effectuer le paiement prescrit par un contrat de pleine assurance, rien ni personne ne peut l'y forcer. Le contrat doit donc stipuler des paiements auto-exécutoires, c'est-à-dire des paiements tels qu'il soit toujours dans l'intéret de l'agent qui doive les effectuer, d'effectivement les effectuer. A chaque période, le surplus qu'un agent anticipe de la relation future conditionne le montant qu'il est prêt à débourser. Si un agent anticipe un surplus futur élevé, il est prêt à payer plus cher aujourd'hui pour ce surplus que s'il anticipe un surplus futur faible.

Nous caractériserons dans les deux premiers essais de la thèse, le contrat optimal de partage de risque lorsque les transferts financiers peuvent être contingents ou non à la réalisation du revenu. On pourra ainsi caractériser la dynamique des emprunts et des remboursements dans le cas de l'endettement des PVD. Dans le troisième essai, nous cherchons à comprendre l'effet d'un surendettement sur l'investissement dans le PVD.

Le problème de partage des risques étudié dans la litérature a généralement la structure suivante [1]. On a deux agents. Un agent riscophobe reçoit une dotation stochastique à chaque période. Cette dotation est indépendamment et identiquement distribuée d'une période à l'autre. Les transferts stipulés par le contrat prennent toujours place en fin de période, lorsque l'état de la nature est réalisé. Dans un environnement où les deux agents ne peuvent s'engager à effectuer les transferts demandés, des contraintes de participation doivent être satisfaites pour chaque agent, à chaque période et pour chaque réalisation possible de la dotation. On qualifie ces contraintes de contraintes ex post puisqu'elles doivent être satisfaites pour chaque niveau possible de réalisation de la dotation. Il est possible que ces contraintes soient serrées éventuellement pour un des agents, disons l'agent 1. Dans ce cas, le montant exigé de l'agent 1 est limité par sa contrainte de participation, et le niveau de partage de risque n'atteint pas le niveau de premier rang. Il serait alors dans l'intéret de l'agent 1 de payer un montant d'argent à l'agent 2 avant la réalisation de l'état de la nature, lorsque les contraintes de participation n'ont à tenir qu'en espérance sur les différentes réalisations possibles de la dotation. L'objectif du premier essai de la thèse est donc d'élargir l'espace contractuel pour permettre un transfert financier avant la réalisation de l'état de la nature et d'étudier comment cette possibilité affecte la dynamique des transferts optimaux entre les agents.

Ce paiement ex ante réduit les paiements futurs à effectuer et relâche les contraintes de participation, ce qui permet d'atteindre un meilleur partage des risques. Lorsqu' un agent effectue un paiement ex ante il relâche ses propres contraintes de participation ex post, mais par contre, il ressere généralement celles de l'autre agent en lui faisant supporter une plus grande partie du fardeau des paiements futurs. Conséquemment, le paiement ex ante doit contrebalancer les contraintes ex post d'un

[1] Par exemple, voir Thomas et Worrall (1998)

agent et celles de l'autre agent.

Dans le contexte étudié ici, l'opportunité de faire un transfert ex ante permet d'augmenter le surplus total espéré dans la relation et ainsi d'améliorer l'ensemble des contrats possibles[2]. Nous analysons l'arbitrage entre les contraintes de participation des deux agents en caractérisant les paiements optimaux ex ante et ex post. Nos principaux résultats sont à l'effet que les paiements ex ante sont utilisés de façon optimale et que ces paiements ne sont pas stationnaires. Ils dépendent de façon optimale du surplus espéré de la relation par chacun des agents. Ce surplus évolue avec l'histoire des réalisations passées des états de la nature. Lorsqu'un agent anticipe une faible part du surplus de la relation, ses contraintes de participation ex post sont relativement serrées, et on ne peut donc exiger de sa part un paiement ex post élevé. Dans ce cas, il est optimal de demander à cet agent un paiement avant la réalisation de l'état de la nature. En général cependant, ces contraintes ne peuvent être complètement éliminées puisqu'un paiement ex ante élevé de la part d'un agent augmente les incitations de l'autre agent à briser le contrat et quitter avec ce paiement. Nous montrons que le montant du paiement ex ante qu'un agent effectue est inversement relié au surplus que cet agent anticipe de la relation. Nous montrons également l'émergence de propriétés dynamiques intéressantes, même si les chocs aléatoires sont indépendamment et identiquement distribués d'une période à l'autre. Par exemple, dans le cas d'un contrat entre un individu riscophobe et une compagnie d'assurance, la prime d'assurance est payée par le client avant la réalisation de l'état de la nature. Cette prime peut donc être vue comme un paiement ex ante. Dans un modèle à deux états, nous montrons que la dynamique du contrat optimal imite une tarification selon l'expérience malgré l'absence d'information privée ou d'apprentissage dans

[2]MacLeod et Malcomson (1989) étudient la structure salariale optimale entre un travailleur et un employeur lorsqu'on ne peut vérifier la performance du travailleur en cour. La possibilité pour le travailleur de donner un montant d'argent à l'employeur avant la réalisation de ses performances ne permet pas d'améliorer l'ensemble des contrats possibles puisque dans leur modèle, il n'y a ni incertitude, ni partage des risques.

notre modèle. En effet, suite à une mauvaise réalisation de l'état de la nature, disons un accident de la part de l'agent 1, la compagnie d'assurance indemnisera son client. En contrepartie, le surplus futur espéré de la compagnie devra augmenter. Ceci sera réalisé de façon optimale par une hausse de la prime d'assurance du client. C'est l'effet de la volonté de lisser la consommation dans le temps: un accident est ainsi amorti sur plusieurs périodes. Une suite d'accidents sera généralement suivie d'une suite de hausses de la prime d'assurance.

Les paiements ex ante sont observés dans plusieurs relations contractuelles différentes. Par exemple, dans une relation financière, la firme s'engage ex ante à donner des garanties financières qu'elle laissera aux financiers en cas de défaut de paiement. Pour leur part, les financiers peuvent s'engager à verser un certain montant à la firme par l'ouverture d'une marge de crédit de laquelle la firme pourra puiser au besoin. Ces paiements peuvent être interprétés comme des transferts ex ante d'un agent à l'autre, effectués dans le but d'alléger le problème d'engagement des parties. Dans un contrat d'assurance, la prime d'assurance est payée avant la réalisation de l'état de la nature, et peut donc être réinterprétée comme un paiement ex ante de l'assuré à l'assureur. Pour ce qui est de la compagnie d'assurance, on exige généralement d'elle un niveau de réserves minimales pour atténuer le problème de responsabilité limitée. Ces paiements non contingents peuvent être réinterprétés comme des transferts ex ante d'un agent à l'autre.

Les manipulations des conditions de premier ordre décrivant le contrat optimal sont très difficiles, étant donné l'existence de plusieurs contraintes d'inégalité. Pour cette raison, certaines caractéristiques du contrat optimal n'ont pu être démontrées que dans un environnement à deux états de la nature. L'objectif du second essai de cette thèse est d'étudier de façon approfondie les propriétés du contrat optimal à l'aide d'un exemple. Cett étude nécessite le développement d'un algorithme permettant l'approximation rapide, sur un micro-ordinateur avec processeur 486, de la fonction de valeur qui décrit la relation dynamique optimale.

Phelan et Townsend (1991) développent un algorithme permettant le calcul d'une solution numérique au problème de planification sociale pour une économie avec un continuum d'agents identiques, ayant tous accès à la même technologie de production indépendante. Cette technologie utilise le travail de l'agent pour produire l'unique bien de consommation de l'économie en fonction du niveau de travail fourni et d'un choc exogène indépendant. Le surplus social de l'économie est défini comme la somme des surplus individuels (définis comme la différence entre la production individuelle et la consommation). Leur méthode pour trouver l'optimum de Pareto de l'économie consiste à maximiser le surplus social sous contrainte que chaque agent recoive un niveau d'utilité spécifié ex ante. Un optimum est ainsi la solution de ce problème pour une distribution initiale des utilités requises qui soit à la fois possible et telle que n'importe laquelle autre distribution d'utilité qui domine au sens de Pareto la distribution initiale ne soit pas une solution réalisable. Phelan et Townsend cherchent à maximiser le surplus social de chaque groupe d'agents correspondant à un niveau d'utilité ex ante donné en choisissant la fraction optimale des agents de ce groupe qui devront choisir un niveau de travail (ou d'effort) fixé et qui auront droit à un niveau fixé de consommation, et ce pour toutes les combinaisons possibles d'effort et de consommation. Phelan et Townsend étudient ensuite le cas où le niveau d'effort n'est observable que par l'agent le fournissant, en rajoutant au problème une contrainte d'incitation telle que le niveau d'effort demandé soit faiblement préféré par l'agent. Les auteurs ramènent ainsi le problème à un problème de programmation linéaire (dans les probabilités) simple. La solution sera d'autant plus précise que la grille des possibilités pour les niveaux de travail et de consommation sera raffinée. Notre première tentative d'approximation fut l'utilisation de la méthode suggérée par Phelan et Townsend. Cette méthode simple à appliquer demande par contre énormément de temps machine. Nous avons estimé, par extrapolation du temps requis pour une approximation grossière de notre fonction, à plusieurs mois le temps machine nécessaire à une approximation raisonnable sur un micro-ordinateur avec processeur 486.

Wang (1994) développe un algorithme pour dériver numériquement l'ensemble, ϕ, des paires d'utilité admissibles dans une relation dynamique d'assurance entre deux agents riscophobes avec asymétrie d'information bilatérale. En utilisant cet ensemble ϕ, la résolution de l'équation de Bellman donne les trajets de consommations optimales et la loi de transition de la variable d'état.

Nous avons obtenu une approximation rapide par une méthode d'approximation polynômiale appelée la méthode de collocation orthogonale. Cette méthode est suggérée dans Judd (1992) et est une application du théorème des projections. Les conditions d'orthogonalité du théorème des projections sont grandement simplifiées lorsque l'ensemble des éléments utilisés pour engendrer l'espace dans lequel on projette l'élément à approximer constitue une base orthogonale. Nous utilisons donc les polynomes de Chebyshev, qui obéissent à certaines relations discrètes d'orthogonalité, pour obtenir une approximation de notre fonction.

D'autre part, on sait par le théorème de Wierstrass que l'on peut obtenir une approximation aussi précise que l'on veut de toute fonction continue sur un intervalle fermé, à l'aide d'un polynôme. La précision de l'approximation est définie comme la distance maximale dans l'intervalle d'approximation entre la fonction à approximer et le polynôme de Chebyshev. Cette distance maximale est appelée l'erreur d'approximation. L'erreur d'approximation diminue avec le degré du polynôme. Nous chercherons donc dans cet essai à obtenir une "bonne" approximation polynômiale de la fonction de valeur, dans l'espace engendré par les polynômes de Chebyshev.

Nous utiliserons de plus deux théorèmes d'analyse numérique. Le premier nous indique que toute approximation polynômiale est caractérisée par l'existence d'au moins deux points distincts dans l'intervalle d'approximation pour lesquels l'erreur d'approximation est minimale (voir Rivlin (1969)). Le second théorème caractérise une approximation à l'aide de polynômes de Chebyshev de degré n, en démontrant l'existence de $n+1$ points dans l'intervalle d'approximation pour lesquels l'erreur d'approximation est nulle. Ces $n+1$ points sont les zéros du polynôme de Chebyshev d'ordre $n + 1$ (voir Rivlin (1990)) et définissent un système d'équations exprimant le fait que la distance entre la fonction que l'on cherche à approximer et le polynôme est nulle en ces points. La méthode de collocation orthogonale consiste à résoudre ce système de $n + 1$ équations linéaires dans les paramètres du polynôme.

L'algorithme d'approximation consiste d'abord à calculer une fonction de valeur de départ. Nous avons obtenu de bons résultats avec la fonction de valeur caractérisant la solution de pleine assurance. La seconde étape consiste à maximiser un problème d'optimisation non-linéaire (à l'aide d'un utilitaire d'optimisation de Matlab) pour $n + 1$ valeurs spécifiques de la variable d'état, où n est le degré de l'approximation polynômiale. Cette étape nous donne $n + 1$ couples (variable d'état, valeur de la fonction). La méthode de collocation orthogonale est ensuite utilisée pour obtenir une nouvelle approximation de la fonction de valeur. Nous répétons les deuxième et troisième étapes jusqu'à ce que l'erreur d'approximation soit inférieure à un seuil prédéterminé.

Cette méthode s'est avérée étonnamment rapide (quelques minutes), pourvu que l'on ait une bonne approximation de la fonction de valeur comme valeur de départ. Nous avons ainsi pu vérifier numériquement pour trois états de la nature, un résultat analytique démontré dans l'essai précédent pour deux états de la nature.

Dans le troisième essai, nous nous intéressons au problème du remboursement de la dette des pays en voie de développement. Deux questions en particulier nous préoccuperont. D'une part, deux propositions sérieuses de règlement du problème du remboursement de la dette ont été formulées dans les Plans Baker et Bradley. Le Plan Baker suggère de continuer à financer les pays endettés sans effacer aucune partie de la dette, ni changer la nature des créances. Le Plan Bradley propose plutôt d'effaçer une partie importante de la dette de manière à rétablir une situation normale. D'autre part, on a observé dans les dernières années une crise d'investissement dans les pays très endettés. Plusieurs études suggèrent que cette baisse des investissements est due au problème d'endettement et que la situation pourrait être rétablie par une diminution de l'endettement.

Le problème peut être vu comme un problème de négotiation entre les créditeurs qui veulent obtenir le plus gros remboursement possible, et le pays qui veut minimiser le transfert de nouvelles ressources. Bulow et Rogoff (1988) étudient cette façette du problème à l'aide d'un modèle de négociation à la Rubinstein. Une approche différente consiste à considérer le montant disponible au remboursement comme connu. Cette approche utilisée par Krugman (1988) simplifie grandement le problème. Krugman étudie l'arbitrage auquel sont confrontés les créditeurs d'un pays dont la dette est tellement élevée qu'il lui est impossible d'attirer volontairement de nouveaux prêteurs. Lorsque les ressources courantes du pays sont insuffisantes pour assurer le service de sa dette, les créditeurs ont deux choix: ils peuvent financer le pays en prêtant à perte dans l'espoir que le pays sera éventuellement capable de rembourser en totalité; ou ils peuvent effacer une partie de la dette. Krugman démontre d'abord, dans un modèle qui fait abstraction des incitations du pays, qu'il serait toujours dans l'intéret des banquiers de donner l'argent nécessaire au service de la dette et d'exiger le plus gros remboursement possible non-contingent à la réalisation du revenu du pays. Supposons maintenant que le fardeau de la dette d'un pays est aussi gros que le montant maximal qu'il puisse obtenir en effectuant l'effort d'ajustement (investissement, politiques monétaires et fiscales, etc.) maximal. Le pays n'a aucun intérêt à effectuer cet effort maximal puisque tous les bénéfices iraient aux créanciers. Lorsqu'il tient compte des incitations du pays à faire un effort de restructuration, Krugman démontre l'existence d'un arbitrage entre demander le plus gros remboursement possible et créer des distortions dans les incitations du pays qui voit s'envoler une part d'autant plus grande des bénéfices de son effort que le remboursement demandé est élevé.

Le modèle de Krugman est un jeu à deux périodes dans lequel la banque choisit en première période le niveau de remboursement à exiger du pays, et le pays choisit ensuite le niveau d'effort d'ajustement à déployer pour améliorer ses perspectives économiques futures. Les deux agents sont neutres au risque, et le jeu commence dans une situation dite de "debt overhang", c'est-à-dire une situation dans laquelle les remboursements espérés par la banque sont inférieurs à la dette accumulée du pays. Dans cet environnement, une hausse du remboursement exigé diminuera le rendement marginal de l'effort puisque le pays devra rembourser plus dans les états où il n'est pas limité par les ressources dont il dispose pour le remboursement. Il réagira de façon optimale à la hausse des exigences bancaires par une baisse de son niveau d'effort de restructuration.

Cette réaction est mise en doute par Warner (1992) dont les résultats empiriques révèlent que la crise d'investissement des pays en voie de développement n'est pas causée par la crise d'endettement. Au contraire, Warner vérifie empiriquement que la chute des investissements peut être prédite par des équations simples du taux d'intérêt mondial et des termes des échanges qui n'incluent pas d'effet dus à la crise d'endettement. Il en conclut que l'hypothèse selon laquelle le déclin des investissements est expliqué par le niveau d'endettement doit être fortement mise en doute. Il rajoute, qu'à tout le moins, l'influence directe des chocs économiques mondiaux sur l'investissement n'a pas reçu suffisamment d'attention.

Dans ce papier, nous cherchons des fondements théoriques aux résultats empiriques de Warner (1994). La différence majeure de notre modèle avec celui de Krugman (1988) réside dans l'hypothèse que les coûts à encourir pour imposer un remboursement du pays sont tellement élevés, que le contrat financier entre les banques et le pays doit être auto-exécutoire. Thomas et Worrall (1994) ont développé un modèle similaire pour étudier le cas d'une firme multinationale qui investit à l'étranger et doit verser une partie de ses dividendes sous forme de taxes au pays hôte. Le pays hôte a le pouvoir de saisir le capital physique de la firme. A l'optimum, la firme investit peu au début de la relation et ne verse aucune taxe au pays hôte en promettant de les lui verser plus tard. Avec un investissement faible au début, la production de la firme est faible et il est donc peu tentant pour le pays hôte de la saisir d'autant plus qu'elle perdrait par le fait même la totalité des taxes promises. A mesure que les taxes futures augmentent, le surplus espéré du pays augmente et la firme peut augmenter son investissement sans crainte d'être saisi.

Dans notre modèle, le pays est riscophobe. Cette hypothèse réflète la difficulté qu'ont les pays en voie de développement à diversifier parfaitement leurs risques sur les marchés internationaux. De plus, toute l'histoire d'endettement du pays y est caractérisée, et les remboursements exigés par les banques sont contingents à la réalisation d'une variable aléatoire qui est fonction du niveau d'investissement choisi par le pays à la période précédente. Dans ce contexte, la réaction du pays suite à une baisse du remboursement exigé est indéterminée. En effet, nous montrons que dans un modèle à deux états, le pays pourrait aussi bien diminuer qu'augmenter son effort d'ajustement suite à une baisse des exigences des banques. L'intuition de ce résultat est la suivante. A l'optimum, le coût marginal de la dette doit être égal au coût marginal de l'investissement. Lorsqu'on efface une partie de la dette du pays, on augmente la valeur marginale en terme d'utilité de cette dette, puisqu'une moins grande part des ressources futures du pays devront êtres utilisées pour le remboursement. Une baisse de l'endettement améliore donc les possibilités de consommation future et diminue le coût marginal de la dette en termes d'utilité marginale de la consommation. Puisque le coût marginal de la dette doit être égal au coût marginal de l'investissement, la pression à la baisse sur le coût marginal de la dette aura un effet négatif sur le coût marginal de l'investissement. Puisque le coût marginal d'investissement est en terme d'utilité marginale de la consommation, une baisse du coût marginal signifie une hausse de la consommation qui sera réalisée par une une baisse du niveau d'investissement. Mais, par la nécessité de maintenir l'investissement à un niveau efficace, le revenu marginal de l'investissement devra lui aussi diminuer, et cette diminution ne peut être accomplie que par une hausse du niveau d'investissement. Lequel de ces deux effets contraires sur le niveau d'investissement sera le plus important dépendra des paramètres du modèle. En particulier, plus le revenu marginal de l'investissement sera sensible à une diminution marginale de la dette, plus l'effet à la baisse sur l'investissement sera important. Une situation dans laquelle l'investissement pourrait diminuer suite à une baisse de l'endettement est donc une situation où le rendement marginal de l'investissement est élevé, c'est-à-dire un environnement dans lequel l'investissement est faible.

1 Using ex ante payments in self-enforcing risk-sharing contracts

1.1 Introduction

Long-term contracts are useful for the governance of long-term relationships. Such contracts can help improve incentives as well as risk-sharing between two agents. An optimal contract will trade-off between incentives and risk-sharing to attain an efficient allocation; however this efficient allocation is often time-inconsistent. For example, an ex ante efficient allocation may not be ex post efficient (once certain actions have been undertaken or some information has been revealed). This time-consistency problem has led to the recent literature on renegotiation. Or, an ex ante profitable contract may not be ex post profitable following a given history. In this case, if enforcement costs are high (or mobility costs are low), agents may be tempted to renege on the contract to seek more profitable opportunities elsewhere. The literature on self-enforcing contracts studies this problem.

Consider two agents that enter into a long-run relationship to share risk and for which enforcement costs are high. Their relationship is governed by a contract that prescribes in every period transfer payments from one agent to the other contingent on the realization of the state of nature (and possibly the complete history of the relationship). If the two agents can commit not to default on any prescribed transfer payment then the optimal contract achieves an efficient risk-sharing allocation; however, if an agent cannot commit not to default, efficient risk sharing may be impeded as the optimal contract is constrained by the possibility of ex post default. The contract should then prescribe payments that are self-enforcing, that is, an agent will make a transfer payment if and only if it is in her interest to do so. In any period, the surplus one expects from the relationship conditions the transfer that she can make in this period. If an agent expects a high surplus in the future, she has low incentives to break the relationship and she is therefore willing to make a high payment to continue the relationship. On the other hand, if her expected surplus is low, she has high incentives to break the relationship and she must therefore be induced not to do so by requiring a low (possibly negative) payment. Self-enforcing constraints generally limit transfer payments and therefore reduce the opportunity for efficient risk sharing.

The risk-sharing problem analysed in the literature usually has the following structure.[1] There are two agents. In every period, a risk-averse agent receives a stochastic endowment. This endowment is independently and identically distributed across periods. Risk-sharing between the two agents is implemented by a contract specifying transfer payments between the two agents. The contracting space is such that all transfers take place at the end of the period once the state of nature has been observed. In an environment in which the two agents cannot commit to making all prescribed payments, ex post self-enforcing constraints must be satisfied, that is, for any realization of the state of nature, the contractually specified payment must satisfy a participation constraint for each agent. It is possible that these constraints be quite stringent for one agent, say agent 1. This effectively limits the payments agent 1 can make to agent 2. In this case, agent 1 would like to make a transfer to agent 2 *before* the state of nature is realized. At this point, agent 1's self-enforcing constraints only have to hold in expectation over all states of nature. Such ex ante transfer would effectively relax agent 1's ex post self-enforcing constraints. When the two agents face self-enforcing constraints, if one agent makes such payment to relax her own self-enforcing constraints it usually makes the other agent's self-enforcing constraints more stringent by leaving the ex post burden to that agent to make the necessary transfers for optimal risk-sharing. Consequently, the ex ante payment must trade-off between the self-enforcing constraints of the two agents.

In this paper we analyse a risk-sharing contract between two risk-averse agents facing self-enforcing constraints. We enlarge the contracting space to allow for an ex ante transfer (at the beginning of the period) before the state of nature is realized. We analyse the trade-off between the self-enforcing constraints of the two agents by characterizing the optimal ex ante and ex post transfer payments.

[1] For example, see Thomas and Worrall (1988).

Our main results are that ex ante payments are optimally used and that these payments are non-stationary. They optimally depend on the surplus from the relationship each agent expects. This expected surplus evolves with the history of past realizations of states of nature. When an agent expects a low share of the surplus of the relationship, her ex post self-enforcing constraints are relatively stringent and she cannot be required to make a high payment. In this case, she is optimally asked to pay up front before the realization of the state of nature. This effectively relaxes her ex post self-enforcing constraints. In general, however, these constraints cannot be completely eliminated because a high ex ante payment by one agent increases the incentives of the other agent to break the relationship and run away with this payment. We show that the size of the ex ante payment an agent makes is inversely related to the surplus she expects to get from the relationship. We can also show that interesting dynamics properties emerge from our model even though shocks are independently and identically distributed across periods. For example, in a two-state example, we show that the dynamics of the optimal contract exhibits experience rating even though there is no private information or learning taking place.

Ex ante payments are observed in many different contractual relationships. For example, in a financing relationship the firm commits ex ante to a certain amount of collateral in case it defaults on the contract. Alternatively, the financier can commit to a given amount of financing by providing the firm with a credit line. Such payments can be interpreted as ex ante transfers from one agent to the other. In other contexts, the ex ante payment can be reinterpreted as a breach-of-contract penalty which is in fact a payment agents are committed to if they default on the contract. In an insurance-contract example, insurance premia are paid before the state of nature is realized as an ex ante payment from the insuree to the insuror. We discuss the economic implications of our model more at length in Section 5. Section 2 presents the basic model. In Section 3 we analyze the role of ex ante payments when only one agent faces self-enforcing constraints. Section 4 presents the main results of the paper when the two agents face self-enforcing constraints. Section 5 provides different economic interpretations of the contractual form with an ex ante payment. A conclusion follows.

1.2 The model

The environment we consider can be described by an infinite sequence of periods, $t = 1, 2, ..., \infty$, and for each period, a finite set of states of nature, $s \in \{1, 2, ..., S\}$, with $S \geq 2$. We assume that the states are distributed independently and identically across all periods, and therefore, in each period, the state of nature s occurs with probability p^s where $\sum_{s=1}^{S} p^s = 1$. It is assumed that each period t is divided into three dates, t_0, t_1, and t_2, where t_1 is the date at which the state of nature is realized; the dates t_0 and t_2 denote respectively the dates preceding and following the realization of the state of nature.

Two infinitely-lived agents evolve in this environment. Both agents are risk averse. In each period, agent 1's preferences over consumption c are represented by a state-independent strictly concave quadratic utility function $u(c)$ for $c \in [0, b]$. In each period, agent 1 obtains a state-contingent endowment y^s. We adopt the convention that $y^s > y^{s-1}$ for all states s. We assume that $0 < y^1 < y^S < b$. In each period, agent 2's preferences over consumption c are given by $v(c)$ which is also a state-independent concave quadratic function. In each period, agent 2 obtains a state-independent endowment e.[2] To insure an interior solution, we assume that $y^S + e < b$ and that $u'(0)$ and $v'(0)$ are sufficiently large. Both agents discount the future by a common factor $\beta \in (0, 1)$.

In such an environment there exists gains from trade to be exploited: both agents are risk averse, and agent 1 obtains a risky endowment, while agent 2 obtains a constant endowment. We assume that there are no contingent markets that would

[2]The analysis can be generalized to the case in which the endowment of agent 2 is stochastic.

allow the agents to diversify their risk and therefore the two agents enter into a risk-sharing relationship. For example, agent 1 may represent a firm and agent 2, a financier; or alternatively, agent 1 may be a sovereign agent and agent 2, a consortium of financiers. A relationship is characterized by transfer payments between the two agents that take place at various time periods and dates. We call the governance of such relationship a contract where the term contract is interpreted in a broad sense, namely it can encompass implicit as well as explicit agreements. A contract then specifies various transfers between the two agents for all periods of the relationship. In each period t, a contract can specify the following structure of transfer payments.

1. A (positive or negative) ex ante transfer B_t from agent 2 to agent 1 at date t_0 (before the state of nature is realized).
2. Ex post (positive or negative) transfers a_t^s from agent 1 to agent 2 at date t_2 (after the state of nature s is realized).

Consumption takes place at the end of the period. Agent 1's consumption in period t if state s is realized is $c_t^s = y_t^s + B_t - a_t^s$; agent 2's consumption is $b - B_t + a_t^s = b + y_t^s - c_t^s$.

In a typical relationship the prescribed transfers can potentially be contingent on the complete past history of the relationship. The history up to period t is the vector of all previous realizations of the state of nature. Let s_t denote the realized state of nature in period t. The history at the end of period $t-1$ (date $(t-1)_2$) or at the beginning of period t (date t_0) is denoted by $h_{t-1} = (s_1, s_2, ..., s_{t-1})$. We assume that $h_0 = \emptyset$. Assume that the two agents enter into a long-term (infinite) contractual relationship. We can then define formally a contract between the two agents.

Definition 1 *A **contract**, δ, is a sequence of two functions: $\{B(h_{t-1}), a(h_t)\}_{t=1}^{\infty}$ where $B_t = B(h_{t-1})$ represents the transfer from agent 2 to agent 1 at the beginning of period t (date t_0) when history is h_{t-1}, and where $a_t^s = a(h_{t-1}, s)$ represents the transfer from agent 1 to agent 2 at the end of period t (date t_2) when history is h_{t-1} up to period t and s is the realized state of nature in period t.*

For any contract, δ, and any history, h_{t-1}, agent 1's expected surplus of the relationship from the beginning of period t onwards is

$$U(\delta; h_{t-1}) \equiv E \sum_{\tau=t}^{\infty} \beta^{\tau-t} \{u(y_\tau^s + B_\tau - a_\tau^s) - u(y_\tau^s)\}$$

where E is the expectation operator taken over all states in all future periods and y_τ^s denotes that the endowment y^s is realized in period τ. Similarly, the expected surplus of agent 2 from the beginning of period t onwards is

$$V(\delta; h_{t-1}) \equiv E \sum_{\tau=t}^{\infty} \beta^{\tau-t} \{v(e - B_\tau + a_\tau^s) - v(e)\}.$$

The surplus of the two agents are measured with respect to autarky, that is, it gives the surplus one agent can get from the relationship over autarky where it would consume its endowment.

The approach we take is to assume that the two agents enter into a relationship at the beginning of a period called period 1. This relationship is governed by a contract. The characterization of the implemented contract depends on the available technology to legally enforce the prescribed payments. The objective of the paper is to study the effects of limited enforceability of payments on optimal contracts. The literature on self-enforcing contracts[3] has tackled such analysis and we will discuss later how our paper relates to some of these papers.

We first establish a benchmark case in which the two agents sign a contract at the beginning of the first period and all prescribed transfers are legally enforceable. We refer to this case as the full-commitment case. In this case, the optimal contract, δ^{fc}, is the solution to the following maximization problem where, for simplicity, it is assumed that agent 1 has the bargaining power and 0 is agent 2's reservation utility.

$$\delta^{fc} = \arg\max_{\delta} \{U(\delta; h_0) \text{ s/t } V(\delta; h_0) \geq 0\} \tag{1}$$

[3]For example, see Harris and Holmström (1982), MacLeod and Malcomson (1988), Thomas and Worrall (1988), and Kletzer and Wright (1990).

This maximization problem simply states that the optimal contract maximizes the discounted expected utility of agent 1 subject to agent 2's participation constraint. This constraint states that the contract must provide agent 2 with a discounted expected surplus of at least 0. A solution to this maximization problem exists and is characterized in the following proposition.

Proposition 1 *When both agents can commit to the terms of the contract, the optimal contract, δ^{fc}, is characterized by the equalization of marginal rates of substitution of consumption of the two agents across all states and periods. Formally, for all periods t, τ, all states s, q, and all histories h_{t-1}, $\frac{u'(y_t^s+B_t-a_t^s)}{u'(y_\tau^q+B_\tau-a_\tau^q)} = \frac{v'(e-B_t+a_t^s)}{v'(e-B_\tau+a_\tau^q)}$.*

The optimal full-commitment contract specifies perfect risk-sharing with a stationary consumption rule. This consumption rule can be written as $c_t^s = c^*(c_{t-1}, y_{t-1}, s)$ where

$$\frac{u'(c^*(c_{t-1}, y_{t-1}, s))}{u'(c_{t-1})} = \frac{v'(e + y_t^s - c^*(c_{t-1}, y_{t-1}, s))}{v'(e + y_{t-1} - c_{t-1})}.$$

Two aspects of this characterization deserve mention. First, in problem (1), the functions $U(\delta; h_0)$ and $V(\delta; h_0)$ depend only on the net transfers $B_t - a_t^s$ and therefore, in each state, only optimal net transfers are determined. This implies that the value of B_t is arbitrary. With full commitment, there is no role for the ex ante transfer B_t in the optimal contract.

Second, in some states of nature, net transfers from agent 1 to agent 2 are positive, and in other states, the reverse is true. Complete legal enforcement of the contract is a sufficient condition to make these transfers feasible. In the next sections we relax the assumption of complete legal enforcement to study the characterization of optimal contracts under incomplete legal enforcement.

1.3 Contracting under one-sided commitment

In this section we consider an environment in which legal enforcement of all prescribed payments is limited. We first examine the situation in which only agent 1 cannot commit to making all transfers prescribed by the contract.[4] We say that agent 1 faces self-enforcing constraints. These constraints impose that, at any point in time, agent 1 should always do as well obeying the contract as reneging on it. When the self-enforcing constraints are satisfied we say that the contract is self-enforcing.

When legal enforcement cannot provide a sufficient incentive for agent 1 to obey the contract, she must be incite to do so differently. In a long term relationship such incentive arises endogenously from the interaction of the two agents over time. One approach to study this incentive would be to model the relationship as a strategic game where each agent's strategy would be a sequence of payments for the complete history and following any history. In this case, the incentive for agent 1 to obey her equilibrium strategy would come from the anticipation of agent 2's response to a deviation. Any payment by agent 1 would therefore be enforced by the strategy of player 2. The more severe would be player 2's punishment, the higher would be cooperation between the two agents. In this case, the Folk theorem states that given a high enough discount factor any individually rational feasible allocation can be sustained in equilibrium. For our purposes, such an approach is unsatisfactory for two reasons. First, as is well known in the theory of supergames, the multiplicity of equilibria creates significant coordination problems between the two agents. Second, we are interested here in characterizing allocations for any value of the discount factor and not just allocations for high values of the discount factor.

We therefore adopt the following approach. We assume that if agent 1 reneges on the contract she suffers maximal punishment in that she must stay in autarky forever after. This punishment strategy by agent 2 allows us to characterize the best

[4]The analysis of the opposite case in which agent 2 can renege on the contract is symmetric.

possible contract satisfying self-enforcing constraints.[5] The optimal contract is then the solution to a well-defined maximization problem. This approach resolves the coordination problem in effectively coordinating the two agents on a Pareto optimal allocation. Furthermore it allows us to characterize optimal allocations for any value of the discount factor.

When agent 1 can renege on the contract at any point in time, she will make a transfer to agent 2 if and only if it is in her interest to do so. Agent 1 will compare the benefit of making the transfer and obeying the contract with the payoff of reneging on the contract and staying in autarky thereafter. For example, suppose the two agents have signed a contract δ prescribing transfers $\{B(h_{t-1}), a(h_t)\}$ for all histories h_t. In period t, agent 1 may decide to renege on the contract at date t_0 before receiving the (possibly negative) transfer B_t. Her surplus from staying in the contract is then $U(\delta; h_{t-1})$. Agent 1 may also decide to renege on the contract after the state of nature has been realized at date t_2. In this case her surplus from staying in the contract is $u(y_t^s + B_t - a_t^s) - u(y_t^s + B_t) + \beta U(\delta; h_{t-1}, s)$ where the first two terms represent her current surplus from the relationship and the last term, her discounted expected future surplus. We can now define a self-enforcing contract for agent 1.

Definition 2 *A contract δ is **self-enforcing for agent 1** if and only if, for all histories h_{t-1}, periods t, and states s, the following constraints hold.*

$$(i)\quad U(\delta; h_{t-1}) \geq 0$$

$$(ii)\quad u(y_t^s + B_t - a_t^s) - u(y_t^s + B_t) + \beta U(\delta; h_{t-1}, s) \geq 0$$

This definition states that a contract is self-enforcing for agent 1 if at any time during the relationship agent 1 prefers making the contractual transfer rather than reneging

[5]MacLeod and Malcomson (1989) model a situation similar to ours as an explicit game and show that the maximal punishment is indeed subgame perfect. Any deviating agent is punished in the future by not being able to enter a successful relationship, all parties expecting the deviating agent to deviate again in the future.

on the contract and be reduced to autarky from then on. Constraint (i) is an ex ante self-enforcing constraint in that it holds at date t_0; constraint (ii) is an ex post self-enforcing constraint in that it holds at date t_2 after the state of nature has been realized. Note that all ex post self-enforcing constraints being satisfied does not necessarily imply that the ex ante self-enforcing constraint is also. For example, if B_t is negative, the ex ante constraint may bind while ex post constraints may not once the ex ante payment B_t has been paid. It is therefore necessary to consider these two sets of constraints.

When designing the optimal contract the two agents will anticipate agent 1's incentive to renege, and the terms of the contract will take into account such incentive. To solve for the optimal contract we must therefore add self-enforcing constraints to the maximization problem (1). The optimal contract with non-commitment by agent 1, δ^1, is then the solution to the following maximization problem where, for simplicity, we assume that agent 2's reservation utility is equal to zero.

$$\begin{aligned} \delta^1 = \quad & \arg\max_{\delta} U(\delta; h_0) \\ & \text{s/t} \quad V(\delta; h_0) \geq 0 \\ & \qquad U(\delta; h_{t-1}) \geq 0 \quad \forall t, h_{t-1} \\ & \qquad u(y_t^s + B_t - a_t^s) - u(y_t^s + B_t) + \beta U(\delta; h_{t-1}, s) \geq 0 \quad \forall s, t, h_{t-1} \end{aligned} \tag{2}$$

The next proposition characterizes the optimal contract δ^1.[6]

Proposition 2 *Suppose that the maximum ex ante payment agent 1 can make is* $\underline{B}$.
(i) For all values of $\beta \in (0,1)$ *the optimal contract with non-commitment by agent 1 is the optimal full-commitment contract, that is,* $\delta^1 = \delta^{fc}$, *if and only if* $\underline{B} \geq y^S - c^{Sfc}$, *where* c^{sfc} *is the optimal consumption in state s under the full-commitment contract. Suppose that* $\underline{B} < y^S - c^{Sfc}$.
(ii) There exists a β_1 *which depends on* $\underline{B}$ *such that for* $\beta \in [\beta_1, 1)$, *the optimal contract with non-commitment by agent 1 is the optimal full-commitment contract, that*

[6] All proofs are relegated to the Appendix A.

is, $\delta^1 = \delta^{fc}$.

(iii) For all $\beta \in (0, \beta_1)$, *the following characterization forms part of an optimal contract* δ^1*:*

1. *Agent 1 makes the highest ex ante payment in every period, that is,* $B_t = -\underline{B}$ *for all periods* t*;*
2. *Agent 2's expected profit is non-increasing in time, that is,* $V(\delta^1; h_{t-1}, s) \leq V(\delta^1; h_{t-1})$ *for all histories* h_{t-1}*, time periods* t*, and states* s*;*
3. *Agent 1's implicit discount factor is no greater than agent 2's, that is,* $\beta \frac{Eu'(c_t^s)}{u'(c_{t-1})} \leq \beta \frac{Ev'(e+y_t^s-c_t^s)}{v'(e+y_{t-1}-c_{t-1})}$ *for all histories* h_{t-1} *and time periods* t *where* $c_t^s = c(h_{t-1}, s)$ *and* $c_{t-1} = c(h_{t-1})$ *and* $c(h)$ *defines the optimal consumption following history* h*.*

This proposition states that if agent 1 can make a high enough ex ante payment ($\underline{B} \geq y^S - c^{Sfc}$), then the optimal full-commitment contract can satisfy agent 1's self-enforcing constraints. A large enough ex ante payment effectively allows all ex post transfers a_t^s to be negative which in turn implies that all ex post self-enforcing constraints are satisfied.[7]

When the maximum ex ante payment agent 1 can make is not high enough, the optimal contract with full-commitment cannot be supported for all values of the discount factor. If the discount factor is high enough, that is, no lower than β_1, then agent 1's ex post self-enforcing constraints are not binding.[8] In this case, the future benefits of perfect risk-sharing exceed the short-run cost of making the prescribed transfer in any state s. Contrary to the full-commitment case however, the transfer B_t is not a matter of indifference. It will optimally be set to the maximum level agent 1 can pay. A maximal ex ante payment reduces ex post transfers a_t^s from agent 1

[7]Note that the transfer $y^S - c^{Sfc}$ is the largest transfer agent 1 makes to agent 2 in the full-commitment contract δ^{fc}.

[8]This result is akin to results in the theory of supergames where any efficient outcome of a static game can be supported as an equilibrium of its associated supergame provided that the discount factor is high enough.

to agent 2 and hence the incentive for the former to renege ex post on the contract. It therefore allows the optimal full-commitment contract δ^{fc} to be supported for the largest interval of discount factors.

When the discount factor is smaller than β_1, the optimal full-commitment contract cannot obtain if $\underline{B} < y^S - c^{Sfc}$. In this case, the transfer that agent 1 must make to agent 2 in state S is large compared to the discounted future benefits of perfect risk-sharing. Agent 1 then has the incentive to renege on the contract. The optimal contract must therefore account for this possibility. In this case, agent 1 can make the maximum ex ante payment B_t in all periods, namely, $B_t = -\underline{B}$.[9] The intuition for this result can be found by looking at the maximization problem (30). Suppose the optimal contract specifies $B_t > -\underline{B}$ and transfer payments a_t^s for some t. It is then possible to construct a contract $\hat{\delta}$ with, in period t, $\hat{B}_t = -\underline{B}$ and $\hat{a}_t^s = a_t^s - B_t - \underline{B} < a_t^s$ for all states s, and leave all other periods unchanged. These modifications leave the two agents' consumption unchanged and therefore do not change the value of agent 2's participation constraint, nor the value of ex ante self-enforcing constraints; however they do relax the ex post self-enforcing constraints of agent 1. When one of these constraints is binding, this new contract (weakly) increases the utility of agent 1.

The optimal contract also specifies that agent 2's expected profit be non-increasing in time. The optimal contract seeks two objectives: (1) to insure agent 1 against shocks to her endowment and (2) to smooth her consumption across periods. When agent 1 cannot commit to make any transfer at the end of a period, risk-sharing and intertemporal smoothing are partially impeded. These objectives can be improved upon by having agent 1 save in the early periods and good states of the world and withdraw her savings in later periods in bad states of the world. This is possible given that agent 2 can commit not to "steal" agent 1's early savings. The optimal contract therefore uses agent 2 as a savings account. The objective of this savings

[9]Although making the largest ex ante payment is not necessary for an optimal contract for all values of the discount factor, it is clearly sufficient.

account is precisely to insure future consumption against bad states of the world. This is optimally achieved by having agent 2's expected profits be non-increasing in time. This can also be seen in the fact that agent 1's implicit discount factor is not larger (and sometimes strictly smaller) than agent 2's which reflects agent 1's relative preference for the future. Self-enforcing constraints force agent 1 to save more than she would in a full-commitment environment. It is therefore as if she could earn a high interest rate on her savings (a low discount factor).

The following corollary gives a more precise characterization of the optimal consumption path.

Corollary 1 *Assume that* $\underline{B} < y^S - c^{Sfc}$ *and that* $\beta < \beta_1$.
(i) For all states s, there exists a time-invariant consumption level $\underline{c}^s$ *such that* $c_t^s \geq \underline{c}^s$ *for all time periods t.*
(ii) The lower bounds of consumption, $\underline{c}^s$*, are increasing in the state of the world, that is,* $k > q \Rightarrow \underline{c}^k > \underline{c}^q$*, and are decreasing in the maximal payment* $\underline{B}$ *that agent 1 can make.*
(iii) For any history (h_{t-2}, q, s)*, optimal consumption at time t is such that:*

$$c(h_{t-2}, q, s) = \begin{cases} \underline{c}^s & \text{if } c^*(c_{t-1}, y_{t-1}^q, s) \leq \underline{c}^s \\ c^*(c_{t-1}, y_{t-1}^q, s) & \text{otherwise} \end{cases}$$

where $c^*(c_{t-1}, y_{t-1}^q, s)$ *is implicitly defined by* $\frac{u'(c^*(c_{t-1}, y_{t-1}^q, s))}{u'(c_{t-1})} = \frac{v'(e + y_t^s - c^*(c_{t-1}, y_{t-1}^q, s))}{v'(e + y_{t-1}^q - c_{t-1})}$.

This corollary shows that, in each state, there exist time-invariant lower bounds on agent 1's consumption. These bounds are increasing with the state of the world and are decreasing in the maximum ex ante payment that agent 1 can make. Thus the higher the payment $\underline{B}$, the larger are the intervals of consumption that can be supported in each state.

If, given consumption in period $t-1$, consumption smoothing satisfies agent 1's ex post self-enforcing constraint, then consumption in period t is equal to $c^*(c_{t-1}, y_{t-1}^q, s)$.

If it does not satisfy agent 1's ex post self-enforcing constraint, then consumption in period t is equal to $\underline{c}^s$. Consumption follows a stationary first-order Markov process where period t consumption depends on period $t-1$ consumption and the realized states in periods $t-1$ and t. The dynamics of consumption also imply that there is convergence to perfect risk-sharing and consumption smoothing. In the steady state, consumption only depends on the current state. Moreover, since optimal risk-sharing at actuarially fair prices is impossible when $\beta < \beta_1$, the steady-state consumption in every state must be higher than optimal consumption in the full-commitment case. This implies that agent 1 gets, in the steady state, optimal risk-sharing at prices lower than actuarially fair prices. This is acceptable to agent 2 because he gets a compensating surplus at the beginning of the relationship in order to make zero-profit overall. In terms of the implicit discount factor the two agents face, agent 1's is lower than agent 2's until the steady state is reached, after which they become equal (as is implied by perfect risk sharing and consumption smoothing).

The results of Proposition 2 and Corollary 1 are similar to results obtained by Harris and Holmström (1982) in a model of labor contracts. They showed that under the assumption of non-commitment by the employee, wages are downward rigid as the risk-neutral employer fully insures the worker against bad states of the world. Our characterization is, first, a generalization to the case of two risk-averse agents. It shows in fact that in this case, consumption can decrease in some states. Secondly, it shows that the non-committed party (agent 1) would like to make, in each period, ex ante transfers to relax the ex post self-enforcing constraints, that is, an optimal characterization sets $B_t = -\underline{B}$.

A similar analysis could be performed for the case in which agent 2 cannot commit not to renege on the contract and similar results would obtain. Having the non-committed agent making the maximal ex ante payment relaxes its ex post self-enforcing constraints. The non-committed agent effectively uses the committed agent as a savings account. This improves risk-sharing and consumption smoothing as consumption eventually achieves perfect risk-sharing. The possibility that the committed agent has of making an ex ante payment allows to shift (some or all) the burden of ex post transfers to the committed agent. However if both agents face self-enforcing constraints, the above characterization may not be feasible. One agent may run away with the ex ante payment of the other agent as its ex post self-enforcing constraints would become too stringent. The next section studies the optimal contract when the two agents face self-enforcing constraints.

1.4 Contracting under no commitment

In this section, we relax the assumption of commitment by either agent and study the properties of the optimal contract. We have seen that when agent 1 cannot commit she makes the maximal ex ante payment at date t_0. Agent 2 does similarly when he cannot commit. This payment reduces the non-committed agent's ex post transfers at date t_2 and thus relaxes its ex post self-enforcing constraints. For constant consumption, the substitution from ex post to ex ante payments does not affect ex ante constraints since they only have to hold in expectation over the possible states of the world. However when both agents simultaneously face self-enforcing constraints, an ex ante payment that relaxes one agent's ex post self-enforcing constraints may strengthen the other agent's constraints. The optimal ex ante payment should therefore trade off between the two sets of self-enforcing constraints. This section studies the details of that trade-off.

We first define the concept of a self-enforcing contract under the non-commitment assumption.

Definition 3 *A contract δ is* ***self-enforcing*** *if and only if, for all histories h_{t-1}, periods t, and states s, the following constraints hold.*

$$(i) \quad U(\delta; h_{t-1}) \geq 0$$

$$(ii) \quad u(y_t^s + B_t - a_t^s) - u(y_t^s + B_t) + \beta U(\delta; h_{t-1}, s) \geq 0$$

$$(iii) \quad V(\delta; h_{t-1}) \geq 0$$

$$(iv) \quad v(e - B_t + a_t^s) - v(e - B_t) + \beta V(\delta; h_{t-1}, s) \geq 0$$

This definition simply states that a contract is self-enforcing if it is self-enforcing for agent 1 (constraints i and ii) as well as for agent 2 (constraints iii and iv).

Before proceeding with the analysis, we assume that there are no exogenous bounds on the ex ante payment B_t. This assumption is motivated by the fact that we want to study how self-enforcing constraints rather than some exogenous bound limit the use of the ex ante payment.

The optimal contract without commitment, δ^{nc}, is the solution to the following maximization problem.

$$\begin{aligned} \delta^{nc} = \quad & \arg\max_{\delta} U(\delta; h_0) \\ \text{s/t} \quad & U(\delta; h_t) \geq 0 \quad \forall t, h_t \\ & u(y_t^s + B_t - a_t^s) - u(y_t^s + B_t) + \beta U(\delta; h_{t-1}, s) \geq 0 \quad \forall s, t, h_{t-1} \\ & V(\delta; h_t) \geq 0 \quad \forall t, h_t \\ & v(e - B_t + a_t^s) - v(e - B_t) + \beta V(\delta; h_{t-1}, s) \geq 0 \quad \forall s, t, h_{t-1} \end{aligned} \tag{3}$$

Following any time period and any history, the optimal contract δ^{nc} will necessarily be efficient, since if it was not it would be possible to replace the non-efficient path by an efficient path thus (weakly) increasing the utility each agent derives from the contract and hence relaxing all previous self-enforcing constraints. This new contract would necessarily be self-enforcing and would dominate the old contract at the beginning of the relationship. This argument implies that the optimal contract from the start of period t onwards is the solution to the following maximization problem.

$$\begin{aligned} f(V_t) = \quad & \max_{B_t, (a_t^s)_s, (V_{t+1}^s)_s} E\left\{u(y_t^s + B_t - a_t^s) - u(y_t^s) + \beta f(V_{t+1}^s)\right\} \\ & \text{s/t} \quad f(V_{t+1}^s) \geq 0 \quad \forall s \\ & \qquad u(y_t^s + B_t - a_t^s) - u(y_t^s + B_t) + \beta f(V_{t+1}^s) \geq 0 \quad \forall s \end{aligned}$$

$$V_{t+1}^s \geq 0 \quad \forall s \tag{4}$$

$$v(e - B_t + a_t^s) - v(e - B_t) + \beta V_{t+1}^s \geq 0 \quad \forall s$$

$$V_t \leq E\left\{v(b - B_t + a_t^s) - v(b) + \beta V_{t+1}^s\right\}$$

where $f(V_t)$ represents the Pareto frontier that can be attained through a self-enforcing contract after an arbitrary history h_{t-1}. Denote by $\Lambda(h_{t-1})$ the set of contracts satisfying the self-enforcing constraints following the history h_{t-1}. The Pareto frontier is then given by a time-independent function

$$f(V_t) = \max_{\delta \in \Lambda(h_{t-1})} \{U(\delta; h_{t-1}) \text{ s/t } V(\delta; h_{t-1}) \geq V_t\}.$$

In problem (4), the variable V_{t+1}^s is to be interpreted as $V(\delta; h_{t-1}, s)$, that is, agent 2's expected surplus from period $t+1$ onwards when contract δ is signed and s is the realized state of nature in period t. The first two constraints represent agent 1's ex ante and ex post self-enforcing constraints respectively. The next two constraints represent agent 2's self-enforcing constraints. The last constraint of the problem ensures that the contract is dynamically consistent. Before characterizing the properties of the optimal contract we need some technical results.

Lemma 1 *(i) The set of self-enforcing contracts following history h_{t-1}, $\Lambda(h_{t-1})$, is convex.*

(ii) The set of values of V_t for which a self-enforcing contract exists is a compact interval $[0, \bar{V}]$.

(iii) The Pareto frontier $f(V_t)$ is decreasing, strictly concave, and continuously differentiable almost everywhere on $(0, \bar{V})$.

(iv) For each value of $V_t \in [0, \bar{V}]$ there exists a unique continuation of the contract δ at time t in which $V(\delta; h_{t-1}) = V_t$ and $U(\delta; h_{t-1}) = f(V_t)$.

To get a better understanding of the role of the ex ante payment in this contracting problem, we will first state the solution to this problem assuming that no ex ante payments are allowed.[10]

Proposition 3 *Assume that $B_t = 0$ for all time periods t.*

(i) For all states s, there exists optimal time-invariant consumption levels $\underline{c}^s$ and $\bar{c}^s$ such that $\underline{c}^s \leq c_t^s \leq \bar{c}^s$ for all time periods t.

(ii) The optimal lower bounds $\underline{c}^s$ and upper bounds $\bar{c}^s$ are increasing with the states of the world, that is, $k > q \Rightarrow \underline{c}^k > \underline{c}^q$ and $\bar{c}^k > \bar{c}^q$.

(iii) For any history (h_{t-2}, q, s), optimal consumption at time t is such that:

$$c(h_{t-2}, q, s) = \begin{cases} \underline{c}^s & \text{if } c^*(c_{t-1}, y_{t-1}^q, s) \leq \underline{c}^s \\ c^*(c_{t-1}, y_{t-1}^q, s) & \text{if } \underline{c}^s < c^*(c_{t-1}, y_{t-1}^q, s) < \bar{c}^s \\ \bar{c}^s & \text{otherwise} \end{cases}$$

(iv) There are no values of β such that the contract with non-commitment, δ^{nc}, is the optimal contract with full commitment, δ^{fc}.

When no ex ante payments are allowed, there are upper and lower bounds on optimal consumption. These bounds are determined by the two agents' ex post self-enforcing constraints. Agent 1's consumption follows a simple stationary first-order Markov process. In period t, consumption depends on period $t-1$ consumption and the states of the world realized in periods $t-1$ and t. This implies that the consumption of the two agents is smoothed as much as possible subject to ex post self-enforcing constraints.

We now characterize the optimal solution when the ex ante payment is chosen optimally. An implication of Lemma 1 is that problem (4) is a concave program and therefore first-order conditions are both necessary and sufficient for a solution. Denote respectively by $\beta p^s \alpha^s$, $p^s \theta^s$, $\beta p^s \phi^s$, $p^s \lambda^s$, and ψ the multipliers of the five

[10]This corresponds to the generalization to the case of bilateral risk aversion of Thomas and Worrall's (1988).

constraints in problem (4). The first-order conditions are

$$
\begin{aligned}
B_t: \quad & \sum_s p^s u'(y_t^s + B_t - a_t^s) + \sum_s p^s \theta^s \left(u'(y_t^s + B_t - a_t^s) - u'(y_t^s + B_t)\right) \\
& - \sum_s p^s (\lambda^s + \psi) v'(e - B_t + a_t^s) + \sum_s p^s \lambda^s v'(e - B_t) = 0 \qquad (5) \\
a_t^s: \quad & -p^s(1+\theta^s) u'(y_t^s + B_t - a_t^s) + p^s(\lambda^s + \psi) v'(e - B_t + a_t^s) = 0 \quad \forall s \qquad (6) \\
V_{t+1}^s: \quad & (1 + \alpha^s + \theta^s) f'(V_{t+1}^s) + \lambda^s + \phi^s + \psi = 0 \quad \forall s \qquad (7)
\end{aligned}
$$

and the envelope condition is $f'(V_t) = -\psi$. Lemma 2 provides some basic properties of the solution.

Lemma 2 *(i) There exists a β_{nc} such that the optimal non-commitment contract, δ^{nc}, yields the same consumption as the optimal full-commitment contract, δ^{fc}, if and only if $\beta \in [\beta_{nc}, 1)$.*
Suppose that $\beta < \beta_{nc}$.
(ii) For $i = 1, 2$, there exists a state s_i in which agent i's ex post self-enforcing constraint is binding.

This lemma shows that if the discount factor is high enough, the optimal full-commitment contract is feasible with non-commitment and is therefore optimal. Agent 2 pays up front a high enough payment ($B_t = c^{1fc} - y^1$) such that the resulting ex post payments, a_t^s, are all positive. These payments yield zero expected utility to agent 2 in every period and therefore his self-enforcing constraints are all satisfied. If the discount factor is high enough, agent 1 prefers to make the ex post payments in all states of nature and be optimally insured in the future rather than keep the up front payment, renege on the contract, and revert to autarky thereafter. The critical discount factor β_{nc} is defined as the lowest discount factor such that agent 1 does not renege on the contract in all states of nature. This contrasts with the case $B_t = 0$ where optimal risk-sharing is not feasible with non-commitment for any value of the discount factor. The intuition for the difference in these two results is the following. Optimal risk-sharing yields zero expected utility to agent 2 in every period. Ex post self-enforcing constraints then hold if and only if $a_t^s \geq 0$ for all states s. But this is incompatible with optimal risk-sharing and hence there is no value of the discount factor such that optimal risk-sharing is feasible. Lemma 2 provides a first indication that the use of ex ante payments can strictly improve the utility of the two agents (at least for some values of the discount factor).

When the discount factor is such that optimal risk-sharing is not feasible, at least one agent is constrained by its ex post self-enforcing constraints. The second result of Lemma 2 states that each agent always has at least one ex post self-enforcing constraint binding. The intuition for this result is the following. Suppose only one agent is constrained. This implies that the constrained agent could increase marginally its up front payment, adjust its ex post payments to maintain its level of consumption, and hence relax its ex post self-enforcing constraints. At the margin, this would not violate the other agent's ex post self-enforcing constraints which were not binding before the increase in the ex ante payment. This would therefore increase the utility of a least one agent. Such increase in the ex ante payment by one agent is possible until one of the other agent's self-enforcing constraint becomes binding, in which case further increases may not be self-enforcing anymore. Therefore, in the optimal contract each agent always has at least one ex post self-enforcing constraint binding.

The next proposition provides a characterization of the optimal ex ante payment when the discount factor does not allow optimal risk-sharing.

Proposition 4 *Assume that $\beta < \beta_{nc}$.*
(i) The optimal value of the ex ante payment in period t is strictly decreasing in the expected surplus that agent 1 has to concede to agent 2 in period t, that is, $V_t' > V_t'' \Rightarrow B_t' < B_t''$ where B_t' (B_t'') is optimal if the expected surplus in period t is V_t' (V_t'').
(ii) The optimal ex ante payment is strictly positive when agent 2 has a zero expected surplus and negative when agent 2 has maximal expected surplus, that is, $B_t > 0$ if $V_t = 0$ and $B_t < 0$ if $V_t = \bar{V}$.

This result states that the ex ante payment is used optimally to relax ex post self-enforcing constraints. Furthermore, the optimal ex ante payment is decreasing in the expected surplus of agent 2. Suppose that, following a given history, the contract promises a low expected surplus to agent 2. This makes the contract not much more profitable than autarky and thus agent 2's ex post self-enforcing constraints are likely to be more constraining than agent 1's ex post self-enforcing constraints. In this case, agent 2 optimally pays out a relatively large ex ante payment to relax his ex post self-enforcing constraints. The size of the optimal ex ante payment is inversely related to the expected surplus of agent 2. This logic can easily be extended to show that the optimal ex ante payment is negative when agent 2 expects a high surplus from the relationship, that is, agent 1 pays out to agent 2 a high ex ante transfer.

It is difficult to provide a more complete characterization of the solution in the general case given the number of inequality constraints; however, we can do so in a special case in which there are only two states. This simple example is sufficient to illustrate the role of the ex ante payment. We then compare our results with the case in which no ex ante payments are allowed. Suppose that $S = 2$ and assume that the discount factor is such that optimal risk-sharing is not self-enforcing.

Proposition 5 *Suppose that* $S = 2$ *and* $\beta < \beta^{nc}$.
(i) The expected profit of agent 2 for period $t+1$ *is larger (smaller) than that of period* t *if state 1 (2) occurs in period* t*, that is,* $V(\delta^{nc}; h_{t-1}, 2) \leq V(\delta^{nc}; h_{t-1}) \leq V(\delta^{nc}; h_{t-1}, 1)$*, with strict inequality if* $V(\delta^{nc}; h_{t-1}) \notin \left\{0, \bar{V}\right\}$.
Suppose that $0 < V_t < \bar{V}$ *and define* $c_t^s = c(h_{t-1}, s)$ *and* $c_{t-1} = c(h_{t-1})$.
(ii) Agent 1's intertemporal marginal rate of substitution is larger (smaller) than agent 2's if state 1 (2) occurs, that is, $\frac{u'(c_t^1)}{u'(c_{t-1})} > \frac{v'(e+y_t^1-c_t^1)}{v'(e+y_{t-1}-c_{t-1})}$ *and* $\frac{u'(c_t^2)}{u'(c_{t-1})} < \frac{v'(e+y_t^2-c_t^2)}{v'(e+y_{t-1}-c_{t-1})}$.
(iii) Agent 1's consumption in period t *is smaller (larger) than* $c^*(c_{t-1}, y_{t-1}^q, s)$ *if* $s = 1$ *(if* $s = 2$*) in period* t *where* q *is the realized state in period* $t-1$*, that is,* $c_t^1 < c^*(c_{t-1}, y_{t-1}^q, 1)$ *and* $c_t^2 > c^*(c_{t-1}, y_{t-1}^q, 2)$.

This proposition states that in a two-state example if the bad state occurs (state 1) agent 1 borrows from agent 2 to smooth her consumption, that is, $V_{t+1}^1 \geq V_t$. Alternatively, agent 1 lends (or reimburses) to agent 2 if state 2 occurs, that is $V_{t+1}^2 \leq V_t$. This allows the best consumption smoothing possible for agent 1. This same result can be translated in terms of each agent's intertemporal marginal rate of substitution. In state 1, self-enforcing constraints are such that agent 1 has stronger preferences than agent 2 for the present and she therefore borrows from agent 2. The opposite is true in state 2 and agent 2 lends to agent 1. The last result of the proposition states that in general the contract cannot achieve optimal risk-sharing because of self-enforcing constraints. Consequently the two agents bear more risk than they do in the full-commitment case.

These results may seem quite similar to those one would obtain when no ex ante payments are allowed. The dynamics of consumption is however quite different in the two cases. Consider first the case where no ex ante payments are allowed. The results of Proposition 3 imply that optimal consumption takes place at $\bar{c}^1$ ($\underline{c}^2$) if state 1 (2) occurs. These consumption levels are time-invariant and therefore consumption can only take one of these two values depending on the realized state. At any given period, expected consumption for next period is the same regardless of the history. Now consider the case where ex ante payments are allowed. Proposition 5 states that $V_{t+1}^2 \leq V_t \leq V_{t+1}^1$. Proposition 4 states that the ex ante payment from agent 1 to agent 2 will decrease (increase) in period $t+1$ compared to that of period t if state 1 (2) occurs. Since the optimal consumption bounds $\bar{c}^1$ and $\underline{c}^2$ are increasing with the ex ante payment,[11] then if state 1 occurs in period t, agent 2 is promised a higher expected surplus for period $t+1$ and this implies that he will make a lower ex ante payment in period $t+1$ thus reducing the two consumption bounds for that period. Expected consumption will then be lower in period $t+1$ than in period t. The opposite holds if state 2 occurs in period t. Expected consumption

[11] This is a straight extension of Proposition 3 and Corollary 1.

rises in period $t+1$ compared to period t. The consumption pattern with an ex ante payment looks like "experience rating", that is, average consumption in one period is inversely related to the state variable and thus is positively related to the previous realization of the state of nature. This implies that the complete history up to period t may be potentially relevant in explaining period t consumption. Our model can therefore generate higher order correlation in consumption even though endowments are independently distributed. In fact, the introduction of the ex ante payment allows better risk-sharing across states within a period at the expense of worse consumption smoothing across periods. This shows that neither asymmetric information nor uncertainty and learning are necessary to explain experience rating in insurance contracts. This simple example shows that allowing for an ex ante payment yields predictions that are significantly different from those without ex ante payments.

1.5 Conclusion

We develop a model of contracting for risk-sharing purposes. Complete insurance is impeded by ex post opportunism in that agents can break the relationship at any time if it is in their own interest to do so. However, agents can commit partially by making payments at the beginning of a period before the state of nature is realized. We show that these payments can increase the potential gains from trade but cannot generally restore perfect risk-sharing.

The ex ante payment can be interpreted as a bond. It is well known that posting a bond is one way of avoiding self-enforcing constraints, namely, the agent that cannot commit or that must be disciplined simply posts a bond that it may lose if it does not perform satisfactorily. For example, Williamson (1983) illustrates how the use of a bond can promote efficient trade. In this context, our model can be reinterpreted as modeling a situation in which the two agents must be disciplined and each agent can run away with the bond that has been posted by the other agent. This puts endogenous limits to the size of the bond that each agent can post and therefore bonding becomes a non-trivial solution to the incentive problem. Our model would therefore predict that the net bond posted by one agent is inversely proportional to its expected surplus from the relationship.

2 Approximation of the value function in self-enforcing risk-sharing contract and simulations

2.1 Introduction

The goal of this paper is to simulate the optimal risk-sharing contract in a non-commitment environment when more than two states of the world are possible[12]. To this end, the first and most complicated step is to get a good approximation of the optimal contract. Judd (1992) describes a general numerical approach, the projection method, to solve operator equations which arises in economic models. Principles from numerical analysis are then used to develop efficient implementations of the projection method for solving aggregate growth models. In the present paper, we will use one of these implementations called orthogonal collocation, which has turned out to be surprisingly good. This method is an application of the projection theorem in which the orthogonal basis is composed of Chebyshev polynomials evaluated at specific points. We will use orthogonal collocation to approximate the value function associated with the optimal self-enforcing risk-sharing contract with bonding studied in Gauthier and Poitevin (1994). This environment is described in Section 1. In Section 2 we present an overview of the projection theorem. Section 3 introduces polynomial approximation. Section 4 presents Chebyshev polynomials. Finally, we describe the orthogonal collocation's method and the approximation's algorithm in Section 5, and presents the simulation's results obtained with the approximated value function.

2.2 The contractual environment

The optimal contract in the general case in which bonding is possible is the solution to the following maximisation problem

$$g(V^t) = \max_{B_t,(a^{s_t})_s,(V^{s_{t+1}})_s} E\left\{u(y^{s_t} + B_t - a^{s_t}) - u(y^{s_t}) + \beta g(V^{s_{t+1}})\right\}$$

[12]The reader is refered to Gauthier and Poitevin (1994) for the theoritical characterization of the optimal contract. Some of the results are proved in a two-states world because of the complexity of the problem.

$$
\begin{aligned}
& g(V^{s_{t+1}}) \geq 0 \quad \forall s \\
& u(y^{s_t} + B_t - a^{s_t}) - u(y^{s_t} + B_t) + \beta g(V^{s_{t+1}}) \geq 0 \quad \forall s \qquad (8) \\
& V^{s_{t+1}} \geq 0 \quad \forall s \\
& a^{s_t} + \beta V^{s_{t+1}} \geq 0 \quad \forall s \\
& V^t \leq E\left\{-B_t + a^{s_t} + \beta V^{s_{t+1}}\right\}
\end{aligned}
$$

The optimal contract is thus the sequences of ex ante payments, (B_t), ex post payments, (a^{s_t}), and promised expected surplus to the risk-neutral agent, $(V^{s_{t+1}})$, which maximize the expected surplus of the risk-averse agent[13]. For the contract to be self-enforcing, it must be in the interest of both agents to comply to the contract. We must therefore add the self-enforcing constraints. The last constraint of the problem ensures that the contract is dynamically consistent.

The consumption and ex ante payment paths has been characterized in Gauthier and Poitevin (1994) in a two-states world. With a good approximation of $g(V^t)$, it will become possible to verify numerically those results when the number of possible states of nature is higher than two. Judd (1992) describes a general method to obtain a good approximation. This method relies on the projection theorem. We will thus present first, this important theorem.

2.3 The projection theorem

The approximation method used in this paper relies on the projection theorem. This theorem tells us that the best approximation[14], $P_\mu x$, of an element, x, in a specific space, μ, (generated by all the linear combinations of the elements of the basis) is such that the distance between x and $P_\mu x$, that is the approximation error, is orthogonal to μ. We now give the formal result and suggest Brockwell and Davis (1991) as reference

[13]The reader is refered to Gauthier and Poitevin (1994) for a detailed description of this problem.

[14]The best approximation is defined as the one which minimizes the approximation error.

to the reader interested in a very well done presentation of Hilbert spaces and proof of the theorem.

Theorem 1 *If μ is the span of $\{x_1, \ldots, x_n\}$, then for any given $x \in$ a Hilbert Space, $P_\mu x$ is the unique element of the form*

$$P_\mu x = \alpha_1 x_1 + \ldots + \alpha_n x_n, \tag{9}$$

such that

$$< x - P_\mu x, y >= 0, \quad y \in \mu. \tag{10}$$

To get an intuition of what this theorem says, let the basis be composed of two independent elements of $\Re^2$. Hence, μ is $\Re^2$. Suppose now that we want to approximate an element $x \in \Re^3$, by a projection in $\Re^2$, that is, we search $P_\mu x$. The best approximation is defined as the one which minimizes the distance $||x - P_\mu x||$. Equation (10) means that $x - P_\mu x$ must be orthogonal to every element in μ. In our case, it is clear that the shortest way from x to μ is the vector orthogonal to μ. Any vector which is not orthogonal to μ is clearly longer and thus, can not be the one which minimizes the distance between x and μ. It is also clear in this example, that there is only one vector from x to μ which is orthogonal to the plane μ. This gives an insight for the unicity's proof.

Equation (10) implies

$$< P_\mu x, x_j >=< x, x_j >, \quad j = 1, \ldots, n \tag{11}$$

where y can be replaced by $x_j \in \mu$ since orthogonality to a linear combination of elements implies orthogonality to each element in the linear combination. This implies that

$$\sum_{i=1}^{n} \alpha_i < x_i, x_j >=< x, x_j >, \quad j = 1, \ldots, n. \tag{12}$$

This theorem tells us that we need to solve a system of n equations in n unknows, the α_i's, in order to obtain the best projection of x in μ. The system to be solved is as big as the number of elements in the basis. We can simplify greatly by choosing an orthonormal basis, which is composed of elements with inner product equal to 1. In fact, if the x_i are orthonormal, then (12) becomes $\alpha_i =< x, x_j >$. This implies that the projection is represented by

$$P_\mu x = \sum_{i=1}^{n} < x, x_i > x_i. \tag{13}$$

Thus, with an orthonormal basis, the projection equations are simpler. Orthogonal collocation will use an orthogonal basis.

What is the best space in which the projection should be done? We will use the power of the famous Weierstrass theorem, which is presented in the next section, to study this question.

2.4 An introduction to approximation of functions

We will now introduce some basic notions in approximation theory. First the Weierstras theorem is presented. Then we present a second theorem which gives some intuition of the idea behind orthogonal collocation[15].

Theorem 2 *If V is a normed linear space and W a finite dimensional subspace of V, then, given $v \in V$, there exists $w^* \in W$ such that*

$$||v - w^*|| \leq ||v - w|| \quad \forall w \in W. \tag{14}$$

As an example, the set of continuous functions on a given closed interval $[a, b]$, which we denote $C[a, b]$, is a linear space. If $f \in C[a, b]$, we can define a norm in $C[a, b]$ by

$$||f|| = \max_{a \leq x \leq b} |f(x)| \tag{15}$$

This norm is called the uniform or Chebyshev norm. As an example of Theorem 2 take V to be $C[a, b]$ and W to be spanned by the functions $\{1, x, \ldots, x^n\}$. That is, W

[15]The interested reader is refered to Rivlin (1965) for the proofs of these theorems.

consists of all polynomials of degree at most n. We call this particular subspace P^n. Theorem 2 says that every continuous function, $f(x)$, on $[a,b]$ has a best approximation out of the polynomials of degree at most n in the uniform norm. That is, given $f \in C[a,b]$, there exists $p^* \in P^n$ such that

$$\max_{a \leq x \leq b} |f(x) - p^*(x)| \leq \max_{a \leq x \leq b} |f(x) - p(x)| \quad \forall p \in P^n. \tag{16}$$

Define $E_n(f;[a,b]) = E_n(f) = ||f - p_n^*||$. What is the behaviour of $E_n(f)$ as $n \to \infty$? We can show that $E_n(f) \to 0$ as $n \to \infty$ for each function $f(x)$ continuous on $[a,b]$. That is, a continuous function on a finite interval can be approximated uniformly within any preassigned error by polynomials. This result is the famous Weierstrass approximation theorem, which we state as follows.

Theorem 3 *(Weierstrass) Given $f(x)$ continuous on $[a,b]$ and $\epsilon > 0$, there exists a polynomial, $p(x)$, such that*

$$||f(x) - p(x)|| < \epsilon. \tag{17}$$

How can be characterized the best approximation p^*? Let $e(x) = f(x) - p^*(x)$; then $||e(x)|| = E_n(f;[a,b])$. The next theorem will give us an important property of the best approximation p^*. We will also present the proof of this theorem which yields some intuition about orthogonal collocation.

Theorem 4 *There exists (at least) two distincts points $x_1, x_2 \in [a,b]$ such that $|e(x_1)| = |e(x_2)| = E_n(f;[a,b])$ and $e(x_1) = -e(x_2)$.*

proof (Rivlin (1965))

The continuous curve $y = e(x)$ is constrained to lie between the lines $y = -E_n(f)$ and $y = E_n(f)$ for $a \leq x \leq b$ and touches at least one of these lines. We wish to prove that it must touch both of them. If it does not, then a better approximation to f than p^* exists. For assume that $e(x) > -E_n(f)$ throughout $[a,b]$; then $\min_{a \leq x \leq b} e(x) = m \geq -E_n(f)$ and $c = \frac{E_n(f)+m}{2} > 0$. Since $q_n = p_n^* + c \in P^n$, $f(x) - q(x) = e(x) - c$, and $-(E_n(f)-c) = m-c \leq e(x)-c \leq E_n(f)-c$; we thus have $||f-q_n|| = E_n(f)-c$, contradicting the definition of $E_n(f)$. Thus, there exists a point of $[a,b]$, call it x_1, such that $e(x_1) = -E_n(f)$. A similar argument establishes the existence of $x_2 \in [a,b]$ such that $e(x_2) = E_n(f)$, and the theorem is proved. *Q.E.D.*

This Theorem is just a foreshadowing of the true state of affairs, however. As can be shown, the curve $y = e(x)$ must touch the lines $y = -E_n(f)$ and $y = E_n(f)$ alternatively at least $n+2$ times, and this property characterizes the best uniform approximation of a continuous function by a polynomial of degree at most n. This suggests the existence of at least $n+1$ points where $e(x) = 0$. We will use this result in the method of orthogonal collocation which is the described in the next section.

2.5 Chebyshev Polynomials

Because of their usefullness, we will review the key properties of Chebyshev polynomials. They are defined over $[-1,1]$ by the formula $T_n(x) \equiv cos(n\, arccos(x))$. They are generated by the recursive scheme $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$, which is initialized by $T_0(x) = 1$ and $T_1(x) = x$. The restriction to $[-1,1]$ is inessential since one can define Chebyshev polynomials over any bounded interval by a linear transformation. The ability of Chebyshev polynomials to approximate smooth functions is summarized in the following properties[16].

The Chebyshev polynomials obey the discrete orthogonality relationship

$$\sum_{l=1}^{n} T_i(Z_l^n) T_j(Z_l^n) = 0 \quad i \neq j \tag{18}$$

where we define the Z_l^n, $l = 1, \ldots, n$, to be the zeroes of T_n, given by

$$Z_l^n \equiv cos(\frac{(2l-1)\pi}{2n}), \quad l = 1, \ldots, n. \tag{19}$$

Thus, the Chebyshev polynomials are orthogonals when evaluated at the zeroes of

[16]See Judd (1992) for more details and Rivlin (1991) for proofs.

T_n. The following result characterizes a polynomial of degree $n-1$ which interpolates $f(x)$ at the Z_l^n.

Suppose $f(x)$ is a continuous function. Define

$$c_j \equiv \frac{2}{n}\sum_{l=1}^{n} f(Z_l^n)T_j(Z_l^n), \quad j = 0, 1, \ldots, n-1. \tag{20}$$

Then the function

$$I_{n-1}^f(x) = -\frac{1}{2}c_0 + \sum_{k=1}^{n-1} c_k T_k(x) \tag{21}$$

agrees with $f(x)$ on the Z_l^n[17], thereby being a polynomial of degree $n-1$ which interpolates $f(x)$ at the Z_l^n. We call I_{n-1}^f the degree $n-1$ Chebyshev interpolant of f. In this way we can approximate a function by interpolation at the Z_l^n. One can immediately remark the analogy between the c_j's defined here and the α's generated by the projection theorem when the basis is orthogonal. We will now present the collocation method and the approximation's algorithm.

2.6 The collocation method and the approximation's algorithm.

Let define

$$\epsilon_n(f(V)) \equiv f(V) - \sum_{i=0}^{n} \alpha_i T_i(V).$$

We know from the Chebyshev interpolation result that $\epsilon_n(f(Z_l^{n+1})) = 0 \quad \forall l = 1, \ldots, n+1$. Solving this system of $n+1$ equations for α_i, $i = 0, \ldots n$ is called the orthogonal collocation method. As long as $\epsilon(V;\alpha)$ is smooth in V, the Chebyshev interpolation result says that these zero conditions force $\epsilon(f(V))$ to be close to zero for all V and that these are the best possible points to use if we want to force $\epsilon(f(V))$ to be close to zero. The performance of orthogonal collocation turns out to be very good. This method is very fast since it only uses the value of $\epsilon(f(V))$ at n

[17]This is that agreement with $f(x)$ on the Z_l^n which leads directly to what is called orthogonal collocation.

points. We now have all the necessary tools to construct the algorithm which will give us our approximated value function.

We want to approximate the value function $f(V)$, described at the begining of the present paper. The constructed algorithm can be divided into three parts:

1. Find good starting values for $f_0(V) \equiv \sum_{i=1}^{n} \alpha_i T_i(V)$.
2. Solve[18]

$$\max_{B_t,(a^{s_t})_s,(V^{s_{t+1}})_s} E\left\{u(y^{s_t} + B_t - a^{s_t}) - u(y^{s_t}) + \beta \sum_{i=1}^{n} \alpha_i T_i(V^{s_{t+1}})\right\}$$

$$\sum_{i=1}^{n} \alpha_i T_i(V^{s_{t+1}}) \geq 0 \quad \forall s$$

$$u(y^{s_t} + B_t - a^{s_t}) - u(y^{s_t} + B_t) + \beta \sum_{i=1}^{n} \alpha_i T_i(V^{s_{t+1}}) \geq 0 \quad \forall s$$

$$V^{s_{t+1}} \geq 0 \quad \forall s$$

$$a^{s_t} + \beta V^{s_{t+1}} \geq 0 \quad \forall s$$

$$V^t \leq E\left\{-B_t + a_t^s + \beta V_{t+1}^s\right\}$$

   for $V^t = Z_l^{n+1}$, $l = 1, \dots n+1$ where Z^{n+1} are the zeroes of the degree $n+1$ Chebyshev polynomial and n is the chosen degree of approximation[19].

3. Apply the orthogonal collocation method to interpolate the new value function $f_1(V^t)$ at the zeroes evaluated in step 1. Define this new value function as $f_1(V^t) \equiv \sum_{i=1}^{n} \hat{\alpha}_i T_i(V^t)$.
4. Substitute $f_1(V^t)$ obtained in step 2 for $f_0(V^t)$ and repeat step 1 if f_1 and f_0 are not close enough. In our application, f_1 and f_0 are defined close enough when $\sum_{i=1}^{n} |\hat{\alpha}_i - \alpha_i| \leq 10e-4$.

[18]This is done using the Optimization toolbox of MATLAB.

[19]We obtained a very good approximation with $n = 2$.

2.7 Simulations

The characterization of the optimal paths of ex ante payments, B_t, consumptions, c^{s_t}, and expected profits of agent 2, $V^{s_{t+1}}$ of Problem (8) is done in Gauthier and Poitevin (1994). However, given the increasing number of inequality constraints as the number of states increases, some results has been demonstrated in a two-states world. In this section, we look at the results of a simulation in a three-states world, step by step, and note the presence of a surprising point in the optimal paths which may be the source of a future new theoretical result.

We did simulations of Problem (8) in a three-states world. We first have calculated an approximation of the value function with the orthogonal collocation algorithm described in Section 5. The discount rate in the simulated model is $\beta = .85$; the three possible states of nature are $y = [3, 4, 5]$; each state has a probability of realization of 1/3, and the quadratic utility function of agent 1 is $-2 * c^2 + 25 * c + 4$. We used the value function of the full-insurance contract as the starting function for $f_0(V)$. We then did simulations with the approximated value function.

Let's now have a look at the results of our simulations, step by step, for a sequence of 20 periods. The starting value for the expected surplus of the risk neutral agent (agent 1) is $V_0 = .8$. Since the value for which agent 2's ex ante constraint is binding is 0.8754, agent 2's ex post self-enforcing contraints are relatively binding. Agent 2 is thus willing to pay an amount of money ex ante to agent 1 in order to relax his ex post self-enforcing constraints. Graphic 1 illustrates this transfer from agent 2 to agent 1 which is precisely $B = -.24$. The first random revenue generated has been $y_0 = 5$. This is the best state of the world, and agent 2 should be expected to pay back some of it's debt to agent 1. This is the case as can be seen in Graphic 3 in which the first element of agent 2's optimal path of consumption is $c_0 = 3.96$ which is much lower than his revenue. This is in fact the optimal lower bound of consumption in that state given the expected surplus of agent 1. What is paid now won't have to be paid tomorrow. Thus, this ex post transfer from agent 2 decreases the expected future surplus of agent 1 (This is illustraded in Graphic 1 where $V_1 = .15$). This in turn will increase the ex ante payment at the beginning of period 1 relatively to the ex ante payment done at the beginning of period 0. Graphic 1 shows this increase in B which is now positive to reflect the fact that agent 1's expected surplus is now close to it's self-enforcing limit of 0. Agent 1 is thus interested in making a transfer to agent 2 in order to relax his ex post self-enforcing constraints.

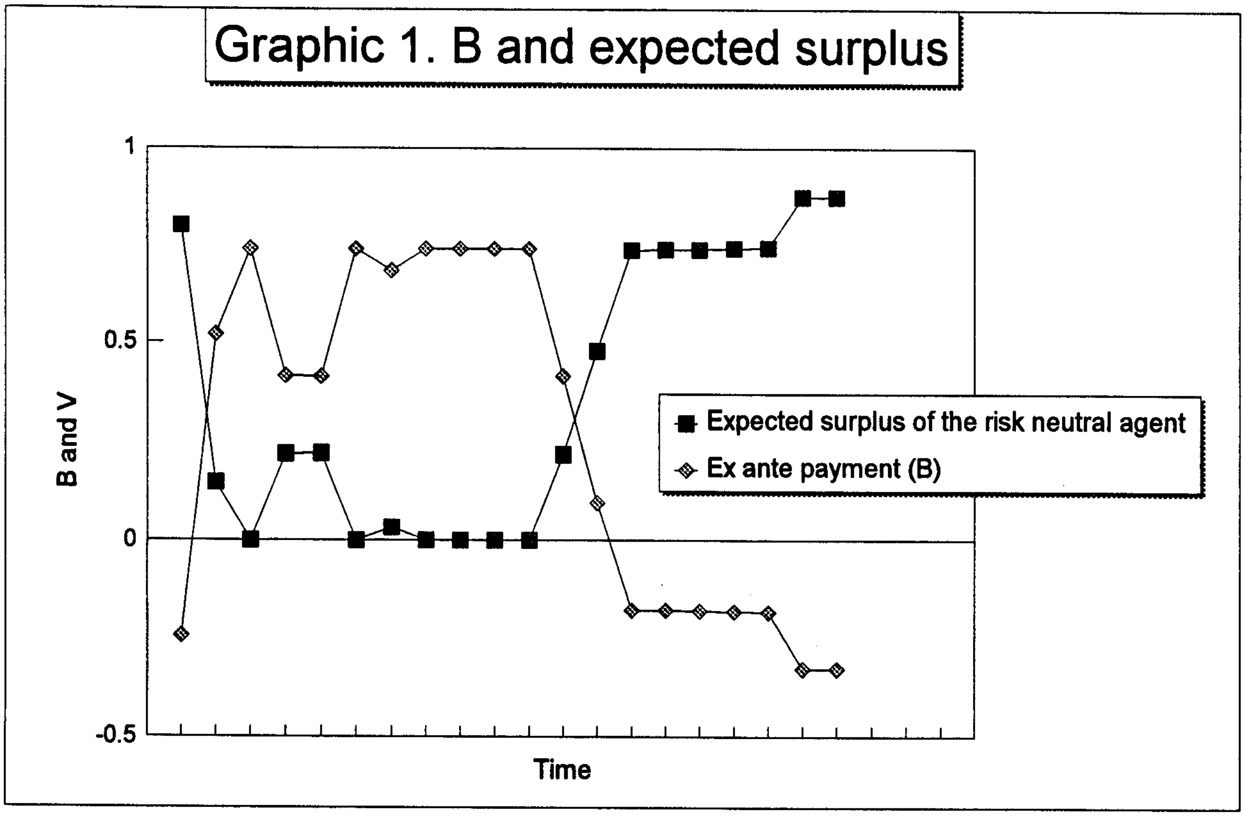
The next random revenue is $y_1 = 5$. We have the same optimal reactions in that agent 2's consumption is under his revenu and agent 1's expected surplus is depressed. This is illustrated in Graphic 1 and Graphic 3 by $\underline{c}_1^5 = 4.22$ and $V_2^5 = 0$. Note that the lower bound of consumption is higher than it was in the last period. This is the result of the positive ex ante payment which has relaxed agent 1's ex post constraint but strengthened those of agent 2. Agent 1's lower level of expected surplus will induce a higher level of ex ante payment. This is reflected by $B_1 = .5201$ and $B_2 = .7386$.

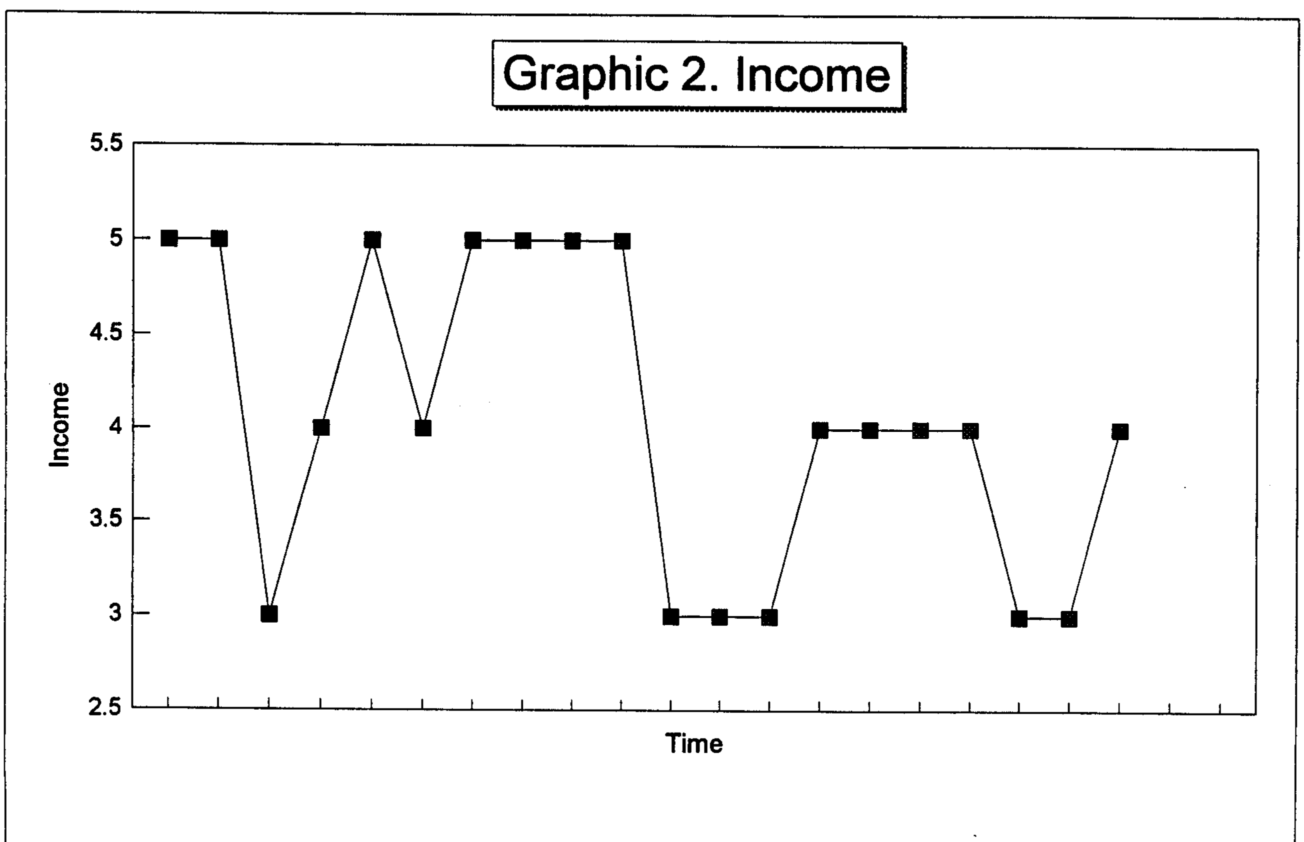
The third random revenue is 3. This is the worst possible revenu in our world. The optimal transfer is such that agent 2's consumption is higher than the revenue ($\bar{c}_2^3 = 3.93$ in Graphic 3), and agent 1's expected surplus is higher ($V_3 > 0$ in Graphic 1). This induce a fall in the ex ante payment (from $B = .7386$ to $B = .4161$). The next generated revenu is $y = 4$. Since no ex post constraints are binding, we have perfect smoothing from last period, that is neither agent 2's consumption or agent 1's expected surplus changes, and $\Delta B = 0$. The fifth revenue is $y = 5$. The expected surplus of the risk neutral agent goes down to zero and agent 2's consumption improves to 4.1804 as predicted by the theoretical results.

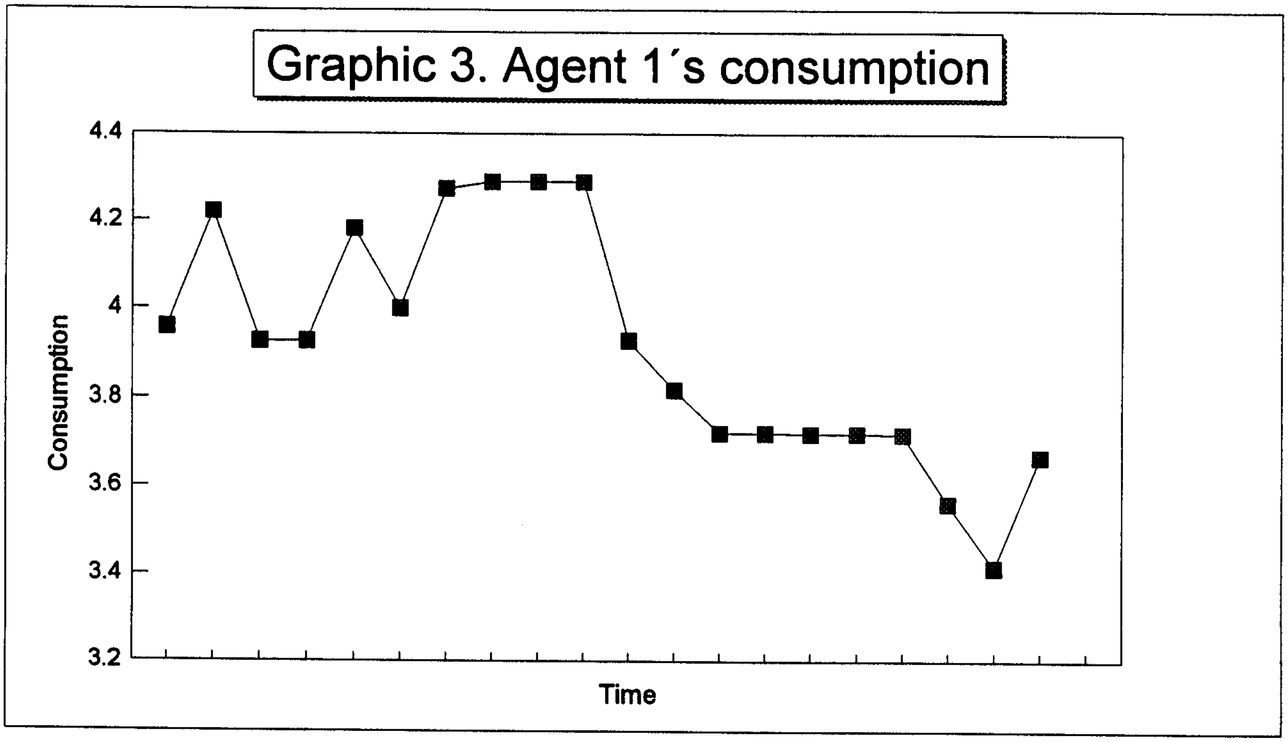
Then we obtain a curious result. This may be an error due to the approximation of the value function or this may be a good result which has not been demonstrated yet in our theoretical work. The generated revenu has been $y = 4$. The optimal level of agent 2's consumption is lower than it was in the previous good state of the world. This would not be curious if agent 2's ex post constraint was binding, but this is not the case. More work should be done to understand this result. But we already know that this is not at odd with the first-order conditions. This should help to improve our understanding of the optimal paths in a world with more than two states. The next four generated revenues has been $y = 5$. As expected, agent 2's consumption is increasing as much as possible, that is up to the point where agent 1's expected surplus can no more decrease. The next three revenues are $y = 3$. We observe successive improvements in the expected surplus of agent 1 and the corresponding successive drops in the level of agent 2's optimal consumption. Note that agent 1's ex post self-enforcing constraints are always binding when $y = 3$. The optimal maximal bounds of consumption are decreasing here, since the successive improvements in agent 1's expected surplus has induced successive drops in the ex ante payment. A lower ex ante payment means a lower bound of consumption. Note that in a contract without ex ante payment, those bounds would not have changed and thus the consumption and agent 1's expected surplus would have been constant. The remaining of the simulation is exactly in line with the anticipated results as described in the theoretical characterization.

2.8 conclusion

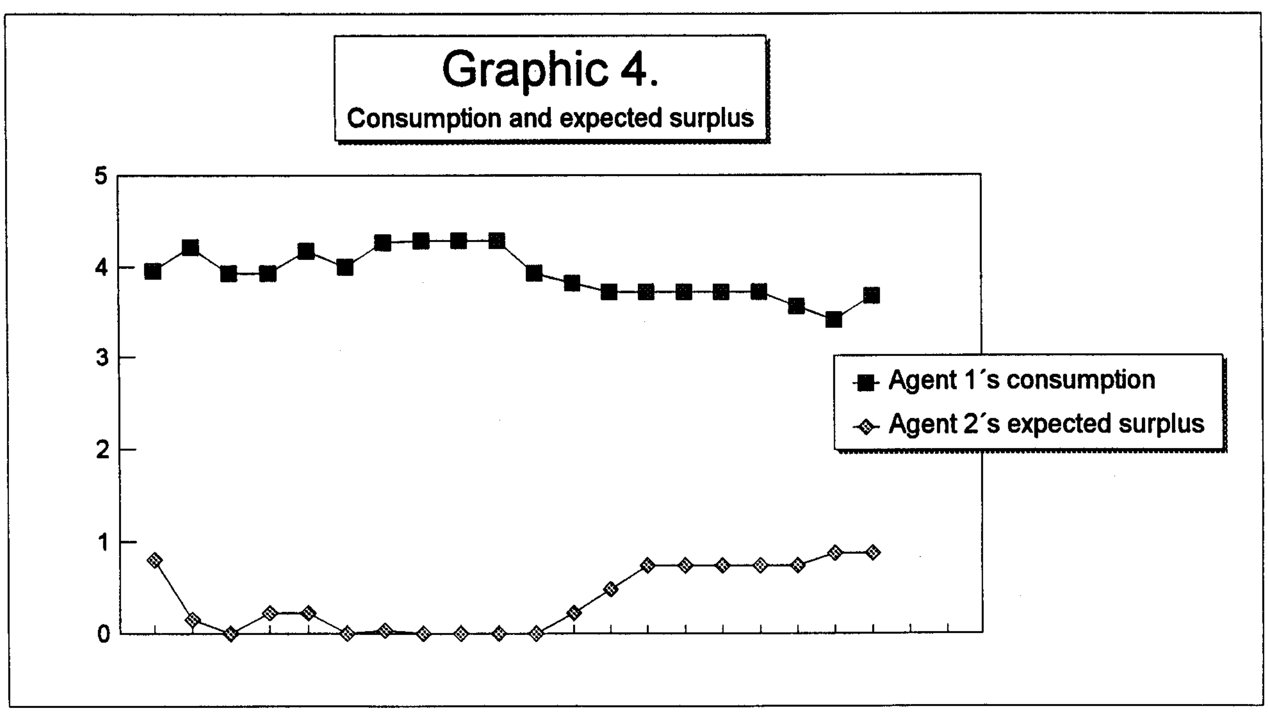
Simulations are very usefull when theoretical characterization of a model is difficult. We have used here one method of approximation derived from the projection method suggested by Judd (1992). This simple and fast approximation method turns out to be very good when the starting guess for the value function is not too far from the solution. More work remains to be done to improve the calculation of starting values.


Graphic 1. B and expected surplus
1
0.5
0
-0.5
B and V
Time
Expected surplus of the risk neutral agent
Ex ante payment (B)


Graphic 2. Income
5.5
5
4.5
4
3.5
3
2.5
Income
Time

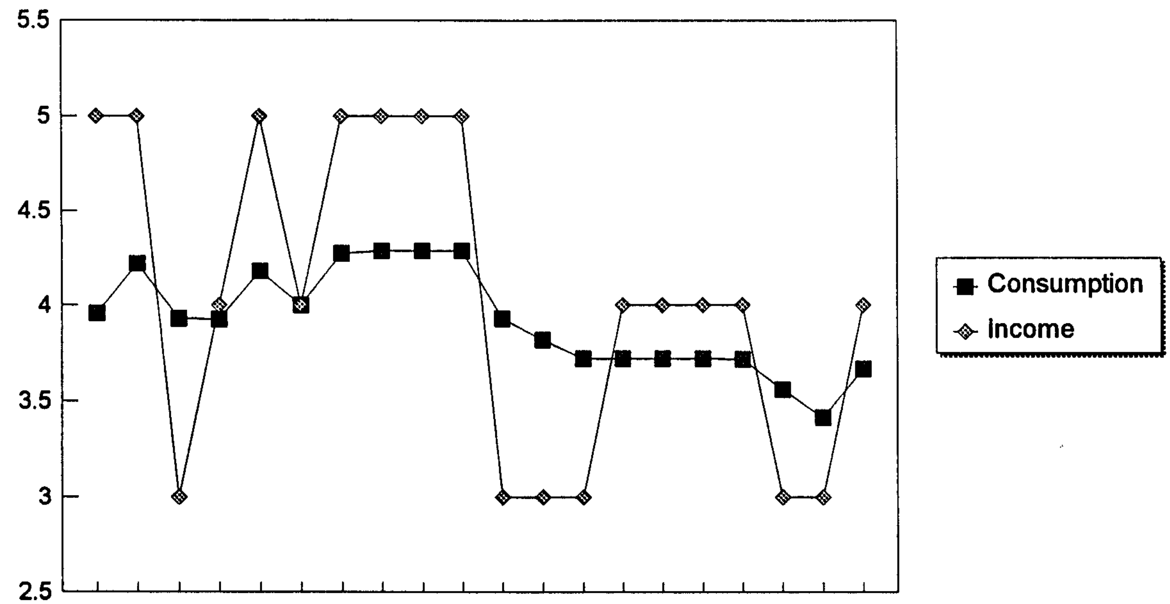


Graphic 3. Agent 1´s consumption
4.4
4.2
4
3.8
3.6
3.4
3.2
Consumption
Time



Graphic 4.
Consumption and expected surplus
5
4
3
2
1
0
Agent 1´s consumption
Agent 2´s expected surplus

Graphic 5. Agent 1´s consumption and income



3 Investment in self-enforcing risk-sharing contracts and the existence of debt overhang

3.1 Introduction

The recent literature on less developed country's debt has introduced the notion of debt overhang, which proposes that an excessively high debt works as a tax on new investment (see Krugman (1988)). This result follows from the assumption that creditors obtain a share in the new proceeds from new investment, which acts as an effective tax on new investment. The cure generally proposed in the case of a debt overhang is to erase a part of the country's outstanding debt, in order to stimulate investment.

On the other hand, in an important empirical paper, Warner (1992) finds that the debt crisis has not caused the investment crisis in the less developed countries. The fall in investment would rather be explained by conjectural variables such as falling export prices and high interest rates. As Warner (1992) concludes: "Yet the evidence casts doubt on the existence of debt effects... (on investment)... Instead, the investment decline in many of the countries on the heavily endebted list can be forecast... by simple terms of trade and world interest rate equations that do not include debt-crisis effects... The presumption that debt-crisis effects are needed to explain the investment crisis decline in heavily indebted countries is strongly reduced by the fact that simple forecasts without debt-crisis effects can explain much of the declines. At the very least, the direct influence of world economic shocks in the 1980's on investment in heavily endebted countries has received insufficient attention."

In this paper, we are searching for theoritical grounds for Warner's empirical results. An important aspect of a country's financial contracts is precisely the country's sovereignty. Sovereignty may be defined as the power to break a promise without the intervention of an international court. In that context, every contract must be self-enforcing, that is, it must be in the interest of both parties to respect their part of the contract, in any state and at any time. We analyse here the optimal financial contract between a sovereign country and a bank (or a consortium of banks) when the country cannot be forced to reimburse ex post all payments prescribed by the contract.

One feature of the model is that the country also has the opportunity to invest[20]in order to increase its expected revenue. In each period, investment is financed with production proceeds as well as with financial transfers from banks. In this context. we show that a lower level of debt does not necessarely increase investment at steady state. We demonstrate, in fact, that in a two-state world, erasing part of the country's debt can induce a fall in investment, that is, the correlation between the stock of debt and investment may be positive. The intuition for this result being that, in the steady state, the marginal cost of debt must be equal to the marginal cost of investment. When we erase a part of the country's debt, we increase the value (in terms of utility) of debt, since less of the country's future revenues would have to be used to pay back the debt. That means that a lower level of debt would increase the future consumption possibilities and thus decrease the marginal cost of debt in terms of marginal utility of consumption. Since the marginal cost of debt must be equal to the marginal cost of investment, the lower marginal cost of debt would push down the level of investment. But to maintain the level of investment at its efficient level, its marginal revenue also need to be depressed, and this would be realized by a higher level of investment.

Which one of those two effects will determine the net optimal impact of a diminution of debt on investment depends on the parameters of the model. In particular, the more the marginal revenue of investment will react to a change in investment, the less the level of investment would be pushed up following a decrease in the level of debt, and thus, the negative effect on investment would be relatively more important. A conjecture in which investment could decrease following the decrease in investment is thus a situation in which marginal returns on investment are high, that is a situation in which investment is low.

[20]See Gauthier and Poitevin (1994) for the characterization of the optimal risk-sharing contract with bonding, when enforcement costs are high.

Moreover, in the steady state, investment is uncorrelated with gross revenue, that is, whatever is the country's economic performance in terms of production, the country's willingness to invest in order to improve its expected future wealth, doesn't change at steady state. This result does not contradict Warner's results. If, for example, the world interest rate was aloud to change in our model, the discount rate would change and with it, the consumption and investment levels would be different at steady state. These results are in line with Warner's (1992) empirical results and suggest that a country's sovereignty and risk-sharing motivation for endebtness may help to explain the causal links between investment and financial obligations. In the next section, the model is presented and the autarky situation and the first-best financial contract are characterized. In Section 3 we study the financial relation in which the country cannot commit not to renege on the contract at any time.

3.2 The model

The environment we consider can be described by an infinite sequence of periods, $t = 1, 2, ..., \infty$, and for each period, a finite set of states of nature, $s \in \{1, 2, ..., S\}$, with $S > 2$. Two infinitely-lived agents evolve in this environment. Agent 1, say a sovereign country, is risk-averse. In each period, agent 1's preferences over consumption c are represented by a state-independent quadratic utility function $u(c)$ for $c \in [a, b]$. In each period, agent 1 obtains a state-contingent endowment y^s. We adopt the convention that $y^s > y^{s-1}$ for all states s. We assume that $a < y^1 < y^S < b$. Agent 1 discounts the future by factor $\beta \in (0, 1)$. In each period, agent 1 can allocate a part of his revenue to investment.

Assumption 1

States follow a conditional distribution $F(y^{s_t}|I)$ where y^{s_t} and I denote, respectively, the realized endowment in state s and period t, and the level of investment chosen in period $t-1$. We denote by $f(y^{s_t}|I)$, the conditional density function associated with this distribution function and assume that this conditional density function is i.i.d..

A decrease in I is a mean-preserving increase in risk, which implies that $\frac{\partial \sum_{s=1}^{s=S} y^{s_t} f(y^{s_t}|I)}{\partial I} = 0$.

It is assumed that each period t is divided into two dates, t_1, and t_2, where t_1 is the date at which the state of nature is realized; the date t_2 denotes the date following the realization of the state of nature. In each period, the level of investment is chosen after the realization of the state of nature, in t_2.

The history up to period t is the vector of all previous realizations of the state of nature. Let s_t denote the realized state of nature in period t. The history at the end of period $t-1$ (date $(t-1)_2$) or at the beginning of period t (date t_1) is denoted by $h_{t-1} = (s_1, s_2, ..., s_{t-1})$. We assume that $h_0 = \emptyset$.

We first establish a benchmark case, autarky, in which the country finances its investment from its endowment. In each period and state, agent 1 must decide upon an investment level. An investment's path is a function $I(h_t)$, that is, the optimal choice of investment in period t may depend on complete past history. The consumption in autarky in period t if state s has occured in t_1, is thus $c^{s_t} \equiv y^{s_t} - I^{s_t}$.

The optimal investment rule is given by solution to the following maximization problem

$$I^a(h_t) = \arg\max_{\{I^{s_t}\}_{s,t}} u(c^{k_1}) + \beta \sum_{s_2} f(y^{s_2}|I^{k_1}) u(c^{s_2}) + \tag{22}$$

$$\beta^2 \sum_{s_2} f(y^{s_2}|I^{k_1}) \sum_{s_3} f(y^{s_3}|I^{s_2}) u(c^{s_3}) + \tag{23}$$

$$\beta^3 \sum_{s_2} f(y^{s_2}|I^{k_1}) \sum_{s_3} f(y^{s_3}|I^{s_2}) \sum_{s_4} f(y^{s_4}|I^{s_3}) u(c^{s_4})$$

$$+ \cdots + \beta^t \sum_{s_2} f(y^{s_2}|I^{k_1}) \sum_{s_3} f(y^{s_3}|I^{s_2}) \tag{24}$$

$$\cdots \sum_{s_{t-1}} f(y^{s_{t-1}}|I^{s_{t-2}}) \sum_{s_t} f(y^{s_t}|I^{s_{t-1}}) u(c^{s_t}) + \cdots$$

This maximization problem simply states that the optimal investment's rule under autarky, $I^a(.)$, maximizes the discounted expected utility of agent 1. We make implicitly the assumption that agent 1 commits to an infinite investment path chosen after the realization of state k in period 1. A solution to this maximization problem exists and is characterized in the following proposition.[21]

Proposition 6 *(i) The optimal investment level only depends on the current state of nature, that is* $I^a(h_{t-1}, s_t)$ *is independent of* h_{t-1}*;*
(ii) The level of investment is increasing with endowment;
(iii) Consumption is increasing with endowment, that is, if $y^k > y^q$*, then* $y^k - I^{ak} > y^q - I^{aq}$*.*

Note that this characterization is independent of time. By Bellman's principle, after every history the investment rule is optimal, and therefore, it depends only on the current state variable. Since capital depreciates completely in one period, the state variable in this model is the current endowment. The contemporaneous choice rule for investment is thus time-invariant since it is not affected by history.

The marginal cost of investment is in terms of lost consumption. This cost is lower as I increases since consumption is lower when investment is higher. Marginal revenue of investment is in terms of marginal change in expected utility of consumption and is also an increasing function of investment. When revenue is higher, the marginal cost of investment is pushed down for a fixed level of I, while the marginal revenue does not change for a fixed level of I. Investment must then be higher when revenue is higher. Investment is thus used to smooth somewhat consumption. But smoothing is not perfect. Investment is not a perfect substitute to asset markets, since consumption is increasing in the better states of the world.

Suppose now that to improve smoothing and risk-sharing agent 1 can borrow from and/or lend to a risk-neutral financier, say agent 2. In each period, agent 2's preferences over consumption c are given by $v(c)$ which is assumed to be a state independent linear function. In each period, agent 2 obtains a state-independent endowment b. Agent 2 has the same discount factor as agent 1, namely, β.

[21]All proofs are relegated to the Appendix B.

A financial relationship between agents 1 and 2 is characterized by transfer payments between the two agents and investment level by agent 1 at various states and dates. We call the governance of such relationship a contract where the term contract is interpreted in a broad sense, namely it can encompass implicit as well as explicit agreements. In each period t, a contract can specify a state-contingent (positive or negative) transfers a^{s_t} from agent 1 to agent 2 and a level of investment by agent 1 at date t_2 (after the state of nature s is realized).For this structure of contract, agent 1's consumption in period t if state s is realized is $c^{s_t} \equiv y^{s_t} - I^{s_t} - a^{s_t}$; agent 2's consumption is $b + a^{s_t}$.

In a typical relationship the prescribed transfers and investment levels can potentially be contingent on the complete past history of the relationship. Assume that the two agents enter into a long-term (infinite) relationship. We can then define formally a contract between the two agents.

Definition 4 *A* **contract**, δ, *is a sequence of couples of functions:* $\{a(h_t), I(h_t)\}_{t=1}^{\infty}$ *where* $a^{s_t} \equiv a(h_{t-1}, s)$ *represents the transfer from agent 1 to agent 2 at the end of period t (date* t_2*), and* $I^{s_t} = I(h_{t-1}, s)$ *represents the level of investment chosen by agent 1 at the end of period t (date* t_2*) when history is* h_{t-1} *up to period t and s is the realized state of nature in period t.*

We suppose here that the financier can monitor the level of investment chosen by the country. This can be interpreted as the presence of a third party (like the FMI) which supervises the restructuration's policies of the country. In that sense, every contingent transfer of money is acompanied by a specific investment level by the country.

We will now define the utility function of both agents under an arbitrary contract, δ following any history, h_{t-1}. Agent 1's expected utility from date t_2 onwards is

$$U(\delta; h_{t-1}, k) \equiv u(c^{k_t}) + \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} f(y^{s_\tau} | I^{s_{\tau-1}}) u(c^{s_\tau})$$

Similarly, agent 2's expected revenue from period t_2 onwards is

$$V(\delta; h_{t-1}, k) \equiv a^{k_t} + \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} f(y^{s_\tau} | I^{s_{\tau-1}}) a^{s_\tau}$$

The approach we take is to assume that the two agents begin a relationship at the beginning of a period called period 1. This relationship is governed by a contract. The characterization of the implemented contract and the associated investment path depends on the available technology to legally enforce the prescribed payments. The objective of the paper is to study the effects of limited enforceability of payments on the optimal contract and the optimal path of investment.

In order to get a comparison point, we will first characterize the optimal contract when both parties can commit to the contract. We will refer to this contract as the full-commitment contract. The full-commitment contract solves the following maximization problem.

$$\left\{\delta^{fc}\right\} = \arg\max_{\delta} \left\{U(\delta; h_0) \text{ s/c } V(\delta; h_0) \geq 0\right\} \tag{25}$$

When both agents commit to the contract, the optimal full-commitment contract maximizes the expected utility of agent 1 subject to agent 2 having its reservation utility which is set to the competitive level of zero for simplicity.

Proposition 7 *(i) The full-commitment contract fully insures agent 1.*
(ii) The full-commitment investment level is constant in every period t and state s.

With complete enforcement, every agent can commit to any payment prescribed by the contract. In this case, the optimal contract fully insures agent 1. This is the first-best solution and, therefore, this is the optimal contract. Hence, agent 1's opportunity to borrow and/or lend on financial markets, provides agent 1 with optimal risk sharing.

With a full-insurance contract, the marginal cost of investment in terms of foregone consumption is constant in every period and state. Furthermore, the expected future revenue is a time-invariant function of investment. This implies that the optimal level of investment is state and time independent. Thus, investment is such that agent 1 smooths his consumption perfectly over time.

In the next section, we turn to the case in which agent 1 cannot commit to make all payments prescribed by the contract. In the sovereign-debt context, this assumption of non-commitment reflects the absence of international courts which could force the country, in some way, to meet her obligation whenever required.

3.3 Contracting under one-sided commitment

In this section we consider an environment in which legal enforcement of all prescribed payments is limited. We examine the situation in which agent 1 cannot commit to make all transfers prescribed by the contract. Specifically, at any date, agent 1 will agree to make a specified transfer or invest at specified level, only if it is in its interest to do so. A contract that specifies transfers and investments such that it is always in the interest of agent 1 of respecting it, is called self-enforcing. We assume that the agent reneging on the contract suffers maximal punishment in that it must remain in autarky forever after.[22] This maximal punishment allows us to support the highest

[22]At first sight, this punition may seem uncredible and thus the equilibrium may seem not to be renegotiation-proof. Farrell and Maskin (1989) have defined a set of weakly renegotiation-proof equilibrium payoffs (WRP), Q, as one in which no equilibrium payoff is Pareto dominated by the payoffs of another equilibrium in Q. The optimal contract induced by this environment would then be WRP if the set of possible equilibrium payoffs was defined as the set of payoffs on the Pareto frontier (which will be characterized in this section). This will be the case as will be demonstrated in the remainder of the paper. The threat of imposing autarky level of utility to a deviant would then have to be interpreted as the autarky payoff on the Pareto frontier, that is, as the strategy profile which gives the cheater its autarky payoff and, consequently, gives to the punishing party its best Pareto-optimal payoff. The interested reader is referred to Fudenberg and Tyrole (1991) for a short survey on renegotiation-proofness in infinitely repeated games and to Kletzer and Wright (1991) for a proof of the renegotiation-proofness of the optimal contract induced by this model in the case without investment.

level of cooperation in a self-enforcing contract.

An arbitrary contract generally prescribes agent 1 to make a transfer to agent 2 and invest at a specific level in some period t. Under our assumption of non-commitment by agent 1, these transfer and investment will be made only if it is in the interest of agent 1 to do so. Agent 1 will compare the benefit of making the transfer and investment and obeying the contract with the payoff of reneging on the contract and remaining in autarky thereafter. For example, suppose the two agents have signed a contract δ prescribing transfers $\{a(h_t)\}$ and investment $I(h_t)$ for all histories h_t. In period t, agent 1's utility from staying in the contract is $u(y^{s_t} - I^{s_t} - a^{s_t}) + \beta \sum_{s_{t+1}} f(y^{s_{t+1}}|I^{s_t})U(\delta; h_t, s_{t+1})$ where $a^{s_t} = a(h_{t-1}, s)$ and $I^{s_t} = I(h_{t-1}, s)$. The first term represent its current utility from the relationship and the last term, its expected future surplus. We can now define a self-enforcing contract for agent 1.

Definition 5 *A contract δ is* **self-enforcing for agent 1** *if, for all histories h_t, all periods t and states s, the following constraint holds.*

$$u(y^{s_t} - I^{s_t} - a^{s_t}) + \beta \sum_{s_{t+1}} f(y^{s_{t+1}}|I^{s_t})U(\delta; h_t, s_{t+1}) \geq U^{as} \tag{26}$$

where U^{as} is the maximum of Problem (1) when state s is the realized state in period 1.

This definition states that a contract is self-enforcing for agent 1 if at any time during the relationship agent 1 prefers making the contractual transfer and investment rather than reneging on the contract and be reduced to autarky from then on.

When designing the optimal contract the two agents will anticipate agent 1's incentive to renege, and the terms of the contract will take into account such incentive. To solve for the optimal contract we must therefore add self-enforcing constraints to the maximization problem (28). This constraint will limit the set of feasible contracts. The optimal contract with non-commitment by agent 1, δ^1 are then the solution to the following maximization problem where, for tractability, we assume that agent 2's reservation utility is equal to zero.

$$\begin{aligned} \delta^1 = \ & \arg\max_{\delta} U(\delta; h_0) \\ & \text{s/t} \quad V(\delta; h_0) \geq 0 \\ & \qquad u(y^{s_t} - I^{s_t} - a^{s_t}) + \beta \sum_{y^{s_{t+1}}} f(y^{s_{t+1}} | I^{s_t}) U(\delta; h_t, s_{t+1}) \geq U^{as} \quad \forall s, t, h_{t-1} \end{aligned} \tag{27}$$

The next two propositions characterize the optimal contract δ^1.

Proposition 8 *(i) For $\beta \in [\beta_1, 1)$, where $\beta_1 = \frac{U^{aS} - u(c^{fc})}{U^{aS}}$, the optimal contract with non-commitment by agent 1 is the optimal contract with full commitment, that is, $\delta_1 = \delta^{fc}$;*

(ii) For all states s, there exists a time-invariant optimal level of consumption $\underline{c}^s$ such that $c^{s_t} \geq \underline{c}^s$;

(iii) The optimal lower bounds of consumption, $\underline{c}^s$, are increasing in the state of the world that is, $k > q \Rightarrow \underline{c}^k > \underline{c}^q$;

(iv) For any history (h_{t-1}, s), the contract at time t is such that:

$$c(h_{t-1}, s) = \begin{cases} c(h_{t-1}) & \text{if } c(h_{t-1}) \geq \underline{c}^s \\ \underline{c}^s & \text{if } c(h_{t-1}) < \underline{c}^s \end{cases}$$

This proposition first shows that there exists a critical value of β under which the full-insurance contract is not self-enforcing for agent 1. For those low values of β, the expected benefits that agent 1 would receive by complying to the contract are not sufficient to compensate the cost incurred in the best state of the world, that is, the state in which agent 1 is asked to make the biggest transfer to agent 2. The second part of the proposition shows that there exist time-invariant lower bounds on agent 1's consumption. These bounds are increasing with the state of the world.

The proposition provides a precise characterization of the downward rigidity in the path of consumption of agent 1. This rigidity is introduced because agent 1 cannot commit to make large transfers. There is however no upward rigidity because agent 2 can commit to make any transfer to agent 1. An immediate consequence of this proposition is that c_t^s eventually converges to a steady state. Since the bounds on consumption are increasing in the state of the world, the evolution of c_t^s given in (iv) implies that the optimal path of c_t^s eventually converges to $\underline{c}^S$ for all states and periods. Moreover, since optimal risk-sharing with full-commitment at actuarially fair prices is impossible when $\beta < \beta_1$, the steady-state $\underline{c}^S$ must be higher than the steady-state optimal consumption with full commitment. This implies that agent 1 gets, in the steady state, optimal risk-sharing at prices lower than actuarially fair prices. This is acceptable to agent 2 because he gets a compensating surplus at the beginning of the relationship in order to make zero-profit overall. This contract is possible because agent 2 can commit to provide agent 1 with insurance at prices yielding negative profits given that agent 1 has provided positive profits early in the relationship. This is resumed in the next corollary[23].

Corollary 2 *For $\beta < \beta_1$, at steady state, $\underline{c}^S = y^s - I^{nc} - a^{nc} > c^{fc} = y^s - I^{fc} - a^{fc}$.*

The results of Proposition 10 are similar to results obtained by Harris and Holmström (1982) in a model of labor contracts. They showed that under the assumption of non-commitment by the employee, wages are downward rigid as the risk neutral employer fully insures the worker against bad states of the world. We show here that the opportunity to invest does not eliminate this downward rigidity in consumption.

The innovation of this paper is the study of the optimal path of investment in the context of a risk-sharing contract where one party cannot commit. In order to characterize the optimal path of investment when commitment is not possible by agent 1, we will limit ourselves to the two-state case. The next proposition presunts the characteristics of the optimal investment path in a partial commitment framework.

[23]In order to distinguish the environment with full commitment from the environment without agent 1's commitment, the notation in the non-commitment case and the full-commitment one, will be, respectively, nc, and fc.

Proposition 9 *At the steady state, in a two-state world where $y^k > y^q$,*
(i) $V^k > V^q$ and $I^k = I^q$;
(ii) the investment level I, is increasing with the expected utility of agent 2 in the bad state of the world if $\frac{\partial f(y^k|I)}{\partial I}[\bar{V}^k - V^q] < 1$.[24]

In the steady state, the marginal cost of investment in terms of foregone consumption is constant in every period and state. Furthermore, the expected revenue is a time-invariant function of investment. This implies that the optimal level of investment is state and time independant in the steady state. The second part of the proposition teaches us that it may not be right to think that forgiveness of a part of agent 1's debt would improve investment. We show that, in a two-state world, erasing a part of the country's debt when the country is in bad economic situation, could rather depress investment. The intuition for this result being that, in the steady state, the marginal cost of debt must be equal to the marginal cost of investment. If we erase a part of the country's debt, we increase the value (in terms of utility) of debt, since less of the country's future revenues would have to be used to pay back the debt. That means that a lower level of debt would increase the future consumption possibilities and thus decrease the marginal cost of debt in terms of marginal utility of consumption. Since the marginal cost of debt must be equal to the marginal cost of investment, the lower marginal cost of debt would push down the level of investment. But to maintain the level of investment at its efficient level, its marginal revenue also need to be depressed, and this would be realized by a higher level of investment.

Which one of those two effects will determine the net optimal impact of a diminution of debt on investment depends on the parameters of the model. In particular, the more the marginal revenue of investment will react to a change in investment, the less the level of investment would be pushed up following a decrease in the level of debt, and thus, the negative effect on investment would be relatively more important.

[24]Note that V^q is exogenous at t and $\bar{V}^k \equiv g^{-1}(U^{ak})$ which is thus a function of an exogenous variable.

A conjecture in which investment could decrease following the decrease in investment is thus a situation in which marginal returns on investment are high.

3.4 Conclusion

In Krugman's (1988) model, the negative correlation between investment and endebtment of a sovereign country is a consequence of the assumption that the debt's burden of the country has become so cumbersome that the expected present value of potential future payments to the bank(s) is lower than the level of debt. In our model, we assume that imperfect commitment possiblity by the country is an important aspect of its international financial relations.

When a contract takes into account that lack of commitment possibilities by the country, a self-enforcing relation emerges. That is, a relation in which anticipated benefits in the future are always sufficient to induce the current prescribed optimal outflow by the country. These net transfers can be interpreted as the combination of new money and repayments observed in the international loan market.

In a model of risk-sharing with non commitment constraint, a diminution of the endebtment's level of the country does not necessarily induce a higher level of investment. In fact, at least in some contingencies, erasing a part of the country's endebtment would rather depress investment. If sovereignty of the country is an important aspect of the financial relation between a bank and a country, then a lower level of debt may not necessarily induce higher investment.

CONCLUSION GÉNÉRALE

Cette thèse est composée de trois essais sur les contrats de long terme de partage de risques avec contraintes de banqueroute. La problématique générale est la suivante. Le revenu d'un agent riscophobe est risqué. Il cherche donc à diversifier le plus possible son risque à chaque période en transigeant sur les marchés financiers. De plus, il voudra lisser autant que possible sa consommation dans le temps. Nous étudions dans cette thèse le contrat financier d'un agent riscophobe dans un contexte où les pouvoir légaux sont limités, c'est-à-dire un environnement économique dans lequel on ne peut forçer le respect des termes du contrat (ou dans lequel les coûts nécessaires à l'imposition du respect du contrat sont excessivement élevés). Par exemple, dans le cas de la dette des pays en voie de développement (PVD), il n'existe pas d'instance supranationale qui puisse imposer au PVD le remboursement de sa créance. En l'absence d'une telle instance, si le remboursement exigé du PVD devient très élevé, il peut décider de ne plus rembourser sans qu'on puisse l'y forçer. Le contrat optimal dans un tel environnement doit faire un compromis entre le partage des risques et le lissage de la consommation d'une part, et donner les incitations à ne pas déclarer faillite d'autre part. Le but de la thèse est précisément d'étudier l'effet des contraintes de banqueroute sur les contrats optimaux de partage de risques, et ce dans différents environnements économiques.

Nous avons caractérisé, dans les deux premiers essais de la thèse, le contrat optimal de partage de risque lorsque les transferts financiers peuvent être contingents ou non à la réalisation du revenu. Nous avons démontré que l'opportunité de faire un transfert ex ante permet d'augmenter le surplus total espéré dans la relation et ainsi d'améliorer l'ensemble des contrats possibles. Nous avons ensuite analysé l'arbitrage entre les contraintes de participation des deux agents en caractérisant les paiements optimaux ex ante et ex post. Nos principaux résultats sont que les paiements ex ante sont utilisés de façon optimale et que ces paiements ne sont pas stationnaires. Ils dépendent de façon optimale du surplus espéré de la relation par chacun de agents. Ce surplus évolue avec l'histoire des réalisations passées des états de la nature. Nous avons montré que le montant du paiement ex ante qu'un agent effectue est inversement relié au surplus que cet agent anticipe de la relation. Nous anons montré également l'émergence de propriétés dynamiques intéressantes, même si les chocs aléatoires sont indépendamment et identiquement distribués d'une période à l'autre. Par exemple, dans un modèle à deux périodes, nous montrons que la dynamique du contrat optimal imite une tarification selon l'expérience malgré l'absence d'information privée ou d'apprentissage dans notre modèle. En effet, suite à une mauvaise réalisation de l'état de la nature, disons un accident de la part de l'agent 1, la compagnie d'assurance indemnisera son client. En contrepartie, le surplus futur espéré de la compagnie devra augmenter. Ceci sera réalisé de façon optimale par une hausse de la prime d'assurance du client. Une suite d'accident sera généralement suivie d'une suite de hausse de la prime d'assurance.

Les manipulations des conditions de premier ordre décrivant le contrat optimal sont très difficiles, étant donné l'existence de plusieurs contraintes d'inégalité. Pour cette raison, certaines caractéristiques du contrat optimal n'ont pu être démontrées que dans un environnement à deux états de la nature. L'objectif du second essai de cette thèse est d'étudier de façon approfondie les propriétés du contrat optimal à l'aide d'un exemple. Cett étude nécessite le développement d'un algorithme permettant l'approximation rapide, sur un micro-ordinateur avec processeur 486, de la fonction de valeur qui décrit la relation dynamique optimale.

Nous avons obtenu une approximation rapide par une méthode d'approximation polynomiale appelée la méthode de collocation orthogonale. Cette méthode est suggérée dans Judd (1992) et est une application du théorème des projections.Cette méthode s'est avérée étonnamment rapide (quelques minutes), pourvu que l'on ait une bonne approximation de la fonction de valeur comme valeur de départ. Nous avons ainsi pu vérifier numériquement pour trois états de la nature, un résultat analytique démontré dans l'essai précédent pour deux états de la nature.

Dans le troisième essai, nous nous sommes intéressés au problème du remboursement de la dette des pays en voie de développement. Deux questions en particulier nous ont préoccupés. D'une part, deux propositions sérieuses de règlement du problème du remboursement de la dette ont été formulées dans les Plans Baker et Bradley. Le Plan Baker suggère de continuer à financer les pays endettés sans effacer aucune partie de la dette, ni changer la nature des créances. Le Plan Bradley propose plutôt d'effaçer une partie importante de la dette de manière à rétablir une situation normale. D'autre part, on a observé dans les dernières années une crise d'investissement dans les pays très endettés. Plusieurs études suggèrent que cette baisse des investissements est due au problème d'endettement et que la situation pourrait être rétablie par une diminution de l'endettement.

Dans notre modèle, le pays est riscophobe. Cette hypothèse réflète la difficulté qu'ont les pays en voie de développement à diversifier parfaitement leurs risques sur les marchés internationaux. De plus, toute l'histoire d'endettement du pays y est caractérisée, et les remboursements exigés par les banques sont contingents à la réalisation d'une variable aléatoire qui est fonction du niveau d'investissement choisi par le pays à la période précédente. Dans ce contexte, la réaction du pays suite à une baisse du remboursement exigé est indéterminée. En effet, nous avons montré que dans un modèle à deux états, le pays pourrait aussi bien diminuer qu'augmenter son effort d'ajustement suite à une baisse des exigences des banques.

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APPENDICE A

Proof of Proposition 1 Consider the maximization

$$\max_{\{(\delta_t^s)_{s=1,\ldots,S}\}_{t=1,\ldots,\infty}} E\sum_{t=1}^{\infty}\beta^t\left\{u(y_t^s+\delta_t^s)-u(y_t^s)\right\}$$
$$E\sum_{t=1}^{\infty}\beta^t\left\{v(e-\delta_t^s)-v(e)\right\}\geq 0$$

Denote λ the multiplier of the constraint. The first-order conditions are

$$u'(y_t^s+\delta_t^s)-\lambda v'(e-\delta_t^s)=0 \quad \forall\, s,t$$

This implies that

$$\frac{u'(y_t^s+\delta_t^s)}{v'(e-\delta_t^s)}=\lambda \quad \forall\, s,t$$

Q.E.D.

Proof of Proposition 2 (i) Consider the optimal full-commitment contract δ^{fc} characterized in Proposition 1. The per-period surplus to agent 1 is $Eu(c^{qfc})-Eu(y^q)$ which is positive. Hence $U(\delta^{fc};h_{t-1})>0$ for all histories h_{t-1} and periods t. This implies that all ex ante self-enforcing constraints are satisfied. Suppose that $\underline{B}\geq y^S-c^{Sfc}$ and that agent 1 makes the maximum ex ante payment, namely, $B_t=-\underline{B}$. From Proposition 1 we know that $a_t^s=y_t^s+B_t-c^{sfc}$. If $B_t=-\underline{B}$, then $a_t^s\leq y_t^s-y^S+c^{Sfc}-c^{sfc}$. Since the transfer from agent 1 to agent 2 is largest when $s=S$, we have that $a_t^s\leq 0$ for all s, and hence no ex post self-enforcing constraints are binding. The contract δ^{fc} can then be supported as the optimal contract δ^1.

(ii) Assume that $\underline{B}<y^S-c^{Sfc}$. Consider the optimal full-commitment contract δ^{fc}. As argued above, all ex ante self-enforcing constraints are satisfied. Ex post self-enforcing constraints are satisfied if and only if

$$u(c^{sfc})-u(y_t^s+B_t)+\frac{\beta}{1-\beta}\left(Eu(c^{qfc})-Eu(y^q)\right)\geq 0 \quad \forall\, s.$$

These constraints become less binding if B_t is set at its lowest level, namely, $B_t=-\underline{B}$ and net transfers are adjusted such that agent 1's consumption be c^{sfc}. Setting $B_t=-\underline{B}$ and solving for β in the ex post self-enforcing constraint yields

$$\beta\geq\frac{u(y_t^s-\underline{B})-u(c^{sfc})}{u(y_t^s-\underline{B})-Eu(y^q)+Eu(c^{qfc})-u(c^{sfc})} \text{ for all } s.$$

that the contract is dynamically consistent. We will first show that problem (1) is a concave program and then use its first-order conditions to characterize the optimal contract δ^1.

To show that problem (1) is a concave program, we show that the Pareto frontier is strictly concave and continuously differentiable.[4] We first have to show that the set $\Omega(h_{t-1})$ is convex. Consider two self-enforcing contracts for agent 1, δ' and δ'' with associated payments $\{a'(h_t), B'(h_t)\}$ and $\{a''(h_t), B''(h_t)\}$ respectively. It is easily shown that because the utility function $u(\cdot)$ is concave and quadratic, any linear combination of these two contracts δ^λ where $a^\lambda(h_t) = \lambda a'(h_t) + (1-\lambda)a''(h_t)$ and $B^\lambda(h_t) = \lambda B'(h_t) + (1-\lambda)B''(h_t)$ is also self-enforcing for agent 1. Secondly, following history h_{t-1}, the set of V_t such that a self-enforcing contract for agent 1 exists is a compact interval $[-K_1, \bar{V}]$ where $-K_1$ is the discounted utility of agent 2 when it pays out to agent 1 transfers yielding her a consumption of b in every state and period.[5] Such transfers are obviously self-enforcing for agent 1. There exists an upper bound on the surplus agent 1 can concede to agent 2 in a self-enforcing contract. Denote this upper bound by $\bar{V}$. If $\bar{V}$ is attainable by a self-enforcing contract, then any $V_t \in [-K_1, \bar{V}]$ is also. The closedness of this interval can be shown by constructing a sequence of self-enforcing contracts yielding some utility level to agent 2 converging to $\bar{V}$. Since $u(\cdot)$ is continuous and $\beta \in (0,1)$, the Dominated Convergence Theorem implies that the limiting contract is also self-enforcing and hence $\bar{V}$ is included in the interval. Finally, we show that the Pareto frontier is decreasing, strictly concave and continuously differentiable. It is obvious that the function $g(\cdot)$ is decreasing. The strict concavity property follows from the strict concavity of $u(\cdot)$, the concavity of $v(\cdot)$, and the convexity of $\Omega(\cdot)$. The differentiability property follows from the continuity and differentiability of $u(\cdot)$. Consider an efficient self-enforcing contract δ such that $V(\delta; h_{t-1}) = V_t \in (-K_1, \bar{V})$. Construct a contract δ^γ which differs from the contract δ in that $a^\gamma(h_{t-1}, s) = a(h_{t-1}, s) + \gamma$. The state s is chosen such that agent 1's ex post self-enforcing constraint is not strictly binding. The contract δ^γ is therefore self-enforcing for γ small enough. Define the function $\hat{g}$ such that $U(\delta^\gamma; h_{t-1}) = \hat{g}(V(\delta^\gamma; h_{t-1})) \leq g(V(\delta; h_{t-1}))$ with equality if $\gamma = 0$. As γ is varied, it ca be showned that the function $\hat{g}$ is concave and differentiable at V_t. Therefore it satisfies Lemma

[4]Most of the arguments used here follow those of Lemma 1 of Thomas and Worrall (1988).

[5]Remember that agent 1's utility function is defined over the interval $[a, b]$.

1 reported in Benveniste and Scheinkman (1979). The function $g(\cdot)$ is then differentiable. Since it is monotonic, it is also continuously differentiable almost everywhere. This implies that for any value $V_t \in [-K_1, \bar{V}]$, there exists a unique efficient continuation of the contract δ at time t in which $V(\delta; h_{t-1}) = V_t$ and $U(\delta; h_{t-1}) = g(V_t)$. Existence is guaranteed by the compactness of the interval $[-K_1, \bar{V}]$; uniqueness is guaranteed by the convexity of $\Omega(\cdot)$ and the strict concavity of $u(\cdot)$. These results effectively imply that problem (1) is a concave program, and therefore its first-order conditions are both necessary and sufficient for a solution. Let $\beta p^s \alpha^s$, $p^s \theta^s$ and ψ be the respective multipliers of the constraints in problem (1), and μ_1 and μ_2 the multipliers for the constraints on the lower and upper bounds of B_t. The first-order conditions are then

$$\begin{aligned} B_t: \quad & \sum_s p^s u'(y_t^s + B_t - a_t^s) + \sum_s p^s \theta^s \left(u'(y_t^s + B_t - a_t^s) - u'(y_t^s + B_t)\right) \\ & - \sum_s \psi p^s v'(e - B_t + a_t^s) + \mu_1 - \mu_2 = 0 \end{aligned} \tag{2}$$

$$a_t^s: \quad -p^s(1+\theta^s)u'(y_t^s + B_t - a_t^s) + p^s \psi v'(e - B_t + a_t^s) = 0 \quad \forall s \tag{3}$$

$$V_{t+1}^s: \quad (1 + \alpha^s + \theta^s) g'(V_{t+1}^s) + \psi = 0 \quad \forall s \tag{4}$$

and the envelope condition is $g'(V_t) = -\psi$.

1. Summing over s all conditions (3) and substituting in condition (2) yield $-\sum_s p^s \theta^s u'(y_t^s + B_t) + \mu_1 - \mu_2 = 0$. If there is at least one self-enforcing constraint that binds, the multiplier $\mu_1 > 0$ and therefore $B_t = -\underline{B}$. If no self-enforcing constraint binds, only net payments matter and hence B_t can be set arbitrarily at its highest level $\underline{B}$, namely the level for which ex post self-enforcing constraints are the least binding.

2. Condition (4) and the envelope condition jointly imply that $(1+\alpha^s+\theta^s)g'(V_{t+1}^s) = g'(V_t)$. Because $\alpha^s + \theta^s \geq 0$ and the Pareto frontier $g(\cdot)$ is decreasing and concave, it implies that $V_{t+1}^s \leq V_t$ with strict inequality when $\alpha^s + \theta^s > 0$.

3. We now show that agent 1's implicit discount factor is not larger than agent 2's. Suppose first that $V_t < \bar{V}$. The above result implies that $\alpha^s = 0$ for all states s. Conditions (3) and (4) then imply that $\frac{u'(y_t^s - \underline{B} - a_t^s)}{v'(e + \underline{B} + a_t^s)} = -g'(V_{t+1}^s)$. Furthermore, efficiency of the optimal contract in period $t-1$ implies that $\frac{u'(y_{t-1} - \underline{B} - a_{t-1})}{v'(e + \underline{B} + a_{t-1})} = -g'(V_t)$. Using the result that $V_{t+1}^s \leq V_t$, we then have that $\frac{u'(y_{t-1} - \underline{B} - a_{t-1})}{v'(b + \underline{B} + a_{t-1})} \geq \frac{u'(y_t^s - \underline{B} - a_t^s)}{v'(e + \underline{B} + a_t^s)}$ for all states s. Rearranging terms and taking expectation over the states s then prove the result. Suppose now that $V_t = \bar{V}$. When

$V^s_{t+1} = \bar{V}$, the condition $(1 + \alpha^s + \theta^s)g'(V^s_{t+1}) = g'(V_t)$ implies that $\alpha^s + \theta^s = 0$. We then have that $\frac{u'(y^s_t - \underline{B} - a^s_t)}{v'(e+\underline{B}+a^s_t)} = -g'(V^s_{t+1}) = -g'(V_t)$. A similar argument for period $t-1$ implies that $\frac{u'(y_{t-1} - \underline{B} - a_{t-1})}{v'(e+\underline{B}+a_{t-1})} = -g'(V_t)$. When $V^s_{t+1} < \bar{V}$, the multiplier $\alpha^s = 0$. We then have that $\frac{u'(y^s_t - \underline{B} - a^s_t)}{v'(e+\underline{B}+a^s_t)} = -g'(V^s_{t+1}) < g'(V_t)$. This implies that $\frac{u'(y_{t-1} - \underline{B} - a_{t-1})}{v'(e+\underline{B}+a_{t-1})} \geq \frac{u'(y^s_t - \underline{B} - a^s_t)}{v'(e+\underline{B}+a^s_t)}$ for all states s and the result follows. Finally note for future reference that these arguments imply that $\alpha^s = 0$ for all states s and time periods t. *Q.E.D.*

Proof of Corollary 1 (i) Denote by $\bar{V}^s$ and $\underline{c}^s$ the optimal maximum and minimum values for V^s_{t+1} and c^s_t respectively such that there exists a self-enforcing contract for agent 1. These values are implicitly defined by

$$\begin{aligned} \frac{u'(\underline{c}^s)}{v'(e + y^s - \underline{c}^s)} &= -g'(\bar{V}^s) \\ u(\underline{c}^s) - u(y^s - \underline{B}) + \beta g(\bar{V}^s) &= 0. \end{aligned}$$

The first equation follows from first-order conditions to problem (1) and the fact that $\alpha^s = 0$ for all s while the second represents agent 1's ex post self-enforcing constraint in state s. Note that these equations are time-independent. After substituting for $\bar{V}^s$ in agent 1's ex post self-enforcing constraint, the optimal bound on consumption, $\underline{c}^s$, is then implicitly defined by

$$u(\underline{c}^s) - u(y^s - \underline{B}) + \beta g\left(g'^{-1}\left(-\frac{u'(\underline{c}^s)}{v'(e + y^s - \underline{c}^s)}\right)\right) = 0. \tag{5}$$

The left-hand side of the ex post self-enforcing constraint is increasing in $\underline{c}^s$ which implies that it is satisfied for $c^s_t \geq \underline{c}^s$ for all t and s.

(ii) Differentiating along equation (5) yields

$$\frac{d\underline{c}^s}{dy^s} = -\frac{\beta g' g'^{-1'} * \left(\frac{u'(\underline{c}^s)v''(e+y^s-\underline{c}^s)}{v'(e+y^s-\underline{c}^s)^2}\right) - u'(y^s - \underline{B})}{u'(\underline{c}^s) + \beta g' g'^{-1'} * \left(-\frac{u''(\underline{c}^s)v'(e+y^s-\underline{c}^s) + u'(\underline{c}^s)v''(e+y^s-\underline{c}^s)}{v'(e+y^s-\underline{c}^s)^2}\right)}$$

which is positive since $g'g'^{-1'} \geq 0$.[6] Hence, $\underline{c}^s$ is increasing in the states of the world, that is, $\underline{c}^k > \underline{c}^q$ if and only if $y^k > y^q$.

[6]Since the function g is continuously differentiable and concave we know that $g'^{-1'}$ exists almost everywhere. Where it does not exists, we know that the right-hand and left-hand derivatives are negative.

Finally, differentiating along equation (5) yields

$$\frac{d\underline{c}^s}{d\underline{B}} = -\frac{u'(y^s - \underline{B})}{u'(\underline{c}^s) + \beta g' g'^{-1\prime} * \left(-\frac{u''(\underline{c}^s)v'(e+y^s-\underline{c}^s)+u'(\underline{c}^s)v''(e+y^s-\underline{c}^s)}{v'(e+y^s-\underline{c}^s)^2}\right)} < 0.$$

(iii) From the first-order conditions we know that

$$(1+\theta^s)\left(u'(c_t^s)/v'(e+y_t^s-c_t^s)\right) = u'(c_{t-1})/v'(e+y_{t-1}-c_{t-1}).$$

Suppose first that $c^*(c_{t-1},q,s) \geq \underline{c}^s$. This condition is satisfied when $c_t^s = c^*(c_{t-1},q,s) \geq \underline{c}^s$ and $\theta^s = 0$. Now suppose that $c^*(c_{t-1},q,s) < \underline{c}^s$. This condition is satisfied when $c_t^s = \underline{c}^s$ and $\theta^s > 0$. *Q.E.D.*

Proof of Lemma 1 (i) From the proof of Proposition 2 we know that $\Omega(h_{t-1})$ is convex. By symmetry, the set of self-enforcing contracts for agent 2 is also convex. The set $\Lambda(h_{t-1})$ is the intersection of these two convex sets and is therefore convex.

(ii), (iii), (iv) The rest of the proof follows that of Proposition 2 with minor modifications. *Q.E.D.*

Proof of Proposition 3 Denote respectively by $\beta p^s\alpha^s$, $p^s\theta^s$, $\beta p^s\phi^s$, $p^s\lambda^s$, and ψ the multipliers of the five constraints in problem (4) in the main text. The first-order conditions when $B_t = 0$ for all time periods are

$$a_t^s: \quad -p^s(1+\theta^s)u'(y_t^s + B_t - a_t^s) + p^s(\lambda^s+\psi)v'(e - B_t + a_t^s) = 0 \quad \forall s \tag{6}$$

$$V_{t+1}^s: \quad (1+\alpha^s+\theta^s)f'(V_{t+1}^s) + \lambda^s + \phi^s + \psi = 0 \quad \forall s \tag{7}$$

and the envelope condition is $f'(V_t) = -\psi$.

(i) First-order conditions imply that

$$(1+\theta^s)\frac{u'(c_t^s)}{v'(e+y_t^s-c_t^s)} = -(1+\alpha^s+\theta^s)f'(V_{t+1}^s) - \phi^s. \tag{8}$$

We first define the optimal lower bounds on consumption. Suppose first that $\phi^s = \alpha^s = 0$. Condition (8) then implies $u'(c_t^s)/v'(e+y_t^s-c_t^s) = -f'(V_{t+1}^s)$. This can be rewritten as $V_{t+1}^s = f'^{-1}(-u'(c_t^s)/v'(e+y_t^s-c_t^s))$. It is easily shown that the right-hand-side of this expression is decreasing in c_t^s. A lower bound on consumption is therefore associated with an upper bound on V_{t+1}^s. If $\alpha^s > 0$, then $V_{t+1}^s = \bar{V}$. The lower bound on consumption, $\underline{c}^s$, is defined by the intersection of this expression and agent 1's ex post self-enforcing constraint. More formally,

$$u(\underline{c}^s) - u(y^s) + \beta f\left(\min\left\{f'^{-1}\left(-u'(\underline{c}^s)/v'(e + y^s - \underline{c}^s)\right), \bar{V}\right\}\right) = 0 \tag{9}$$

where $f(\bar{V}) = 0$. This expression states that optimal lower bounds on consumption are defined by the intersection of first-order conditions and agent 1's ex post self-enforcing constraint to the extent that they respect the ex ante self-enforcing constraints; otherwise the expression reduces to $\underline{c}^s = y^s$. It is clear from expression (9) that $\underline{c}^s \leq y^s$. Note that these optimal lower bounds are time-independent.

We now define the optimal upper bounds on consumption. The expression

$$V_{t+1}^s = f'^{-1}\left(-u'(c_t^s)/v'(e + y_t^s - c_t^s)\right)$$

is substituted in agent 2's ex post self-enforcing constraint. More formally,

$$v(e + y^s - \bar{c}^s) - v(b) + \beta\left(\max\left\{f'^{-1}\left(-u'(\bar{c}^s)/v'(e + y^s - \bar{c}^s)\right), 0\right\}\right) = 0. \tag{10}$$

This expression states that optimal upper bounds on consumption are defined by the intersection of first-order conditions and agent 2's ex post self-enforcing constraint to the extent that they respect the ex ante self-enforcing constraints; otherwise the expression reduces to $\bar{c}^s = y^s$. It is clear from expression (10) that $\bar{c}^s \geq y^s$. Again, note that these optimal upper bounds are time-independent.

The preceding arguments show that in any time period t and state s, consumption c_t^s must be included in the interval $[\underline{c}^s, \bar{c}^s]$; otherwise one of the self-enforcing constraints or first-order conditions would be violated.

(ii) We now show that the optimal lower bounds are increasing in the states of the world. The optimal lower bounds are implicitly defined as a function of y^s in expression (9). This expression is continuous in $\underline{c}^s$ and y^s but is not differentiable at one point (where the minimum switches from $f'^{-1}\left(-u'(\underline{c}^s)/v'(e + y^s - \underline{c}^s)\right)$ to $\bar{V}$). When the minimum equals $\bar{V}$, $\underline{c}^s = y^s$ and clearly the optimal bound is increasing in the state of the world. When the minimum equals the first expression, total differentiation of the implicit function yields

$$\frac{d\underline{c}^s}{dy^s} = -\frac{\beta f' f'^{-1\prime} * \left(\frac{u'(\underline{c}^s)v''(e+y^s-\underline{c}^s)}{v'(e+y^s-\underline{c}^s)^2}\right) - u'(y^s)}{u'(\underline{c}^s) + \beta f' f'^{-1\prime} * \left(-\frac{u''(\underline{c}^s)v'(e+y^s-\underline{c}^s)+u'(\underline{c}^s)v''(e+y^s-\underline{c}^s)}{v'(e+y^s-\underline{c}^s)^2}\right)}$$

which is positive since $f'f'^{-1'} > 0$.[7] Hence, because $\underline{c}^s$ is a continuous implicit function of y^s, these results imply that $\underline{c}^s$ is increasing in the states of the world, that is, $\underline{c}^k > \underline{c}^q$ if and only if $y^k > y^q$.

We now show that the optimal upper bounds are increasing in the states of the world. The optimal upper bounds are implicitly defined as a function of y^s in expression (10). This expression is continuous in $\bar{c}^s$ and y^s but is not differentiable at one point (where the maximum switches from $f'^{-1}(-u'(\bar{c}^s)/v'(e+y^s-\bar{c}^s))$ to 0). When the maximum equals 0, $\bar{c}^s = y^s$ and clearly the optimal bound is increasing in the state of the world. When the maximum equals the first expression, total differentiation of the implicit function yields

$$\frac{d\bar{c}^s}{dy^s} = -\frac{v'(e+y^s-\bar{c}^s) + \beta f'^{-1'} * \left(\frac{u'(\bar{c}^s)v''(e+y^s-\bar{c}^s)}{v'(e+y^s-\bar{c}^s)^2}\right)}{-v'(e+y^s-\bar{c}^s) + \beta f'^{-1'} * \left(-\frac{u''(\bar{c}^s)v'(e+y^s-\bar{c}^s)+u'(\bar{c}^s)v''(e+y^s-\bar{c}^s)}{v'(e+y^s-\bar{c}^s)^2}\right)}$$

which is positive since $f'^{-1'} \leq 0$. Hence, because $\bar{c}^s$ is a continuous implicit function of y^s, these results imply that $\bar{c}^s$ is increasing in the states of the world, that is, $\bar{c}^k > \bar{c}^q$ if and only if $y^k > y^q$.

(iii) The proof of this part of the proposition requires proving the following preliminary result.

Lemma 1 *The multipliers $\alpha^s = \phi^s = 0$ for all s and t.*

Proof of Lemma 3 This proof consists of two parts. First we show that $\alpha^s = 0$ for all s. Consider the optimal lower bounds $\underline{c}^s$. From agent 1's self-enforcing constraints, we know that $y^s \geq \underline{c}^s$. Furthermore it is easy to show that $0 < d\underline{c}^s/dy^s < 1$ when $V^s_{t+1} < \bar{V}$. Since $\underline{c}^s$ is continuous in y^s it must be the case that if $\underline{c}^s = y^s$ then $s = 1$. In this case if $c^s_t = \underline{c}^1$, then $V^1_{t+1} = \bar{V}$. Given that the optimal lower bounds are increasing in the states of nature and that first-order conditions imply an inverse relationship between c^s_t and V^s_{t+1}, it must be the case that $\alpha^s = 0$ for all $s \geq 2$. Now suppose that in period $t-1$ we had $c_{t-1} = \underline{c}^1$ and $V_t = \bar{V}$. First-order conditions in periods $t-1$ and t imply that

$$\frac{u'(c_{t-1})}{v'(e+y^1-c_{t-1})} = \frac{(1+\alpha_{t-1}+\theta_{t-1})}{1+\theta_{t-1}}\left\{(1+\theta^s)\frac{u'(c^s_t)}{v'(e+y^s_t-c^s_t)} - \lambda^s\right\} - \frac{\phi_{t-1}}{1+\theta_{t-1}}$$

[7] Since the function f is continuously differentiable and concave we know that $f'^{-1'}$ exists almost everywhere. Where it does not exists, we know that the right-hand and left-hand derivatives are negative.

$$f'(V_t) = (1+\alpha^s+\theta^s)f'(V_{t+1}^s)+\phi^s+\lambda^s$$

for all states s. Take $s=1$ and suppose that $\alpha_{t-1}>0$. This implies that $\phi_{t-1}=0$ and $V_t=\bar{V}$. The solution to these two equations must then include consumption $c_t^1=\underline{c}^1$ which implies that $\lambda^1=0$. But then the first equation cannot be satisfied. It must then be the case that $\alpha_{t-1}=0$ and hence $\alpha^1=0$ in all periods.

The second part shows that $\phi^s=0$ for all s. The argument is similar as above. We know that $y^s\le\bar{c}^s$. It is easy to show that $0<d\bar{c}^s/dy^s<1$ when $V_{t+1}^s>0$. Since $\bar{c}^s$ is continuous in y^s it must be the case that if $\bar{c}^s=y^s$ then $s=S$. In this case if $c_t^s=\bar{c}^S$, then $V_{t+1}^S=0$. Given that the optimal upper bounds are increasing in the states of nature and that first-order conditions imply an inverse relationship between c_t^s and V_{t+1}^s, it must be the case that $\alpha^s=0$ for all $s\le S-1$. Suppose that in period $t-1$ we had $c_{t-1}=\bar{c}^S$ and $V_t=0$. Use the above relationships between c_{t-1} and c_t^s, and V_t and V_{t+1}^s, take $s=S$, and suppose that $\phi_{t-1}>0$. This implies that $\alpha_{t-1}=0$ and $V_t=0$. The solution must then include consumption $c_t^S=\bar{c}^S$ which implies that $\theta^S=0$. But then the above equation cannot be satisfied. It must then be the case that $\phi_{t-1}=0$ and hence $\phi^S=0$ in all periods. *Q.E.D.*

The results of this lemma are now used to show that the optimal consumption path follows that stated in part (iii) of the proposition. First-order conditions and the envelope condition imply that

$$\begin{aligned}(1+\theta^s)\frac{u'(c_t^s)}{v'(e+y_t^s-c_t^s)}-\lambda^s &= -f'(V_t)\\ \frac{u'(c_t^s)}{v'(e+y_t^s-c_t^s)} &= -f'(V_{t+1}^s)\end{aligned}$$

First-order conditions in period $t-1$ then imply that

$$(1+\theta^s)\frac{u'(c_t^s)}{v'(e+y_t^s-c_t^s)}-\lambda^s=\frac{u'(c_{t-1})}{v'(e+y_{t-1}-c_{t-1})}.$$

Suppose that $\underline{c}^s\le c^*(c_{t-1},y_{t-1},s)\le\bar{c}^s$. The solution must then be $c_t^s=c^*(c_{t-1},y_{t-1},s)$ with $\theta^s=\lambda^s=0$. If $c^*(c_{t-1},y_{t-1},s)>\bar{c}^s$, then the solution must be $\lambda^s>0$ and $c_t^s=\bar{c}^s$. If $c^*(c_{t-1},y_{t-1},s)<\underline{c}^s$, then the solution must be $\theta^s>0$ and $c_t^s=\underline{c}^s$.

(iv) The contract δ^{fc} is such that $V_{t+1}^{sfc}=0$ for all states and periods. This implies that agent 2's ex post self-enforcing constraints can be satisfied if and only if $a_t^s\ge0$ in all states and periods. But this is inconsistent with the payments prescribed by the contract δ^{fc}, hence it cannot be self-enforcing when $B_t = 0$. *Q.E.D.*

Proof of Lemma 2 (i) We know that $V(\delta^{fc}; h_t) = 0$ for all histories h_t. The contract δ^{fc} is self-enforcing if and only if $a_t^s \geq 0$ for all states s and periods t. Consider the following contract $\hat{\delta}$: the ex ante payment is set at $\hat{B}_t = c^{1fc} - y^1$ and contingent payments at $\hat{a}_t^s = y_t^s - y^1 + c^{1fc} - c^{sfc} \geq 0$ in all states and periods. This contract yields for both agents the same consumption as under the contract δ^{fc}. The contract $\hat{\delta}$ is self-enforcing for agent 2. The ex ante self-enforcing constraints are trivially satisfied, that is, $\hat{V}_{t+1}^s = 0$ for all states and periods; the ex post self-enforcing constraints are also satisfied by construction since $\hat{a}_t^s \geq 0$ for all states and periods. It is self-enforcing for agent 1 if and only if all her ex post self-enforcing constraints are satisfied, that is,

$$u(c^{sfc}) - u(y^s + c^{1fc} - y^1) + \frac{\beta}{1-\beta}\left\{Eu(c^{qfc}) - Eu(y^q)\right\} \geq 0 \text{ for all } s.$$

Note that agent 1's ex ante self-enforcing constraints are satisfied by $\hat{\delta}$. Define β_{nc} as the smallest discount factor that satisfies the above equation in all states. This shows that $\beta \geq \beta_{nc}$ is a sufficient condition for the contract $\hat{\delta}$ to be self-enforcing. It is also necessary since a contract with a smaller ex ante payment would not be self-enforcing for agent 2 as it would require at least one ex post payment to be negative; a contract with a larger ex ante payment would be self-enforcing for larger values of the discount factor than β_{nc}.

(ii) When $\beta < \beta_{nc}$, there is at least one ex post self-enforcing constraint that binds. Adding up all conditions (6) to condition (5) in the main text yields

$$\sum_s p^s \left\{\lambda^s v'(e - B_t) - \theta^s u'(y_t^s + B_t)\right\} = 0. \tag{11}$$

It therefore follows that there must be a s_1 for which $\theta^{s_1} > 0$ and a s_2 for which $\lambda^{s_2} > 0$. *Q.E.D.*

Proof of Proposition 4 Consider the optimal solution to maximization (4) in the main text as a function of the state variable V_t. By the theorem of the maximum we know that the solution is continuous in the state variable over the interval $[0, \bar{V}]$. Consider a marginal increase in the value of the state variable. We want to show that the optimal ex ante payment is strictly decreasing in the state variable. The proof goes by contradiction. Suppose that the optimal value of the ex ante payment is left unchanged following a marginal increase in the state variable. The envelope condition implies that $f''(V_t)dV_t = -d\psi < 0$. Consider all ex post constraints that are satisfied at equality before the increase in the state variable. Of these constraints, we choose all those that become strictly binding following the increase in the state variable. These are the only constraints that bind following the increase in V_t. Consider the first-order conditions (6) in the main text for all states s for which one self-enforcing constraint becomes binding. In these states, consumption is left unchanged following the small increase in V_t.[8] For all those ex post self-enforcing constraints, for first-order conditions to continue to hold we have that $d\lambda^s = -d\psi$ if the binding constraint is that of agent 2, and $(u'(c_t^s)/v'(e + y_t^s - c_t^s))\, d\theta^s = d\psi$ if it is that of agent 1. If we substitute these changes in condition (11), we have

$$\sum_s p^s \left\{ -d\psi u'(y_t^s + B_t) * \frac{v'(e + y_t^s - c_t^s)}{u'(c_t^s)} - d\psi v'(e - B_t) \right\} = 0$$

for this condition to continue to hold. Since $d\psi > 0$, this expression cannot be equal to 0 if B_t remains constant. This implies that for any marginal change in the state variable V_t, the ex ante payment B_t must change and therefore B_t is monotonic in the state variable V_t over the range $[0, \bar{V}]$.

Suppose that $V_t = \bar{V}$ and fix $B_t = 0$. In this case, only agent 1 has some ex post self-enforcing constraints that bind. We know from Proposition 2 that agent 1 pays the maximum ex ante payment. Since our problem is a concave problem, this implies that at $V_t = \bar{V}$, the optimal value of B_t is negative. A similar argument shows that the optimal B_t is positive at $V_t = 0$. Since B_t is monotonic in V_t the relationship between B_t and V_t must be decreasing. *Q.E.D.*

Proof of Proposition 5 We know from Lemma 2 that there exist states s_1 and s_2 such that $\theta^{s_1} > 0$ and $\lambda^{s_2} > 0$. Suppose that $\lambda^2 > 0$ and $\theta^1 > 0$. This implies that $\lambda^1 = 0$ and $\theta^2 = 0$. First-order conditions then imply

$$-(1 + \theta^1)\frac{u'(c_t^1)}{v'(e + y_t^1 - c_t^1)} = -f'(V_t)$$

[8] This follows from the results of Proposition 3 which show that for $B_t = 0$, the optimal lower (and upper) optimal consumption bounds that satisfy the ex post self-enforcing constraints are time-invariant. This result can easily be generalized to any fixed value of B_t.

$$\frac{u'(c_t^2)}{v'(e+y_t^2-c_t^2)} - \lambda^2 = -f'(V_t)$$

which yields

$$(1+\theta^1)\frac{u'(c_t^1)}{v'(e+y_t^1-c_t^1)} = \frac{u'(c_t^2)}{v'(e+y_t^2-c_t^2)} - \lambda^2.$$

But this implies that $c_t^1 = \underline{c}^1 > c_t^2 = \bar{c}^2$ which is inconsistent with the results of Proposition 3.[9] Consequently it must be the case that $\lambda^1 > 0$ and $\theta^2 > 0$.

(i) First-order conditions imply that, for state 1,

$$(1+\alpha^1)f'(V_{t+1}^1) + \lambda^1 + \phi^1 = f'(V_t).$$

If $\alpha^1 > 0$, then $V_{t+1}^1 = \bar{V} \geq V_t$. If $\alpha^1 = 0$, the above expression reduces to

$$f'(V_{t+1}^1) + \lambda^1 + \phi^1 = f'(V_t)$$

which implies that $V_{t+1}^1 > V_t$ by the concavity of the Pareto frontier $f(\cdot)$. In state 2, we have that

$$(1+\alpha^2+\theta^2)f'(V_{t+1}^2) + \phi^2 = f'(V_t).$$

If $\phi^2 > 0$, then $V_{t+1}^2 = 0 \leq V_t$. If $\phi^2 = 0$, the above expression reduces to

$$(1+\alpha^2+\theta^2)f'(V_{t+1}^2) = f'(V_t)$$

which implies that $V_{t+1}^2 < V_t$ by the concavity of the Pareto frontier $f(\cdot)$. We then have $V_{t+1}^2 \leq V_t \leq V_{t+1}^1$ which proves the result. For future reference, note that these equations imply that $\phi^1 = \alpha^2 = 0$.

(ii) If $0 < V_t < \bar{V}$, first-order conditions in period $t-1$ imply that $u'(c_{t-1})/v'(e+y_{t-1} - c_{t-1}) = -f'(V_t)$. We then have

$$(1+\theta^2)\frac{u'(c_t^2)}{v'(e+y_t^2-c_t^2)} = \frac{u'(c_t^1)}{v'(e+y_t^1-c_t^1)} - \lambda^1 = \frac{u'(c_{t-1})}{v'(e+y_{t-1}-c_{t-1})}.$$

This yields the following inequalities.

$$\frac{u'(c_t^2)}{v'(e+y_t^2-c_t^2)} < \frac{u'(c_{t-1})}{v'(e+y_{t-1}-c_{t-1})} < \frac{u'(c_t^1)}{v'(e+y_t^1-c_t^1)}$$

[9]The results of Proposition 3 to the effect that the optimal bounds on consumption are increasing in the states of the world hold for $B_t = 0$. In any period this can be easily generalized to any value of the ex ante payment, namely the optimal value.

Rearranging terms gives the result.

$$\frac{u'(c_t^2)}{u'(c_{t-1})} < \frac{v'(e + y_t^2 - c_t^2)}{v'(e + y_{t-1} - c_{t-1})} \text{ and } \frac{u'(c_t^1)}{u'(c_{t-1})} > \frac{v'(e + y_t^1 - c_t^1)}{v'(e + y_{t-1} - c_{t-1})}$$

(iii) This result is an immediate consequence of the above inequalities. *Q.E.D.*

APPENDICE B

Proof of Proposition 8 (i) First-order conditions are $\forall\, t,\, q$:

$$-u'(c^{q_t})+\beta\frac{\sum_{s_{t+1}}\partial f(s_{t+1}|I^{q_t})[u(c^{s_{t+1}})+\beta\sum_{s_{t+2}}f(s_{t+2}|I^{s_{t+1}})u(c^{s_{t+2}})+\beta^2\cdots]}{\partial I^{aq_t}}=0. \quad (12)$$

The optimal choice of investment is clearly independent of history. We may thus write the problem as the following dynamic problem [10]

$$g_s(y^{k_t})=\max_{I^{ak_t}}u(c^{k_t})+\beta\sum_{s_{t+1}}f(y^{s_{t+1}}|I^{ak_t})g(y^{s_{t+1}})\quad \forall\, t \tag{13}$$

The optimal investment choice is characterized by

$$-u'(c^{k_t})+\beta\frac{\partial\sum_{s_{t+1}}f(y^{s_{t+1}}|I^{ak_t})g_s(y^{s_{t+1}})}{\partial I^{ak_t}}=0\quad\forall\, t \tag{14}$$

which implies that $I^{k_t}=I^k$.

(ii) If we use the Implicit Function Theorem and differentiate (14), we obtain

$$\frac{\partial I^{ak_t}}{\partial y^{k_t}}=-\frac{-u''(c^{k_t})}{u''(c^{k_t})+\beta\frac{\partial^2 E_{I^{ak_t}}g_s(y^{s_{t+1}})}{\partial {I^{ak_t}}^2}}>0 \tag{15}$$

(iii) The convexity in I of the distribution function of y implies that $\frac{\partial I^{ak}}{\partial y^k}<1$. This in turn implies that $\frac{\partial(y^k-I^{ak})}{\partial y^k}>0$. Thus $c^k>c^q\quad\forall\, k>q$. *Q.E.D.*

Proof of Proposition 9 The next formulation of the problem is motivated in Proposition 3 as is given the proof of existence and unicity of the solution when the problem is more constrained. We omit to repeat it unnecessarily here. Problem (29) in the main text can be written, $\forall t$, k, as:

$$g_k(V^{k_t})=\max_{I^{k_t},a^{k_t},(V^{s_{t+1}})_s}u(y^{k_t}-I^{k_t}-a^{k_t})+\beta\sum_{s_{t+1}}f(s_{t+1}|I^{k_t})g_s(V^{s_{t+1}})$$
$$V^{k_t}\le a^{k_t}+\beta\sum_{s_{t+1}}f(s_{t+1}|I^{k_t})V^{s_{t+1}}$$

With λ^{k_t} as the multiplier of the constraint, the first-order conditions are:

$$I^{k_t}:\quad -u'(c^{k_t})+\beta\sum_s\frac{\partial f(s_{t+1}|I^{k_t})}{\partial I^{k_t}}g_s(V^{s_{t+1}})+\lambda^{k_t}\beta\sum_s\frac{\partial f(s_{t+1}|I^{k_t})}{\partial I^{k_t}}V^{s_{t+1}}=0 \tag{16}$$

$$a^{k_t}:\quad -u'(c^{k_t})+\lambda^{k_t}=0 \tag{17}$$

$$V^{s_{t+1}}:\quad g_s'(V^{s_{t+1}})+\lambda^{k_t}=0\quad\forall\, s \tag{18}$$

[10]The existence, unicity and concavity of the following value function is proved for the most constrained case in Propoposition 3. We omit to repeat it unnecessarily here.

and the envelope condition is

$$g'_k(V^{k_t}) = -\lambda^{k_t}. \tag{19}$$

(i) Equations (17), (18), and (19) implies that $c^{k_t} = c^{q_{t+i}} \equiv c^{fc} \quad \forall\, i,\, k,\, q.$

(ii) Equations (18) and (19) $\Rightarrow g'_s(V^{s_{t+1}}) = g'_s(V^{s_t}) \Rightarrow V^{s_{t+1}} = V^s \quad \forall t,\, s.$ Equations (16), (17) and (i) implies

$$-u'(c^{fc}) + \beta \frac{\partial \sum_{s_{t+1}} f(y^{s_{t+1}}|I^{q_t})[g_s(V^s) + u'(c^{fc})V^s]}{\partial I^{q_t}} = 0 \quad \forall\, q, t. \tag{20}$$

Since the expression in brackets is independent of time and current state, this implies that $I^{q_t} = I^q = I^k \equiv I^{fc} \quad \forall\, q, t.$ *Q.E.D.*

Proof of Proposition 10 (i) Consider the optimal contract with full commitment δ^{fc} characterized in Proposition 9. The per-period utility of agent 1 in state s is $u(c^{fc})$. We can easily show, by using the results of Proposition 1, that U^{as} is the biggest in state S. Thus, it is sufficient for δ^{fc} to be self-enforcing that $u(c^{fc}) + \beta \sum_{s_{t+1}} f(s_{t+1}|I^{S_t}) U(\delta; h_t, s_{t+1}) \geq U^{aS}$. This is equivalent to $\frac{u(c^{fc})}{1-\beta} \geq U^{aS}$. Since the left-hand side of the last inequality is increasing in β, the critical β over which δ^{fc} is self- enforcing, β_1, is

$$\beta_1 = \frac{U^{aS} - u(c^{fc})}{U^{aS}}.$$

(ii) The proof of this part of the proposition is more involved and we need to introduce some notation. Define by $\Omega(h_t)$ the set of contracts satisfying the self-enforcing constraints for agent 1 following history h_t. The Pareto frontier is then given by a time-independent function

$$g_s(V^{s_t}) = \max_{\delta \in \Omega(h_t)} \{U(\delta; h_{t-1}, s) \text{ s/t } V(\delta; h_{t-1}, s) \geq V^{s_t}\}.$$

The frontier is time-independent since the functions $U(.)$, $V(.)$, and the set $\omega(.)$ are all forward looking. This frontier implicitly defines the set of all efficient self-enforcing contracts for agent 1. Following any history, the optimal contract δ^1 will necessarily be efficient, since if it was not it would be possible to replace the non-efficient path by an efficient path thus relaxing all previous self-enforcing constraints. This new contract would necessarily be self-enforcing and would dominate the old contract at the beginning of the relationship. This argument implies that the optimal contract from the start of period t onwards is the solution to the following maximization problem.

$$
\begin{aligned}
g_k(V^{k_t}) = \max_{I^{k_t}, a^{k_t}, (V^{s_{t+1}})_s} \quad & u(y^{k_t} - I^{k_t} - a^{k_t}) + \beta \sum_{s_{t+1}} f(y^{s_{t+1}} | I^{k_t}) g_s(V^{s_{t+1}}) \\
& g_s(V^s_{t+1}) \geq U^{as} \quad \forall\, s \\
& V^k_t \leq a^k_t + \beta \sum_{s_{t+1}} f(y^{s_{t+1}} | I^{k_t}) V^{s_{t+1}}
\end{aligned} \tag{21}
$$

In this problem, the variable V^s_{t+1} is to be interpreted as $V(\delta; h_{t-1}, k, s)$, that is, agent 2's expected profit from period $t+1$ onwards when contract δ is signed, k is the realized state of nature in period t, and s is a realized state in period $t+1$. The last constraint of the problem ensures that the contract is dynamically consistent. We will first show that problem (21) is a concave program and then use its first-order conditions to characterize the optimal contract δ[11].

To show that problem (21) is a concave program, we have to show that the set $\Omega(h_t)$ is convex, that is, that the self-enforcing constraints are concave. The first term in the self-enforcing constraints is $u(.)$ which is concave. We will now see that the third and following terms in these constraints can be simplified. The third term is

$$
\sum_{s_{t+1}} f(y^{s_{t+1}} | I^{s_t}) \sum_{s_{t+2}} f(y^{s_{t+2}} | I^{s_{t+1}}) u(c^{s_{t+2}}). \tag{22}
$$

By the assumption of independence of the conditional density functions, $f(y^{s_{t+2}} | I^{s_{t+1}})$ is independent of $f(y^{s_{t+1}} | I^{s_t})$ and therefore, the first summation in equation (22) is equal to 1. Thus (22) is equivalent to

$$
\sum_{s_{t+2}} f(y^{s_{t+2}} | I^{s_{t+1}}) u(c^{s_{t+2}}).
$$

Of course, this can be generalized to every following terms in the self-enforcing constraints. Hence, it is sufficient to show that the self-enforcing constraints are concave, to show that

$$
\sum_{s_t} f(y^{s_{t+1}} | I^{s_t}) u(c^{s_{t+1}})
$$

[11]The arguments showing that problem (21) is a concave program can be proven formally by using a proof similar to that employed in Lemma 1 of Thomas and Worrall (1988).

is concave. That is, we have to show that:[12]

$$\int u(c^\lambda)dF(y|I_t^\lambda) \geq \lambda u(c')dF(y|I_t') + (1-\lambda)\int u(c'')dF(y|I_t'') \tag{23}$$

where $t+1$ is meant when no time subscript is specified, and where $k^\lambda = \lambda k' + (1-\lambda)k''$ for $k \in \{c, I\}$. If we use integration by parts the inequation (23) can be written

$$u(c_S^\lambda) - \lambda u(c_S') - (1-\lambda)u(c_S'') - \int u'(c^\lambda)F(y|I^\lambda) \geq -\lambda\int u'(c')F(y|I') - (1-\lambda)\int u'(c'')F(y|I'') \tag{24}$$

By concavity of $u(.)$, we know that the first three terms are positive. If we integrate by parts the three last terms, it is now sufficient to show that

$$\begin{aligned} u'(c^\lambda)\int F(y|I^\lambda)|_{y^1}^{y^S} - \int\int F(y|I^\lambda)u''(c^\lambda) \leq \lambda[u'(c')\int F(y|I')|_{y^1}^{y^S} - \int\int F(y|I')u''(c')] \\ +(1-\lambda)[u'(c'')\int F(y|I'')|_{y^1}^{y^S} - \int\int F(y|I'')u''(c'')] \end{aligned} \tag{25}$$

The three terms in $u''(.)$ can be written as

$$-\int u''(.)\left\{\int F(y|I^\lambda) - \lambda\int F(y|I') - (1-\lambda)\int F(y|I'')\right\} \tag{26}$$

by the assumption of quadratic utility function. These three terms sums to zero by the mean-preserving spread assumption which implies that $\int F(y|I)$ is invariant in I. By Taylor's theorem we now can write the three other terms of equation (25) as:

$$u''(.)c^\lambda|_{y^1}^{y^S}\int F(y|I^\lambda) - \lambda\left\{u''(.)c'|_{y^1}^{y^S}\int F(y|I')\right\} - (1-\lambda)\left\{u''(.)c''|_{y^1}^{y^S}\int F(y|I'')\right\} \tag{27}$$

This last expression also sums to zero by the mean-preserving spread assumption. This proves the concavity of the self-enforcing constraints. Second, we argue that the Pareto frontier is decreasing, strictly concave and continuously differentiable. The strict concavity property follows from the strict concavity of $u(\cdot)$ and the convexity of $\Omega(\cdot)$. The differentiability property follows from the continuity and differentiability of $u(\cdot)$. This could be shown by constructing a differentiable concave function that lies below the function $g(\cdot)$ in the neighbourhood of one point. Lemma 1 reported in Benveniste and Scheinkman (1979) then

[12]We suppose here a continuum of states in order to simplify the proof by the use of integration by parties. Moreover, we omit the integration bounds which are always the same, namely y^1 and y^S.

proves the result. Finally, following history h_{t-1}, the set of V_t^s such that a self-enforcing contract for agent 1 exists is the compact interval $[-K_1, \bar{V}^s]$ where $\bar{V}^s = g_s^{-1}(U^{as})$ and $-K_1$ is the discounted utility of agent 2 when it pays out the maximum feasible transfer to agent 1 in every period, namely $a^{s_t} = -b < 0$, and investment is I^{a1} for all dates and periods. Such transfer and investment are obviously self-enforcing for agent 1. The closedness of this interval can be shown by constructing a sequence of self-enforcing contracts yielding some utility level to agent 2 converging to $\bar{V}^s$. Since $u(\cdot)$ is continuous, the Dominated Convergence Theorem implies that the limiting contract is also self-enforcing and hence $\bar{V}^s$ is included in the interval. This implies that for any value $V_t^s \in [-K_1, \bar{V}^s]$, there exists a unique efficient continuation of the contract δ at time t in which $V(\delta; h_{t-1}) = V_t$ and $U(\delta; h_{t-1}) = g(V_t)$. Existence is guaranteed by the compactness of the interval $[-K_1, \bar{V}^s]$; uniqueness is guaranteed by the convexity of $\Omega(\cdot)$ and the concavity of $u(\cdot)$. These results effectively imply that problem (21) is a concave program, and therefore its first-order conditions are both necessary and sufficient for a solution.

Let $\beta\alpha^s$ and ψ^k be the respective multipliers of the constraints in problem (21). The first-order conditions are then

$$I^{k_t}: \quad -u'(y^{k_t} - I^{k_t} - a^{k_t}) + \beta \frac{\partial \sum_{s_{t+1}} f(y^{s_{t+1}}|I^{k_t})}{\partial I^{k_t}} \left\{ g_s(V^{s_{t+1}}) + \psi^k V^{s_{t+1}} \right\} = 0 \quad (28)$$

$$a^{k_t}: \quad -u'(y^{k_t} - I^{k_t} - a^{k_t}) + \psi^k = 0 \quad (29)$$

$$V^{s_{t+1}}: \quad (1 + \frac{\alpha^s}{f(y^{s_{t+1}}|I^{s_t})}) g_s'(V^{s_{t+1}}) + \psi^k = 0 \quad \forall s \quad (30)$$

and the envelope condition is

$$g_k'(V^{k_t}) = -\psi^k. \quad (31)$$

Define $\underline{c}^s$ by $-u'(\underline{c}^s) = g_s'(\bar{V}^s)$. By concavity of $g(.)$ and by the definition of $\bar{V}^s$, we must have that $c^{s_t} \geq \underline{c}^s$. (iii) Define $v^{s_t} \equiv y^{s_t} - a^{s_t}$. We have that

$$\begin{aligned} g_k(V^{k_t}) = \quad & \max_{I^{k_t}, v^{k_t}, (V^{s_{t+1}})_s} u(v^{k_t} - I^{k_t}) + \beta \sum_{s_{t+1}} f(y^{s_{t+1}}|I^{k_t}) g_s(V^{s_{t+1}}) \\ & g_s(V_{t+1}^s) \geq U^{as} \quad \forall s \\ & V^{k_t} \leq y^{k_t} - v^{k_t} + \beta \sum_{s_{t+1}} f(y^{s_{t+1}}|I^{k_t}) V^{s_{t+1}} \end{aligned} \quad (32)$$

In the same way, in state q we can write

$$
\begin{aligned}
g_q(V^{q_t}) = \quad & \max_{I^{q_t}, v^{q_t}, (V^{s_{t+1}})_s} u(v^{q_t} - I^{q_t}) + \beta \sum_{s_{t+1}} f(y^{s_{t+1}} | I^{q_t}) g_s(V^{s_{t+1}}) \\
& g_s(V^s_{t+1}) \geq U^{as} \quad \forall s \\
& V^{q_t} \leq y^{q_t} - v^{q_t} + \beta \sum_{s_{t+1}} f(y^{s_{t+1}} | I^{q_t}) V^{s_{t+1}}
\end{aligned}
\tag{33}
$$

Written this way, it is clear that

$$
g_k(V^k_t) \equiv g_q(V^k_t + y^q_t - y^k_t). \tag{34}
$$

This implies that $g'_k(V^{k_t}) \equiv g'_q(V^{k_t} + y^{q_t} - y^{k_t})$. When $V^{k_t} = \bar{V}^k$, we have that $U^{ak} = g_q(\bar{V}^k + y^q - y^k)$. Moreover, $g_q(\bar{V}^q) = U^{aq} < U^{ak}$ by Proposition 8. The concavity of $g_q(V^q)$ implies that $g'_q(\bar{V}^k + y^q - y^k) > g'_q(\bar{V}^q)$. Thus by (34), $g'_k(\bar{V}^k) > g'_q(\bar{V}^q)$. Since $u'(\underline{c}^{s_t}) = -g'_s(\bar{V}^s) \Rightarrow \underline{c}^s = u'^{-1}(-g'_s(\bar{V}^s)) \Rightarrow \underline{c}^k > \underline{c}^q$. *Q.E.D.*

(iv) Equations (29) and (31) $\Rightarrow u'(c^s_t) = -g'(V^s_t)$. Since the first-order conditions are verified $\forall t$ and $\forall s$, $u'(c^q_{t+1}) = -g'_q(V^q_{t+1})$. Equations (30) and (31) $\Rightarrow (1 + \frac{\alpha^q}{f(y^{q_{t+1}}|I^{s_t})}) g'_q(V^q_{t+1}) = g'_s(V^s_t)$. This implies that $-(1+\frac{\alpha^q}{p^q_{I^{s_t}}}) u'(c^q_{t+1}) = -u'(c^s_t)$. Suppose that $c^s_t \geq \underline{c}^q$ and $c^q_{t+1} \neq c^s_t$. If $c^q_{t+1} > c^s_t$ then, the above expression implies $\alpha^q > 0 \Rightarrow c^q_{t+1} = \underline{c}^q \Rightarrow \underline{c}^q > c^s_t$ which contradicts the initial assumption. If $c^q_{t+1} < c^s_t$, then $\alpha^q < 0$ which is impossible. Thus $c^s_t \geq \underline{c}^q \Rightarrow c^q_{t+1} = c^s_t$. Now suppose that $c^s_t < \underline{c}^q$ and $c^q_{t+1} > \underline{c}^q$. If $c^q_{t+1} > \underline{c}^q$ then $\alpha^q = 0 \Rightarrow c^q_{t+1} = c^s_t \Rightarrow c^s_t > \underline{c}^q$ which contradicts the initial assumption. *Q.E.D.*

Proof of Proposition 11 (i) At the steady state, we have $u'(\underline{c}^S) = -g'(\bar{V}^S)$ by Proposition 10; This implies that $V^{s_t} = V^s$ and $\psi^{k_t} = \psi^k$. Thus the marginal revenue of investment is equal to $\sum_s f(y^s|I^k)\{g_s(V^s) + \psi^q\}$ which is equal to the constant marginal cost, that is $u'(\underline{c}^k)$. That implies that $I^q = I, \forall q$ at steady state. Moreover, $c^k = c^q$ implies that $y^k - a^k - I = y^q - a^q - I$ which implies that $a^k > a^q \Rightarrow V^k > V^q$ at the steady state.

(ii) With two states of the world, $y^k > y^q$, we have $\frac{\partial f(y^k|I)}{\partial I} = -\frac{\partial f(y^q|I)}{\partial I} > 0$; Using (31), equation (28) becomes

$$
-u'(\underline{c}^k) + \beta \frac{\partial f(y^k|I)}{\partial I} [g_k(V^k) + \psi^q V^k - g_q(V^q) - \psi^q V^q] = 0 \tag{35}
$$

Using equations (29) and (31), and using the fact that $g'_q(V^{q_{t+1}}) = g'_q(V^{q_t})$ implies that

$\frac{dV^{q_{t+1}}}{dV^{q_t}} = 1$, we can differentiate (35):

$$\frac{\partial I}{\partial V^q} = -\frac{g_q''(V^q)\left\{1 - \beta\frac{\partial f(y^k|I)}{\partial I}[V^k - V^q]\right\} + \beta\frac{\partial f(y^k|I)}{\partial I}(-2g_q'(V^q))}{(-)} \tag{36}$$

where we use the fact that $g_k'(V^k) = g_q'(V^q)$ at steady state. The sufficient condition for this derivative to be positive follows.