Strategy-Proofness and Essentially Single-Valued Cores Revisited*

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Abstract

We consider general allocation problems with indivisibilities where agents' preferences possibly exhibit externalities. In such contexts many different core notions were proposed. One is the γ-core whereby blocking is only allowed via allocations where the non-blocking agents receive their endowment. We show that if there exists an allocation rule satisfying *individual rationality*, *efficiency*, and *strategy-proofness*, then for any problem for which the γ-core is non-empty, the allocation rule must choose a γ-core allocation and all agents are indifferent between all allocations in the γ-core. We apply our result to housing markets, coalition formation and networks.

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1 Introduction

In an influential paper, Sönmez (1999) established an important result for rules satisfying *individual rationality*, *efficiency*, and *strategy-proofness* in various allocation problems. More precisely he showed that if there exists a rule satisfying these three properties, then for any problem having a non-empty core, the core must be essentially single-valued (meaning that all agents are indifferent between all allocations belonging to the core) and the rule must choose a core allocation.

This unifies several well-known results in the allocation literature. In marriage problems (Gale and Shapley, 1962) it has been shown that no rule satisfies *strategy-proofness* and stability (Roth, 1982a) and no rule satisfies *individual rationality*, *efficiency*, and *strategy-proofness* (Alcalde and Barberà, 1994). Since in marriage problems the core is non-empty and not necessarily essentially single-valued, these results follow from Sönmez (1999). In housing markets (Shapley and Scarf, 1974) the core is always single-valued (Roth and Postlewaite, 1977) and the core is the unique rule satisfying *individual rationality*, *efficiency*, and *strategy-proofness* (Ma, 1994). Again those two results follow from Sönmez (1999).[1]

An important feature of Sönmez's result is that it applies to allocation problems both *with* externalities and *without* externalities. Since Sönmez's result only holds for one direction, subsequent literature established if-and-only-if versions of it in certain contexts. Indeed, already Sönmez (1999) showed that when the core is essentially single-valued and externally stable[2], the core is weakly coalitionally strategy-proof.[3] Related results were established by Ma (1998) in coalition formation models and by Takamiya (2003). The latter showed in environments where the core is essentially single-valued for all profiles and agents care only about their own consumption and have strict preferences, then any selection from the core satisfies *strategy-proofness*.

[1]Roth (1982b) has shown *strategy-proofness* of the core and it is straightforward that the core satisfies *individual rationality* and *efficiency*.

[2]Any allocation not belonging to the core is weakly dominated by a core element.

[3]This is a variant of a theorem in Demange (1987).

Pápai (2004) characterizes the coalition formation models having for each profile a single-valued core and then shows in any such model the core is the only rule satisfying *individual rationality*, *efficiency*, and *strategy-proofness*. Rodriguez-Alvarez (2009) shows that this result continues to hold on smaller domains which are minimally rich.

The results in the previous paragraph apply to contexts without externalities. In reality externalities prevail and it is important to understand the consequences of them. Sönmez (1999) did not rule out externalities but his blocking notion is very strong in those environments. Namely a coalition can block an allocation via another allocation irrespectively of what is happening to the other agents' allotments. For example, a coalition may block via allocations which are not *individually rational* and/or where some of the undesired endowments of the blocking coalition are consumed by other agents. Since the (optimistic) core is often empty in environments with externalities, alternative core notions have been proposed. One is the γ-core (Hart and Kurz, 1983; Chander and Tulkens, 1997) whereby coalitions can only block via allocations where each agent not belonging to the coalition receives his endowment (and such allocations are individually rational for the non-blocking agents). The interpretation is that any allocation is a global agreement and once a coalition deviates (or forms) all non-deviating agents receive their endowments. The core is always a subset of the γ-core and the γ-core is more frequently non-empty than the core.

We show that if there exists a rule satisfying *individual rationality*, *efficiency*, and *strategy-proofness*, then for any problem having a non-empty γ-core, the γ-core must be essentially single-valued (meaning that all agents are indifferent between all allocations belonging to the γ-core) and the rule must choose a γ-core allocation. Sönmez's result follows from our result (but not vice versa). In addition, our proof is extremely short and straightforward. We show the importance of our result by demonstrating that the γ-core is the "largest" core notion whereby such a result can be obtained. The "larger" the core, the more powerful a result in the sense of Sönmez (1999) becomes.

The note is organized as follows. In Section 2 we present the model and the main result of Sönmez (1999). In Section 3 we give our main results for essentially single-valued γ-cores. In Section 4 we apply our results to several environments with externalities.

2 The Model

A generalized indivisible goods allocation problem, or simply a problem, is a 4-tuple $(N, w, \mathcal{A}^f, R)$ where $N = \{1, \ldots, n\}$ is a finite set of agents, $w = (w(1), \ldots, w(n))$ is an initial endowment, $\mathcal{A}^f$ is a set of feasible allocations, and $R = (R_1, \ldots, R_n)$ is a list of preference relations. Each agent $i \in N$ is endowed with the (finite) set of indivisible goods $w(i)$. For all $T \subseteq N$, let $w(T) = \cup_{i \in T} w(i)$. An allocation is a mapping $a : N \twoheadrightarrow w(N)$ such that for all $x \in w(N)$, $|a^{-1}(x)| = 1$. Then $a(i)$ is the set of indivisible goods assigned to i. Let $\mathcal{A}$ denote the set of all allocations.

We exogenously specify a set $\mathcal{A}^f \subseteq \mathcal{A}$ as the set of feasible allocations. We require that $w \in \mathcal{A}^f$. Each agent i is endowed with a set of preference relations $\mathcal{R}_i$. Any preference relation $R_i \in \mathcal{R}_i$ is a complete, transitive and reflexive binary relation on $\mathcal{A}^f$. Let P_i denote the strict preference relation and I_i denote the indifference relation induced by R_i.

Throughout the set $\mathcal{R}_i$ is supposed to satisfy the following two conditions.

ASSUMPTION A: For all $R_i \in \mathcal{R}_i$ and all $a \in \mathcal{A}^f$, $aI_iw \Leftrightarrow a(i) = w(i)$.

Thus, an agent is indifferent between an allocation and the initial endowment allocation if and only if he keeps his initial endowment.

ASSUMPTION B: For all $R_i \in \mathcal{R}_i$ and all $a \in \mathcal{A}^f$ with aR_iw, there exists $\tilde{R}_i \in \mathcal{R}_i$ such that

1. for all $b \in \mathcal{A}^f \backslash \{a\}$, $bR_i a \Leftrightarrow b\tilde{R}_i a$,

2. for all $b \in \mathcal{A}^f \backslash \{a\}$, $aR_i b \Leftrightarrow a\tilde{R}_i b$, and

3. for all $b \in \mathcal{A}^f \backslash \{a\}$, $aP_i b \Leftrightarrow a\tilde{P}_i b$ and $a\tilde{R}_i w \tilde{R}_i b$.

Let $\mathcal{R} = \times_{i \in N} \mathcal{R}_i$. For all $R \in \mathcal{R}$ and all $T \subseteq N$, let R_T denote the restriction of R to T and let R_{-T} denote the restriction of R to $N \backslash T$. If $T = \{i\}$, then we write R_{-i} instead of $R_{-\{i\}}$. Throughout N, w and $\mathcal{A}_f$ remain fixed and for short a problem is a profile $R \in \mathcal{R}$.

A coalition is a non-empty subset of N. An allocation $a \in \mathcal{A}^f$ weakly dominates the allocation $b \in \mathcal{A}^f$ via the coalition T under R if (i) $a(i) \subseteq w(T)$ for all $i \in T$, (ii) $aR_i b$ for all $i \in T$, and (iii) $aP_j b$ for some $j \in T$. We also write $a\ wdom_T\ b$. The core consists of all undominated allocations. For all $R \in \mathcal{R}$, let

$$C(R) = \{b \in \mathcal{A}^f : \nexists T \text{ and } a \in \mathcal{A}^f \text{ with } a\ wdom_T\ b\}.$$

The core is *essentially single-valued* if for all $R \in \mathcal{R}$ and all $a, b \in C(R)$, we have $aI_i b$ for all $i \in N$.

An allocation rule is a function $\varphi : \mathcal{R} \to \mathcal{A}^f$. Given profile R, let $\varphi(R)$ denote the allocation chosen by φ for profile R. We are interested in the following three properties: (i) *individual rationality*: for any profile each agent weakly prefers the chosen allocation to his endowment; (ii) *efficiency*: no other feasible allocation Pareto dominates the chosen allocation; (iii) *strategy-proofness*: reporting the truth is a weakly dominant strategy.

Individual Rationality: For all $R \in \mathcal{R}$ and all $i \in N$, $\varphi(R) R_i w$.

Efficiency: For all $R \in \mathcal{R}$, there exists no $a \in \mathcal{A}^f$ such that $aR_i\varphi(R)$ for all $i \in N$ and $aP_j\varphi(R)$ for some $j \in N$.

Strategy-Proofness: For all $R \in \mathcal{R}$, all $i \in N$, and all $R'_i \in \mathcal{R}_i$, $\varphi(R) R_i \varphi(R'_i, R_{-i})$.

The principal result of Sönmez (1999) relates the existence of rules satisfying the above three properties with essentially single-valued cores.

Theorem 1 (Sönmez, 1999) *If there exists an allocation rule $\varphi : \mathcal{R} \to \mathcal{A}^f$ that is individually rational, efficient, and strategy-proof, then N, w, $\mathcal{A}^f$, $\mathcal{R}$ are such that*

1. *For all $R \in \mathcal{R}$, for all $a, b \in C(R)$, we have $a I_i b$ for all $i \in N$;*

2. *For all $R \in \mathcal{R}$ with $C(R) \neq \emptyset$, we have $\varphi(R) \in C(R)$.*

Sönmez (1999) also established a converse result. For this the core needs to be *externally stable*: for any R and any $b \in \mathcal{A}^f \backslash C(R)$, there exists $a \in C(R)$ and $\emptyset \neq T \subseteq N$ such that $a \; wdom_T \; b$.

Weak Coalitional Strategy-Proofness: For all $R \in \mathcal{R}$, all $T \subseteq N$, and all $R'_T \in \mathcal{R}_T$, there exists $i \in T$ with $\varphi(R) R_i \varphi(R'_T, R_{-T})$.

Proposition 1 (Sönmez, 1999) *Let N, w, $\mathcal{A}^f$, $\mathcal{R}$ be such that the core correspondence is essentially single-valued and the core of each problem is externally stable. Then any selection from the core correspondence is weakly coalitionally strategy-proof.*

3 The γ-Core

Many papers have shown if-and-only-if versions of the result by Sönmez (1999) in environments where agents are selfish and have strict preferences over their own assignments, and allocations are "separable". However, his result allows for externalities in the sense that an agent might not be indifferent between two allocations where he receives the same set of objects.

Once externalities enter the environment, there are different notions of the core. In particular, in the above dominance relation, a dominates b via some coalition T

under R even if a is not individually rational for the agents outside of T and/or $a(T) \subsetneq w(T)$ meaning that some of the undesired endowments of coalition T are consumed by non-blocking agents under a. Under externalities, the core may be frequently empty and Theorem 1 does not apply. This is the optimistic notion of the core.

Our notion below will only allow blocking for coalitions with some allocation where all other agents receive their endowment. This corresponds to Hart and Kurz (1983)'s and Chander and Tulken (1997)'s γ-stability of agreements whereby an agreement (or allocation) is disbanded once a coalition deviates and the others stay put with their endowment. In the γ-core blocking is only allowed with allocations where the non-blocking agents receive their endowments. For any $R \in \mathcal{R}$, let

$$C_\gamma(R) = \{b \in \mathcal{A}^f : \nexists T \text{ and } a \in \mathcal{A}^f \text{ with } a \; wdom_T \; b \text{ and } a(i) = w(i) \text{ for all } i \in N \backslash T\}.$$

Obviously, $C(R) \subseteq C_\gamma(R)$ for any profile R. Assumption A guarantees that for the γ-core blocking only occurs via allocations which are individually rational for the non-blocking agents. In addition, the blocking coalition needs to consume all of their endowments, i.e. $a(T) = w(T)$.

Let $\mathcal{IR}(R)$ denote the set of feasible allocations which are individually rational under R.

Theorem 2 *If there exists an allocation rule $\varphi : \mathcal{R} \to \mathcal{A}^f$ that is individually rational, efficient, and strategy-proof, then N, w, $\mathcal{A}^f$, $\mathcal{R}$ are such that*

1. *For all $R \in \mathcal{R}$, for all $a, b \in C_\gamma(R)$, we have aI_ib for all $i \in N$;*
2. *For all $R \in \mathcal{R}$ with $C_\gamma(R) \neq \emptyset$, we have $\varphi(R) \in C_\gamma(R)$.*

Proof. Let $R \in \mathcal{R}$. If $C_\gamma(R) = \emptyset$, then there is nothing to show. Let $C_\gamma(R) \neq \emptyset$. If 1. or 2. does not hold, then there exists $a \in C_\gamma(R)$ such that not $aI_i\varphi(R)$ for all $i \in N$.[4] Because both a and $\varphi(R)$ are efficient, there exists $i \in N$ with $aP_i\varphi(R)$. By individual rationality, $\varphi(R)R_iw$. We distinguish two cases: $\varphi(R)P_iw$ or $\varphi(R)I_iw$.

[4]If $aI_i\varphi(R)$ for all $i \in N$, then $\varphi(R) \in C_\gamma(R)$.

First, let $\varphi(R)P_i w$. Note that $aR_i w$. Let $\tilde{R}_i \in \mathcal{R}_i$ be such that

1. for all $b \in \mathcal{A}^f \backslash \{a\}$, $bR_i a \Leftrightarrow b\tilde{R}_i a$,

2. for all $b \in \mathcal{A}^f \backslash \{a\}$, $aR_i b \Leftrightarrow a\tilde{R}_i b$, and

3. for all $b \in \mathcal{A}^f \backslash \{a\}$, $aP_i b \Leftrightarrow a\tilde{P}_i b$ and $a\tilde{R}_i w\tilde{R}_i b$.

Now observe that Assumption A and $\varphi(R)P_i w$ imply $\varphi(R)(i) \neq w(i)$. Also $aP_i\varphi(R)$ implies $a\tilde{P}_i\varphi(R)$ and $a\tilde{R}_i w\tilde{R}_i\varphi(R)$. Thus, by Assumption A and $\varphi(R)(i) \neq w(i)$, we have $w\tilde{P}_i\varphi(R)$. Now $\varphi(R) \notin \mathcal{IR}(\tilde{R}_i, R_{-i})$ and $\mathcal{IR}(\tilde{R}_i, R_{-i}) \subsetneq \mathcal{IR}(R)$[5].

By strategy-proofness, $\varphi(R)R_i\varphi(\tilde{R}_i, R_{-i})$. Hence, $aP_i\varphi(\tilde{R}_i, R_{-i})$. Now by 3., $a\tilde{P}_i\varphi(\tilde{R}_i, R_{-i})$ and $a\tilde{R}_i w\tilde{R}_i\varphi(\tilde{R}_i, R_{-i})$. Now this together with individual rationality and Assumption A implies $\varphi(\tilde{R}_i, R_{-i})(i) = w(i)$. Note that $a \in C_\gamma(\tilde{R}_i, R_{-i})$ and again 1. or 2. does not hold for $(\tilde{R}_i, R_{-i})$, and $\mathcal{IR}(\tilde{R}_i, R_{-i}) \subsetneq \mathcal{IR}(R)$.

Second, let $\varphi(R)I_i w$. By Assumption A, $\varphi(R)(i) = w(i)$. Since $a \in C_\gamma(R)$, we cannot have that $\varphi(R)$ weakly dominates a via $N\backslash\{i\}$. If $\varphi(R)I_j a$ for all $j \in N\backslash\{i\}$, then by $aP_i\varphi(R)$, a Pareto dominates $\varphi(R)$, a contradiction to efficiency of φ. Thus, there exists $j \in N\backslash\{i\}$ with $aP_j\varphi(R)$. If $\varphi(R)(j) \neq w(j)$, then by Assumption A, $\varphi(R)P_j w$, and we do the same as above. Otherwise, $\varphi(R)(j) = w(j)$ and again we cannot have that $\varphi(R)$ weakly dominates a via $N\backslash\{i,j\}$ (noting that $\varphi(R)(i) = w(i)$ and $\varphi(R)(j) = w(j)$). If $\varphi(R)I_h a$ for all $h \in N\backslash\{i,j\}$, then by $aP_i\varphi(R)$ and $\varphi(R)(j) = w(j)$, a Pareto dominates $\varphi(R)$, a contradiction to efficiency of φ. Note that we cannot have $\varphi(R) = w$ because otherwise a Pareto dominates w and φ is not efficient. Because N is finite, we eventually find l with $aP_l\varphi(R)$ and $\varphi(R)(l) \neq w(l)$. Then we do the same construction as above.

Because N and $\mathcal{A}^f$ are finite and the number of individually rational allocations becomes smaller at each step, this is a contradiction. □

[5]For all $b \in \mathcal{IR}(\tilde{R}_i, R_{-i})$, $b\tilde{R}_i w$; if $wP_i b$, then $b \neq a$ and by Assumption A, $b(i) \neq w(i)$; now $aP_i b$ and by construction and Assumption A, $w\tilde{P}_i b$, a contradiction to $b\tilde{R}_i w$.

Since $C(R) \subseteq C_\gamma(R)$ for any profile R, $C(R) \neq \emptyset$ implies $C_\gamma(R) \neq \emptyset$ and Theorem 1 follows from Theorem 2. Therefore, for any environment for which Theorem 1 is conclusive (i.e. for any profile R with $C(R) \neq \emptyset$), Theorem 2 is conclusive as well (i.e. $C_\gamma(R) \neq \emptyset$ and both 1. and 2. must hold for the γ-core and the allocation rule φ (and therefore, 1. and 2. must hold for the core and the allocation rule φ)). Of course, for any profile R where $C(R) = \emptyset$ and $C_\gamma(R) \neq \emptyset$, Theorem 1 has no bite whereas 1. and 2. of Theorem 2 must hold.

Using the proof of Proposition 1, we obtain the following conditional converse of Theorem 2.

Proposition 2 *Let N, w, $\mathcal{A}^f$, $\mathcal{R}$ be such that the γ-core correspondence is essentially single-valued and the γ-core of each problem is externally stable. Then any selection from the γ-core correspondence is weakly coalitionally strategy-proof.*

In the following we will argue that the γ-core is the "largest core notion" for which 1. and 2. of Theorem 2 hold in a preference domain free sense.

Given allocation a, we say that coalition T is *effective* for a if (i) $a(i) \subseteq w(T)$ for all $i \in T$ and (ii) if $|T| \geq 2$, then $a(i) \neq w(i)$ for all $i \in T$. Note that for any profile R, all $a, b \in \mathcal{A}^f$ and all $S \subseteq N$, if a weakly dominates b via S, then there exists $T \subseteq S$ which is effective for a and a weakly dominates b via T. For a coalition to block individually rational allocations, we only need to consider allocations which are effective for this coalition.

Fixing N, w, and $\mathcal{A}^f$, we define for each coalition a set of feasible allocations via which this coalition can weakly dominate other allocations *and* for which this coalition is effective: formally, let

$$B^f(T) = \{a \in \mathcal{A}^f : T \text{ is effective for } a\}.$$

Let $B^f = (B^f(T))_{T \subseteq N}$. In general, let $B(T) \subseteq B^f(T)$ for all $T \subseteq N$ and $B = (B(T))_{T \subseteq N}$. Now B records for each coalition T the set of feasible allocations via which T can weakly dominate other allocations and for which T is effective. The B-core is defined as follows: for all $R \in \mathcal{R}$, let

$$C_B(R) = \{b \in \mathcal{A}^f : \nexists T \text{ and } a \in B(T) \text{ with } a \; wdom_T \; b\}.$$

The B^f-core coincides with the core. Setting $B^\gamma(T) = \{a \in B^f(T) : a(i) = w(i)$ for all $i \in N \backslash T\}$ and $B^\gamma = (B^\gamma(T))_{T \subseteq N}$, the B^γ-core coincides with the γ-core.

Obviously, for any B such that for all $T \subseteq N$, $B^f(T) \supseteq B(T) \supseteq B^\gamma(T)$, we have for any domain $\mathcal{R}$ and any profile $R \in \mathcal{R}$, $C(R) \subseteq C_B(R) \subseteq C_\gamma(R)$. Then by Theorem 2, both 1. and 2. of Theorem 2 must hold for the B-core for any profile $R \in \mathcal{R}$ with $C_B(R) \neq \emptyset$.

We call a domain $\mathcal{R}$ solvable if there exists an allocation rule $\varphi : \mathcal{R} \to \mathcal{A}^f$ that is individually rational, efficient, and strategy-proof. Of course, if a domain is not solvable, then Theorem 2 holds for all B-cores and no comparison with the γ-core is possible. Indeed, we show that 1. and 2. of Theorem 2 hold for the B-core for any solvable preference domain if and only if the B-core is contained in the γ-core.

Let $\mathcal{PO}(R)$ denote the set of feasible allocations which are *efficient* under R.

Theorem 3 *Fix $(N, w, \mathcal{A}^f)$ and $B \subseteq B^f$. Then the following are equivalent:*

(i) For any solvable domain $\mathcal{R}$ and any allocation rule $\varphi : \mathcal{R} \to \mathcal{A}^f$ that is individually rational, efficient, and strategy-proof, N, w, $\mathcal{A}^f$, $\mathcal{R}$ are such that

1. *For all $R \in \mathcal{R}$, for all $a, b \in C_B(R)$, we have aI_ib for all $i \in N$;*
2. *For all $R \in \mathcal{R}$ with $C_B(R) \neq \emptyset$, we have $\varphi(R) \in C_B(R)$.*

(ii) For any solvable domain $\mathcal{R}$, $C_B \subseteq C_\gamma$.

Proof. (ii)⇒(i): Let $\mathcal{R}$ be a solvable domain. If $C_B \subseteq C_\gamma$, then by Theorem 2, 1. and 2. hold for the B-core, i.e. (i) holds.

(i)⇒(ii): In showing the other direction, let $\mathcal{R}$ be a solvable domain. Let $\varphi : \mathcal{R} \to \mathcal{A}^f$ be an allocation rule that is individually rational, efficient, and strategy-proof.

Suppose that 1. and 2. hold for the B-core but for some $R \in \mathcal{R}$, $C_B(R) \nsubseteq C_\gamma(R)$. Then $C_B(R) \neq \emptyset$ and 1. and 2. hold for $C_B(R)$, i.e. $\varphi(R) \in C_B(R)$.

We show $C_\gamma(R) = \emptyset$. If $C_\gamma(R) \neq \emptyset$, then by 2. of Theorem 2, $\varphi(R) \in C_\gamma(R)$. Thus, by 1. for the B-core and 1. of Theorem 2, for all $a \in C_B(R)$ and all $b \in C_\gamma(R)$, aI_ib for all $i \in N$. But then $C_B(R) \subseteq C_\gamma(R)$, a contradiction.

Hence, $C_\gamma(R) = \emptyset$. By Assumption A and individual rationality of φ, for all $i \in N$, if $\varphi(R)(i) \neq w(i)$, then $\varphi(R)P_iw$. Let $S = \{i \in N : \varphi(R)(i) \neq w(i)\}$. If $S = \emptyset$, then $\varphi(R) = w$ and by efficiency of φ, $w \in C_\gamma(R)$, a contradiction. Thus, $S \neq \emptyset$. Let $\tilde{R} \in \mathcal{R}$ be such that (i) for all $i \in N \backslash S$, $\tilde{R}_i = R_i$ and (ii) for all $i \in S$, $\tilde{R}_i \in \mathcal{R}_i$ is such that

1. for all $b \in \mathcal{A}^f \backslash \{\varphi(R)\}$, $bR_i\varphi(R) \Leftrightarrow b\tilde{R}_i\varphi(R)$,
2. for all $b \in \mathcal{A}^f \backslash \{\varphi(R)\}$, $\varphi(R)R_ib \Leftrightarrow \varphi(R)\tilde{R}_ib$, and
3. for all $b \in \mathcal{A}^f \backslash \{\varphi(R)\}$, $\varphi(R)P_ib \Leftrightarrow \varphi(R)\tilde{P}_ib$ and $\varphi(R)\tilde{R}_iw\tilde{R}_ib$.

Note that Assumption B guarantees the existence of $\tilde{R}$. It is straightforward that $\varphi(R) \in C_B(\tilde{R})$. Thus, by 1. and 2. for the B-core, $\varphi(\tilde{R}) \in C_B(\tilde{R})$ and for all $i \in N$, $\varphi(\tilde{R})\tilde{I}_i\varphi(R)$. Since $\varphi(R) \notin C_\gamma(R)$, we have for some $\tilde{a} \in \mathcal{A}^f$ and $T \subseteq N$, $\tilde{a}\ wdom_T\ \varphi(R)$ under R, $\tilde{a}(i) \neq w(i)$ for all $i \in T$, and $\tilde{a}(i) = w(i)$ for all $i \in N \backslash T$. By Assumption A, $\tilde{a} \in \mathcal{IR}(\tilde{R})$. Now by construction of $\tilde{R}$, $\tilde{a}\ wdom_T\ \varphi(R)$ under $\tilde{R}$. If $C_\gamma(\tilde{R}) \neq \emptyset$, then by 2. of Theorem 2, $\varphi(\tilde{R}) \in C_\gamma(\tilde{R})$. Now by $\varphi(R)\tilde{I}_i\varphi(\tilde{R})$ for all $i \in N$, we then also have $\varphi(R) \in C_\gamma(\tilde{R})$, a contradiction. Thus, $C_\gamma(\tilde{R}) = \emptyset$.

Note that $\tilde{a}\ wdom_T\ \varphi(R)$ under $\tilde{R}$, $\tilde{a}\ wdom_T\ \varphi(\tilde{R})$ under $\tilde{R}$, and $\tilde{a} \in \mathcal{IR}(\tilde{R})$. Without loss of generality, $\tilde{a} \in (\mathcal{IR} \cap \mathcal{PO})(\tilde{R})$: if not, then for some $c \in (\mathcal{IR} \cap \mathcal{PO})(\tilde{R})$, $c\ wdom_N\ \tilde{a}$. Let $W = \{i \in N : c(i) \neq w(i)\}$. Then $c\ wdom_W\ \tilde{a}$ under $\tilde{R}$ and by construction, $c\tilde{R}_i\varphi(R)$ for all $i \in W$. Thus, $c\ wdom_W\ \varphi(\tilde{R})$ under $\tilde{R}$.

For all $i \in N$, let $\hat{R}_i$ be such that

1. for all $b \in \mathcal{A}^f$, $b\hat{I}_iw \Leftrightarrow b(i) = w(i)$,

2. if $\tilde{a}(i) \neq w(i)$, then for all $b \in \mathcal{A}^f \backslash \{\varphi(\tilde{R})\}$, $b\hat{I}_i\tilde{a} \Leftrightarrow b = \tilde{a}$,

3. if $\varphi(\tilde{R})(i) \neq w(i)$, then for all $b \in \mathcal{A}^f \backslash \{\tilde{a}\}$, $b\hat{I}_i\varphi(\tilde{R}) \Leftrightarrow b = \varphi(\tilde{R})$,

4. $\tilde{a}\hat{R}_i\varphi(\tilde{R}) \Leftrightarrow \tilde{a}\tilde{R}_i\varphi(\tilde{R})$, and

5. for all $b \in \mathcal{A}^f$, if $b(i) \neq w(i)$, $b \neq \tilde{a}$ and $b \neq \varphi(\tilde{R})$, then $w\hat{R}_ib$.

Note that by construction of $\hat{R} = (\hat{R}_i)_{i\in N}$, we have $\varphi(\tilde{R}) \in C_B(\hat{R})$, $\tilde{a} \in (\mathcal{IR} \cap \mathcal{PO})(\hat{R})$, and $\tilde{a}$ $wdom_T$ $\varphi(\tilde{R})$ under $\hat{R}$.[6]

Given $i \in N$, let $\hat{\mathcal{R}}_i$ be defined as follows: $R_i \in \hat{\mathcal{R}}_i \Leftrightarrow$ (i) $R_i = \hat{R}_i$ or (ii) R_i satisfies Assumption A and there exists $a \in \mathcal{A}^f$ with $a(i) \neq w(i)$ and $a\hat{R}_iw$ such that

1. for all $b \in \mathcal{A}^f \backslash \{a\}$, $b\hat{R}_ia \Leftrightarrow bR_ia$,

2. for all $b \in \mathcal{A}^f \backslash \{a\}$, $a\hat{R}_ib \Leftrightarrow aR_ib$, and

3. for all $b \in \mathcal{A}^f \backslash \{a\}$, $a\hat{P}_ib \Leftrightarrow aP_ib$ and aR_iwR_ib.

It is easy to check that $\hat{\mathcal{R}}_i$ satisfies Assumptions A and B. Let $\hat{\mathcal{R}} = \times_{i\in N}\hat{\mathcal{R}}_i$. We define $\hat{\varphi}$ on the domain $\hat{\mathcal{R}}$ as follows: for all $R \in \hat{\mathcal{R}}$, (i) if $\tilde{a} \in (\mathcal{IR} \cap \mathcal{PO})(R)$, then $\hat{\varphi}(R) = \tilde{a}$, (ii) if $\tilde{a} \notin (\mathcal{IR} \cap \mathcal{PO})(R)$ and $\varphi(\tilde{R}) \in (\mathcal{IR} \cap \mathcal{PO})(R)$, then $\hat{\varphi}(R) = \varphi(\tilde{R})$, and (iii) otherwise, $\hat{\varphi}(R) = w$. Note that $\hat{R} \in \hat{\mathcal{R}}$, $\tilde{a} \in (\mathcal{IR} \cap \mathcal{PO})(\hat{R})$, $\hat{\varphi}(\hat{R}) = \tilde{a}$, and for some $i \in N$, we do not have $\tilde{a}\hat{I}_i\varphi(\tilde{R})$. Because $\varphi(\tilde{R}) \in C_B(\hat{R})$, now 1. or 2. does not hold for C_B and $\hat{\varphi}$.

It remains to show that $\hat{\varphi}$ is individually rational, efficient and strategy-proof on the domain $\hat{\mathcal{R}}$ (and therefore, $\hat{\mathcal{R}}$ is solvable). Individual rationality and efficiency are obvious. In order to demonstrate that $\hat{\varphi}$ is strategy-proof, let $R \in \hat{\mathcal{R}}$, $i \in N$, and $R'_i \in \hat{\mathcal{R}}_i$. By contradiction, suppose that $\hat{\varphi}(R'_i, R_{-i})P_i\hat{\varphi}(R)$. Let $\hat{\varphi}(R'_i, R_{-i}) = d$. Then by

[6] Note that here it is not possible to define a "one-profile" domain (à la Takamiya (2003)) whereby $\tilde{a}$ and w are the only individually rational allocations because for some $U \subsetneq T$ we may have $\tilde{a} \in B(U)$ and $\tilde{a}$ weakly dominates w via U (and one could let the rule choose $\tilde{a}$ for this profile without violating (i)).

individual rationality of $\hat{\varphi}$ and Assumption A, $d(i) \neq w(i)$. Thus, by construction, $d = \tilde{a}$ or $d = \varphi(\tilde{R})$.

If $d = \tilde{a}$, then by construction of $\hat{\mathcal{R}}$, we must have $\tilde{a} \in (\mathcal{IR} \cap \mathcal{PO})(R)$. By definition, then $\hat{\varphi}(R) = \tilde{a}$, which is a contradiction.

If $d = \varphi(\tilde{R})$, then by construction we must have $\hat{\varphi}(R) \in \{w, \tilde{a}\}$. If $\hat{\varphi}(R) = w$, then $dP_i w$ and for all $j \in N$, $\varphi(\tilde{R})R_j w$, which is a contradiction to efficiency of $\hat{\varphi}$. If $\hat{\varphi}(R) = \tilde{a}$ and $\tilde{a}(i) \neq w(i)$, then by construction of $\hat{\mathcal{R}}_i$, $\tilde{a}R_i d$, which is a contradiction. If $\hat{\varphi}(R) = \tilde{a}$ and $\tilde{a}(i) = w(i)$, then $\tilde{a} \in (\mathcal{IR} \cap \mathcal{PO})(R'_i, R_{-i})$ and by definition, $\hat{\varphi}(R'_i, R_{-i}) = \tilde{a}$, a contradiction. $\square$

Remark 1 If (i) $\mathcal{A}^f$ is separable in the following sense: for all $a, b \in \mathcal{A}^f$ and all $T \subseteq N$ such that $a(T) = w(T) = b(T)$, there exists $c \in \mathcal{A}^f$ such that for all $i \in T$, $c(i) = a(i)$, and for all $i \in N \backslash T$, $c(i) = b(i)$ and (ii) preferences are externality-free (or selfish) (for all $R \in \mathcal{R}$, all $i \in N$, and all $a, b \in \mathcal{A}^f$, if $a(i) = b(i)$, then $aI_i b$), then the core and the γ-core coincide. It is easy to see that both (i) and (ii) are necessary for the core to coincide with the γ-core, i.e. if (i) or (ii) does not hold, then the core does not necessarily coincide with the γ-core. In particular, selfish preferences do not suffice.

4 Applications

4.1 Global Trades

One application are global trades where either all agents receive their endowments or no agent receives his endowments. Let

$$\mathcal{A}^g = \{a \in \mathcal{A} : a(i) \neq w(i) \text{ for all } i \in N\}.$$

Now note that if $\mathcal{A}^f \subseteq \mathcal{A}^g \cup \{w\}$, then C_γ coincides with the set of individually rational and efficient allocations.

We obtain the following corollary from our main result.

Corollary 1 *Let $\mathcal{A}^f \subseteq \mathcal{A}^g \cup \{w\}$. There exists an allocation rule $\varphi : \mathcal{R} \to \mathcal{A}^f$ that is individually rational, efficient, and strategy-proof if and only if N, w, $\mathcal{A}^f$, $\mathcal{R}$ are such that*

1. *For all $R \in \mathcal{R}$, for all $a, b \in (\mathcal{IR} \cap \mathcal{PO})(R)$, we have aI_ib for all $i \in N$;*

2. *For all $R \in \mathcal{R}$, we have $\varphi(R) \in (\mathcal{IR} \cap \mathcal{PO})(R)$.*

Proof. (Only if) This follows directly from $C_\gamma = \mathcal{IR} \cap \mathcal{PO}$ and Theorem 2.

(If) It is easy to see that for any R, $(\mathcal{IR} \cap \mathcal{PO})(R)$ is externally stable and Proposition 2 yields the desired conclusion. □

Note that Corollary 1 does not follow from Theorem 1 because $C \neq \mathcal{IR} \cap \mathcal{PO}$. Furthermore, C may often be empty (because coalitions $T \subsetneq N$ may block with allocations $a \in \mathcal{A}^f$ for which $a(T) = T$) and Theorem 1 often does not apply whereas above C_γ is always non-empty.

Example 1 Let $N = \{1, 2, 3, 4\}$ and for all $i \in N$, $w(i) = i$. Let $\mathcal{R} = \{R\}$ and $\mathcal{A}^f = \{w, (2, 1, 3, 4), (2, 1, 4, 3), (1, 2, 4, 3)\}$ and where

R_1	R_2	R_3	R_4
$(2, 1, 4, 3)$	$(2, 1, 4, 3)$	$(1, 2, 4, 3)$	$(1, 2, 4, 3)$
$(1, 2, 4, 3), w$	$(1, 2, 4, 3), w$	$(2, 1, 3, 4), w$	$(2, 1, 3, 4), w$
$(2, 1, 3, 4)$	$(2, 1, 3, 4)$	$(2, 1, 4, 3)$	$(2, 1, 4, 3)$

Note that $\mathcal{R}$ satisfies Assumptions A and B and that $(\mathcal{IR} \cap \mathcal{PO})(R) = \{(1, 2, 4, 3)\} = C_\gamma(R)$ and $C(R) = \emptyset$. Even though there is a unique individually rational and efficient allocation the core fails to identify it because $(2, 1, 4, 3)$ $wdom_{\{1,2\}}$ $(1, 2, 4, 3)$. Of course, any individually rational and efficient rule must choose this allocation (which is also the unique γ-core allocation).

4.2 Constrained Efficiency

In many environments there does not exist any rule satisfying *individual rationality*, *efficiency*, and *strategy-proofness*. Then in order to obtain a positive result, one must weaken one of the properties. One possible route is to require constrained efficiency instead of (full) efficiency: the chosen allocation should be efficient on the range of the rule. Below we show that Theorem 2 has implications for such an approach, and that this allows us to identify "largest" sets of feasible allocations for which we may obtain a positive result.

Given $(N, w, \mathcal{A}^f)$ and any $R_i \in \mathcal{R}_i$, for $A \subseteq \mathcal{A}^f$ with $w \in A$, let $R_i|_A$ denote the restriction of R_i to A[7] Similarly, let $\mathcal{R}_i|_A = \{R_i|_A : R_i \in \mathcal{R}_i\}$. Obviously, if $\mathcal{R}_i$ satisfies Assumptions A and B, then $\mathcal{R}_i|_A$ satisfies Assumptions A and B. Let $\mathcal{R}|_A = \times_{i\in N}\mathcal{R}_i|_A$.

Let $\varphi : \mathcal{R} \to \mathcal{A}^f$ be an allocation rule. Let $A(\varphi) = \{\varphi(R) : R \in \mathcal{R}\} \cup \{w\}$.

Constrained Efficiency: For all $R \in \mathcal{R}$, there exists no $a \in A(\varphi)$ such that $aR_i\varphi(R)$ for all $i \in N$ and $aP_j\varphi(R)$ for some $j \in N$.

Note that in the corollary below, in 1. and 2. the γ-core is defined for the constrained set of allocations $A(\varphi)$.

Corollary 2 *If there exists an allocation rule $\varphi : \mathcal{R} \to \mathcal{A}^f$ that is individually rational, constrained efficient, and strategy-proof, then N, w, $A(\varphi)$, $\mathcal{R}|_{A(\varphi)}$ are such that*

1. *For all $R \in \mathcal{R}$, for all $a, b \in C_\gamma(R|_{A(\varphi)})$, we have aI_ib for all $i \in N$;*

2. *For all $R \in \mathcal{R}$ with $C_\gamma(R|_{A(\varphi)}) \neq \emptyset$, we have $\varphi(R) \in C_\gamma(R|_{A(\varphi)})$.*

Proof. Given $\varphi : \mathcal{R} \to \mathcal{A}^f$, we define $\phi : \mathcal{R}|_{A(\varphi)} \to A(\varphi)$ as follows: for any $R_i \in \mathcal{R}_i|_{A(\varphi)}$ fix some $l_i(R_i) \in \mathcal{R}_i$ such that $l_i(R_i)|_{A(\varphi)} = R_i$; let $l(R) = (l_i(R_i))_{i\in N}$

[7] For all $a, b \in A$, we have $aR_i|_Ab \Leftrightarrow aR_ib$.

and set $\phi(R) = \varphi(l(R))$ for all $R \in \mathcal{R}|_{A(\varphi)}$. Now for N, w, $A(\varphi)$ and $\mathcal{R}|_{A(\varphi)}$ the rule ϕ satisfies *individual rationality*, *efficiency* (because constrained efficiency of φ is equivalent to *efficiency* on $A(\varphi)$), and *strategy-proofness*. Thus, Theorem 2 applies and 1. and 2. hold for ϕ. Note that this is true for any ϕ constructed in this way, i.e. for any $R \in \mathcal{R}$ we may choose the functions l_i such that $l_i(R_i|_{A(\varphi)}) = R_i$ (and $\phi(R|_{A(\varphi)}) = \varphi(R)$). □

Corollary 2 allows us in addition to construct largest sets of feasible allocations such that there may exist a rule satisfying *individual rationality*, *efficiency*, and *strategy-proofness*: in situations where for $\mathcal{A}^f$ there does not exist any such rule, at the extreme by setting $A = \{w\}$ there trivially exists a rule satisfying our properties. Now a set $w \in A \subseteq \mathcal{A}^f$ is a largest set of allocations (for our properties) if there exists a rule $\varphi : \mathcal{R} \to A$ (with $(A \backslash \{w\}) \subseteq A(\varphi)$) satisfying *individual rationality*, *constrained efficiency*, and *strategy-proofness*, and for any set $A' \supsetneq A$ there does not exist a rule $\varphi : \mathcal{R} \to A'$ (with $(A' \backslash \{w\}) \subseteq A(\varphi)$). Now of course, any such largest set A must be such that for any $R \in \mathcal{R}$, $C_\gamma(R|_A)$ is either essentially single-valued or empty. Hence, we may look for largest sets of allocations $w \in A \subseteq \mathcal{A}^f$ such that for all $R \in \mathcal{R}$, $C_\gamma(R|_A)$ is either essentially single-valued or empty (and for any set $A' \supsetneq A$ there exists $R \in \mathcal{R}$ such that $C_\gamma(R|_{A'})$ is multi-valued). Note that this construction applies to environments *with* and *without* externalities.

In the context of coalition formation, Pápai (2004) characterizes the coalition formation models which have a singleton core for any preference profile. Equivalently she identifies largest sets of coalitions which can be formed such that the core is a singleton. Pápai (2007) answers a similar question in exchange markets with multiple individual endowments.

Remark 2 Sönmez (1999) has shown that special cases of his model are (hedonic) coalition formation and (hedonic) network formation. For both these models, we have for all $i \in N$, $w(i) = \{w_{ij} : j \in N \backslash \{i\}\}$ where w_{ij} is the permit for agent j to join a coalition with agent i (or w_{ij} is the permit for agent j to form a link with agent i). In the coalition formation model a feasible allocation is a partition of the set of agents, i.e.

$$\begin{aligned} \mathcal{A}_c^f &= \{a \in \mathcal{A} : \text{ for all distinct } i,j \in N, w_{ij} \in a(j) \Leftrightarrow w_{ji} \in a(i), \\ &\quad \text{and for all distinct } i,j,k \in N, w_{ij} \in a(j) \ \& \ w_{jk} \in a(k) \Rightarrow w_{ik} \in a(k)\}. \end{aligned}$$

In the network formation model a feasible allocation is a network consisted of a set of pairwise (undirected) links, i.e.

$$\mathcal{A}_c^f = \{a \in \mathcal{A} : \text{ for all distinct } i,j \in N, w_{ij} \in a(j) \Leftrightarrow w_{ji} \in a(i)\}.$$

For the coalition formation model, one may consider for any agent all strict rankings over all coalitions to which he belongs to (and preferences over partitions are selfish). For the network formation model, one may consider for any agent all strict rankings over all components (which is a set of links) to which he belongs to (and preferences over networks are selfish). In both of these cases on the full domain there does not exist an allocation rule satisfying *individual rationality*, *efficiency*, and *strategy-proofness*. This is not entirely surprising because both coalition formation and network formation are generalizations of two-sided marriage markets, and if there would be a rule satisfying *individual rationality*, *efficiency*, and *strategy-proofness* in these models, then there would exist one satisfying these properties in marriage markets[8] (but the γ-core may be multi-valued in marriage markets). However, in applications we may use Corollary 2 to restrict preference domains and/or the set of feasible coalitions/links (or equivalently weaken *efficiency* to *constrained efficiency* in order to obtain a positive result (and this approach allows for externalities)).

4.3 Housing Markets with Indifferences

Using our main result it follows that on the domain where indifferences with the endowment are excluded, all the rules of Alcalde-Unzu and Molis (2011) and Jaramillo

[8]Simply take the restriction of this rule to all marriage problems.

and Manjunath (2012) are core selections (when the core is non-empty) and that the core is essentially single-valued when non-empty (Wako, 1991; Ma, 1994).

For all $i \in N$, let $w(i) = i$ and $\tilde{\mathcal{A}} = \{a : N \to N : a(i) \neq a(j) \text{ for all } i \neq j\}$, and let $\mathcal{R}_i$ consist of all weak selfish preference relations on N such that $aI_iw \Leftrightarrow a(i) = w(i)$ and for all $a, b \in \tilde{\mathcal{A}}$, $a(i) = b(i) \Rightarrow aI_ib$. Let $\mathcal{P}_i \subseteq \mathcal{R}_i$ consist of all strict selfish preference relations on N such that for all $a, b \in \tilde{\mathcal{A}}$, $a(i) \neq b(i) \Leftrightarrow$ not aI_ib. Both Alcalde-Unzu and Molis (2011) and Jaramillo and Manjunath (2012) construct "large" families of rules satisfying individual rationality, efficiency and strategy-proofness on the domain $\mathcal{R}$ with $\tilde{\mathcal{A}}$ being the set of feasible allocations. Therefore, the domain $\mathcal{R}$ is solvable while for some preference profiles the γ-core is empty and those rules may choose non-equivalent (in terms of welfare) allocations. Now Theorem 2 generalizes these results from the core to the γ-core and both for any domain in between the strict and the weak one, and for any subset of $\tilde{\mathcal{A}}$ being the set of feasible allocations.

Corollary 3 *For all $i \in N$, let $\mathcal{P}_i \subseteq \mathcal{D}_i \subseteq \mathcal{R}_i$ satisfy Assumption B, $\mathcal{D} = \times_{i \in N} \mathcal{D}_i$, and $w \in \mathcal{A}^f \subseteq \tilde{\mathcal{A}}$. Then the following holds:*

(i) For any allocation rule $\varphi : \mathcal{D} \to \mathcal{A}^f$ satisfying individual rationality, efficiency and strategy-proofness, we have for all $R \in \mathcal{D}$ such that $C_\gamma(R) \neq \emptyset$, $\varphi(R) \in C_\gamma(R)$.

(ii) For all $R \in \mathcal{D}$ and all $a, b \in C_\gamma(R)$, we have aI_ib for all $i \in N$.

One extreme case of Corollary 3 is the strict domain where $\tilde{\mathcal{A}}$ is the set of feasible allocations, and Corollary 3 together with Proposition 2 corresponds to Ma (1994)'s result. The other one is the weak domain and $\tilde{\mathcal{A}}$ is the set of feasible allocations, and Corollary 3 generalizes the results of Alcalde-Unzu and Molis (2011) and Jaramillo and Manjunath (2012). Note that Corollary 3 holds for any domain in between the strict and the weak domain and/or the set of feasible allocations does not necessarily satisfy separability (as defined in Remark 1). When $\tilde{\mathcal{A}}$ is the set of feasible allocations, all these rules are extensions of the top trading cycles algorithm from the strict domain to the domain $\mathcal{D}$.

Often in one-sided assignment problems we face logistical constraints, as for example in kidney exchange (Roth, Sönmez and Ünver, 2004) only a certain number of kidneys can be transplanted, or geographical moving constraints whereby agents can only move to houses which are not "too far away". Given $1 \leq k \leq |N|$, let $\mathcal{A}^k = \{a \in \mathcal{A}^f : |\{i \in N : a(i) \neq w(i)\}| \leq k\}$ denote the set of feasible allocations where at most k agents move. Obviously, $\mathcal{A}^k$ is not separable and the core is a subsolution of the γ-core. In addition, in kidney exchange externalities prevail because often "shorter" chains or cycles are preferred to "longer" ones. Another source for externalities in housing markets may be the fact that an agent does not care only about which good he consumes but also about the identity of the agent who consumes his/her endowment. In other words agent i's preference relation on $\mathcal{A}^f$ ranks indifferent all allocations $a, b \in \mathcal{A}^f$ such that $a(i) = b(i)$ and $a^{-1}(i) = b^{-1}(i)$.

Below we show that Assumption B is necessary for deriving the conclusion of Theorem 2. We know from Ma (1994) that on the strict domain for any profile there exists a unique core allocation which can be derived via the top trading cycles algorithm (TTC), and TTC is the unique allocation rule satisfying individual rationality, efficiency and strategy-proofness.[9] We describe a subdomain of the "strict" house exchange domain where for any profile there exists a unique core allocation (and the core is externally stable) but there are allocation rules which are individually rational, efficient and strategy-proof *and do not always choose core allocations.*

Example 2 [10]Let $N = \{1, 2, 3\}$, $w = (1, 2, 3)$, and for all $i \in N$, let $\hat{\mathcal{P}}_i$ consist of all strict preferences over N which are single-peaked with respect to the line $1-2-3$. For any R_i, let $top(R_i)$ denote the most preferred object under R_i, i.e. then $top(R_3) = 1$

[9]In a model with multiple individual endowments Pápai (2007) develops a notion of "exchanges along deals" such that this result carries over to her setting.

[10]Kasajima (2013) first considered problem of assigning heterogenous indivisible goods with single-peaked preferences.

and single-peakedness imply $1P_32P_33$. Obviously $\times_{i\in N}\hat{\mathcal{P}}_i = \hat{\mathcal{P}} \subseteq \mathcal{P} = \times_{i\in N}\mathcal{P}_i$. We define φ as follows: for any $R \in \hat{\mathcal{P}}$, (i) if $top(R_1) = 3$, $top(R_3) = 1$, $top(R_2) = 1$, then $\varphi(R) = (3, 1, 2)$; (ii) if $top(R_1) = 3$, $top(R_3) = 1$, $top(R_2) = 3$, then $\varphi(R) = (2, 3, 1)$; and (iii) otherwise $\varphi(R) = C(R)$. By single-peakedness, for profiles of type (i) and (ii) we must have $3P_12P_11$ and $1P_32P_33$. Now it is straightforward that φ is individually rational and efficient. In showing strategy-proofness[11] it suffices to consider profiles R of type (i) or (ii), say R is a profile of type (i). Obviously agents 1 and 2 cannot manipulate. If agent 3 reports some other $R'_3 \in \hat{\mathcal{P}}_3$, then by single-peakedness, $top(R'_3) \neq 1$. If $top(R'_3) = 2$, then $\varphi(R_1, R_2, R'_3) = \varphi(R)$, and if $top(R'_3) = 3$, then by individual rationality, $\varphi(R_1, R_2, R'_3) = 3$. In both cases agent 3 does not profitably manipulate.

References

Alcalde, J., and S. Barberà (1994): "Top Dominance and the Possibility of Strategy-Proof Stable Solutions to Matching Problems," *Economic Theory 4*, 417–435.

Alcalde-Unzu, J., and E. Molis (2011): Exchange of Indivisible Goods and Indifferences: the Top Trading Absorbing Sets Mechanisms, *Games and Economic Behavior* 73, 1–16.

Chander, P., and H. Tulkens (1997): "The Core of an Economy with Multilateral Externalities," *International Journal of Game Theory 26*, 379–401.

Demange, G. (1987): "Nonmanipulable Cores," *Econometrica 55*, 1057–1074.

Gale, D., and L. Shapley (1962): "College Admissions and the Stability of Marriage," *American Mathematical Monthly 69*, 9–15.

[11]Note that only agent 2 can induce a change from profiles of type (i) to type (ii) (and vice versa), and for profiles of type (iii), either agent 2 belongs to the top cycle and no agent has an incentive to go to type (i)/(ii) or is not able to induce a change to (i)/(ii), or agent 2 receives his endowment and does not want to change to type (i)/(ii).

Jaramillo, P., and V. Manjunath (2012): The Difference Indifference Makes in Strategy-Proof Allocation of Objects, *Journal of Economic Theory 147*, 1913–1946.

Kasajima, Y. (2013): Probabilistic assignment of indivisible goods with single-peaked preferences, *Social Choice and Welfare 41*, 203–215.

Ma, J. (1994): "Strategy Proofness and the Strict Core in a Market with Indivisibilities," *International Journal of Game Theory 23*, 75–83.

Ma, J. (1998): "Strategic Formation of Coalitions," Discussion Paper No. 1225, Northwestern University.

Pápai, S. (2004): "Unique Stability in Simple Coalition Formation Games," *Games and Economic Behavior 48*, 337–354.

Pápai, S. (2007): "Exchange in a General Market with Indivisible Goods," *Journal of Economic Theory 132*, 208–235.

Rodríguez-Álvarez, C. (2009): "Strategy-Proof Coalition Formation," *International Journal of Game Theory 38*, 431–452.

Roth, A.E. (1982a): "The Economics of Matching: Stability and Incentives," *Mathematics of Operations Research 7*, 617–628.

Roth, A.E. (1982b): "Incentive Compatibility in a Market with Indivisible Goods," *Economics Letters 9*, 127–132.

Roth, A.E., and A. Postlewaite (1977): "Weak versus Strong Domination in a Market with Indivisible Goods," *Journal of Mathematical Economics 4*, 131–137.

Roth, A., and M. Sotomayor (1990). *Two-Sided Matching: A Study in Game Theoretic Modeling and Analysis.* London/New York: Cambridge University Press.

Roth, A.E., T. Sönmez, and U. Ünver (2004): "Kidney Exchange," *Quarterly Journal of Economics 119*, 457–488.

Shapley, L., and H. Scarf (1974): "On Cores and Indivisibility," *Journal of Mathematical Economics 1*, 23–37.

Sönmez, T. (1999): "Strategy-Proofness and Essentially Single-Valued Cores," *Econometrica 67*, 677–689.

Takamiya, K. (2003): "On Strategy-Proofness and Essentially Single-Valued Cores: A Converse Result," *Social Choice and Welfare 20*, 77–83.

Wako, J. (1991): "Some Properties of Weak Domination in an Exchange Market with Indivisible Goods," *Econom. Stud. Quart. 42*, 303–314.

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