An Algorithm for Identifying Agent-k-linked Allocations in Economies with Indivisibilities*

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Abstract

We consider envy-free (and budget-balanced) rules that are least manipulable with respect to agents counting or with respect to utility gains. Recently it has been shown that for any profile of quasi-linear preferences, the outcome of any such least manipulable envy-free rule can be obtained via agent-k-linked allocations. This note provides an algorithm for identifying agent-k-linked allocations.

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Keywords: least manipulable envy-free rules; algorithm.

1 Introduction

Policy makers often adopt social choice rules and matching mechanisms that are vulnerable to manipulation by strategic misrepresentation (e.g., voting rules, school choice mechanisms, and auction procedures). This has motivated researchers to identify rules and mechanisms that are "least manipulable" according to some predetermined measure. Two prominent measures are (i) counting the number of profiles at which a rule is manipulable (Maus et al., 2007a,b) and (ii) comparing via set inclusion the preference domains where different rules are manipulable (Pathak and Sönmez, 2013). Even those measures are natural, Andersson et al. (2010) demonstrated that in the context of assigning indivisible objects with monetary compensations among a set of agents, they do not distinguish envy-free and budget-balanced rules, and a "finer" measure is needed to identify "least manipulable" rules among envy-free and budget-balanced rules.

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In Andersson et al. (2010), rule φ is judged to be more manipulable with respect to agents counting than rule ψ if, for each preference profile, the number of agents that can manipulate φ is larger than or equal to the number of agents that can manipulable ψ. Andersson et al. (2012) and Fujinaka and Wakayama (2012) considered a different approach and calculated the maximal amount by which an agent can gain from manipulating a given rule. In this case, rule φ is defined to be more manipulable with respect to utility gains than rule ψ if, for each preference profile, the maximal gain that any agent can obtain by manipulating φ is weakly larger than the maximal gain that any agent can obtain by manipulating ψ. Even though these two "finer" measures appear to be quite different, they share one important feature. Namely, for any given preference profile, the outcome of least manipulable envy-free rules can be identified via agent-k-linked allocations. Here, an allocation is agent-k-linked if for each agent i, there is a sequence of agents from i to k such that any agent in the sequence is indifferent between his consumption bundle and the consumption bundle of the next agent in the sequence. These allocations are not only important when identifying least manipulable rules, they have also played an important role in other contexts (see, e.g., Alkan et al., 1991; Velez, 2011; Fujinaka and Wakayama, 2012). This note provides an algorithm for finding envy-free and budget-balanced agent-k-linked allocations under quasi-linear preferences.

2 The Model and Basic Definitions

Let $N = \{1, \ldots, n\}$ and $M = \{1, \ldots, m\}$ denote the sets of agents and objects, respectively, with $|N| = |M|$. Each agent $i \in N$ consumes one bundle $(j, x_j) \in M \times \mathbb{R}$ containing one object $j \in M$ and some amount of money $x_j \in \mathbb{R}$. For each $i \in N$, i's preferences over bundles (j, x_j) are represented by a quasi-linear utility function u_i:

$$u_i(j, x_j) = v_{ij} + x_j \text{ for some } v_{ij} \in \mathbb{R}.$$

A list of utility functions $u = (u_1, \ldots, u_n)$ is a (preference) profile. Let $\mathcal{U}$ denote the set of profiles.

An allocation (a, x) is a list of $|N|$ bundles where $a : N \to M$ assigns object a_i to $i \in N$ and $x : M \to \mathbb{R}$ assigns monetary compensation x_j to $j \in M$. An allocation (a, x) is *feasible* if $a_i \neq a_j$ whenever $i \neq j$ for $i, j \in N$, and $\sum_{j \in M} x_j \leq \alpha$ for some $\alpha \in \mathbb{R}_+$. If $\sum_{j \in M} x_j = \alpha$, allocation (a, x) is *budget-balanced*. Let $\mathcal{A}$ denote the set of feasible and budget-balanced allocations. For convenience, we write "allocation" instead of "feasible allocation satisfying budget-balance". At profile $u \in \mathcal{U}$, allocation (a, x) is *envy-free* if $u_i(a_i, x_{a_i}) \geq u_i(a_j, x_{a_j})$ for all $i, j \in N$. Let $\mathcal{F}(u)$ denote the set of envy-free allocations at profile $u \in \mathcal{U}$.

A *rule* is a non-empty correspondence φ choosing for each $u \in \mathcal{U}$ a non-empty set of allocations $\varphi(u)$ such that $u_i(a_i, x_{a_i}) = u_i(b_i, y_{b_i})$ for all $i \in N$ and all $(a, x), (b, y) \in \varphi(u)$. A rule φ is *envy-free* if $\varphi(u) \subseteq \mathcal{F}(u)$ for each $u \in \mathcal{U}$. Given $u \in \mathcal{U}$, a rule φ is *manipulable* at u by agent $i \in N$ if there exists $(\hat{u}_i, u_{-i}) \in \mathcal{U}$ and two allocations $(a, x) \in \varphi(u)$ and $(b, y) \in \varphi(\hat{u}_i, u_{-i})$ such that $u_i(b_i, y_{b_i}) > u_i(a_i, x_{a_i})$. If rule φ is not manipulable by any agent at u, then φ is *non-manipulable* at u.

We use the following concepts for describing indifference relations at any allocation (Andersson et al., 2010).

Definition 1. Let $(a, x) \in \mathcal{A}$ and $u \in \mathcal{U}$.

(i) For any $i, j \in N$, we write $i \rightarrow_{(a,x)} j$ if $u_i(a_i, x_{a_i}) = u_i(a_j, x_{a_j})$,

(ii) An *indifference chain* at (a, x) consists of a tuple of distinct agents $g = (i_0, \ldots, i_k)$ such that $i_0 \rightarrow_{(a,x)} \cdots \rightarrow_{(a,x)} i_k$,

(iii) Agent $i \in N$ is *linked* to agent $k \in N$ at (a, x) if there exists an indifference chain $(i_0, \ldots, i_t)$ at (a, x) with $i = i_0$ and $i_t = k$,

(iv) Allocation (a, x) is *agent-k-linked* if each agent $i \in N$ is linked to agent $k \in N$.

Definition 2. Let $(a, x) \in \mathcal{A}$. An *indifference component* at (a, x) is a non-empty set $G \subseteq N$ such that for all $i, k \in G$ there exists an indifference chain at (a, x) in G, say $g = (i_0, ..., i_k)$ with $\{i_0, \ldots, i_k\} \subseteq G$, such that $i = i_0$ and $i_k = k$, and there exists no $G' \supsetneq G$ satisfying the previous property at (a, x).

Lemma 1 (Svensson, 2009)**.** Let $u \in \mathcal{U}$. If $(a, x), (b, y) \in \mathcal{F}(u)$, then $(a, y), (b, x) \in \mathcal{F}(u)$.

3 Least Manipulable Envy-Free Rules

Note the following two facts for any $k \in N$ and any $u \in \mathcal{U}$,

(1) there exist allocations in $\mathcal{F}(u)$ maximizing k's utility in $\mathcal{F}(u)$ (and such allocations will be called agent-k-preferred) (Alkan et al., 1991); and $(a^*, x^*) \in \mathcal{F}(u)$ is agent-k-linked if and only if (a^*, x^*) maximizes k's utility in $\mathcal{F}(u)$ (Andersson et al., 2010, Theorem 6);

(2) for any envy-free rule φ, there exists $(\hat{u}_i, u_{-i}) \in \mathcal{U}$ such that some $(a^*, x^*) \in \varphi(\hat{u}_i, u_{-i})$ is agent-k-linked (under u) (Andersson et al., 2010).

Given a rule φ and $u \in \mathcal{U}$, let $P^{\varphi}(u)$ denote the set of agents who can manipulate φ at u. Rule φ is non-manipulable at u if

$$|P^{\varphi}(u)| = 0. \tag{1}$$

Because (1) is never satisfied for all profiles by envy-free (and budget-balanced) rules (Green and Laffont, 1979), one approach is to search for rules where $|P^{\varphi}(u)|$ is minimized for each profile u.

Definition 3. Envy-free rule φ is least manipulable with respect to agents counting if for any envy-free rule ψ, we have $|P^{\varphi}(u)| \leq |P^{\psi}(u)|$ for all $u \in \mathcal{U}$.

Andersson et al. (2010) show the following: first, by their Lemma 2, the set of indifference components is invariant for any two envy-free allocations; and second, agent k cannot manipulate an envy-free rule iff all allocations chosen by the rule are agent-k-linked (or equivalently, agent-k-preferred). An immediate consequence is now Andersson et al. (2010, Theorem 3) which states that the least manipulable envy-free rules with respect to agents counting are exactly "maximally preferred" envy-free rules: for each profile u we choose some agent k belonging to an indifference component with maximal cardinality and then a non-empty subset of agent-k-linked allocations. Note that such allocations are agent-i-linked for any agent i belonging to the same indifference component as agent k. Hence, to identify the outcome of a least manipulable envy-free rule with respect to agents counting, envy-free agent-k-linked allocations must be identified (and then indifference components with maximal cardinality may be found). Here it suffices to identify one agent-k-linked allocation for each $k \in N$.

Andersson et al. (2012) and Fujinaka and Wakayama (2012) determine the maximal utility gain which each agent can obtain by manipulating an envy-free rule. For any $u \in \mathcal{U}$ and any $(a,x) \in \varphi(u)$, let

$$f_k(\varphi, u) = \sup_{(\hat{u}_k, u_{-k}) \in \mathcal{U}} \max_{(b,y) \in \varphi(\hat{u}_k, u_{-k})} u_k(b_k, y_{b_k}) - u_k(a_k, x_{a_k})$$

denote agent k's maximal gain from manipulating φ at u.

Let φ be an envy-free rule, $u \in \mathcal{U}$ and $(a,x) \in \varphi(u)$. Let $k \in N$. By Fact (1) there exist agent-k-linked $(a^*, x^*) \in \mathcal{F}(u)$. By Lemma 1, now $(a, x^*) \in \mathcal{F}(u)$ and (by envy-freeness) $u_k(a_k, x^*_{a_k}) = u_k(a^*_k, x^*_{a^*_k})$ implying that (a, x^*) is agent-k-linked. Observing Fact (2), Andersson et al. (2012, Theorem 2) under quasi-linearity implies

$$f_k(\varphi, u) = v_{ka_k} + x^*_{a_k} - (v_{ka_k} + x_{a_k}) = x^*_{a_k} - x_{a_k}. \tag{2}$$

Hence, $f_k(\varphi, u)$ represents the maximal amount of money that agent k can obtain by manipulating φ at u.

Definition 4. Envy-free rule φ is least manipulable with respect to utility gains if for any envy-free rule ψ, we have $\max_{i \in N} f_i(\varphi, u) \leq \max_{i \in N} f_i(\psi, u)$ for all $u \in \mathcal{U}$.

Andersson et al. (2012) show that (a) there exist least manipulable envy-free rules φ with respect to utility gains, (b) any such φ satisfies $f_i(\varphi, u) = f_j(\varphi, u)$ for all $i, j \in N$ and all $u \in \mathcal{U}$, and (c) the allocations chosen by any such φ can be identified via agent-k-linked allocations for any profile u. More explicitly, given $u \in \mathcal{U}$ start by identifying one agent-k-linked allocation in $\mathcal{F}(u)$, say (a^k, x^k), for any $k \in N$. Using Lemma 1 and the above argument, we may suppose that $a^1 = \cdots = a^n \equiv a$ (and $(a, x^k) \in \mathcal{F}(u)$ is agent-k-linked). By (2), for all $k \in N$, $x^k_{a_k} \geq x_{a_k}$ (where $(a,x) \in \mathcal{F}(u)$), and $\sum_{k \in N} x^k_{a_k} \geq \alpha$. Thus, the compensations $(x^1_{a_1}, \ldots, x^n_{a_n})$ need to be reduced by $\beta \geq 0$ in order to satisfy budget-balance, i.e. we choose $\beta \geq 0$ such that $\sum_{k \in N}(x^k_{a_k} - \beta) = \alpha$. Andersson et al. (2012, Theorem 5) show that the allocation $(a, (x^k_{a_k} - \beta)_{k \in N})$ is envy-free. Now obviously, for profile u, any envy-free rule φ choosing $(a, (x^k_{a_k} - \beta)_{k \in N})$ satisfies by (2), $f_i(\varphi, u) = \beta = f_j(\varphi, u)$

for all $i, j \in N$. Hence, the outcome of a least manipulable rule with respect to utility gains may be found via identifying envy-free agent-k-linked allocations (and again it suffices to identify one agent-k-linked allocation for each $k \in N$).

4 Identification of Agent-k-linked Allocations

Fix $u \in \mathcal{U}$ and $k \in N$. Similarly to Aragones (1995), our algorithm starts with an arbitrary envy-free allocation, say $(a, x) \in \mathcal{F}(u)$. This assumption is not restrictive since such allocations can be easily found in polynomial time (Klijn, 2000; Haake et al., 2003). In every step of the algorithm we keep the object assignment a fixed.

Definition 5. A group of agents $C \subsetneq N$ is *isolated* at (a, x) if $i \not\to_{(a,x)} j$ for all $i \in N \setminus C$ and all $j \in C$.

An allocation cannot be agent-k-linked if agent k belongs to an isolated group $C \subsetneq N$ because then at least one agent is not linked to agent k. The termination criterion for our algorithm will be the non-existence of an isolated group containing agent k.

Algorithm 1. Let $(a, x) \in \mathcal{F}(u)$ and set $K^0 = \{k\}$. For each iteration $t = 1, \ldots$:

Step t. Define $K^t \equiv K^{t-1} \cup \{i \in N \setminus K^{t-1} \mid i \to_{(a,x)} j \text{ for some } j \in K^{t-1}\}$. If $K^t = K^{t-1}$, then stop. Otherwise, continue with Step $t + 1$.

Lemma 2. Algorithm 1 identifies an isolated group containing agent k in at most $|N|$ iterations.

Proof. Let Algorithm 1 terminate at Step T. If $K^T \neq N$, then $i \not\to_{(a,x)} j$ for all $i \in N \setminus K^T$ and all $j \in K^T$ by construction. Thus, K^T is isolated and $k \in K^T$ since $\{k\} = K^0 \subseteq K^T$.

Furthermore, note that $|K^t| - |K^{t-1}| \geq 1$ for all $t \in \{1, \ldots, T - 1\}$, and Algorithm 1 terminates in at most $|N|$ iterations. □

Algorithm 2. Let $(a, x) \in \mathcal{F}(u)$ and set $K^0 = \{k\}$ and $x^0 = x$. Let x^t denote the compensations determined in iteration t. For each iteration $t = 1, \ldots$:

Step t. Run Algorithm 1 for (a, x^{t-1}) and let N^t denote the output of Algorithm 1. If $N \setminus N^t = \emptyset$, then stop (with output (a, x^{t-1})). Otherwise, let $\lambda^t_{ij} \equiv u_i(a_i, x^{t-1}_{a_i}) - u_i(a_j, x^{t-1}_{a_j})$ for each $i \in N \setminus N^t$ and each $j \in N^t$. Define $\lambda^t \equiv \min_{i \in N \setminus N^t, j \in N^t} \lambda^t_{ij}$. Define x^t by

$$\begin{aligned} x^t_{a_i} &\equiv x^{t-1}_{a_i} - \frac{|N^t|}{|N|} \cdot \lambda^t \text{ for each } i \in N \setminus N^t, \\ x^t_{a_j} &\equiv x^{t-1}_{a_j} + \frac{|N \setminus N^t|}{|N|} \cdot \lambda^t \text{ for each } j \in N^t, \end{aligned}$$

and continue with Step $t + 1$.

Theorem 1. Algorithm 2 identifies an agent-k-linked envy-free allocation in at most $|N|$ iterations.

Proof. Note that the adjustment of compensations in Step t from x^{t-1} to x^t respects budget-balance because (a, x^0) is budget-balanced, and by induction, if (a, x^{t-1}) is budget-balanced, then

$$\sum_{i\in N} x^t_{a_i} = \sum_{i\in N} x^{t-1}_{a_i} - \frac{|N^t|}{|N|} \cdot \lambda^t \cdot |N \setminus N^t| + \frac{|N \setminus N^t|}{|N|} \cdot \lambda^t \cdot |N^t| = \sum_{i\in N} x^{t-1}_{a_i} = \alpha.$$

Note that $(a, x^0) \in \mathcal{F}(u)$. By induction, we show that if $(a, x^{t-1}) \in \mathcal{F}(u)$, then $(a, x^t) \in \mathcal{F}(u)$. Equivalently, we show for all $i, j \in N$,

$$\text{if } u_i(a_i, x^{t-1}_{a_i}) \geq u_i(a_j, x^{t-1}_{a_j}), \text{ then } u_i(a_i, x^t_{a_i}) \geq u_i(a_j, x^t_{a_j}). \quad (3)$$

If $i, j \in N^t$ or $i, j \in N \setminus N^t$, then (3) is true because $(a, x^{t-1}) \in \mathcal{F}(u)$ and the adjustments of $x^{t-1}_{a_i}$ and $x^{t-1}_{a_j}$ are identical. If $i \in N^t$ and $j \in N \setminus N^t$, then (3) is true because $(a, x^{t-1}) \in \mathcal{F}(u)$ and $x^{t-1}_{a_i}$ is increased and $x^{t-1}_{a_j}$ is decreased. If $i \in N \setminus N^t$ and $j \in N^t$, then (3) is true because $(a, x^{t-1}) \in \mathcal{F}(u)$ and by definition of λ^t, $\lambda^t \leq \lambda^t_{ij} = u_i(a_i, x^{t-1}_{a_i}) - u_i(a_j, x^{t-1}_{a_j})$, i.e.,

$$\begin{aligned}
u_i(a_i, x^t_{a_i}) &= v_{ia_i} + x^t_{a_i} = v_{ia_i} + x^{t-1}_{a_i} - \frac{|N^t|}{|N|} \cdot \lambda^t \geq v_{ia_i} + x^{t-1}_{a_i} - \frac{|N^t|}{|N|} \cdot \lambda^t_{ij} = \\
&= u_i(a_i, (x^{t-1}_{a_i}) - \lambda^t_{ij} + \frac{|N \setminus N^t|}{|N|} \cdot \lambda^t_{ij} = u_i(a_j, x^{t-1}_{a_j}) + \frac{|N \setminus N^t|}{|N|} \cdot \lambda^t_{ij} \\
&\geq v_{ia_j} + x^{t-1}_{a_j} + \frac{|N \setminus N^t|}{|N|} \cdot \lambda^t = v_{ia_j} + x^t_{a_j} = u_i(a_j, x^t_{a_j}).
\end{aligned}$$

Because $(a, x^0) = (a, x) \in \mathcal{F}(u)$, now (3) yields $(a, x^t) \in \mathcal{F}(u)$.

Finally, we show that Algorithm 2 terminates in at most $|N|$ iterations. By construction of N^t, each agent $i \in N^t$ must belong to an indifference chain $g = (i, \ldots, k)$ at (a, x^{t-1}). Note that at Step t, for $i \in N \setminus N^t$ and $j \in N^t$ such that $\lambda^t_{ij} = \lambda^t$, all the above inequalities become equalities and we obtain $u_i(a_i, x^t_{a_i}) = u_i(a_j, x^t_{a_j})$, $i \rightarrow_{(a,x^t)} j$ and $i \in N^{t+1}$. Note that $N^t \subseteq N^{t+1}$ because for any $i, j \in N^t$ such that $i \rightarrow_{(a,x^{t-1})} j$, the adjustments of $x^{t-1}_{a_i}$ and $x^{t-1}_{a_j}$ are identical and we also have $i \rightarrow_{(a,x^t)} j$. Thus, $|N^{t+1}| - |N^t| \geq 1$ as long as $N \setminus N^t \neq \emptyset$. Hence, Algorithm 2 terminates in at most $|N|$ iterations. □

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