# A representation of risk measures

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ABSTRACT. We provide a representation theorem for risk measures satisfying (i) monotonicity; (ii) positive homogeneity; and (iii) translation invariance. As a simple corollary to our theorem, we obtain the usual representation of coherent risk measures (i.e., risk measures that are, in addition, sub-additive; see Artzner et al. [**2**]).

## 1. Introduction

Let $(\Omega, \Sigma)$ be a measurable space and let $B(\Sigma)$ denote the Banach space of bounded, $\Sigma$-measurable functions on $\Omega$ equipped with the sup-norm. $\Omega$ is the set of states of nature and $B(\Sigma)$ is the set of all (measurable) risks (see Artzner et al. [**2**]). A measure of risk is a mapping $\rho : B(\Sigma) \longrightarrow \mathbb{R}$. Coherent risk measures were introduced in [**5**] (under the name of "upper expectations") and further studied in [**2**]. These are risk measures that satisfy the following four properties:

(1) Translation invariance: for all $f \in B(\Sigma)$ and for all $\alpha \in \mathbb{R}$,

$$\rho(f + \alpha \mathbf{1}) = \rho(f) - \alpha$$

(2) Positive homogeneity: for all $f \in B(\Sigma)$ and for all $\lambda \geq 0$

$$\rho(\lambda f) = \lambda \rho(f)$$

(3) Monotonicity:

$$f, g \in B(\Sigma) \quad and \quad f \leq g \quad \Longrightarrow \qquad \rho(g) \leq \rho(f)$$

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(4) Sub-additivity: For all $f, g \in B(\Sigma)$

$$\rho(f+g) \leq \rho(f) + \rho(g)$$

Our formulation of property (1) differs slightly from the one in [**2**]. We use the normalization $\rho(\mathbf{1}) = -1$, where $\mathbf{1}$ is the function identically equal to 1 on $\Omega$. Artzner et al. [**2**] use the normalization $\rho(\mathbf{r}) = -1$, where $\mathbf{r}$ is the function identically equal to $r$ on $\Omega$, $r > 0$ (see [**2**], p. 208). Clearly, in view of property (2), this is inconsequential.

A representation theorem for coherent risk measures was proved in [**2**]. This was extended in [**6**], who requires sub-additivity for comonotonic risks only. Here, we are concerned with risk measures satisying the first three properties only.

Recall that the norm dual of $B(\Sigma)$ is (isometrically isomorphic to) $ba(\Sigma)$, the space of bounded charges on $\Sigma$ equipped with the variation norm. For $\mathcal{C}$ a convex, weak*-compact set of probability charges in $ba(\Sigma)$, we denote by $A(\mathcal{C})$ the space of all weak*-continuous affine mappings $\mathcal{C} \longrightarrow \mathbb{R}$. The canonical mapping $\kappa : B(\Sigma) \longrightarrow A(\mathcal{C})$ is the mapping $\kappa : f \longmapsto \psi_f$, where $\psi_f : \mathcal{C} \longrightarrow \mathbb{R}$ is given by $\psi_f(P) = \int_\Omega f dP$, $P \in \mathcal{C}$.

THEOREM 1. *A risk measure $\rho : B(\Sigma) \longrightarrow \mathbb{R}$ satisfies properties (1), (2) and (3) if and only if for all $f \in B(\Sigma)$*

$$\rho(f) = \int_{\mathcal{C}} -\kappa(f) d\nu$$

*where $\mathcal{C}$ is a convex, weak*-compact set of probability charges in $ba(\Sigma)$, $\nu$ is a capacity on the Borel field on $\mathcal{C}$ generated by the weak*-topology, and the integral is taken in the sense of Choquet.*

Thus, the theorem says that every risk measure satisfying (1), (2) and (3) corresponds to an integration over a set measures, but integration is in the sense of Choquet. Clearly, in the special case where $\nu$ is a measure, integration is Lebesgue integration and one obtains risk measures that are linear, i.e. $\rho(f+g) = \rho(f) + \rho(g)$, for all $f, g \in B(\Sigma)$. The proof of the theorem is based on the following two results. The first was proved in [**1**, Theorem 2 and Corollary 1]. The second was essentially proved in [**4**]. We include its proof here for completeness.

THEOREM 2 (Amarante [**1**]). *Let $\mathcal{C}$ be a convex, weak*-compact set of probability charges in $ba(\Sigma)$. A functional $V : A(\mathcal{C}) \longrightarrow \mathbb{R}$ is isotonically* *additive and satisfies $V(\psi) \geq V(\varphi)$ whenever $\psi \geq \varphi$ if and only if there is a capacity $\nu$ on the Borel field on $\mathcal{C}$ such that for all $\psi \in A(\mathcal{C})$*

$$V(\psi) = \int_{\mathcal{C}} \psi d\nu$$

LEMMA 1. *Let $\tau : B(\Sigma) \longrightarrow \mathbb{R}$ satisfy the following two properties:*
*(1') $\tau(\lambda f + \alpha \mathbf{1}) = \lambda\tau(f) + \alpha$; $\lambda \geq 0$ and $\alpha \in \mathbb{R}$*
*(2') $f \leq g \quad \Longrightarrow \quad \tau(f) \leq \tau(g)$.*

*Then, there exists a weak*-compact, convex set $\mathcal{C}$ of probability charges on $\Sigma$ and a mapping $a : B(\Sigma) \longrightarrow [0,1]$ such that $\tau$ admits the representation*

$$\tau(f) = a(f) \min_{P \in \mathcal{C}} \int_{\Omega} f dp + (1 - (a(f)) \max_{P \in \mathcal{C}} \int_{\Omega} f dp \tag{1.1}$$

PROOF. First, notice that $\tau$ is sup-norm continuous: From

$$\begin{aligned} f &= g + f - g \leq g + \|f - g\| \\ g &= f + g - f \leq f + \|f - g\| \end{aligned}$$

by using (2') and (1'), we get

$$|\tau(f) - \tau(g)| \leq \|f - g\| \, \tau(\mathbf{1}) = \|f - g\|$$

which is the sup-norm continuity of $\tau$. Next, define a binary relation $\gtrsim$ on $B(\Sigma)$ by

$$f \gtrsim g \qquad iff \qquad \tau(\lambda f + h) \geq \tau(\lambda f + h)$$

for all $\lambda \geq 0$ and for all $h \in B(\Sigma)$. By construction, this binary relation is *conic* (i.e. $f \gtrsim g \Longrightarrow \lambda f + h \gtrsim \lambda f + h$ for all $\lambda \geq 0$ and for all $h \in B(\Sigma)$), and it is easy to see that it is reflexive and transitive. Moreover, property (2') of $\tau$ implies that $\gtrsim$ is *non-trivial* (i.e., there exist $f, g \in B(\Sigma)$ such that $f \gtrsim g$ but not $g \gtrsim f$) and has the property $f \geq g \Longrightarrow f \gtrsim g$ . Finally, property (2') and the sup-norm continuity of $\tau$ easily imply that $\gtrsim$ is *continuous* in the sense that $f_i \to f$, $g_i \to g$ and $f_i \gtrsim g_i$ imply $f \gtrsim g$. As it is well-known (see [**4**, Proposition 22]), given a binary relation $\gtrsim$ with these properties, there exists a (unique) weak*-compact, convex set $\mathcal{C}$ of probability charges on $\Sigma$ such that

$$f \gtrsim g \qquad iff \qquad \int f dP \geq \int g dP \qquad for\ all\ P \in \mathcal{C} \tag{1.2}$$

Now, let $f \in B(\Sigma)$ and let $\mathcal{C}$ be the set determined by $\gtrsim$. Let

$$\bar{x} = \min_{P \in \mathcal{C}} \int f dP$$

($\bar{x}$ exists because the mapping $P \longmapsto \displaystyle\int f dP$ is weak*-continuous and $\mathcal{C}$ is weak*-compact). Then, by (1.2), $f \gtrsim \bar{x}\mathbf{1}$. By definition of $\gtrsim$, this implies that

$$\tau(\lambda f + h) \geq \tau(\lambda \bar{x}\mathbf{1} + h)$$

for all $\lambda > 0$ and for all $h \in B(\Sigma)$. In turn, by property (1') of $\tau$, this implies

$$\bar{x} = \min_{P \in \mathcal{C}} \int f dP \leq \inf_{\lambda > 0; h \in B(\Sigma)} \tau(f + \frac{1}{\lambda}h) - \tau(\frac{1}{\lambda}h)$$

Hence,

$$\bar{x} = \min_{P \in \mathcal{C}} \int f dP \leq \tau(f)$$

Similarly, one shows the inequality

$$\max_{P \in \mathcal{C}} \int f dP \geq \tau(f)$$

and the statement in the lemma follows at once from these two inequalities.

□

Proof of Theorem 1. Given a risk measure $\rho$, define $\tilde{\rho} : B(\Sigma) \longrightarrow \mathbb{R}$ by $\tilde{\rho}(f) = \rho(-f)$. Then, $\tilde{\rho}$ has the properties (1') and (2') in Lemma 1. Hence,

$$\tilde{\rho}(f) = a(f) \min_{P \in \mathcal{C}} \kappa(f)(P) + (1 - (a(f)) \max_{P \in \mathcal{C}} \kappa(f)(P) \tag{1.3}$$

where $\kappa$ canonical linear mapping $\kappa : B(\Sigma) \longrightarrow A(\mathcal{C})$. If $f, g \in B(\Sigma)$ are such that $\kappa(f) = \kappa(g)$, then by (1.2) in the proof of Lemma 1 we have that $f \gtrsim g$ and $g \gtrsim f$, which imply $\tilde{\rho}(f) = \tilde{\rho}(g)$. We conclude that if $f, g \in B(\Sigma)$ are such that $\kappa(f) = \kappa(g)$, then $a(f) = a(g)$. It follows that the mapping $\tilde{a} : A(\mathcal{C}) \longrightarrow [0, 1]$ defined by $\tilde{a}(\kappa(f)) = a(f)$ is well-defined, and that the functional $\tilde{\rho}$ factors as $\tilde{\rho} = V \circ \kappa$

$$\tilde{\rho}(f) = V \circ \kappa(f) = \tilde{a}(\kappa(f)) \min_{P \in \mathcal{C}} \kappa(f)(P) + (1 - (\tilde{a}(\kappa(f))) \max_{P \in \mathcal{C}} \kappa(f)(P)$$

Hence, from the linearity of $\kappa$ and property (1') of $\tilde{\rho}$, it follows that

$$V(a\psi + b\mathbf{1}) = aV(\psi) + b$$

for all $a \geq 0$, $b \in \mathbb{R}$ and for all $\psi \in A(\mathcal{C})$. In particular, if $\psi, \varphi \in A(\mathcal{C})$ are isotonic (i.e., $\psi(P) \geq \psi(P') \Longleftrightarrow \varphi(P) \geq \varphi(P')$), then there exist $a \geq 0$ and

$b \in \mathbb{R}$ such that $\psi = a\varphi + b$ and

$$V(\psi + \varphi) = V(\psi) + V(\varphi)$$

that is, $V$ is additive on isotonic functions.

Let $\psi, \varphi \in A(\mathcal{C})$ be such that $\psi \geq \varphi$. Since the canonical mapping is onto, there exist $f, g \in B(\Sigma)$ such that $\psi = \kappa(f)$ and $\varphi = \kappa(g)$. By(1.2) in the proof of Lemma 1, $\psi \geq \varphi$ is equivalent to $f \gtrsim g$. In turn, this implies $\tilde{\rho}(f) \geq \tilde{\rho}(g)$ and, by the factorization above, $V \circ \kappa(f) \geq V \circ \kappa(f)$. That is,

$$\psi \geq \varphi \Longrightarrow V(\psi) \geq V(\varphi)$$

By Theorem 2, $V$ admits a representation as a Choquet integral. We then conclude that

$$\rho(f) = \tilde{\rho}(-f) = \int_{\mathcal{C}} -\kappa(f) d\nu$$

where $\nu$ is a capacity on the Borel field on $\mathcal{C}$ generated by the weak*-topology, and the integral is a Choquet.

Conversely, it follows immediately from the properties of the Choquet integral that any functional $\rho : B(\Sigma) \longrightarrow \mathbb{R}$ defined by $\rho(f) = \int_{\mathcal{C}} -\kappa(f) d\nu$ – $\mathcal{C}$ convex and weak*-compact, $\nu$ a capacity on the Borel field on $\mathcal{C}$ – satisfies properties (1), (2) and (3) above. □

## 2. Examples

It is clear that the risk measures characterized in the theorem are not necessarily coherent: coherence obtains if and only if the capacity is submodular (i.e., for all $A$ and $B$ in the Borel field on $\mathcal{C}$, $\nu(A \cup B) + \nu(A \cap B) \leq \nu(A) + \nu(B)$; see below). Below, we give a few examples of risk measures that can be defined starting from Theorem 1. For $\mathcal{C}$ a convex, weak*-compact set of probability charges in $ba(\Sigma)$, let $\mathcal{B}$ denote the Borel field on $\mathcal{C}$ generated by the weak*-topology.

EXAMPLE 1. *Let $\alpha$ be a number in $[0,1]$. Define a capacity $\nu : \mathcal{B} \longrightarrow [0,1]$ by $\nu(A) = \alpha$ for all $A \in \mathcal{B} \backslash \{\varnothing, \mathcal{C}\}$, $\nu(\varnothing) = 0$ and $\nu(\mathcal{C}) = 1$. If $\alpha$ is neither 0 nor 1, and if $\mathcal{C}$ contains more than two elements, this capacity gives rise to a risk measure that is neither sub-additive nor super-additive.*

EXAMPLE 2 (Distortion of a probability measure). *Let $\mu$ be a probability measure on $\mathcal{B}$. Let $\varphi : [0,1] \longrightarrow [0,1]$ be an increasing function with the property that $\varphi(0) = 0$ and $\varphi(1) = 1$. Define a capacity $\nu$ on $\mathcal{B}$ by $\nu = \varphi \circ \mu$.*

*If $\varphi$ is neither concave nor convex, $\nu$ gives rise to a risk measure that is neither sub-additive nor super-additive.*

EXAMPLE 3 (Quantile functions). *Let $(T,\Theta)$ be a measurable space, and let $B(\Theta)$ denote the Banach space (sup-norm) of bounded, $\Theta$-measurable real-valued functions on $T$. Let $p$ be a probability measure on $\Theta$. A functional $F : B(\Theta) \longrightarrow \mathbb{R}$ is a lower quantile with respect to $p$ if there exists $\alpha \in [0,1)$ such that*

$$F(f) = \inf\{x \mid p(\{t : f(t) \geq \alpha\}) \leq \alpha\}$$

*$F$ is an upper quantile if there exists $\alpha \in (0,1]$ such that*

$$F(f) = \sup\{x \mid p(\{t : f(t) \geq \alpha\}) \geq \alpha\}$$

*$F$ is a quantile function if it is either a lower quantile or an upper quantile. Quantile functions can be represented by means of Choquet integrals (see [**3**]). Thus, it follows from Theorem 1 that every quantile function $F : A(\mathcal{C}) \longrightarrow \mathbb{R}$ defines a risk measure satisfying (1), (2) and (3) by means of $\rho(f) = F(-\kappa(f))$, for all $f \in B(\Sigma)$.*

As a corollary to Theorem 1, we obtain the representation of coherent risk measures given by Artzner et al. [**2**]. To this end, we recall that given a compact, convex subset $\mathcal{C}$ of a locally convex space $E$ and a probability measure $\mu$ on $\mathcal{C}$, a barycenter of $\mu$ is a point $P \in \mathcal{C}$ such that $\psi(P) = \int \psi d\mu$ for every continuous linear functional $\psi$ on $E$.

COROLLARY 1. *A risk measure $\rho : B(\Sigma) \longrightarrow \mathbb{R}$ is coherent if and only if there exists a unique convex, weak*-compact set $\mathcal{B} \subset ba(\Sigma)$ such that*

$$\rho(f) = \max_{P \in \mathcal{B}} \int_{\Omega} - f dp$$

PROOF. Let $\rho$ be a risk measure satisfying (1), (2) and (3), and let $\tilde{\rho}$ and $V$ be the functionals defined in the proof of Theorem 1. It is easy to see that $\rho$ is subadditive iff $\tilde{\rho}$ is subadditive iff $V$ is subadditive. Thus, let $V$ be subadditive. By a theorem of Schmeidler [**9**, Proposition 3], there exists a unique weak*-compact, convex set $\Gamma$ of charges on the Borel field of $\mathcal{C}$ such that for all $\psi_f \in A(\mathcal{C})$

$$V(\psi_f) = \int_{\mathcal{C}} \psi_f d\nu = \max_{\mu \in \Gamma} \int_{\mathcal{C}} \psi_f d\mu \tag{2.1}$$

We can assume that each $\mu$ is a regular Borel measure on $\mathcal{C}$. In fact, for each $\mu$, $\int_{\mathcal{C}} \cdot \, d\mu$ is a continuous linear functional on $A(\mathcal{C})$. By Hahn-Banach, this can be extended to a continuous linear functional on $C(\mathcal{C})$, the Banach space of all continuous functions on $\mathcal{C}$ equipped with sup-norm, and (via the Riesz theorem) there exists a unique regular Borel measure representing it. It follows from [**8**, Proposition 1.1] that each $\mu \in \Gamma$ has a unique barycenter $P_\mu \in \mathcal{C}$, and that the mapping $\mu \longmapsto P_\mu$ is weak*-continuous. Let us denote by $\mathcal{B} \subset \mathcal{C}$ the image of $\Gamma$ under such a mapping. Then, we can rewrite (2.1) as

$$V(\psi_f) = \max_{\mu \in \Gamma} \int_{\mathcal{C}} \psi_f d\mu = \max_{\mu \in \Gamma} \psi_f(P_\mu) = \max_{P \in \mathcal{B}} \psi_f(P) = \max_{P \in \mathcal{B}} \int f dP$$

Thus,

$$\rho(f) = \max_{P \in \mathcal{B}} \int_{\Omega} - f dp$$

□

We conclude by observing the well-known fact (see [**7**, Theorem 35]) that a Choquet integral is subadditive if and only if the capacity that defines it is submodular.

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