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Aversion Analysis

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AVERSION ANALYSIS

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ABSTRACT

Due to their underlying assumptions, the standard concepts of risk aversion and preference for the present are generally defined separately and represented by scalar measures, and this implies many shortcomings. More specifically, if measured by a scalar, the risk aversion remains unchanged, whatever type of risk is considered. Consequently, the main purpose of this paper is to provide a more complete analysis of aversions, which clearly emphasizes the multidimensionality of risk aversion and the necessity for the measures of risk aversion and preference for the present to be defined jointly.

This will be done by considering a general framework allowing not only to address these important issues, but also to discuss other basic concepts such as the certainty direction and the preference for liquidity. Our model also allows to analyze income shocks in two different settings, that is, when the individual can financially adjust himself and when he cannot. These two settings lead to the definition of various generalized aversions and to how they are linked together. Our main findings are that these generalized aversion measures are multidimensional and invariant with respect to monotonic transformations of the utility function.

Keywords : Risk aversion, aversion to impatience, illiquidity aversion, multidimensional aversions, financial premiums, Antonelli matrix, asset substituability, Drèze-Modigliani decomposition, subjective certainty, sure and risky assets, incomplete markets.

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1. Introduction

The seminal work of Arrow (1965) and Pratt (1964) has stimulated an important litterature on risk aversion.[1] The underlying assumptions of their definition imply shortcomings that are now well-known. First, the definition has been derived in a static context, which does not take into consideration the concept of psychological preference for the present over the future (also called time impatience). In other words, their static context doesn't allow for time and risk aversions to be defined jointly. Second, the definition relies on the expected utility hypothesis, which implies that the measure is only invariant with respect to linear transformations of the (expected) utility function. Moreover, an indirect utility function is more often used, which also means that the measure is not invariant with respect to a change in the monetary unit. Third, it is a local measure derived in the neighborhood of certainty, which is often represented by a vector e of unitary components. Fourth, the considered shocks are different. In their framework, wealth can involve two components : income and porfolio value. However, shocks are never decomposed into income shock and porfolio reallocation, since only "aggregated" or wealth shocks are considered. As a result, the risk aversion measure is a scalar, the measure being absolute or relative depending on the additive or multiplicative nature of the shocks (lotteries).

On the other hand, the definition of preference for the present[2] also relies on restrictive assumptions, which imply shortcomings. First, it is derived in a certainty setting : whereas it uses utility functions allowing for intertemporal analysis, all the uncertainty is resolved in the future by assuming that the individual consumes the same commodity, whatever state is realized. Second, it assumes temporal separability of the utility function, for instance, $u(c_0, c_1) = u(c_0) + \delta u(c_1)$, where c_0 and c_1 represent present and future consumption, respectively. However, for δ to be interpreted as the preference for the present,

[1] Yaari (1969), Diamond and Stiglitz (1974), Kihlstrom and Mirman (1974), Duncan (1977), Ross (1981), Karni (1983), and Machina (1987) are important references on this subject.

[2] The origin of this concept can be found in the pioneer work of Fisher (1930) and Allais (1947). Their work has also stimulated a litterature parallel to that on risk aversion. For example, the explanation of the positivity of the interest rate in Malinvaud (1972) completes their own explanation.

usually measured by the marginal rate of substitution between present and future consumption, a third assumption has still to be made, that is, the stationary-growth-path assumption ($c_0 = c_1$ in the above two-period example). If the future consisted of many dates, the latter assumption would be expressed in a more complicated way, still meaning, however, a sort of "temporal certainty". Fourth, it assumes that a single commodity is consumed in each period, and any kind of shocks can be considered. As a result, the measure of preference for the present is a scalar, which should not be the case if many commodities were considered, due to substitution effects. Indeed, even in a simple two-period setting, involving two commodities in each period, the concept of preference for the present itself becomes rather difficult to define.

Because of their underlying assumptions, the classical concepts of risk aversion and preference for the present are usually defined separately and represented by scalar measures. A natural way to analyse together uncertainty and time effects is to extend the von Neumann-Morgenstern utility function accordingly. This approach, which amounts to considering two-parameter settings, neglects, however, two of the four behavioral cases that are possible in such a setting. It predicts that individuals are either time and risk averse or time and risk lovers, but never in a mixed situation. Consequently, more complex utility functions are needed to solve this puzzle.[3] The latter was first recognized in Epstein and Zin (1989), who propose a recursive utility function involving a supplementary parameter. They also abandon the separability with respect to time and uncertainty, and this is done, in their paper, by introducing an aggregator. From our point of view, their solution is not completely satisfactory, since it yields a measure of risk aversion that is still a scalar, which means that the risk aversion remains unchanged, whatever type of risk is considered. Our feeling is that an individual could, for example, both dislike financial risks and love risky sports.

The main purpose of this paper is to provide a more complete analysis of aversions, which clearly emphasizes the multidimensionality of risk aversion and the necessity for the measures of risk aversion and preference for the present to be defined jointly. This will be done by considering a general framework that allows not only to address these important issues, but also to discuss other basic

[3] Other puzzles or paradoxes have also been raised, whose solutions can be found in related litteratures on state-dependent or non-expected utility functions. More references on these functions are given in subsection 2.1.

concepts such as the preference for liquidity and the certainty direction. Our model also allows to analyse shocks in two different settings, that is, when the individual can financially adjust himself, and when he cannot. These analysis lead to the definition of various generalized aversions and to how they are linked together, more specifically, how they can be deduced from each other. Our main findings are that these generalized aversion measures are multidimensional and invariant with respect to monotonic transformations of the utility function. Moreover, the generalized risk aversion is linked to the Arrow-Pratt measure. The multidimensionality feature appears especially important, since it stresses the fact that a unique risk aversion cannot account for different types of risks.

The consumer's problem is presented in Section 2. In a two-period economy, the second period falling into S states, a consumption unit maximizes a general utility function (defined up to an increasing monotonic transformation) that depends directly on L commodities and indirectly on $N \leq S$ financial assets. Commodity demands, asset demands and desirabilities of incomes (virtual Arrow prices) exist and are continuously differentiable (Proposition 1). The virtual Arrow prices will be transformed into bona fide subjective Arrow prices (or marginal willingnesses to pay for contingent elementary assets) in Section 3, which is devoted to the indirect utility function. At this point, the "demand system" involves commodity demands, tradable asset demands and subjective or individual Arrow prices.

In Section 4, we focus on the third part of the demand system, namely the system of subjective Arrow prices. When expanding it, it is soon realized that a consumption theory extended to financial assets necessarily contains a generalized theory of aversions. Our general purpose, in this section, is to make explicit such a theory. To do so, we have to break up this general purpose into more specific purposes, each one having its own interest. We first define and characterize a disutility-premium function (Proposition 5). The latter premium will be called residual, meaning that the individual can use financial markets to adjust himself to an income shock. However, this financial adjustment is not necessarily complete, if the asset structure is incomplete.

An alternative setting will also be considered : the individual still faces an income shock, but is no more able to adjust financially. This can be the case, for example, if he is endowed with an illiquid portfolio. This setting will first be used to define and characterize the corresponding disutility premium, to be called fundamental (Proposition 6). We will then introduce an appropriate normalization (for the subjective forward Arrow prices and the subjective discount factor) in order to completely disentangle time and uncertainty. As a result, the fundamental disutility premium will be decomposed into an impatience and a risk premium (Proposition 8). This two-component decomposition has a natural counterpart in terms of aversion : the fundamental joint time-risk aversion will be decomposed into an aversion to impatience and a risk aversion (Proposition 10). These two decompositions require that the certainty direction be redefined. The definition that we propose for the subjective certainty direction (or subjective certainty space) appears as a natural extension of the standard definition. However, while the latter has no interpretation in real terms (since it is not invariant under a change in the value of the numeraire of the different states), the former is locally invariant.

The difference between the fundamental and the residual disutility premiums appears as a liquidity premium. It is defined and characterize (Proposition 11) by using a natural extension of the Drèze-Modigliani decomposition (1972). Accordingly, the measure of the fundamental joint time-risk aversion can be decomposed into a measure of residual aversion and a measure of aversion to illiquidity (Proposition 11). If financial markets are complete, an income shock has no residual effect (the individual can adjust himself by using the S tradable assets), and the fundamental joint time-risk aversion reduces to an aversion to illiquidity. Finally, we refine the two-component decomposition of Drèze and Modigliani into a three-component decomposition. As a result, the measure of the fundamental joint time-risk aversion is decomposed into i) a measure of aversion to impatience, ii) a measure of residual (risk) aversion, and iii) a measure of aversion to the illiquidity of risky assets. This three-component decomposition supposes the existence of a sure asset (reallocation), which, in general, cannot be reduced to the standard certainty direction.

Most of the proofs are gathered in the appendices.

2. The model

2.1. Basic concepts

The model involves two periods, 0 and 1. At period 0, there is uncertainty as to the state of nature s, $s = 1, \cdots, S$, that will occur at date 1. All uncertainty is resolved in period 1. The consumer has to choose his present commodity vector x_0 and establish a plan (or strategy) regarding his future one for each possible state, denoted $x_{1s},\ s = 1, \cdots, S$. x_0 has $L_0 \geq 1$ components, x_{1s} has $L_{1s} \geq 1$ components, $s = 1, \cdots, S$. The corresponding price vectors are denoted p_0 and $p_{1s},\ s = 1, \cdots, S$, respectively. The corresponding (scalar) incomes are denoted w_0 and $\mathsf{w}_{1s},\ s = 1, \cdots, S$, respectively. Prices and incomes are expressed in relevant money units (p_0 and w_0 are expressed in a money unit specific to period 0, p_{1s} and w_{1s} are expressed in a money unit specific to period 1 and state s). This convention corresponds to financial practice, respects available data and, as we shall see in the sequel, is reconcilable with the traditional framework of price theory where intertemporal prices and incomes are expressed in a same account unit (the present one, say).

The consumer can transfer revenue between periods and states by selecting a portfolio of financial assets. We shall suppose the existence of N assets. They are traded in the first period where the corresponding vector of asset prices is denoted q_0; they pay off in the second period where the vector of asset values (pay-offs) in the state s is denoted q_{1s}. A portfolio is denoted y.

The budget constraints are[4] :

$$p_0' x_0 + q_0' y = \mathsf{w}_0 \quad \text{in period } 0, \tag{2.1}$$

$$p_{1s}' x_{1s} - q_{1s}' y = \mathsf{w}_{1s} \quad \text{in period 1 and state } s, s = 1, \cdots, S\ . \tag{2.2}$$

Assuming that the consumer's preferences are representable by a utility function $\mathcal{U}$, his optimization problem is :

$$\max_{x_0, \ldots, x_{1S}, y} \mathcal{U}(x_0, x_{11}, \cdots, x_{1S}) \tag{2.3}$$

subject to (2.1) and (2.2).

Matrix notation will be used in the sequel.

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[4] The symbol $'$ (prime) is used to denote transposition.

$x = [x_0', x_{11}', ..., x_{1S}']'$ is the L - dimensional vector of commodity quantities where $L = L_0 + \sum_s L_{1s}$;

$$P = \begin{bmatrix} p_0 & & & \\ & & 0 & \\ & p_{11} & & \\ 0 & & \ddots & \\ & & & p_{1S} \end{bmatrix}$$

is the $L \times (S+1)$ - dimensional matrix of commodity prices;

$Q = [q_0 \quad -q_{11} \quad \cdots \quad -q_{1S}]$ is the $N \times (S+1)$ - dimensional matrix of asset prices and pay-offs;

$\mathsf{w} = [\mathsf{w}_0, \ \mathsf{w}_{11}, \ \cdots, \ \mathsf{w}_{1S}]'$ is the $(S+1)$ - dimensional vector of incomes.

With the help of this notation the previous consumer's problem can be written :

$$\left.\begin{array}{ll} \max_{x,y} \ \mathcal{U}(x) & \\ \text{subject to} & P'x + Q'y = \mathsf{w} \end{array}\right\} . \tag{2.4}$$

In order to distinguish between present and future variables (a usual distinction), it is also convenient to consider the following partition :

$$x = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}, P' = \begin{bmatrix} p_0' & 0 \\ 0 & P_1' \end{bmatrix}, Q' = \begin{bmatrix} q_0' \\ -Q_1' \end{bmatrix}, \mathsf{w} = \begin{bmatrix} \mathsf{w}_0 \\ \mathsf{w}_1 \end{bmatrix} .$$

The individual decision problem is then

$$\left.\begin{array}{ll} \max_{x_0,x_1,y} \ \mathcal{U}(x_0, x_1) & \\ \text{subject to} & p_0'x_0 + q_0'y = \mathsf{w}_0 \\ & P_1'x_1 - Q_1'y = \mathsf{w}_1 \end{array}\right\} . \tag{2.5}$$

Remark 1 : a) The consumer's problem considered here encompasses the usual framework where the utility function would be written as :

$$\begin{aligned} \mathcal{U}(x_0, x_{11}, \cdots, x_{1S}) &= u(x_0) + \delta \underset{\pi}{E} u(x_{1s}) \\ &= u(x_0) + \delta \sum_s \pi_s u(x_{1s}) \ , \end{aligned} \tag{2.6}$$

where π_s is a subjective probability and δ a psychological discount factor. However, such a specification seems very restrictive since additive separability is assumed not only between periods, but also across states. Moreover, state-dependency of the utility function is obtained in an ad hoc way via the probability π_s. Consequently, in spite of its computational advantages, the functional form (2.6) inherited from the von Neumann-Morgenstern tradition has been challenged in order to eradicate its paradoxical implications [see, for instance, Allais (1953) and Ellsberg (1961)] while keeping the computational burden as low as possible.

b) Early examples of the utility function which do not satisfy the von Neumann-Morgenstern tradition can be found in Machina (1987). More recent ones (or more appropriate to our purposes) are :

$$\mathcal{U}(x) = \sum_s \pi_s u(x_0, x_{1s}) \;, \tag{2.7}$$

$$\mathcal{U}(x) = u(x_0) + \delta \sum_s \pi_s u_s(x_{1s}) \;, \tag{2.8}$$

$$\mathcal{U}(x) = u(x_0) + \delta \sum_s \sum_\sigma \varphi(x_{1s}, x_{1\sigma}) \pi_s \pi_\sigma \;, \tag{2.9}$$

$$\mathcal{U}(x) = u(x_0) + \delta \left\{ \min_{\pi \in \prod} \sum_s \pi_s u(x_{1s}) \right\} \;, \tag{2.10}$$

$$\mathcal{U}(x) = u(x_0) + \delta \left\{ \sum_s \pi_s u(x_{1s}) - \frac{\eta}{2} \sum_s \pi_s \left[u(x_{1s}) - \sum_s \pi_s u(x_{1s}) \right]^2 \right\} \;, \tag{2.11}$$

$$\mathcal{U}(x) = u(x_0) + \delta v(x, \pi) \;. \tag{2.12}$$

The specification (2.7), which is not necessarily time-separable, was used by Drèze and Modigliani (1972). The specification (2.8), where the elementary utilities u_s are state-dependent, was for instance considered by Cook and Graham (1977). The "quadratic" form (2.9) was axiomatized by Chew, Epstein and Segal (1991). The specification (2.10) was axiomatized by Gilboa and Schmeidler (1989) and used in finance by Dow and Werlang (1992). In this form, the probability π is not exactly known, allowing for "ambiguity" (Knightian uncertainty) [see Epstein (1999), Epstein and Schneider (2001, 2002), Epstein and Zhang (2001), and Machina (2001)]. The specification (2.11) extends (2.8) by introducing a risk correction that accounts for the uncertainty of utility levels $u(x_{1s})$. The specification (2.12) was proposed by Samuelson (1960), used by Machina (1995, 2000), and axiomatized by Machina and Schmeidler (1992, 1995). This specification generalizes all the other ones except (2.7).

2.2. Assumptions

The consumer's problem will be studied under the following assumptions :

A.1 The utility function $\mathcal{U}$ is defined on the consumption set $\mathcal{X} \subset \mathsf{R}^L$. This set is open, convex, bounded below and non-empty. The utility function is twice continuously differentiable. The gradient $\mathcal{U}_x$ is strictly positive (strong monotonicity); the Hessian matrix $\mathcal{U}_{xx'}$ is such that the quadratic form $\zeta' \mathcal{U}_{xx'} \zeta$ is strictly negative whenever $\zeta \neq 0$ and $\zeta' \mathcal{U}_x = 0$ (strong quasi-concavity).

Assumption A.1 concerns the individual's characteristics and is standard in economic theory since Debreu (1972) [see, for instance, Balasko (1988), Mas-Colell (1985), Mas-Colell et al. (1995)]. The consumption set is not necessarily the strictly positive orthant of R^L, since a negative component of x can be a quantity of labor. Furthermore, the general form of the utility function is general enough to admit, in particular, state dependency of the utility of consumption.

A.2 The budgetary constraints are known to the consumer.

In Assumption A.2, it is implicitly assumed that prices and incomes are exogenous data imposed on the consumer. This is not surprising for p_0, q_0 and w_0, but amounts to imposing a specific (though usual) assumption on p_1, q_1 and w_1. Even in this case the possible values of these variables should not be seen as determined by the state of nature (war does not suffice to explain the formation of war prices). On the one hand the space of the possible prices and incomes cannot be spanned from the (small and discrete) space of states; on the other hand, the possible states of the world are given a priori, once and for all, while the considered prices and incomes are in principle endogenous at the aggregate level. This point is important because usual financial approaches identify states and price evolutions [see, for instance, the option pricing model of Black and Scholes (1973)] and, by doing so, are led into error on the degree of market incompleteness.

A.3 In the budgetary constraints, P, Q and w are such that :

a) $p_0, p_{11}, \cdots, p_{1S}$ are strictly positive;

b) rank $Q_1 = N \leq S$;

c) $Q'y \ngeq 0$ for any y;

d) $\mathcal{X} \cap \{x \mid P'x + Q'y = \mathsf{w}\} \neq \emptyset$ for at least a y.

Assumption A.3 concerns the budget constraints faced by the individual. Under these constraints a portfolio does not necessarily belong to the positive orthant of R^N : a negative component of y may indicate a bank borrowing as well as a short sale transaction. Assumption A.3a implies, in particular, that the Jacobian matrix of the $S+1$ budget constraints has rank $S+1$. Assumption A.3b eliminates redundant assets ; if $N = S$, the asset structure is complete ; if $N < S$, the asset structure is incomplete, and this incompleteness has dimension $S-N$. Asset prices are arbitrage-free by A.3c (without this assumption the consumer's wealth could increase indefinitely). Assumption A.3d is the survival condition.

From a mathematical viewpoint, financial assets can be conceived as a way to reduce the number of constraints faced by the consumer. Indeed, since Q_1 has rank N, it is always possible to choose N out of $S+1$ constraints in order to express y in terms of revenues and expenditures, and to substitute these expressions in the residual constraints : we then have $S - N + 1$ independent constraints. If $N = 0$, $S+1$ constraints are effective. If $N = S$, the future constraints of (2.5) can be written as $y = [Q_1']^{-1}[P_1'x_1 - \mathsf{w}_1]$ which, in turn, can be substituted in the present constraints ; thus one has

$$p_0'x_0 + q_0'[Q_1']^{-1}P_1'x_1 = \mathsf{w}_0 + q_0'[Q_1']^{-1}\mathsf{w}_1 \ ,$$

and this unique effective constraint is, up to the notation, the constraint considered in the standard case of consumption theory. If, moreover, $Q_1' = I_S$ the financial assets reduce to the elementary contingent assets of Arrow (1953). Since the asset structure is complete, standard results apply.

Since there are actually $S-N+1$ constraints the budget set is a linear manifold of dimension $L - S + N - 1$ in R^L. Under Assumption A.3, it intersects the consumption set $\mathcal{X}$. Let x^B be a point in this intersection and $x^A < x^B$ be another point in $\mathcal{X}$: the consumption plans that are simultaneously preferred to x^A and less expensive than x^B give a non-empty and compact subset of $\mathcal{X}$. Maximizing $\mathcal{U}(x)$ on this set leads to a solution which is unique from strong monotonicity and strong quasi-concavity assumptions.

2.3. First-order conditions and demand systems

Under the previous assumptions, the consumer's problem may be solved with the help of the Lagrangean :

$$L(x, y; \lambda) = \mathcal{U}(x) - \lambda'[P'x + Q'y - \mathsf{w}] \ ,$$

where $\lambda' = [\lambda_0, \lambda_{11}, \cdots, \lambda_{1S}]$ is a vector of Lagrange multipliers. This vector exists and is unique. The first-order conditions are necessary and sufficient. They are written :

$$\left.\begin{array}{rcl} \mathcal{U}_x - P\lambda & = & 0 \\ -Q\lambda & = & 0 \\ -P'x - Q'y + \mathsf{w} & = & 0 \end{array}\right\} , \qquad (2.13)$$

when the whole structure is considered;

$$\left.\begin{array}{l} \mathcal{U}_0 = \lambda_0 p_0, \ \mathcal{U}_{1s} = \lambda_{1s} p_{1s}, \ s = 1, \cdots, S \\ \lambda_0 q_0 = \sum_s \lambda_{1s} q_{1s} \\ p_0' x_0 + q_0' y = \mathsf{w}_0, \ p_{1s}' x_{1s} - q_{1s} y = \mathsf{w}_{1s}, \ s = 1, \cdots, S \end{array}\right\} , \qquad (2.14)$$

when time and uncertainty are explicited.

The conditions $\mathcal{U}_x - P\lambda = 0$ are internal to the considered period $\times$ state. For instance, the conditions $\mathcal{U}_0 = \lambda_0 p_0$ imply that marginal rates of substitution are equal to relative prices <u>within</u> period 0, but do not give rise to any link across periods or states. Those links are established by the N conditions $Q\lambda = 0$. If $N = S$ and $Q_1 = I_S$, one has $\dfrac{\lambda_1}{\lambda_0} = q_0$ and the links are complete. If $N < S$, both intertemporal and contingent substitution among commodities are imperfect. If $N = 0$, no such substitution is allowed, but the Lagrange multipliers are still defined and can be used to study the desirability of transfers over time and across states. Finally, since A.3c implies that the consumer's wealth cannot be increased indefinitely, the conditions $Q'y = \mathsf{w} - P'x$ express, under A.3, the fact that, for a utility-maximizing consumer, some kind of hedging is a necessity. Deeper interpretations are possible if some relevant tools are first developed.

Proposition 1 : Let us set $p = [p_0', p_{11}', ..., p_{1S}']'$, $q = [q_0', -q_{11}', ..., -q_{1S}']'$. Under Assumptions A.1, A.2 and A.3, there exist solutions of the consumer optimization problem :

a) $x = x(p, q, \mathsf{w})$ (commodity demands),

b) $y = y(p, q, \mathsf{w})$ (asset demands),

c) $\lambda = \lambda(p, q, \mathsf{w})$ (desirabilities of incomes),
that are continuously differentiable. ■

Proof : See Appendix A.

As already mentioned, if the asset structure is complete $(N = S)$, the $S + 1$ budgetary constraints faced by the consumer reduce to a unique constraint. As a result, commodity demands x, asset demands y and the desirabilities of incomes λ will depend on (p, q, w) in a specific way. In particular, the desirabilities of future incomes can be written as $\lambda_1(\cdot) = \lambda_0(\cdot) Q_1^{-1} q_0$, since Q_1 is invertible.

The desirabilities of income are not independent of a monotonic transformation of the utility function $\mathcal{U}$. In the next section, we shall transform them into subjective Arrow prices.

3. Subjective Arrow prices and indirect utility function

Since demand functions are already known to exist, an indirect utility function can be defined by the relation :

$$v(p, q, \mathsf{w}) \equiv \mathcal{U}[x(p, q, \mathsf{w})] \ , \tag{3.1}$$

where $\mathcal{U}[x(p, q, \mathsf{w})]$ is the maximal value of the Lagrangean $L = \mathcal{U}(x) - \lambda'[P'x + Q'y - \mathsf{w}]$. By the envelope theorem, the partial derivatives of (3.1) can be computed with the help of this Lagrangean. For instance, one has :

$$\frac{\partial v}{\partial \mathsf{w}} \equiv \lambda(p, q, \mathsf{w}) \ , \tag{3.2}$$

or equivalently $\lambda_0 \equiv \dfrac{\partial v}{\partial \mathsf{w}_0}$, $\lambda_{1s} \equiv \dfrac{\partial v}{\partial \mathsf{w}_{1s}}$, $s = 1, \cdots, S$. The multiplier λ_0 is the (present) marginal utility of present income. The multiplier λ_{1s} is the (present) marginal utility of future income in state s. These desirabilities will be converted into subjective Arrow prices in subsection 3.1. These Arrow prices will be utilized to reinterpret our first-order conditions in subsection 3.2 and to characterize the dual first-order conditions in subsection 3.3.

3.1. Arrow prices

A ratio of marginal utilities is a price. The ratio $\dfrac{\lambda_{1s}}{\lambda_0} = \dfrac{\partial v/\partial \mathsf{w}_{1s}}{\partial v/\partial \mathsf{w}_0}$ is the amount of present money a consumer is willing to pay for an additional unit of future revenue in state s. Such a (present) marginal willingness to pay, or subjective Arrow price of an elementary contingent asset, will be denoted $\tilde{\mu}_{1s}$. Before being actualized, this value (price) is denoted μ_{1s} so that one has $\tilde{\mu}_{1s} = \beta\mu_{1s}$ with the normalization $\sum_{s=1}^{S} \mu_{1s} = 1$. Consequently, $\beta = \dfrac{\sum_s \lambda_{1s}}{\lambda_0}$ is a subjective discount factor often written $\dfrac{1}{1+r}$ where r is the nominal rate of interest, and $\mu_{1s} = \dfrac{\lambda_{1s}}{\sum_s \lambda_{1s}}$ is a subjective forward Arrow price. As $\mu_{1s} > 0$ and $\sum_s \mu_{1s} = 1$, μ_{1s} can also be interpreted as an Arrow risk-neutral subjective probability. Note that μ_{1s} does exist whatever the number of assets ($N = 0$ is admissible).

3.2. Arrow prices and first-order conditions

We now reconsider the interpretation of the first order conditions (2.14) of the consumer's problem. With the previous interpretations in mind, $\mathcal{U}_0/\lambda_0 = p_0$ is an equality between a marginal willingness to pay and an (official) price. So is $\mathcal{U}_{1s}/\lambda_{1s} = p_{1s}$. In each case the marginal willingness to pay is expressed in the specific unit of account of the corresponding price. Let us now consider the present marginal willingness to pay $\mathcal{U}_{1s}/\lambda_0$ expressed in the present account unit. We have $\mathcal{U}_{1s}/\lambda_0 = (\lambda_{1s}/\lambda_0)\, p_{1s} = \tilde{\mu}_{1s}p_{1s}$. It can be compared to p_{1s} because the latter has been converted (via $\tilde{\mu}_{1s}$) into a subjective Arrow-Debreu price. In the same way, the condition $\sum_s \lambda_{1s}q_{1s} = \lambda_0 q_0$ can be written $\sum_s \tilde{\mu}_{1s}q_{1s} = \beta \sum_s \mu_{1s}q_{1s} = q_0$, and expresses that, from the consumer's viewpoint, his own marginal valuation of assets is equal to their given prices.

Proposition 2 : The asset prices, at date 0, can be written as a mathematical expectation of discounted future cash-flows : $q_0 = \beta \sum_s \mu_{1s} q_{1s} = \dfrac{1}{1+r} \sum_s \mu_{1s} q_{1s}$ where $\mu_{1s} = \dfrac{\lambda_{1s}}{\sum_s \lambda_{1s}}$ defines a subjective probability measure on the set of states, and where $\beta = \dfrac{1}{1+r} = \dfrac{\sum_s \lambda_{1s}}{\lambda_0}$ is a subjective discount factor. ¥

Remark 2 : The probability measure and the discount factor are a priori subjective, but can become "objective" according to the set of exchangeable assets. For instance, if this set contains an Arrow security associated with state s, its price is $\beta\mu_{1s} = \tilde{\mu}_{1s}$. Similarly, if there is a zero-coupon ($q_{1s} = 1,\ \forall s$), its price at date 0 is $\beta = \sum_s \tilde{\mu}_{1s}$. In the general case $\beta\mu_{1s} = \tilde{\mu}_{1s}$ is the subjective price of an Arrow security, $\mu_1 = (\mu_{11}, \cdots, \mu_{1S})$ a risk-neutral subjective probability measure and β a subjective valuation of a zero-coupon [See Harrison and Kreps (1979), Duffie (1992)].

Let us consider the no-arbitrage condition. The Assumption $Q'y$ ¤ 0 is equivalent to the existence of a $(S+1)-$ dimensional vector $\alpha > 0$ such that $Q\alpha = 0$ [by Stiemke's lemma, see Mangasarian (1969)[5]]. The relation $Q\alpha = 0$ is usually linked to Farkas' lemma and thus written $q_0 = Q_1 (\alpha_1/\alpha_0)$. In this case, α_1/α_0 is a non-negative-S-dimensional vector (but not necessarily strictly positive). When the asset structure is incomplete, such a vector α_1/α_0 is not necessarily unique : there exists a positive cone of dimension $S-N$ (the dimension of incompleteness) of such vectors [Ross (1978), Breeden and Litzenberger (1978), Varian (1987)]. The vector $\tilde{\mu}_1 = \lambda_1/\lambda_0$ is an element of the previous cone. When the asset structure is complete, the vector α_1/α_0 becomes unique and $\tilde{\mu}_1$ is equal to this unique α_1/α_0. As a result, a vector α_1/α_0 is often seen as a vector of shadow prices associated with elementary contingent assets and consistent with the prices of traded assets. The first-order conditions $Q\lambda = 0$ and the positivity of the $\lambda_{1s}/\lambda_0 = \tilde{\mu}_{1s}$ express the no-arbitrage condition. As already seen in Proposition 1, by solving his optimization problem, the individual chooses a vector of shadow prices among the multiplicity of shadow prices of elementary contingent assets, and the chosen vector depends on his preferences, his income and the set of given prices.

[5] Mangasarian, chap. 2, p. 32.

Remark 3 : Suppose the consumer's problem is solved sequentially. After solving for x and λ (first step), the second step of the optimization consists of $\underset{y}{Max}\ \bar{v}(p,\mathsf{w}-Q'y)$, where $\bar{v}$ is an indirect utility function. This specific objective function is typical of the characteristic approach [Becker (1965), Lancaster (1966) and Rosen (1974)]. To each financial asset is naturally associated a column of Q' and such a column (whose components are initial cost and possible future pay-offs) can be seen as a vector of characteristics. $Q'y$ is then the vector of total amounts of characteristics. Therefore, it should not be surprising that, while solving the consumer's problem, one obtains naturally these characteristic shadow (or hedonic) prices (as seen in Proposition 2).

3.3. Arrow prices and the indirect utility function

The Lagrangean $L = \mathcal{U}(x) - \lambda'[P'x + Q'y - \mathsf{w}]$ can also be written $L = \mathcal{U}(x) - \lambda_0\,(x_0'p_0 + y'q_0 - \mathsf{w}_0) - \sum_s \lambda_{1s}\,(x_{1s}'p_{1s} - y'q_{1s} - \mathsf{w}_{1s})$. By applying the envelope theorem, we get the partial derivatives of $v(p,q,\mathsf{w})$ with respect to p, q and w :

$$\frac{\partial v}{\partial p_0} = -\lambda_0 x_0, \quad \frac{\partial v}{\partial p_{1s}} = -\lambda_{1s} x_{1s}, \quad s = 1, \quad \cdots, \quad S\ , \tag{3.3}$$

$$\frac{\partial v}{\partial q_0} = -\lambda_0 y, \quad \frac{\partial v}{\partial(-q_{1s})} = -\lambda_{1s} y, \quad s = 1, \quad \cdots, \quad S\ , \tag{3.4}$$

$$\frac{\partial v}{\partial \mathsf{w}_0} = \lambda_0, \quad \frac{\partial v}{\partial \mathsf{w}_{1s}} = \lambda_{1s}, \quad s = 1, \quad \cdots, \quad S\ , \tag{3.5}$$

where any term of these equalities should be seen as a function.

Now, suppose the consumer is facing shocks $dp,\ dq,\ d\mathsf{w}$. At first order, the impact on his utility level is :

$$\begin{aligned} dv &= \lambda_0[d\mathsf{w}_0 - x_0'dp_0 - y'dq_0] + \sum_s \lambda_{1s}[d\mathsf{w}_{1s} - x_{1s}'dp_{1s} + y'dq_{1s}]\ , \\ \frac{dv}{\lambda_0} &= [d\mathsf{w}_0 - x_0'dp_0 - y'dq_0] + \sum_s \tilde{\mu}_{1s}[d\mathsf{w}_{1s} - x_{1s}'dp_{1s} + y'dq_{1s}]\ . \end{aligned} \tag{3.6}$$

In the above decomposition, each bracket represents a real variation of income within a given period and state.

Proposition 3 : The first-order variations of utility (or standard of living) dv are linked to the real variations of incomes by the formulas :

$$\begin{aligned}\frac{dv}{\lambda_0} &= [d\mathsf{w}_0 - x_0' dp_0 - y' dq_0] + \sum_s \tilde{\mu}_{1s}[d\mathsf{w}_{1s} - x_{1s}' dp_{1s} + y' dq_{1s}] \\ &= [d\mathsf{w}_0 - x_0' dp_0 - y' dq_0] + \beta \sum_s \mu_{1s}[d\mathsf{w}_{1s} - x_{1s}' dp_{1s} + y' dq_{1s}] \\ &= [d\mathsf{w}_0 - x_0' dp_0 - y' dq_0] + \beta E_{\mu_1}[d\mathsf{w}_1 - x_1' dp_1 + y' dq_1] ,\end{aligned}$$

where E_{μ_1} denotes the expectation with respect to the probability distribution $\mu_{1s}, s = 1, ..., S$. ¥

In other words, after aggregating the various shocks $dp, dq, d\mathsf{w}$ into spot variations of real incomes, one can go on, even if the asset structure is incomplete, and aggregate over time and across states. When doing so, subjective Arrow prices are used. The final result is that utility variations are proportional to intertemporal real variations of wealth. This last concept contains a present and a future. The future real wealth is locally seen as a mathematical expectation where the probabilities utilized are the consumer's risk-neutral probabilities.

Remark 4 : Let us consider equation (3.6) when $dp = 0, d\mathsf{w} = 0$. One has $dv = \lambda_0 y' [\sum_s \tilde{\mu}_{1s} dq_{1s} - dq_0]$, where the expression between brackets is a vector of subjective excess return changes. The consumer summarizes in a simple N-dimensional index the $N(S+1)$ financial price variations.

Proposition 4 : Roy identities are given by :

$$\begin{aligned} x_0 &= -\frac{\partial v / \partial p_0}{\partial v / \partial \mathsf{w}_0}, x_{1s} = -\frac{\partial v / \partial p_{1s}}{\partial v / \partial \mathsf{w}_{1s}}, s = 1, \cdots, S , \\ y &= -\frac{\partial v / \partial q_0}{\partial v / \partial \mathsf{w}_0} = -\frac{\partial v / \partial (-q_{1s})}{\partial v / \partial \mathsf{w}_{1s}}, s = 1, \cdots, S .\end{aligned}$$

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Remark 4 explains the rather peculiar look of the financial components y of the Roy identities.

4. Premiums and aversions

Arrow prices $\tilde{\mu}_{1s} = \lambda_{1s}/\lambda_0$ have been defined and linked to the indirect utility function in the previous section. In this section, the local structure of Arrow prices will be used to define premiums and aversion measures. In subsection 4.1, we will study the local structure of Arrow prices (with the help of an Antonelli matrix) which, in turn, will be used to define a disutility premium. The latter will be seen as a residual premium which means that, while facing a shock, the consumer uses the financial markets to adjust himself. An alternative setting will be considered in subsection 4.2 : the individual still faces a shock, but is no more able to adjust financially. In such a context, the resulting premium and risk aversion will be called fundamental. In subsection 4.3, we show how residual and fundamental premiums are linked together via the Drèze-Modigliani decomposition. For convenience, only income shocks will be considered. This is consistent with the bulk of the literature and, in particular, it is the natural context of the Drèze-Modigliani decomposition.

4.1. The local structure of Arrow prices and the disutility premium

We start from Proposition 3. Let us set :

$$\tilde{\mu} = \lambda/\lambda_0 = \begin{bmatrix} 1 \\ \tilde{\mu}_1 \end{bmatrix} = [1, \beta\mu_{11}, ..., \beta\mu_{1S}]' \ .$$

Under this convention, and when commodity and asset prices are kept constant, the first-order variation of utility is linked to the variation of incomes by the relation :

$$\frac{dv}{\lambda_0} = \tilde{\mu}' d\mathsf{w} \ .^6 \tag{4.1}$$

Both sides of the relation are invariant under a monotonic transformation of the utility function. The expansion of the indirect utility function can also be considered at order two :

$$\Delta v = dv + \frac{1}{2} d^2 v + o\left(\|d\mathsf{w}\|^2\right) \ . \text{ (say)}$$

Then by differentiating (4.1), one has :

$$\frac{d^2 v}{\lambda_0} - \frac{d\lambda_0}{\lambda_0} \tilde{\mu}' d\mathsf{w} = d\mathsf{w}_1^{\prime} d\tilde{\mu}_1 \ . \tag{4.2}$$

[6] If income shocks $d\mathsf{w}$ were coupled with commodity and asset price shocks dp and dq, relation (4.1) would be replaced by Proposition 3.

We shall study : a) the decomposition of $d\tilde{\mu}_1$ into substitution and wealth effects, b) the decomposition of d^2v, c) the corresponding formation of a disutility-premium formula, and d) an example which illustrates the Antonelli matrix.

a) The local structure of Arrow prices

The local structure of Arrow prices is studied by examining the coefficients of $d\tilde{\mu}_1$. Their main characteristics are summarized in the lemma below.

Lemma 1 : i) We get :

$$d\tilde{\mu}_1 = A_{11} d\mathsf{w}_1 + \frac{\partial \tilde{\mu}_1}{\partial \mathsf{w}_0} \tilde{\mu}' d\mathsf{w} \ ,$$

where A_{11} is an Antonelli matrix which measures the effects on Arrow prices of a compensated income shock :

$$A_{11} = \left[\frac{\partial \tilde{\mu}_1}{\partial \mathsf{w}_1'}\right]_{\tilde{\mu}^{\theta} d\mathsf{w} = 0} = \left[\frac{\partial \tilde{\mu}_1}{\partial \mathsf{w}_1'} - \frac{\partial \tilde{\mu}_1}{\partial \mathsf{w}_0} \tilde{\mu}_1'\right] = [-\tilde{\mu}_1 \ \ I_s] \frac{\partial \lambda / \partial \mathsf{w}'}{\lambda_0} \begin{bmatrix} -\tilde{\mu}_1' \\ I_s \end{bmatrix}$$

$$= [-\tilde{\mu}_1 \ \ I_s] \frac{\partial^2 v / \partial \mathsf{w} \partial \mathsf{w}'}{\lambda_0} \begin{bmatrix} -\tilde{\mu}_1' \\ I_s \end{bmatrix} \ ;$$

ii) A_{11} is a symmetric matrix which is independent of a monotonic transformation of the utility function, negative semi-definite, with rank $S - N$ (the dimension of incompleteness), and ker A_{11} = range Q_1'. ¥

Proof : See Appendix B.

An Antonelli matrix is analogous to a Slutsky matrix : the latter characterizes variations in commodity demands following compensated price shocks while the former characterizes variations in the (corresponding) prices following compensated quantity shocks. Both matrices can be used to characterize substitution and complementarity [see, for instance, Samuelson (1950) and his survey (1974)]. By analogy, A_{11} is called an Antonelli matrix since it also characterizes variations in the subjective prices. The matrix A_{11}, however, measures variations in the subjective prices of the implied (or virtual) Arrow assets following a compensated income shock[7], and the assets can be tradable

[7] Recall that $P_1^{\theta} x_1 = \mathsf{W}_1 + Q_1^{\theta} y$ can reduce to $P_1^{\theta} x_1 = \mathsf{W}_1 + y$. Therefore, incomes can be seen as Arrow assets.

or not. The decomposition of $d\tilde{\mu}_1$ (given by Lemma 1) involves substitution and wealth effects, and the Antonelli matrix A_{11} is used to characterize the substitution-complementarity among (virtual) assets. Concerning the rank of A_{11}, the intuition is easier to get if one first considers the case where the asset structure is complete. In such a case, the $N = S$ tradable assets are Arrow assets and Arrow prices are exogenous or fixed variables, which implies that $\left[\frac{\partial \tilde{\mu}_1}{\partial \mathsf{w}_1}\right] = 0$. Consequently, the rank of A_{11} reduces to zero. When the asset structure is incomplete $N < S$, the existence of N tradable assets amounts to having N fixed combinations of the S Arrow prices $\tilde{\mu}_{1s}$ which, in turn, reduces the rank of A_{11} to $S - N$.

b) The decomposition of $d^2 v$

Lemma 2 : We get :

$$d^2 v = \lambda_0 \left[d\mathsf{w}_1^{\prime} A_{11} d\mathsf{w}_1 + 2 d\mathsf{w}_1^{\prime} \frac{\partial \tilde{\mu}_1}{\partial \mathsf{w}_0} \tilde{\mu}' d\mathsf{w} + \frac{\partial \lambda_0}{\partial \mathsf{w}_0} \left(\tilde{\mu}' d\mathsf{w}\right)^2 \right] .$$

¥

Proof : see Appendix C.

The decomposition of $d^2 v$ involves a substitution effect $\lambda_0 \left(d\mathsf{w}_1^{\prime} A_{11} d\mathsf{w}_1\right) = [d^2 v]_{\tilde{\mu}' d\mathsf{w} = 0}$ and a wealth effect, the latter being of degree two. It is important to note that neither $d^2 v$ nor $d^2 v / \lambda_0$ is invariant with respect to increasing transformation of the utility function. However, the relation of Lemma 2 can also be written as :

$$\frac{d^2 v}{\lambda_0} - \frac{\partial \lambda_0 / \partial \mathsf{w}_0}{\lambda_0} \left(\tilde{\mu}' d\mathsf{w}\right)^2 = d\mathsf{w}_1^{\prime} A_{11} d\mathsf{w}_1^{\prime} + 2 d\mathsf{w}_1^{\prime} \frac{\partial \tilde{\mu}_1}{\partial \mathsf{w}_0} \tilde{\mu}' d\mathsf{w} , \tag{4.3}$$

where both sides satisfy the invariance property.

We deduce :

$$\frac{dv}{\lambda_0} + \frac{1}{2} \left[\frac{d^2 v}{\lambda_0} - \frac{\partial \lambda_0 / \partial \mathsf{w}_0}{\lambda_0} \left(\tilde{\mu}' d\mathsf{w}\right)^2 \right] = \left(1 + d\mathsf{w}_1^{\prime} \frac{\partial \tilde{\mu}_1}{\partial \mathsf{w}_0}\right) \tilde{\mu}' d\mathsf{w} + \frac{1}{2} d\mathsf{w}_1^{\prime} A_{11} d\mathsf{w}_1 . \tag{4.4}$$

Our next purpose is to show that such a relation defines a premium and still a welfare criterion.

c) The disutility premium

Let $v(\mathsf{w}^0)$ be the value of the indirect utility function at some point w^0, and $v(\mathsf{w})$ its value after the income shock $d\mathsf{w}{=}\mathsf{w}{-}\mathsf{w}^0$ (prices are deleted to signal their constancy). Then, there exists a twice continuously differentiable function ρ such that

$$v(\mathsf{w}) = v\left[\left(\mathsf{w}_0^0 - \rho\left(\mathsf{w}, \mathsf{w}^0\right), \mathsf{w}_1^0\right)\right] \ , \tag{4.5}$$

everywhere on the domain of v. In this relation, v and $-\rho$ are monotonic transformations of each other. Therefore, ρ may be seen as a disutility function. The value $-\rho(\mathsf{w}, \mathsf{w}^0)$ is expressed in present money. In the language of welfare measures, it is analogous to an equivalent variation. Its opposite, $\rho(\mathsf{w}, \mathsf{w}^0)$, is a premium.

Let us differentiate (4.5). This gives :

$$\begin{aligned} dv &= -\lambda_0\left[\mathsf{w}_0^0 - \rho(\cdot), \mathsf{w}_1^0\right] d\rho \ , \\ d^2 v &= -\lambda_0\left[\mathsf{w}_0^0 - \rho(\cdot), \mathsf{w}_1^0\right] d^2\rho + \frac{\partial \lambda_0}{\partial \mathsf{w}_0}\left[\mathsf{w}_0^0 - \rho(\cdot), \mathsf{w}_1^0\right](d\rho)^2 \ . \end{aligned}$$

At the reference point $\mathsf{w}^0, \rho(\mathsf{w}^0, \mathsf{w}^0) = 0$. The previous relations become :

$$\begin{aligned} -d\rho &= \frac{dv}{\lambda_0} \ , \\ -d^2\rho &= \frac{d^2 v}{\lambda_0} - \frac{\partial \lambda_0/\partial \mathsf{w}_0}{\lambda_0}\left(\tilde{\mu}' d\mathsf{w}\right)^2 \ , \end{aligned}$$

where $\lambda_0 = \lambda_0(\mathsf{w}_0^0, \mathsf{w}_1^0) = \lambda_0(\mathsf{w}^0)$.

This yields :

$$-d\rho - \frac{1}{2}d^2\rho = \frac{dv}{\lambda_0} + \frac{1}{2}\left[\frac{d^2 v}{\lambda_0} - \frac{\partial \lambda_0/\partial \mathsf{w}_0}{\lambda_0}\left(\tilde{\mu}' d\mathsf{w}\right)^2\right] \ ,$$

which is exactly the left-hand side of (4.4). Finally, the Taylor expansion of ρ may be written $\Delta\rho = d\rho + \frac{1}{2}d^2\rho + o\left(\|d\mathsf{w}\|^2\right)$. At the reference point w^0,

$\Delta\rho = \rho(\mathsf{w}, \mathsf{w}^0) - \rho(\mathsf{w}^0, \mathsf{w}^0) = \rho(\mathsf{w}, \mathsf{w}^0)$. So, we have [by using (4.1) and Lemma 2] :

$$\begin{aligned} -\rho &= \frac{dv}{\lambda_0} + \frac{1}{2}\left[\frac{d^2v}{\lambda_0} - \frac{\partial\lambda_0/\partial\mathsf{w}_0}{\lambda_0}(\tilde{\mu}'d\mathsf{w})^2\right] - o\left(\|d\mathsf{w}\|^2\right) \\ &= \left(1 + d\mathsf{w}_1^{\prime}\frac{\partial\tilde{\mu}_1}{\partial\mathsf{w}_0}\right)\tilde{\mu}'d\mathsf{w} + \frac{1}{2}d\mathsf{w}_1^{\prime}A_{11}d\mathsf{w}_1 - o\left(\|d\mathsf{w}\|^2\right) . \end{aligned} \tag{4.6}$$

The assertion concerning relation (4.4), that is, it defines a premium and is still a welfare criterion, is proved since ρ is both a disutility function and a premium. Moreover, remark that ρ is also independent of a monotonic transformation of the utility function, as is the left-hand side of (4.4). From now on, we shall refer to ρ as a disutility premium.

Proposition 5 : The welfare effect of an income shock $d\mathsf{w}{=}\mathsf{w}{-}\mathsf{w}^0$ may be measured with the help of a disutility-premium function ρ characterized by the relation $v(\mathsf{w}) = v\left[\mathsf{w}_0^0 - \rho(\mathsf{w}, \mathsf{w}^0), \mathsf{w}_1^0\right]$. At the reference point w^0, this function is such that

$$-\rho = \left(1 + d\mathsf{w}_1^{\prime}\frac{\partial\tilde{\mu}_1}{\partial\mathsf{w}_0}\right)\tilde{\mu}'d\mathsf{w} + \frac{1}{2}d\mathsf{w}_1^{\prime}A_{11}d\mathsf{w}_1 - o\left(\|d\mathsf{w}\|^2\right) ,$$

where A_{11} is an Antonelli matrix. Locally, the disutility premium involves both substitution and wealth effects; at first order, $-d\rho = \tilde{\mu}'d\mathsf{w}$ and the welfare effect is a wealth variation; at the second order, however, $-\frac{1}{2}d^2\rho = \frac{1}{2}d\mathsf{w}_1^{\prime}A_{11}d\mathsf{w}_1 + d\mathsf{w}_1^{\prime}\frac{\partial\tilde{\mu}_1}{\partial\mathsf{w}_0}\tilde{\mu}'d\mathsf{w}$ and the welfare effect involves both substitution and wealth effects. ¥

Corollary 1 : If financial markets are complete $(N = S)$, Arrow prices $\tilde{\mu}_1$ coincide with market prices in the sense that $\tilde{\mu}_1 = Q_1^{-1}q_0$: there are no substitution effects $A_{11} = 0$, and a wealth variation has no effect on (individual) Arrow prices $\frac{\partial\tilde{\mu}_1}{\partial\mathsf{w}_0} = 0$. Therefore, the disutility premium reduces to the first-order wealth effect :

$$\rho = d\rho + o\left(\|d\mathsf{w}\|^2\right) = -\tilde{\mu}'d\mathsf{w} + o\left(\|d\mathsf{w}\|^2\right) . \tag{4.7}$$

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In the general case, if $\tilde{\mu}'d\mathsf{w} \neq 0$, the first-order term (that is, the wealth effect of the shock) is not reduced to zero and is still the dominant term of the Taylor's expansion of ρ, even if the shock also involves a reallocation over time and across states. Consequently, the interesting case to be studied is when $\tilde{\mu}' d\mathsf{w} = 0$.

The condition $\tilde{\mu}' d\mathsf{w} = 0$ also means $d\rho = 0$ and $\frac{dv}{\lambda_0} = 0$, that is, first-order compensation. As $\tilde{\mu}' d\mathsf{w} = \sum_s \mu_{1s} [d\mathsf{w}_0 + \beta d\mathsf{w}_{1s}]$, it also means that the mathematical expectation of wealth changes vanishes. In such a case, the shock "only" involves a reallocation over time and across states, the case we shall focus on from now on.

Corollary 2 : If there is first-order compensation $(\tilde{\mu}' d\mathsf{w} = 0)$, the disutility premium is equivalent to :

$$\begin{aligned} \rho &= -\frac{1}{2} d\mathsf{w}_1' A_{11} d\mathsf{w}_1 + o\left(\|d\mathsf{w}\|^2\right) \\ &= -\frac{1}{2} d\mathsf{w}_1' d\tilde{\mu}_1 + o\left(\|d\mathsf{w}\|^2\right) = -\frac{1}{2} d\mathsf{w}' \frac{\partial\lambda/\partial\mathsf{w}'}{\lambda_0} d\mathsf{w} + o\left(\|d\mathsf{w}\|^2\right) \\ &= -\frac{1}{2}\frac{d^2 v}{\lambda_0} + o\left(\|d\mathsf{w}\|^2\right) = -\frac{\Delta v}{\lambda_0} + o\left(\|d\mathsf{w}\|^2\right) . \end{aligned} \tag{4.8}$$

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The formulas of Proposition 5 and Corollary 2 take into account the consumer's adjustments (to $d\mathsf{w}$) in commodities and assets. In a financial economy, the consumer adjusts himself by making income transfers which, in turn, are made possible through portfolio selection. Facing an S-dimensional income shock, he uses the N-tradable assets to ajust himself and to satisfy the new budget constraints. This is why the previous disutility premium is necessarily a residual premium, meaning that it takes into account financial adjustment. The extent of this residual effect depends on the number and characteristics of financial assets, in particular, through the rank $S - N$ of the Antonelli matrix A_{11}. We shall have a closer look at this residual effect in subsection 4.3.

d) Example

We now provide a simple example to illustrate graphically the effect measured by the Antonelli matrix. More specifically, we consider the consumer's optimization problem when the future period consists of only one state $(S = 1)$ and when no tradable asset exists $(N = 0)$. The optimization problem is

$$\left.\begin{array}{l}\max\limits_{x_0,x_{11}} \mathcal{U}(x_0,x_{11}) \\ \text{subject to } p_0x_0 = \mathsf{w}_0 \\ \qquad\qquad\quad p_{11}x_{11} = \mathsf{w}_{11}\end{array}\right\} \text{P1} .$$

The solution is straightforward, given by $x_0 = \mathsf{w}_0$, $x_{11} = \mathsf{w}_{11}$, where, for simplicity, we have assumed $p_0 = p_{11} = 1$. Moreover, if solved by means of the Lagrangean : $\mathcal{L}(x_0,x_{11},\lambda_0,\lambda_{11}) = \mathcal{U}(x_0,x_{11}) - \lambda_0 [x_0 - \mathsf{w}_0] - \lambda_{11}[x_{11} - \mathsf{w}_{11}]$, problem P1 allows to define the Arrow price $\tilde{\mu}_{11} = \frac{\lambda_{11}}{\lambda_0} = \frac{\partial\mathcal{U}/\partial x_{11}}{\partial\mathcal{U}/\partial x_0}$, and since $S = 1$, we have $\sum_s \lambda_{1s} = \lambda_{11}$ which implies $\beta = \frac{\sum\lambda_{1s}}{\lambda_0} = \frac{\lambda_{11}}{\lambda_0} = \tilde{\mu}_{11}$. In such a case, the Antonelli matrix has rank $S - N = 1$, consists of only one coefficient and is given by :

$$A_{11} = \left[\frac{\partial\tilde{\mu}_{11}}{\partial\mathsf{w}_{11}}\right]_{d\mathsf{w}_0+\tilde{\mu}_{11}d\mathsf{w}_{11}=0} = \frac{\partial\tilde{\mu}_{11}}{\partial\mathsf{w}_{11}} - \frac{\partial\tilde{\mu}_{11}}{\partial\mathsf{w}_0}\tilde{\mu}_{11} \text{ or, equivalently,}$$
$$A_{11} = \left[\frac{\partial\beta}{\partial\mathsf{w}_{11}}\right]_{d\mathsf{w}_0+\beta d\mathsf{w}_{11}=0} = \frac{\partial\beta}{\partial\mathsf{w}_{11}} - \frac{\partial\beta}{\partial\mathsf{w}_0}\beta .$$

Note that problem P1 is equivalent to

$$\begin{array}{l}\max\limits_{x_0,x_{11}} \mathcal{U}(x_0,x_{11}) \\ \text{subject to } x_0 + \beta(\mathsf{w}_0,\mathsf{w}_{11})\,x_1 = \mathsf{w}_0 + \beta(\mathsf{w}_0,\mathsf{w}_{11})\,\mathsf{w}_{11} = \overline{\mathsf{w}} ,\end{array}$$

where $\overline{\mathsf{w}}$ is the consumer's present wealth.

Let us consider given values of incomes. The solution of P1 is represented by point A on Figure 1, where $\beta(\mathsf{w}_0,\mathsf{w}_{11}) = \beta^A$, say, is the slope of the tangent of the indifference curve $\mathcal{U}(x_0,x_{11}) = \mathcal{U}^A$ and $\overline{\mathsf{w}}$ is the x_0-intercept of the same tangent line.

Let us then consider an income shock $(d\mathsf{w}_0, d\mathsf{w}_{11})$. The new solution, represented by point B, is $x_0^B = \mathsf{w}_0 + d\mathsf{w}_0$, $x_{11}^B = \mathsf{w}_{11} + d\mathsf{w}_{11}$, and involves a change in both the optimal value of β and in the present wealth. These new values are

$\beta(\mathsf{w}_0 + d\mathsf{w}_0, \mathsf{w}_{11}, d\mathsf{w}_{11}) = \beta^B$, say, and $\overline{\mathsf{w}}+d\overline{\mathsf{w}}= (\mathsf{w}_0 + d\mathsf{w}_0) + \beta^B (\mathsf{w}_{11} + d\mathsf{w}_{11})$, respectively. The wealth variation $d\overline{\mathsf{w}}$ has two components :

$$d\overline{\mathsf{w}} = \underbrace{d\mathsf{w}_0 + \beta^A d\mathsf{w}_{11}}_{1} + \underbrace{\left(\frac{\partial \beta^A}{\partial \mathsf{w}_0} d\mathsf{w}_0 + \frac{\partial \beta^A}{\partial \mathsf{w}_{11}} d\mathsf{w}_{11}\right) \mathsf{w}_{11}}_{2} ,$$

where $\frac{\partial \beta^A}{\partial \mathsf{w}_0} = \frac{\partial \beta(\mathsf{w}_0, \mathsf{w}_{11})}{\partial \mathsf{w}_0}$. The first one refers to the variation when β is unchanged (a parallel movement of the tangent line $x_0 + \beta^A x_{11} = \overline{\mathsf{w}}$) while the second one is due to the change in β.

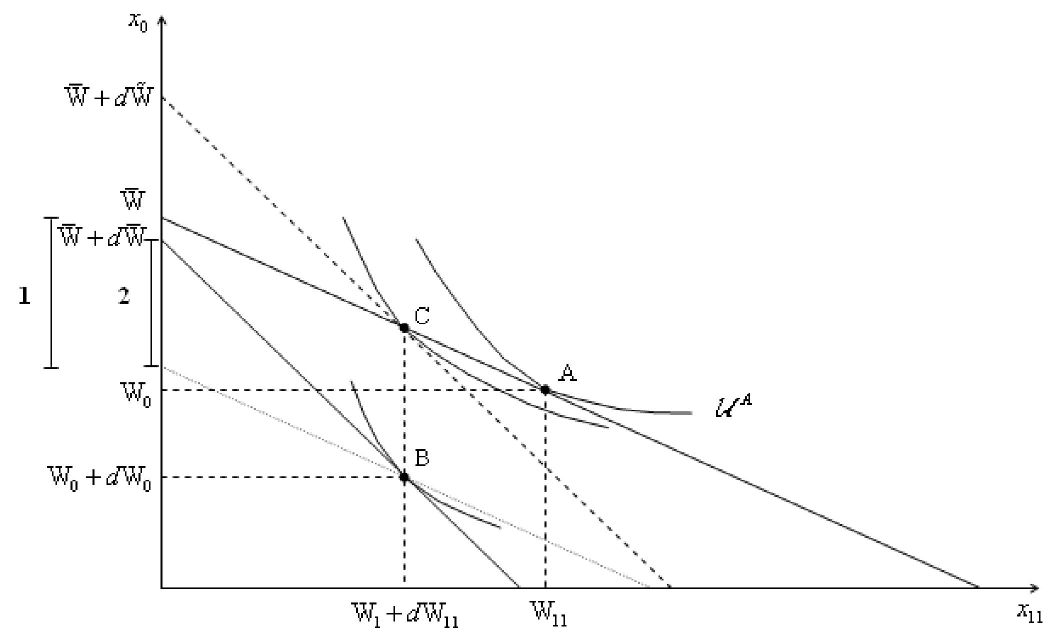

Figure 1 : The effect measured by the Antonelli matrix

In order to obtain the effect measured by the Antonelli matrix, we need to identify a third point, that is, what the consumer would choose if the income shock were compensated. Let $\left(x_0^C, x_{11}^C\right) = (x_0 + d\tilde{\mathsf{w}}_0, x_{11} + d\tilde{\mathsf{w}}_{11})$ (represented by point C) be such a candidate. It should satisfy the two following conditions :

1) $d\tilde{\mathsf{w}}_0$ should compensate for $d\tilde{\mathsf{w}}_1$ so as to leave the present wealth $\overline{\mathsf{w}}$ unchanged, that is $d\tilde{\mathsf{w}}_0 + \beta(\mathsf{w}_0, \mathsf{w}_{11})\, d\tilde{\mathsf{w}}_1 = 0$; and 2) $\beta^C = \beta(\mathsf{w}_0 + d\tilde{\mathsf{w}}_0, \mathsf{w}_{11} + d\tilde{\mathsf{w}}_{11})$ should be equal to $\beta^B = \beta(\mathsf{w}_0 + d\mathsf{w}_0, \mathsf{w}_{11} + d\mathsf{w}_{11})$. It is easily checked that point C is located on the tangent line $x_0 + \beta^A x_{11} = \overline{\mathsf{w}}$ and is such that $d\tilde{\mathsf{w}}$ is equal to the second component of $d\overline{\mathsf{w}}$.

Note that in this particular example $\frac{\partial \beta^A}{\partial \mathsf{w}_0} = 0$. Therefore, the compensated and the non compensated income shock both involve the same effect on the slope β. More precisely, we get $A_{11} = \left[\frac{\partial \beta}{\partial \mathsf{w}_{11}}\right]_{\mathsf{w}_0 + \beta d\mathsf{w}_{11} = 0} = \frac{\partial \beta}{\partial \mathsf{w}_{11}}$, and the effect of the Antonelli matrix corresponds to the change of slope from point A to point C.

This example also provides the intuition of why, in our model, the indirect utility function is not quasi-convex in w as usual, but quasi-concave in w. Indeed, we have : $v(p_0, p_{11}, \mathsf{w}_0, \mathsf{w}_{11}) = v(1, 1, \mathsf{w}_0, \mathsf{w}_{11}) = u(\mathsf{w}_0, \mathsf{w}_{11})$, and consequently, the indirect utility function behaves (roughly speaking) like a direct utility function.

In the next subsection, we define a fundamental disutility premium as well, when the individual does not use the financial markets to adjust himself. In subsection 4.3, we see how residual and fundamental premiums are tied together via the Drèze-Modigliani decomposition.

4.2. The fundamental premium and the fundamental risk aversion

In order to define and characterize the fundamental disutility premium at w^0 (and, ultimately, a corresponding measure of fundamental risk aversion), one considers the following set-up. The consumer still faces the income shock $d\mathsf{w}$ at the same reference point w^0, but is not able to adjust financially or, equivalently, is endowed with an illiquid portfolio. This setting will first be used to study the fundamental disutility premium. We then introduce a new normalization (for μ_{1s} and β) in order to completely disentangle time and uncertainty. This will allow us to decompose the fundamental disutility premium into an impatience and a risk premium. The latter decomposition has a natural counterpart in terms of aversion : the fundamental joint time-risk aversion will be decomposed into an aversion to impatience and a risk aversion.

a) The fundamental premium

As in the case with financial adjustment, the income shock is locally analyzed around the optimal portfolio y^* (the selection of which is made without any restriction). For instance, the income shock could here be analyzed by using the indirect utility function $v(\mathsf{w}) = \bar{v}(\mathsf{w} - Q'\bar{y})$ where $\bar{y} = y^*$ (see Section 3, Remark 3). Compared with the case with financial adjustment, the Arrow prices take the same initial (optimal) values, but their changes are, in principle, different. Let $d\bar{\beta}$, $d\bar{\tilde{\mu}}$, $d\bar{\mu}$, and $\bar{A}_{11}$ denote, respectively, the discount factor variation, the Arrow prices variations, the forward Arrow prices variations, and the Antonelli matrix corresponding to this case. Let also $\bar{\rho}$ denote the new disutility premium. If N were interpreted as the number of liquid assets, the analysis of the income shock in the present case amounts to rewriting Proposition 5 in the particular case where N vanishes locally. This leads to the proposition below :

Proposition 6 : If there is first-order compensation $(\tilde{\mu}' d\mathsf{w} = 0)$, the welfare effect of an income shock $d\mathsf{w} = \mathsf{w} - \mathsf{w}^0$ without financial adjustment may be measured with a disutility premium $\bar{\rho}$ as follows :

$$\begin{aligned} -\bar{\rho} &= \frac{1}{2} d\mathsf{w}_1' \bar{A}_{11} d\mathsf{w}_1 - o\left(\| \; d\mathsf{w} \; \|^2\right) \\ &= \frac{1}{2} d\mathsf{w}_1' d\bar{\tilde{\mu}}_1 - o\left(\|d\mathsf{w}\|^2\right) = \frac{1}{2} d\mathsf{w}' \frac{\partial\bar{\lambda}/\partial\mathsf{w}'}{\bar{\lambda}_0} d\mathsf{w} - o\left(\|d\mathsf{w}\|^2\right) \\ &= \frac{1}{2}\frac{d^2\bar{v}}{\bar{\lambda}_0} - o\left(\|d\mathsf{w}\|^2\right) = \frac{\Delta\bar{v}}{\bar{\lambda}_0} - o\left(\|d\mathsf{w}\|^2\right) \;, \end{aligned} \tag{4.9}$$

where $\bar{A}_{11}$ has rank S.

Proof : The rank of $\bar{A}_{11}$ can be deduced from the rank of the matrix A_{11} (see Lemma 1) when $N = 0$. ¥

From now on, we shall refer to $\bar{\rho}$ as the fundamental disutility premium, meaning that it does not take into account financial adjustment. The aim of the next subsection is to decompose the fundamental disutility premium into an impatience and a risk premium.

b) Impatience and risk premiums

Before going into the details of the decomposition, we first give the intuition of how it will be done. Let us consider the S-dimensional vector of future incomes and interpret it as a portfolio composed of S contingent Arrow assets w_{1s}, $s = 1, ..., S$. In this framework, $\tilde{\mu}_1' \mathsf{w}_1$ represents the present price (at date 0) of this portfolio. Its price admits different components which refer to the cost of intertemporal transfers, the cost of insurance, and a cost for cross-effects. If we used the normalized Arrow prices μ_{1s} and the discount factor β, the choice of the normalization (through the choice of β) will obviously affect the decomposition of the portfolio price since $\tilde{\mu}_{1s} = \beta \mu_{1s}$ and $\tilde{\mu}_1' \mathsf{w}_1 = \beta \mu_1' \mathsf{w}_1$. The number of admissible normalizations is infinite. Until now, we have used the canonical normalization of the literature. It allows to interpret μ_{1s} as an Arrow risk-neutral subjective probability. The different results of subsection 4.1 were established from the Arrow prices $\tilde{\mu}_{1s}$. They are therefore independent of the chosen normalization for the discount factor β and the $\mu_{1s} = \frac{\tilde{\mu}_{1s}}{\beta}$. As mentioned above, this will not be the case for the decomposition of the portfolio value. We would like to restrict our choice to a set of normalizations that completely disentangle time and uncertainty. In such a case, the portfolio price would be made-up of only two components, that is, a temporal and an insurance component. The normalization used so far does not allow for such a "strict" decomposition. Let us now introduce the new normalization.

Let us define β and $\mu_1 = \frac{\tilde{\mu}_1}{\beta} = \frac{\lambda_1}{\beta \lambda_0}$ such that

$$\mu_1' \bar{B}_{11} \mu_1 = 1 \ , \tag{4.10}$$

where $\bar{B}_{11} = -\bar{A}_{11}^{-1} \left(\mathsf{w}^0 \right)$ is a positive definite matrix. We deduce

$$\mu_1 = \frac{\lambda_1}{\left(\lambda_1' \bar{B}_{11} \lambda_1 \right)^{1/2}} \ , \tag{4.11}$$

$$\beta = \frac{\left(\lambda_1' \bar{B}_{11} \lambda_1 \right)^{1/2}}{\lambda_0} \ . \tag{4.12}$$

From the normalization restriction (4.10) and $\tilde{\mu}_1 = \beta \mu_1$, we deduce the differential restrictions :

$$d\bar{\tilde{\mu}}_1 = d\bar{\beta} \mu_1 + d\bar{\mu}_1 \beta \tag{4.13}$$

$$\mu_1' \bar{B}_{11} d\bar{\mu}_1 = 0 \ . \tag{4.14}$$

The conditions above can be interpreted in the following way. The Arrow price variations $d\bar{\tilde{\mu}}_1$ can be decomposed as the sum of $d\bar{\beta}\mu_1$ and $d\bar{\mu}_1\beta$. When $d\mathsf{w}$ varies, $d\bar{\beta}\mu_1$, $d\bar{\mu}_1\beta$, $d\bar{\tilde{\mu}}_1$ generate vector spaces E^I, E^{II}, E, respectively. They are such that

$$E = E^I + E^{II} \ ,$$

and E^I and E^{II} are orthogonal for the scalar product $\bar{B}_{11}$. Indeed,

$$d\bar{\beta}\mu_1' \bar{B}_{11} \beta d\bar{\mu}_1 = 0 \ ,$$

by relation (4.14). This can be summarized in the following lemma :

Lemma 3 : Decomposition of the space of price variations. The S-dimensional vector space E spanned by the Arrow prices variations $d\bar{\tilde{\mu}}_1$ is the direct sum of the one-dimensional vector subspace E^I spanned by the vector of normalized Arrow prices μ_1 and the $(S-1)$-dimensional vector subspace E^{II} spanned by the normalized Arrow price variations $d\bar{\mu}_1$. The subspaces E^I and E^{II} are orthogonal for the scalar product $\bar{B}_{11}$. The projector T on E^I along E^{II} is denoted by :

$$T = \mu_1 \mu_1' \bar{B}_{11} \ .$$

¥

We can also introduce a dual decomposition in the space of income shocks. Indeed, under first-order compensation, $\bar{A}_{11} d\mathsf{w}_1 = d\bar{\tilde{\mu}}_1$. Since $\bar{A}_{11}$ is invertible, one also has $d\mathsf{w}_1 = \bar{A}_{11}^{-1} d\bar{\tilde{\mu}}_1 = -\bar{B}_{11} d\bar{\tilde{\mu}}_1$. Using (4.13), we get :

$$d\mathsf{w}_1 = -\bar{B}_{11} d\bar{\beta}\mu_1 - \bar{B}_{11}\beta d\bar{\mu}_1 \ . \tag{4.15}$$

Moreover, the orthogonal projector T satisfies : $\bar{B}_{11}T = T'\bar{B}_{11}$. We deduce

$$\begin{aligned} T' d\mathsf{w}_1 &= -T'\bar{B}_{11} d\bar{\beta}\mu_1 - T'\bar{B}_{11}\beta d\bar{\mu}_1 \\ &= -\bar{B}_{11} T d\bar{\beta}\mu_1 - \bar{B}_{11} T \beta d\bar{\mu}_1 \\ &= -\bar{B}_{11} d\bar{\beta}\mu_1 \ , \end{aligned}$$

since $T\beta d\bar{\mu}_1 = 0$ and $T d\bar{\beta}\mu_1 = d\bar{\beta}\mu_1$ by the definition of T, and therefore :

$$(I - T') \, d\mathsf{w}_1 = -\bar{B}_{11}\beta d\bar{\mu}_1 \ .$$

Note also that $\bar{B}_{11}^{-1}T' = (T')' \bar{B}_{11}^{-1}$, which means that T' is an orthogonal projector for the scalar product $\bar{B}_{11}^{-1} = -\bar{A}_{11}$.

In relation (4.15), $-\bar{B}_{11} = \bar{A}_{11}^{-1}$ is a Slutsky matrix (deleted of its first line and column). $d\mathsf{w}_1$ can be decomposed into $d\mathsf{w}_1 = d\mathsf{w}_1^I + d\mathsf{w}_1^{II}$ where $d\mathsf{w}_1^I = -\bar{B}_{11}d\bar{\beta}\mu_1$ and $d\mathsf{w}_1^{II} = -\bar{B}_{11}\beta d\bar{\mu}_1$ are intertemporal and across state reallocations, respectively. Thus, $-\bar{B}_{11}\mu_1$ may be interpreted as intertemporal substitution effects and $-\bar{B}_{11}\beta$ as substitution effects across states.

The results can be summarized in the following lemma :

Lemma 4 : Decomposition of the space of income reallocations. The S-dimensional vector space $W = \bar{A}_{11}^{-1}E$ spanned by the income reallocations $d\mathsf{w}_1$ is the direct sum of the one-dimensional subspace $W^I = \bar{A}_{11}^{-1}E^I$ of intertemporal reallocations and the $(S-1)$-dimensional subspace $W^{II} = \bar{A}_{11}^{-1}E^{II}$ of across state reallocations. The subspaces W^I, W^{II} are orthogonal for the scalar product $-\bar{A}_{11} = \bar{B}_{11}^{-1}$. The orthogonal projector on W^I along W^{II} is T'. ¥

The interpretations of Lemmas 3 and 4 are easier to get if, once again, one interprets w_1 as a portfolio of contingent Arrow assets and $d\mathsf{w}_1$ as its reallocation. First, the duality between the two vector spaces, E and W, becomes more apparent. The space of Arrow prices variations is the dual space of the space of Arrow portfolio reallocations. Second, starting with an Arrow portfolio w_1, if Arrow prices change, the portfolio value becomes $\left(\tilde{\mu}_1 + d\bar{\tilde{\mu}}_1\right)' \mathsf{w}_1$. If there is a reallocation of the Arrow portfolio, the readjustment value is $\left(\tilde{\mu}_1 + d\bar{\tilde{\mu}}_1\right)' d\mathsf{w}_1$. Using the orthogonality condition (4.14), it is easily checked that the readjustment value can be written as :

$$\left(\tilde{\mu}_1 + d\bar{\tilde{\mu}}_1\right)' d\mathsf{w}_1 = \left(\beta + d\bar{\beta}\right) d\mathsf{w}_1'\mu_1 + \beta d\mathsf{w}_1' d\bar{\mu}_1 \ ,$$

where $\left(\beta + d\bar{\beta}\right) d\mathsf{w}_1'\mu_1$ is the cost of intertemporal transfers (saving or credit) and $\beta d\mathsf{w}_1' d\bar{\mu}_1$ is the cost of insurance. As already mentioned, the decomposition involves two components only, due to the selected normalization. Third, if one considers a reallocation of the Arrow portfolio such that $d\mathsf{w}_1 = d\mathsf{w}_1^I$ (also named intertemporal reallocations), Lemmas 3 and 4 imply the following equivalent statements (see Appendix D) :

i) the insurance price associated with an intertemporal portfolio reallocation is equal to zero $(d\mathsf{w}_1' d\bar{\mu}_1 = 0)$;

ii) the implied modification of Arrow prices reduced to a readjustment of the discount factor $\left(d\bar{\tilde{\mu}}_1 = d\bar{\beta}\mu_1\right)$;

iii) the intertemporal portfolio reallocation implies no change in the forward Arrow prices $(d\bar{\mu}_1 = 0)$.

Finally, we can interpret the space W^I (respectively, the direction $\bar{B}_{11}\mu_1$) as a subjective certainty space (respectively, subjective certainty direction), as shown by considering the special Arrow-Pratt framework. This framework mainly assumes :

i) a Von Neumann Morgenstern (VNM) utility function $u(x_0) + \delta \underset{s}{\Sigma} \pi_s u(x_s)$ [see (2.6)],

ii) which is strictly concave, and

iii) an adjustment in a neighborhood of a standard certainty point : $\mathsf{w}_1 + Q_1'\bar{y}$ proportional to $e = (1, ..., 1)'$.[8]

In the Arrow-Pratt framework, $\bar{B}_{11}\mu_1 = \frac{e}{e^{\theta}\mu_1}$ (the S-dimensional vector whose components are all equal to $\frac{1}{e^{\theta}\mu_1}$). Moreover, the projector T' on W^I along W^{II} is equal to $T' = \bar{B}_{11}\mu_1\mu_1' = e\pi'$ where π is the S-dimensional vector of probabilities involved in the VNM utility function (see Appendix D). $T'd\mathsf{w}_1 = \left(\underset{s}{\Sigma}\pi_s d\mathsf{w}_{1s}\right) e$ measures the expected income shock whereas $[I - T']\, d\mathsf{w}_1$ provides the demeaned shock, that is the risky components of the shock. Thus, in the general framework, $\bar{B}_{11}\mu_1$ is the natural extension of the standard certainty direction e, whereas $T'd\mathsf{w}_1 = d\mathsf{w}_1^I$ is a sure reallocation and $[I - T']\, d\mathsf{w}_1 = d\mathsf{w}_1^{II}$ the risky reallocation.

[8] In fact, the Arrow-Pratt framework assumes more specific assumptions. The additional assumptions will become explicit in Appendix E, where a complete example is presented.

Proposition 7 : In the Arrow-Pratt framework, the space W^I is spanned by the vector e with unitary components. An income reallocation $d\mathsf{w}_1$ is decomposed into its expected value and its demeaned value. In the general case, the space W^I will be called the sure reallocation space. ¥

In the literature, e is considered as the standard certainty direction. It is not invariant under a change of the value of the numeraire in the different states and, therefore, it has no interpretation in real terms. At the opposite, its natural extension $\bar{B}_{11}\mu_1$ is locally invariant. Some confusion could be possible owing to the proportionality between $\bar{B}_{11}\mu_1$ and e in the Arrow-Pratt framework.

An alternative rationale can be given for interpreting W^I as the subjective certainty or sure reallocation space. Let us consider the reallocation $d\mathsf{w}_1^I = T'd\mathsf{w}_1 = -\bar{B}_{11}d\bar{\beta}\mu_1$. It is easily checked that $d\mathsf{w}_1^I$ is the optimal $d\mathsf{w}_1$ an individual would choose to minimize the disutility premium (loss) given by $\bar{\rho} = -\frac{1}{2}d\mathsf{w}_1'\bar{A}_{11}d\mathsf{w}_1$, when $\mu_1'd\mathsf{w}_1 = -\frac{d\mathsf{w}_0}{\beta}$ with $d\mathsf{w}_0$ being exogenous.[9] As a result, the reallocation $d\mathsf{w}_1^I$ (respectively, the direction $\bar{B}_{11}\mu_1$) appears as the best or preferred reallocation (respectively, best or preferred direction) in the uncertain world considered.[10] Finally, note that $-\bar{B}_{11}\mu_1$ has already been interpreted as a vector of intertemporal substitution effects. Both interpretations, that is subjective certainty direction and intertemporal substitution effects, are thus equivalent.

The decomposition of the income reallocations space and the orthogonality condition $\left(d\mathsf{w}_1^I\right)'\bar{A}_{11}d\mathsf{w}_1^{II} = 0$ will now be used to obtain a decomposition of the fundamental disutility premium.

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9 The corresponding auxiliary problem can be written as :

$$\min \mathcal{L} = -\frac{1}{2}d\mathsf{w}_1'\bar{A}_{11}d\mathsf{w}_1 + \gamma\left[\mu_1'd\mathsf{w}_1 + \frac{d\mathsf{w}_0}{\beta}\right],$$

where γ is a Lagrange multiplier.

10 This interpretation is analogous to that of the least risky portfolio in the standard mean-variance framework without risk-free asset.

Proposition 8 : Decomposition of the fundamental disutility premium. Under the assumptions of Proposition 6, the fundamental disutility premium $\bar{\rho}$ decomposes into an impatience and a risk premium. More formally,

$$\bar{\rho} = \bar{\rho}^{I} + \bar{\rho}^{II} + o\left(\| d\mathsf{w} \|^2\right) \ ,$$

where $\bar{\rho}^{I} = -\frac{1}{2}\left(d\mathsf{w}_1^{I}\right)' \bar{A}_{11} d\mathsf{w}_1^{I}$ which depends on the sure reallocation $d\mathsf{w}_1^{I}$ is an impatience premium, and $\bar{\rho}^{II} = -\frac{1}{2}\left(d\mathsf{w}_1^{II}\right)' \bar{A}_{11} d\mathsf{w}_1^{II}$ which depends on the risky reallocation $d\mathsf{w}_1^{II}$ is a risk premium. ¥

c) Impatience and risk aversions

The fundamental disutility premium leads naturally to a measure of aversion which, for obvious reasons, will be called the "fundamental joint time-risk aversion".

The decomposition of the fundamental disutility premium leads to a similar decomposition of the "fundamental joint time-risk aversion" into an aversion to impatience and a risk aversion, and this is what we intend to show. Before doing so, it shall be helpful to reconsider the risk premium of Proposition 8 and its formulation in the Arrow-Pratt framework.

In our model, the Arrow-Pratt framework leads to :

i) $\bar{v}\left(\mathsf{w} - Q'\bar{y}\right) = \bar{v}\left(\mathsf{w}_0 - q_0'\bar{y}, \mathsf{w}_1 + Q_1'\bar{y}\right) = u\left(\mathsf{w}_0 - q_0'\bar{y}\right) + \delta \sum_s \pi_s u\left(\mathsf{w}_{1s} + q_{1s}'\bar{y}\right)$ which implies

$$\begin{aligned}
\lambda_0 &= \frac{\partial \bar{v}}{\partial \mathsf{w}_0}\left(\mathsf{w}_0 - q_0'\bar{y}\right) = \frac{\partial u}{\partial x_0}\left(\mathsf{w}_0 - q_0'\bar{y}\right) = u_0' \ , \text{ say,} \\
\frac{\partial \bar{\lambda}}{\partial \mathsf{w}_0} &= \frac{\partial^2 \bar{v}}{\partial \mathsf{w}_0^2}\left(\mathsf{w}_0 - q_0'\bar{y}\right) = \frac{\partial^2 u}{\partial x_0^2}\left(\mathsf{w}_0 - q_0'\bar{y}\right) = u_0'' \ , \text{ say,} \\
\lambda_{1s} &= \frac{\partial \bar{v}}{\partial \mathsf{w}_{1s}}\left(\mathsf{w}_{1s} + q_{1s}'\bar{y}\right) = \delta \pi_s \frac{\partial u}{\partial x_{1s}}\left(\mathsf{w}_{1s} + q_{1s}'\bar{y}\right), \quad s = 1, ..., S \ , \\
\frac{\partial \bar{\lambda}}{\partial \mathsf{w}'} &= \begin{bmatrix} \frac{\partial^2 u}{\partial x_0^2}\left(\mathsf{w}_0 - q_0'\bar{y}\right) & & 0 \\ & \ddots & \\ 0 & & \delta \pi_s \frac{\partial^2 u}{\partial x_{1s}^2}\left(\mathsf{w}_{1s} + q_{1s}'\bar{y}\right) \end{bmatrix} ;
\end{aligned}$$

ii) all derivatives of $\bar{v}$ with respect to future incomes are taken at a certainty point : $\mathsf{w}_1 + Q_1' \bar{y}$ proportional to $e = (1, ..., 1)'$, say $(a, ..., a)'$. This implies $\frac{\partial \bar{\lambda}_{1s}}{\partial \mathsf{w}_{1s}} = \frac{\partial^2 \bar{v}}{\partial \mathsf{w}_{1s}^2}(a) = \delta \pi_s \frac{\partial^2 u}{\partial x_{1s}^2}(a) = \delta \pi_s u''(a)$, $s = 1, ..., S$, so that one can write

$$\frac{\partial \bar{\lambda} / \partial \mathsf{w}'}{\lambda_0} = \begin{bmatrix} \frac{u_0^{00}}{u_0^0} & 0 \\ 0 & \frac{\delta u^{00}(a)}{u_0^0} \hat{\pi} \end{bmatrix},$$

where $\hat{\pi}$ is the diagonal matrix diag π.

Proposition 9 : In the Arrow-Pratt framework, the risk premium reduces to :

$$\begin{aligned} \bar{\rho}^{II} = \frac{1}{2} d\mathsf{w}_1' [I - T] \left[-\bar{A}_{11}\right] [I - T'] d\mathsf{w}_1 &= -\frac{1}{2} \delta \frac{u''(a)}{u_0'} d\mathsf{w}_1' [I - \pi e'] \hat{\pi} [I - e\pi'] d\mathsf{w}_1 \\ &= -\frac{1}{2} \delta \frac{u''(a)}{u'(a)} \frac{u'(a)}{u_0'} \sigma_{d\mathsf{w}_1}^2 , \end{aligned} \tag{4.16}$$

where $\sigma_{d\mathsf{w}_1}^2 = \sum_s \pi_s (d\mathsf{w}_{1s} - E_\pi d\mathsf{w}_1)^2$ is the variance of the income shock $d\mathsf{w}_1$ (see Appendix E). ¥

The right-hand side of (4.16) coincides up to the scalar $u'(a)/u_0'$ with the Arrow-Pratt risk premium. This is basically why the matrix $[I - T]\left[-\bar{A}_{11}\right][I - T']$ will appear as a generalized measure of risk aversion. The scalar $u'(a)/u_0'$ is also a discount factor.

The orthogonality condition $\left(d\mathsf{w}_1^I\right)' \bar{A}_{11} d\mathsf{w}_1^{II} = 0$ also yields the next proposition.

Proposition 10 : Decomposition of the fundamental joint time-risk aversion. Under the assumptions of Proposition 6, the fundamental joint time-risk aversion decomposes into an aversion to impatience and a risk aversion. More formally,

$$\left[-\bar{A}_{11}\right] = T \left[-\bar{A}_{11}\right] T' + [I - T] \left[-\bar{A}_{11}\right] [I - T'] ,$$

where $\left[-\bar{A}_{11}\right]$ can be interpreted as a joint time-risk aversion, $T \left[-\bar{A}_{11}\right] T'$ as an aversion to impatience and $[I - T] \left[-\bar{A}_{11}\right] [I - T']$ as a risk aversion. Moreover, each of these fundamental aversion measures is an invariant measure. ■

Besides their invariance, the main feature of the generalized aversion measures is their multidimensionality. While the invariance should be clear from what has been said up to now, the multidimensionality requires some explanation.

Let us consider the risk premium

$$\bar{\rho}^{II} = \frac{1}{2} d\mathsf{w}_1^{\prime} \left[I - T\right] \left[-\bar{A}_{11}\right] \left[I - T'\right] d\mathsf{w}_1 \ .$$

In this relation, $[I - T] \bar{A}_{11} [I - T'] d\mathsf{w}_1 = \beta d\bar{\mu}_1$, since $\bar{A}_{11} [I - T'] d\mathsf{w}_1 = \beta d\bar{\mu}_1$ and $\bar{A}_{11} [I - T'] = [I - T] \bar{A}_{11} [I - T']$ by the orthogonality condition. We deduce

$$\begin{aligned} \bar{\rho}^{II} &= -\frac{1}{2} d\mathsf{w}_1^{\prime} \beta d\bar{\mu}_1 = \frac{1}{2} d\mathsf{w}_1^{\prime} \left[I - T\right] \left[-\bar{A}_{11}\right] \left[I - T'\right] d\mathsf{w}_1 \ , \\ \bar{\rho}^{II} &= -\frac{1}{2} \beta \sum_s d\mathsf{w}_{1s} d\bar{\mu}_{1s} = \frac{1}{2} \sum_s \sum_\sigma \left(-\alpha_{s\sigma}^{II}\right) d\mathsf{w}_{1s} d\mathsf{w}_{1\sigma} \ , \end{aligned} \tag{4.17}$$

where $\left[\left(-\alpha_{s\sigma}^{II}\right)\right] = [I - T] \left[-\bar{A}_{11}\right] \left[I - T'\right]$, s, $\sigma = 1, ..., S$. We shall come back to the interpretation of the coefficients $\left(-\alpha_{s\sigma}^{II}\right)$ in a moment.

Let us set

$$\bar{\rho}_s^{II} = -\frac{1}{2} \beta d\mathsf{w}_{1s} d\bar{\mu}_{1s} \ , \ s = 1, ..., S \ , \tag{4.18}$$

where $\bar{\rho}_s^{II}$ is the risk premium specific to state s, meaning that it is associated with the specific shock $d\mathsf{w}_{1s}$, a change in the quantity of the s-elementary Arrow asset. In fact, while $\beta d\mathsf{w}_1^{\prime} d\bar{\mu}_1$ is the cost of insurance in the readjustment value of a reallocation of the Arrow portfolio [see subsection 4.2b)], $\beta d\mathsf{w}_{1s} d\bar{\mu}_{1s}$ can be seen as the cost of insurance in the readjustment value of a specific component (asset) of this portfolio. Moreover, if we write $-\beta d\bar{\mu}_{1s} = \frac{2\bar{\rho}_s^{II}}{d\mathsf{w}_{1s}}$, $-d\bar{\mu}_{1s}$ is the rate of this specific premium.

Let us now come to the interpretation of the coefficients $\alpha_{s\sigma}^{II}$. We first note from relation (4.17) that

$$-\beta d\bar{\mu}_{1s} = \sum_\sigma \left(-\alpha_{s\sigma}^{II}\right) d\mathsf{w}_{1\sigma} \ , \ s = 1, ..., S \ . \tag{4.19}$$

Now, let us suppose $d\mathsf{w}_{1\sigma} = 0$ for $\sigma \neq s$ and set $\bar{\rho}_{ss}^{II} = -\frac{1}{2} \alpha_{ss}^{II} \left(d\mathsf{w}_{1s}\right)^2$. This yields

$$\frac{2\bar{\rho}_{ss}^{II}}{\left(d\mathsf{w}_{1s}\right)^2} = -\alpha_{ss}^{II} \ ,$$

where $\left(-\alpha_{ss}^{II}\right)$ is a direct elementary coefficient of risk aversion, which, apart from being elementary, is very similar to what we get in the usual case (the general coefficient of absolute risk aversion is twice the risk premium per unit of variance). By (4.19) and (4.18) the specific risk premium can also be written

$$\bar{\rho}_s^{II} = \frac{1}{2} d\mathsf{w}_{1s} \sum_{\sigma} \left(-\alpha_{s\sigma}^{II}\right) d\mathsf{w}_{1\sigma} = \frac{1}{2} \sum_{\sigma} \bar{\rho}_{s\sigma}^{II} \ , \tag{4.20}$$

whose summation over s, $s = 1, ..., S$, gives the risk premium $\bar{\rho}^{II}$.

Whereas $-\alpha_{ss}^{II}$ is a direct elementary coefficient, $-\alpha_{s\sigma}^{II} \, (\sigma \neq s)$ is a cross elementary coefficient. These cross elementary aversions represent aversions to cross elementary risks that sum up with the direct elementary risk to give the specific risk. As can be seen from (4.20), the same rationale applies to elementary and specific premiums.

It is noteworthy that, in the Arrow-Pratt case, the general (aggregated) aversion measure is, in fact, the summation of specific aversion measures. Indeed, in this case, one has :

$$[I - T] \left[-\bar{A}_{11}\right] [I - T'] = [I - \pi e'] \left[-\delta\hat{\pi} \frac{u''(a)}{u_0'}\right] [I - e\pi'] \ ,$$

and the standard Arrow-Pratt coefficient is the trace of the matrix of specific risk aversions $\left[-\delta\hat{\pi} \frac{u''(a)}{u'(a)} \frac{u'(a)}{u_0'}\right]$.

Let us now turn to the impatience aversion and consider the impatience premium

$$\bar{\rho}^I = \frac{1}{2} d\mathsf{w}_1' T \left[-\bar{A}_{11}\right] T' d\mathsf{w}_1 \ .$$

In this relation, $T\bar{A}_{11}T'd\mathsf{w}_1 = d\bar{\beta}\mu_1$ since $\bar{A}_{11}T'd\mathsf{w}_1 = d\bar{\beta}\mu_1$ and $\bar{A}_{11}T' = T\bar{A}_{11}T$ by definition of the orthogonal projector T'. We deduce

$$\begin{aligned} \bar{\rho}^I &= -\frac{1}{2} d\mathsf{w}_1' d\bar{\beta}\mu_1 = \frac{1}{2} d\mathsf{w}_1' T \left[-\bar{A}_{11}\right] T' d\mathsf{w}_1 \ , \\ \bar{\rho}^I &= -\frac{1}{2} \sum_s d\mathsf{w}_{1s} d\bar{\beta}\mu_{1s} = \frac{1}{2} \sum_s \sum_\sigma \left(-\alpha_{s\sigma}^I\right) d\mathsf{w}_{1s} d\mathsf{w}_{1\sigma} \ , \end{aligned} \tag{4.21}$$

where $\left[\left(-\alpha_{s\sigma}^I\right)\right] = T\left[-\bar{A}_{11}\right]T'$, $s, \sigma = 1, ..., S$. The interpretation of these coefficients can be obtained in a very similar way to that used to interpret $\left(-\alpha_{s\sigma}^{II}\right)$. While $-\alpha_{ss}^I$ is a direct elementary coefficient of impatience aversion, $-\alpha_{s\sigma}^I \, (\sigma \neq s)$ is the corresponding cross elementary coefficient.

4.3. The residual risk aversion and the liquidity premium

The fundamental disutility premium $\bar{\rho}$ of Proposition 6 and the residual disutility premium ρ of Corollary 2 are tied together in this subsection. Their difference $\bar{\rho}-\rho$ will appear as a liquidity premium. Accordingly, the fundamental Antonelli matrix $-\bar{A}_{11}$ of Proposition 6 and the residual Antonelli matrix $-A_{11}$ of Corollary 2 (respectively, the fundamental and the residual joint time-risk aversion measures) are also linked together in this subsection. Their difference $\bar{A}_{11} - A_{11}$ will appear as a measure of aversion to illiquidity. The main tool to show these links is the Drèze-Modigliani decomposition and its invariant generalization. Therefore, we shall first recall the Drèze-Modigliani decomposition and give its reformulation in our setting. The latter will be used for the computation of the liquidity premium $\bar{\rho} - \rho$. A generalization of the Drèze-Modigliani decomposition together with the assumption that their exists an tradable risk-free asset will then be used to decompose the fundamental joint time-risk aversion into an aversion to impatience, a residual aversion and a financial aversion.

a) Liquidity premium and aversion to illiquidity

The results of subsections 4.2 and 4.3 are naturally linked together by the indirect utility function $\bar{v}$ (see Section 3, Remark 3). After optimizing $\bar{v}\left(\mathsf{w} - Q'y\right)$ with respect to y, we have :

$$v\left(\mathsf{w}\right) = \bar{v}\left(\mathsf{w} - Q'y\left(\mathsf{w}\right)\right) \ . \qquad (4.22)$$

By differentiating this relation with respect to w, we have :

$$\lambda = \bar{\lambda} - \frac{\partial y'}{\partial \mathsf{w}} Q\bar{\lambda} = \bar{\lambda} \quad \left(\text{since } Q\bar{\lambda} = 0\right) ,$$

where $\lambda\left(\mathsf{w}\right) = \frac{\partial v}{\partial \mathsf{w}}\left(\mathsf{w}\right)$ and $\bar{\lambda}\left(\mathsf{w}\right) = \frac{\partial \bar{v}}{\partial \mathsf{w}}\left(\mathsf{w} - Q'y\left(\mathsf{w}\right)\right)$ are the desirabilities of incomes when the income variation is analyzed with and without financial adjustments respectively. A second differentiation gives :

$$\frac{\partial \lambda}{\partial \mathsf{w}'} = \frac{\partial \bar{\lambda}}{\partial \mathsf{w}'}\left[I - Q'\frac{\partial y}{\partial \mathsf{w}'}\right] \ . \qquad (4.23)$$

Since $Q\frac{\partial \lambda}{\partial \mathsf{w}^{\theta}} = 0$, we get by premultiplying (4.23) by $\frac{\partial y^{\theta}}{\partial \mathsf{w}} Q$:

$$\frac{\partial y'}{\partial \mathsf{w}} Q \frac{\partial \bar{\lambda}}{\partial \mathsf{w}'} = \frac{\partial y'}{\partial \mathsf{w}}\left[Q\frac{\partial \bar{\lambda}}{\partial \mathsf{w}'}Q'\right]\frac{\partial y}{\partial \mathsf{w}'} = \frac{\partial \bar{\lambda}}{\partial \mathsf{w}'}Q'\frac{dy}{\partial \mathsf{w}'} \ , \qquad (4.24)$$

due to the symmetry of the second term. Substituting into the relation (4.23) and rearranging terms give the Drèze-Modigliani decomposition.

Lemma 5 : Drèze-Modigliani decomposition. The variations of the desirabilities of incomes when financial adjustments are taken into account $\frac{\partial\lambda/\partial \mathsf{w}^{\theta}}{\lambda_0}$ are the sum of i) the variations of the desirabilities of incomes when financial adjustments are not taken into account $\frac{\partial\bar{\lambda}/\partial \mathsf{w}^{\theta}}{\lambda_0}$ and ii) a corrective term $\frac{\partial y^{\theta}}{\partial \mathsf{w}}\left[Q\frac{\partial\bar{\lambda}/\partial \mathsf{w}^{\theta}}{\lambda_0}Q'\right]\frac{\partial y}{\partial \mathsf{w}^{\theta}}$:

$$-\frac{\partial\lambda/\partial \mathsf{w}'}{\lambda_0} = -\frac{\partial\bar{\lambda}/\partial \mathsf{w}'}{\lambda_0} + \frac{\partial y'}{\partial \mathsf{w}}\left[Q\frac{\partial\bar{\lambda}/\partial \mathsf{w}'}{\lambda_0}Q'\right]\frac{\partial y}{\partial \mathsf{w}'} ,$$

where the corrective term is invariant under a monotonic transformation of the utility function. ■

The decomposition of Lemma 5 is very similar to the formula derived by Drèze and Modigliani. The latter, however, was obtained in a rather different context. Their formula is not multidimensional and refers to a specific state of the world. It is used for comparing a situation where there exists a nominal sure asset (risk-free asset) with a situation where there exist N tradable assets. Finally, their corrective term measures "twice the expected value of perfect information per unit of variance (...) for infinitesimal risks" [see Drèze and Modigliani (1972), p. 314]. As mentioned above, Lemma 5 will now be used to link together the fundamental and the residual disutility premiums.

By premultiplying and postmultiplying the Drèze-Modigliani equation by $d\mathsf{w}'$ and $d\mathsf{w}$, and by using Corollary 2, Proposition 6 and the fact that $\lambda_0 = \bar{\lambda}_0$, we get the following proposition.

Proposition 11 : If the income shock is compensated $(\tilde{\mu}' d\mathsf{w} = 0)$,

i) the liquidity premium is :

$$\bar{\rho} - \rho = -\frac{1}{2}\left(\frac{d^2\bar{v}}{\lambda_0} - \frac{d^2 v}{\lambda_0}\right) = -\left(\frac{\Delta\bar{v}}{\lambda_0} - \frac{\Delta v}{\lambda_0}\right) = -\frac{1}{2}d\mathsf{w}_1' C_2' K_{22}^{-1} C_2 d\mathsf{w}_1 \geq 0 ,$$

where $C_2 = \frac{\partial y}{\partial \mathsf{w}_1^{\theta}} - \frac{\partial y}{\partial \mathsf{w}_0}\tilde{\mu}_1'$ is a matrix of compensated income effects and $K_{22} = \left[Q\frac{\partial\bar{\lambda}/\partial \mathsf{w}^{\theta}}{\lambda_0}Q'\right]^{-1} = \left[Q_1\bar{A}_{11}Q_1'\right]^{-1}$ is a symmetric negative definite matrix with rank N which characterizes substitution and complementarity among assets;

ii) the fundamental joint time-risk aversion $\bar{A}_{11}$ can be decomposed into a residual aversion A_{11} and a corrective term $C_2' K_{22}^{-1} C_2$ which measures the aversion to illiquidity :

$$-\bar{A}_{11} = -A_{11} - C_2' K_{22}^{-1} C_2 \gg -A_{11} \ ,$$

¥

where $\gg$ denotes the usual ordering on symmetric matrices. Following a compensated income shock, the disutility premium (which measures the welfare loss) will be greater when financial adjustments are not possible, and this welfare effect is a second order effect. In other words, the fundamental disutility premium $\bar{\rho}$ will, in general, be greater than the residual disutility premium ρ due to portfolio reallocations. In a similar way, the fundamental aversion $\left[-\bar{A}_{11}\right]$ will, in general, be greater than the residual aversion $[-A_{11}]$, since the individual is naturally more averse to a compensated income shock when his portfolio is illiquid than when financial adjustments are possible. Finally, note that if financial markets are complete, there is no residual effect. When facing an S-dimensional compensated income shock, the individual can adjust himself by using the S-tradable assets (there is no lack of substitution among assets). As a result, there is no residual disutility premium $\left(\rho = -\frac{1}{2} d\mathsf{w}_1' A_{11} d\mathsf{w}_1 = 0\right)$, the fundamental disutility premium is a liquidity premium, and the fundamental joint time-risk aversion is an aversion to illiquidity.

b) Residual risk aversion and aversion to illiquidity of risky assets

The fundamental joint time-risk aversion can be decomposed into an aversion to impatience and a risk aversion (see Proposition 10), and it can also be decomposed into a residual aversion and an aversion to illiquidity (see Proposition 11). We are now looking for a three-component decomposition, that is, an aversion to impatience, a residual aversion and an aversion to the illiquidity of risky assets. For this purpose, we shall "mix" both decompositions of Propositions 10 and 11 as follows. These propositions obviously lead to the relation

$$\left[-\bar{A}_{11}\right] = T\left[-\bar{A}_{11}\right]T' + [I-T]\left[-A_{11}\right][I-T'] - [I-T]\,C_2' K_{22}^{-1} C_2\,[I-T'] \ ,$$

which, in turn, gives the decomposition we are looking for, provided the following condition is satisfied :

$$[I-T]\left[-A_{11}\right][I-T'] = \left[-A_{11}\right] \ . \tag{4.25}$$

Thus, we get :

Proposition 12 : Under condition (4.25), the fundamental joint time-risk aversion $\left[-\bar{A}_{11}\right]$ can be decomposed as :

$$\left[-\bar{A}_{11}\right] = T\left[-\bar{A}_{11}\right]T' + \left[-A_{11}\right] + \left[I-T\right]C_2'\left[-K_{22}\right]^{-1}C_2\left[I-T'\right] \ ,$$

where each component of the decomposition is a symmetric positive semi-definite matrix, respectively with rank 1, $S-N$ and $N-1$. In the decomposition, $T\left[-\bar{A}_{11}\right]T'$ can be interpreted as a measure of aversion to impatience, $\left[-A_{11}\right]$ as a measure of residual (risk) aversion, and $\left[I-T\right]C_2'\left[-K_{22}\right]^{-1}C_2\left[I-T'\right]$ as a measure of aversion to the illiquidity of risky assets. ■

Once again, if financial markets are complete, there is no residual effect. Consequently, the fundamental risk aversion $\left[I-T\right]\left[-\bar{A}_{11}\right]\left[I-T\right]$ is just an aversion to the illiquidity of risky assets, which implies that the fundamental joint time-risk aversion decomposes into an aversion to impatience and an aversion to the illiquidity of risky assets.

We now turn to the condition (4.25) required to obtain the decomposition of Proposition 12. The meaning of this condition becomes more explicit in the next proposition.

Proposition 13 : The condition $\left[I-T\right]\left[-A_{11}\right]\left[I-T'\right] = \left[-A_{11}\right]$ is satisfied if and only if a sure asset (in real terms) can be built as a portfolio of existing assets, meaning that it belongs to the set of tradable assets. ■

Proof : One has : $\left[I-T\right]\left[-A_{11}\right]\left[I-T'\right] = \left[-A_{11}\right] \Leftrightarrow T\left[-A_{11}\right] = 0 \Leftrightarrow \left[-A_{11}\right]T' = 0 \Leftrightarrow \left[-A_{11}\right]T'd\mathsf{w}_1 = 0 \Leftrightarrow \left[-A_{11}\right]d\mathsf{w}_1^I = 0 \Leftrightarrow d\mathsf{w}_1^I = Q_1'dy$, since $A_{11}Q_1' = 0$ (see Lemma 1). Therefore, $d\mathsf{w}_1^I$ which is a sure reallocation belongs to the set of tradable assets (see Proposition 7). ■

Is this a natural condition? This assumption, in some sense, replaces the assumption made by Tobin (1958) on the existence of a zero-coupon bond (often seen as a risk-free asset). The role of this assumption is most easily understood by looking at the opposite situation. Suppose a sure asset or reallocation (also defined as a best or preferred reallocation in the uncertain world considered) is not tradable. In order to make intertemporal transfers, the individual would try to get as close as possible to the subjective certainty direction, but this could only be done by using risky assets.

5. Conclusion

Most of the literature on consumption behaviors and portfolio management relies on preferences that are representable by separable von Neumann-Morgenstern utility functions. Such is also the case of the standard measures of risk aversion and preference for the present. Due to their underlying assumptions, these measures are generally defined separately and represented by using scalars. The purpose of this paper was to provide a general framework allowing to emphasize the multidimensionality of the risk aversion measure and to show the necessity for risk and time aversions to be defined jointly. Our analysis naturally led to the discussion of other important concepts, such as the preference for liquidity and the certainty. It is important to stress again that all of these concepts depend on the considered individual. Moreover, the different decomposition formulas we derived and the corresponding aversion measures can be utilized in different ways.

In a standard framework for analysing individual behaviors, those formulas and measures should allow to match each individual with different parameters that measure risk aversion, different types of illiquidity and preference for the present. The individuals could thus be compared by using these quantities, which, in turn, should lead to segmentations of the population into homogeneous classes with respect to those criteria. The resulting segmentations could then be used to offer made-to-measure financial products for the various categories, and to set prices more or less high, depending on the importance of the underlying demands.

This classification of individuals and of their corresponding needs is also important for market analysis. The asset pricing equilibrium models should take account of these more detailled measures of individual heterogeneity. As a result, one should get a finer market segmentation with respect to the usual specifications, which, in turn, should imply different analysis of the multiplier effects of shocks, and of their transmissions to the several markets.

Appendix A

Proof of Proposition 1

The first-order conditions of the consumer optimization problem [(see (2.7)] define the following function

$$F(a,b) = \begin{bmatrix} \mathcal{U}_x - P\lambda \\ -Q\lambda \\ P'x - Q'y + \mathsf{w} \end{bmatrix},$$

where $a = (p, q, \mathsf{w})$, $b = (x, y, \lambda)$. An optimum is a point $(a^0,\ b^0)$ such that $F(a^0,\ b^0) = 0$. If the Jacobian matrix

$$F_b = \begin{bmatrix} \mathcal{U}_{xx'} & 0 & -P \\ 0 & 0 & -Q \\ -P' & -Q' & 0 \end{bmatrix}$$

has full rank, Proposition 1 follows by the implicit function theorem.

Lemma A : F_b has full rank at $(a^0,\ b^0)$.

Proof : Assume the contrary : there exists a vector $(\phi_1',\ \phi_2',\ \phi_3')' \neq 0$ such that a) $\phi_1'\ \mathcal{U}_{xx'} = \phi_3'\ P'$, b) $\phi_3'\ Q' = 0$, c) $\phi_1'\ P + \phi_2'\ Q = 0$. Then $\phi_1'\ P\ \phi_3 = 0$ and, by the first-order conditions $\phi_1'\ P\lambda = \phi_1'\ \mathcal{U}_x = 0$. So, one has simultaneously $\phi_1'\ \mathcal{U}_{xx'}\ \phi_1 = 0$ and $\phi_1'\ \mathcal{U}_x = 0$, which contradicts the strong quasi-concavity assumption unless $\phi_1 = 0$. But $\phi_1 = 0$ implies $\phi_3 = 0$ [since, in a), P' has rank $S+1$] and $\phi_2 = 0$ [since, in c), Q has rank N]. One has $(\phi_1',\ \phi_2',\ \phi_3') = 0$, hence a contradiction. ¥

Appendix B

Proof of Lemma 1

i) Decomposition of $d\tilde{\mu}_1$

Since $\tilde{\mu}_1 = \frac{\lambda_1}{\lambda_0}$, the income effects (on Arrow prices) can be written as :

$$\frac{\partial \tilde{\mu}_1}{\partial \mathsf{w}'} = [-\tilde{\mu}_1 \quad I_S] \begin{bmatrix} \frac{\partial \lambda_0 / \partial \mathsf{w}'}{\lambda_0} \\ \frac{\partial \lambda_1 / \partial \mathsf{w}'}{\lambda_0} \end{bmatrix} = [-\tilde{\mu}_1 \quad I_S] \frac{\partial \lambda / \partial \mathsf{w}'}{\lambda_0} . \tag{B.1}$$

Let us postmultiply (B.1) by $\begin{bmatrix} -\tilde{\mu}_1' \\ I_S \end{bmatrix}$ and rearrange terms. This yields

$$\frac{\partial \tilde{\mu}_1}{\partial \mathsf{w}_1'} = A_{11} + \frac{\partial \tilde{\mu}_1}{\partial \mathsf{w}_0} \tilde{\mu}_1' , \tag{B.2}$$

where $A_{11} = [-\tilde{\mu}_1 \quad I_S] \frac{\partial \lambda / \partial \mathsf{w}'}{\lambda_0} \begin{bmatrix} -\tilde{\mu}_1' \\ I_S \end{bmatrix}$ is an Antonelli matrix. Postmultiplying (B.2) by $d\mathsf{w}_1$ and adding $\frac{\partial \tilde{\mu}_1}{\partial \mathsf{w}_0} d\mathsf{w}_0$ to both sides of the equality, one obtains :

$$d\tilde{\mu}_1 = \frac{\partial \tilde{\mu}_1}{\partial \mathsf{w}'} d\mathsf{w} = A_{11} d\mathsf{w}_1 + \frac{\partial \tilde{\mu}_1}{\partial \mathsf{w}_0} \tilde{\mu}' d\mathsf{w} . \tag{B.3}$$

ii) Properties of the Antonelli matrix

a) A_{11} is symmetric

Consider $A_{11} = [-\tilde{\mu}_1 \quad I_S] \frac{\partial \lambda / \partial \mathsf{w}'}{\lambda_0} \begin{bmatrix} -\tilde{\mu}_1' \\ I_S \end{bmatrix}$. The symmetry of A_{11} follows from the symmetry of the matrix $\partial \lambda / \partial \mathsf{w}'$. Indeed, since $\lambda = \partial v / \partial \mathsf{w}$ is continuously differentiable (see Proposition 1), one has $\partial \lambda / \partial \mathsf{w}' = \partial^2 v / \partial \mathsf{w} \partial \mathsf{w}'$ where the Hessian matrix is symmetric.

b) A_{11} is negative semi-definite :

Lemma B.1 : 1) $A_{11} = C_1' \frac{\mathcal{U}_{xx'}}{\lambda_0} C_1$, where $C_1 = \begin{bmatrix} \frac{\partial x}{\partial \mathsf{w}_0} & \frac{\partial x}{\partial \mathsf{w}_1'} \end{bmatrix} \begin{bmatrix} -\tilde{\mu}_1' \\ I_S \end{bmatrix}$;

2) $\mathcal{U}_x' C_1 = 0$.

Proof : 1) Proposition 1 provides solutions of the consumer optimization problem $x = x(p, q, \mathsf{w})$, $y = y(p, q, \mathsf{w})$ and $\lambda = \lambda(p, q, \mathsf{w})$, which are continuously differentiable functions. By substituting these functions in the first-order conditions of the consumer optimization problem, and by using the implicit function theorem, we get the following identities

$$\left.\begin{array}{rcl} \mathcal{U}_x(p, q, \mathsf{w}) & = & P\lambda(p, q, \mathsf{w}) \\ -Q\lambda(p, q, \mathsf{w}) & = & 0 \\ P'_x(p, q, \mathsf{w}) + Q'y(p, q, \mathsf{w}) & = & \mathsf{w} \end{array}\right\} . \tag{B.4}$$

By differentiating (B.4) with respect to w, we obtain

$$\left.\begin{array}{rcl} \mathcal{U}_{xx'}\frac{\partial x}{\partial \mathsf{w}'} & = & P\frac{\partial \lambda}{\partial \mathsf{w}'} \\ Q\frac{\partial \lambda}{\partial \mathsf{w}'} & = & 0 \\ P'\frac{\partial x}{\partial \mathsf{w}'} + Q'\frac{\partial y}{\partial \mathsf{w}'} & = & I_{S+1} \end{array}\right\} . \tag{B.5}$$

Premultiplying the first relation of (B.5) by $\frac{\partial x'}{\partial \mathsf{w}}$ leads to

$$\frac{\partial x'}{\partial \mathsf{w}}\mathcal{U}_{xx'}\frac{\partial x}{\partial \mathsf{w}'} = \frac{\partial \lambda}{\partial \mathsf{w}'} , \tag{B.6}$$

since $(\partial x'/\partial \mathsf{w})\, P\,(\partial\lambda/\partial \mathsf{w}') = [I_{S+1} - (\partial y'/\partial \mathsf{w})]\, \partial\lambda/\partial \mathsf{w}' = \partial\lambda/\partial \mathsf{w}'$. Then by premultiplying and postmultiplying (B.6) respectively by $[-\tilde{\mu}_1 \; I_S]$ and $\begin{bmatrix} -\tilde{\mu}_1 \\ I_S \end{bmatrix}$, and by dividing the same relation by λ_0, one has :

$$C_1'\frac{\mathcal{U}_{xx'}}{\lambda_0}C_1 = A_{11} . \tag{B.7}$$

2) By premultiplying the third relation of (B.5) by λ', one has : $\lambda' P'(\partial x/\partial \mathsf{w}') = \lambda'$, or equivalently, $\mathcal{U}'_x \partial x/\partial \mathsf{w}' = \lambda'$. We then postmultiply the latter relation by $\begin{bmatrix} -\tilde{\mu}_1' \\ I_S \end{bmatrix}$ and get

$$\mathcal{U}'_x C_1 = 0 , \tag{B.8}$$

since $\lambda' \begin{bmatrix} -\tilde{\mu}_1' \\ I_S \end{bmatrix} = 0$. ¥

The negative semi-definiteness of A_{11} follows from Lemma B.1 and the strong quasi-concavity of the utility function (see Assumption A.1). By premultiplying and postmultiplying (B.7) respectively by $d\mathsf{w}'$ and $d\mathsf{w}_1$, and by postmultyplying (B.8) by $d\mathsf{w}_1$, one has : $d\mathsf{w}_1' C_1' \frac{\mathcal{U}_{xx^0}}{\lambda_0} C_1 d\mathsf{w}_1 = d\mathsf{w}_1' A_{11} d\mathsf{w}_1$ and, simultaneously, $\mathcal{U}_x' C_1 d\mathsf{w}_1 = 0$. Thus, by the strong quasi-concavity of the utility function, $d\mathsf{w}_1' A_{11} d\mathsf{w}_1 < 0$ whenever $C_1 d\mathsf{w}_1 \neq 0$.

c) ker A_{11} = range Q_1'

Lemma B.2 : ker C_1 = range Q_1'.

Proof : Postmultiplying the first relation of (B.5) by Q' yields $\mathcal{U}_{xx^0}\,(\partial x/\partial \mathsf{w}')\ Q' = 0$ which can also be written as

$$\mathcal{U}_{xx^0} C_1 Q_1' = 0 \ , \tag{B.9}$$

since $\begin{bmatrix} -\tilde{\mu}_1' \\ I_S \end{bmatrix} Q_1' = -Q'$. Let us set $\zeta = C_1 Q_1' \phi$, where ϕ is an N-dimensional arbitrary vector. Relation (B.9) implies $\zeta' \mathcal{U}_{xx^0} \zeta = 0$ for any vector ϕ while relation (B.8) implies $\mathcal{U}_x' \zeta = 0$ for any vector ϕ. But this is impossible (again from the strong quasi-concavity assumption) unless $\zeta = 0$ for any ϕ, and this implies that $C_1 Q_1' = 0$, or equivalently, $Q_1 C_1' = 0$. It remains to show that the kernel of C_1 is spanned by the range of Q_1'. Suppose the converse. There exists a $S \times \tilde{N}$-dimensional matrix $\tilde{Q}_1'$ with rank $\tilde{Q}_1' = \tilde{N} > N$ such that $C_1 \tilde{Q}_1' = 0$. Postmultiplying the third relation of (B.5) by $\begin{bmatrix} -\tilde{\mu}_1' \\ I_S \end{bmatrix} \tilde{Q}_1'$, one has

$$Q' C_2 \tilde{Q}_1' = \begin{bmatrix} -\tilde{\mu}_1' \\ I_S \end{bmatrix} \tilde{Q}_1' \ , \tag{B.10}$$

where $C_2 = \begin{bmatrix} \frac{\partial y}{\partial \mathsf{w}_0} & \frac{\partial y}{\partial \mathsf{w}_1^0} \end{bmatrix} \begin{bmatrix} -\tilde{\mu}_1' \\ I_S \end{bmatrix}$. The relation (B.10) can also be written as

$$-\begin{bmatrix} -\tilde{\mu}_1' \\ I_S \end{bmatrix} Q_1' C_2 \tilde{Q}_1' = \begin{bmatrix} -\tilde{\mu}_1' \\ I_S \end{bmatrix} \tilde{Q}_1' \ ,$$

but involves a contradiction since its left-hand side has rank smaller or equal to N while the right-hand side has rank strictly greater than N. Therefore, C_1 has rank $S - N$ and ker C_1 = range Q_1'. ¥

Consequently, ker A_{11} = range Q_1' (and rank $A_{11} = S - N$) by using the first property of Lemma B.1 and Lemma B.2. ¥

Appendix C

Proof of Lemma 2

By (B.1) and the symmetry of $\partial\lambda/\partial \mathsf{w}'$, one has :

$$\begin{aligned}
\frac{\partial\tilde{\mu}_1}{\partial \mathsf{w}_0} &= [-\tilde{\mu}_1 \quad I_S]\,\frac{\partial\lambda/\partial \mathsf{w}_0}{\lambda_0} = [-\tilde{\mu}_1 \quad I_S]\,\frac{\partial\lambda_0/\partial \mathsf{w}'}{\lambda_0}\ , \text{ and} \\
d\mathsf{w}_1'\frac{\partial\tilde{\mu}_1}{\partial \mathsf{w}_0} &= \frac{\partial\lambda_0/\partial \mathsf{w}_1'}{\lambda_0}d\mathsf{w}_1 - \frac{\partial\lambda_0/\partial \mathsf{w}_0}{\lambda_0}\tilde{\mu}_1' d\mathsf{w}_1\ , \text{ where} \\
\tilde{\mu}_1' d\mathsf{w}_1 &= \tilde{\mu}' d\mathsf{w} - d\mathsf{w}_0. \text{ This gives :} \\
\frac{d\lambda_0}{\lambda_0} &= d\mathsf{w}_1^{\prime}\frac{\partial\tilde{\mu}_1}{\partial \mathsf{w}_0} + \frac{\partial\lambda_0/\partial \mathsf{w}_0}{\lambda_0}\tilde{\mu}' d\mathsf{w}\ . \qquad \text{(C.1)}
\end{aligned}$$

The result follows by substituting (C.1) and (B.3) into (4.2), and rearranging terms.

Appendix D

Implications of Lemmas 3 and 4

i) If $d\mathsf{w}_1 = d\mathsf{w}_1^I$, then $d\mathsf{w}_1' d\bar{\mu}_1 = (T' d\mathsf{w}_1)' \, d\bar{\mu}_1 = d\mathsf{w}_1' T d\bar{\mu}_1 = 0$ by definition of T. Thus, the insurance price $(d\mathsf{w}_1' d\bar{\mu}_1)$ associated with an intertemporal portfolio reallocation $\left(d\mathsf{w}_1^I\right)$ is equal to zero.

ii) Since $d\bar{\tilde{\mu}}_1 = \bar{A}_{11} d\mathsf{w}_1$, if $d\mathsf{w}_1 = d\mathsf{w}_1^I$, we deduce $d\bar{\tilde{\mu}}_1 = \bar{A}_{11} d\mathsf{w}_1^I = \bar{A}_{11} \left(-\bar{B}_{11} d\bar{\beta}\mu_1\right) = \bar{A}_{11} \bar{A}_{11}^{-1} d\bar{\beta}\mu_1 = d\bar{\beta}\mu_1$. Thus, if $d\mathsf{w}_1 = d\mathsf{w}_1^I$, the implied modification of Arrow prices reduces to a readjustment of the discount factor $\left(d\bar{\tilde{\mu}}_1 = d\bar{\beta}\mu_1\right)$.

iii) Since $d\bar{\tilde{\mu}}_1 = d\bar{\beta}\mu_1 + \beta d\bar{\mu}_1$ and $d\bar{\tilde{\mu}}_1 = d\bar{\beta}\mu_1$, when $d\mathsf{w}_1 = d\mathsf{w}_1^I$, we deduce $\beta d\bar{\mu}_1 = d\bar{\mu}_1 = 0$. Thus, the intertemporal portfolio reallocation implies no change in the forward Arrow prices $(d\bar{\mu}_1 = 0)$.

Appendix E

The Arrow-Pratt framework without financial adjustment

i) The consumer optimization problem

For simplicity, let us assume that the individual consumes only one commodity in each period $\times$ state. His preferences are represented by a VNM utility function

$$\mathcal{U}(x_0, x_{11}, ..., x_{1S}) = u(x_0) + \delta\Sigma_s\pi_s u(x_{1s}) \ .$$

The consumer's problem can be solved by means of the Lagrangean :

$$\mathcal{L}(x;\lambda) = u(x_0) + \delta\Sigma_s\pi_s u(x_{1s}) - \lambda_0(p_0x_0 + q_0'\bar{y} - \mathsf{w}_0) - \Sigma_s\lambda_{1s}(p_{1s}x_{1s} - q_{1s}'\bar{y} - \mathsf{w}_{1s}) \ ,$$

where $\bar{y}$ is the illiquid portfolio of the consumer. The first-order conditions are :

$$\begin{aligned} \frac{\partial u}{\partial x_0}(x_0) - \lambda_0 p_0 &= 0, \quad \frac{\partial u}{\partial x_{1s}}(x_{1s}) - \lambda_{1s}p_{1s} = 0, \quad s = 1, ..., S \ , \\ p_0x_0 + q_0'\bar{y} &= \mathsf{w}_0, \quad p_{1s}x_{1s} - q_{1s}'\bar{y} = \mathsf{w}_{1s}, \quad s = 1, ..., S \ , \end{aligned}$$

which yield the following solution :

$$\begin{aligned} x_0 &= \frac{\mathsf{w}_0 - q_0'\bar{y}}{p_0}, \quad x_{1s} = \frac{\mathsf{w}_{1s} + q_{1s}'\bar{y}}{p_{1s}}, \quad s = 1, ..., S \ , \\ \lambda_0 &= \frac{1}{p_0}\frac{\partial u}{\partial x_0}\left(\frac{\mathsf{w}_0 - q_0'\bar{y}}{p_0}\right), \quad \lambda_{1s} = \frac{\pi_s\delta}{p_{1s}}\frac{\partial u}{\partial x_{1s}}\left(\frac{\mathsf{w}_{1s} + q_{1s}'\bar{y}}{p_{1s}}\right), \quad s = 1, ..., S \ . \end{aligned}$$

The indirect utility function is given by

$$\bar{v}(p, \mathsf{w} - Q'\bar{y}) = u\left[(\mathsf{w}_0 - q_0'\bar{y})/p_0\right] + \delta\Sigma_s\pi_s u\left[(\mathsf{w}_{1s} + q_{1s}'\bar{y})/p_{1s}\right] \ .$$

It is noteworthy that if the direct utility function is VNM, the indirect utility function is, in general, period $\times$ state separable, but not necessarily VNM.[11] To get a VNM indirect utility function, one should also assume $p_0 = p_{11} = ... = p_{1S}$, which is implicitly done in the standard Arrow-Pratt framework. For convenience,

[11] One could write : $\bar{v}(p, \mathsf{w} - Q^{\emptyset}\bar{y}) = \bar{v}_0(\cdot) + \delta\Sigma_s\pi_s\bar{v}_s(\cdot)$, with $\bar{v}_0 \neq \bar{v}_1 \neq ... \neq \bar{v}_S$.

we shall here assume $p_0 = p_{11} = ... = p_{1S} = 1$, and write the indirect utility function as follows :

$$\bar{v}\left(\mathsf{w} - Q'\bar{y}\right) = u\left(\mathsf{w}_0 - q_0'\bar{y}\right) + \delta \sum_s \pi_s u\left(\mathsf{w}_{1s} + q_{1s}'\bar{y}\right) \ . \tag{E.1}$$

ii) The certainty direction

Let us first consider the derivatives of the indirect utility function (E.1) and introduce some notation :

$$\begin{aligned}
\lambda_0 &= \frac{\partial \bar{v}}{\partial \mathsf{w}_0}\left(\mathsf{w}_0 - q_0'\bar{y}\right) = \frac{\partial u}{\partial x_0}\left(\mathsf{w}_0 - q_0'\bar{y}\right) = u_0' \ , \qquad \text{(E.2)}\\
\frac{\partial \bar{\lambda}_0}{\partial \mathsf{w}_0} &= \frac{\partial^2 \bar{v}}{\partial \mathsf{w}_0^2}\left(\mathsf{w}_0 - q_0'\bar{y}\right) = \frac{\partial^2 u}{\partial x_0^2}\left(\mathsf{w}_0 - q_0'\bar{y}\right) = u_0'' \ ,\\
\lambda_{1s} &= \frac{\partial \bar{v}}{\partial \mathsf{w}_{1s}}\left(\mathsf{w}_{1s} + q_{1s}'\bar{y}\right) = \delta\pi_s \frac{\partial u}{\partial x_{1s}}\left(\mathsf{w}_{1s} + q_{1s}'\bar{y}\right), \quad s = 1, ..., S \ ,\\
\frac{\partial \bar{\lambda}_{1s}}{\partial \mathsf{w}_{1s}} &= \frac{\partial^2 \bar{v}}{\partial \mathsf{w}_{1s}^2}\left(\mathsf{w}_{1s} + q_{1s}'\bar{y}\right) = \delta\pi_s \frac{\partial^2 u}{\partial x_{1s}^2}\left(\mathsf{w}_{1s} + q_{1s}'\bar{y}\right), \quad s = 1, ..., S \ .
\end{aligned}$$

We deduce

$$\partial\bar{\lambda}/\partial\mathsf{w}' = \begin{bmatrix} \partial\bar{\lambda}_0/\partial\mathsf{w}_0 & \cdots & \partial\bar{\lambda}_0/\partial\mathsf{w}_{1S} \\ \vdots & \ddots & \vdots \\ \partial\bar{\lambda}_{1S}/\partial\mathsf{w}_0 & \cdots & \partial\bar{\lambda}_{1S}/\partial\mathsf{w}_{1S} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 u}{\partial x_0^2}\left(\mathsf{w}_0 - q_0'\bar{y}\right) & & 0 \\ & \ddots & \\ 0 & & \delta\pi_S \frac{\partial^2 u}{\partial x_{1S}^2}\left(\mathsf{w}_{1S} + q_{1S}'\bar{y}\right) \end{bmatrix},$$

and $\tilde{\mu}' = (1, \tilde{\mu}_1') = \left(1, \frac{\lambda_{11}}{\lambda_0}, ..., \frac{\lambda_{1S}}{\lambda_0}\right)' = \left[1, \frac{\delta\pi_1 \frac{\partial u}{\partial x_{11}}\left(\mathsf{w}_{11} + q_{11}^{0}\bar{y}\right)}{\frac{\partial u}{\partial x_0}\left(\mathsf{w}_0 - q_0^{0}\bar{y}\right)}, ..., \frac{\delta\pi_S \frac{\partial u}{\partial x_{1S}}\left(\mathsf{w}_{1S} + q_{1S}^{0}\bar{y}\right)}{\frac{\partial u}{\partial x_0}\left(\mathsf{w}_0 - q_0^{0}\bar{y}\right)}\right].$

In the Arrow-Pratt framework, all derivatives with respect to future incomes are taken at a certainty point : $\mathsf{w}_1 + q_1'\bar{y}$ proportional to $e = (1, ..., 1)'$, say $(a, ..., a)'$. The derivatives can thus be written : $\lambda_{1s} = \frac{\partial \bar{v}}{\partial \mathsf{w}_{1s}}(a) = \delta\pi_s \frac{\partial u}{\partial x_{1s}}(a) = \delta\pi_s u'(a)$, $s = 1, ..., S$ and $\lambda_1' = \delta u'(a)\pi'$; $\frac{\partial\bar{\lambda}_{1s}}{\partial \mathsf{w}_{1s}} = \frac{\partial^2 \bar{v}}{\partial \mathsf{w}_{1s}^2}(a) = \delta\pi_s \frac{\partial^2 u}{\partial x_{1s}^2}(a) = \delta\pi_s u''(a)$, $s = 1, ..., S$ and $\frac{\partial\bar{\lambda}_1}{\partial \mathsf{w}_1^{0}} = \delta u''(a)\hat{\pi}$, where $\hat{\pi}$ is the diagonal matrix diag π.

In this case, the matrix $\partial\bar{\lambda}/\partial\mathsf{w}'$ and the vector $\tilde{\mu}$ are respectively written :

$$\partial\bar{\lambda}/\partial\mathsf{w}' = \begin{bmatrix} \partial\bar{\lambda}_0/\partial\mathsf{w}_0 & 0 \\ 0 & \partial\bar{\lambda}_1/\partial\mathsf{w}_1' \end{bmatrix} = \begin{bmatrix} u_0'' & 0 \\ 0 & \delta u''(a)\hat{\pi} \end{bmatrix}, \tag{E.3}$$

$$\text{and } \tilde{\mu} = [1, \tilde{\mu}_1']' = \left[1, \frac{\delta\pi_1 u'(a)}{u_0'}, ..., \frac{\delta\pi_S u'(a)}{u_0'}\right]' = \left[1, \frac{\delta u'(a)}{u_0'}\pi'\right]' . \tag{E.4}$$

The certainty direction is defined as $\bar{B}_{11}\mu_1$ where $\bar{B}_{11} = -\bar{A}_{11}^{-1}$. Let us derive the expression of the Antonelli matrix $\bar{A}_{11}$ and then find its inverse. By using the notation introduced in (E.2) and (E.3), one has

$$\begin{aligned}
\bar{A}_{11} &= [-\tilde{\mu}_1 \quad I_S] \frac{\partial\bar{\lambda}/\partial \mathsf{w}'}{\lambda_0} \begin{bmatrix} -\tilde{\mu}_1' \\ I_S \end{bmatrix} \\
&= \frac{u_0''}{u_0'}\tilde{\mu}_1\tilde{\mu}_1' + \frac{\delta u''(a)}{u_0'}\hat{\pi} && \text{(E.5)} \\
&= \frac{\delta u''(a)}{u_0'}\hat{\pi}\left[I_S + b\tilde{\mu}_1'\right] && \text{(E.6)}
\end{aligned}$$

where $b = [\delta u''(a)\hat{\pi}]^{-1} u_0''\tilde{\mu}_1$. From (E.6), it is easily checked that the inverse of $-\bar{A}_{11}$ satisfies

$$\begin{aligned}
-\bar{A}_{11}^{-1} &= \bar{B}_{11} = [I_S + b\tilde{\mu}_1']^{-1}\left[\frac{-\delta u''(a)}{u_0'}\hat{\pi}\right]^{-1} \\
&= \left[\frac{-\delta u''(a)}{u_0'}\hat{\pi}\right]^{-1}\left[I_S - \frac{\tilde{\mu}_1 b'}{1 + b'\tilde{\mu}_1}\right]
\end{aligned}$$

(recall that $\bar{A}_{11}$ and $-\bar{A}_{11}^{-1}$ are symmetric matrices), which implies

$$\bar{B}_{11}\mu_1 = \left[\frac{-\delta u''(a)}{u_0'}\hat{\pi}\right]^{-1}\mu_1\frac{1}{1 + b'\tilde{\mu}_1} . \tag{E.7}$$

We then use the normalization restriction

$$\mu_1'\bar{B}_{11}\mu_1 = 1 \Leftrightarrow \mu_1'\left[\frac{-\delta u''(a)}{u_0'}\hat{\pi}\right]^{-1}\mu_1\frac{1}{1 + b'\tilde{\mu}_1} = 1$$

to deduce $(1 + b'\tilde{\mu}_1) = \mu_1'\left[\frac{-\delta u^{00}(a)}{u_0^0}\hat{\pi}\right]^{-1}\mu_1$. Substituting the latter relation in (E.7) and using the notation introduced in (E.4) yields the certainty direction

$$\bar{B}_{11}\mu_1 = \frac{[(-\delta u''(a)/u_0')\hat{\pi}]^{-1}\mu_1}{\mu_1'[(-\delta u''(a)/u_0')\hat{\pi}]^{-1}\mu_1} = \frac{\hat{\pi}^{-1}\mu_1}{\mu_1'\hat{\pi}^{-1}\mu_1} = \frac{e}{e'\mu_1} .$$

Moreover, the projector $T' = \bar{B}_{11}\mu_1\mu_1'$ can be written as

$$\bar{B}_{11}\mu_1\mu_1' = \frac{e\mu_1'}{e'\mu_1} = e\pi' \ . \tag{E.8}$$

iii) Proof of Proposition 9

The formula of the fundamental risk premium $\bar{\rho}^{II}$ is easily derived from the expressions of the matrix $\bar{A}_{11}$ and of the projector T [given respectively by (E.5) and (E.8)] :

$$\begin{aligned}
\bar{\rho}^{II} &= \frac{1}{2} d\mathsf{w}_1' \left[I - T\right] \left[-\bar{A}_{11}\right] \left[I - T'\right] d\mathsf{w}_1 \\
&= \frac{1}{2} d\mathsf{w}_1' \left[I - T\right] \left[-\left(u_0''/u_0'\right) \tilde{\mu}_1 \tilde{\mu}_1' - \left(\delta u''\left(a\right)/u_0'\right) \hat{\pi}\right] \left[I - T'\right] d\mathsf{w}_1 \\
&= \frac{1}{2} d\mathsf{w}_1' \left[I - T\right] \left[-\left(\delta u''\left(a\right)/u_0'\right) \hat{\pi}\right] \left[I - T'\right] d\mathsf{w}_1 \quad \text{(since } T\tilde{\mu}_1 = \tilde{\mu}_1\text{)} \\
&= \frac{1}{2} \left(-\delta u''\left(a\right)/u_0'\right) \sigma^2_{d\mathsf{w}_1} \\
&= \frac{1}{2} \left(-\delta u''\left(a\right)/u'\left(a\right)\right) \left(u'\left(a\right)/u_0'\right) \sigma^2_{d\mathsf{w}_1} \ .
\end{aligned}$$

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