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RÉSUMÉ

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Mots clés : dominance parétienne

ABSTRACT

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Key words : Pareto dominance

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Abstract

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Résumé

Nous caractérisons les ordres partiels parétiens pour le cas de deux agents et d'un continuum de choix. *Mots-clés:* Dominance parétienne.

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1. Introduction

Consider a fixed set of agents having preferences over some set of alternatives. The Pareto relation generated by their preference profile deems an alternative a weakly better than an alternative b if and only if all agents weakly prefer a over b. If individual preferences are complete and transitive, this relation is a quasi-order, i.e., a reflexive and transitive relation. What kind of quasi-order is it?

This question has not been directly addressed in the literature. It is well known that every quasi-order is the Pareto relation generated by some preference profile *if* no restriction is imposed on the number of agents. The seminal article on this subject is Szpilrajn (1930); for two recent contributions (and further references), see Donaldson and Weymark (1998) and Duggan (1998). Aizerman and Aleskerov (1995) offer a theorem on the minimal number of agents needed —a problem formulated by Dushnik and Miller (1941)— and discuss the various upper bounds established in the literature. By contrast, we are interested in identifying necessary and sufficient conditions for representing a quasi-order as the Pareto relation of a preference profile of a *given* number of agents.

Our motivation is the following. In many economic contexts, the number of agents is known. Their preferences, however, are not. This poses the problem of identifying the restrictions imposed by various collective choice theories on the agents' observed joint behavior in the absence of information on their preferences. In particular, it is of interest to identify the restrictions imposed by Pareto efficiency: this calls for a better understanding of Pareto relations.

To gain some intuition about the problem under study, suppose there are only two agents and six alternatives: a, b, c, x, y, z. Consider the binary relation R described in Figure 1, where the symbol R is written at the intersection of row m and column n if and only if mRn. It is a quasi-order. Yet we cannot find two complete and transitive preferences whose Pareto relation would coincide with R. For suppose $\succcurlyeq_1, \succcurlyeq_2$ were such preferences. Since a, b, c are maximal elements of R, $\succcurlyeq_1$ and $\succcurlyeq_2$ must be completely opposite over those three alternatives. To fix ideas, suppose

$$a \succcurlyeq_1 b \succcurlyeq_1 c \text{ and, therefore, } c \succcurlyeq_2 b \succcurlyeq_2 a. \tag{1.1}$$

Since aRy and cRy, we must have $a \succcurlyeq_2 y$ and $c \succcurlyeq_1 y$. It follows that $b \succcurlyeq_1 y$ and $b \succcurlyeq_2 y$, hence bRy, a contradiction to the definition of R. A similar contradiction arises if we replace (1.1) with any other profile of opposite preferences over the three alternatives a, b, c.

Thus, two-agent "Paretian quasi-orders" do possess specific properties. The property suggested by this example is that, for any three noncomparable alternatives, one is "between" the other two, in the sense that its lower contour set includes the intersection of the other two lower contour sets. Building on this observation, we will present a complete characterization of Paretian quasi-orders for the case of two agents and a continuum of alternatives. This, obviously, is only a first —hopefully useful— step.

2. A formal statement of the problem

The set of alternatives is a nonempty, compact, connected set $X \subseteq \mathbb{R}^m$. It is fixed throughout the paper. There are two agents, $i = 1, 2$, whose preferences are preorders — i.e., complete and transitive binary relations — on X, denoted $\succcurlyeq_1$ and $\succcurlyeq_2$. We sometimes write $y \preccurlyeq_i x$ instead of $x \succcurlyeq_i y$: both notations mean that agent i weakly prefers x to y, and $\sim_i$ denotes indifference. We call $(\succcurlyeq_1, \succcurlyeq_2)$ a *preference profile.* The *Pareto relation* generated by $(\succcurlyeq_1, \succcurlyeq_2)$ on X, or simply the Pareto relation of $(\succcurlyeq_1, \succcurlyeq_2)$, is the binary relation $R^*(\succcurlyeq_1, \succcurlyeq_2)$ on X defined by

$$xR^*(\succcurlyeq_1, \succcurlyeq_2)y \Leftrightarrow x \succcurlyeq_1 y \text{ and } x \succcurlyeq_2 y. \tag{2.1}$$

A maximal element of that binary relation in X is a *Pareto-efficient* alternative. The Pareto relation $R^*(\succcurlyeq_1, \succcurlyeq_2)$ is clearly a quasi-order, i.e., a reflexive and transitive binary relation. As mentioned in the Introduction, our purpose is to identify the characteristic properties of such quasi-orders. We will do so under some regularity restrictions.

Definition 1. A preference profile $(\succcurlyeq_1, \succcurlyeq_2)$ is *regular* if it satisfies the following three conditions:

1) both agents' preferences are continuous (in the usual sense that all weak upper and lower contour sets are closed);

2) for every alternative x there exist Pareto-efficient alternatives x', x'' such that $x' \sim_1 x \sim_2 x''$ and, conversely, for any Pareto-efficient alternatives x', x'' there exists some alternative x such that $x' \sim_1 x \sim_2 x''$;

3) the correspondence associating with each alternative x the set of alternatives weakly preferred to x by both agents is lower-hemicontinuous.

The question we answer in this paper is the following: if R is a quasi-order on X, under what conditions does there exist a regular preference profile $(\succcurlyeq_1, \succcurlyeq_2)$

whose Pareto relation $R^*(\succcurlyeq_1, \succcurlyeq_2)$ coincides with R? If such a profile exists, we call R a *regular Paretian quasi-order.*

The following straightforward observation will be helpful. The preference pre-orders $\succcurlyeq_1, \succcurlyeq_2$ meet conditions 1) and 2) of Definition 1 if and only if they admit continuous numerical representations u_1, u_2 generating a *complete* utility set: that is, letting $u = (u_1, u_2)$, writing vector inequalities $\geq, >, \gg$, and defining the upper-frontier of the utility set $u(X)$ by $\partial u(X) = \{x \in X : \text{there is no } y \in X \text{ such that } u(y) > u(x)\}$,

$$w \in u(X) \Leftrightarrow \exists w', w'' \in \partial u(X) : w = w' \wedge w'', \tag{2.2}$$

where $\wedge$ denotes the coordinate-by-coordinate minimum operator.

3. Basic conditions

Let R be a quasi-order on X. Denote by P, I, and N, the asymmetric, symmetric, and noncomparable factors of R, respectively. Denote by M the union of I and N: xMy thus means that x is indifferent or noncomparable to y. If $y \in X$, let $W_R(y) = \{x \in X : yRx\}$ and $B_R(y) = \{x \in X : xRy\}$. These are, respectively, the sets of alternatives that are weakly worse and weakly better than y. To simplify notations, we often write $W(y)$ and $B(y)$ instead of $W_R(y)$ and $B_R(y)$, especially in the proofs. Here are three basic properties that every regular Paretian quasi-order R possesses.

Property 1: Complete Intermediateness. Let $x, y, z \in X$ be pairwise indifferent or noncomparable under R, i.e., xMy, xMz, yMz. Then at least one of the following statements is true: i) $W_R(y) \cap W_R(z) \subseteq W_R(x)$, ii) $W_R(x) \cap W_R(z) \subseteq W_R(y)$, iii) $W_R(x) \cap W_R(y) \subseteq W_R(z)$. Moreover, i) and ii) are both true only if xIy; similarly, i) and iii) together imply xIz, and ii) and iii) together imply yIz.

Note that xIy implies i) and ii) for any quasi-order and any $x, y \in X$. It follows that if i) does not hold, x must be noncomparable to both y and z. Note also that the regularity of a preference profile is important to make sure that its Pareto relation satisfies the second part of Property 1.

Property 2: Transitive Intermediateness. Let $w, x, y, z \in X$ be pairwise indifferent or noncomparable under R, and xNy. If $W_R(w) \cap W_R(y) \subseteq W_R(x)$ and $W_R(x) \cap W_R(z) \subseteq W_R(y)$, then $W_R(w) \cap W_R(z) \subseteq W_R(x)$.

Note that a regular Paretian quasi-order need not satisfy the stronger property obtained by dropping the proviso xNy in Property 2: suppose, e.g., that $R = R^*(\succcurlyeq_1, \succcurlyeq_2)$, where $\succcurlyeq_1, \succcurlyeq_2$ are represented by u_1, u_2 and $u_1(x) = u_1(y) < u_1(w) < u_1(z)$ and $u_2(x) = u_2(y) > u_2(w) > u_2(z)$.

Property 3: Exclusive Intermediateness. Let $w, x, y, z \in X$ be pairwise indifferent or noncomparable under R, and xNy and xNz. If $W_R(w) \cap W_R(y) \subseteq W_R(x)$ and $W_R(w) \cap W_R(z) \subseteq W_R(x)$, then $W_R(y) \cap W_R(z) \nsubseteq W_R(x)$.

The stronger properties obtained by dropping either of the provisos xNy or xNz are violated by some regular Paretian quasi-orders: suppose, e.g., that $R = R^*(\succcurlyeq_1, \succcurlyeq_2)$,where $\succcurlyeq_1, \succcurlyeq_2$ are represented by u_1, u_2 with $u_1(w) < u_1(x) = u_1(y) < u_1(z)$ and $u_2(w) > u_2(x) = u_2(y) > u_2(z)$.

We conclude this section by noting a useful consequence of the three above properties.

Lemma 1. *Let R be a quasi-order on X satisfying Properties 1, 2, 3, and let $w, x, y, z \in X$ be noncomparable or indifferent under R. If $W_R(w) \cap W_R(y) \subseteq W_R(x)$ and $W_R(w) \cap W_R(x) \subseteq W_R(z)$, then $W_R(y) \cap W_R(z) \subseteq W_R(x)$.*

Proof. Assume Properties 1, 2, 3, let w, x, y, z be noncomparable or indifferent, and suppose

$$W(w) \cap W(y) \subseteq W(x), \tag{3.1}$$

$$W(w) \cap W(x) \subseteq W(z). \tag{3.2}$$

Suppose, contrary to the claim, that $W(y) \cap W(z) \nsubseteq W(x)$. By Property 1, xNy, xNz, and at least one of the following statements is true:

$$W(x) \cap W(z) \subseteq W(y), \tag{3.3}$$

$$W(x) \cap W(y) \subseteq W(z). \tag{3.4}$$

Suppose (3.3) is true. By Property 2, (3.1) and (3.3) imply that $W(w) \cap W(z) \subseteq W(x)$, which together with (3.2) implies xIz by Property 1. This contradicts the fact that xNz.

Suppose next that (3.4) is true. If zIy, rewrite (3.1) as $W(w) \cap W(z) \subseteq W(x)$: this, along with (3.2), implies that xIz, a contradiction. If zIw, rewrite (3.1) as $W(z) \cap W(y) \subseteq W(x)$: this, along with (3.4) implies again, by Property 1, that xIz. We conclude that z is noncomparable to both w and y. By Property 3, then, (3.2) and (3.4) imply that $W(w) \cap W(y) \nsubseteq W(z)$, while (3.1) and (3.4) trivially imply that $W(w) \cap W(y) \subseteq W(z)$. This contradiction completes the proof. ■

4. Regularity conditions and characterization theorem

Denote by $\partial_R X$, or simply ∂X, the set of maximal elements of R in X, i.e., $\partial_R X = \{x \in X : \text{there is no } y \in X \text{ such that } yPx\}$. For each $x \in X$, define the *exterior set of* x as $E_R(x) = \{(x', x'') \in X \times X : W_R(x') \cap W_R(x'') \subseteq W_R(x)\}$; we also write it $E(x)$ to alleviate notations.

Consider the following restrictions on R.

Property 4: Continuity.
1) The set $\partial_R X$ is a closed subset of X.
2) For each $x \in X$, $B_R(x)$ is a closed subset of X.
3) The correspondence $B_R : X \rightarrow\rightarrow X$ is continuous.
4) For each $x \in X, E_R(x)$ is a closed subset of $X \times X$.

Property 5: Richness.
1) There exists $x_0 \in X$ such that, for every $x \in X$, xRx_0.
2) For all $x, z \in X$ such that xPz, there exists some $y \in X$ such that $xPyPz$.
3) Let $x, y \in X$. If for every $z \in \partial_R X$, zRy implies zRx, then yRx.

We are now in a position to state our main result.

Theorem. *A quasi-order on* X *satisfies Properties 1 to 5 if and only if it is a regular Paretian quasi-order.*

The proof of the "if" part is almost straightforward. In checking Property 4.3, note that if $(\succcurlyeq_1, \succcurlyeq_2)$ is a regular profile, the correspondence $B_{R^*(\succcurlyeq_1, \succcurlyeq_2)}$, which we assumed to be lower-hemicontinuous in Definition 1, is in fact also upper-hemicontinuous by continuity of $\succcurlyeq_1$ and $\succcurlyeq_2$.

The proof of the "only if" part of the theorem relies on two lemmata.

Lemma 2. *If a quasi-order* R *on* X *satisfies Property 4.2 and* Y *is a nonempty compact subset of* X*, then* R *has a maximal element in* Y*. In particular,* $\partial_R X$ *is nonempty.*

Proof. Let R and Y satisfy the assumptions of the lemma. Let $Z \subseteq Y$ be an arbitrary R-chain in Y: for all $x, y \in Z, xRy$ or yRx. For each $y \in Z$, define $B(y; Y) = B(y) \cap Y$. For every finite set $Z' \subseteq Z$, $\cap_{y \in Z'} B(y; Y) \neq \emptyset$ since the sets $B(y; Y)$ are nonempty and nested. By Property 4.2, each of these sets is compact (recall that X itself is compact). By a standard result (see, e.g., Rudin 1964, Theorem 2.36), it follows that the set $\cap_{y \in Z} B(y; Y)$ is nonempty. Any element of that set is an R-upper bound for Z. Invoking Zorn's lemma completes the proof. ■

In what follows, we will be concerned with binary relations on $\partial_R X$. A binary relation $\preccurlyeq$ on $\partial_R X$ is *continuous* if, for every $x \in \partial_R X$, $\{y \in \partial_R X : y \preccurlyeq x\}$ and $\{y \in \partial_R X : x \preccurlyeq y\}$ are closed subsets of $\partial_R X$. It is *R-consistent* if

$$\forall x, y, z \in \partial_R X, W_R(x) \cap W_R(z) \subseteq W_R(y) \Leftrightarrow x \preccurlyeq y \preccurlyeq z \text{ or } z \preccurlyeq y \preccurlyeq x. \tag{4.1}$$

Finally, two binary relations $\preccurlyeq, \preccurlyeq'$ are *opposite* if, for all x and $y \in \partial_R X$, $x \preccurlyeq y \Leftrightarrow y \preccurlyeq' x$.

Lemma 3. *If a quasi-order R on X satisfies Properties 1,2,3, and 4, then there exist two continuous R-consistent opposite preorders $\preccurlyeq_l, \preccurlyeq_r$ on $\partial_R X$.*

Proof. Let R satisfy Properties 1 to 4. Call (z_l, z_r) an *extreme pair* (of ∂X) if

$$\forall x \in \partial X, W(z_l) \cap W(z_r) \subseteq W(x). \tag{4.2}$$

Step 1. An extreme pair of ∂X exists.

Define the binary relation e on $\partial X \times \partial X$ by

$$(x, y)e(x', y') \Leftrightarrow W(x) \cap W(y) \subseteq W(x') \cap W(y'). \tag{4.3}$$

We claim, first, that e has a maximal element in $\partial X \times \partial X$. If $x, y \in \partial X$, let $E^*(x, y) = E(x) \cap E(y) \cap (\partial X \times \partial X)$. This set is nonempty since it contains (x, y) and it is compact by Property 4.1 and 4.2. Pick now an arbitrary e-chain Z in $\partial X \times \partial X$. For any finite set $Z' \subseteq Z, \cap_{(x,y)\in Z'} E^*(x, y) \neq \emptyset$ because the sets $E^*(x, y)$ are nested. It follows that $\cap_{(x,y)\in Z} E^*(x, y)$ is also nonempty. Any element of that set is an e-upperbound for Z in $\partial X \times \partial X$. Applying Zorn's lemma establishes our claim.

Let (z_l, z_r) be a maximal element of e in $\partial X \times \partial X$. We argue now that it is an extreme pair of ∂X. Suppose, by contradiction, that

$$W(z_l) \cap W(z_r) \nsubseteq W(x) \tag{4.4}$$

for some $x \in \partial X$. According to Property 1, $W(z_l) \cap W(x) \subseteq W(z_r)$ or $W(z_r) \cap W(x) \subseteq W(z_l)$. Assume the former inclusion; the argument is similar if the latter holds. It implies that $W(z_l) \cap W(x) \subseteq W(z_l) \cap W(z_r)$, i.e., $(z_l, x)e(z_l, z_r)$. On the other hand, (4.4) implies that $W(z_l) \cap W(z_r) \nsubseteq W(z_l) \cap W(x)$, i.e., we do not have $(z_l, z_r)e(z_l, x)$. This contradicts the fact that (z_l, z_r) is a maximal element of e, thereby completing the proof of Step1.

Call two extreme pairs $(y_l, y_r), (z_l, z_r)$ *R-equivalent* if at least one of the following statements is true:

$$y_l I z_l \text{ and } y_r I z_r, \tag{4.5}$$

$$y_l I z_r \text{ and } y_r I z_l. \tag{4.6}$$

Step 2. All extreme pairs of ∂X are R-equivalent.

Let $(y_l, y_r), (z_l, z_r)$ be two extreme pairs. We claim, first, that

$$y_l I z_l \text{ or } y_l I z_r. \tag{4.7}$$

Suppose, on the contrary, that $y_l N z_l$ and $y_l N z_r$. By the definition of extreme pairs,

$$W(z_l) \cap W(z_r) \subseteq W(y_l), \tag{4.8}$$

$$W(y_l) \cap W(y_r) \subseteq W(z_r), \tag{4.9}$$

$$W(y_l) \cap W(y_r) \subseteq W(z_l). \tag{4.10}$$

By Property 2, (4.8) and (4.9) imply $W(z_l) \cap W(y_r) \subseteq W(y_l)$ while (4.8) and (4.10) imply $W(y_r) \cap W(z_r) \subseteq W(y_l)$. These two inclusions imply by Property 3 that (4.8) does not hold. This contradiction establishes (4.7). The same argument, *mutatis mutandis,* yields that

$$y_r I z_l \text{ or } y_r I z_r. \tag{4.11}$$

The claim follows now easily from (4.7) and (4.11). Notice that if $y_l I z_l$ and $y_r I z_l$, then $y_l I y_r$ and, using (4.9), we conclude that y_l, y_r, z_l, z_r are all indifferent. A similar argument applies if $y_l I z_r$ and $y_r I z_r$. Step 2 is proved.

Step 3. If (z_l, z_r) is an extreme pair, then, for all $x, y \in \partial X$,

$$W(x) \cap W(y) \subseteq W(z_l) \textit{ only if } xIz_l \textit{or } yIz_l, \tag{4.12}$$

and this statement is also true if we replace z_l with z_r.

Assume, by way of contradiction, that

$$W(x) \cap W(y) \subseteq W(z_l),\ xNz_l \text{ and } yNz_l. \tag{4.13}$$

Since (z_l, z_r) is an extreme pair, $W(z_l) \cap W(z_r) \subseteq W(x)$ and $W(z_l) \cap W(z_r) \subseteq W(y)$. These two inclusions and (4.13) imply by Property 2 that $W(y) \cap W(z_r) \subseteq W(x)$ and $W(x) \cap W(z_r) \subseteq W(y)$. Property 1 now implies that xIy, hence $W(x) = W(y) = W(x) \cap W(y)$, contradicting (4.13).

Step 4. There exist two R-consistent opposite preorders on ∂X.

Let (z_l, z_r) be an extreme pair of ∂X. Define the binary relations $\preccurlyeq_l, \preccurlyeq_r$ on ∂X as follows: for $x, y \in \partial X$,

$$x \preccurlyeq_l y \Leftrightarrow W(z_l) \cap W(y) \subseteq W(x), \tag{4.14}$$

$$x \preccurlyeq_r y \Leftrightarrow W(z_r) \cap W(y) \subseteq W(x). \tag{4.15}$$

We claim that these relations are R-consistent opposite preorders.

First of all, observe that the (unordered) pair of relations $(\preccurlyeq_l, \preccurlyeq_r)$ is not affected by the choice of the extreme pair (z_l, z_r) because of Step 2.

Next, let us check that these relations are preorders. Transitivity of $\preccurlyeq_l$ is straightforward: if $x \preccurlyeq_l y \preccurlyeq_l z$, then $W(z_l) \cap W(y) \subseteq W(x)$ and $W(z_l) \cap W(z) \subseteq W(y)$, automatically implying that $W(z_l) \cap W(z) \subseteq W(x)$, hence $x \preccurlyeq_l z$. If $\preccurlyeq_l$ were not complete, there would exist $x, y \in \partial X$ such that $W(z_l) \cap W(y) \not\subseteq W(x)$ and $W(z_l) \cap W(x) \not\subseteq W(y)$. By Property 1, $W(x) \cap W(y) \subseteq W(z_l)$ and xNz_l and yNz_l, contradicting Step 3. This shows that $\preccurlyeq_l$ is a preorder; the proof for $\preccurlyeq_r$ is similar.

Let us now prove R-consistency. We must show that (4.1) holds true if $\preccurlyeq$ is $\preccurlyeq_l$ or $\preccurlyeq_r$. We focus on $\preccurlyeq_l$; the proof for $\preccurlyeq_r$ is again similar.

Assume, first, that $x \preccurlyeq_l y \preccurlyeq_l z$. Then $W(z_l) \cap W(y) \subseteq W(x)$ and $W(z_l) \cap W(z) \subseteq W(y)$. By Lemma 1, we conclude that $W(x) \cap W(z) \subseteq W(y)$, as desired. If $z \preccurlyeq_l y \preccurlyeq_l x$, the same conclusion follows by exchanging x and z in the argument.

Conversely, suppose now that

$$W(x) \cap W(z) \subseteq W(y). \tag{4.16}$$

Since $\preccurlyeq_l$ is complete, $x \preccurlyeq_l y$ or $y \preccurlyeq_l x$. Consider the former case first. By definition,

$$W(z_l) \cap W(y) \subseteq W(x). \tag{4.17}$$

If xNy, (4.16) and (4.17) imply by Property 2 that $W(z_l) \cap W(z) \subseteq W(y)$, hence $y \preccurlyeq_l z$, as desired. On the other hand, if xIy, then $x \sim_l y$ and we necessarily have $x \sim_l y \preccurlyeq_l z$ or $z \preccurlyeq_l y \sim_l x$.

Next, consider the case where $y \preccurlyeq_l x$. By definition,

$$W(z_l) \cap W(x) \subseteq W(y). \tag{4.18}$$

If yNz and yNz_l, (4.16) and (4.18) imply through Property 3 that $W(z_l) \cap W(z) \not\subseteq W(y)$. Hence, by definition and completeness of $\preccurlyeq_l$, $z \preccurlyeq_l y$, as desired. If yIz, then $z \sim_l y$ and we are done again. Finally, if yIz_l, (4.16) yields $W(x) \cap W(z) \subseteq W(z_l)$, implying by Step 3 that $z_l I x$ or $z_l I z$. If $z_l I x$, then xIy, hence $x \sim_l y$: again, $x \sim_l y \preccurlyeq_l z$ or $z \preccurlyeq_l y \sim_l x$, as desired. If $z_l I z$, then yIz and we are done again.

Now, let us check that $\preccurlyeq_l$ and $\preccurlyeq_r$ are opposite relations. If $x \preccurlyeq_l y$, then $W(z_l) \cap W(y) \subseteq W(x)$. Since (z_l, z_r) is an extreme pair, $W(z_l) \cap W(z_r) \subseteq W(y)$. By Lemma 1, these two inclusions imply $W(z_r) \cap W(x) \subseteq W(y)$, hence $y \preccurlyeq_r x$, as desired. The same argument, *mutatis mutandis*, shows that $y \preccurlyeq_r x$ implies $x \preccurlyeq_l y$.

Finally, let us check that $\preccurlyeq_l$ and $\preccurlyeq_r$ are continuous. We fix $x \in \partial X$ and show that $\{y \in \partial X : x \preccurlyeq_l y\}$ is a closed set. Consider a sequence $\{y_n\}$, $y_n \in \partial X$ and $x \preccurlyeq_l y_n$ for all n, converging to some y^0. Since ∂X is closed by Property 4.1, $y^0 \in \partial X$. Moreover, $W(z_l) \cap W(y_n) \subseteq W(x)$, i.e., $(z_l, y_n) \in E(x)$, for all n. Since $E(x)$ is closed by Property 4.4, $(z_l, y^0) \in E(x)$. Therefore $W(z_l) \cap W(y^0) \subseteq W(x)$, hence $x \preccurlyeq_l y^0$, proving our claim. The proof that $\{y \in \partial X : y \preccurlyeq_l x\}$ is closed is similar. Thus $\preccurlyeq_l$ is continuous. Since $\preccurlyeq_r$ is opposite to $\preccurlyeq_l$, it too is continuous. This completes the proof of Step 4 and the lemma. ■

We may now prove our theorem.

Proof of the Theorem.

We have mentioned that a regular Paretian quasi-order satisfies Properties 1 to 5. Conversely, fix a quasi-order R on X satisfying those properties.

Step 1: Constructing the preference profile $(\succcurlyeq_1, \succcurlyeq_2)$.

For each $x \in X$, define $B^*(x) = B(x) \cap \partial X$. This set is compact by Property 4.1 and 4.2. Because R is a quasi-order, $B^*(x)$ coincides with the set of maximal elements of R in $B(x)$. It is therefore nonempty by Lemma 2. Using the same type of argument as in the proof of Lemma 2, one easily shows that $\succcurlyeq_l$ has a minimal element $l(x)$ and a maximal element $r(x)$ in $B^*(x)$. In fact,

$$B^*(x) = \{y \in \partial X : l(x) \preccurlyeq_l y \preccurlyeq_l r(x)\}. \tag{4.19}$$

Indeed, if $y \in \partial X$ and $l(x) \preccurlyeq_l y \preccurlyeq_l r(x)$, then $x \in W(l(x)) \cap W(r(x)) \subseteq W(y)$ since $l(x), r(x) \in B(x)$ and $\preccurlyeq_l$ is R-consistent. This means that $x \in W(y)$, i.e., $y \in B^*(x)$.

If $Y, Y' \subseteq \partial X$ are such that $x \sim_l y$ for all $x, y \in Y$ and $x' \sim_l y'$ for all $x', y' \in Y'$, the notation $Y \preccurlyeq_l Y'$ has an obvious unambiguous meaning. We use a similar notation for $\preccurlyeq_r$. Denote by $\lambda(x)$ and $\rho(x)$ the (compact) sets of all minimal and maximal elements of $\succcurlyeq_l$ in $B^*(x)$, respectively. Define the binary relations $\preccurlyeq_1$ and $\preccurlyeq_2$ on (the entire set) X as follows:

$$x \preccurlyeq_1 y \Leftrightarrow \lambda(x) \preccurlyeq_l \lambda(y), \tag{4.20}$$

$$x \preccurlyeq_2 y \Leftrightarrow \rho(x) \preccurlyeq_r \rho(y). \tag{4.21}$$

Clearly, these relations are preorders.

Step 2: Checking that the Pareto relation of $(\succcurlyeq_1, \succcurlyeq_2)$ *coincides with* R.

If yRx, then $B^*(y) \subseteq B^*(x)$. Recalling (4.19), this implies that $\lambda(x) \preccurlyeq_l \lambda(y)$ and $\rho(x) \preccurlyeq_r \rho(y)$. Therefore $x \preccurlyeq_1 y$ and $x \preccurlyeq_2 y$. Conversely, suppose $x \preccurlyeq_1 y$ and $x \preccurlyeq_2 y$. Then $\lambda(x) \preccurlyeq_l \lambda(y)$ and $\rho(x) \preccurlyeq_r \rho(y)$ and, by (4.19), $B^*(y) \subseteq B^*(x)$. By Property 5.3, yRx.

Step 3: Checking regularity.

3.1) First, let us establish that the two binary relations are continuous. We start off by noting that the correspondences λ and ρ from X to ∂X are upper-hemicontinuous. For any $x \in X$, $\lambda(x)$ is the set of maximal elements of a continuous preorder over the nonempty compact set $B^*(x)$. Since by Property 4.1 and 4.3, $B^* : X \rightarrow\rightarrow X$ is a continuous correspondence, λ is upper-hemicontinuous by (a version of) the maximum theorem. A similar argument applies to ρ.

Fix now $x \in X$ and check that $\{y \in X : x \preccurlyeq_1 y\}$ is closed. Consider a sequence $\{y_n\}$ in that set converging to some y^0. Since X is closed, $y^0 \in X$. For each n, $\lambda(x) \preccurlyeq_l \lambda(y_n)$. Recalling the definition of $\preccurlyeq_l$ from (4.14), we can choose some $l(x) \in \lambda(x)$ and, for each n, some $l(y_n) \in \lambda(y_n)$, such that $W(z_l) \cap W(l(y_n)) \subseteq W(l(x))$, i.e., $(z_l, l(y_n)) \in E(l(x))$ for all n. Since λ is compact-valued and upper-hemicontinuous, some subsequence of $\{l(y_n)\}$ converges to a limit $l^0 \in \lambda(y^0)$. Since $E(l(x))$ is closed by Property 4.4, $(z_l, l^0) \in E(l(x))$. So $l(x) \preccurlyeq_l l^0$, hence $\lambda(x) \preccurlyeq_l \lambda(y^0)$, implying that $x \preccurlyeq_1 y^0$, as desired. A similar argument establishes that $\{y \in X : y \preccurlyeq_1 x\}$ is also closed. This proves that $\succcurlyeq_1$ is continuous; a similar argument applies to $\succcurlyeq_2$.

3.2) Since $\succcurlyeq_1$ and $\succcurlyeq_2$ are continuous preorders on a connected set, they admit continuous numerical representations u_1, u_2. Since X is compact and connected, so is $u(X)$. We now show that $u(X)$ is also complete in the sense defined by (2.2).

First, let $w', w'' \in \partial u(X)$, say, $w'_1 \leq w''_1$ and $w'_2 \geq w''_2$, and let us check that $w' \wedge w'' \in u(X)$. Suppose not. The set

$$Z_1^+ = \{w \in u(X) : w_1 = w'_1 \text{ and } w_2 \geq w''_2\}$$

is compact, nonempty (since it contains w'), and does not contain $w' \wedge w''$. Hence, there exists $w_2^* > w''_2$ such that $(w'_1, w_2^*) \in Z_1^+$ and $w_2 \geq w_2^*$ for all $(w_1, w_2) \in Z_1^+$. The set

$$Z_1^- = \{w \in u(X) : w_1 = w'_1 \text{ and } w_2 \leq w''_2\}$$

is also compact and does not contain $w' \wedge w''$. We claim that $Z_1^- = \emptyset$. Otherwise, there exists $w_2^{**} < w_2''$ such that $(w_1', w_2^{**}) \in Z_1^-$ and $w_2 \leq w_2^{**}$ for all $(w_1, w_2) \in Z_1^-$. By Property 5.2, there exists some w_2 such that $w_2^{**} < w_2 < w_2^*$ and $(w_1', w_2) \in u(X)$, contradicting the definitions of w_2^* and w_2^{**}. The same argument, *mutatis mutandis*, shows that the set

$$Z_2^- = \{w \in u(X) : w_1 \leq w_1' \text{ and } w_2 = w_2''\}$$

is also empty. By Property 5.1, however, there exists some $w_0 \in u(X), w_0 \ll w' \wedge w''$. This contradicts the fact that $u(X)$ is connected.

Conversely, let us now fix $w \in u(X)$ and show that there exist $w', w'' \in \partial u(X)$ such that $w = w' \wedge w''$. Let $x \in X$ be such that $u(x) = w$. From Step 1, $B^*(x) = \{y \in \partial X : \lambda(x) \preccurlyeq_l y \preccurlyeq_l \rho(x)\}$. Recalling that $\preccurlyeq_l$ and $\preccurlyeq_r$ are opposite, observing that $\lambda(y) \sim_l y \sim_r \rho(y)$ for any $y \in \partial X$ and using the definitions of $\succcurlyeq_1, \succcurlyeq_2$, we see that

$$B^*(x) = \{y \in \partial X : u_1(y) \geq u_1(l(x)) \text{ and } u_2(y) \geq u_2(r(x))\}. \tag{4.22}$$

We claim that

$$u(x) = u(l(x)) \wedge u(r(x)).$$

Since $l(x), r(x) \in B^*(x)$, we obviously have $u(x) \leq u(l(x)) \wedge u(r(x))$. Suppose $u(x) < u(l(x)) \wedge u(r(x))$. As we have shown, the right-hand side of this inequality belongs to $u(X)$. Let thus $x' \in X$ be such that $u(x') = u(l(x)) \wedge u(r(x)) = (u_1(l(x)), u_2(r(x)))$. By (4.22), $B^*(x) = B^*(x')$. This contradicts Property 5.3 since $x'Px$. ■

5. References

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Figure 1: A quasi-order that is not a two-agent Pareto relation

	a	b	c	x	y	z
a	R				R	R
b		R		R		R
c			R	R	R	
x				R		
y					R	
z						R