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Probabilistic and Analytic Aspects of Ruin Theory

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SOMMAIRE

Les modèles du risque généralisés, qui permettent de faire plus d'hypothèses sur la distribution du temps d'attente que le modèle du risque classique, sont considérés par des approches probabiliste et analytique. En utilisant respectivement les variables de record dans le modèle probabiliste et la transformation de Laplace dans le modèle analytique, des expressions alternatives de probabilité de la ruine sont tirées dans chaque modèle et ses liens avec le modèle classique sont étudiés.

Mot clé: probabilité de la ruine

ABSTRACT

Generalized risk models, which permit much more assumptions on the distribution of the waiting times than the classical risk model, are considered through probabilistic and analytic approaches. Employing respectively the ladder variables in probabilistic models and Laplace transforms as well as complex variables in analytic models, alternative expressions of ruin probability are derived in each model and studied in connection with the classical model.

Key word: ruin probability

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Contents

Chapter 1

INTRODUCTION

A mathematically convenient definition of insurer's surplus is the excess of the initial fund plus premiums collected over claims paid. Since the amount of surplus changes through time, let us consider the surplus at time t, namely, $U(t)$, in the period of interest and define the surplus process as $\{U(t), t \geq 0\}$. To model the amount of surplus, we study how this surplus process fluctuates in the period of interest. We are concerned especially with the probability that this surplus process becomes strictly negative at some point in time, which is called the ruin probability. In the classical risk theory, the surplus process, especially based upon the compound Poisson process, is used to derive an explicit formula for the ruin probability.

However, when we want to calculate the ruin probability in diverse cases, use of the classical model becomes limited owing to the assumption that the distribution of waiting times between claims is exponential. This problem may be solved by building a more general model which has a wider assumption on the distribution of waiting times. As a matter of fact, this kind of work has already been done by some authors. For example, Sparre Andersen (1957) proposed the model in which the distribution of waiting times is arbitrary. Dickson (1998) derived some explicit results when claims occur as an Erlang process. Borovkov (1976) also worked on similar cases, but by the use of queueing theory and so on.

The present study will look at the case when the distribution of waiting times is other than exponential via two approaches. One is probabilistic and the other is analytic. In both approaches, the random walk whose summands consist of the difference between the waiting time and the claim, will be used to solve the problem with help of random walk theory. Thus, in these approaches, we consider that ruin is the event when a random walk crosses a barrier. With this new concept of ruin, some formulations of ruin probability and their interpretations in connection with the classical model will be carried out.

In Chapter 2 we deal with the classical risk theory: relevant definitions and assumptions, as well as fundamental theorems in the compound Poisson process case are introduced. In the next two chapters, the random walk is used to construct more general models than the classical one.

In Chapter 3 we use a probabilistic approach to study random walks, especially by means of ladder variables. Ladder variables will be defined and their properties described in Section 1. In Section 2 the duality lemma is introduced and used for proving the relevant theorems therein. Derivations of the formulas for the ladder variable distributions are given in Section 3, which relate them to Wiener–Hopf type factorization and the distribution of maxima. Section 4 is for the applications to the risk theory. An alternative expression to the ruin probability in the classical model is presented. In Section 5 the ruin probability of the classical model case is calculated, using the probabilistic model, and compared with the ruin probability calculated in the classical model.

In Chapter 4 we look at the problem associated with the random walk from an analytical point of view, by means of the Laplace transform. The Laplace transform and, in particular, its features for the survival probability are given in some detail in Section 1. Section 2 defines the class of finite rational distribution $\mathcal{R}_+^f$ and explains some associated properties by making use of a few examples. An explicit expression is derived in Section 3 for the survival probability in the case of the distribution of waiting times being in $\mathcal{R}_+^f$, whereas Section 4 is for the derivation in the case of the distribution of claims being in $\mathcal{R}_+^f$. In Section 5 the ruin probability of the classical model case is calculated, using the analytic model, and compared with the ruin probability calculated in the classical model.

Concluding remarks are made in Chapter 5.

Chapter 2

THE CLASSICAL RISK MODEL

2.1. Collective Risk Model For A Single Period

2.1.1. Notion of Aggregate Claims

As explained in Bowers *et al.* (1997), the collective risk model concentrates on the total claim amount for a portfolio rather than on an individual claim amount. According to this approach, we consider a random process that represents a portfolio as a whole. For that purpose, we need to consider two factors. One is the random variable that counts the number of claims, which is denoted by N. The other is the random variable that represents the i^{th} claim amount. We denote the latter by X_i. The total claim amount S is given by $S = X_1 + X_2 + \cdots + X_N$. We call it the aggregate claims generated by the portfolio for the period of interest. In this model, the two assumptions are usually made to make this model tractable, namely,

(1) $X_1, X_2, \ldots$ are independent and identically distributed (i.i.d.) random variables.

(2) $N, X_1, X_2, \ldots$ are mutually independent.

2.1.2. Distribution of aggregate claims

By simple calculations, we can find out the following:

$$(1)\ \mathsf{E}(S) = \mathsf{E}\left[\mathsf{E}\left(S \mid N\right)\right] = \mathsf{E}(X_1)\mathsf{E}(N),$$

$$(2)\ \mathsf{Var}(S) = \mathsf{E}\left[\mathsf{Var}\left(S \mid N\right)\right] + \mathsf{Var}\left[\mathsf{E}(S \mid N)\right] = \mathsf{E}(N)\mathsf{Var}(X_1) + [\mathsf{E}(X_1)]^2\mathsf{Var}(N),$$

(3) The moment generating function (m.g.f.) of S is

$$M_S(t) = \mathsf{E}(e^{tS}) = M_N\left[\log M_{X_1}(t)\right]. \tag{2.1.1}$$

One of the most useful assumptions may be that the distribution of N may be described by the Poisson distribution given by

$$\mathsf{P}(N = n) = \frac{\lambda^n e^{-\lambda}}{n!}, \qquad n = 0, 1, 2, \ldots$$

In this case, the distribution of S is called compound Poisson. An advantage of the compound Poisson distribution lies in its mathematical properties that make the calculations involved easier than for other types of distributions.

By applying the law of total probability, the distribution function of S can be written as

$$\begin{aligned} F_S(x) &= \mathsf{P}(S \le x) = \sum_{n=0}^{\infty} \mathsf{P}(S \le x \mid N = n)\mathsf{P}(N = n) \\ &= \sum_{n=0}^{\infty} \mathsf{P}(X_1 + X_2 + \cdots + X_n \le x)\mathsf{P}(N = n) \\ &= \sum_{n=0}^{\infty} \mathsf{P}^{n*}(x)\mathsf{P}(N = n). \end{aligned}$$

2.2. Collective Risk Model Over An Extended Period

2.2.1. Surplus Process

Let a deterministic process $c(t)$ denote the premiums collected over the time period $[0, t]$, and $S(t)$ denote the aggregate claims paid over t. If u is an initial surplus at $t = 0$, then the surplus at t, $U(t)$ can be expressed as

$$U(t) = u + c(t) - S(t), \qquad t \ge 0. \tag{2.2.1}$$

We will consider the distribution of $U(t)$ for $t \ge 0$, which is different from the previous section where we are concerned with the case for a single value of t. Let

$N(t)$ be the number of claims up to time t. Then we can express $S(t)$ as follows:

$$S(t) = X_1 + X_2 + \cdots + X_{N(t)}, \tag{2.2.2}$$

where $\{N(t), t \geq 0\}$ is called the claim number process. If $\{N(t), t \geq 0\}$ is a Poisson process which is defined as

$$\mathsf{P}\left[N(t+h) - N(t) = k \mid N(s) \text{ for all } s \leq t\right] = \frac{e^{-\lambda h}\,(\lambda h)^k}{k!}$$

$$k = 0, 1, 2, \ldots \text{ for all } t \geq 0 \text{ and } h > 0,$$

and independent, identically distributed random variables $X_1, X_2, \ldots$ are also independent of $\{N(t), t \geq 0\}$, then $S(t)$ is called a compound Poisson process.

2.2.2. Ruin Probability

Definition 2.2.1. *Ruin is the event "the surplus becomes strictly negative at some point in time."*

Definition 2.2.2. *The time of ruin T is defined as*

$$T = \begin{cases} \min\{t : t \geq 0 \text{ and } U(t) < 0\}, \\ \\ \infty \quad \text{if} \quad U(t) \geq 0, \quad \text{for all } t \geq 0. \end{cases}$$

Definition 2.2.3. *The probability of ruin with initial surplus u is*

$$\psi(u) = \mathsf{P}(T < \infty).$$

Let us assume that premiums are received continuously at a constant rate $c > 0$. Then the previously mentioned process $c(t)$ becomes ct. Consider a compound Poisson process with parameter λ. We usually assume that the premium collection rate c exceeds the expected claim payments per time unit, which is $\lambda\mathsf{E}(X_1)$. Therefore we express the premium c by

$$c = (1 + \theta)\lambda\mathsf{E}(X_1),$$

where θ is a positive number called the *relative security loading.*

Definition 2.2.4. *The Adjustment coefficient R is the smallest positive root r of the equation*

$$\mathsf{E}(e^{-rU(t)}) = e^{-ru}, \tag{2.2.3}$$

if it exists.

Based on this definition, we can derive an equivalent relation in the case of compound Poisson process. Since

$$\begin{aligned}
\mathsf{E}(e^{-rU(t)}) &= \mathsf{E}(e^{-r(u+ct-S(t)}) \\
&= e^{-ru}e^{-rct}\mathsf{E}(e^{rS(t)}) \\
&= e^{-ru}e^{-rct}M_{N(t)}\left[\log M_{X_1}(r)\right] \\
&= e^{-ru}e^{-rct}e^{\lambda t[\mathsf{E}(e^{rX_1})-1]} \\
&= e^{-ru}\exp\left[-rct + \lambda t\left(\mathsf{E}(e^{rX_1}) - 1\right)\right],
\end{aligned}$$

Eq.(2.2.3) becomes

$$e^{-ru}\exp\left[-rct + \lambda t\left(\mathsf{E}(e^{rX_1}) - 1\right)\right] = e^{-ru}$$

$$\Longleftrightarrow -rct + \lambda t\left(\mathsf{E}(e^{rX_1}) - 1\right) = 0$$

$$\Longleftrightarrow -rc + \lambda\left(\mathsf{E}(e^{rX_1}) - 1\right) = 0$$

$$\Longleftrightarrow 1 + \frac{rc}{\lambda} = \mathsf{E}(e^{rX_1}).$$

By substituting $(1+\theta)\lambda\mathsf{E}(X_1)$ for c in the above equation, we obtain

$$1 + (1+\theta)\mathsf{E}(X_1)r = \mathsf{E}(e^{rX_1}). \tag{2.2.4}$$

The reason that the adjustment coefficient R is important is that it is very closely related to the ruin probability. This can be demonstrated in the following theorem.

Theorem 2.2.1. *Suppose that $U(t)$ is the surplus process based upon a compound Poisson aggregate claims process $S(t)$ with positive relative security loading and the adjustment coefficient $R > 0$ if it exists. Then, for $u \geq 0$,*

$$\psi(u) = \frac{\exp(-Ru)}{\mathsf{E}[\exp(-RU(T)) \mid T < \infty]}. \tag{2.2.5}$$

Since

$$U(T) < 0 \ \text{ if } \ T < \infty \quad \text{and} \quad \mathsf{E}[\exp(-RU(T)) \mid T < \infty] > 1,$$

we get *Lundberg's inequality*:

$$\psi(u) < e^{-Ru}.$$

The following theorem is proved in Bowers *et al.* (1997, p.427), by means of an integral equation.

Theorem 2.2.2. *For a compound Poisson process, the probability that the surplus will ever fall below its initial level u and will be between $u - y$ and $u - y - dy$ when it happens for the first time is given by*

$$\frac{\lambda}{c}\left[1 - F_X(y)\right] dy = \frac{1 - F_X(y)}{(1+\theta)\mathsf{E}(X_1)} dy, \qquad \textit{for} \ \ y > o, \tag{2.2.6}$$

where θ is the relative security loading.

According to Theorem 2.2.2, the probability that the surplus will ever fall below its original level is

$$\frac{1}{(1+\theta)\,\mathsf{E}(X_1)} \int_0^{\infty} \left[1 - F_X(y)\right] dy = \frac{1}{1+\theta}, \tag{2.2.7}$$

which is the same as

$$\psi(0) = \frac{1}{1+\theta}. \tag{2.2.8}$$

2.2.3. Maximal Aggregate Loss

Definition 2.2.5. *The maximal aggregate loss is defined by*

$$L = \max_{t \geq 0} \{S(t) - ct\}.$$

Using this definition, for $u \geq 0$, the distribution function of the random variable L may be expressed as

$$\begin{aligned} 1 - \psi(u) &= \mathsf{P}[U(t) \geq 0 \text{ for all } t] \\ &= \mathsf{P}[u + ct - S(t) \geq 0 \text{ for all } t] \\ &= \mathsf{P}[S(t) - ct \leq u \text{ for all } t] \\ &= \mathsf{P}[L \leq u]. \end{aligned} \tag{2.2.9}$$

Let τ_1 be the first instant t such that $U(t) < u$, provided that it ever happens. We define a random variable L_1 denoting the amount by which the surplus falls below the initial level for the first time, granted that this ever happens, as

$$L_1 = u - U_{\tau_1}.$$

Again, let τ_2 be the first instant t after τ_1 such that $U(t) < U_{\tau_1}$, given that this ever happens. We define a random variable L_2 in the same manner as for L_1, namely,

$$L_2 = U_{\tau_2} - U_{\tau_1}.$$

Repeating in the same way, we can express L in terms of L_i. That is,

$$L = L_1 + L_2 + \cdots + L_N, \qquad \text{where } N \text{ is the number of new record lows.}$$

Here $L_1, L_2, \ldots, L_N$ are i.i.d. and independent of N which has a geometric distribution, that is,

$$\mathsf{P}(N=n) = [1-\psi(0)][\psi(0)]^n = \theta\left(\frac{1}{1+\theta}\right)^{n+1}, \quad \text{for } n = 0, 1, \ldots,$$

and the m.g.f. of N is

$$M_N(r) = \frac{\theta}{1+\theta-e^r}.$$

Thus Eq.(2.2.9) becomes

$$1-\psi(u) = \mathsf{P}(L_1 + L_2 + \cdots + L_N \leq u). \tag{2.2.10}$$

Equation (2.2.6) is not a probability density function (p.d.f.), because it does not integrate to 1. But, by setting a condition that the ruin ever happens, we can make it a p.d.f. of random variable which is identical to L_1. Since

$$\mathsf{P}(\tau_1 < \infty) = \psi(0) = \frac{1}{1+\theta},$$

the p.d.f. for L_1 is

$$\begin{aligned} f_{L_1}(y) &= \frac{\frac{1}{\mathsf{E}(X_1)(1+\theta)}\left[1-F_X(y)\right]}{\frac{1}{1+\theta}} \\ &= \frac{1}{\mathsf{E}(X_1)}\left[1-F_X(y)\right]. \end{aligned}$$

Hence the m.g.f. of L_1 is

$$M_{L_1}(r) = \frac{1}{\mathsf{E}(X_1)}\int_0^\infty e^{ry}\left[1-F_X(y)\right]dy. \tag{2.2.11}$$

By performing the integration by parts, the (2.2.11) becomes

$$M_{L_1}(r) = \frac{1}{\mathsf{E}(X_1)}\left[\frac{e^{ry}}{r}(1-F_X(y))\right]_0^\infty + \frac{1}{r\mathsf{E}(X_1)}\int_0^\infty e^{ry} f_X(y)dy$$

$$= \frac{1}{r\mathsf{E}(X_1)}\left[M_X(r)-1\right]. \tag{2.2.12}$$

According to (2.1.1), the m.g.f. of L is

$$M_L(r) = \frac{\theta}{1+\theta-M_{L_1}(r)}.$$

Therefore by (2.2.12), we have

$$M_L(r) = \frac{\theta\mathsf{E}(X_1)r}{1+(1+\theta)\mathsf{E}(X_1)r - M_X(r)}. \tag{2.2.13}$$

By taking (2.2.9) into consideration, since $L \geq 0$, we can say that L has a point mass of $1-\psi(0)$ at the origin and is distributed continuously for the strictly positive L. Thus we obtain

$$M_L(r) = 1-\psi(0) + \int_0^\infty e^{ur}[-\psi'(u)]du$$

$$= \frac{\theta}{1+\theta} + \int_0^\infty e^{ur}[-\psi'(u)]du. \tag{2.2.14}$$

Hence from (2.2.13) and (2.2.14), we have

$$\int_0^\infty e^u r[-\psi'(u)]du = \frac{1}{1+\theta}\cdot\frac{\theta[M_X(r)-1]}{1+(1+\theta)\mathsf{E}(X_1)-M_X(r)}. \tag{2.2.15}$$

Let us suppose that the distribution of claims is a mixture of exponentials, that is,

$$f_X(x) = \sum_{i=1}^n A_i\beta_i e^{-\beta_i x}, \text{ for } x>0, \text{ where } \sum_{i=1}^n A_i = 1,\ A_i>0,\ \beta_i>0.$$

Then the m.g.f. of X is

$$M_X(r) = \sum_{i=1}^{n} A_i \frac{\beta_i}{\beta_i - r}. \tag{2.2.16}$$

By replacing $M_X(r)$ in (2.2.15) with (2.2.16) and applying the method of partial fractions, we have

$$\int_0^\infty e^u r[-\psi'(u)]du = \sum_{i=1}^{n} \frac{C_i r_i}{r_i - r}, \quad \text{where } C_i \text{ and } r_i \text{ are some constants,}$$

which implies that

$$\psi(u) = \sum_{i=1}^{n} C_i e^{-r_i u}.$$

Hence we can conclude that when the distribution of claims is a mixture of exponential distributions, the probability of ruin is a series of exponential functions multiplied by some constant.

Example 2.2.1. *Let*

$$f_X(x) = \frac{1}{4} \cdot 2e^{-2x} + \frac{3}{4} \cdot 4e^{-4x} \quad \textit{and} \quad \theta = \frac{3}{5}.$$

Then we have

$$\begin{aligned} \mathsf{E}(X) &= \frac{1}{4} \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{4} = \frac{5}{16}, \\ M_X(r) &= \frac{1}{4} \cdot \frac{2}{2-r} + \frac{3}{4} \cdot \frac{4}{4-r}. \end{aligned}$$

Thus

$$\begin{aligned}
M_{L_1}(r) &= \frac{1}{r\mathsf{E}(X_1)}\left[M_X(r)-1\right] \\
&= \frac{1}{5r/16}\left(\frac{1}{4}\cdot\frac{2}{2-r}+\frac{3}{4}\cdot\frac{4}{4-r}-1\right) \\
&= \frac{16}{5}\left(\frac{1}{4}\cdot\frac{1}{2-r}+\frac{3}{4}\cdot\frac{1}{4-r}\right) \\
&= \frac{4}{5}\cdot\frac{1}{2-r}+\frac{12}{5}\cdot\frac{1}{4-r} \\
&= \frac{2}{5}\cdot\frac{2}{2-r}+\frac{3}{5}\cdot\frac{4}{4-r}.
\end{aligned}$$

Since $\psi(0) = 1/(1+\theta) = 5/8$,

$$\begin{aligned}
M_L(r) &= M_N\left(\log M_{L_1}(r)\right) \\
&= \frac{3/8}{1-\frac{5}{8}\left(\frac{2}{5}\cdot\frac{2}{2-r}+\frac{3}{5}\cdot\frac{4}{4-r}\right)} \\
&= \frac{\frac{3}{5}(2-r)(4-r)}{\frac{8}{5}(2-r)(4-r)-\left[\frac{4}{5}(4-r)+\frac{12}{5}(2-r)\right]} \\
&= \frac{24-18r+3r^2}{24-32r+8r^2} \\
&= \frac{3}{8}+\frac{9}{16}\cdot\frac{1}{1-r}+\frac{1}{16}\cdot\frac{3}{3-r}.
\end{aligned}$$

Thus

$$-\psi'(u) = \frac{d}{du}\mathsf{P}(L \le u) = \frac{9}{16}e^{-u}+\frac{3}{16}e^{-3u}.$$

Hence

$$\begin{aligned}
\psi(u) &= \mathsf{P}(L > u) \\
&= \int_u^\infty \frac{d}{dx}\mathsf{P}(L \leq x)dx \\
&= \frac{9}{16}e^{-u} + \frac{1}{16}e^{-3u}.
\end{aligned}$$

Chapter 3

PROBABILISTIC APPROACH

In the previous chapter, we looked at the problem of ruin probability in the classical model. In this chapter, on the contrary, we would like to take a general modeling approach, especially by means of the combinatorial methods and ladder variables. As for the method used in the present study, one is referred to Feller (1971, Chapter 12). Some properties mentioned without proofs are proved here, and detailed explanations are added to the existing proofs, summarizing the content of Chapter 12 of Feller(1971). In the general model to be studied in this work, we look at the ruin probability problem from the perspective of random walks crossing a barrier.

3.1. Definitions and Properties of Ladder Variables

Suppose that mutually independent random variables $X_1, X_2, \ldots$ have common distribution F. Let $S_0 = 0$ and $S_n = X_1 + X_2 + \cdots + X_n$. Then the sequence $\{S_n\}$ constitutes the random walk generated by F. We say that S_n is the position, at epoch n, of the random walk.

Consider the sequence of point (n, S_n) for $n = 1, 2, \ldots$. By observing the graph of the sequence, one may notice that S_n reaches record values at some epochs. We call these record values *ladder points*. These epochs and the record values have very useful features for the calculation of ruin probabilities. For example, they have the property that the sections of the random walk between the ladder points are independent and identically distributed (i.i.d.). This is one of the reasons that study of the first ladder point is so important.

Definition 3.1.1. *The first strict ascending ladder point* $(\mathcal{T}_1, \mathcal{H}_1)$ *is the first term in the sequence of points* $\{(n, S_n)\}_{n=1}^{\infty}$ *for which* $S_n > 0$. *That is,* $\mathcal{T}_1$ *is the epoch of the first entry into the strictly positive half-axis defined by*

$$\{\mathcal{T}_1 = n\} = \{S_1 \leq 0, \ldots, S_{n-1} \leq 0, S_n > 0\}, \tag{3.1.1}$$

or $\mathcal{T}_1 = \infty$ *if* $S_n \leq 0$ *for all* n. *We denote*

$$\mathcal{H}_1 = S_{\mathcal{T}_1} \cdot 1_{\{\mathcal{T}_1 < \infty\}} + \infty \cdot 1_{\{\mathcal{T}_1 = \infty\}}.$$

We call $\mathcal{T}_1$ *the first ladder epoch and* $\mathcal{H}_1$ *the first ladder height. Furthermore, we write*

$$H(x) = \mathsf{P}(\mathcal{H}_1 \leq x).$$

Following the first ladder point, the first point which satisfies

$$S_n > S_0, \ldots, S_n > S_{n-1},$$

is called the second ladder point and denoted by $(\mathcal{T}_1 + \mathcal{T}_2, \mathcal{H}_1 + \mathcal{H}_2)$. In general, the rth ladder point is of the form $(\mathcal{T}_1 + \cdots + \mathcal{T}_r, \mathcal{H}_1 + \cdots + \mathcal{H}_r)$. The pairs $(\mathcal{T}_k, \mathcal{H}_k)$ are clearly i.i.d.

Definition 3.1.2. *The renewal measure for the strict ascending ladder height process is defined as*

$$\psi = \sum_{n=0}^{\infty} H^{n*}, \qquad H^{0*} = \psi_0 \tag{3.1.2}$$

and its improper distribution function is given by

$$\psi(x) = \psi\{(-\infty, x]\},$$

where ψ_0 *is the atomic distribution with unit mass at the origin. That is, for any interval* I

$$\psi_0(I) = 1 \ \text{ if } \ 0 \in I, \ \psi_0(I) = 0 \ \text{ otherwise.}$$

Proposition 3.1.1. *Let $N_{(0,x]}$ be the number of the strict ascending ladder points in $(0,x]$. Then*

$$\psi(x) = 1 + \mathsf{E}(N_{(0,x]}).$$

Proof. For $x \geq 0$,

$$\begin{aligned}
\mathsf{E}(N_{(0,x]}) &= \int_0^\infty \mathsf{P}(N_{(0,x]} \geq y)dy = \sum_{k=1}^\infty \mathsf{P}(N_{(0,x]} \geq k) \\
&= \sum_{k=1}^\infty \mathsf{P}(\mathcal{H}_1 + \cdots + \mathcal{H}_k \leq x) = \sum_{k=1}^\infty H^{k*}(x) \\
&= \sum_{k=0}^\infty H^{k*}(x) - \psi_0(x) = \psi(x) - 1. \qquad \square
\end{aligned}$$

Proposition 3.1.2. *The following relation holds:*

$$\psi(\infty) = \sum_{n=0}^\infty H^{n*}(\infty) = \frac{1}{1 - H(\infty)}.$$

Proof. If $\mathcal{H}_1$, $\mathcal{H}_2$ are proper,

$$\mathsf{P}(\mathcal{H}_1 + \mathcal{H}_2 \leq \infty) = [\mathsf{P}(\mathcal{H}_1 \leq \infty)]^2 = 1.$$

If not, let

$$H(\infty) = \lim_{x \to \infty} H(x) \quad \text{and} \quad \widetilde{H}(x) = \frac{H(x)}{H(\infty)}.$$

Since $\widetilde{H}(x) \to 1$ as $x \to \infty$, $\widetilde{H}(x)$ is proper. Using this notation, we prove this proposition as follows.

By the definition of convolution,

$$H^{2*}(\infty) = \lim_{y \to \infty} \int_0^y H(y - z)dH(z).$$

Dividing two sides of the equation by $H^2(\infty)$, we obtain

$$\begin{aligned}\frac{H^{2*}(\infty)}{H^2(\infty)} &= \lim_{y\to\infty}\int_0^y \widetilde{H}(y-z)d\widetilde{H}(z)\\ &= \lim_{y\to\infty} \mathsf{P}(\widetilde{\mathcal{H}}_1+\widetilde{\mathcal{H}}_2 \le y) = 1.\end{aligned}$$

Hence

$$H^{2*}(\infty) = [H(\infty)]^2,$$

and more generally

$$H^{n*}(\infty) = [H(\infty)]^n.$$

Therefore,

$$\psi(\infty) = \sum_{n=0}^{\infty} H^{n*}(\infty) = \sum_{n=0}^{\infty}[H(\infty)]^n = \frac{1}{1-H(\infty)}. \qquad \square$$

Definition 3.1.3. *The first weak ascending ladder point $(\bar{\mathcal{T}}_1, \bar{\mathcal{H}}_1)$ is the first term in the sequence of points $\{(n, S_n)\}_{n=1}^{\infty}$ for which $S_n \ge 0$. That is, $\bar{\mathcal{T}}_1$ is defined by*

$$\{\bar{\mathcal{T}}_1 = n\} = \{S_1 < 0, \dots, S_{n-1} < 0,\ S_n \ge 0\},$$

or $\bar{\mathcal{T}}_1 = \infty$ if $S_n < 0$ for all n. And we define

$$\bar{\mathcal{H}}_1 = S_{\bar{\mathcal{T}}_1}\cdot 1_{\{\bar{\mathcal{T}}_1<\infty\}} + \infty\cdot 1_{\{\bar{\mathcal{T}}_1=\infty\}}.$$

We denote $\bar{H}(x) = \mathsf{P}(\bar{\mathcal{H}}_1 \le x)$.

Proposition 3.1.3. *Let*

$$\begin{aligned}\zeta &= \mathsf{P}(\bar{\mathcal{H}}_1 = 0)\\ &= \sum_{n=1}^{\infty}\mathsf{P}(S_1<0,\dots,S_{n-1}<0, S_n = 0).\end{aligned}$$

Then

$$\bar{H} = \zeta\psi_0 + (1-\zeta)H.$$

PROOF. We suppose that $\mathcal{T}_1 < \infty$. Then for $x > 0$,

$$\begin{aligned}
\bar{H}(x) &= \mathsf{P}(\bar{\mathcal{H}}_1 \le x) \\
&= \mathsf{P}(\bar{\mathcal{H}}_1 \le x, \bar{\mathcal{H}}_1 = 0) + \mathsf{P}(\bar{\mathcal{H}}_1 \le x, \bar{\mathcal{H}}_1 > 0) \\
&= \zeta + \mathsf{P}(\bar{\mathcal{H}}_1 \le x \mid \bar{\mathcal{H}}_1 > 0)\mathsf{P}(\bar{\mathcal{H}}_1 > 0) \\
&= \zeta + \mathsf{P}(\mathcal{H}_1 \le x \mid \bar{\mathcal{H}}_1 > 0)(1-\zeta).
\end{aligned} \tag{3.1.3}$$

However,

$$\mathsf{P}(\mathcal{H}_1 \le x, \bar{\mathcal{H}}_1 = 0)$$

$$= \mathsf{P}(\mathcal{T}_1 < \infty, \mathcal{H}_1 \le x, \bar{\mathcal{H}}_1 = 0)$$

$$= \mathsf{P}(S_{\bar{\mathcal{T}}_1} = 0, \mathcal{T}_1 < \infty, S_{\mathcal{T}_1} \le x)$$

$$= \sum_{n=1}^{\infty} \mathsf{P}(\bar{\mathcal{T}}_1 = n, S_{\bar{\mathcal{T}}_1} = 0, \mathcal{T}_1 < \infty, S_{\mathcal{T}_1} \le x)$$

$$= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \mathsf{P}(\bar{\mathcal{T}}_1 = n, S_{\bar{\mathcal{T}}_1} = 0, \mathcal{T}_1 = n+k, S_{\mathcal{T}_1} \le x)$$

$$= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \mathsf{P}(\mathcal{T}_1 = n+k, S_{\mathcal{T}_1} \le x \mid \bar{\mathcal{T}}_1 = n, S_{\bar{\mathcal{T}}_1} = 0)\mathsf{P}(\bar{\mathcal{T}}_1 = n, S_{\bar{\mathcal{T}}_1} = 0). \tag{3.1.4}$$

By the properties of Markov Chain, (3.1.4) becomes

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \mathsf{P}(\mathcal{T}_1 = k, S_{\mathcal{T}_1} \le x)\mathsf{P}(\bar{\mathcal{T}}_1 = n, S_{\bar{\mathcal{T}}_1} = 0)$$

$$= \sum_{n=1}^{\infty} \mathsf{P}(\mathcal{T}_1 < \infty, S_{\mathcal{T}_1} \le x)\mathsf{P}(\bar{\mathcal{T}}_1 = n, S_{\bar{\mathcal{T}}_1} = 0)$$

$$= \mathsf{P}(\mathcal{T}_1 < \infty, S_{\mathcal{T}_1} \le x)\mathsf{P}(\bar{\mathcal{T}}_1 < \infty, S_{\bar{\mathcal{T}}_1} = 0)$$

$$= \mathsf{P}(\mathcal{H}_1 \le x)\mathsf{P}(\bar{\mathcal{H}}_1 = 0).$$

Hence we can say that $\{\mathcal{H}_1 \le x\}$ is independent from $\{\bar{\mathcal{H}}_1 = 0\}$. This also implies that $\{\mathcal{H}_1 \le x\}$ is independent from $\{\bar{\mathcal{H}}_1 > 0\}$. Therefore for $x > 0$, (3.1.3) becomes

$$\begin{aligned} \bar{H}(x) &= \zeta + \mathsf{P}(\mathcal{H}_1 \le x)(1-\zeta) \\ &= \zeta\psi_0(x) + (1-\zeta)H(x). \end{aligned}$$

□

We shall denote by $\bar{\psi}$ the renewal measure for the weak ascending ladder height process and define it in the same way as for ψ, namely,

$$\bar{\psi} = \sum_{n=0}^{\infty} \bar{H}^{n*}, \quad \text{where} \quad \bar{H}^{0*} = \psi_0.$$

The weak and strict descending ladder variables are defined by reversing the inequality in the definitions (3.1.3) and (3.1.1). They are denoted by adding a superscript minus, for example, $\bar{H}^-$ and H^- respectively.

Proposition 3.1.4. *There holds*

$$\bar{\psi} = \frac{1}{1-\zeta}\psi.$$

Proof. Suppose that there are N_0 weak ladder points whose values are equal to 0 in interval $[0, \mathcal{H}_1)$ and there are N_1 weak ladder points whose record values are equal to $\mathcal{H}_1$ in interval $[\mathcal{H}_1, \mathcal{H}_2)$ and so on. Then we notice that N_i are i.i.d. and have a geometric distribution with parameter $1-\zeta$. Thus $\mathsf{E}(N_i) = \frac{\zeta}{1-\zeta}$. Let $\bar{N}_{[0,x]}$ be the number of weak ascending ladder points in $[0, x]$, $N_{(0,x]}$ be the number of strict ascending ladder points in $(0,x]$ and $\mathcal{T}_n$ be the last epoch where $S_{\mathcal{T}_n} \leq x$. Then for $x \geq 0$, we have

$$\bar{N}_{[0,x]} = N_0 + (1+N_1) + (1+N_2) + \cdots + (1+N_n)\,.$$

Thus

$$\begin{aligned}
\mathsf{E}(\bar{N}_{[0,x]}) &= \mathsf{E}(N_0) + [1+\mathsf{E}(N_1)]\,\mathsf{E}\left(N_{(0,x]}\right) \\
&= \frac{\zeta}{1-\zeta} + \frac{1}{1-\zeta}\,[\psi(x)-1] \\
&= \frac{\zeta}{1-\zeta} - \frac{1}{1-\zeta} + \frac{1}{1-\zeta}\psi(x) \\
&= \frac{1}{1-\zeta}\psi(x) - 1.
\end{aligned}$$

Hence

$$1 + \mathsf{E}(\bar{N}_{[0,x]}) = \bar{\psi}(x) = \frac{1}{1-\zeta}\psi(x). \qquad \square$$

3.2. Duality and Types of Random Walks

In this section we present a duality lemma. It is essential to prove the associated theorems that follow.

Lemma 3.2.1. *(Duality) For every finite interval* $I \subset (0,\infty)$, $\psi(I)$ *is equal to*

(a) the expected number of ladder points in I.

(b) the expected number of visits $S_n \in I$ *such that* $S_k > 0$ *for* $k = 1, 2, \ldots, n$.

Proof. For a fixed n, let

$$X_1^* = X_n, \ldots, X_n^* = X_1,$$

$$S_k^* = X_1^* + \cdots + X_k^*.$$

Since the joint distribution of $(S_0, \dots, S_n)$ and that of $(S_0^*, \dots, S_n^*)$ are the same, the event A in $(S_0, \dots, S_n)$ is mapped to the event A^* in $(S_0^*, \dots, S_n^*)$ with the same probability by a correspondence $X_k \to X_k^*$. Hence the event

$$S_n > S_0, \dots, S_n > S_{n-1},$$

can be mapped with equal probability to the event

$$S_n^* > S_0^*, \dots, S_n^* > S_{n-1}^*. \tag{3.2.1}$$

But the event $S_n^* > S_{n-k}^*$ implies that $S_k > 0$, because

$$S_k^* = S_n - S_{n-k}.$$

Thus the event (3.2.1) can be rewritten as

$$S_1 > 0, \dots, S_n > 0.$$

Hence, for every finite interval $I \subset (0, \infty)$,

$$\mathsf{P}(S_n > S_j, j = 0, \dots, n-1, S_n \in I) = \mathsf{P}(S_j > 0, j = 1, \dots, n, S_n \in I). \tag{3.2.2}$$

Summed over n, the left side is the expected number of ladder points in I and the right side the expected number of visits $S_n \in I$ such that $S_k > 0$, where $k = 1, 2, \dots, n$. □

Theorem 3.2.1. *There exist only two types of random walks.*
(i) The oscillating type: S_n oscillates with probability 1 between $-\infty$ and ∞, and $\mathsf{E}(\mathcal{T}_1) = \infty$, $\mathsf{E}(\mathcal{T}_1^-) = \infty$.
(ii) Drifts to $-\infty$ or ∞.

PROOF. When $>$ is replaced by $\geq$, Eq.(3.2.2) still holds. Then, for $I = [0, \infty)$,

$$\mathsf{P}(S_n \geq S_k, \quad 0 \leq k \leq n) = \mathsf{P}(S_k \geq 0, \quad 0 \leq k \leq n). \tag{3.2.3}$$

The left side of Eq.(3.2.3) is equal to

$$\mathsf{P}\left\{(n, S_n) \text{ is a weak ascending ladder point with ordinate in } [0, \infty)\right\}.$$

Summing over n, we obtain

$$1+\sum_{n=1}^{\infty} \mathsf{P}\left\{(n, S_n) \text{ is a weak ascending ladder point with ordinate in } [0,\infty)\right\}$$

$$= 1 + \text{the expected number of weak ascending ladder points in } [0,\infty)$$

$$= \bar{\psi}(\infty) = \frac{1}{1-\zeta}\psi(\infty)$$

$$= \frac{1}{(1-\zeta)(1-H(\infty))}. \tag{3.2.4}$$

On the other hand, the right side of Eq.(3.2.3) is equal to

$$\mathsf{P}(\mathcal{T}_1^- > n).$$

If $\mathcal{T}_1^-$ is proper, summing over n, we obtain

$$\sum_{n=0}^{\infty} \mathsf{P}(\mathcal{T}_1^- > n) = \mathsf{E}(\mathcal{T}_1^-). \tag{3.2.5}$$

Therefore, from two Eqs. (3.2.4) and (3.2.5), we can write

$$\mathsf{E}(\mathcal{T}_1^-) = \frac{1}{(1-\zeta)(1-H(\infty))}. \tag{3.2.6}$$

Thus

$$\mathsf{E}(\mathcal{T}_1^-) < \infty \Longleftrightarrow H(\infty) < 1 \Longleftrightarrow \mathcal{T}_1 \text{ is defective.} \tag{3.2.7}$$

If $\mathcal{T}_1$ is proper, for the same reason, we can also write

$$\mathsf{E}(\mathcal{T}_1) = \psi^-(\infty) = \frac{1}{1-H^-(\infty)}. \tag{3.2.8}$$

Thus

$$\mathsf{E}(\mathcal{T}_1) < \infty \Longleftrightarrow H^-(\infty) < 1 \Longleftrightarrow \mathcal{T}_1^- \text{ is defective.} \tag{3.2.9}$$

We divide the cases depending on whether $\mathsf{E}(\mathcal{T}_1^-) < \infty$ or $\mathsf{E}(\mathcal{T}_1) < \infty$, or both $\mathsf{E}(\mathcal{T}_1^-) = \infty$ and $\mathsf{E}(\mathcal{T}_1) = \infty$. In the first case, let us suppose that $\mathsf{E}(\mathcal{T}_1^-) < \infty$.

Then, by Eq.(3.2.7), $\mathcal{T}_1$ is defective, which implies P(the number of pair $(\mathcal{T}_k,\mathcal{H}_k)$ is finite) $= 1$. Therefore the random walk drifts to $-\infty$. Similarly if we suppose that $\mathsf{E}(\mathcal{T}_1) < \infty$, then $\mathcal{T}_1^-$ is defective and the random walk drifts to ∞. The second case is obviously the case where S_n oscillates with probability 1 between $-\infty$ and ∞. □

Theorem 3.2.2. *(i) If* $\mathsf{E}(X_1) = 0$, *then* $\mathcal{H}_1$ *and* $\mathcal{T}_1$ *are proper and* $\mathsf{E}(\mathcal{T}_1) = \infty$.
(ii) If $\mathsf{E}(X_1)$ *is finite and positive, then* $\mathcal{H}_1$ *and* $\mathcal{T}_1$ *are proper, have finite expectations, and*

$$\mathsf{E}(\mathcal{H}_1) = \mathsf{E}(\mathcal{T}_1)\,\mathsf{E}(X_1).$$

The random walk drifts to ∞.
(iii) If $\mathsf{E}(X_1) = \infty$, *then* $\mathsf{E}(\mathcal{H}_1) = \infty$ *and the random walk drifts to* ∞.
(iv) Otherwise either the random walk drifts to $-\infty$ *(in case either* $\mathcal{H}_1$ *is defective, or* $\mathcal{H}_1$ *is proper and* $\mathsf{E}(\mathcal{H}_1) < \infty$*) or else* $\mathsf{E}(\mathcal{H}_1) = \infty$.

PROOF. (i) In view of Theorem 3.2.1, let us suppose that there exists only two types of random walks and let

$$n_k = \mathcal{T}_1 + \mathcal{T}_2 + \cdots + \mathcal{T}_k.$$

Then $n_k \to \infty$ as $k \to \infty$. By the strong law of large numbers,

$$\frac{S_{n_k}}{n_k} = \frac{(\mathcal{H}_1 + \cdots + \mathcal{H}_k)/k}{(\mathcal{T}_1 + \mathcal{T}_2 + \cdots + \mathcal{T}_k)/k} \to 0, \qquad \text{as } k \to \infty.$$

Since $(\mathcal{H}_1 + \cdots + \mathcal{H}_k)/k$ converges to a positive number,

$$(\mathcal{T}_1 + \mathcal{T}_2 + \cdots + \mathcal{T}_k)/k \to \infty, \quad \text{as } k \to \infty.$$

Again, according to the strong law of large numbers, this implies $\mathcal{T}_1$ is proper and $\mathsf{E}(\mathcal{T}_1) = \infty$. Thus $\mathcal{H}_1$ is also proper.

(ii) If $0 < \mathsf{E}(X_1) < \infty$, then the random walk drifts to ∞. (As a reference, see Loève(1977, vol.1, p.384).) This implies that $\mathcal{T}_1^-$ is defective. Therefore, by

Eq.(3.2.9), $\mathsf{E}(\mathcal{T}_1) < \infty$, $\mathcal{T}_1$ is proper, and so is $\mathcal{H}_1$. By the strong law of large numbers, we have

$$\frac{S_{n_k}}{n_k} \to \mathsf{E}(X_1), \quad \text{as} \quad n_k \to \infty.$$

And we know that as $k \to \infty$,

$$\frac{S_{n_k}}{n_k} \to \frac{\mathsf{E}(\mathcal{H}_1)}{\mathsf{E}(\mathcal{T}_1)}.$$

Therefore we obtain

$$\mathsf{E}\left(\mathcal{H}_1\right) = \mathsf{E}\left(\mathcal{T}_1\right)\mathsf{E}(X_1).$$

(iii) In this case, the same argument as in proof (ii) can be done. Hence we have $\mathsf{E}(\mathcal{H}_1) = \infty$ since $\mathsf{E}(X_1) = \infty$.

(iv) On one hand, if $\mathcal{H}_1$ is improper, then the number of $\mathcal{T}_k$ is finite and $\mathsf{P}(\mathcal{T}_1 = \infty) > 0$. From Eq.(3.2.6),

$$\mathsf{E}\left(\mathcal{T}_1^-\right) = \frac{1}{(1-\zeta)\,\mathsf{P}\left(\mathcal{T}_1 = \infty\right)} < \infty.$$

Thus, the first case of Theorem 3.2.1 is impossible, namely, the random walk does not oscillate between $-\infty$ and ∞. Therefore the random walk drifts to ∞ or $-\infty$. If we assume that it drifts to ∞, the number of $\mathcal{T}_k$ is infinite. However, this contradicts the fact that the number of $\mathcal{T}_k$ is finite. Hence the random walk drifts to $-\infty$.

On the other hand, if $\mathcal{H}_1$ is proper, we can say that for $x > 0$,

$$\mathsf{P}(\mathcal{H}_1 > x) \geq \mathsf{P}(X_1 > x). \tag{3.2.10}$$

By integrating this with respect to x from 0 to ∞, we obtain

$$\mathsf{E}(\mathcal{H}_1) \geq \mathsf{E}(X_1).$$

If $\mathsf{E}\left(\mathcal{H}_1\right) < \infty$, then $\mathsf{E}(X_1) < \infty$. In the case $0 \leq \mathsf{E}(X_1) < \infty$, one is referred to the proof of (i) and (ii). Therefore, if $\mathsf{E}(X_1) < 0$, the random walk drifts to $-\infty$. This can also be confirmed by Loève(1977, vol.1, p.384). □

Corollary 3.2.1. *If both* $\mathcal{H}_1$ *and* $\mathcal{H}_1^-$ *are proper and have finite expectations, then* $\mathsf{E}(X_1) = 0$.

PROOF. From Eq.(3.2.10), we can derive an analogous inequality for $x < 0$. Thus if $\mathcal{H}_1$ and $\mathcal{H}_1^-$ are proper and have finite expectations, then $\mathsf{P}(\ |X_1| > x)$ is integrable and consequently $\mathsf{E}(X_1)$ exists. However, if $\mathsf{E}(X_1) > 0$, then the random walk drifts to ∞ and $\mathcal{H}_1^-$ is improper. If $\mathsf{E}(X_1) < 0$, then the random walk drifts to $-\infty$ and $\mathcal{H}_1$ is improper. Therefore both cases contradict the fact that both $\mathcal{H}_1$ and $\mathcal{H}_1^-$ are proper. Hence $\mathsf{E}(X_1) = 0$. □

3.3. Distribution of Ladder Heights

3.3.1. Derivation of formula

Consider a modified random walk which terminates at the first entry into $(-\infty, 0]$. For an arbitrary interval I and $n = 1, 2, \ldots$, we denote by $\psi_n\{I\}$ the defective probability that the position of this modified random walk at epoch n is within I, namely,

$$\psi_n\{I\} = \mathsf{P}\left(S_1 > 0, \ldots, S_n > 0, S_n \in I\right),$$

where

$$\psi_n\{(-\infty, 0]\} = 0.$$

From Eq.(3.2.2), we can write

$$\psi_n\{I\} = \mathsf{P}(S_n > S_j \text{ for } j = 0, \ldots, n-1 \text{ and } S_n \in I).$$

In other words, $\psi_n\{I\}$ represents the probability that (n, S_n) be a strict ascending ladder point with $S_n \in I$. Summed over n, it becomes

$$\sum_{n=0}^{\infty} \psi_n\{I\} = \psi\{I\}, \tag{3.3.1}$$

where $\psi\{I\}$ is the expected number of strict ascending ladder points with the ordinate I. As proved in Feller(1971, p.185), $\psi(x) = \psi\{(-\infty, x]\} < \infty$ for all x. Therefore the series in Eq.(3.3.1) converges for every bounded interval I.

Consider the distribution of the weak descending ladder process denoted by $\bar{H}_1^-$. For typographical convenience, we denote it by ρ from now on. Also we denote by $\rho_n\{I\}$ the probability that the first entry to $(-\infty, 0]$ takes place at epoch n and within the interval I, namely,

$$\rho_n\{I\} = \mathsf{P}(S_1 > 0, \dots, S_{n-1} > 0, S_n \leq 0, S_n \in I),$$

where

$$\rho_n\{(0, \infty)\} = 0.$$

By summing over n, we obtain

$$\sum_{n=1}^{\infty} \rho_n\{I\} = \rho\{I\}, \tag{3.3.2}$$

which represents the possibly defective distribution of the point of the first entry. Let

$$A_n = (S_1 > 0, \dots, S_{n-1} > 0, S_n \leq 0, S_n \in I).$$

Then

$$A_n \cap A_p = \emptyset, \quad \text{where } n \neq p.$$

Therefore

$$\sum_{n=1}^{\infty} p(A_n) \leq 1.$$

Hence the series in Eq.(3.3.2) converges. With the notations introduced so far, we can derive recurrence relations for ψ_n and ρ_n. For $I \subset R_- = (-\infty, 0]$,

$$\begin{aligned}\rho_{n+1}\{I\} &= \mathsf{P}\left(S_1 > 0, \ldots, S_n > 0, S_{n+1} \leq 0, S_{n+1} \in I\right) \\ &= \int_{0-}^{\infty} \mathsf{P}\left(S_1 > 0, \ldots, S_n > 0, S_n \in dy, S_{n+1} \leq 0, S_{n+1} \in I\right).\end{aligned}$$

Since S_n and S_{n+1} are independent,

$$\begin{aligned}\rho_{n+1}\{I\} &= \int_{0-}^{\infty} \mathsf{P}(S_1 > 0, \ldots, S_n > 0, S_n \in dy)\mathsf{P}(S_{n+1} - y \in I - y) \\ &= \int_{0-}^{\infty} \mathsf{P}(S_1 > 0, \ldots, S_n > 0, S_n \in dy) F(I - y) \\ &= \int_{0-}^{\infty} \psi_n\{dy\} F(I - y).\end{aligned}$$

For $I \subset R_+ = (0, \infty)$,

$$\begin{aligned}\psi_{n+1}\{I\} &= \mathsf{P}(S_1 > 0, \ldots, S_{n+1} > 0, S_{n+1} \in I) \\ &= \int_{0-}^{\infty} \mathsf{P}(S_1 > 0, \ldots, S_n > 0, S_n \in dy, S_{n+1} \in I).\end{aligned}$$

Since S_n and S_{n+1} are independent,

$$\begin{aligned}\psi_{n+1}\{I\} &= \int_{0-}^{\infty} \mathsf{P}(S_1 > 0, \ldots, S_n > 0, S_n \in dy)\mathsf{P}(S_{n+1} - y \in I - y) \\ &= \int_{0-}^{\infty} \psi_n\{dy\} F(I - y).\end{aligned}$$

By summing over n, we obtain

$$\rho\{I\} = \int_{0-}^{\infty} \psi\{dy\} F(I-y), \qquad \text{if } I \subset (-\infty, 0], \tag{3.3.3}$$

$$\psi\{I\} = \int_{0-}^{\infty} \psi\{dy\} F(I-y), \qquad \text{if } I \subset (0, \infty). \tag{3.3.4}$$

Let $\rho(x) = \rho\{(-\infty, x]\}$ and $\psi(x) = \psi\{(-\infty, x]\}$, then Eqs. (3.3.3) and (3.3.4) are

$$\rho(x) = \int_{0-}^{\infty} \psi\{dy\} F(x-y), \qquad x \leq 0 \tag{3.3.5}$$

$$\begin{aligned}
\psi(x) &= 1 + \psi\{(0, x)\}, \qquad x > 0 \\
&= 1 + \int_{0-}^{\infty} \psi\{dy\} [F(x-y) - F(-y)] \\
&= 1 - \rho(0) + \int_{0-}^{\infty} \psi\{dy\} F(x-y), \qquad x \geq 0.
\end{aligned} \tag{3.3.6}$$

Proposition 3.3.1.

$$\begin{aligned}
\rho + \psi &= \psi_0 + \psi * F \\
&= \psi_0 + \sum_{n=0}^{\infty} \psi_n * F.
\end{aligned} \tag{3.3.7}$$

PROOF. For $x < 0$, the left side of Eq.(3.3.7) becomes

$$\rho\{(-\infty, x]\} + \psi\{(-\infty, x]\} = \rho(x) + 0.$$

Furthermore, the right side of Eq.(3.3.7) can be written as

$$\psi_0\{(-\infty, x]\} + \psi * F(x) = 0 + \int_{0-}^{\infty} \psi\{dy\} F(x-y).$$

Then, by Eq.(3.3.5), it is possible to show that the proposition is true.

For $x \geq 0$, the left side of Eq.(3.3.7) becomes

$$\rho\{(-\infty,x]\} + \psi\{(-\infty,x]\} = \rho\{(-\infty,0]\} + \psi\{(-\infty,x]\} = \rho(0) + \psi(x).$$

Furthermore, the right side of Eq.(3.3.7) may be expressed by

$$\psi_0\{(-\infty,x]\} + \psi * F(x) = 1 + \int_{0-}^{\infty} \psi\{dy\}\, F(x-y).$$

Then, by Eq.(3.3.6), this proposition is also shown to be true. □

3.3.2. Wiener-Hopf Type Factorization

From Eq.(3.1.2), we write

$$\psi * H = \left(\sum_{n=0}^{\infty} H^{n*}\right) * H.$$

Since all the terms are positive,

$$\begin{aligned} \psi * H &= \sum_{n=0}^{\infty} (H^{n*} * H) \\ &= \sum_{n=1}^{\infty} H^{n*}. \end{aligned}$$

Therefore

$$\begin{aligned} \psi_0 + \psi * H &= \psi_0 + \sum_{n=1}^{\infty} H^{n*} = H^{0*} + \sum_{n=1}^{\infty} H^{n*} \\ &= \sum_{n=0}^{\infty} H^{n*} = \psi. \end{aligned} \tag{3.3.8}$$

By convolving Eq.(3.3.7) with H, we obtain

$$\rho * H + \psi * H = \psi_0 * H + \psi * H * F.$$

By Eq.(3.3.8),

$$\iff \quad \rho * H + \psi - \psi_0 = H + (\psi - \psi_0) * F$$

$$\iff \quad \rho * H + \psi - \psi_0 = H + \psi * F - F. \tag{3.3.9}$$

On subtracting Eq.(3.3.9) from Eq.(3.3.7), we obtain

$$\rho - \rho * H \quad = \quad F - H$$

$$\iff \quad F = H + \rho - H * \rho. \tag{3.3.10}$$

Since an analogue to Proposition 3.1.3 is

$$\rho = \zeta\psi_0 + (1 - \zeta)\, H^-,$$

where ζ is defined as in Proposition 3.1.3 with the inequalities reversed, Equation (3.3.10) can be expressed as

$$\begin{aligned}
F &= H + \zeta\psi_0 + (1-\zeta)\, H^- - H * \left[\zeta\psi_0 + (1-\zeta)\, H^-\right] \\
&= \psi_0 + H - (1-\zeta)\,\psi_0 + (1-\zeta)\, H^- - \zeta H * \psi_0 - (1-\zeta)\, H * H^- \\
&= \psi_0 + (1-\zeta)\, H - (1-\zeta)\,\psi_0 + (1-\zeta)\, H^- - (1-\zeta)\, H * H^- \\
&= \psi_0 + (1-\zeta)\left[H - \psi_0 + H^- - H * H^-\right] \\
&= \psi_0 - (1-\zeta)\left[(\psi_0 - H) * \left(\psi_0 - H^-\right)\right].
\end{aligned}$$

3.3.3. Distribution of Maxima

Let us assume that the distribution F has a density f and a negative expectation. Thus the random walk drifts to $-\infty$ and we can define a finite valued random variable M as

$$M = \max\,[0, S_1, S_2, \dots]\,. \tag{3.3.11}$$

Then for $x \geq 0$, by conditioning on X_1, the probability distribution function of M, that is, $M(x)$ is given by

$$\begin{aligned} M(x) &= \mathsf{P}(M \leq x) \\ &= \int_{-\infty}^{x} M(x-y) f(y) dy. \end{aligned} \tag{3.3.12}$$

With $s = x - y$, Eq.(3.3.12) becomes

$$M(x) = \int_{0}^{\infty} M(s) f(x-s) ds. \tag{3.3.13}$$

On the other hand, given that the n^{th} ladder point occurred, the probability for being the last one equals $1 - H(\infty)$. And we know that

$$H^{n*}(x) = \mathsf{P}(\mathcal{H}_1 + \cdots + \mathcal{H}_n \leq x).$$

Thus we can derive another equation as follows:

$$M(x) = [1 - H(\infty)] \sum_{n=0}^{\infty} H^{n*}(x) \tag{3.3.14}$$

$$= [1 - H(\infty)]\, \psi(x), \tag{3.3.15}$$

where $\psi(x)$ satisfies Eq.(3.3.6) with $\rho(0) = 1$, because

$$\begin{aligned} \rho(0) &= \rho\{(-\infty, 0]\} \\ &= \text{probability that the first entry to } (-\infty, 0] \text{ takes place within } (-\infty, 0] \\ &= 1. \end{aligned}$$

Therefore, for $x \geq 0$, by replacing $\psi(x)$, Eq.(3.3.15) becomes

$$\begin{aligned} M(x) &= [1 - H(\infty)] \int_{0-}^{\infty} \psi\{dy\} F(x-y) \\ &= \int_{0-}^{\infty} F(x-y)\, d\{[1-H(\infty)]\, \psi(y)\} \\ &= \int_{0-}^{\infty} F(x-y)\, dM(y) \\ &= [F(x-y)\, M(y)]_{0-}^{\infty} + \int_{0-}^{\infty} M(y) d_y F(x-y) \\ &= \lim_{y\to\infty} [F(x-y)\, M(y)] - F(x-0^-)M(0^-) + \int_{0-}^{\infty} M(y) f(x-y) dy. \end{aligned}$$

Since $\lim_{y\to\infty} F(x-y) = 0$ and $M(0^-) = 0$,

$$M(x) = \int_{0}^{\infty} M(y) f(x-y) dy, \qquad \text{for } x \geq 0.$$

Hence we can conclude that the two equations (3.3.13) and (3.3.15) are identical. Eq.(3.3.13) is said to be a standard form of the Wiener-Hopf integral equation.

3.3.4. Example

Suppose that F has an expectation $\mu \neq 0$ and that there exists a number $\kappa \neq 0$ such that

$$\int_{-\infty}^{\infty} e^{\kappa y} F\{dy\} = 1.$$

For an arbitrary measure r, we define a new modified measure by

$${}^a\gamma\{dy\} = e^{\kappa y} r\{dy\}.$$

The measure ${}^aF$ is said to be associated with F, and ${}^aF$ is a proper probability distribution. Then we say that the random walks generated by ${}^aF$ and F are associated with each other.

Let

$$\phi(t) = \int_{-\infty}^{\infty} e^{yt} F\{dy\}, \qquad \text{for } 0 \le t \le \kappa.$$

Since $\phi''(t) > 0$, ϕ is a convex function. Thus $\phi(0) = \phi(\kappa)$ implies that $\phi'(0)$ and $\phi'(\kappa)$ have opposite signs. Because $\phi'(0) = \mathsf{E}(Y)$ if F is the distribution function of Y, and $\phi'(\kappa) = \mathsf{E}({}^aY)$ if ${}^aF$ is the distribution function of ${}^aY$, the random walks induced by ${}^aF$ and F have drifts in opposite directions from each other. Hence it is useful when we want to translate the facts about a random walk with $\mu < 0$ into results for a random walk with $\mu > 0$ and vice versa.

3.4. Application

3.4.1. Khintchine-Pollaczek formula

When we mention the distribution M shown in Section 3.3.3 in connection with the ruin problems in compound Poisson process, the underlying distribution is of the form

$$F = A * B, \tag{3.4.1}$$

where A is the probability distribution function of waiting time A_j, concentrated on $(0, \infty)$ and B is that of claim B_j, concentrated on $(-\infty, 0)$. This means that the summands of the random walk can be expressed as

$$X_j = A_j + B_j, \quad \text{for } \ j = 1, 2, \ldots$$

Assume that the right tail of F is exponential, that is,

$$F(x) = 1 - pe^{-\alpha x}, \qquad \text{for } x > 0, \tag{3.4.2}$$

where p is constant. If F is of the form of (3.4.1) with

$$A(x) = 1 - e^{-\alpha x}, \quad \text{for } x > 0, \tag{3.4.3}$$

then

$$\begin{aligned}\int_{-\infty}^{0} A(x-y)B\{dy\} &= \int_{-\infty}^{0} \left[1-e^{-\alpha(x-y)}\right] B\{dy\} \\ &= \int_{-\infty}^{0} B\{dy\} - e^{-\alpha x}\int_{-\infty}^{0} e^{\alpha y} B\{dy\} \\ &= 1 - e^{-\alpha x}\int_{-\infty}^{0} e^{\alpha y} B\{dy\} \\ &= 1 - pe^{-\alpha x}.\end{aligned}$$

Thus

$$p = \int_{-\infty}^{0} e^{\alpha y} B\{dy\}.$$

That is, the condition of the expression (3.4.2) holds.

As we noted in the previous chapter, we generally assume that, in the risk model, a premium collection c exceeds the expected claim payments per unit time. This assumption corresponds to the random walk with $\mu > 0$. Thus, the classical model in the previous chapter corresponds to the model in the general framework that we have seen so far in this chapter, with the condition that F in (3.4.2) has a positive expectation. Let us suppose that F is of the form expressed by Eq.(3.4.1) with A given by Eq.(3.4.3), B having a finite expectation $-b$ and $\mu = \frac{1}{\alpha} - b > 0$. We also suppose that F is continuous. As shown in Feller(1971, p.405), the ladder height distribution H has a density proportional to $e^{-\alpha x}$ (This is a consequence of our Eq.(3.3.5) with the two half-axes interchanged). Since $\mu > 0$, H is proper and the random walk drifts to ∞. Hence for $x > 0$,

$$H(x) = 1 - e^{-\alpha x}.$$

Considering the analogue to Eq.(3.3.14)(again interchanging the roles of the two half-axes), H in the expression (3.3.14) should be replaced by ρ in (3.3.5) which is equivalent to

$$\rho(x) = F(x) + \alpha \int_{-\infty}^{x} F(s)ds, \quad \text{for} \quad x < 0, \tag{3.4.4}$$

and also M in (3.3.14) by $m = \min[0, S_1, S_2, \ldots]$. That is, for $x < 0$,

$$\mathsf{P}(m \le x) = [1 - \rho(\infty)] \sum_{0}^{\infty} \rho^{n*}(x), \qquad \text{where } m = \min[0, S_1, S_2, \ldots]. \tag{3.4.5}$$

According to (ii) in Theorem 3.2.2 and Eq.(3.2.8),

$$\begin{aligned} \mathsf{E}(\mathcal{T}_1) &= \frac{\mathsf{E}(\mathcal{H}_1)}{\mathsf{E}(X_1)} \\ \\ \Longleftrightarrow \quad \frac{1}{1 - H^{-}(\infty)} &= \frac{1/\alpha}{\mu}. \end{aligned} \tag{3.4.6}$$

Since F is continuous, $H^{-}(\infty) = \rho(\infty)$. Thus Eq.(3.4.6) implies

$$\rho(\infty) = 1 - \alpha\mu.$$

Since $\mu = \frac{1}{\alpha} - b$,

$$\rho(\infty) = \alpha b.$$

Because $\rho\{(0, \infty)\} = 0$, we note that $\rho(\infty) = \rho(0)$. Hence Eq.(3.4.5) becomes

$$\mathsf{P}(m \le x) = [1 - \alpha b] \sum_{0}^{\infty} \rho^{n*}(x), \qquad \text{for } x < 0. \tag{3.4.7}$$

This is the probability that the minimum position of the random walk is less than or equal to a certain level. Since ruin means that a random walk crossing a barrier, the above equation is another expression of ruin probability. Thus we can say that calculating ruin probability is the same as finding the distribution of $m = \min[0, S_1, S_2, \ldots]$. Using the notations introduced earlier in Chapter 2, we can rewrite (3.4.7) as follows:

$$\psi(u) = \mathsf{P}(m \le -u),$$

or

$$\psi(-x) = C\sum_{0}^{\infty} \rho^{n*}(x), \qquad \text{for } x < 0, \tag{3.4.8}$$

where C is positive constant. Here we can actually notice that Eq.(3.4.8) has the similar pattern as Eq.(2.2.10) except that ρ is defective for $\rho(0) = \alpha b < 1$. But L_1 is proper since it has a conditional distribution, given that $\tau_1 < \infty$. And the defective part of $\rho, 1 - \alpha b$ can be compared to the point mass, $1 - \psi(0)$ of L at the origin. Therefore Eq.(3.4.8) can be regarded as an alternative expression to Eq.(2.2.10) by means of ladder variables.

3.4.2. Asymptotic Estimate

To find out the tail behavior of the distribution of M in (3.3.15), let us get the estimates for the tail of the distribution of M. As we mentioned in Section 3.3.4., by means of the associated measure, ${}^a\psi\{dx\} = e^{\kappa x}\psi\{dx\}$, Eq.(3.3.15) can be rewritten as follows:

$$M\{dx\} = [1 - H(\infty)]\, e^{-\kappa x} \cdot {}^a\psi\{dx\},$$

which is equivalent to

$$dM(x) = [1 - H(\infty)]\, e^{-\kappa x} d\, {}^a\psi(x). \tag{3.4.9}$$

Integrating from t to ∞, the left side of (3.4.9)is

$$\int_t^\infty dM(x) = 1 - M(t).$$

Suppose that $\beta < \infty$, then by the Renewal Theorem(see Feller(1971, p.360)),

$${}^a\psi(dx) \sim \frac{1}{\beta}, \quad \text{as } x \to \infty.$$

Therefore integrating from t to ∞, the right side of (3.4.9) becomes as $t \to \infty$

$$[1 - H(\infty)] \int_t^\infty e^{-\kappa x} \cdot d\,^a\psi(x) \;\sim\; [1 - H(\infty)] \int_t^\infty e^{-\kappa x} \cdot \frac{1}{\beta} dx$$

$$= [1 - H(\infty)] \frac{e^{-\kappa t}}{\beta \kappa}, \qquad (3.4.10)$$

where

$$\beta = \int_0^\infty x e^{\kappa x} H\{dx\}. \qquad (3.4.11)$$

3.4.3. Cramér's estimate for probabilities of ruin

Let us apply the preceding results to the ruin problem. Suppose that $\mu > 0$. To get the asymptotic estimate about the distribution of m in (3.4.5), we first integrate ρ in (3.4.4) by parts. This gives us

$$\rho(x) = \alpha \int_{-\infty}^x B(y) dy. \qquad (3.4.12)$$

Then this ρ is substituted for H in (3.4.11), and the negative κ for positive κ since $\mu > 0$. Thus β in (3.4.11) becomes

$$\beta = \alpha \int_{-\infty}^0 e^{-|\kappa| y} \, |y| \, B(y) dy.$$

Hence the analogue of (3.4.10) is of the form

$$\mathsf{P}(m \le x) \sim \frac{1 - \alpha b}{|\kappa| \, \beta} e^{|\kappa| x}, \qquad \text{as} \quad x \to -\infty. \qquad (3.4.13)$$

The above formula is called Cramér's estimate for the probability of ruin. Cramér's original derivation used Wiener-Hopf techniques. As a reference, see Cramér(1954).

3.5. Interpretation of the classical model in the probabilistic approach

Let us consider the classical model case, where the distribution of waiting times is exponential with parameter α, and then calculate the ruin probability of this case using probabilistic model with an assumption that the distribution of claims is exponential with parameter β. In the notation of Section 3.4.1 in this chapter, it is the case where

$$f_{A_j} = \alpha e^{-\alpha x}, \quad \text{for } x > 0, \quad \alpha > 0,$$

$$f_{B_j} = \beta e^{\beta x}, \quad \text{for } x < 0, \quad \beta > 0,$$

and

$$A(x) = 1 - e^{-\alpha x}, \quad \text{for } x > 0, \quad \alpha > 0,$$

$$B(x) = e^{\beta x}, \quad \text{for } x < 0, \quad \beta > 0.$$

Thus we have

$$\mu = \mathsf{E}(A_j + B_j) = \frac{1}{\alpha} - \frac{1}{\beta}. \tag{3.5.1}$$

Since $\mu > 0$, let us suppose that $\beta - \alpha > 0$. Since we have Eq.(3.4.12), for $x < 0$, we obtain

$$\rho(x) = \alpha \int_{-\infty}^{x} e^{\beta y} dy = \frac{\alpha}{\beta} e^{\beta x}.$$

Moreover

$$\rho'(x) = \frac{\alpha}{\beta} \cdot \beta e^{\beta x}, \quad \text{for } x < 0.$$

Since this is an exponential density function of ρ, with parameter β, multiplied by constant α/β and concentrated on $(-\infty, 0)$, let us first consider the same exponential density function $g(x)$, but concentrated on $(0, \infty)$, namely,

$$g(x) = \frac{\alpha}{\beta} \cdot \beta e^{-\beta x} \cdot 1_{\{x>0\}}.$$

Then its Laplace transform(see the next chapter) is

$$\widetilde{g}(s) = \frac{\alpha}{\beta} \cdot \frac{\beta}{s + \beta}.$$

Furthermore Laplace transform of the n^{th} convolution of $g(x)$ is

$$\widetilde{g}^{n*}(s) = \left(\frac{\alpha}{\beta}\right)^n \left(\frac{\beta}{s+\beta}\right)^n .$$

This implies that for $x > 0$, the density function of the n^{th} convolution of $g(x)$ is

$$\begin{aligned} g^{n*}(x) &= \left(\frac{\alpha}{\beta}\right)^n \frac{\beta^n x^{n-1} e^{-\beta x}}{(n-1)!} \\ &= \frac{\alpha^n x^{n-1} e^{-\beta x}}{(n-1)!}. \end{aligned}$$

Thus the density function of the n^{th} convolution of $\rho(x)$ is

$$(\rho^{n*}(x))' = \frac{\alpha^n (-x)^{n-1} e^{\beta x}}{(n-1)!} 1_{(-\infty,0)}(x),$$

and for $y < 0$,

$$\begin{aligned} \sum_{n=0}^{\infty} \rho^{n*}(y) &= \sum_{n=1}^{\infty} \int_{-\infty}^{y} \frac{\alpha^n (-x)^{n-1} e^{\beta x}}{(n-1)!} 1_{(-\infty,0)}(x) dx \\ &= \int_{-\infty}^{y} \alpha e^{\beta x} \sum_{n=1}^{\infty} \frac{(-\alpha x)^{n-1}}{(n-1)!} 1_{(-\infty,0)}(x) dx \\ &= \int_{-\infty}^{y} \alpha e^{\beta x} e^{-\alpha x} 1_{(-\infty,0)}(x) dx \\ &= \frac{\alpha}{\beta - \alpha} e^{(\beta-\alpha) y}. \end{aligned} \tag{3.5.2}$$

Since we have (3.5.1), by replacing b and $\sum_{n=0}^{\infty} \rho^{n*}(x)$ in (3.4.7) with $1/\beta$ and (3.5.2) respectively, we obtain

$$\begin{aligned} \mathsf{P}(m \leq x) &= \left(1 - \frac{\alpha}{\beta}\right) \frac{\alpha}{\beta - \alpha} e^{(\beta-\alpha)x} \\ &= \frac{\alpha}{\beta} e^{(\beta-\alpha)x}, \quad \text{for } x < 0. \end{aligned} \tag{3.5.3}$$

Let us check if this result is equal to the result calculated in the classical model with the same assumption about the claim distribution. As written in Bowers *et* *al*(1997, p.414), when the claim distribution is exponential with parameter $\beta > 0$, for $u \geq 0$, the ruin probability is

$$\psi(u) = \frac{1}{1+\theta} \exp\left(\frac{-\theta\beta}{1+\theta} u\right). \tag{3.5.4}$$

Since

$$\theta = \frac{\mathsf{E}(A_j + B_j)}{\mathsf{E}(B_j)} = \frac{\beta}{\alpha} - 1,$$

Eq.(3.5.4) is rewritten as

$$\begin{aligned} \psi(u) &= \frac{1}{\beta/\alpha} \exp\left(\frac{-\left(\frac{\beta}{\alpha} - 1\right)\beta}{\frac{\beta}{\alpha}} u\right) \\ &= \frac{\alpha}{\beta} e^{-(\beta-\alpha)u}, \quad \text{for} \quad u \geq 0, \end{aligned} \tag{3.5.5}$$

which gives us the same result as (3.5.3).

Chapter 4

ANALYTIC APPROACH

In the previous chapters, the ruin problems in the classical model have been introduced and explained from the view point of the random walk, which provides us with an idea about how the ruin problems can be understood from their general perspectives. Another general approach will be presented in this chapter on the risk model with waiting times (or claims) of which Laplace transform is a rational function. Using Laplace transform techniques and complex variables, we will seek explicit expressions for non-ruin probability. For more details on the present method, the reader is referred to the monograph by Dufresne (2001).

4.1. Laplace Transforms

4.1.1. Definitions and Properties

It often happens that the transform of a certain quantity may be found relatively easily, where perhaps the original problem can be solved only with difficulty, if at all, in the original coordinate. Then, the inverse transform returns the solution of interest from the transform coordinates to the original system. Laplace transform is one such method for various problems.

Definition 4.1.1. *If $f(x)$ is a p.d.f. of X concentrated on $[0,\infty)$, then its Laplace transform $\widetilde{f}(r)$ is defined by*

$$\widetilde{f}(r) = \int_0^\infty e^{-rx} f(x)dx, \qquad \text{for } r \geq 0.$$

The interval of integration may be extended to $(-\infty, \infty)$. The Laplace transform of a random variable X means the Laplace transforms of the distributions of X. With the usual notation for expectation, we may write

$$\widetilde{f}(r) = \mathsf{E}(e^{-rX}).$$

Notice that the Laplace transform of $f(x)$ is the moment-generating function of X with r being replaced by $-r$. Therefore it also has the associated properties of a moment-generating function, as shown in Feller(1971). Here we would like to introduce some of them as examples.

Theorem 4.1.1. *The Laplace transform of a convolution of two distributions is the multiplication of each Laplace transform.*

PROOF. Let X, Y, Z be the random variables such that $Z = X + Y$, where X and Y are independent. If we denote $g(x)$, $f(x)$ and $u(x)$ by the corresponding p.d.f. functions of X, Y and Z, respectively, then we can write

$$u(x) = \int_0^\infty g(x-y)f(y)dy = g * f(x).$$

Since X and Y are independent,

$$\mathsf{E}(e^{-rZ}) = \mathsf{E}(e^{-r(X+Y)}) = \mathsf{E}(e^{-rX})\mathsf{E}(e^{-rY}),$$

which gives rise to

$$\widetilde{u}(r) = \widetilde{g}(r)\widetilde{f}(r).$$

□

Theorem 4.1.2. *The following identity holds*

$$\widetilde{f^{(n)}}(r) = r^n \widetilde{f}(r) - r^{n-1}f(0) - r^{n-2}f'(0) - \cdots - rf^{(n-2)}(0) - f^{(n-1)}(0).$$

PROOF. Since

$$\begin{aligned}\widetilde{f'}(r) &= \int_0^\infty e^{-rx} f'(x)dx \\ &= \left[e^{-rx} f(x)\right]_0^\infty + r\int_0^\infty e^{-rx} f(x)dx \\ &= -f(0) + r\widetilde{f}(r), \qquad (4.1.1)\end{aligned}$$

similarly for the second derivative, it can be expressible by

$$\widetilde{f''}(r) = -f'(0) + r\widetilde{f'}(r).$$

By Eq.(4.1.1), therefore, we obtain

$$\begin{aligned}\widetilde{f''}(r) &= -f'(0) + r\left\{-f(0) + r\widetilde{f}(r)\right\} \\ &= r^2\widetilde{f}(r) - rf(0) - f'(0).\end{aligned}$$

By repeating the same way for the higher orders, we can conclude, in general,

$$\widetilde{f^{(n)}}(r) = r^n\widetilde{f}(r) - r^{n-1}f(0) - r^{n-2}f'(0) - \cdots - rf^{(n-2)}(0) - f^{(n-1)}(0). \qquad \square$$

Theorem 4.1.3. *Distinct probability distributions have distinct Laplace transforms.*

The following two theorems are proved by Doetsch(1974).

Theorem 4.1.4. *(Initial Value Theorem) If the indicated limits exist, then*

$$\lim_{x\to 0} f(x) = \lim_{r\to\infty} r\widetilde{f}(r).$$

Theorem 4.1.5. *(Final Value Theorem) If the indicated limits exist, then*

$$\lim_{x\to\infty} f(x) = \lim_{r\to 0} r\widetilde{f}(r).$$

Example 4.1.1. *If*

$$f(x) = e^{ax},$$

then

$$\begin{aligned}\widetilde{f}(r) &= \int_0^\infty e^{-rx}e^{ax}dx \\ &= \frac{1}{r-a}, \qquad \textit{for} \;\; r > a.\end{aligned}$$

Example 4.1.2. *When*

$$f(x) = x^n, \qquad n = 0, 1, 2, \ldots$$

its transform is given by

$$\begin{aligned}\widetilde{f}(r) &= \int_0^\infty e^{-rx}x^n dx \\ &= \frac{n!}{r^{n+1}}, \qquad \textit{for} \;\; r > 0.\end{aligned}$$

The reader is referred to Spiegel (1965) for some more exemplary properties associated with the Laplace transforms such as introduced above. As to how to invert the transformed functions, see the literature, for example, Doetsch(1974), and Panjer and Doray (1988).

4.1.2. Laplace Transform of Survival Function and Its Property

The probability of ruin with an initial surplus u, as introduced in the previous chapter, is denoted by $\psi(u)$. We define $\varphi(u)$ as

$$\varphi(u) = 1 - \psi(u), \quad \text{where} \;\; \varphi(u) = 0, \quad \text{for} \;\; u < 0.$$

and call it the probability of survival or the probability of non-ruin with the initial surplus u. For a complex number s, the Laplace transform of $\varphi(u)$ is given by

$$\widetilde{\varphi}(s) = \int_0^\infty du e^{-su}\varphi(u). \tag{4.1.2}$$

From the equations (2.2.1) and (2.2.2) in Chapter 2, by assuming the rate at which the premiums are received per time unit to be 1, the classical risk process is written by

$$U(t) = u + t - \sum_{i=1}^{N(t)} X_i,$$

where $N(t)$ is the counting process such that

$$N(t) = \begin{cases} \max\{n \mid W_1 + \cdots + W_n \leq t\}, & \text{where } \{W_i\} \text{ are waiting times,} \\ 0, \quad \text{if the above set is empty.} & \end{cases}$$

We assume that the sequences $\{X_i\}$ and $\{W_i\}$ are independent of each other and that all those variables are i.i.d., non-negative. If the occurrence time of the n^{th} claim is denoted by T_n, by using the waiting times $\{W_i\}$, it can be written as

$$T_n = W_1 + \cdots + W_n, \quad \text{where } T_0 = 0.$$

Since there are n claims until T_n, we express U_{T_n} by

$$\begin{aligned} U_{T_n} &= u + T_n - \sum_{i=1}^{n} X_i \\ &= u + \sum_{i=1}^{n} (W_i - X_i). \end{aligned} \tag{4.1.3}$$

With the definition of Y_i given by

$$Y_i = W_i - X_i,$$

it is possible to express equation (4.1.3) in the following fashion:

$$U_{T_n} = u + \sum_{i=1}^{n} Y_i.$$

By making use of $\{Y_i\}$, we define a random walk in the same form as given in the previous chapter, namely,

$$S_0 = 0, \quad S_n = \sum_{i=1}^{n} Y_i, \quad n \geq 1.$$

Hence, in order for ruin not to occur, this random walk should be always positive , namely,

$$\varphi(u) = \mathsf{P}(M \geq -u), \qquad \text{where} \qquad M = \inf_{0 \leq n < \infty} S_n.$$

Moreover, by conditioning on Y_1, we also get

$$\varphi(u) = \int dF_Y(y)\varphi(u+y) = \mathsf{E}\varphi(u+Y). \tag{4.1.4}$$

Since $\{W_i\}$ and $\{X_i\}$ are i.i.d. and independent, $\{Y_i\}$ is i.i.d. Let $Y_1, Y_2, \ldots$ have common distribution Y and define similarly W and X for $\{W_i\}$ and $\{X_i\}$, respectively. Since $Y = W - X$, equation (4.1.4) becomes

$$\mathsf{E}\left[\varphi(u + W - X)\right] = \int dF_W(t) \int dF_X(v)\varphi(u + t - v). \tag{4.1.5}$$

For a complex number s, we use the following notations:

$$\widetilde{w}(s) = \mathsf{E}e^{-sW}, \quad \widetilde{x}(s) = \mathsf{E}e^{-sX} \quad \text{and} \quad \widetilde{y}(s) = \mathsf{E}s^{-sY}.$$

Definition 4.1.2. *The abscissa of holomorphy h_V of a random variable V is defined as*

$$h_V = \inf\left\{s \in \mathbb{R} \mid \mathsf{E}(e^{-sV}) < \infty\right\}.$$

Definition 4.1.3. *The positive part of a is denoted by* $a^+ = \max(a, 0)$. *Also, the positive real number and the positive integer are denoted by $\mathbb{R}^+$ and $\mathbb{Z}^+$, respectively.*

Theorem 4.1.6. *Suppose $h_W < 0$. Then, for $0 < \mathfrak{Re}(s) < -h_W$, we obtain*

$$\widetilde{\varphi}(s) = \frac{n(s)}{d(s)}, \tag{4.1.6}$$

where

$$n(s) = \mathsf{E}\int_0^\infty dv e^{sv}\varphi(Y-v) = \mathsf{E}\int_0^{Y^+} dv e^{sv}\varphi(Y^+-v) \qquad (4.1.7)$$

$$= \mathsf{E}\int_0^W dv e^{s(W-v)}\varphi(v-X) \qquad (4.1.8)$$

$$d(s) = \widetilde{y}(-s) - 1. \qquad (4.1.9)$$

PROOF. Since

$$\widetilde{\varphi}(s) = \int_0^\infty du e^{-su}\varphi(u),$$

in order to obtain the Laplace transform of $\varphi(u)$, we multiply Eq.(4.1.4) by e^{-su} and integrate with respect to u from 0 to ∞, which yields, for $0 < \mathfrak{Re}(s) < -h_W$,

$$\widetilde{\varphi}(s) = \mathsf{E}\int_0^\infty du\, e^{-su}\varphi(u+Y)$$

$$= \mathsf{E}e^{sY}\int_0^\infty du\, e^{-s(u+Y)}\varphi(u+Y).$$

Let $v = u + Y$. Since $\varphi(u) = 0$, for all $u < 0$, we obtain

$$\widetilde{\varphi}(s) = \mathsf{E}e^{sY}\int_{Y^+}^\infty dv e^{-sv}\varphi(v)$$

$$= \mathsf{E}e^{sY}\left[\int_0^\infty dv e^{-sv}\varphi(v) - \int_0^{Y^+} dv e^{-sv}\varphi(v)\right]$$

$$= \widetilde{y}(-s)\widetilde{\varphi}(s) - \mathsf{E}\int_0^{Y^+} dv e^{s(Y-v)}\varphi(v)$$

$$= \widetilde{y}(-s)\widetilde{\varphi}(s) - \mathsf{E}\int_0^{Y^+} du e^{su}\varphi(Y^+-u). \qquad (4.1.10)$$

Let

$$n(s) = \mathsf{E}\int_0^{Y^+} due^{su}\varphi(Y^+ - u).$$

Then $\widetilde{\varphi}(s)$ can be written in the following form:

$$\widetilde{\varphi}(s) = \widetilde{y}(-s)\widetilde{\varphi}(s) - n(s). \tag{4.1.11}$$

Since

$$\left|\int_0^{Y^+} dve^{sv}\varphi(Y^+ - v)\right| \leq Y^+ e^{\mathfrak{Re}(s)Y^+}, \tag{4.1.12}$$

the function $n(s)$ is finite for $0 < \mathfrak{Re}(s) < -h_W$. Hence, from Eq.(4.1.11), it is possible to write

$$\widetilde{\varphi}(s) = \frac{n(s)}{\widetilde{y}(-s) - 1}.$$

Letting $d(s) = \widetilde{y}(-s) - 1$, we obtain

$$\widetilde{\varphi}(s) = \frac{n(s)}{d(s)},$$

which proves (4.1.6), (4.1.7) and (4.1.9). Now, similarly for $0 < \mathfrak{Re}(s) < -h_W$, we will get another expression of the Laplace transform of $\varphi(u)$ from Eq.(4.1.5). That is,

$$\begin{aligned}\widetilde{\varphi}(s) &= \mathsf{E}\int_0^{\infty} du\, e^{-su}\varphi(W - X + u) \\ &= \mathsf{E}e^{sW}\int_0^{\infty} du\, e^{-s(u+W)}\varphi(W - X + u).\end{aligned}$$

With $v = u + W$, we get

$$\begin{aligned}\widetilde{\varphi}(s) &= \mathsf{E}e^{sW}\int_W^{\infty} dv\, e^{-sv}\varphi(v - X) \\ &= \mathsf{E}e^{sW}\left[\int_0^{\infty} dv\, e^{-sv}\varphi(v - X) - \int_0^{W} dv\, e^{-sv}\varphi(v - X)\right].\end{aligned}$$

Since W and X are independent, we can write

$$\begin{aligned}
\mathsf{E}e^{sW}\int_0^\infty dv\, e^{-sv}\varphi(v-X) &= \widetilde{w}(-s)\int_0^\infty dve^{-sv}\mathsf{E}\varphi(v-X) \\
&= \widetilde{w}(-s)\int_0^\infty dve^{-sv}\int_0^v dF_X(x)\varphi(v-x).
\end{aligned}$$

By Theorem 4.1.1, the above equation may be expressed as

$$\mathsf{E}e^{sW}\int_0^\infty dv\, e^{-sv}\varphi(v-X) = \widetilde{w}(-s)\widetilde{x}(s)\widetilde{\varphi}(s).$$

Hence

$$\begin{aligned}
\widetilde{\varphi}(s) &= \widetilde{w}(-s)\widetilde{x}(s)\widetilde{\varphi}(s) - \mathsf{E}\int_0^W dv\, e^{s(W-v)}\varphi(v-X) \\
&= \widetilde{y}(-s)\widetilde{\varphi}(s) - \mathsf{E}\int_0^W dv\, e^{s(W-v)}\varphi(v-X),
\end{aligned}$$

which proves (4.1.8). □

4.2. Rational Laplace Transforms

4.2.1. Definitions and Examples

Formulating the ruin probability in a general framework is rather difficult. However, it becomes easier once we specify the area where we will construct the model. It may appear to make the applications of the model rather limited in its extent. Nevertheless, this model can be applicable to a much wider class of distributions in comparison with the case for the classical model. Thus, first of all, we consider a class of distributions called $\mathcal{R}^f$, and then we try to formulate an explicit formula on the Laplace transforms of ruin probability in $\mathcal{R}^f$.

Definition 4.2.1. *A rational function is a ratio of two polynomials.*

Definition 4.2.2. *A probability distribution μ on $\mathbb{R}$ is said to belong to $\mathcal{R}^f$ if its Laplace transform is a rational function. If μ is concentrated on $\mathbb{R}^+$, then it is said to belong to $\mathcal{R}^f_+$. In either case the distribution will be said to be rational.*

The class $\mathcal{R}^f$ includes all finite combinations of Erlang densities and also contains phase-type distributions. Even if the distributions may have a mass at origin, it is possible to bring them in $\mathcal{R}^f$ by transforming the given distributions into the non-zero claims or waiting-times distributions by means of the zero removal techniques.

Example 4.2.1. *Suppose that X has an Erlang (3,1) distribution, where the density function is given by*

$$f(x) = \frac{x^2 e^{-x}}{2} 1_{\{x>0\}}.$$

Then its Laplace transform can be written as

$$\begin{aligned}
\widetilde{f}(r) &= \frac{1}{2}\int_0^\infty e^{-rx} x^2 e^{-x} dx \\
&= \frac{1}{2}\int_0^\infty e^{-(r+1)x} x^2 dx \\
&= \frac{1}{(r+1)^3}.
\end{aligned}$$

Since $f(x)$ is non-negative and its Laplace transform $\widetilde{f}(r)$, is the ratio of two polynomials, this distribution belongs to $\mathcal{R}^f_+$.

Example 4.2.2. *Let*

$$f(x) = e^{-2x}(\frac{8}{5} + \sin x)1_{\{x>0\}},$$

where $\sin x$ *is a complex trigonometric function. Since*

$$\begin{aligned}
&\int_0^\infty \left(\frac{8}{5}e^{-2x} + e^{-2x}\sin x\right) dx \\
= \;& \frac{8}{5}\cdot\frac{1}{(-2)}\left[e^{-2x}\right]_0^\infty + \int_0^\infty e^{-2x}\left(\frac{e^{ix}-e^{-ix}}{2i}\right) dx \\
= \;& \frac{4}{5} + \frac{1}{2i}\int_0^\infty \left(e^{-(2-i)x} - e^{-(2+i)x}\right) dx \\
= \;& \frac{4}{5} + \frac{1}{2i}\left(\frac{1}{2-i} - \frac{1}{2+i}\right) \\
= \;& 1,
\end{aligned}$$

the function $f(x)$ *is a density function. Furthermore, its Laplace transform is given by*

$$\begin{aligned}
\widetilde{f}(r) &= \int_0^\infty e^{-rx}\left(\frac{8}{5}e^{-2x} + e^{-2x}\sin x\right) dx \\
&= \int_0^\infty \left[\frac{8}{5}e^{-(r+2)x} + \frac{e^{-(2+r)x}}{2i}\left(e^{ix} - e^{-ix}\right)\right] dx \\
&= \frac{8}{5}\cdot\frac{1}{r+2} + \frac{1}{\left[(r+2)^2+1\right]} \\
&= \frac{8r^2+37r+50}{5(r+2)\left[(r+2)^2+1\right]},
\end{aligned}$$

which shows that $\mathcal{R}_+^f$ *contains the cases where the density has damped sine or cosine functions.*

As can already be noticed, all the measures in $\mathcal{R}_+^f$ are in the form of

$$d\mu(t) = a_0\delta(dt) + \left[\sum_{j=1}^{n}\sum_{k=1}^{d_j} a_{jk}\frac{b_j^{c_{jk}}t^{c_{jk}-1}e^{-b_jt}}{(c_{jk}-1)!}1_{(0,\infty)}(t)\right]dt, \tag{4.2.1}$$

where

$$\begin{aligned} a_0 + \sum_{j,k} a_{jk} &= 1, \\ \{b_j\} &\in \mathbb{R}^+ \text{ or } \{\text{complex number with positive real part}\}, \\ \{c_{jk}\} &\in \mathbb{Z}^+. \end{aligned}$$

4.2.2. Waiting Times in $\mathcal{R}_+^f$

Let us prove that it is possible to find the Laplace transform of the ruin probability in an explicit form in $\mathcal{R}_+^f$. First, we consider the case where the distribution of waiting times belongs to $\mathcal{R}_+^f$ while the distribution of claims is arbitrary. To get the Laplace transform of $\varphi(u), \widetilde{\varphi}(s)$ in this case, it is enough to show that $n(s)$ is a rational function. But this does not always provide us with the probability of ruin because the inversion of $\widetilde{\varphi}(s)$ is not always easy in the cases where the distribution of claims is arbitrary.

Theorem 4.2.1. *Suppose that the distribution of W is in $\mathcal{R}_+^f$, that is, in the form of (4.2.1). Let $\{b_j\}$ be complex numbers with positive real part, and $\{c_{jk}, d_j\}$ $\in \mathbb{Z}^+$ with $c_{jm} \le c_{jd_j}$ for all $m \le d_j$, $a_{jd_j} \neq 0$. Then for $s \in \mathbb{C} - \{b_1, b_2, \ldots, b_n\}$,*

$$n(s) = \sum_{j=1}^{n}\sum_{k=1}^{d_j}\sum_{m=0}^{c_{jk}-1} \frac{a_{jk}b_j^{c_{jk}}(-1)^m\left(\widetilde{x}\widetilde{\varphi}\right)^{(m)}(b_j)}{m!(b_j-s)^{c_{jk}-m}}, \tag{4.2.2}$$

where

$$\left(\widetilde{x}\widetilde{\varphi}\right)^{(m)}(s) = \left(d^m/ds^m\right)\left[\widetilde{x}(s)\widetilde{\varphi}(s)\right].$$

PROOF. Since the distribution of W is in the form of (4.2.1), for $\Re\mathfrak{e}(s) < -h_W$ and $v > 0$,

$$\mathsf{E}e^{sW}1_{\{W>v\}} = \int_v^\infty dt \left[\sum_{j=1}^{n} \sum_{k=1}^{d_j} a_{jk} \frac{b_j^{c_{jk}} t^{c_{jk}-1} e^{-(b_j-s)t}}{(c_{jk}-1)!} \right].$$

Letting $u = (b_j - s)t$, we obtain

$$\mathsf{E}e^{sW}1_{\{W>v\}} = \sum_{j=1}^{n} \sum_{k=1}^{d_j} \frac{a_{jk} b_j^{c_{jk}}}{(c_{jk}-1)!\,(b_j-s)^{c_{jk}}} \int_{v(b_j-s)}^\infty du\ u^{c_{jk}-1} e^{-u}. \qquad (4.2.3)$$

With the use of the incomplete gamma function, equation (4.2.3) can be expressed in another way. According to Abramowitz & Stegun(1972), the incomplete gamma function is defined by

$$\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} dt, \qquad x \geq 0.$$

Furthermore, for $c = 1, 2, \dots$, we have

$$\Gamma(c, x) = \Gamma(c)\varepsilon(c-1, x)e^{-x}, \qquad \text{where} \quad \varepsilon(c-1, x) = \sum_{m=0}^{c-1} \frac{x^m}{m!}. \qquad (4.2.4)$$

Thus, making use of the relation(4.2.4), the integral in Eq.(4.2.3) can be rewritten as

$$\begin{aligned} \int_{v(b_j-s)}^\infty du u^{c_{jk}-1} e^{-u} &= \Gamma(c_{jk}, v(b_j - s)) \\ &= \Gamma(c_{jk})\varepsilon(c_{jk}-1, v(b_j-s))e^{-v(b_j-s)} \\ &= (c_{jk}-1)!\varepsilon(c_{jk}-1, v(b_j-s))e^{-v(b_j-s)}. \end{aligned}$$

Therefore, the equation (4.2.3) becomes

$$\begin{aligned} \mathsf{E}e^{sW}1_{\{W>v\}} &= \sum_{j=1}^{n}\sum_{k=1}^{d_j} \frac{a_{jk}b_j^{c_{jk}}}{(b_j-s)^{c_{jk}}}\varepsilon(c_{jk}-1, v(b_j-s))e^{-v(b_j-s)} \\ &= \sum_{j=1}^{n}\sum_{k=1}^{d_j}\sum_{m=0}^{c_{jk}-1} \frac{a_{jk}b_j^{c_{jk}}[v(b_j-s)]^m}{(b_j-s)^{c_{jk}}\, m!}e^{-v(b_j-s)}. \end{aligned} \tag{4.2.5}$$

From Eq.(4.1.8), we can express $n(s)$ as

$$\begin{aligned} n(s) &= \mathsf{E}\int_0^W dv e^{s(W-v)}\varphi(v-X) \\ &= \mathsf{E}\int_0^\infty dv e^{-sv}\varphi(v-X)e^{sW}1_{\{W>v\}}. \end{aligned}$$

Since W and X are independent, we obtain

$$n(s) = \int_0^\infty dv e^{-sv}\mathsf{E}\varphi(v-X)\mathsf{E}e^{sW}1_{\{W>v\}}.$$

By replacing $\mathsf{E}e^{sW}1_{\{W>v\}}$ in the above expression with (4.2.5), we write

$$\begin{aligned} n(s) &= \int_0^\infty dv e^{-sv}\mathsf{E}\varphi(v-X)\sum_{j=1}^{n}\sum_{k=1}^{d_j}\sum_{m=0}^{c_{jk}-1} \frac{a_{jk}b_j^{c_{jk}}}{(b_j-s)^{c_{jk}}}\frac{[v(b_j-s)]^m}{m!}e^{-v(b_j-s)} \\ &= \sum_{j=1}^{n}\sum_{k=1}^{d_j}\sum_{m=0}^{c_{jk}-1} \frac{a_{jk}b_j^{c_{jk}}}{m!\,(b_j-s)^{c_{jk}}}(b_j-s)^m\int_0^\infty dv e^{-vb_j}v^m\mathsf{E}\varphi(v-X) \\ &= \sum_{j=1}^{n}\sum_{k=1}^{d_j}\sum_{m=0}^{c_{jk}-1} \frac{a_{jk}b_j^{c_{jk}}}{m!\,(b_j-s)^{c_{jk}-m}}\int_0^\infty dv e^{-vb_j}v^m\int_0^v dF_X(x)\,\varphi(v-x) \\ &= \sum_{j=1}^{n}\sum_{k=1}^{d_j}\sum_{m=0}^{c_{jk}-1} \frac{a_{jk}b_j^{c_{jk}}(-1)^m}{m!\,(b_j-s)^{c_{jk}-m}}\frac{d^m}{dr^m}\left[\int_0^\infty dv e^{-vr}\int_0^v dF_X(x)\,\varphi(v-x)\right]_{r=b_j} \\ &= \sum_{j=1}^{n}\sum_{k=1}^{d_j}\sum_{m=0}^{c_{jk}-1} \frac{a_{jk}b_j^{c_{jk}}(-1)^m}{m!\,(b_j-s)^{c_{jk}-m}}\frac{d^m}{dr^m}\left[\widetilde{x}(r)\widetilde{\varphi}(r)\right]_{r=b_j}. \end{aligned}$$

Here $d(s) = \widetilde{w}(-s)\widetilde{x}(s) - 1$ is analytic, except for poles, in $\{\mathfrak{Re}(s) > 0\}$, and so is $n(s) = \widetilde{\varphi}(s)d(s)$. □

Theorem 4.2.2. *(Rouché's Theorem) Suppose $f(z)$ and $g(z)$ are analytic on and within a closed contour Γ in $\mathbb{C}$. Suppose $f(z)$ does not vanish on Γ, and $|g(z)| < |f(z)|$ on Γ. Then $f(z)$ and $f(z) + g(z)$ have the same number of zeros within Γ.*

Theorem 4.2.3. *If $h_W < 0$, then*

$$n(0) = \mathsf{E} \int_0^{Y^+} \varphi(Y^+ - u)du = \mathsf{E}(Y).$$

PROOF. We know that

$$1 = \varphi(\infty). \tag{4.2.6}$$

In view of (4.1.2), $\widetilde{\varphi}(s)$ is analytic in $\{s \in \mathbb{C} \mid \mathfrak{Re}(s) > 0\}$. Thus, by the Final Value Theorem for Laplace transforms, we obtain

$$\varphi(\infty) = \lim_{s \to 0^+} s\widetilde{\varphi}(s) = \frac{n(0)}{\lim_{s \to 0^+} \frac{d(s)}{s}} = \frac{n(0)}{\mathsf{E}\,(Y)}. \tag{4.2.7}$$

Hence from (4.2.6) and (4.2.7), we conclude that

$$n(0) = \mathsf{E}\,(Y)\,.$$ □

Theorem 4.2.4. *Suppose $h_X < 0$ and that W has the rational distribution of Theorem 4.2.1. Let*

$$\pi(s) = \prod_{j=1}^{n} (b_j - s)^{c_{jd_j}}\,.$$

Then the polynomial

$$N(s) = \pi(s)n(s),$$

has degree $\nu = c_{1d_1} + \cdots + c_{nd_n} - 1$, and the function

$$D(s) = \pi(s)d(s),$$

is analytic in $\{\Re\mathfrak{e}(s) > 0\}$. *Moreover, there is* $R > \max_j |b_j|$ *such that*

$$|\widetilde{w}(-s)| < 1 \quad \forall s \text{ with } |s| = R,\ \Re\mathfrak{e}(s) > 0.$$

Let C_R *be the closed path consisting of the half-circle* $\{s \mid |s| = R, \Re\mathfrak{e}(s) > 0\}$ *and the line segment going from* $-Ri$ *to* $+Ri$. *Then* $N(s)$ *and* $D(s)$ *have exactly* ν *zeros inside* C_R. *Moreover,*

$$N(s) = \sum_{j=0}^{\nu} p_j s^j \text{ with } p_0 = \mathsf{E}(Y) \prod_{j=1}^{n} b_j^{c_{jd_j}} \text{ and } p_\nu = \varphi(0)(-1)^\nu. \tag{4.2.8}$$

Proof. From Theorem 4.2.1, we know that $n(s)$ is in the form of (4.2.2) when the distribution of W is given by (4.2.1). Thus

$$\begin{aligned} N(s) &= \pi(s)n(s) \\ &= \prod_{j=1}^{n} (b_j - s)^{c_{jd_j}}\, n(s), \end{aligned}$$

has degree $\nu = c_{1d_1} + \cdots + c_{nd_n} - 1$.

From the assumption made in this theorem, we have

$$\begin{aligned} D(s) &= \pi(s)d(s) \\ &= \prod_{j=1}^{n} (b_j - s)^{c_{jd_j}} \left[\widetilde{w}(-s)\widetilde{x}(s) - 1\right]. \end{aligned}$$

Since the distribution of W is in (4.2.1), $\widetilde{w}(-s)$ has the factors $(b_j - s)^{c_{jd_j}}$, where $1 \le j \le n$, in denominator. Thus they are canceled by the factors $(b_j - s)^{c_{jd_j}}$, where $1 \le j \le n$, in $\pi(s)$. In addition, $\widetilde{x}(s)$ is obviously analytic in $\{s \in \mathbb{C} \mid \Re\mathfrak{e}(s) > 0\}$. Consequently, $D(s)$ is analytic in $\{s \in \mathbb{C} \mid \Re\mathfrak{e}(s) > 0\}$. Let us express the Laplace transform of W as follows:

$$\widetilde{w}(s) = a_0 + \frac{P_1(s)}{P_2(s)},$$

where the degree of $P_1(s)$ is strictly smaller than that of $P_2(s)$ and $a_0 < 1$. Then we have

$$\frac{P_1(s)}{P_2(s)} \to 0, \quad \text{as } |s| \to \infty.$$

Thus

$$\widetilde{w}(-s) \to a_0 < 1, \quad \text{as } |s| \to \infty.$$

Hence there exists some constant $R < \infty$ such that

$$R > \max_j |b_j| \quad \text{and} \quad |\widetilde{w}(-s)| < 1 \quad \forall s \text{ with } |s| = R,\ \mathfrak{Re}(s) > 0.$$

Since $\mathsf{E}(Y) > 0$, it is possible to choose some $q > 0$ such that $h_X < -q < 0$ and $\widetilde{y}(-q) < 1$. Now using R and q, let us construct a closed path $C_{R,q}$ which is composed of the half circle $\{s \mid |s| = R, \mathfrak{Re}(s) > 0\}$, the line segment from Ri to $-q$, and the other line segment from $-q$ to $-Ri$. Since we have $s = u + iv, u < 0, v \in \mathbb{R}$, for all s on the line segment, we obtain

$$\begin{aligned} |\widetilde{y}(u+iv)| &= \left|\mathsf{E}e^{-(u+iv)Y}\right| \le \mathsf{E}\left|e^{-(u+iv)Y}\right| \\ &= \mathsf{E}\left|e^{-uY}\right| = \mathsf{E}e^{-uY} = \widetilde{y}(u) < 1. \end{aligned}$$

Furthermore $|\widetilde{y}(\pm iR)| < 1$, since Y does not have an arithmetic distribution as shown in Feller(1971, p.501). Therefore it leads to

$$|\widetilde{y}(-s)| < 1 \quad \forall s \in C_{R,q}.$$

Moreover, because $R > \max_j |b_j|$, $\pi(s) \neq 0$ on $C_{R,q}$. Thus we can conclude that for $s \in C_{R,q}$,

$$|\pi(s)\widetilde{y}(-s)| < |\pi(s)|. \tag{4.2.9}$$

By applying Rouché's Theorem to (4.2.9), it is noticeable that $\pi(s)\widetilde{y}(-s) - \pi(s)$ and $\pi(s)$ have the same number of zeros inside $C_{R,q}$. Since $\pi(s)$ has $c_{1d_1} + \cdots + c_{nd_n} = \nu + 1$ zeros, so does $\pi(s)\widetilde{y}(-s) - \pi(s) = \pi(s)d(s) = D(s)$. Let us move q

to the origin. Then we have a closed path C_R which is composed of the half circle $\{s \mid |s| = R, \mathfrak{Re}(s) > 0\}$ and the line segment from $-Ri$ to Ri instead of $C_{R,q}$. At this closed path C_R, $d(s)$ has a zero with multiplicity one at $s = 0$, because $d'(0) = \widetilde{y}'(0) = \mathsf{E}(Y) \neq 0$. Hence $D(s)$ and $N(s)$ have ν zeros inside C_R.

Since $N(s)$ is the polynomial with degree ν, it can be expressed as

$$N(s) = p_0 + p_1 s + \cdots + p_\nu s^\nu, \text{ where } p_0, \ldots, p_\nu \text{ are constants.} \tag{4.2.10}$$

Thus by Theorem 4.2.3, we have

$$p_0 = N(0) = \pi(0)n(0) = \pi(0)\mathsf{E}(Y) = \mathsf{E}(Y)\prod_{j=1}^{n} b_j^{c_j d_j}.$$

Moreover (4.2.10) can be rewritten as

$$\frac{N(s)}{s^\nu} = \frac{p_0}{s^\nu} + \frac{p_1}{s^{\nu-1}} + \cdots + p_\nu.$$

Hence

$$\begin{aligned} p_\nu &= \lim_{s\to\infty} \frac{N(s)}{s^\nu} \\ &= \lim_{s\to\infty} s\widetilde{\varphi}(s)\frac{\pi(s)\left[\widetilde{y}(-s) - 1\right]}{s^{\nu+1}}. \end{aligned} \tag{4.2.11}$$

By the Final Value Theorem, (4.2.11) becomes

$$\begin{aligned} p_\nu &= \varphi(0) \lim_{s\to\infty} \frac{\pi(s)}{s^{\nu+1}} \lim_{s\to\infty} \left[\widetilde{w}(-s)\widetilde{x}(s) - 1\right] \\ &= -\varphi(0) \lim_{s\to\infty} \prod_{j=1}^{n} \left(\frac{b_j}{s} - 1\right)^{c_j d_j} \\ &= -\varphi(0)\left(-1\right)^{\nu+1}. \end{aligned}$$

□

4.2.3. Claims in $\mathcal{R}_+^f$

Now, we look at the problem where the distribution of claims is in $\mathcal{R}_+^f$. Contrary to the case where the distribution of waiting times is in $\mathcal{R}_+^f$, a more explicit result can be obtained in a closed-form when the distribution of claims is in $\mathcal{R}_+^f$.

Theorem 4.2.5. *If the function f is analytic in $\mathbb{C}$ except for a finite number of poles, then f is a ratio of two polynomials, that is,*

$$f(z) = \frac{p(z)}{q(z)}, \qquad \text{where } p(z) \text{ and } q(z) \text{ are polynomials.}$$

In this case, the degree of $q(z)$ equals the number of poles of $f(z)$.

Theorem 4.2.6. *If $h_W < 0$ and X has the rational distribution as W of Theorem 4.2.1 then $\widetilde{\varphi}(s)$ is a rational function. Moreover, if no zero of $\widetilde{w}(-s)$ is a pole of $\widetilde{x}(s)$, then the non-zero roots of $d(s)$ are all in $\{\mathfrak{Re}(s) < 0\}$, and their number is equal to the number of poles of $\widetilde{x}(s)$. Especially, if the distribution of X is a combination of Erlang (m_j, β) distributions (possibly with a mass at the origin), then no zero of $\widetilde{w}(-s)$ can be a pole of $\widetilde{x}(s)$.*

Proof. We know that $\widetilde{\varphi}(s)$ is analytic in $\{s \in \mathbb{C} \mid \mathfrak{Re}(s) > 0\}$ and $\widetilde{w}(s)$ and $\widetilde{x}(s)$ are also analytic in this area when $W, X \geq 0$. Furthermore, according to the function $n(s)$ of Theorem 4.1.6, $n(s)$ is analytic in $\{s \in \mathbb{C} \mid \mathfrak{Re}(s) < -h_W\}$, and $d(s) = \widetilde{w}(-s)\widetilde{x}(s) - 1$ is analytic in $\{s \in \mathbb{C} \mid \mathfrak{Re}(s) < -h_W\}$ except for a finite number of poles in $\{s \in \mathbb{C} \mid \mathfrak{Re}(s) < 0\}$, because the distribution of X is in the form of (4.2.1). Thus $\widetilde{\varphi}(s)$ is analytic in $\mathbb{C}$ except for a finite number of poles. Hence by Theorem 4.2.5, $\widetilde{\varphi}(s)$ is the ratio of the two polynomials, that is, rational function.

If the distribution of X is a combination of Erlang (m_j, β) distributions with a mass at the origin, all the poles of its Laplace transform are real. And any zero of $\widetilde{w}(-s)$ in $\{s \in \mathbb{C} \mid \mathfrak{Re}(s) < 0\}$ is complex, because the distribution of X has a non-negative and real measure. Thus no zero of $\widetilde{w}(-s)$ can be a pole of $\widetilde{x}(s)$.

The rest of the theorem can be proved by constructing a closed contour in the left-half plane. First of all, notice that $d(s)$ has a root at $s = 0$, but there are no other zeros of $d(s)$ on the imaginary axis, since Y cannot have an arithmetic distribution (see Fellers(1971, p.501)). Now, let us make a closed contour which include all the poles of $\widetilde{x}(s)$ and all the zeros of $d(s)$ in $\{s \in \mathbb{C} \mid \mathfrak{Re}(s) < 0\}$. For $X \geq 0$, there exists $\delta_0 > 0$ such that

$$|\widetilde{x}(s)| = \left|\mathsf{E}e^{-sX}\right| \leq \mathsf{E}e^{\mathfrak{Re}(s)X} \leq \mathsf{E}e^{\delta_0 X} < \infty, \quad \text{for } \mathfrak{Re}(s) \in (-\delta_0, 0)\,.$$

Since $\mathsf{E}Y > 0$, there exists $\delta_1 > 0$ such that

$$|\widetilde{y}(-s)| = \left|\mathsf{E}e^{sY}\right| \leq \mathsf{E}e^{-\delta_1 Y} < 1, \quad \text{for } \mathfrak{Re}(s) \in (-\delta_1, 0)\,,$$

because $\left[\frac{d}{dr}\mathsf{E}e^{-ry}\right]_{r=0} = -\mathsf{E}Y < 0$. Let $\delta = \min(\delta_0, \delta_1)$. Thus, for $\mathfrak{Re}(s) \in (-\delta, 0)$, $\widetilde{x}(s)$ has no poles and $|\widetilde{y}(-s)| < 1$. On the other hand, for $\mathfrak{Re}(s) < 0$, $|\widetilde{w}(-s)| < 1$, and for $|s|$ larger than some number R_0, $|\widetilde{x}(s)| < 1$. That is,

$$|\widetilde{w}(-s)| < \frac{1}{|\widetilde{x}(s)|}, \qquad \textit{for } \; \mathfrak{Re}(s) < 0 \text{ and } |s| \text{ larger than some number } R_0, \tag{4.2.12}$$

which implies that $d(s)$ does not vanish in this area. Now we define the half circle in the left-half plane as

$$C_R = \left\{s = -\frac{\delta}{2} + iy, -R \leq y \leq R; s = -\frac{\delta}{2} + R \cdot e^{i\theta}, \frac{\pi}{2} < \theta < \frac{3\pi}{2}\right\},$$

where R is chosen to let all the poles of $\widetilde{x}(s)$ and all the zeros of $d(s)$ be in this half circle and to make Eq.(4.2.12) satisfy on this half circle. Since $X \in R_+^f$, $\widetilde{x}(s)$ can be expressed as a ratio of two polynomials, that is,

$$\widetilde{x}(s) = \frac{P_1(s)}{P_2(s)},$$

where $P_1(s)$, $P_2(s)$ are polynomials and $P_1(s)/P_2(s)$ is irreducible. Since $1/\widetilde{x}(s)$ is not analytic in C_R, let

$$g(s) = P_1(s)\widetilde{w}(-s),$$

$$f(s) = -\frac{P_1(s)}{\widetilde{x}(s)} = -P_2(s).$$

Now we can apply Rouché's Theorem here. Therefore

$$h(s) = f(s) + g(s) = P_1(s)\widetilde{w}(-s) - P_2(s) = P_2(s)d(s), \tag{4.2.13}$$

and $f(s) = -P_2(s)$ have the same number of zeros in C_R. Since every zero of $d(s)$ is also a zero of $h(s)$, now let us check if every zero of $h(s)$ is a zero of $d(s)$. If s is a zero of $h(s)$, but not a zero of $P_2(s)$, then it is a zero of $d(s)$. If s is a zero of both $h(s)$ and $P_2(s)$, then s becomes one of the poles of $\widetilde{x}(s)$. Since no zero of $\widetilde{w}(-s)$ is a pole of $\widetilde{x}(s)$ and $P_1(s)/P_2(s)$ is irreducible, this leads to $P_1(s)\widetilde{w}(-s) \neq 0$. It contradicts (4.2.13). Therefore every zero of $h(s)$ is a zero of $d(s)$ and vice versa. Hence the number of zeros of $d(s)$ is equal to the number of poles of $\widetilde{x}(s)$. □

Corollary 4.2.1. *If $h_W < 0$ and $X \in \mathcal{R}_+^f$, then*

$$\varphi(u) = 1 - \sum_{k=1}^{m} f_k(u)e^{-r_k u},$$

where each $f_k(u)$ is a polynomial, and $\{-r_k; k = 1, \ldots, m\}$ are the zeros of $d(s)$ in $\{\mathfrak{Re}(s) < 0\}$.

Using the notation we have seen so far in this chapter, the adjustment coefficient is expressible as the smallest $r_1 > 0$ such that

$$d(-r_1) = \widetilde{w}(r_1)\widetilde{x}(-r_1) - 1 = 0, \tag{4.2.14}$$

if it exists. Let us check if (4.2.14) gives us the same equation as (2.2.4) in the classical model based on a compound Poisson process with the parameter λ.

Since,

$$\widetilde{w}(r_1) = \frac{\lambda}{\lambda + r_1},$$

$$\widetilde{x}(-r_1) = \mathsf{E}e^{r_1 X},$$

the equation (4.2.14) becomes

$$\frac{\lambda}{\lambda + r_1}\mathsf{E}e^{r_1 X} = 1$$

$$\Longleftrightarrow \quad 1 + \frac{r_1}{\lambda} = \mathsf{E}e^{r_1 X}. \qquad (4.2.15)$$

On the assumption that the rate at which the premiums are received is 1, we obtain

$$1 = \lambda(1+\theta)\mathsf{E}(X).$$

Hence, by replacing $1/\lambda$ in (4.2.15) with $(1+\theta)\mathsf{E}(X)$, Eq.(4.2.15) yields

$$1 + r_1(1+\theta)\mathsf{E}(X) = \mathsf{E}e^{r_1 X},$$

which is identical to (2.2.4).

Taking the fact that r_1 is the smallest root of $d(-s)$ and Corollary 4.2.1 into consideration, it is possible to say that

$$\psi(u) \sim Ce^{-r_1 u}, \qquad \text{as } u \to \infty, \qquad \text{where } C \text{ is a constant.} \qquad (4.2.16)$$

Eq.(4.2.16) is also feasible from Eq.(2.2.5) in Theorem 2.2.1 of Chapter 2, because the denominator of (2.2.5) becomes some constant as $u \to \infty$. And it is actually the same pattern with (3.4.13) in Section 3.4.3.

4.3. Interpretation of the classical model in the analytic approach

In the previous sections, we have considered the analytic model in which the Laplace transform of waiting times(or claims) is a rational function. Here we will see how the classical model can be expressed in the frame of the analytic model.

Let us suppose that waiting times has the exponential density function with parameter $\alpha > 0$. Then its Laplace transform is

$$\widetilde{w}(s) = \int_0^\infty e^{-sx} \alpha e^{-\alpha x} dx = \frac{\alpha}{\alpha + s}.$$

Needless to say, it is a rational function. Thus it belongs to $\mathcal{R}_+^f$. Hence the classical model cases can be expressed as particular cases with waiting times being in $\mathcal{R}_+^f$ in the analytic model by means of Theorem 4.2.4. That is, by looking at the form of (4.2.1), the exponential density is the case where $b_1 = \alpha$, $n = 1$ $d_1 = 1$ and $c_{11} = 1$. Therefore

$$\pi(s) = (\alpha - s) \text{ and } N(s) = p_0 = \alpha \mathsf{E}(Y).$$

Since

$$\begin{aligned} d(s) &= \widetilde{y}(-s) - 1 \\ &= \widetilde{w}(-s)\widetilde{x}(s) - 1 \\ &= \frac{\alpha}{\alpha - s}\widetilde{x}(s) - 1 \\ &= \frac{\alpha \widetilde{x}(s) - (\alpha - s)}{\alpha - s}, \end{aligned}$$

$$D(s) = \pi(s) d(s) = \alpha \widetilde{x}(s) - (\alpha - s).$$

Hence

$$\widetilde{\varphi}(s) = \frac{N(s)}{D(s)} = \frac{\alpha \mathsf{E}(Y)}{\alpha\widetilde{x}(s) - (\alpha - s)}. \tag{4.3.1}$$

Therefore the inversion of (4.3.1) totally depends on what kind of claim distribution we have. With luck, the inversion can be performed easily for certain families of claim distributions, for example, in the case of exponential distribution or a mixture of exponential distributions.

Let us suppose that the distribution of claims is exponential with parameter $\beta > 0$. Then

$$\widetilde{x}(s) = \frac{\beta}{\beta + s},$$

$$\mathsf{E}(Y) = \mathsf{E}(W) - \mathsf{E}(X) = \frac{1}{\alpha} - \frac{1}{\beta}.$$

Thus (4.3.1) becomes

$$\begin{aligned}
\widetilde{\varphi}(s) &= \frac{\alpha\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)}{\alpha\left(\frac{\beta}{\beta+s}\right) - (\alpha - s)} \\
&= \frac{\frac{\beta-\alpha}{\beta}}{\frac{\alpha\beta-(\alpha-s)(\beta+s)}{\beta+s}} \\
&= \frac{(\beta - \alpha)(\beta + s)}{\beta\left[\alpha\beta - (\alpha - s)(\beta + s)\right]} \\
&= \frac{(\beta - \alpha)(\beta + s)}{\beta s\left[s + (\beta - \alpha)\right]}. \qquad (4.3.2)
\end{aligned}$$

Since the degree of numerator is 1 and that of denominator is 2, by applying the method of partial fractions to (4.3.2), it becomes

$$\begin{aligned}
\widetilde{\varphi}(s) &= \frac{1}{\beta}\left[\frac{\beta}{s} + \frac{-\alpha}{s + (\beta - \alpha)}\right] \\
&= \frac{1}{s} - \frac{\frac{\alpha}{\beta}}{\left[s + (\beta - \alpha)\right]}. \qquad (4.3.3)
\end{aligned}$$

By inverting (4.3.3), for $x > 0$, we have

$$\varphi(x) = 1 - \frac{\alpha}{\beta} e^{-(\beta-\alpha)x}.$$

Hence for $x > 0$,

$$\psi(x) = \frac{\alpha}{\beta} e^{-(\beta-\alpha)x},$$

which is identical to (3.5.5).

Chapter 5

CONCLUSION

In this thesis, we have shown two generalized models in which the extent of the distribution of waiting times is wider than the classical model. Both models use a random walk whose summands are expressed as the difference between the waiting time and the claim contrary to the classical model in which the surplus process based upon the compound Poisson process is employed.

The model presented in Chapter 3 has formulated the distribution of the maximum position M, including 0, of the random walk using ladder variables and their properties. By taking the analogue to the formula presented in this model for the case of minimum position, we have derived the expression for the probability that this minimum position falls below a certain level, which is equal to the ruin probability of interest. An explicit formula for the ruin probability is obtained under the assumption about the common distribution of X_i, F that the right tail of F is exponential. Since there is no condition imposed on the distribution of waiting times, it is a more general and flexible model than the classical one. However, it is usually difficult to get the explicit expressions for the distributions of ladder variables when F is arbitrary.

In the model presented in Chapter 4, a general class, which permits much more assumption about waiting times than the case in the classical model, called $\mathcal{R}_+^f$, is defined. By expressing the Laplace transforms of non-ruin probability in integral function, we have obtained them in closed forms for two cases. The first case is when the distribution of waiting times is in $\mathcal{R}_+^f$ and that of claims is arbitrary. The second case is when the distribution of claims is in $\mathcal{R}_+^f$ and that of waiting times is arbitrary. It is found that the second case gives rise to a simpler yet more explicit result than the first case. This is due to the fact that the Laplace transform itself of non-ruin probability in the second case is rational. Consequently, the inversion of this Laplace transform gets easier than the first case.

From the formulas of ruin probability derived, it is possible to say that these two general models include the classical model cases as shown in Sections 3.5 and 4.3. Hence if the present models are observed from the view point of the classical model, it leads to the same results as those calculated in the classical model. But those models presented in this work may be applicable to much more cases than the classical model, since they are not only another way of writing known results in the classical model, but also are obtained from more generalized approaches.

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