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Finite Cyclicity of Graphics Through a Nilpotent Singularity of Elliptic or Saddle Type

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To my parents, my wife and my son

Sommaire

Dans cette thèse on considère le problème de cyclicité finie des graphiques les plus génériques passant par un point singulier de codimension 3 de type elliptique ou selle. On donne des théorèmes de cyclicité finie pour plusieurs tels graphiques dans des familles C^∞ de champs de vecteurs. Dans plusieurs cas nos résultats sont indépendants de la codimension exacte du point et dépendent seulement du fait que le point nilpotent est de multiplicité 3. En utilisant des formes normales adéquates et un éclatement de la famile de champs de vecteurs on établit la liste de tous les ensembles limites périodiques dans la famille éclatée. On calcule les deux types d'applications de Dulac pour la famille éclatée et on développe une méthode générale pour montrer que certaines transitions régulières ont une dérivée d'ordre supérieur non nulle en un point. En analysant le nombre de solutions d'un système d'équations par un algorithme de dérivation-division généralisé on montre les théorèmes suivants:

Théorème	Type de graphique	Condition	Codimension	Cyclicité
Th. 5.5	Sxhh convexe	$P'(0) \neq 1$ *le point de cod* 3	4	finie
Th. 6.3	Epp	$R^{(n)}(0) \neq 0$ $(n \geq 2)$	$n+1$	n
Th. 6.6	Ehp	*une conjecture*	3	finie
Th. 6.10	Ehh	$P'(0) \neq 1$ *le point de cod* 3	4	finie

On applique ces théorèmes pour montrer la cyclicité finie de certains graphiques passant par un point singulier nilpotent triple dans les systèmes quadratiques.

Summary

In this thesis we consider the problem of finite cyclicity of the most generic graphics through a nilpotent point of saddle or elliptic type of codimension 3. We give theorems of finite cyclicity for several such graphics inside C^∞ families of vector fields. In some case our results are independent of the exact codimension of the point and dependent only on the fact that the nilpotent point has multiplicity 3. Using adequate normal forms and blow-up of a family of vector fields, we list all the limit periodic sets of the blown-up family. We calculate two different types of Dulac maps in the blown-up family and develop a general method to prove that some regular transition maps have a nonzero higher derivative at a point. By the analysis of the number of solutions of systems of equations via a generalized derivation-division method, we prove the following theorems:

Theorem	Type of graphic	Condition	Codimension	Cyclicity
Th. 5.5	Sxhh convex	$P'(0) \neq 1$ *the point of cod* 3	4	finite
Th. 6.3	Epp	$R^{(n)}(0) \neq 0$ $(n \geq 2)$	$n+1$	n
Th. 6.6	Ehp	A conjecture	3	finite
Th. 6.10	Ehh	$P'(0) \neq 1$ *the point of cod* 3	4	finite

We apply these theorems to prove some graphics with a nilpotent saddle or elliptic point in quadratic systems have finite cyclicity.

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Chapter 1

Introduction

A graphic (singular cycle, limit periodic set, polycycle) of a planar vector field is an invariant set of the vector field involving regular orbits and singular points. We are interested in the graphics of generic families of vector fields depending on a small number of parameters and their cyclicity, i.e., the maximum number of limit cycles that may appear by perturbation inside the family. A simpler problem than the problem of finding the cyclicity of a graphic is to prove that the graphic has finite cyclicity. The question of finding the number of limit cycles which appear by perturbation of a graphic in a generic family and the problem of finite cyclicity is closely related to Hilbert-Arnold Problem ([AI88], [IY95]):

Hilbert-Arnold Problem. *Prove that for any n, the bifurcation number $B(n)$ is finite, where $B(n)$ is the maximum cyclicity of nontrivial polycycles occurring in generic $n-$parameter families.*

A graphic of planar vector field can be elementary or non-elementary in the sense that its singular points are elementary (hyperbolic or semi-hyperbolic, i.e. at least one nonzero eigenvalue) or non-elementary. Some essential steps have been made towards the understanding of the bifurcation of elementary graphics through the works of Roussarie [R86], Mourtada [Mou90], [Mou94], [Mou97], Il'yashenko-Yakovenko [IY95], Dumortier, Roussarie and Rousseau [DRR94], Kotova and Stanzo [KS95], Dumortier, El Morsalani and Rousseau [DER96], etc. As for the non-elementary graphics of a planar analytic vector, the study cannot be fully reduced to the analysis of singularities and zeroes of algebraic equations, particularly when the number of parameters is larger than or equal to three.

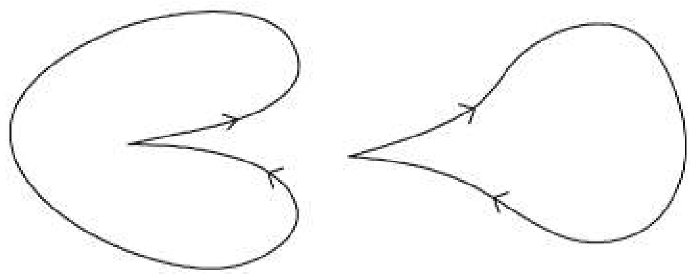
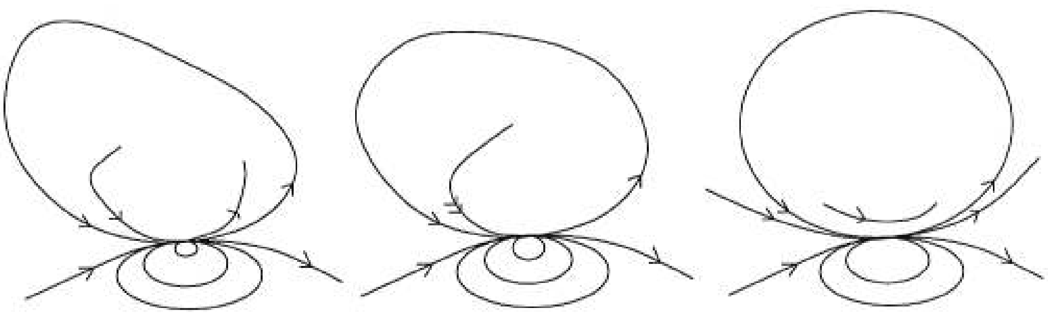


Figure 1.1: Cuspidal loop

For the graphics with a nilpotent singular point of multiplicity 2 or 3, they can be one of the following types:

- cuspidal loop: Fig 1.1
- graphic through a nilpotent elliptic point: Fig 1.2
- graphic through a nilpotent saddle: Fig 1.3

In [DRS97], by an analytic and geometric method based on the blowing up for the unfoldings, the authors studied the simplest case, the bifurcation diagram of a cuspidal loop of codimension 3. They give a complete answer for the cyclicity and bifurcation diagram up to a conjecture. The study of the unfolding of codimension 3 nilpotent singular point is still not finished. As for the problem concerning the graphics through a nilpotent singular point of codimension 3, in [KS95], when the authors tried to list the set of all these graphics, they have the following comments:



(a) Epp graphic (b) Ehp graphic (c) Ehh graphic

Figure 1.2: Graphics through a nilpotent elliptic point

(3) A_0 is the degenerate cusp, cod(A_0) = 3: this case is the most difficult one. Two possible subcases can be distinguished:

(3a) The polycycle consists of a singular point.

(3b) If the singularity is of an elliptic type in the terminology of [D93], then the singularity point has two parabolic sectors of opposite attractivity. Hence a pp-loop can occur without increasing the codimension of the polycycle.

Even the first subcase is not yet completely investigated, to say nothing about the much more difficult loop subcase. We simply label subcase (3.3), without going deeply into the subject.

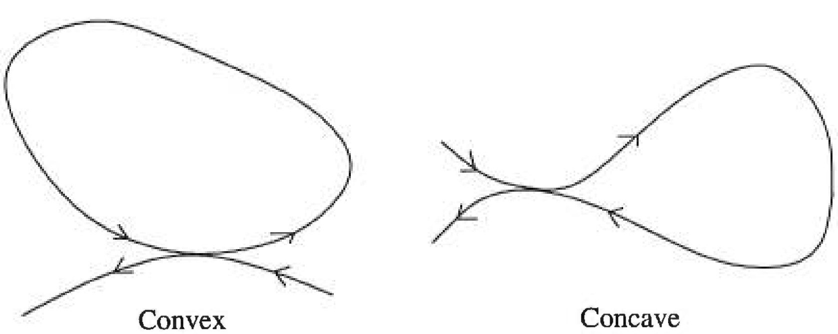


Figure 1.3: Graphics through a nilpotent saddle

From this and the complexity of the bifurcation diagram in the case of the cuspidal loop, it seems hopeless to find a complete solution to solve the similar question with triple nilpotent points. Fortunately we will show that the question of proving the finite cyclicity of a graphic is much simpler and that indeed we can give a complete answer to this question for several graphics of codimension 3 and 4. This means in particular that we do not consider the birth of small limit cycles from the singularities but only the large limit cycles which coalesce with the graphic when the parameters vanish.

In this thesis, we study the finite cyclicity of graphics with a nilpotent singularity of saddle or elliptic type, i.e., the existence of a bound for the number of limit cycles which can bifurcate from such graphics. In some of the finite cyclicity theorems, we will only use the multiplicity of the nilpotent point and not its codimension, the finite cyclicity following from a global genericity assumption. The precise definition of cyclicity for a limit periodic set was given by Roussarie ([R86]).

Definition 1.1. *A limit periodic set* Γ *of a vector field* X_{μ_0} *inside a family* X_μ *has finite cyclicity in* X_μ *if there exist* $N \in \mathbb{N}$ *and* $\varepsilon, \delta > 0$*, such that any* X_μ *with* $|\mu - \mu_0| < \delta$ *has at most* N *limit cycles* γ_i *such that* $dist_H(\Gamma, \gamma_i) < \varepsilon$*. The minimum of such* N *when* ε *and* δ *tend to zero is called the cyclicity of* Γ *in* X_μ*, which we denote by* $Cycl(\Gamma)$*.*

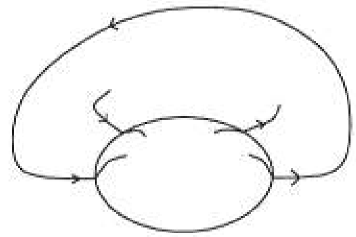
Let X be a smooth vector field on $\mathbb{R}^2$. A singular point p (i.e. $X(p) = 0$) is said to be a triple nilpotent point of saddle or elliptic type if there is a local chart

$$(x, y) : (\mathbb{R}^2, p) \longrightarrow (\mathbb{R}^2, 0)$$

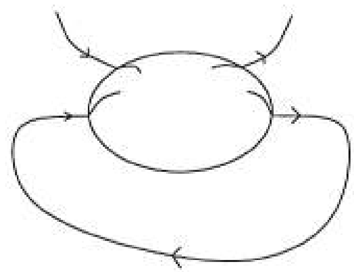
in which the vector field has the form ([T74])

$$X = y\frac{\partial}{\partial x} + (\varepsilon_1 x^3 + dx^4 + bxy + ax^2y + ex^3y)\frac{\partial}{\partial y} + O\Big(|(x, y)|^5\Big) \tag{1.1}$$

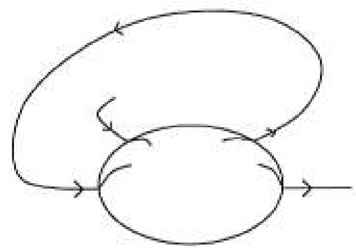
where, for the saddle case, $\varepsilon_1 = 1$; for the elliptic case $\varepsilon_1 = -1$, $b > 2\sqrt{2}$. Denoting the graphic with a nilpotent singularity by (X, p, Γ), we are going to study the cyclicity of Γ by considering a codimension 3 unfolding X_μ of X.



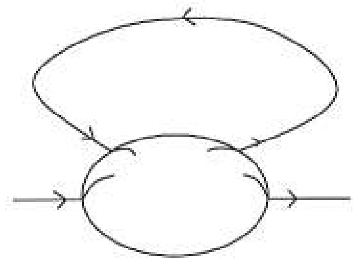
(a) Epp upper



(b) Epp lower



(c) Ehp upper



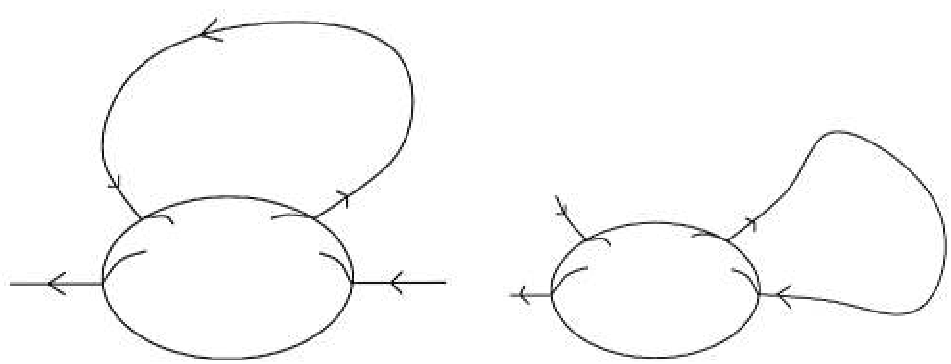
(d) Ehh upper

Figure 1.4: Pp, hp and hh-graphics of elliptic type

Following the convention in [KS95], we use pp to denote a graphic going out of a parabolic sector to a parabolic sector, hp to denote a graphic going out of a hyperbolic sector to a parabolic sector, and hh to denote the graphic going out of a hyperbolic sector to a hyperbolic sector. Then, after the global blow up, a graphic through a elliptic point can happen in three cases (Fig. 1.4):

- pp graphic: Epp,
- hp graphic: Ehp, this cod 3 type of graphic was not mentioned in [KS95],
- hh graphic: Ehh.

Each graphic can occur in two versions: upper and lower (see one example in Fig. 1.4(b)). Although the upper and lower graphics may have different bifurcation diagrams, the proofs of their finite cyclicity are the same.



(a) Sxhh (b) Sahh

Figure 1.5: Convex and concave graphics of saddle type

A graphic through a nilpotent saddle can happen in two cases (Fig. 1.5):

- hh graphic convex: Sxhh,
- hh graphic concave: Sahh.

Due to the technical difficulties, we do not consider the concave graphic of saddle type. For other graphics listed in Fig. 1.4 and Fig. 1.5, we have proved three main theorems which we list in Tab. 1.

Theorem	Type of graphic	Condition	Codimension	Cyclicity
Th. 5.5	Sxhh convex	$P'(0) \neq 1$ *the point of cod* 3	4	finite
Th. 6.3	Epp	$R^{(n)}(0) \neq 0$ $(n \geq 2)$	$n+1$	n
Th. 6.6	Ehp	*A conjecture*	3	finite
Th. 6.10	Ehh	$P'(0) \neq 1$ *the point of cod* 3	4	finite

Table 1.1: Main results concerning the finite cyclicity

To prove the finite cyclicity theorems listed above, one basic ingredient is the blow-up of families developed in [DRr96] and [DRS97]. Around that we set up a machinery which can be used for other similar graphics. Some of these tools have been introduced for the study of the cuspidal loop [DRS97]. These tools include:

1. Normal form for a family with a nilpotent singularity: we develop a special normal form different from the classical one and allowing:
   - to use the special properties of quadratic systems:
     - some transitions occur along straight lines,
     - convexity of some trajectories,
     - knowledge of the center conditions;
   - to be easily applicable to graphics inside quadratic systems.
2. Blow-up of the family to allow the calculations of the passage maps near the nilpotent singularity.
3. The list of limit periodic sets appearing in the blown-up family of vector fields and which must be proved to have finite cyclicity.
4. The calculations of the different types of Dulac maps in the neighborhood of the singular points of the blown-up sphere.
5. To derive finite cyclicity property, we consider systems whose number of solutions bounds the number of fixed points of the return map in the neighborhood of a limit periodic set under a small perturbation of the blown-up vector field. We derive bounds for the number of solutions of these systems by a generalized derivation-division method.
6. We introduce a general method to prove that some regular transitions have a nonzero higher derivative at a given point.

For Hilbert's 16th problem for quadratic systems which consists in "*finding the maximum number $H(2)$ and relative positions of limit cycles of a quadratic* *vector field*", till now we only know that $H(2) \geq 4$ ([Sh80]). In [DRR94], the authors launched a program aiming at solving the finiteness part of Hilbert's 16th problem for quadratic vector fields (i.e., $H(2) < \infty$). The paper listed all the 121 limit periodic sets surrounding the origin in a family of quadratic vector fields and reduced the finiteness problem for quadratic systems to the proof that all of these graphics have finite cyclicity. Up to now, about 50 elementary graphics have been proved to have finite cyclicity. By the results of this work, we will be able to prove that more than 20 non-elementary graphics have finite cyclicity.

The thesis is organized as follows: In Ch.2, we develop a new general normal form unfolding the nilpotent singularity of saddle or elliptic type of codimension 3. For further application to quadratic systems, we also discuss the normal form for unfolding the nilpotent singularity of saddle or elliptic type for quadratic systems. In Ch.3, we make the global blow-up for the family. By using the properties of quadratic systems we give the bifurcation diagrams for the rescaled family and list all the possible limit periodic sets. In order to prove the cyclicity of the limit periodic sets, in Ch.4, we study two types of Dulac maps. We prove the main finite cyclicity theorems for saddle and elliptic cases respectively in Ch.5 and Ch.6. As applications of the main theorems, in Ch.7 we prove that the graphics (H_6^1), (F_{6a}^1) listed in [DRR94] have finite cyclicity and that the graphics (I_{13}^1), (I_{12}^1), (I_{9b}^1), (I_{11b}^1) have finite cyclicity when the nilpotent point has codimension 3.

Chapter 2

Normal Forms Unfolding the Nilpotent Singularity of Saddle or Elliptic type

In this chapter, we will first develop a new normal form unfolding the codimension 3 nilpotent singularity of saddle or elliptic type different from the standard unfolding used in [DRS91]. Then we discuss the corresponding normal forms unfolding the nilpotent singularity of saddle or elliptic type for the quadratic systems.

2.1 Normal forms unfolding the nilpotent singularity of saddle or elliptic type

We know by [T73] that the germs of C^∞ vector fields at $0 \in \mathbb{R}^2$ whose 1-jet is nilpotent and 2-jet is $C^\infty-$conjugate to a vector field with a 2-jet $y\frac{\partial}{\partial x} + \beta xy\frac{\partial}{\partial y}$, is $C^\infty-$conjugate to a vector field with 4-jet

$$y\frac{\partial}{\partial x} + (\varepsilon_1 x^3 + dx^4 + bxy + ax^2y + ex^3y)\frac{\partial}{\partial x} \tag{2.1}$$

where $\varepsilon_1 = 0, \pm 1$ and $a, b, c, d \in \mathbb{R}$.

It was shown in [D77] and [D78] that the topological type of such a germ is determined by its 3-jet, if $\varepsilon_1 \neq 0$.

The codimension of the point is determined by looking at b and the quantity

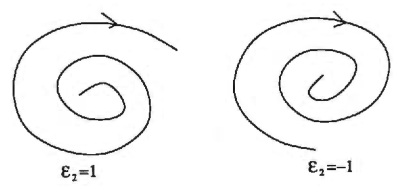
$$Q := 5\varepsilon_1 a - 3bd \tag{2.2}$$

associated with the 4-jet (2.1).

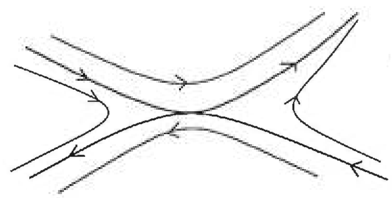
By [DRS91], the vector field is C^∞-equivalent to a vector field with a 4-jet

$$y\frac{\partial}{\partial x} + (\varepsilon_1 x^3 + bxy + \varepsilon_2 x^2 y + f x^3 y)\frac{\partial}{\partial x} \tag{2.3}$$

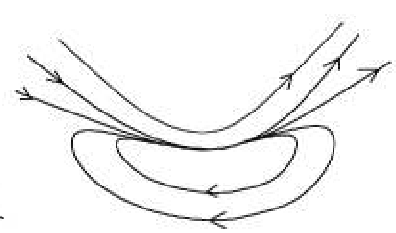
where $\varepsilon_1 = \pm 1$ and ε_2 is a multiple of $\mathcal{Q}$.



(a) Focus cases



(b) Saddle case



(c) Elliptic case

Figure 2.1: The different topological types

The topological type falls into one of the following categories (Fig. 2.1):

1). The saddle case: $\varepsilon_1 = 1$, any ε_2 and b (a topological saddle).

2). The focus case: $\varepsilon_1 = -1$ and $0 < b < 2\sqrt{2}$ (a topological focus).

3). The elliptic case: $\varepsilon_1 = -1$ and $b \geq 2\sqrt{2}$ (an elliptic point).

For saddle (resp. elliptic) cases, for $\varepsilon_2 = 1$ and $\varepsilon_2 = -1$, they have the same topological type.

For $\varepsilon_1 = -1$, the nilpotent singularity is of codimension 3 if $\varepsilon_2 \neq 0$, $b \neq 0$ and $b \neq 2\sqrt{2}$; it is of codimension ≥ 4 if $\varepsilon_2 = 0$, or $b = 0$, or $b = 2\sqrt{2}$.

We are interested only in the vector fields with a triple nilpotent singularity of saddle or elliptic type with $\varepsilon_1 = \pm 1$ and $b > 2\sqrt{2}$ if $\varepsilon_1 = -1$. A family containing this singularity can be brought to ([DRS91])

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= \varepsilon_1 x^3 + \lambda_2 x + \lambda_1 + y(\lambda_3 + bx + \varepsilon_2 x^2 + x^3 h(x,\lambda)) + y^2 Q(x,y,\lambda) \end{cases} \tag{2.4}$$

where for the saddle case $\varepsilon_1 = 1$; for the elliptic case $\varepsilon_1 = -1, b > 2\sqrt{2}$, $\lambda = (\lambda_1, \lambda_2, \lambda_3, \overline{\lambda})$ are the parameters, $Q(x,y,\lambda)$ is C^∞ in (x,y,λ) and of arbitrarily high order in (x,y,λ). For any value of ε_2, they have the same topological type.

Remark 2.1. *Some of the work done here will also be useful for higher codimension nilpotent saddle or elliptic singularities in the case $\varepsilon_2 = 0$ and/or $b = 0$.*

In this normal form (2.4), the "principal" part (the remaining part on the blown-up sphere) will be cubic. We develop a new normal form for the unfolding of the nilpotent singularity of the saddle and elliptic type, so that the principal part becomes quadratic.

Theorem 2.2. *The family (2.4) is C^∞-equivalent to*

$$\begin{cases} \dot{X} &= Y + \mu_2 + aX^2 \\ \dot{Y} &= \mu_1 + Y(\mu_3 + X + \bar{\varepsilon}_2 X^2 + X^3 h_1(X,\mu)) + X^4 h_2(X,\mu) + Y^2 Q(X,Y,\mu) \end{cases} \tag{2.5}$$

where $\bar{\varepsilon}_2 = -a\varepsilon_1\varepsilon_2$, and

- *for the saddle case: $a(0) \in (-\frac{1}{2}, 0)$;*

  if $a(0) = -\frac{1}{2}$, the unfolding is of codimension 4 which corresponds to the case $b = 0$.

- *for the elliptic case: $a(0) \in (0, \frac{1}{2})$;*

  if $a(0) = \frac{1}{2}$, the unfolding is of codimension 4, type 1, which corresponds to the case $b = 2\sqrt{2}$ (the two characteristic trajectories coalesce into one).

$\mu = (\mu_1, \mu_2, \mu_3, \hat{\mu})$ is the parameter, $h_1(X,\mu)$, $h_2(X,\mu) = \bar{\varepsilon}_2 a + O(\mu) + O(X)$ and $Q(X,Y,\mu)$ are C^∞ and $Q(X,Y,\mu)$ is of arbitrary high order in (X,Y,μ).

Proof. In the family (2.4), we make the transformation

$$\begin{cases} x &= m_1 + X \\ y &= m_2 + Y + a_2 X^2 \end{cases} \tag{2.6}$$

Then we have

$$\begin{cases} \dot{X} &= Y + m_2 + a_2 X^2 \\ \dot{Y} &= \lambda_1 + m_1\lambda_2 + m_2\lambda_3 + O(|(m_1, m_2)|^2) \\ & +[\lambda_2 + (b - 2a_2)m_2 + O(|(m_1, m_2)|^2)]X \\ & +[a_2\lambda_3 + (a_2 b + 3\varepsilon_1)m_1 + \varepsilon_2 m_2 + O(|(m_1, m_2)|^2)]X^2 \\ & +[\varepsilon_1 + a_2 b - 2a_2^2 + O(|(m_1, m_2)|)]X^3 \\ & +Y\Big[\lambda_3 + bm_1 + O(m_1^2) + (b - 2a_2 + O(m_1))X \\ & +(\varepsilon_2 + O(m_1))X^2 + X^3 h_{11}(X, m_1)\Big] \\ & +(a_2\varepsilon_2 + O(m_1) + O(X))X^4 + Y^2 Q_1(X, Y, m_1, m_2, \lambda) \end{cases} \tag{2.7}$$

where $h_{11}(X, m_1)$ and $Q_1(X, Y, m_1, m_2, \lambda)$ are C^∞. Also Q_1 is of arbitrarily high order in its variables. To eliminate the terms X, X^2, X^3 in the second equation of (2.7), let

$$F(m_1, m_2, a_2, \lambda) := \begin{pmatrix} \lambda_2 + (b - 2a_2)m_2 + d_1(m_1, m_2) \\ a_2\lambda_3 + (a_2 b + 3\varepsilon_1)m_1 + \varepsilon_2 m_2 + d_2(m_1, m_2) \\ \varepsilon_1 + a_2 b - 2a_2^2 + d_3(m_1, m_2) \end{pmatrix} = 0 \tag{2.8}$$

where for $i = 1, 2$, $d_i(m_1, m_2) = o(|(m_1, m_2)|)$ and $d_3(m_1, m_2) = O(|(m_1, m_2)|)$.

For $\lambda = 0$ and $m_1 = m_2 = 0$, by (2.8) we have a equation for $a_2(0)$

$$2a_2^2(0) - b(0)a_2(0) - \varepsilon_1 = 0. \tag{2.9}$$

To solve the equation (2.9), we have the following subcases

- In the elliptic case $\varepsilon_1 = -1$, since $b(0) > 2\sqrt{2}$, then
  - $a_2^+(0) = \frac{1}{4}[b(0) + \sqrt{b^2(0) - 8}] \in (\frac{\sqrt{2}}{2}, \infty)$

- $a_2^-(0) = \frac{1}{4}[b(0) - \sqrt{b^2(0) - 8}] \in (0, \frac{\sqrt{2}}{2})$.

  In this case, we choose $a(0) = a_2^-(0)$.

- In the saddle case

  - if $b(0) > 0$, then

    * $a_2^+(0) = \frac{1}{4}[b(0) + \sqrt{b^2(0) + 8}] \in (\frac{\sqrt{2}}{2}, \infty)$
    * $a_2^-(0) = \frac{1}{4}[b(0) - \sqrt{b^2(0) + 8}] \in (-\frac{\sqrt{2}}{2}, 0)$.

  - if $b(0) < 0$, then

    * $a_2^+(0) = \frac{1}{4}[b(0) + \sqrt{b^2(0) + 8}] \in (0, \frac{\sqrt{2}}{2})$
    * $a_2^-(0) = \frac{1}{4}[b(0) - \sqrt{b^2(0) + 8}] \in (-\infty, -\frac{\sqrt{2}}{2})$.

Consider $F(m_1, m_2, a_2, \lambda) = 0$. Note that for the saddle case and the elliptic case with $a(0) \in (0, \frac{\sqrt{2}}{2})$, we have

$$\begin{aligned} &F(0, 0, a_2(0), 0) = 0, \\ &\det\left(\left.\frac{\partial F(m_1, m_2, a_2, \lambda)}{\partial(m_1, m_2, a_2)}\right|_{(0,0,a_2(0),0)}\right) = \frac{2\varepsilon_1(a^2(0)+\varepsilon_1)(2a^2(0)+\varepsilon_1)}{a^2(0)} \neq 0. \end{aligned}$$

So by the Implicit function theorem, in the neighborhood of $(0, 0, a_2(0), 0)$, the solution of (2.8) can be written as

$$\left\{\begin{array}{lcl} a_2 &=& a_2(0) + O(|\lambda|) \\ m_1 &=& -\frac{a_2(0)\varepsilon_1}{2(a_2^2(0)+\varepsilon_1)}[\varepsilon_2\lambda_2 + \varepsilon_1\lambda_3] + O(|\lambda|^2) \\ m_2 &=& a_2(0)\varepsilon_1\lambda_2 + O(|\lambda|^2). \end{array}\right.$$

The family has the form

$$\left\{\begin{array}{ll} \dot{X} &= Y + m_2 + a_2X^2 \\ \dot{Y} &= \lambda_1 + m_1\lambda_2 + m_2\lambda_3 + O(|(m_1, m_2)|^2) \\ & \quad +Y[\lambda_3 + bm_1 + O(|(m_1, m_2)|^2) + b_1(\lambda)X + b_2(\lambda)X^2 + X^3h_{11}(X, \lambda)] \\ & \quad +X^4h_{12}(X, \lambda) + Y^2Q_1(X, Y, \lambda) \end{array}\right. \tag{2.10}$$

where

$$\begin{aligned} b_1(\lambda) &= b(0) - 2a_2(0) + O(|\lambda|) = -\tfrac{\varepsilon_1}{a_2(0)} + O(|\lambda|) \\ b_2(\lambda) &= \varepsilon_2 + O(|\lambda|) \\ h_{12}(X,\lambda) &= \varepsilon_2 a_2 + O(\lambda) + O(X). \end{aligned}$$

Rescaling in (2.10) by $(X,Y) = (\frac{\overline{X}}{b_1(\lambda)}, \frac{\overline{Y}}{b_1(\lambda)})$, then we get a new normal form

$$\begin{cases} \dot{\overline{X}} &= \overline{Y} + \mu_2 + a\overline{X}^2 \\ \dot{\overline{Y}} &= \mu_1 + \overline{Y}[\mu_3 + \overline{X} + \overline{\varepsilon}_2\overline{X}^2 + \overline{X}^3 h_1(\overline{X},\mu)] + \overline{X}^4 h_2(\overline{X},\mu) + \overline{Y}^2 Q(\overline{X},\overline{Y},\mu) \end{cases} \tag{2.11}$$

where

$$\begin{aligned} a &= -\varepsilon_1 a_2^2 + O(\lambda) \\ \overline{\varepsilon}_2 &= \varepsilon_2 a_2^2 + O(\lambda) \\ h_2(\overline{X},\lambda) &= -\varepsilon_1\varepsilon_2 a_2^4 + O(\lambda) + O(\overline{X}). \end{aligned}$$

Moreover we have:

- For the saddle case $\varepsilon_1 = 1$, $a(0) = -a_2^2(0)$, also
  - if $b(0) < 0$, then $a(0) \in (-\frac{1}{2}, 0)$;
  - if $b(0) = 0$, then $a(0) = -\frac{1}{2}$, the unfolding is of codimension 4;
  - if $b(0) > 0$, then $a(0) \in (-\infty, -\frac{1}{2})$.
- For the elliptic case $\varepsilon_1 = -1$, $a(0) = a_2^2(0)$, $h_2(\overline{X},\lambda) = \overline{\varepsilon}_2 a + O(\lambda) + O(\overline{X})$. and also
  - if $b(0) > 2\sqrt{2}$, then $a(0) \in (0, \frac{1}{2})$;
  - if $b(0) = 2\sqrt{2}$, then $a(0) = \frac{1}{2}$, the unfolding is of codimension 4, type 1.

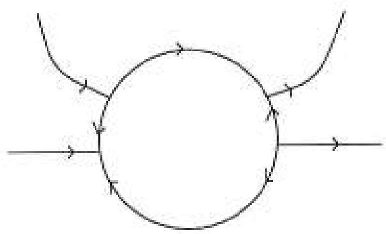
Also $\mu = (\mu_1, \mu_2, \mu_3, \hat{\mu})$ is the new parameter with

$$\begin{cases} \mu_1 &= \frac{\varepsilon_1}{a_2(0)}\lambda_1 + O(|\lambda|^2) \\ \mu_2 &= -\lambda_2 + O(|\lambda|^2) \\ \mu_3 &= -\frac{\varepsilon_1(2a_2(0)-\varepsilon_1)}{2(a_2(0)+\varepsilon_1)}[-\varepsilon_2\lambda_2 + 3\lambda_3] + O(|\lambda|^2) \end{cases}$$

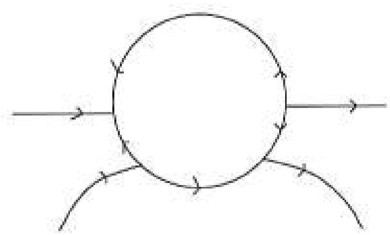
with $\det\left(\frac{\partial(\mu_1,\mu_2,\mu_3)}{\partial(\lambda_1,\lambda_2,\lambda_3)}\right)\Big|_{\lambda=0} \neq 0$.

By using the original coordinates (x, y) and using ε_2 as a parameter, we get the new normal form (2.5) unfolding the nilpotent singularity of saddle or elliptic type.

For the saddle case, family (2.5) with $a \in (-\infty, -\frac{1}{2})$ and $a(0) \in (-\frac{1}{2}, 0)$ are C^∞-equivalent. But for the elliptic case, family (2.5) with $a(0) \in (0, \frac{1}{2})$ and $a(0) \in (\frac{1}{2}, \infty)$ are C^∞-equivalent except for $a(0) = \frac{1}{4}$ (The reason for the difference at $a(0) = \frac{1}{4}$ can be explained by the existence of a Jordan block for two equal eigenvalues. Accordingly, this is reflected in (2.17) and (2.19) below). Fig. 2.2 gives the two equivalent types of nilpotent elliptic singularity (The saddle case has the same kind of equivalence).



(a) $a \in (0, \frac{1}{2})$



(b) $a \in (\frac{1}{2}, \infty)$

Figure 2.2: Two $C^\infty-$equivalent types for the nilpotent elliptic singularity

Indeed, for the saddle case, let $a(0) \in (-\frac{1}{2}, 0)$ and consider the family (2.5). Under the transformation

$$X = \widehat{X}, \quad Y = \widehat{Y} + (\frac{1}{2} - a)\widehat{X}^2 \tag{2.12}$$

the family (2.5) becomes

$$\begin{cases} \dot{\widehat{X}} &= \mu_2 + \widehat{Y} + \frac{1}{2}\widehat{X}^2 \\ \dot{\widehat{Y}} &= \mu_1 - (1-2a)\mu_2\widehat{X} + (\frac{1}{2} - a)\mu_3\widehat{X}^2 + \widehat{Y}(\mu_3 + 2a\widehat{X} + \varepsilon_2\widehat{X}^2 + \widehat{X}^3 h_{12}) \\ & \quad + \widehat{X}^4 h_{22} + \widehat{Y}^2 Q_2(\widehat{X}, \widehat{Y}, \mu) \end{cases} \tag{2.13}$$

To eliminate the terms $\widehat{X}, \widehat{X}^2$ in the second equation of (2.13), we make the following transformation

$$\begin{cases} \widehat{X} &= \widetilde{X} + \beta_1 \\ \widehat{Y} &= \widetilde{Y} + \beta_2 + \beta_3 \widetilde{X} + \beta_4 \widetilde{X}^2 \end{cases} \tag{2.14}$$

then we have

$$\begin{cases} \dot{\widetilde{X}} &= \widetilde{Y} + \mu_2 + \frac{1}{2}\beta_1^2 + \beta_2 + (\beta_1 + \beta_3)\widetilde{X} + (\frac{1}{2} + \beta_4)\widetilde{X}^2 \\ \dot{\widetilde{Y}} &= \mu_1 + O(|\mu, \beta|^2) + ((-1 + 2a)\mu_2 + 2a\beta_2 + O(|\mu, \beta|^2))\widetilde{X} \\ & +((\frac{1}{2} - 1)\mu_3 - \varepsilon_2\beta_2 + 2(a - \frac{1}{4})\beta_3 + O(|\mu, \beta|^2))\widetilde{X}^2 \\ & +(d_1\beta_2 - \varepsilon_2\beta_3 - (1 - 2a)\beta_4 + 4d_2\beta_1 + O(|\beta|^2))\widetilde{X}^3 \\ & +\widetilde{Y}\Big[\mu_3 + 2a\beta_1 - \beta_3 + O(|\mu, \beta|^2) + (2a + O(|\mu, \beta|))\widetilde{X} \\ & \quad +(\varepsilon_2 + 3d_1\beta_1 + O(|\mu, \beta|^2))\widetilde{X}^2 + \widetilde{X}^3 h_{13}(\widetilde{X}, \mu, \beta)\Big] \\ & +\widetilde{X}^4 h_{23}(\widetilde{X}, \mu, \beta) + \widetilde{Y}^2 Q_3(\widetilde{X}, \widetilde{Y}, \mu, \beta). \end{cases} \tag{2.15}$$

Let

$$\begin{cases} \beta_1 + \beta_3 = 0 \\ (-1 + 2a)\mu_2 + 2a\beta_2 + O(|\mu, \beta|^2) = 0 \\ (\frac{1}{2} - 1)\mu_3 - \varepsilon_2\beta_2 + 2(a - \frac{1}{4})\beta_3 + O(|\mu, \beta|^2) = 0 \\ d_1\beta_2 - \varepsilon_2\beta_3 - (1 - 2a)\beta_4 + 4d_2\beta_1 + O(|\beta|^2) = 0. \end{cases} \tag{2.16}$$

Since $a(0) < 0$, we can solve (2.16) for $\beta_i (i = 1, 2, 3, 4)$:

$$\begin{cases} \beta_1 &= \frac{1-2a}{a(4a-1)}(\varepsilon_2\mu_2 + a\mu_3) + O(|\mu|^2) \\ \beta_2 &= \frac{1-2a}{2a} + O(|\mu|) \\ \beta_3 &= -\frac{1-2a}{a(4a-1)}(\varepsilon_2\mu_2 + a\mu_3) + O(|\mu|^2) \\ \beta_4 &= \frac{1}{2a(4a-1)}\Big[(4a-1)d_1 - 2\varepsilon_2^2 + 8\varepsilon_2 d_2)\mu_2 + 2a(4d_2\varepsilon_2)\mu_3\Big] + O(|\mu|^2). \end{cases} \tag{2.17}$$

So family (2.15) becomes

$$\begin{cases} \dot{\widetilde{X}} &= \widetilde{Y} + \tilde{\mu}_2 + \tilde{a}\widetilde{X}^2 \\ \dot{\widetilde{Y}} &= \tilde{\mu}_1 + \widetilde{Y}\Big[\tilde{\mu}_3 + \tilde{b}_1\widetilde{X} + \tilde{\varepsilon}_2\widetilde{X}^2 + \widetilde{X}^3 h_{14}(\widetilde{X}, \mu)\Big] \\ & \quad +\widetilde{X}^4 h_{24}(\widetilde{X}, \mu) + \widetilde{Y}^2 Q_4(\widetilde{X}, \widetilde{Y}, \mu) \end{cases} \tag{2.18}$$

where

$$\begin{cases} \tilde{a} &= \frac{1}{2} + \beta_4 = \frac{1}{2} + O(|\mu|) \\ \tilde{b}_1 &= 2(a + \varepsilon_2\beta_1 - \beta_4) + O(|\mu|^2) \\ \tilde{\varepsilon}_2 &= \varepsilon_2 + 2d_1\beta_1 + O(|\mu|^2) \\ \tilde{\mu}_1 &= \mu_1 + O(|\mu|^2) \\ \tilde{\mu}_2 &= \frac{1}{2a}\mu_2 + O(|\mu|^2) \\ \tilde{\mu}_3 &= \frac{(1-4a^2)\varepsilon_2}{a(4a-1)}\mu_2 + \frac{4a(1-a)}{4a-1}\mu_3 + O(|\mu|^2). \end{cases} \tag{2.19}$$

Rescaling in (2.18) by

$$\widetilde{X} = \frac{\tilde{\tilde{X}}}{\tilde{b}_1}, \qquad \widetilde{Y} = \frac{\tilde{\tilde{Y}}}{\tilde{b}_1}$$

we get a family

$$\begin{cases} \dot{\tilde{\tilde{X}}} &= \tilde{\tilde{Y}} + \tilde{\mu}_2 + a'\tilde{\tilde{X}}^2 \\ \dot{\tilde{\tilde{Y}}} &= \tilde{\mu}_1 + \tilde{\tilde{Y}}\Big[\tilde{\mu}_3 + \tilde{\tilde{X}} + \tilde{\varepsilon}_2\tilde{\tilde{X}}^2 + \tilde{\tilde{X}}^3 h_{14}(\tilde{\tilde{X}}, \tilde{\mu})\Big] \\ & \quad + \tilde{\tilde{X}}^4 h_{24}(\tilde{\tilde{X}}, \mu) + \tilde{\tilde{Y}}^2 Q_4(\tilde{\tilde{X}}, \tilde{\tilde{Y}}, \tilde{\mu}) \end{cases} \tag{2.20}$$

where $a' = \frac{1}{4a} + O(|\tilde{\mu}|)$ and $a'(0) = \frac{1}{4a(0)} \in (-\infty, -\frac{1}{2})$, also for the new parameter $(\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3)$ we have

$$\det\left(\frac{\partial(\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3)}{\partial(\mu_1, \mu_2, \mu_3)}\right)\Big|_{\mu=0} = \frac{4a(0)(1-a(0))}{(4a(0)-1)} \neq 0. \tag{2.21}$$

Hence for the saddle case, family (2.5) with $a(0) \in (-\frac{1}{2}, 0)$ and $a(0) \in (-\infty, -\frac{1}{2})$ are C^∞-equivalent.

By the above process and (2.21), we can see that for the elliptic case, family (2.5) with $a(0) \in (\frac{1}{2}, 0)$ is C^∞-equivalent to family (2.5) with $a(0) \in (\frac{1}{2}, \infty)$ except for $a(0) = \frac{1}{4}$. For the family (2.5) with $a(0) = \frac{1}{4}$, it is C^∞-equivalent to the family

$$\begin{cases} \dot{\widetilde{X}} &= \tilde{\mu}_2 + \widetilde{Y} + \widetilde{X}^2 \\ \dot{\widetilde{Y}} &= \tilde{\mu}_1 + \frac{1}{2}\tilde{\mu}_3\widetilde{X}^2 + \widetilde{Y}\Big[\tilde{\mu}_3 + \widetilde{X} + \tilde{\varepsilon}_2\widetilde{X}^2 + \widetilde{X}^3 h_{12}(\widetilde{X}, \tilde{\mu})\Big] \\ & \quad + \widetilde{X}^4 h_{22}(\widetilde{X}, \tilde{\mu}) + \widetilde{Y}^2 Q_2(\widetilde{X}, \widetilde{Y}, \tilde{\mu}) \end{cases} \tag{2.22}$$

where

$$\begin{cases} \tilde{\mu}_1 &= \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2\mu_3 \\ \tilde{\mu}_2 &= \mu_2 \\ \tilde{\mu}_3 &= \mu_3. \end{cases}$$

So except for $a(0) = \frac{1}{4}$ and $a(0) = 1$ for the elliptic case, we will use the equivalent unfoldings depending on the context. One passes from one unfolding to the other by means of the changes (2.12), (2.14) and the rescaling.

If $\varepsilon_2 = 0$, i.e., if the associated quantity $\mathcal{Q}$ in (2.2) vanishes, the 3-parameter unfolding (2.5) is not universal. In this case, the codimension of the nilpotent singularity is at least 4. Theorem 2.3 in the quadratic systems gives examples. □

2.2 Normal forms unfolding the nilpotent singularity of saddle or elliptic type for quadratic systems

In view of application to Hilbert's 16th problem for quadratic systems, we discuss the normal form of nilpotent singularities of quadratic systems. In [DF91], there is a detailed classification of the nilpotent singularity of saddle or elliptic type for quadratic systems. To have a better understanding of the meaning of the quantity $\mathcal{Q}$ defined in (2.2), we make a little modification of the classification.

By Jordan normal form theorem, we can write the quadratic system with a triple nilpotent singular point of saddle or elliptic type at the origin in the form

$$\begin{cases} \dot{x}_1 &= y_1 + a_1x_1^2 + b_1x_1y_1 + c_1y_1^2 \\ \dot{y}_1 &= e_1x_1y_1 + f_1y_1^2 \end{cases} \tag{2.23}$$

where $a_1e_1 \neq 0$.

By a linear transformation

$$\begin{cases} x_1 &= \frac{1}{e_1}x_2 - \frac{f_1}{e_1^2}y_2 \\ y_1 &= \frac{1}{e_1}y_2 \end{cases}$$

system (2.23) is equivalent to

$$\begin{cases} \dot{x}_2 &= y_2 + a_2 x_2^2 + b_2 x_2 y_2 + c_2 y_2^2 \\ \dot{y}_2 &= x_2 y_2 \end{cases} \tag{2.24}$$

where

$$\begin{aligned} a_2 &= \tfrac{a_1}{e_1} \\ b_2 &= \tfrac{b_1 e_1 + (e_1 - 2a_1) f_1}{e_1^2} \\ c_2 &= \tfrac{c_1}{e_1} - \tfrac{b_1 f_1}{e_1^2} + \tfrac{a_1 f_1^2}{e_1^3} \end{aligned}$$

also $\det(\frac{\partial(a_2,b_2,c_2)}{\partial(a_1,b_1,c_1)}) = \frac{1}{e_1^3} \neq 0$.

Adding an additional singular point on the y-axis, we should have $c_2 \neq 0$. This singular point is an anti-saddle if $c_2 < 0$. By rescaling

$$\begin{cases} x &= \sqrt{-c_2}\, x_2 \\ y &= -c_2 y_2 \\ \tau &= \frac{t}{\sqrt{-c_2}} \end{cases}$$

then we can take $c_2 = -1$. So a quadratic system with a nilpotent singular point at the origin and an anti-saddle is linearly equivalent to

$$\begin{cases} \dot{x} &= y + ax^2 + cxy - y^2 \\ \dot{y} &= xy. \end{cases} \tag{2.25}$$

where $a \neq 0$ and $c \in \mathbb{R}$.

Using the Takens normal form theory [T74] and by a near-identity transformations we obtain a C^∞-equivalent system

$$\begin{cases} \dot{u} &= v \\ \dot{v} &= -au^3 + v\Big[(1+2a)u + \frac{1}{2}cu^2 + O(u^3)\Big] - \frac{1}{2}acu^4 + v^2 O(|u,v|^3). \end{cases} \tag{2.26}$$

For system (2.26), the associated quantity $\mathcal{Q}$ becomes

$$\mathcal{Q} = ac(3a-1).$$

Hence we have the following theorem.

Theorem 2.3. *If $\mathcal{Q} \neq 0$ and $a \neq -\frac{1}{2}$, then the nilpotent singularity is of codimension 3, and system* (2.26) *is C^∞-equivalent to*

$$\begin{cases} \dot{u} & := v \\ \dot{v} & = -au^3 + v\Big[(1+2a)u - \frac{\mathcal{Q}}{5}u^2 + O(u^3)\Big] + v^2 O(|u,v|^3). \end{cases} \tag{2.27}$$

Furthermore, for quadratic system (2.25)*, we have the classification:*

- $a < 0$: *nilpotent saddle*
  - $c \neq 0$
    - $a \neq -\frac{1}{2}$ *nilpotent saddle of codimension 3*
    - $a = -\frac{1}{2}$ *nilpotent saddle of codimension 4* *(corresponds to $b = 0$ in* (2.4)*)*
  - $c = 0$ *nilpotent saddle of codimension ∞* *(corresponds to $\varepsilon_2 = 0$)*
- $a > 0$: *nilpotent elliptic point*
  - $c \neq 0$
    - $a \neq \frac{1}{2}, \frac{1}{3}$ *elliptic singularity of codimension 3*
    - $a = \frac{1}{2}$ *elliptic singularity of codimension 4, type 1* *(corresponds to $b = 2\sqrt{2}$)*
    - $a = \frac{1}{3}$ *elliptic singularity of codimension 4, type 2* *(corresponds to $\varepsilon_2 = 0$)*
  - $c = 0$ *elliptic singularity of codimension ∞* *(corresponds to $\varepsilon_2 = 0$ in* (2.4)*).*

Remark 2.4. *If $\mathcal{Q} = 0$, i.e., $\varepsilon_2 = 0$, then for the nilpotent saddle, the singularity can only be of codimension ∞; for the elliptic case, the singularity is either of codimension 4, type 2, or infinity.*

Proposition 2.5. *For further application to quadratic systems, let us remark that a general 5-parameter quadratic family unfolding a nilpotent singularity of codimension 3 of the saddle or elliptic type at the origin and with an anti-saddle in the upper half plane can be written as*

$$\begin{cases} \dot{x} &= \mu_2 + y + ax^2 + cxy - y^2 \\ \dot{y} &= \mu_1 + (\mu_3 + x)y \end{cases} \qquad a \neq 1 \tag{2.28}$$

or

$$\begin{cases} \dot{x} &= \mu_2 + y + (1+\mu_4)x^2 + cxy - y^2 \\ \dot{y} &= \mu_1 + (\mu_3 + x)y + \frac{1}{2}\mu_3 x^2. \end{cases} \tag{2.29}$$

Hence they have the same blow-up as family (2.5).

Chapter 3

Generalities on the blow-up of the family

3.1 Blow-up of the family

Consider the normal form unfolding the nilpotent singularity of saddle or elliptic type

$$X : \begin{cases} \dot{x} &= y + \mu_2 + ax^2 \\ \dot{y} &= \mu_1 + y(\mu_3 + x + \varepsilon_2 x^2 + x^3 h_1(x,\mu)) + x^4 h_2(x,\mu) + y^2 Q(x,y,\mu) \end{cases} \tag{3.1}$$

where $a \in [-\frac{1}{2}, 0)$ saddle case; $a \in (0, \frac{1}{2}]$ elliptic case, $\mu = (\mu_1, \mu_2, \mu_3)$ is the parameter, and $h_1(x,\mu)$, $h_2(x,\mu) = \varepsilon_1\varepsilon_2 a + O(x)$ and $Q(x,y,\mu)$ are C^∞ and also $Q(x,y,\mu)$ has arbitrarily high order in (x,y,μ).

From now on, we denote $A = (0, \frac{1}{2})$ for the elliptic case, $A = (-\frac{1}{2}, 0)$ for the saddle case.

We are interested in this family for $a \in A$ and $(x,y,\mu) \in U \times \Lambda$, a neighborhood of $(0,0)$ in $\mathbb{R}^2 \times \mathbb{R}^3$. Λ can be identified to $\mathbb{S}^2 \times [0, \nu_0)$ through the change of parameters. Making the change of the parameters

$$\begin{cases} \mu_1 &= \nu^3 \bar{\mu}_1 \\ \mu_2 &= \nu^2 \bar{\mu}_2 \\ \mu_3 &= \nu \bar{\mu}_3 \end{cases} \tag{3.2}$$

where $(\bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3) \in \mathbb{S}^2$ and $\nu \in (0, \nu_0)$, we have a 3-parameter family of vector fields in (x, y, ν) space with parameters $(a, \bar{\mu}) \in A \times \mathbb{S}^2$:

$$\widehat{X} : \begin{cases} \dot{x} &= y + \nu^2 \bar{\mu}_2 + ax^2 \\ \dot{y} &= \nu^3 \bar{\mu}_1 + y \Big[\nu \bar{\mu}_3 + x + \varepsilon_2 x^2 + x^3 h_1(x, \nu\bar{\mu}) \Big] \\ & \qquad + x^4 h_2(x, \nu\bar{\mu}) + y^2 Q(x, y, \nu\bar{\mu}) \\ \dot{\nu} &= 0. \end{cases} \tag{3.3}$$

We then make the (weighted) blow-up of the singular point of (3.3) at the origin by

$$\begin{cases} x &= r\bar{x} \\ y &= r^2 \bar{y} \\ \nu &= r\rho \end{cases} \tag{3.4}$$

where $r > 0$ and $(\bar{x}, \bar{y}, \rho) \in \mathbb{S}^2$.

By the blow up (3.4), we have a C^∞–family $\overline{X} = \frac{1}{r}\widehat{X}$. For each $(a, \bar{\mu}) \in A \times \mathbb{S}^2$, $\overline{X}$ induces a 3-dimensional vector field $\overline{X}_{(a,\bar{\mu})}$ defined in the neighborhood of $\mathbb{S}^2 \times \{0\}$ with parameters $(a, \bar{\mu}) \in A \times \mathbb{S}^2$. In other words, the blow-up (3.4) changes the 4-parameter 2-dimensional family into a 3-dimensional family $\overline{X}_{(a,\bar{\mu})}$ with parameters $(a, \bar{\mu}) \in A \times \mathbb{S}^2$.

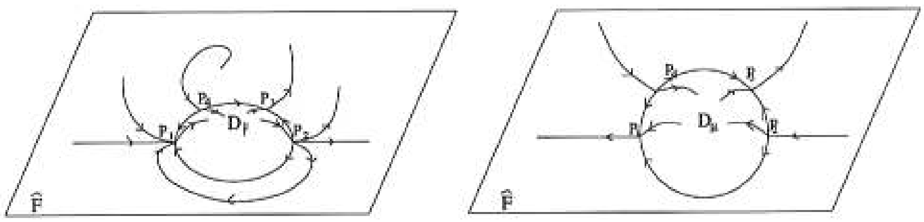
Putting together (3.2) and (3.4), as in [DRS97], for (3.1), at $(x, y, \mu_1, \mu_2, \mu_3) = (0, 0, 0, 0, 0)$, we make the global blow-up

$$\Phi : \ \mathbb{S}^2 \times \mathbb{R}^+ \times \mathbb{S}^2 \longrightarrow \mathbb{R}^5$$

$$((\bar{x}, \bar{y}, \rho), r, (\bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3)) \mapsto (x, y, \mu_1, \mu_2, \mu_3)$$

$$\begin{cases} x &= r\bar{x} \\ y &= r^2\bar{y} \\ \mu_1 &= r^3\rho^3\bar{\mu}_1 \\ \mu_2 &= r^2\rho^2\bar{\mu}_2 \\ \mu_3 &= r\rho\bar{\mu}_3 \end{cases} \tag{3.5}$$

where $\bar{x}^2 + \bar{y}^2 + \rho^2 = 1$.



(a) The elliptic case (b) The saddle case

Figure 3.1: The stratified set $\{r\rho = 0\}$ in the blow-up

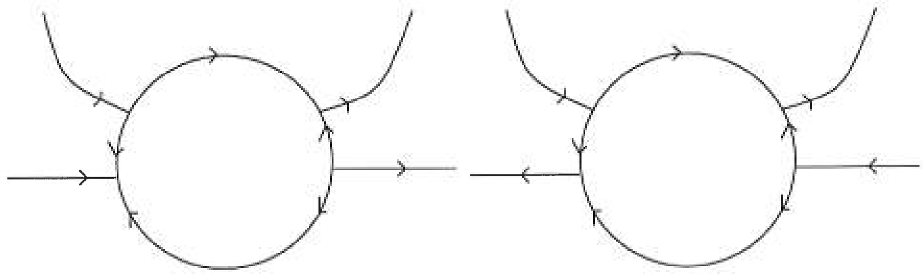
Because of the symmetry, as in Fig. 3.1, we only need to study $\bar{X}$ on $\{\rho \geq 0\}$ to get a complete information for $((\bar{x}, \bar{y}, \rho), r, (\bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3))$ near $0 \in \mathbb{S}^2 \times [0, r_0) \times \mathbb{S}^2$. Note that for each $\bar{\mu}$, the foliation given by $\{\nu = r\rho = const\}$ is preserved by $\overline{X}_{(a,\bar{\mu})}$:

- For $\{r\rho = \nu\}$ with $\nu > 0$, the leaf is a regular manifold of dimension 2.
- For $\{r\rho = 0\}$, we get a stratified set in the critical locus. As shown in Fig. 3.1, there are two strata of 2-dimensional manifolds:
  - $\hat{F}_{\bar{\mu}} \cong S^1 \times R^+$ the blow-up of the fiber $\mu = 0$,
  - $D_{\overline{\mu}} = \{\bar{x}^2 + \bar{y}^2 + \rho^2 = 1,\ \rho \geq 0\}$.

On $\hat{F}_{\bar{\mu}} = \{\rho = 0\}$, (3.5) is just the common blow-up of the nilpotent point:

$$\begin{cases} x = r\bar{x} \\ y = r^2\bar{y} \end{cases} \tag{3.6}$$

and by the blow-up (3.6), we get a vector field with four singular points P_i ($i = 1, 2, 3, 4$). P_3 and P_4 are hyperbolic saddles, P_1 and P_2 are nodes (resp. saddles) in the elliptic (resp. saddle) case (Fig. 3.2).



(a) The elliptic case (b) The saddle case

Figure 3.2: Common blow-up of the nilpotent singularity

To study the objects on and near $\hat{F}_{\bar{\mu}}$, we use the "phase directional rescaling". We use charts

$$\mathbf{P.R.1} \quad \bar{x} = -1, (r_1, \rho_1, \bar{y}_1)$$
$$\mathbf{P.R.2} \quad \bar{x} = \ \ 1, (r_2, \rho_2, \bar{y}_2)$$

which cover the boundary of the half 2-sphere.

In the chart **P.R. 1** and **P.R. 2**, by transformation ($i = 1, 2$)

$$\begin{cases} x &= \mp r_i \\ y &= r_i^2 \bar{y}_i \\ \mu_j &= (r_i \rho_i)^{4-j} \bar{\mu}_j \qquad\qquad (j = 1, 2, 3) \end{cases} \tag{3.7}$$

and after division by r_i, we get a vector field near P_i ($i = 1, 2$)

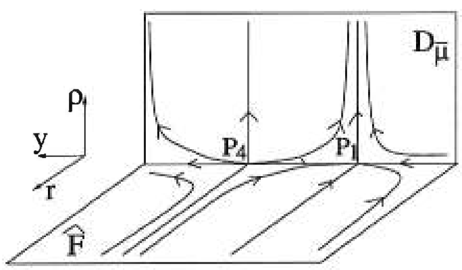
$$\mathbf{\overline{X}_{P_i}} \begin{cases} \dot{r}_i &= \mp(a + \bar{y}_i + \bar{\mu}_2 \rho_i^2) r_i \\ \dot{\rho}_i &= \pm(a + \bar{y}_i + \bar{\mu}_2 \rho_i^2) \rho_i \\ \dot{\bar{y}}_i &= \mp(1 - 2a)\bar{y}_i \pm 2\bar{y}_i^2 + \bar{y}_i[\varepsilon_2 r_i + \bar{\mu}_3 \rho_i \pm 2\bar{\mu}_2 \rho_i^2 \mp r_i^2 \bar{h}_1(\pm r_i, r_i\rho_i, \bar{\mu})] \\ & \quad + \bar{\mu}_1 \rho_i^3 + r_i \bar{h}_2(\pm r_i, r_i \rho_i, \bar{\mu}) + \bar{y}_i^2 \overline{Q}_2(r_i, \rho_i, \bar{y}_i, \bar{\mu}) \end{cases} \tag{3.8}$$

where $\bar{h}_1$ and $\bar{h}_2 = a\varepsilon_2 + O(r)$ are C^∞ in $(r_i, \rho_i, \bar{\mu})$, $\bar{Q}(r_i, \rho_i, \bar{y}_i, \bar{\mu})$ is C^∞ in $(r_i, \rho_i, \bar{y}_i, \bar{\mu})$ and of arbitrarily high order in $(r_i, \rho_i, \bar{y}_i)$.

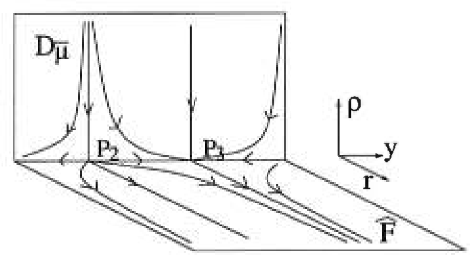
Easily, we see that $\overline{X}_{P_1}$ has two singular points $P_1(0, 0, 0)$ and $P_4(0, 0, \frac{1-2a}{2})$, $\overline{X}_{P_2}$ has two singular points $P_2(0, 0, 0)$ and $P_3(0, 0, \frac{1-2a}{2})$. Each singularity has three real eigenvalues listed in Tab. 3.1. In Fig. 3.3, we draw the phase portraits in the charts **P.R.1**, **P.R.2** for the elliptic and saddle case respectively. The chart **P.R.3** ($\bar{y} = \pm 1$) gives no additional singular points other than those found in the charts **P.R.1** and **P.R.2**.

	r	ρ	y
P_1	$-a$	a	$-(1-2a)$
P_2	a	$-a$	$(1-2a)$
P_3	$1/2$	$-1/2$	$-(1-2a)$
P_4	$-1/2$	$1/2$	$(1-2a)$

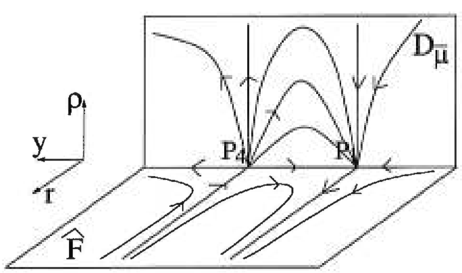
Table 3.1: The eigenvalues at P_i ($i = 1, 2, 3, 4$)



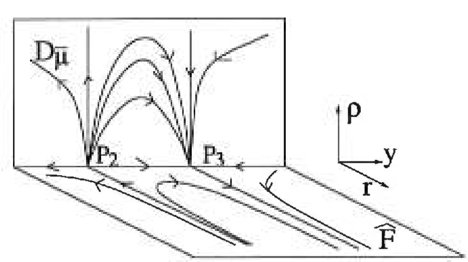
(a) $\overline{X}_{P_1}$, the elliptic case



(b) $\overline{X}_{P_2}$, the elliptic case



(c) $\overline{X}_{P_1}$, the saddle case



(d) $\overline{X}_{P_2}$, the saddle case

Figure 3.3: The phase portraits of $\overline{X}_{P_1}$

To complete the phase portrait on the blown-up sphere $D_{\bar{\mu}}$, we use the family rescaling and use the chart

$$\mathbf{F.R.} \quad \rho = 1, (\bar{x}, \bar{y}, r)$$

yielding

$$\begin{cases} \dot{\bar{x}} &= \bar{\mu}_2 + \bar{y} + a\bar{x}^2 \\ \dot{\bar{y}} &= \bar{\mu}_1 + (\bar{x} + \bar{\mu}_3)\bar{y} + r\bar{h}(\bar{x}, \bar{y}, r, \bar{\mu}) \\ \dot{r} &= 0 \end{cases} \tag{3.9}$$

where $\bar{h}(\bar{x}, \bar{y}, r, \bar{\mu})$ is C^∞ in $(\bar{x}, \bar{y}, r, \bar{\mu})$. Especially, on $\{r = 0\}$, we have

$$\overline{X}_{\rho=1} \begin{cases} \dot{\bar{x}} &= \bar{\mu}_2 + \bar{y} + a\bar{x}^2 \\ \dot{\bar{y}} &= \bar{\mu}_1 + (\overline{\mu}_3 + \bar{x})\bar{y} \end{cases} \tag{3.10}$$

In order to list all the possible limit periodic sets for the family, we have to study the bifurcation diagram of (3.10) for $\bar{\mu} \in \mathbb{S}^2$. Since we use charts, here we give the coordinates changes between the charts P.R. 1, P.R. 2, and F.R.:

$$\Phi_{i0} : \mathbf{P.R.i} \longrightarrow \mathbf{F.R} \begin{cases} \bar{x} &= \mp\dfrac{1}{\rho_i} \\ \bar{y} &= \dfrac{\bar{y}_i}{\rho_i^2} \\ \nu &= r_i\rho_i \end{cases} \quad (i = 1, 2). \tag{3.11}$$

3.2 Bifurcation diagrams for the family rescaling, limit periodic sets

In [KS95], the authors investigate generic 2- and 3-parameter smooth families of vector fields on the 2-dimensional sphere and gives the list of all polycycles of codimension less than 3 (also called "Kotova Zoo") and of their cyclicity, but with the nilpotent elliptic and saddle cases of codimension 3 left unknown. Following the convention of [KS95], we use **pp** to denote a graphic connecting two parabolic sectors, **hp** to denote the graphic coming out of a hyperbolic sector and connecting to a parabolic sector, and **hh** to denote the graphic connecting two hyperbolic sectors. Then to find out all the limit periodic sets after the blow-up, we have to answer the following two questions:

1. In the elliptic case, when does a pp, or hp or hh-graphic exist? i.e., when does a passage from P_1 to P_2, or P_1 to P_3, or P_4 to P_3 exist?

2. In the saddle case, we have two types of hh-graphic, one is a graphic which passes through P_3 and P_4 (or P_1 and P_2), the other type is a graphic which passes through P_4 and P_1 (or P_2 and P_3). For each of the graphics, there is always at least one family of limit periodic sets and there can be two. What are the bordering limit periodic sets for these families?

To answer the above two questions, we need to give the complete bifurcation diagrams of system (3.10). They correspond via $\bar{y}+\bar{\mu}_2+a\bar{x}^2 = Y$ to the bifurcation diagrams for the principal rescalings studied in [DRS91] and [DRc90]. Complete bifurcation diagrams have been given there except for the position of the separatries at infinity which are better studied in the quadratic model given here.

Proposition 3.1. *For system* (3.10), *there holds*

(1) System (3.10) *has an invariant line* $\bar{y} = 0$ *if and only if* $\bar{\mu}_1 = 0$.

- *In the elliptic case, the curve* $\bar{\mu}_1 = 0$ *is a bifurcation curve except when there are two nodes on the line* $\bar{y} = 0$.

- *In the saddle case, the curve* $\bar{\mu}_1 = 0$ *is a bifurcation curve precisely when there are two finite saddles on it.*

(2) If $a \neq \frac{1}{4}$, *system* (3.10) *has an invariant parabola*

$$y = (\frac{1}{2} - a)x^2 + \bar{\mu}_3\frac{1-2a}{1-4a}x + \frac{(1-2a)(2a\bar{\mu}_3^2 + (1-4a)^2\bar{\mu}_2}{a(1-4a)^2} \tag{3.12}$$

if and only if

$$\bar{\mu}_1 = \frac{2a(1-2a)}{(1-4a)^3}\bar{\mu}_3^3 + \frac{2(1-2a)}{1-4a}\bar{\mu}_2\bar{\mu}_3. \tag{3.13}$$

(3) If $a = \frac{1}{4}$, *system* (3.10) *has an invariant parabola if and only if* $\bar{\mu}_3 = 0$. *For* $\bar{\mu}_3 = 0$, *system* (3.10) *has 1, 2 or 3 invariant parabolas*

$$y = \frac{1}{4}x^2 + Bx + \bar{\mu}_2 + 2B^2 \tag{3.14}$$

if $27\bar{\mu}_1^2 + 16\bar{\mu}_3^3 > 0$, $= 0$ *or* < 0, *and* B *is the solution of the algebraic equation*

$$2B^3 + 2\bar{\mu}_2 B - \bar{\mu}_1 = 0.$$

Proof. Direct calculations. □ □

Now we turn to the bifurcation of system (3.10) for elliptic and saddle cases for $\bar{\mu} \in \mathbb{S}^2$.

Theorem 3.2. *The bifurcation diagram of* (3.10) *for the elliptic case* $(0 < a < \frac{1}{2})$ *is in Fig.* 3.4, *the bifurcation diagram for the saddle case* $(-\frac{1}{2} < a < 0)$ *is in Fig.* 3.5, *in both diagrams, we use the following nomenclature:* **BT** - *Bogdanov-Takens bifurcation,* **DH** - *Degenerate Hopf bifurcation,* **H** - *Hopf bifurcation,* **IL** - *Invariant line,* **SL** - *Saddle loop,* **SC** - *Saddle Connection,* **SN** - *Saddle Node bifurcation,* **SNC** - *Saddle Node Connection.*

All the limit periodic sets are listed in Tables 3.2-3.6.

Proof. The proof comes from the bifurcation diagram in [DRc90], from the existence of the invariant line $\bar{y} = 0$ for $\bar{\mu}_1 = 0$ and from the fact that this line becomes without contact for $\bar{\mu}_1 \neq 0$.

Indeed, the phase portraits in Fig. 3.4 and Fig. 3.5 can be completely recovered from the bifurcation diagram in [DRc90] and the phase portraits on $\bar{\mu}_1 = 0$. It is known for quadratic systems that any saddle connection between a finite and an infinite saddle must occur along an invariant line.

The bifurcation diagram of Fig. 3.5 is exactly the bifurcation diagram of [DRc90] with the additional information that the upper saddle connection occurs for $\bar{\mu}_1 = 0$ on the invariant line $\bar{y} = 0$ and the lower saddle connection occurs on the invariant parabola (3.11). □

Note that when we study $\overline{X}_{\rho=1}$ at infinity, we use the quasi-homogeneous compactification:

$$\bar{x} = \pm\frac{1}{z} \qquad \bar{y} = \frac{u}{z^2}. \tag{3.15}$$

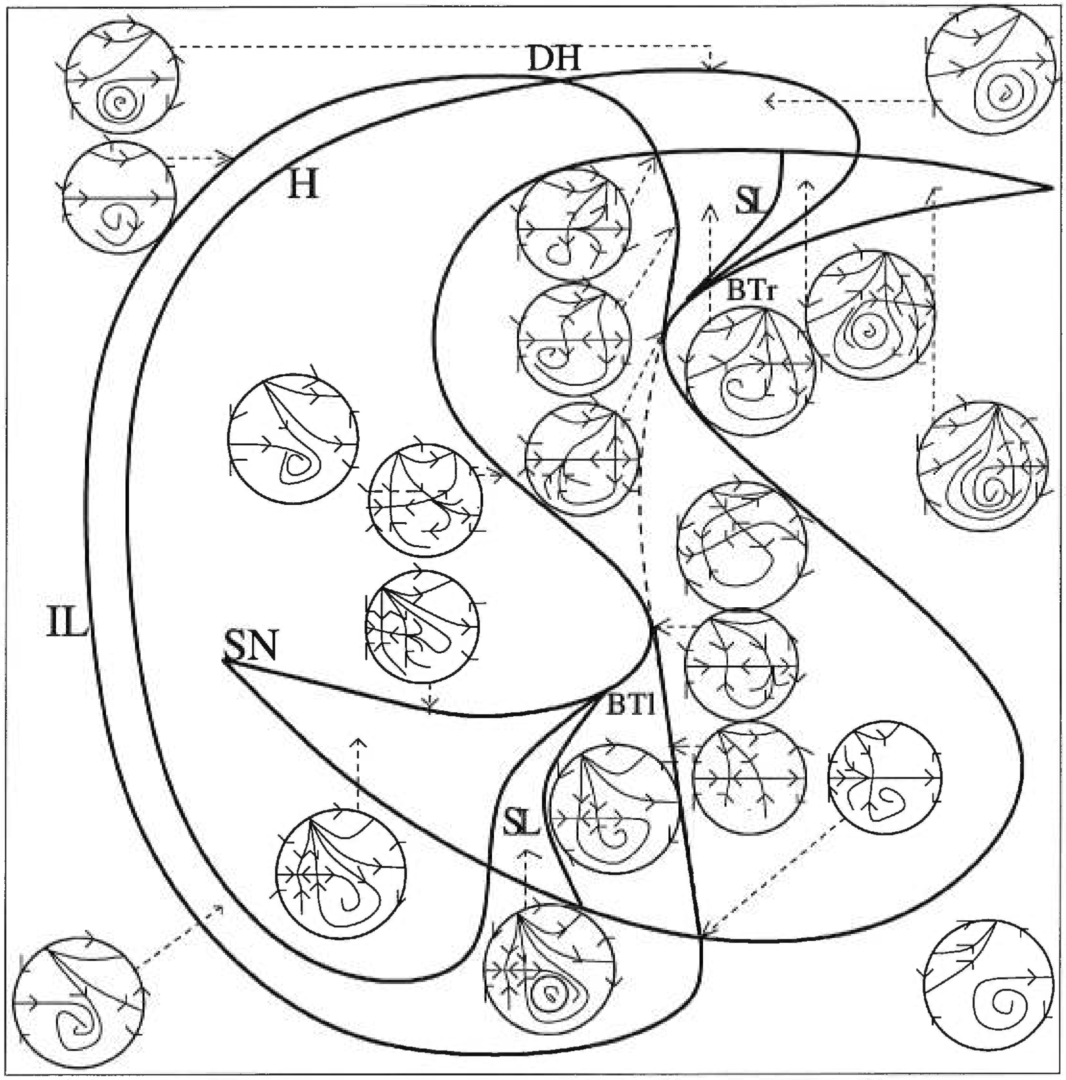


Figure 3.4: The bifurcation diagram of the rescaled family: the elliptic case $a(0) \neq \frac{1}{2}$

These transformations are just the transformations we used in charts **P.R.1** and **P.R.2**. So at infinity, we add what we obtained from these two charts and draw the bifurcation diagrams for the elliptic and saddle cases in Fig. 3.4 and Fig. 3.5.

For the elliptic case, there are 22 limit periodic sets which fall into three types: Epp, Ehp and Ehh. We list all the 22 graphics of the elliptic type in Tab. 3.2, 3.3 and 3.4.

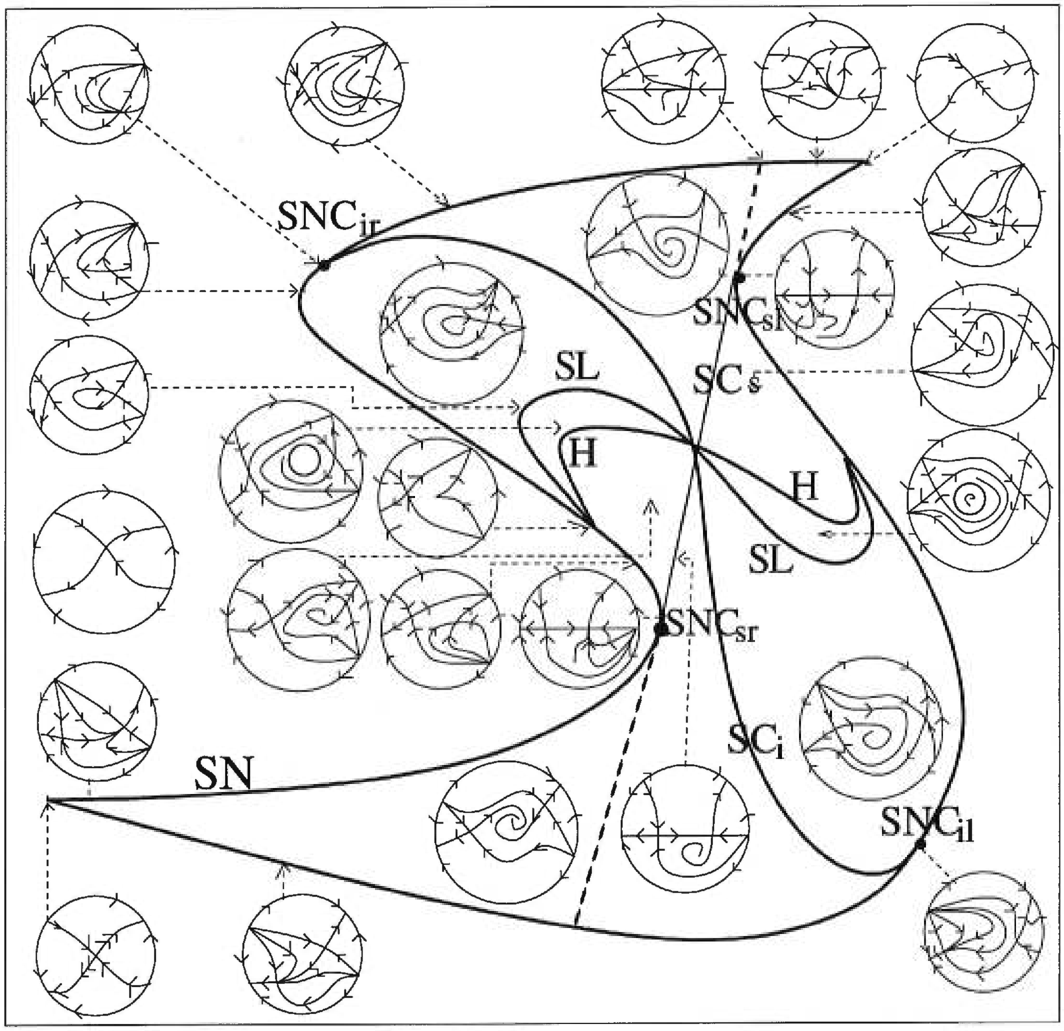


Figure 3.5: The bifurcation diagram of the rescaled family: the saddle case $a(0) \neq -\frac{1}{2}$

For the saddle case, there are two types of limit periodic sets: convex graphic Sxhh and concave graphic Sahh. We list all the possible graphics of saddle type in Tab. 3.5 and Tab. 3.6.

For all the families of graphics listed in the following tables, we use a to denote the upper boundary graphic, b or d to denote the intermediate graphics, c or e to denote the lower boundary graphic.

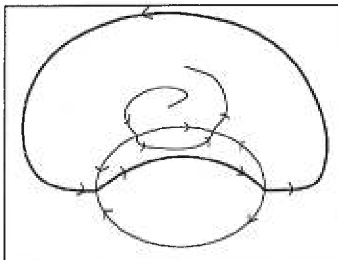
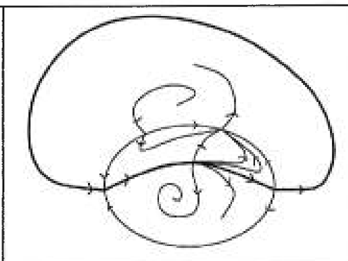
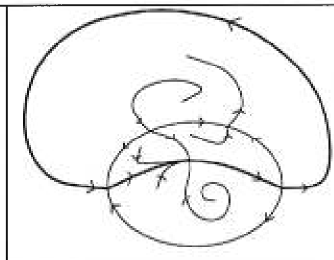
		
graphic Epp1	family of graphics Epp2	graphic Epp3

Table 3.2: Limit periodic sets of pp type for the Elliptic case

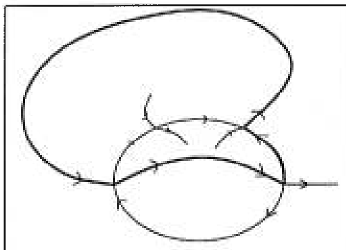
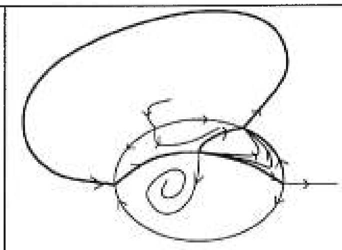
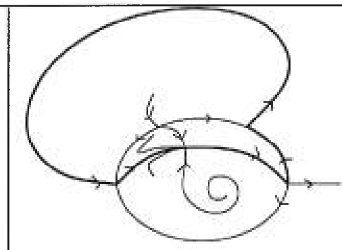
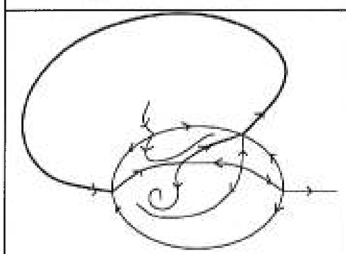
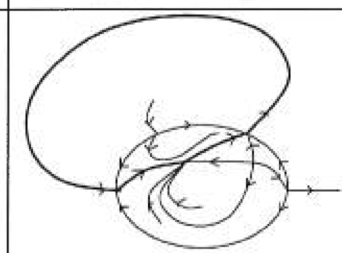
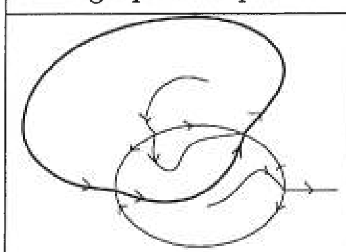
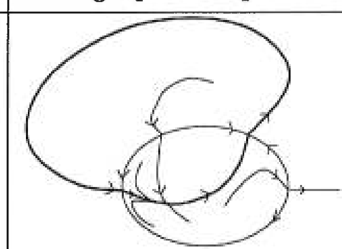
		
graphic Ehp1	graphic Ehp2a, b, c	graphic Ehp3
		
graphic Ehp4		graphic Ehp5
		
graphic Ehp6		graphic Ehp7

Table 3.3: Limit periodic sets of hp-type for the elliptic case

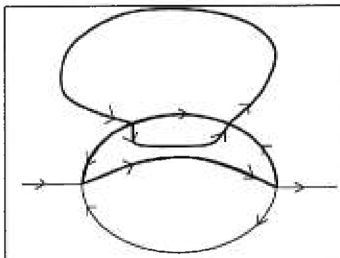
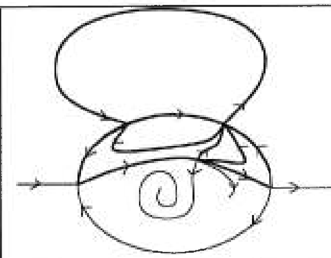
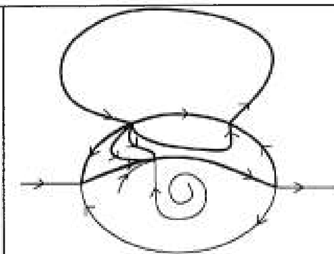
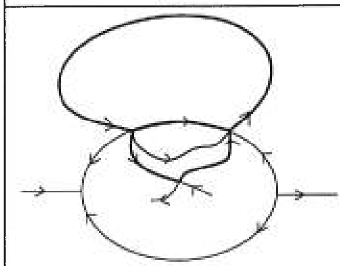
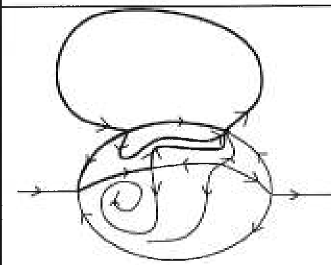
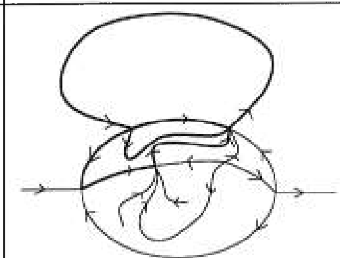
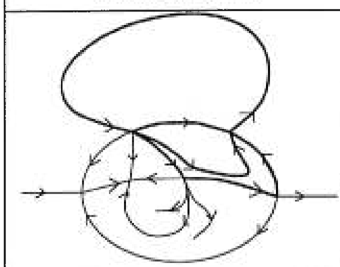
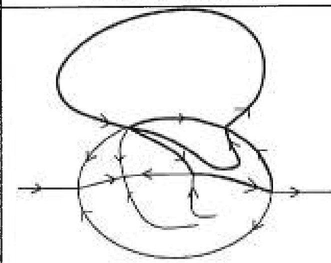
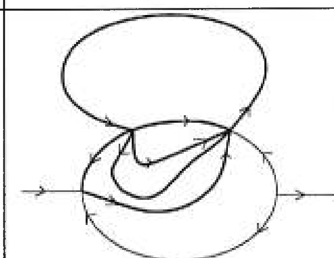
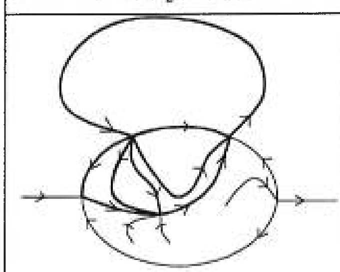
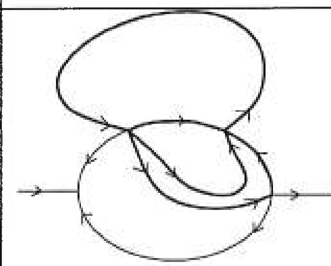
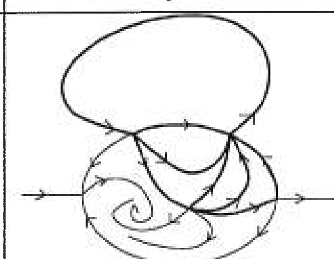
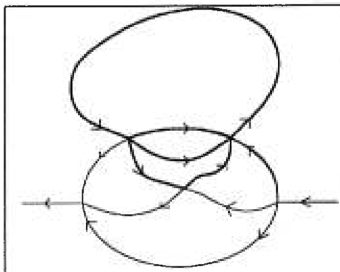
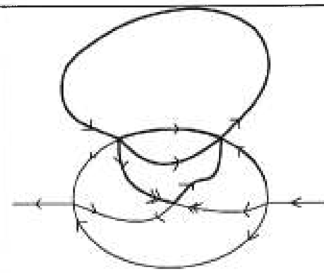
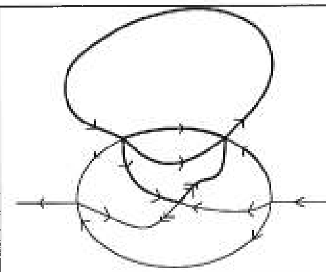
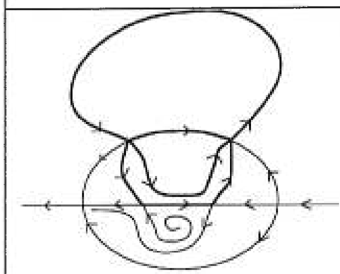
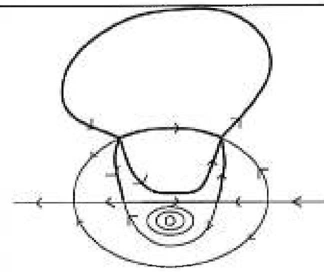
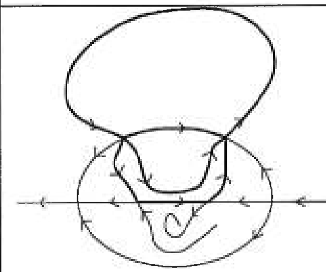
		
family Ehh1	family Ehh2	family Ehh3
		
family Ehh4	family Ehh5	family Ehh6
		
family Ehh7	family Ehh8	family Ehh9
		
family Ehh10	family Ehh11	family Ehh12

Table 3.4: Limit periodic sets of hh-type for the elliptic case

		
Sxhh1	Sxhh2	Sxhh3
		
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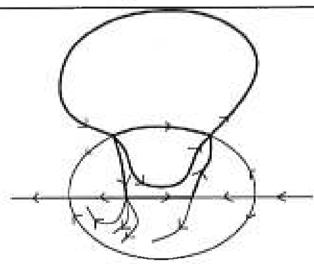		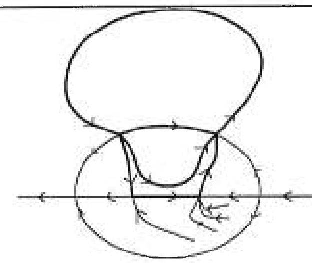
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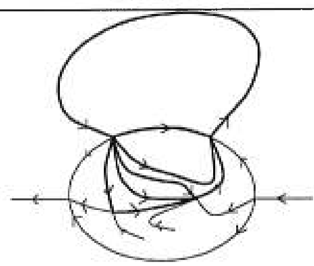
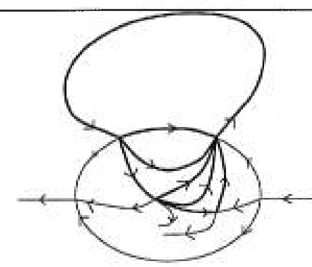
		
Sxhh9		Sxhh10

Table 3.5: Convex limit periodic sets of hh-type for the saddle case

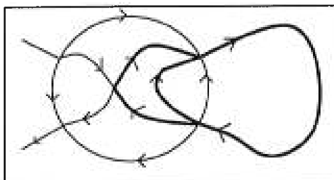
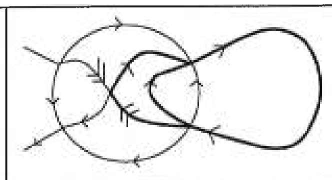
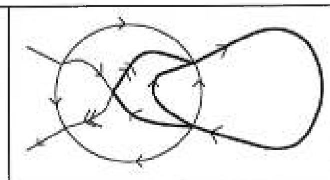
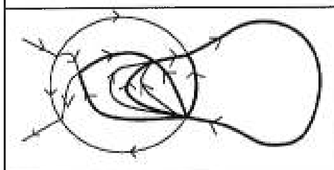
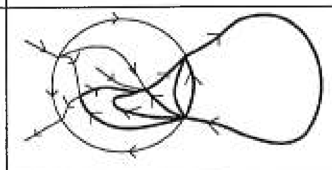
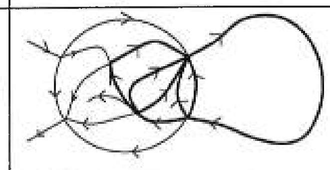
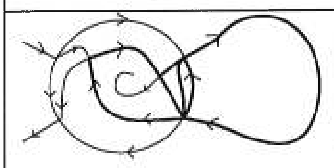
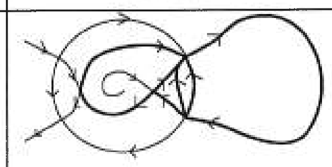
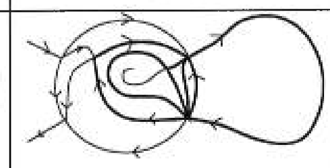
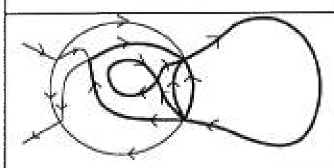
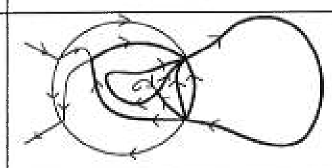
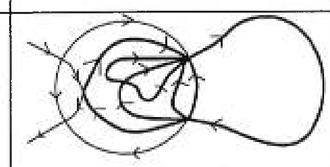
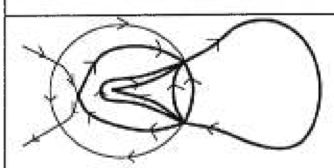
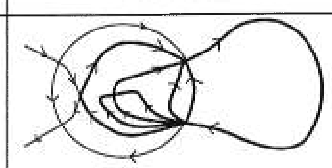
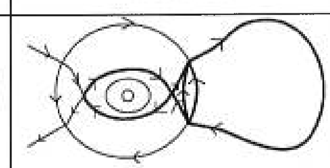
		
Sahh1	Sahh2	Sahh3
		
Sahh4	Sahh5	Sahh6
		
Sahh7	Sahh8	Sahh9
		
Sahh10	Sahh11	Sahh12
		
Sahh13	Sahh14	Sahh15

Table 3.6: Concave limit periodic sets of hh-type for the saddle case

Chapter 4

Dulac maps at the entrance points of the blown-up sphere

To study the cyclicity of the graphics after the global blow-up, we will need some basic properties of the transition maps in the neighborhood of an elementary singular point. First we give some definitions and some results about the transition maps near an elementary singular point. Since we use three dimensional charts to study the object, in §4.2 and §4.3 we will establish two types of Dulac maps in the neighborhood of a three dimensional hyperbolic singular point.

4.1 Transition maps near the elementary singular points in the plane

Definition 4.1. *(1) A singular point is elementary if it has at least one nonzero eigenvalue. It is hyperbolic (resp. semi-hyperbolic) if the two eigenvalues are not on the imaginary axis (resp. exactly one eigenvalue is zero).*

(2) The hyperbolicity ratio at a hyperbolic saddle is the ratio $r = -\frac{\lambda_1}{\lambda_2}$*, where* $\lambda_1 < 0 < \lambda_2$ *are the two eigenvalues.*

Let X_λ, $\lambda \in \Lambda$, be a C^∞ family of vector fields defined in the neighborhood of a hyperbolic saddle at the origin. We also assume that the coordinates axes are the invariant manifolds near the saddle point.

By normal form theory, for any fixed $k \in \mathbb{N}$, up to C^k-equivalence, we can write the vector field X_λ into some explicit expressions of the normal form (cf. [St58], [IY91]). Let r_λ be the hyperbolicity ratio of X_λ at the origin, then

- If r_0 is irrational, then , $\forall k \in \mathbb{N}$, the vector field X_λ is C^k-equialent to

$$\begin{cases} \dot{x} &= x \\ \dot{y} &= -r(\lambda)y \end{cases}$$

for λ in some neighborhood W of the origin in parameter space.

- If $r_0 \in \mathbb{Q}$, let $r_0 = \frac{p}{q}$, $(p,q) = 1$. Then $\forall k \in \mathbb{N}$, X_λ is C^k-equivalent to

$$\begin{cases} \dot{x} &= x \\ \dot{y} &= y\Big[- r_0 + \sum_{i=0}^{N(k)} \alpha_{i+1}(\lambda)(x^p y^q)^i \Big]. \end{cases}$$

for λ in some neighborhood W of the origin in parameter space. In particular, $\alpha_1 = r_0 - r(\lambda)$.

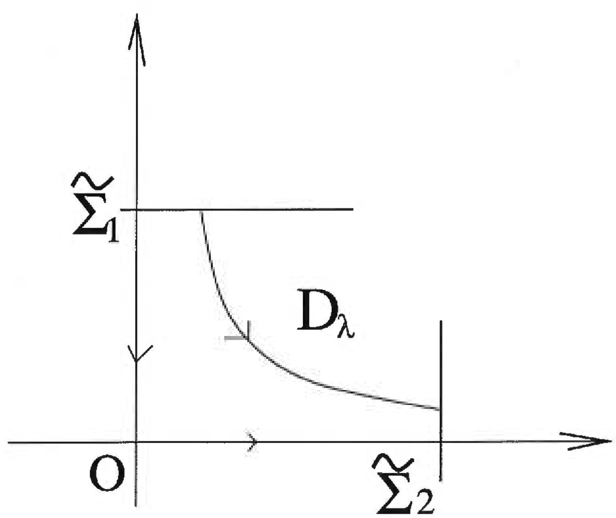


Figure 4.1: Dulac map near a hyperbolic saddle

Let $\widetilde{\Sigma}_1 = \{y = y_0\}$ and $\widetilde{\Sigma}_2 = \{x = x_0\}$ be two sections transverse to the vector field X_λ (Fig. 4.1), where $x_0, y_0 > 0$ constant. The flow of X_λ induces a transition map $D_\lambda(.,\lambda)$, also called a Dulac map:

$$D_\lambda : \widetilde{\Sigma}_1 \longrightarrow \widetilde{\Sigma}_2$$

for all $\lambda \in W$.

The Dulac map is C^∞ for $x > 0$. The following theorem of Mourtada ([Mou90]) describes its behavior near $x = 0$.

Proposition 4.2. *(Mourtada) The Dulac map D_λ can be written as*

$$D_\lambda(x) = x^{r(\lambda)}[c(\lambda) + \psi(x,\lambda)] \tag{4.1}$$

where $c(\lambda) = \frac{y_0}{x_0^{r(\lambda)}}$, $\psi(x,\lambda)$ is C^∞ for $(x,\lambda) \in (0, x_0] \times W$. Furthermore, ψ satisfies the following property (I_0^∞):

$$(I_0^\infty): \qquad \forall n \in \mathbb{N}, \qquad \lim_{x\to 0} x^n \frac{\partial^n \psi}{\partial x^n}(x,\lambda) = 0 \qquad \textit{uniformly for } \lambda \in W. \tag{4.2}$$

(1) If $r_0 \in \mathbb{Q}$, then $\psi \equiv 0$;
(2) If $r_0 = 1$, then the expression (4.1) *is in general not fine enough for proving the cyclicity.*

Definition 4.3. *The Ecalle-Roussarie compensator of the vector field X_λ is defined as*

$$\omega(x,\alpha_1) = \begin{cases} \frac{x^{-\alpha_1}-1}{\alpha_1} & \textit{if } \alpha_1 \neq 0 \\ -\ln x & \textit{if } \alpha_1 = 0. \end{cases} \tag{4.3}$$

By the definition of ω, we can easily check $\omega(x,\alpha_1)$ has the following property:

Proposition 4.4.

$$\omega(ab,\alpha_1) = \omega(a,\alpha_1)(1 + \alpha_1\omega(b,\alpha_1)) + \omega(b,\alpha_1). \tag{4.4}$$

Since the Dulac map in Prop. 4.2 is not fine enough to prove the cyclicity for the case $r_0 = 1$, in [R86], by using the compensator, Roussarie has an additional refinement:

Proposition 4.5. *If $r_0 = 1$, then the Dulac map D_λ has a well-ordered asymptotic expansion:*

$$\begin{aligned} D_\lambda(x) &= \alpha_1(\lambda)[x\omega + \cdots] + \beta_1(\lambda)[x + \cdots] \\ &\quad + \alpha_2(\lambda)[x^2\omega + \cdots] + \alpha_k(\lambda)[x^k\omega + \cdots] + \psi_k(x, \lambda) \end{aligned} \tag{4.5}$$

where $\alpha_1(\lambda) = r(\lambda) - 1$, and ψ_k is a C^k function, k–flat with respect to $x = 0$.

4.2 Normal forms at the entrance points

To study the Dulac maps in the neighborhood of the entrance points, the vector fields should be in the normal form.

For saddle and elliptic case, the family of vector fields at each point P_i (i = 1, 2, 3, 4) has the same form as (3.8), the three eigenvalues not all having the same sign. Due to the special form of the family (3.8), after dividing by a C^∞ positive function, system (3.8) is linear in r and ρ. If necessary we change the time ($t \mapsto -t$), so that we have two negative eigenvalues while the third is positive (Tab. 3.1). So for the three eigenvalues at each point, there are only two possibilities

$$-1, \quad 1, \quad -\sigma(a)$$

or

$$1, \quad -1, \quad -\sigma(a)$$

where $\sigma(a) = \begin{cases} |\frac{1-2a}{a}| & \text{at } P_1 \text{ and } P_2 \\ 2(1-2a) & \text{at } P_3 \text{ and } P_4 \end{cases}$.

By exchanging the roles of r and ρ, we only need to consider one case of system (3.8) which we rewrite as

$$X_{(a,\bar{\mu})} \begin{cases} \dot{r} &= -r \\ \dot{\rho} &= \rho \\ \dot{\bar{y}} &= -\sigma(a)\bar{y} + f_{(a,\bar{\mu})}(r,\rho,\bar{y}) \end{cases} \tag{4.6}$$

where

$$f_{(a,\bar{\mu})}(r,\rho,\bar{y})$$
$$= \sigma(a)\bar{y} + \frac{-(1-2a)\bar{y}+2\bar{y}^2+\bar{y}[\varepsilon_2 r+\bar{\mu}_3\rho+2\bar{\mu}_2\rho^2-r^2h_1(r,r\rho,\bar{\mu})]+\bar{\mu}_1\rho^3+r\bar{h}_2(r,r\rho,\bar{\mu})+\bar{y}^2\overline{Q}_2(r,\rho,\bar{y},\bar{\mu})}{a+\bar{y}+\bar{\mu}_2\rho^2}$$

and the parameters $(a,\mu) \in A \times \mathbb{S}^2$, where for the saddle case $A = (-\frac{1}{2}, 0)$ and for the elliptic case $A = (0, \frac{1}{2})$.

Proposition 4.6. *Consider the family $X_{(a,\bar{\mu})}$ in the form of (4.6) with parameters $(a,\bar{\mu}) \in A \times \mathbb{S}^2$. Then $\forall (a_0,\bar{\mu}) \in A \times \mathbb{S}^2$ and $\forall k \in \mathbb{N}$, there exists $A_0 \subset A$, a neighborhood of a_0, $N(k) \in \mathbb{N}$ and a C^k-transformation*

$$\Psi_{(a,\bar{\mu})} : (r,\rho,y) \longrightarrow (r,\rho,\psi_{(a,\bar{\mu})}(r,\rho,y))$$

where

$$\psi_{(a,\bar{\mu})}(r,\rho,y) = y + o(|(r,\rho,y)|) \tag{4.7}$$

such that $\forall (a,\bar{\mu}) \in A_0 \times \mathbb{S}^2$, the map $\psi_{(a,\bar{\mu})}$ transforms $X_{(a,\bar{\mu})}$ into one of the following normal forms:

- *if $\sigma(a_0) \notin \mathbb{Q}$*

$$\widetilde{X}_{(a,\bar{\mu})} \begin{cases} \dot{r} &= \mp r \\ \dot{\rho} &= \pm\rho \\ \dot{y} &= -\bar{\sigma}(a,\bar{\mu},\nu)y \end{cases} \tag{4.8}$$

- *If $\sigma(a_0) = \frac{p}{q} \in \mathbb{Q}$*

$$\widetilde{X}_{(a,\bar{\mu})} \begin{cases} \dot{r} &= -r \\ \dot{\rho} &= \rho \\ \dot{y} &= \kappa r^p + \frac{1}{q}\Big[-p + \sum_{i=0}^{N(k)} \alpha_{i+1}(a,\bar{\mu},\nu)(\rho^p y^q)^i\Big]y \end{cases} \tag{4.9}$$

where $\nu = r\rho > 0$ *and*

$$\begin{aligned} \bar{\sigma}(a, \bar{\mu}, \nu) &= \sigma(a) - \alpha_0(a, \bar{\mu}, \nu) \\ \alpha_0(a, \bar{\mu}, \nu) &= \sum_{i=1}^{N(k)} \gamma_i \nu^i \\ \alpha_1(a, \bar{\mu}, \nu) &= p - \bar{\sigma}(a, \bar{\mu}, \nu) q \end{aligned} \tag{4.10}$$

where $\gamma_i(a, \bar{\mu})$, α_i *and* κ *are smooth functions defined for* $(a, \bar{\mu}) \in A_0 \times \mathbb{S}^2$. *Especially if* $q \geq 2$, $\kappa = 0$.

Proof. The proof is a straightforward application of normal from theory (see for instance [GH83], [IY91]). Depending on the value of a_0, we have two cases:

Case 1: $\sigma(a_0) \notin \mathbb{Q}$, $X_{(a,\bar{\mu})}$ has only resonant terms $(r\rho)^i y = \nu^i y, \quad i \geq 1$.

Case 2: $X_{(a,\bar{\mu})}$ has eigenvalues $-1, 1, -\sigma(a_0)$ with $\sigma(a_0) = \frac{p}{q}$, $p, q \in \mathbb{N}$, $(p, q) = 1$. Then the resonant term $r^i \rho^j y^l$ will satisfy

$$-i + j - \frac{p}{q} l = -\frac{p}{q}, \qquad i, j, l \geq 0, \quad i + j + l \geq 2. \tag{4.11}$$

- $\sigma(a_0) = \frac{p}{q} \in \mathbb{Q} \setminus \mathbb{N}$, $q \geq 2$. By (4.11), we have

$$j = i + np, \quad l = 1 + nq \qquad n \geq 0, \quad i + n \geq 1.$$

  Then the resonant terms will be

$$(r\rho)^i (\rho^p y^q)^n y, \qquad i + n \geq 1.$$

- $\sigma(a_0) = p \in \mathbb{N}$, then by (4.11)

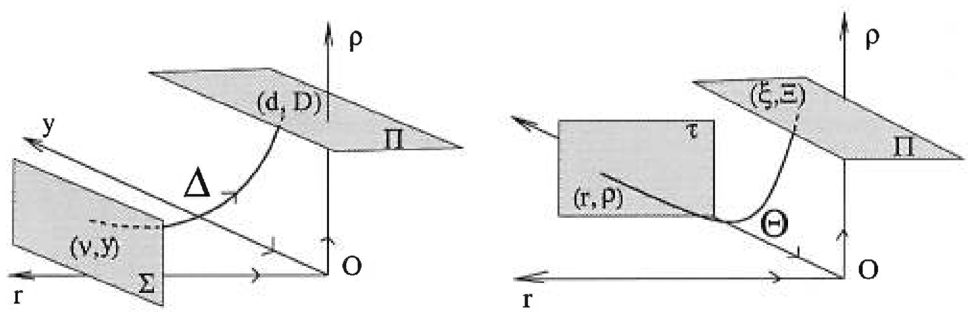
$$-i + j - pl = -p \qquad i, j, l \geq 0, \quad i + j + l \geq 2.$$

  Then the resonant terms will be

  - $r^p (r\rho)^i \qquad i + p \geq 1,$
  - $(r\rho)^i (\rho^p y)^l y \qquad i + l \geq 1.$

□

Remark 4.7. *It seems a priori that the resonant terms cannot be written directly in the case* $\sigma(a_0) = 1$ *since the linear part can have a Jordan normal form. Indeed when applying a polynomial transition* $Y = y + Cr^i\rho^j y^k$ *to get rid of a non-resonant term in* $r^i\rho^j y^k (i+k-j \neq 1)$*, one creates another non-resonant term in* $r^{i-1}\rho^j y^{k+1}$*. Iterating the process we realize that we get exactly the terms which appear in* (4.9)*. Moreover a transformation of the same form when* $i+k-j=1$ *allows to get rid of certain resonant monomials but this refinement is not necessary for our purpose.*



(a) The first type (b) The second type

Figure 4.2: Two types of Dulac map

In order to study the cyclicity of the graphics with a nilpotent singularity of elliptic or saddle type, we only need to consider $\widetilde{X}_{(a,\bar{\mu})}$ with eigenvalues $-1, 1, -\sigma(a)$ in the normal forms (4.8) and (4.9) and consider the following two types of Dulac maps (Fig. 4.2):

$$\Delta_{(a,\bar{\mu})} = (d, D): \quad \Sigma \longrightarrow \Pi$$

$$\Theta_{(a,\bar{\mu})} = (\xi, \Xi): \quad \tau \longrightarrow \Pi$$

where $\Sigma = \{r = r_0\}$, $\Pi = \{\rho = \rho_0\}$ and $\tau = \{y = y_0\}$ are sections in the normal form coordinates, r_0, ρ_0 and y_0 are positive constants.

To simplify the notation, for all the maps and vector fields, we will drop the index $(a, \bar{\mu})$. For example, the Dulac map $\Delta(\nu, \tilde{y}_1)$ means $\Delta_{(a,\bar{\mu})}(\nu, \tilde{y}_1)$.

In the next section, we give some preparation propositions for the proof of the two main theorems about the Dulac maps.

4.3 Preliminaries

Before studying the Dulac maps, we give two propositions as preparation. First we define a notation $I(i_1, i_2, \cdots, i_j;\ m, n)$:

Definition 4.8. *Let $m, n, j \in \mathbb{N} \cup \{0\}$, we define*

$$I(i_1 i_2 \cdots i_j; m, n) = \left\{ i_i, i_2, \cdots, i_j \in \mathbb{N} \cup \{0\} \;\middle|\; \begin{array}{c} i_1 + i_2 + \cdots + i_j = m \\ i_1 + 2i_2 + \cdots + j i_j = n \end{array} \right\}.$$

Proposition 4.9. *Let $f(t, z)$ be a smooth function and consider the initial value problem*

$$\frac{dz}{dt} = f(t, z_0, z), \quad z(0) = z_0.$$

Denote the unique solution as $z = z(t, z_0)$. Then $\forall n \in \mathbb{N}$, the nth derivative $\frac{\partial^n z}{\partial z_0^n}(t, z_0)$ satisfies the linear initial value problem

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial^n z}{\partial z_0^n}\right) & = \dfrac{\partial f}{\partial z}\dfrac{\partial^n z}{\partial z_0^n} + f_n\Big(t, z_0, z, \dfrac{\partial z}{\partial z_0}, \dfrac{\partial^2 z}{\partial z_0^2}, \cdots, \dfrac{\partial^{n-1} z}{\partial z_0^{n-1}}\Big) \\ \frac{\partial^n z}{\partial z_0^n}(0) & = 0, \end{cases}$$

where

$$\begin{aligned} & f_n\Big(t, z_0, z, \tfrac{\partial z}{\partial z_0}, \tfrac{\partial^2 z}{\partial z_0^2}, \cdots, \tfrac{\partial^{n-1} z}{\partial z_0^{n-1}}\Big) \\ & = \frac{\partial^n f}{\partial z_0^n} + \sum_{i=2}^{n} \frac{\partial^i f}{\partial z^i} \sum_{I(i_1 i_2 \cdots i_{n-1}; i, n)} * \prod_{l=1}^{n-1} \Big(\frac{\partial z}{\partial z_0}\Big)^{i_l} \\ & \quad + \sum_{j=1}^{n-1} \sum_{i=1}^{n-j} \frac{\partial^{i+j} f}{\partial z_0^i \partial z^j} \sum_{I(i_1 i_2 \cdots i_{n-1}; j, n-j)} * \prod_{l=1}^{n-1} \Big(\frac{\partial z}{\partial z_0}\Big)^{i_l} \end{aligned}$$

f_n as well as all the partial derivatives are all evaluated at $(t, z_0, z(t, z_0))$ and $*$ denotes a positive integer.

Proof. By induction. ☐

Proposition 4.10. *Let us consider the initial value problem*

$$\begin{cases} \frac{dz}{dt} &= f(t,z) \\ z(0) &= 0 \end{cases}$$

with $t \in [0,T]$. *If there exist two continuous functions* $A(t), B(t)$ *with*

$$|f(t,z)| \leq |A(t)||z| + |B(t)|, \quad (t,z) \in [0,T] \times [0, Z_0],$$

then for $t \in [0,T]$, *the solution of the initial value problem satisfies*

$$|z(t)| \leq e^{\int_0^t |A(\tau)|d\tau} \int_0^t |B(s)|e^{-\int_0^s |A(\tau)|d\tau} ds.$$

Furthermore, if there exist constants $M_1, M_2 > 0$ *such that*

$$\begin{aligned} |A(t)| &\leq M_1 \\ |B(t)| &\leq M_2 \end{aligned}$$

then there exists a constant $K > 0$ *such that for* $t \in [0,T]$, *there holds*

$$|z(t)| \leq K M_2 t.$$

Proof. By

$$\frac{d|z|}{dt} \leq |A(t)||z| + |B(t)|$$

then

$$\left(e^{-\int_0^t |A(\tau)|d\tau}|z(t)|\right)' \leq |B(t)|e^{-\int_0^t |A(\tau)|d\tau} dt$$

Integrating the above inequality from 0 to t with the initial condition $z(0) = 0$, then for $t \in [0,T]$, we have

$$|z(t)| \leq e^{\int_0^t |A(\tau)|d\tau} \int_0^t |B(s)|e^{-\int_0^s |A(\tau)|d\tau} ds.$$

If $|A(t)| \leq M_1, |B(t)| \leq M_2$, immediately we have

$$|z(t)| < K M_2 t$$

where $K = e^{M_1 T}$. ∎

4.4 First type of Dulac map

We now study the first type Dulac map $\Delta = (d, D)$. If we parametrize the sections Σ and Π by (ν, y) with the obvious relation $\rho = \frac{\nu}{r_0}$ on Σ and $r = \frac{\nu}{\rho_0}$ on Π, then we have

Theorem 4.11. *For any $a_0 \in A$ and $\bar{\mu} \in \mathbb{S}^2$, consider the family $\widetilde{X} = \widetilde{X}_{(a,\bar{\mu})}$ with eigenvalues $-1, 1, \sigma(a_0)$ in normal form (4.8) or (4.9). Then $\forall Y_0 \in \mathbb{R}$, there exist $A_0 \subset A$, a neighborhood of a_0, and $\nu_1 > 0$ such that $\forall \nu \in (0, \nu_1)$ and $(a, \bar{\mu}, y) \in A_0 \times \mathbb{S}^2 \times [0, Y_0]$, the Dulac map $\Delta(\nu, y) = (d(\nu, y), D(\nu, y))$ has the form*

$$\begin{cases} d(\nu, y) &= \nu \\ D(\nu, y) &= \eta(\nu, \omega(\frac{\nu}{\nu_0}, -\alpha_1)) + \left(\frac{\nu}{\nu_0}\right)^{\bar{\sigma}} \left[y + \phi(\nu, \omega(\frac{\nu}{\nu_0}, -\alpha_1), y)\right] \end{cases} \tag{4.12}$$

where $\nu_0 = r_0 \rho_0 > 0$ a constant and

If $\sigma(a_0) \notin \mathbb{Q}$

$$\eta = \phi = 0;$$

If $\sigma(a_0) \in \mathbb{Q} \setminus \mathbb{N}$

$$\eta = 0;$$

If $\sigma(a_0) = p \in \mathbb{N}$

$$\eta(\nu, \omega(\tfrac{\nu}{\nu_0}, -\alpha_1)) = \kappa r_0^p \omega(\tfrac{\nu}{\nu_0}, -\alpha_1)(\tfrac{\nu}{\nu_0})^{\bar{\sigma}};$$

If $\sigma(a_0) = \frac{p}{q} \in \mathbb{Q}$, $p, q \in \mathbb{N}$ and $(p, q) = 1$, then $\phi(\nu, \omega(\frac{\nu}{\nu_0}, -\alpha_1), y)$ is C^∞ and

$$\begin{aligned} \phi &= O(\nu^{\bar{p}} \omega^{q+1}(\tfrac{\nu}{\nu_0}, -\alpha_1) \ln \tfrac{\nu}{\nu_0}) \\ \tfrac{\partial \phi}{\partial y} &= O(\nu^{\bar{p}} \omega^{q}(\tfrac{\nu}{\nu_0}, -\alpha_1) \ln \tfrac{\nu}{\nu_0}) \\ \tfrac{\partial^j \phi}{\partial y^j} &= O\Big(\nu^{\bar{p}(1+[\frac{j-2}{q}])} \omega^{q-j+1+q[\frac{j-2}{q}]}(\frac{\nu}{\nu_0}, -\alpha_1) \ln \frac{\nu}{\nu_0}\Big) \quad , j \geq 2 \end{aligned} \tag{4.13}$$

where

$$\bar{p} = \begin{cases} q\bar{\sigma}(a, \nu) & \alpha_1 \geq 0 \\ p & \alpha_1 < 0. \end{cases} \tag{4.14}$$

Also all the partial derivatives with respect to the parameters $(a, \bar{\mu})$ are of order

$$O\Big(\nu^{\bar{p}} \omega^q(\frac{\nu}{\nu_0}, -\alpha_1) \ln \frac{\nu}{\nu_0}\Big).$$

Proof. Since we consider the Dulac map $\Delta(\nu, y)$ in the invariant leaf $\nu = r\rho > 0$, let $\nu_0 = r_0\rho_0$. Then the transition time from Σ to Π is

$$t = \ln \frac{\rho}{\rho_0} = -\ln \frac{\nu}{\nu_0}.$$

The first component $d(\nu, y)$ is easily obtained:

$$d(\nu, y) = \rho_0 r_0 e^{-t} = \rho_0 r_0 \frac{\nu}{\nu_0} = \nu.$$

Now let us consider the second component $D(\nu, y)$.

(1) Case $\sigma(a_0) \notin \mathbb{Q}$:

By the normal form (4.8), we directly have

$$D(\nu, y) = e^{-\bar{\sigma}t} y = \left(\frac{\nu}{\nu_0}\right)^{\bar{\sigma}} y.$$

(2) Case $\sigma(a_0) = \frac{p}{q} \in \mathbb{Q}$, $p, q \in \mathbb{N}$, $(p, q) = 1$. Note that $r(t) = r_0 e^{-t}$, $\rho(t) = \frac{\nu}{r_0} e^t$, so by the third equation of (4.9), we can write the solution of y as

$$y(t) = e^{-\bar{\sigma}t}\Big[y_0 + \kappa r_0^p \Omega(-\alpha_1, t) + U(t)\Big] \tag{4.15}$$

where

$$\Omega(\alpha_1, t) = \begin{cases} \frac{e^{\alpha_1 t} - 1}{\alpha_1} & \alpha_1 \neq 0 \\ t & \alpha_1 = 0. \end{cases}$$

Let

$$W(t) = y_0 + \kappa r_0^p \Omega(-\alpha_1, t) + U(t). \tag{4.16}$$

Then by (4.15), we have

$$y(t) = e^{-\bar{\sigma}t} W(t).$$

Note that if $q \geq 2$, then $\kappa = 0$.

A straightforward calculation shows that $U(t)$ satisfies

$$\begin{cases} \dot{U}(t) & = g(\nu, t, W(t)) \\ U(0) & = 0 \end{cases} \tag{4.17}$$

where $g(\nu,t,W(t)) = \sum_{i=1}^{N} \frac{\alpha_{i+1}}{r_0^{pi}} \nu^{pi} e^{\alpha_1 it} W^{qi+1}(t)$. First we are going to prove that $U(t)$ is bounded for $t \in [0, |\ln \frac{\nu}{\nu_0}|]$.

By the definition of $W(t)$ in (4.16), there exist constants $K_1, K_2 > 0$ such that

$$|W(t)| \leq K_1 + K_2\omega(\frac{\nu}{\nu_0}, -\alpha_1) + |U(t)|, \qquad t \in [0, |\ln \frac{\nu}{\nu_0}|] \tag{4.18}$$

where $K_2 = 0$ as long as $q \geq 2$.

We want to show that $U(t)$ is bounded, so we only need to consider the region where $|U(t)| \geq 1$. In such a region, by the definition of $\bar{p}$ in (4.14), there exists $K_3 > 0$ such that for $t \in [0, |\ln \frac{\nu}{\nu_0}|]$ with ν sufficiently small

$$|g(\nu,t,W)| \leq K_3 \nu^{\bar{p}} \omega^{q+1}(\frac{\nu}{\nu_0}, -\alpha_1)|U|^{Nq+1}. \tag{4.19}$$

Indeed, for ν sufficiently small and for $t \in [0, |\ln \frac{\nu}{\nu_0}|]$,

$$\begin{aligned}
|g(\nu,t,W)| &\leq \sum_{i=1}^{N} \frac{|\alpha_{i+1}|}{r_0^{pi}} \nu^{pi} e^{\alpha_1 it} |W(t)|^{qi+1} \\
&\leq \sum_{i=1}^{N} \frac{|\alpha_{i+1}| \nu_0^{\alpha_1 i}}{r_0^{pi}} \nu^{\bar{p}i} |W(t)|^{qi+1} \\
&\leq \sum_{i=1}^{N} \frac{|\alpha_{i+1}| \nu_0^{\alpha_1 i}}{r_0^{pi}} \nu^{\bar{p}i} [K_1 + K_2\omega(\frac{\nu}{\nu_0}, -\alpha_1) + |U(t)|]^{qi+1} \\
&\leq K_3 \nu^{\bar{p}} \omega^{q+1}(\tfrac{\nu}{\nu_0}, -\alpha_1)|U(t)|^{Nq+1}.
\end{aligned}$$

Hence $U(t)$ stays bounded by the solution of the initial value problem

$$\begin{cases} \dot{Z}(t) &= K_3 \nu^{\bar{p}} \omega^{q+1}(\frac{\nu}{\nu_0}, -\alpha_1) Z(t)^{Nq+1} \\ Z(0) &= 1 \end{cases}$$

so for $t \in [0, |\ln \frac{\nu}{\nu_0}|]$, there exist constants $K_4, K_5 > 0$ such that

$$\begin{aligned}
|U(t)| \leq Z(t) &= \frac{1}{(1 - NqK_3\nu^{\bar{p}}\omega^{q+1}(\frac{\nu}{\nu_0}, -\alpha_1)t)^{\frac{1}{Nq}}} \\
&\leq [1 - K_4 \nu^{\bar{p}} \omega^{q+1}(\frac{\nu}{\nu_0}, -\alpha_1) \ln \frac{\nu}{\nu_0}] \\
&\leq K_5.
\end{aligned}$$

Since $|U(t)|$ is bounded, by (4.17) and (4.19), there exists a constant $K_6 > 0$ such that for $t \in [0, |\ln \frac{\nu}{\nu_0}|]$

$$|\dot{U}(t)| \leq K_6 \nu^{\bar{p}} \omega^{q+1}(\frac{\nu}{\nu_0}, -\alpha_1).$$

So, $\forall (a, \bar{\mu}) \in A_0 \times \mathbb{S}^2$, $\forall \nu \in (0, \nu_0)$ and for $t \in [0, |\ln \frac{\nu}{\nu_0}|]$, we have

$$|U(t)| \leq K_6 \nu^{\bar{p}} \omega^{q+1}(\frac{\nu}{\nu_0}, -\alpha_1)\Big|\ln \frac{\nu}{\nu_0}\Big|. \tag{4.20}$$

Substituting the transition time $t = -\ln \frac{\nu}{\nu_0}$ into (4.15) and letting

$$U(t)|_{t=-\ln \frac{\nu}{\nu_0}} = \phi(\nu, \omega(\frac{\nu}{\nu_0}, -\alpha_1), y),$$

then for the second component of the map Δ, we get

$$\begin{aligned} D(\nu, y) &= \left(\tfrac{\nu}{\nu_0}\right)^{\bar{\sigma}} \Big[y + \kappa r_0^p \omega(\tfrac{\nu}{\nu_0}, -\alpha_1) + \phi(a, \nu, \omega(\tfrac{\nu}{\nu_0}, -\alpha_1), y)\Big] \\ &= \eta(\nu, \omega(\tfrac{\nu}{\nu_0})) + \left(\tfrac{\nu}{\nu_0}\right)^{\bar{\sigma}} \Big[y + \phi(a, \nu, \omega(\tfrac{\nu}{\nu_0}, -\alpha_1), y)\Big] \end{aligned}$$

where ϕ is C^∞ in $(a, \bar{\mu}, \nu, \omega(\frac{\nu}{\nu_0}, -\alpha_1), y)$ and uniformly bounded, i.e., for $(a, \bar{\mu}) \in A_0 \times \mathbb{S}^2$, $\nu \in (0, \nu_1)$ and $y \in [0, Y_0]$, we have

$$\phi(\nu, \omega(\frac{\nu}{\nu_0}, -\alpha_1), y) = O\Big(\nu^{\bar{p}} \omega^{q+1}(\frac{\nu}{\nu_0}, -\alpha_1) \ln \frac{\nu}{\nu_0}\Big).$$

Now we consider the derivatives $\dfrac{\partial^i \phi}{\partial y^i}$ for $i \geq 1$.

For $\frac{\partial \phi}{\partial y}$, since $\frac{\partial W}{\partial y} = 1 + \frac{\partial U}{\partial y}$, so $\frac{\partial W}{\partial y}$ satisfies

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial W}{\partial y}\right) &= g_1(\nu, t, W(t)) \frac{\partial W}{\partial y} \\ \frac{\partial W}{\partial y}(0) &= 1 \end{cases} \tag{4.21}$$

where $g_1(\nu, t, W(t)) = \displaystyle\sum_{i=1}^{N} \frac{(qi+1)\alpha_{i+1}}{r_0^{pi}} \nu^{pi} e^{\alpha_1 i t} W^{qi}(t)$.

For $t \in [0, |\ln \frac{\nu}{\nu_0}|]$, by (4.18), and (4.20), and similar to the proof of (4.19), there exists a $\overline{K}_1 > 0$ such that

$$\Big|g_1(\nu, t, W(t))\Big| \leq \overline{K}_1 \nu^{\bar{p}} \omega^q(\frac{\nu}{\nu_0}, -\alpha_1) \tag{4.22}$$

By (4.21) and (4.22), then for $t \in [0, |\ln \frac{\nu}{\nu_0}|]$ we have

$$e^{-\overline{K}_1 \nu^{\bar{p}} \omega^q (\frac{\nu}{\nu_0}, -\alpha_1)|\ln \frac{\nu}{\nu_0}|} \le \frac{\partial W}{\partial y} \le e^{\overline{K}_1 \nu^{\bar{p}} \omega^q (\frac{\nu}{\nu_0}, -\alpha_1)|\ln \frac{\nu}{\nu_0}|}$$

hence

$$e^{-\overline{K}_1 \nu^{\bar{p}} \omega^q (\frac{\nu}{\nu_0}, -\alpha_1)|\ln \frac{\nu}{\nu_0}|} - 1 \le \frac{\partial U}{\partial y} \le e^{\overline{K}_1 \nu^{\bar{p}} \omega^q (\frac{\nu}{\nu_0}, -\alpha_1)|\ln \frac{\nu}{\nu_0}|} - 1.$$

So there exists $\widehat{K}_1 > 0$ such that for $(a, \bar{\mu}) \in A_0 \times \mathbb{S}^2$ and $\nu \in (0, \nu_0)$, for $0 \le t \le |\ln \frac{\nu}{\nu_0}|$

$$\Big|\frac{\partial U}{\partial y}\Big| \le \widehat{K}_1 \nu^{\bar{p}} \omega^q (\frac{\nu}{\nu_0}, -\alpha_1)\Big|\ln \frac{\nu}{\nu_0}\Big|. \tag{4.23}$$

Thus for $\phi(\nu, \omega(\frac{\nu}{\nu_0}, -\alpha_1), y)$, we have

$$\frac{\partial \phi}{\partial y} = O\Big(\nu^{\bar{p}} \omega^q (\frac{\nu}{\nu_0}, -\alpha_1) \ln \frac{\nu}{\nu_0}\Big).$$

It is clear that the above properties on ϕ also hold for all the partial derivatives with respect to the parameters $(a, \bar{\mu}) \in A_0 \times \mathbb{S}^2$.

For $\dfrac{\partial^i W}{\partial y^i} (i \ge 2)$, we will use induction on i. First show that for $2 \le i \le q+1$, there holds

$$\frac{\partial^i \phi}{\partial y^i} = O(\nu^{\bar{p}} \omega^{q+1-i} (\frac{\nu}{\nu_0}, -\alpha_1) \ln \frac{\nu}{\nu_0}\Big).$$

Assume that for $2 \le i \le q$, we have

$$\Big|\frac{\partial^i W}{\partial y^i}\Big| \le \widehat{K}_i \nu^{\bar{p}} \omega^{q+1-i} (\frac{\nu}{\nu_0}, -\alpha_1)\Big|\ln \frac{\nu}{\nu_0}\Big|. \tag{4.24}$$

Now we turn to $\frac{\partial^{i+1} W}{\partial y^{i+1}}$. By Prop. 4.9, $\frac{\partial^{i+1} W}{\partial y^{i+1}}$ satisfies the following initial value problem

$$\begin{cases} \frac{d}{dt}(\frac{\partial^{i+1} W}{\partial y^{i+1}}) & = g_1(\nu, t, W)\frac{\partial^{i+1} W}{\partial y^{i+1}} + g_{i+1}\Big(\nu, t, W, \frac{\partial W}{\partial y}, \frac{\partial^2 W}{\partial y^2}, \cdots, \frac{\partial^i W}{\partial y^i}\Big) \\ \frac{\partial^{i+1} W}{\partial y^{i+1}}(0) & = 0 \end{cases} \tag{4.25}$$

where

$$g_{i+1}\Big(\nu, t, W, \frac{\partial W}{\partial y}, \frac{\partial^2 W}{\partial y^2}, \cdots, \frac{\partial^i W}{\partial y^i}\Big) = \sum_{j=2}^{i+1} \frac{\partial^j g}{\partial W^j} \sum_{I(j_1 j_2 \cdots j_i; j, i+1)} * \prod_{k=1}^{i} \Big(\frac{\partial W}{\partial y_0}\Big)^{j_k}.$$

We claim that there exists a constant $\overline{K}_{i+1} > 0$ such that for $t \in [0, |\ln \frac{\nu}{\nu_0}|]$

$$\left|g_{i+1}\Big(\nu, t, W, \frac{\partial W}{\partial y}, \frac{\partial^2 W}{\partial y^2}, \cdots, \frac{\partial^i W}{\partial y^i}\Big)\right| \leq \overline{K}_{i+1}\nu^{\bar{p}}\omega^{q+1-(i+1)}(\frac{\nu}{\nu_0}, -\alpha_1). \tag{4.26}$$

Indeed, for $2 \leq j \leq i+1$, similar to the proof of (4.19), there exist constants $\overline{K}_{j1} > 0$ $(j = 2, 3, \cdots, i+1)$ such that

$$\begin{aligned} \left|\frac{\partial^j g}{\partial W^j}\right| &= \left|\sum_{i=1}^{N(k)} j! \binom{qi+1}{j} \frac{\alpha_{i+1}}{(\nu_0^{\alpha_1} r_0^{p})^i} \nu^{pi} e^{\alpha_1 it} W^{qi+1-j}(t)\right| \\ &\leq \overline{K}_{j1}\nu^{\bar{p}}\omega^{q+1-j}(\tfrac{\nu}{\nu_0}, -\alpha_1). \end{aligned} \tag{4.27}$$

Note that by (4.23), we have $|\frac{\partial W}{\partial y}| \leq \overline{K}_0$, so by (4.27) and by the induction assumption (4.24), there exists a constant $\overline{K}_{i+1} > 0$ such that

$$\begin{aligned} &\left|g_{i+1}\left(\nu, t, W(t), \tfrac{\partial W}{\partial y}, \tfrac{\partial^2 W}{\partial y^2}, \cdots, \tfrac{\partial^i W}{\partial y^i}\right)\right| \\ &\leq \sum_{j=2}^{i+1} \overline{K}_{j1}\nu^{\bar{p}}\omega^{q+1-j} \sum_{I(j_1 j_2 \cdots j_i; j, i+1)} * \prod_{l=2}^{i}\Big(\nu^{\bar{p}}\omega^{q-(l-1)}|\ln\frac{\nu}{\nu_0}|\Big)^{j_l} \\ &= \sum_{j=2}^{i+1} \overline{K}_{j1}\nu^{\bar{p}}\omega^{q+1-j} \sum_{I(j_1 j_2 \cdots j_i; j, i+1)} * \Big(\nu^{\bar{p}}\omega^{q}|\ln\frac{\nu}{\nu_0}|\Big)^{j_2+j_3+\cdots+j_i} \omega^{-j_2-2j_3-(i-1)j_i} \\ &= \sum_{j=2}^{i+1} \overline{K}_{j1}\nu^{\bar{p}}\omega^{q+1-j} \sum_{I(j_1 j_2 \cdots j_i; j, i+1)} * \Big(\nu^{\bar{p}}\omega^{q}|\ln\frac{\nu}{\nu_0}|\Big)^{j-j_1} \omega^{j-(i+1)} \\ &\leq \overline{K}_{i+1}\nu^{\bar{p}}\omega^{q+1-(i+1)}(\tfrac{\nu}{\nu_0}, -\alpha_1) \end{aligned}$$

where in the final sum the dominant term is the term with $j = j_1 = q+1$.

By (4.22), (4.26) and Prop. 4.10, for the solution of (4.25), there exists a constant $\widehat{K}_{i+1}$ such that for $t \in [0,\ |\ln \frac{\nu}{\nu_0}|]$

$$\left|\frac{\partial^{i+1} W}{\partial y^{i+1}}\right| \leq \widehat{K}_{i+1}\nu^{\bar{p}}\omega^{q+1-(i+1)}(\frac{\nu}{\nu_0}, -\alpha_1)\left|\ln\frac{\nu}{\nu_0}\right|, \qquad 2 \leq i \leq q. \tag{4.28}$$

Therefore, for $2 \leq i \leq q+1$, it follows from (4.24), (4.28) and by induction that we have

$$\frac{\partial^i \phi}{\partial y_0^i} = O(\nu^{\bar{p}}\omega^{q+1-i}(\frac{\nu}{\nu_0}, -\alpha_1)|\ln\frac{\nu}{\nu_0}|).$$

Generally, $\forall \mathbf{j} \geq \mathbf{2}$, we can decompose j as $\mathbf{j} - \mathbf{2} = \mathbf{lq} + \mathbf{i}$ with $0 \leq i \leq q-1, l \geq 0$. Then in the same way as for the case $l = 0$, we can prove that

$$\frac{\partial^j \phi}{\partial y_0^j} = O\Big(\nu^{\bar{p}(1+[\frac{j-2}{q}]}\omega^{q-j+1+q[\frac{j-2}{q}]}(\frac{\nu}{\nu_0}, -\alpha_1)\ln\frac{\nu}{\nu_0}\Big).$$

□

Remark 4.12. *In Theorem* 4.11, *if* $q = 1$, *then* $\forall j \geq 2$, *we have*

$$\frac{\partial^j \phi}{\partial y_0^j} = O(\nu^{(j-1)\bar{p}} \ln \frac{\nu}{\nu_0}).$$

Remark 4.13. *By Theorem* 4.11, *the properties of the Dulac map* Δ *are valid on a compact set* $\mathbb{K}_1$ *of* Σ. *When we want to analyse a graphic intersecting* Σ *at* (r_0, y^*), *we will of course choose* $\mathbb{K}_1$ *so that* $(r_0, y^*) \in \mathbb{K}_1$.

4.5 Dulac map of the second type

Now we consider the second type Dulac map $\Theta = (\xi, \Xi)$ (Fig. 4.2(b)). If we parameterize τ by (r, ρ), Π by (ν, y) with the relation $r\rho = \nu$ and $r = \frac{\nu}{\rho_0}$ on the two sections respectively, then we have

Theorem 4.14. *For any* $a_0 \in A$, *consider* $\widetilde{X}_\lambda$ *with eigenvalues* $-1, 1, -\sigma(a_0)$ *in the normal form* (4.8) *and* (4.9). *Then for* $r, \rho > 0$ *sufficiently small, there exist* $A_0 \subset A$, *a neighborhood of* a_0, *and* $\nu_1 > 0$ *such that* $\forall (a, \bar{\mu}) \in A_0 \times \mathbb{S}^2$ *and* $\nu \in (0, \nu_1)$, *the Dulac map* $\Theta(r, \rho)$ *has the form*

$$\begin{cases} \xi(r, \rho) & = \nu \\ \Xi(r, \rho) & = \eta(\nu, \omega(\frac{\rho}{\rho_0}, \alpha_1)) + (\frac{\rho}{\rho_0})^{\bar{\sigma}}[y_0 + \theta(r, \rho, \omega(\frac{\rho}{\rho_0}, -\alpha_1))] \end{cases} \tag{4.29}$$

where

- *if* $\sigma(a_0) \notin \mathbb{N}$, *then* $\eta = 0$; *if* $\sigma(a_0) = p \in \mathbb{N}$, *then*

$$\eta(\nu, \omega(\rho, \alpha_1)) = \frac{\kappa}{\rho_0^p} \nu^p \omega(\frac{\rho}{\rho_0}, \alpha_1),$$

- *if* $\sigma(a_0) \notin \mathbb{Q}$, *then* $\theta = 0$; *if* $\sigma(a_0) = \frac{p}{q} \in \mathbb{Q}$, *then* $\theta(r, \rho, \omega(\frac{\rho}{\rho_0}, -\alpha_1))$ *is* C^∞ *in* $(a, \bar{\mu})$ *and* $(r, \rho, \omega(\frac{\rho}{\rho_0}, -\alpha_1))$, *and also satisfies*

$$\begin{aligned} \theta & = O\left(\rho^p \omega(\tfrac{\rho}{\rho_0}, \alpha_1)\left[1 + \kappa r^p \omega^2(\tfrac{\rho}{\rho_0}, -\alpha_1)\right]\right) \\ \rho^j \frac{\partial^j \theta}{\partial \rho^j} & = O\left(\rho^p \omega(\tfrac{\rho}{\rho_0}, \alpha_1))\left[1 + \kappa r^p \omega^2(\tfrac{\rho}{\rho_0}, -\alpha_1)\right]\right), \quad j \geq 1. \end{aligned} \tag{4.30}$$

which are uniformly valid for $(a, \bar{\mu}) \in A_0 \times \mathbb{S}^2$ *and* $r, \rho > 0$ *sufficiently small.*

Proof. $\xi(r,\rho)=\nu$ follows from the invariance of $r\rho=\nu$.

For the second component $\Xi(r,\rho)$, the transition time from τ to Π is

$$t=|\ln\frac{\rho}{\rho_0}|.$$

So for the case $\sigma(a_0)\notin\mathbb{Q}$ by (4.8), we directly have

$$\Xi(r,\rho)=y_0e^{-\sigma t}\Big|_{t=|\ln\frac{\rho}{\rho_0}|}=y_0(\frac{\rho}{\rho_0})^{\sigma}.$$

Now we consider the case $\sigma(a_0)=\frac{p}{q}\in\mathbb{Q}$. Note that $r(t)=re^{-t}$ and $\rho(t)=\rho e^{t}$. Hence, by the third equation of (4.9), we have a first order differential equation about y

$$\dot{y}=-\bar{\sigma}y+\kappa r^{p}e^{-pt}+\frac{1}{q}\sum_{i=1}^{N(k)}\alpha_{i+1}(\rho e^{t})^{pi}y^{qi+1}. \tag{4.31}$$

Let the solution of (4.31) with the initial value $y(0)=y_0$ be

$$y(t)=e^{-\bar{\sigma}t}\Big[y_0+\kappa r^{p}\Omega(t,-\alpha_1)+V(t)\Big]. \tag{4.32}$$

Then $V(t)$ satisfies the initial value problem

$$\begin{cases}\dot{V}(t) &= h(\nu,t,\rho,E(t))\\ V(0) &= 0\end{cases} \tag{4.33}$$

where

$$\begin{aligned}E(t)&=y_0+\kappa r^{p}\Omega(t,-\alpha_1)+V(t),\\ h(\nu,t,\rho,E(t))&=\sum_{i=1}^{N}\alpha_{i+1}\rho^{pi}e^{i\alpha_1 t}E^{qi+1}(t).\end{aligned} \tag{4.34}$$

So for $\Xi(r,\rho)$, substituting the transition time $t=|\ln\frac{\rho}{\rho_0}|$ into (4.32), and letting

$$\theta(r,\rho,\omega(\frac{\rho}{\rho_0},-\alpha_1))=V(t)\Big|_{t=|\ln\frac{\rho}{\rho_0}|}, \tag{4.35}$$

then we have

$$\begin{aligned}\Xi(r,\rho) &= \Big(\frac{\rho}{\rho_0}\Big)^{\bar\sigma}\Big[y_0+\kappa r^p\omega(\frac{\rho}{\rho_0},-\alpha_1)+\theta(r,\rho,\omega(\frac{\rho}{\rho_0},-\alpha_1))\Big]\\ &= \eta(\nu,\omega(\frac{\rho}{\rho_0},\alpha_1))+\Big(\frac{\rho}{\rho_0}\Big)^{\bar\sigma}\Big[y_0+\theta(r,\rho,\omega(\frac{\rho}{\rho_0},-\alpha_1))\Big]\end{aligned}$$

where $\theta(r,\rho,\omega(\frac{\rho}{\rho_0},-\alpha_1))$ is C^∞ and

$$\begin{aligned}\eta(\nu,\omega(\tfrac{\rho}{\rho_0},\alpha_1)) &= \kappa r^p(\tfrac{\rho}{\rho_0})^{\bar\sigma}\omega(\tfrac{\rho}{\rho_0},-\alpha_1)\\ &= \kappa r^p(\frac{\rho}{\rho_0})^p\Big[(\frac{\rho}{\rho_0})^{-\alpha_1}\omega(\frac{\rho}{\rho_0},-\alpha_1)\Big]\\ &= \tfrac{\kappa}{\rho_0^p}\nu^p\omega(\tfrac{\rho}{\rho_0},\alpha_1).\end{aligned}$$

(1). Bound for $\theta(r,\rho,\omega(\frac{\rho}{\rho_0},-\alpha_1))$.

First we prove that $V(t)$ is bounded for $t\in[0,|\ln\frac{\rho}{\rho_0}|]$.

By the definition of $E(t)$ in the first equation of (4.34), if we denote $M_0=|y_0|$, then, for $t\in[0,|\ln\frac{\rho}{\rho_0}|]$, we have

$$|E(t)|\le M_0+|\kappa|r^p\Omega(t,-\alpha_1)+|V(t)|. \tag{4.36}$$

Note that for $t\in[0,|\ln\frac{\rho}{\rho_0}|]$, $\Omega(t,-\alpha_1)\le\omega(\frac{\rho}{\rho_0},-\alpha_1)$, so, if we restrict to the region where $V(t)\ge 1$, for $t\in[0,|\ln\frac{\rho}{\rho_0}|]$, by (4.34) and (4.36), there exists a constant $M_1>0$ such that

$$\begin{aligned}&|h(\nu,t,\rho,E(t))|\\ &\le\sum_{i=1}^{N}|\alpha_{i+1}|\rho^{pi}e^{i\alpha_1t}|E^{qi+1}(t)|\\ &\le\rho^pe^{\alpha_1t}\sum_{i=1}^{N}|\alpha_{i+1}|\rho^{p(i-1)}e^{\alpha_1(i-1)t}[M_0+\kappa r^p\omega(\frac{\rho}{\rho_0},-\alpha_1)+|V(t)|]^{qi+1}\\ &=\rho^pe^{\alpha_1t}\Big[|\alpha_2|[M_0+\kappa r^p\omega(\tfrac{\rho}{\rho_0},-\alpha_1)+|V(t)|]^{q+1}\\ &\qquad+|\alpha_3|\rho^pe^{\alpha_1t}[M_0+\kappa r^p\omega(\tfrac{\rho}{\rho_0},-\alpha_1)+|V(t)|]^{2q+1}]\\ &\qquad+\cdots\\ &\qquad+|\alpha_{N+1}|\rho^{p(N-1)}e^{\alpha_1(N-1)t}[M_0+\kappa r^p\omega(\tfrac{\rho}{\rho_0},\alpha_1)+|V(t)|]^{Nq+1}\Big]\\ &\le\rho^pe^{\alpha_1t}\Big[*\,\omega^{q+1}(\tfrac{\rho}{\rho_0},-\alpha_1)|V(t)|^{q+1}+*|V(t)|]^{2q+1}+\cdots+*|V(t)|]^{Nq+1}\Big]\\ &\le M_1\rho^pe^{\alpha_1t}\omega^{q+1}(\tfrac{\rho}{\rho_0},-\alpha_1)|V(t)|^{Nq+1}.\end{aligned}$$

(4.37)

Hence by (4.33) and (4.37), $V(t)$ stays bounded by the solution of the initial value problem

$$\begin{cases} \dot{Z}(t) &= M_1 \rho^p e^{\alpha_1 t} \omega^{q+1}(\frac{\rho}{\rho_0}, -\alpha_1) Z^{Nq+1} \\ Z(0) &= 1 \end{cases}$$

So there exists a constant $M_2 > 0$ such that for $t \in [0, |\ln \frac{\rho}{\rho_0}|]$,

$$\begin{aligned} |V(t)| &\leq Z(t) \\ &= \frac{1}{[1 - qNM_1 \rho^p \omega^{q+1}(\frac{\rho}{\rho_0}, -\alpha_1) \Omega(t, \alpha_1)]^{\frac{1}{Nq}}} \\ &\leq M_2. \end{aligned} \tag{4.38}$$

Again by (4.36) and (4.38), there exists a constant $M_3 > 0$ such that for $t \in [0, |\ln \frac{\rho}{\rho_0}|]$, there holds

$$|E(t)| \leq M_3 + \kappa r^p \omega(t, -\alpha_1). \tag{4.39}$$

We will prove that there exist constants $M_4, M_5 > 0$ such that

$$|h(\nu, t, \rho, E(t))| \leq \rho^p e^{\alpha_1 t} \Big[M_4 + M_5 \kappa r^p \omega^2 (\frac{\rho}{\rho_0}, -\alpha_1) \Big]. \tag{4.40}$$

Indeed, we have two cases: $q = 1$ and $q \geq 2$.

For the case $q = 1$, there exist $M_{43}, M_5, M_{5i} > 0$ $(i = 3, 4, \cdots N+1)$ such that

$$\begin{aligned} & |h(\nu, t, \rho, E(t))| \\ & \leq \sum_{i=1}^{N} |\alpha_{i+1}| \rho^{pi} e^{i\alpha_1 t} |E^{i+1}(t)| \\ & \leq \rho^p e^{\alpha_1 t} \Big[|\alpha_2| [M_3 + \kappa r^p \omega(\tfrac{\rho}{\rho_0}, -\alpha_1)]^2 \\ & \qquad + |\alpha_3| \rho^p e^{\alpha_1 t} [M_3 + \kappa r^p \omega(\tfrac{\rho}{\rho_0}, -\alpha_1)]^3] \\ & \qquad + \cdots \\ & \qquad + |\alpha_{N+1}| \rho^{pN} e^{\alpha_1 N t} [M_3 + \kappa r^p \omega(\tfrac{\rho}{\rho_0}, -\alpha_1)]^{N+1} \Big] \\ & \leq \rho^p e^{\alpha_1 t} \Big[|\alpha_2| (M_3^2 + 2\kappa M_3 r^p \omega(\tfrac{\rho}{\rho_0}, -\alpha_1) + \kappa^2 r^{2p} \omega^2(\tfrac{\rho}{\rho_0}, -\alpha_1)) \\ & \qquad + |\alpha_3| M_{53} + \cdots + |\alpha_{N+1}| M_{5(N+1)} \Big] \\ & \leq \rho^p e^{\alpha_1 t} [M_{43} + M_5 \kappa r^p \omega^2(\tfrac{\rho}{\rho_0}, -\alpha_1)]. \end{aligned}$$

For the case $q \geq 2$, $\kappa = 0$, so by (4.39) there exists a constant $M_{4i} > 0$ ($i = 1, 2, \cdots, N+1$) such that

$$\begin{aligned}
&|h(\nu, t, \rho, E(t))| \\
&\leq \sum_{i=1}^{N} |\alpha_{i+1}| \rho^{pi} e^{i\alpha_1 t} |E^{qi+1}(t)| \\
&\leq \rho^p e^{\alpha_1 t} \Big[|\alpha_2| M_3^{q+1} + |\alpha_3| \rho^p e^{\alpha_1 t} M_3^{2q+1} + \cdots + |\alpha_{N+1}| \rho^{pN} e^{\alpha_1 N t} M_3^{Nq+1} \Big] \\
&\leq \rho^p e^{\alpha_1 t} \Big[|\alpha_2| M_{42} + |\alpha_3| M_{43} + \cdots + |\alpha_{N+1}| M_{4(N+1)} \Big] \\
&\leq M_{41} \rho^p e^{\alpha_1 t}
\end{aligned}$$

Combining the two cases $q = 1$ and $q \geq 2$ together, letting $M_4 = \max(M_{41}, M_{43})$, yields (4.40) for $t \in [0, |\ln \frac{\rho}{\rho_0}|]$.

So for the solution of the equation (4.33), for $t \in [0, |\ln \frac{\rho}{\rho_0}|]$, by (4.40) we have

$$\begin{aligned}
|V(t)| &\leq \int_0^t \Big| h(s, \rho, E(s)) \Big| ds \\
&= \int_0^t \rho^p e^{\alpha_1 s} \Big[M_4 + M_5 \kappa r^p \omega^2 (\frac{\rho}{\rho_0}, -\alpha_1) \Big] ds \\
&= \rho^p [M_4 + \kappa M_5 r^p \omega^2 (\frac{\rho}{\rho_0}, -\alpha_1)] \int_0^t e^{\alpha_1 s} ds \\
&= \rho^p \Omega(t, \alpha_1) [M_4 + \kappa M_5 r^p \omega^2 (\tfrac{\rho}{\rho_0}, -\alpha_1)] \\
&\leq \rho^p \omega(\tfrac{\rho}{\rho_0}, \alpha_1) [M_4 + \kappa M_5 r^p \omega^2 (\tfrac{\rho}{\rho_0}, -\alpha_1)].
\end{aligned} \tag{4.41}$$

Hence for $\theta(r, \rho, \omega(\frac{\rho}{\rho_0}, -\alpha_1))$ given in (4.35), for $(a, \bar{\mu}) \in A_0 \times \mathbb{S}^2$ and $r, \rho > 0, r\rho = \nu$ sufficiently small, we have

$$\theta(r, \rho, \omega(\frac{\rho}{\rho_0}, -\alpha_1)) = O\left(\rho^p \omega(\frac{\rho}{\rho_0}, \alpha_1) \Big[1 + \kappa r^p \omega^2 (\frac{\rho}{\rho_0}, -\alpha_1) \Big] \right).$$

We will use induction on i ($i \geq 1$) to study $\dfrac{\partial^i \theta}{\partial \rho^i}$.

(2). Bound for $\frac{\partial \theta}{\partial \rho}$.

By the first equation of (4.34), we have $\frac{\partial V}{\partial \rho} = \frac{\partial E}{\partial \rho}$, so $\frac{\partial E}{\partial \rho}$ satisfies the following linear equation

$$\begin{cases} \frac{d}{dt}(\frac{\partial E}{\partial \rho}) &= h_0(t, \rho, E) \frac{\partial E}{\partial \rho} + h_1(t, \rho, E) \\ \frac{\partial E}{\partial \rho}(0) &= 0 \end{cases} \tag{4.42}$$

where

$$\begin{aligned} h_0(t,\rho,E) &= \tfrac{\partial h}{\partial E}(t,\rho,E) \\ &= \sum_{i=1}^{N(k)} (qi+1)\alpha_{i+1}\rho^{pi}e^{\alpha_1 it}E^{qi}(t), \\ h_1(t,\rho,E) &= \tfrac{\partial h}{\partial \rho}(t,\rho,E) \\ &= \sum_{i=1}^{N(k)} pi\alpha_{i+1}\rho^{pi-1}e^{\alpha_1 it}E^{qi+1}(t). \end{aligned}$$

By (4.39) and similar to the proof of (4.40), we can prove that there exist constants $\overline{M}_{1i} > 0$ $(i = 1,2,3,4)$ such that

$$\begin{aligned} |h_0(t,\rho,E)| &\le \rho^p e^{\alpha_1 t}\Big[\overline{M}_{11} + \kappa\overline{M}_{12}r^p\omega(\frac{\rho}{\rho_0},-\alpha_1)\Big] \\ |h_1(t,\rho,E)| &\le \rho^{p-1}e^{\alpha_1 t}\Big[\overline{M}_{13} + \kappa\overline{M}_{14}r^p\omega^2(\frac{\rho}{\rho_0},-\alpha_1))\Big]. \end{aligned} \tag{4.43}$$

Then by (4.43) and Prop. 4.10, there exist constants $\widehat{M}_{11}, \widehat{M}_{12} > 0$ such that for $t \in [0, |\ln \frac{\rho}{\rho_0}|]$, there holds

$$\Big|\frac{\partial E}{\partial \rho}\Big| \le \rho^{p-1}\omega(\frac{\rho}{\rho_0},\alpha_1)\Big[\widehat{M}_{11} + \kappa\widehat{M}_{12}r^p\omega^2(\frac{\rho}{\rho_0},-\alpha_1)\Big]. \tag{4.44}$$

Indeed, let

$$\begin{aligned} \overline{A}(r,\rho) &= \rho^p\Big[\overline{M}_{11} + \kappa\overline{M}_{12}r^p\omega(\frac{\rho}{\rho_0},-\alpha_1)\Big] \\ \overline{B}(r,\rho) &= \rho^{p-1}\Big[\overline{M}_{13} + \kappa\overline{M}_{14}r^p\omega^2(\frac{\rho}{\rho_0},-\alpha_1)\Big] \end{aligned}$$

and let

$$\begin{aligned} A(t,r,\rho) &= \overline{A}(r,\rho)e^{\alpha_1 t} \\ B(t,r,\rho) &= \overline{B}(r,\rho)e^{\alpha_1 t}. \end{aligned}$$

Then by (4.43), for $(a,\bar{\mu}) \in A_1 \times \mathbb{S}^2$ and for $r,\rho > 0, r\rho = \nu$ sufficiently small, we have

$$\begin{aligned} \Big|h_0(t,\rho,E)\Big| &\le A(t,r,\rho) \\ \Big|h_1(t,\rho,E)\Big| &\le B(t,r,\rho). \end{aligned}$$

So by Prop. 4.10, for $t \in [0, |\frac{\rho}{\rho_0}|]$, there exist constants $\overline{M}_1, \widehat{M}_{11}, \widehat{M}_{12} > 0$ such that

$$
\begin{aligned}
\left|\frac{\partial E}{\partial \rho}\right| &\le e^{\int_0^t A(s,r,\rho)ds}\int_0^t B(u,r,\rho)e^{-\int_0^u A(s,r,\rho)ds}du \\
&= e^{\int_0^t \overline{A}(r,\rho)e^{\alpha_1 t}dt}\int_0^t \overline{B}(r,\rho)e^{\alpha_1 u}e^{-\int_0^u \overline{A}(r,\rho)e^{\alpha_1 s}ds}du \\
&= e^{\int_0^t \overline{A}(r,\rho)e^{\alpha_1 s}ds}\Big[-\frac{\overline{B}(r,\rho)}{\overline{A}(r,\rho)}\int_0^t d\Big(e^{-\int_0^u \overline{A}(r,\rho)e^{\alpha_1 s}ds}\Big)\Big] \\
&= \frac{\overline{B}(r,\rho)}{\overline{A}(r,\rho)}(e^{\int_0^t \overline{A}(r,\rho)e^{\alpha_1 s}ds}-1) \\
&\le \frac{\overline{B}(r,\rho)}{\overline{A}(r,\rho)}\Big[e^{\overline{A}(r,\rho)\int_0^t e^{\alpha_1 s}ds}-1\Big] \\
&= \frac{\overline{B}(r,\rho)}{\overline{A}(r,\rho)}\Big[e^{\overline{A}(r,\rho)e^{\Omega(t,\alpha_1)-1}}\Big] \\
&\le \frac{\overline{B}(r,\rho)}{\overline{A}(r,\rho)}\Big[e^{\overline{A}(r,\rho)\omega(\frac{\rho}{\rho_0},\alpha_1)}-1\Big] \\
&= \frac{\overline{B}(r,\rho)}{\overline{A}(r,\rho)}\Big[e^{\xi_1}\overline{A}(r,\rho)\omega(\frac{\rho}{\rho_0},\alpha_1)\Big] \\
&\le \overline{M}_1\rho^{p-1}\omega(\tfrac{\rho}{\rho_0},\alpha_1)\Big[\overline{M}_{13}+\kappa\overline{M}_{14}r^p\omega^2(\tfrac{\rho}{\rho_0},-\alpha_1)\Big] \\
&= \rho^{p-1}\omega(\tfrac{\rho}{\rho_0},\alpha_1)\Big[\widehat{M}_{11}+\kappa\widehat{M}_{12}r^p\omega^2(\tfrac{\rho}{\rho_0},-\alpha_1)\Big]
\end{aligned}
\tag{4.45}
$$

where $\xi_1 \in (0,\overline{A}_1(r,\rho)\omega(\frac{\rho}{\rho_0},\alpha_1))$, $\overline{M}_1 = e^{\overline{A}_1(r,\rho)\omega(\frac{\rho}{\rho_0},\alpha_1)}$, and $\widehat{M}_{11} = \overline{M}_1\overline{M}_{13}$, $\widehat{M}_{12} = \overline{M}_1\overline{M}_{14}$.

So for $(a,\bar{\mu}) \in A_0 \times \mathbb{S}^2$ and $r,\rho > 0$ sufficiently small, we have uniformly

$$\rho\frac{\partial\theta}{\partial\rho} = O\Big(\rho^p\omega(\frac{\rho}{\rho_0},\alpha_1)(1+\kappa r^p\omega^2(\frac{\rho}{\rho_0},-\alpha_1)\Big).$$

(3). Bound for $\frac{\partial^i\theta}{\partial\rho^i}$, $(i \ge 2)$.

Assume that there exist constants $\widehat{M}_{j1}, \widehat{M}_{j2} > 0$ such that for $2 \le j \le i$, there holds

$$\left|\frac{\partial^j E}{\partial\rho^j}\right| \le \rho^{p-j}\omega(\frac{\rho}{\rho_0},\alpha_1)\widehat{L}_j(r,\omega(\frac{\rho}{\rho_0},-\alpha_1)) \tag{4.46}$$

where

$$\widehat{L}_j(r,\omega(\frac{\rho}{\rho_0},-\alpha_1)) = \widehat{M}_{j1}+\kappa\widehat{M}_{j2}r^p\omega^2(\frac{\rho}{\rho_0},-\alpha_1).$$

Now let us consider $\frac{\partial^{j+1}E}{\partial\rho^{j+1}}$. By Prop. 4.9, it satisfies

$$\begin{cases} \frac{d}{dt}\Big(\frac{\partial^{i+1}E}{\partial\rho^{i+1}}\Big) & = h_0(t,\rho,E)\frac{\partial^{i+1}E}{\partial\rho^{i+1}} + h_{i+1}\Big(t,\rho,E,\frac{\partial E}{\partial\rho},\frac{\partial^2 E}{\partial\rho^2},\cdots,\frac{\partial^i E}{\partial\rho^i}\Big) \\ \frac{\partial^{i+1}E}{\partial\rho^{i+1}}(0) & = 0 \end{cases} \tag{4.47}$$

where

$$\begin{aligned} & h_{i+1}\Big(t,\rho,E,\frac{\partial E}{\partial\rho},\frac{\partial^2 E}{\partial\rho^2},\cdots,\frac{\partial^i E}{\partial\rho^i}\Big) \\ & = \frac{\partial^{i+1}h}{\partial\rho^{i+1}} + \sum_{j=2}^{i}\frac{\partial^j h}{\partial E^j}\sum_{I(j_1j_2\cdots j_i;j,i+1)} * \prod_{k=1}^{i}\Big(\frac{\partial^k E}{\partial\rho^k}\Big)^{j_k} \\ & \quad + \sum_{l=1}^{i}\sum_{j=1}^{i+1-l}\frac{\partial^{j+l}h}{\partial\rho^j\partial E^l}\sum_{I(l_1l_2\cdots l_i;l,i+1-j)} * \prod_{k=1}^{i}\Big(\frac{\partial^k E}{\partial\rho^k}\Big)^{l_k} \end{aligned} \tag{4.48}$$

Lemma 4.15. *There exist constants $\overline{M}_{i+1,1},\overline{M}_{i+1,2} > 0$ such that for $\forall(a,\bar{\mu}) \in A_0 \times \mathbb{S}^2$ and for $r,\rho > 0$ sufficiently small, there holds*

$$\Big|h_{i+1}(t,\rho,E)\Big| \le \rho^{p-(i+1)}e^{\alpha_1 t}\omega(\frac{\rho}{\rho_0},\alpha_1)\Big[\overline{M}_{i+1,1} + \kappa\overline{M}_{i+1,2}\omega(\frac{\rho}{\rho_0},-\alpha_1)\Big]. \tag{4.49}$$

Proof. Let us denote the first and the second sum in (4.48) by h_I and h_{II}, i.e.

$$h_{i+1} = \frac{\partial^{i+1}h}{\partial\rho^{i+1}} + h_I + h_{II}. \tag{4.50}$$

For $\frac{\partial^{i+1}h}{\partial\rho^{i+1}}$, by (4.39) and the definition of h in (4.34), there exist constants $M_{i+1,1}$, $M_{i+1,2} > 0$ such that

$$\Big|\frac{\partial^{i+1}h}{\partial\rho^{i+1}}\Big| \le \rho^{p-(i+1)}e^{\alpha_1 t}\Big[M_{i+1,1} + \kappa M_{i+1,2}r^p\omega^2(\frac{\rho}{\rho_0},\alpha_1)\Big]. \tag{4.51}$$

Similarly, there exist constants $\widetilde{M}_{j1}$, $\widetilde{M}_{j2}$, $\widetilde{M}_{jl1}$ and $\widetilde{M}_{jl2} > 0$ such that

$$\Big|\frac{\partial^j h}{\partial E^j}\Big| \le \rho^p e^{\alpha_1 t}\overline{L}_j(r,\omega(\frac{\rho}{\rho_0},-\alpha_1)) \tag{4.52}$$

$$\Big|\frac{\partial^{j+l}h}{\partial\rho^j\partial E^l}\Big| \le \rho^{p-j}e^{\alpha_1 t}\widetilde{L}_{jl}(r,\omega(\frac{\rho}{\rho_0},-\alpha_1)) \tag{4.53}$$

where

$$\begin{aligned} \overline{L}_j(r,\omega(\tfrac{\rho}{\rho_0},-\alpha_1)) & = \widetilde{M}_{j1} + \kappa\widetilde{M}_{j2}r^p\omega(\tfrac{\rho}{\rho_0},-\alpha_1) \\ \widetilde{L}_{jl}(r,\omega(\tfrac{\rho}{\rho_0},-\alpha_1)) & = \widetilde{M}_{jl1} + \kappa\widetilde{M}_{jl2}r^p\omega(\tfrac{\rho}{\rho_0},-\alpha_1). \end{aligned}$$

So for h_I, by (4.52), (4.45) and assumption (4.46), for $t \in [0, |\ln \frac{\rho}{\rho_0}|]$ we have

$$
\begin{aligned}
&\left|h_I(t, \rho, E, \tfrac{\partial E}{\partial \rho}, \tfrac{\partial^2 E}{\partial \rho^2}, \cdots, \tfrac{\partial^i E}{\partial \rho^i})\right| \\
&\leq \sum_{j=2}^{i} \left|\frac{\partial^j h}{\partial E^j}\right| \sum_{I(j_1 j_2 \cdots j_i; j, i+1)} * \left|\frac{\partial E}{\partial \rho}\right|^{j_1} \left|\frac{\partial^2 E}{\partial \rho^2}\right|^{j_2} \cdots \left|\frac{\partial^i E}{\partial \rho^i}\right|^{j_i} \\
&\leq \sum_{j=2}^{i} \rho^p e^{\alpha_1 t} \overline{L}_j(r, \omega) \sum_{I(j_1 j_2 \cdots j_i; j, i+1)} * \prod_{k=1}^{i} \left|\rho^{p-k} \omega \widehat{L}_k(r, \omega)\right|^{j_k} \\
&\leq \sum_{j=2}^{i} \rho^p e^{\alpha_1 t} \overline{L}_j(r, \omega) \sum_{I(j_1 j_2 \cdots j_i; j, i+1)} * \rho^{p \sum_{k=1}^{i} j_k - \sum_{k=1}^{i} k j_k} \omega^{\sum_{k=1}^{i} j_k} \prod_{k=1}^{i} \widehat{L}_k^{j_k}(r, \omega) \\
&\leq \sum_{j=2}^{i} \rho^p e^{\alpha_1 t} \overline{L}_j(r, \omega) \sum_{I(j_1 j_2 \cdots j_i; j, i+1)} * \rho^{pj-(i+1)} \omega^j \prod_{k=1}^{i} \widehat{L}_k^{j_k}(r, \omega) \\
&\leq \sum_{j=2}^{i} \rho^{p(1+j)-(i+1)} e^{\alpha_1 t} \overline{L}_j(r, \omega) \sum_{I(j_1 j_2 \cdots j_i; j, i+1)} * \omega^j \prod_{k=1}^{i} \widehat{L}_k^{j_k}(r, \omega) \\
&\leq \rho^{p-(i+1)} e^{\alpha_1 t} \sum_{j=2}^{i} \rho^{pj} \omega^j \overline{L}_j(r, \omega) \sum_{I(j_1 j_2 \cdots j_i; j, i+1)} * \prod_{k=1}^{i} \widehat{L}_k^{j_k}(r, \omega) \\
&\leq o(\rho^{2p-1}) \rho^{p-(i+1)} e^{\alpha_1 t}.
\end{aligned}
\tag{4.54}
$$

Similarly, for h_{II}, by (4.53) and (4.45) and (4.46), we have

$$
\begin{aligned}
&\left|h_{II}(t, \rho, E, \tfrac{\partial E}{\partial \rho}, \tfrac{\partial^2 E}{\partial \rho^2}, \cdots, \tfrac{\partial^i E}{\partial \rho^i})\right| \\
&\leq \sum_{l=1}^{i} \sum_{j=1}^{i+1-l} \left|\frac{\partial^{j+l} h}{\partial \rho^j \partial E^l}\right| \sum_{I(l_1 l_2 \cdots l_i; l, i+1-j)} * \prod_{k=1}^{i} \left|\frac{\partial E}{\partial \rho}\right|^{l_k} \\
&\leq \sum_{l=1}^{i} \sum_{j=1}^{i+1-l} \rho^{p-j} e^{\alpha_1 t} \widetilde{L}_{jl}(r, \omega) \sum_{I(l_1 l_2 \cdots l_i; l, i+1-j)} * \prod_{k=1}^{i} \left[\rho^{p-k} \omega \widehat{L}_k(r, \omega)\right]^{l_k} \\
&\leq \sum_{l=1}^{i} \sum_{j=1}^{i+1-l} \rho^{p-j} e^{\alpha_1 t} \widetilde{L}_{jl}(r, \omega) \sum_{I(l_1 l_2 \cdots l_i; l, i+1-j)} * \rho^{pl-(i+1-j)} \omega^l \prod_{k=1}^{i} \widehat{L}_k^{l_k}(r, \omega) \\
&\leq \sum_{l=1}^{i} \sum_{j=1}^{i+1-l} \rho^{p-j+pl-(i+1-j)} e^{\alpha_1 t} \omega^l \widetilde{L}_{jl}(r, \omega) \sum_{I(l_1 l_2 \cdots l_i; l, i+1-j)} * \prod_{k=1}^{i} \widehat{L}_k^{l_k}(r, \omega) \\
&\leq \rho^{p-(i+1)} e^{\alpha_1 t} \sum_{l=1}^{i} \rho^{pl} \omega^l \sum_{j=1}^{i+1-l} \widetilde{L}_{jl}(r, \omega) \sum_{I(l_1 l_2 \cdots l_i; l, i+1-j)} * \prod_{k=1}^{i} \widehat{L}_k^{l_k}(r, \omega) \\
&\leq O\Big(\rho^p \omega(\frac{\rho}{\rho_0}, \alpha_1)\Big) \rho^{p-(i+1)} e^{\alpha_1 t}.
\end{aligned}
\tag{4.55}
$$

Then by (4.50), (4.51), (4.54) and (4.55), there exist positive constants $\overline{M}_{i+1,1}$ and $\overline{M}_{i+1,2}$ such that

$$\begin{aligned}|h_{i+1}| &\leq \left|\tfrac{\partial^{i+1}h}{\partial\rho^{i+1}}\right| + |h_I| + |h_{II}| \\ &\leq \rho^{p-(i+1)}e^{\alpha_1 t}[M_{i+1,1} + \kappa M_{i+1,2}r^p\omega^2(\tfrac{\rho}{\rho_0},\alpha_1)] + o(\rho^{2p-1})\rho^{p-(i+1)}e^{\alpha_1 t} \\ &\quad +O(\rho^p\omega(\tfrac{\rho}{\rho_0},\alpha_1))\rho^{p-(i+1)}e^{\alpha_1 t} \\ &\leq \rho^{p-(i+1)}e^{\alpha_1 t}\omega(\tfrac{\rho}{\rho_0},\alpha_1)[\overline{M}_{i+1,1} + \kappa\overline{M}_{i+1,2}\omega^2(\tfrac{\rho}{\rho_0},-\alpha_1)].\end{aligned}$$

□

End of proof of Theorem 4.14

Then for the initial value problem (4.47) with the estimations (4.43) and (4.49), similar to the proof (4.45), again by Prop. 4.10, for $t \in [0, |\frac{\rho}{\rho_0}|]$, there exist constants $\widehat{M}_{i+1,1}, \widehat{M}_{i+1,2} > 0$ such that for $t \in [0, |\frac{\rho}{\rho_0}|]$ we have

$$\left|\frac{\partial^{i+1}E}{\partial\rho^{i+1}}\right| \leq \rho^{p-(i+1)}\omega(\frac{\rho}{\rho_0},\alpha_1)[\widehat{M}_{i+1,1} + \kappa\widehat{M}_{i+1,2}r^p\omega^2(\frac{\rho}{\rho_0},-\alpha_1)]. \tag{4.56}$$

Hence for $(a,\bar{\mu}) \in A_0 \times \mathbb{S}^2$ and $r, \rho > 0$ sufficiently small, we have

$$\rho^{i+1}\frac{\partial^{i+1}\theta}{\partial\rho^{i+1}} = O\Big(\rho^p\omega(\frac{\rho}{\rho_0},\alpha_1)(1+\kappa r^p\omega^2(\frac{\rho}{\rho_0},-\alpha_1)\Big).$$

□

Chapter 5

Finite cyclicity of convex graphics with a nilpotent singularity of saddle type

In this chapter, we study the finite cyclicity of convex graphics with a nilpotent singularity of saddle type. In §5.2, we discuss the generic properity of the graphics. We claim the main theorem in §5.3. The finite cycilcity theorem is proved in §5.4 and §5.5. In proving the finite cyclicity theorems on graphics of saddle as well as elliptic type, we will have to calculate the derivatives of regular transition maps, so we begin with the preliminaries on derivatives of regular transition maps.

5.1 Preliminaries on derivatives of regular transition maps

First we recall briefly the formula of [ALGM].

Proposition 5.1 (ALGM). *Consider the vector field*

$$X = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}. \tag{5.1}$$

Let $\Sigma = \{(x,y) = (f_1(s), g_1(s))\}$ *and* $\widetilde{\Sigma} = \{(x,y) = (f_2(s), g_2(s))\}$ *be two arcs transverse to the same orbit. Let* $R(s)$ *be the transition map from* Σ *to* $\widetilde{\Sigma}$*. Then*

$$R'(s) = \frac{\Delta(s)}{\widetilde{\Delta}(R(s))} \exp\left(\int_0^{T(s)} div\ X(\gamma(t))dt\right) \tag{5.2}$$

where $T(s)$ is the transition time from $(f_1(s), g_1(s))$ to $(f_2(R(s)), g_2(R(s)))$ along the orbit $\gamma(t)$ starting at $(f_1(s), g_1(s))$ for $t = 0$ and

$$\Delta(s) = \begin{vmatrix} P(f_1(s), g_1(s)) & f_1'(s) \\ Q(f_1(s), g_1(s)) & g_1'(s) \end{vmatrix}$$

$$\widetilde{\Delta}(\tilde{s}) = \begin{vmatrix} P(f_2(\tilde{s}), g_2(\tilde{s})) & f_2'(\tilde{s}) \\ Q(f_1(\tilde{s}), g_2(\tilde{s})) & g_2'(\tilde{s}) \end{vmatrix}.$$

It is not easy to use Prop. 5.1 to calculate the higher order derivatives of a regular transition map. The following proposition will be very useful.

Proposition 5.2. *We consider the transition map $R(x)$ of the vector field (5.1) between two arcs without contact: $\Sigma = \{(x, y) = (x, f_1(x))\}$ and $\widetilde{\Sigma} = \{(x, y) = (x, f_2(x))\}$, in a region where $Q(x, y) \neq 0$. Let $x = x(x_0, y_0, y)$ be the solution with initial condition $x(x_0, y_0, y_0) = x_0$. Then*

$$\frac{dR}{dx_0}(x_0) = \exp\left(\int_{f_1(x_0)}^{f_2(R(x_0))} \left(\frac{P_x'Q - PQ_x'}{Q^2}\right)\Big|_{x=x(x_0, f_1(x_0), y)} dy\right) \frac{1 - \left(\frac{P}{Q}\right)(x_0, f_1(x_0))f_1'(x_0)}{1 - \left(\frac{P}{Q}\right)(x_0, f_2(R(x_0)))f_2'(R(x_0))}. \tag{5.3}$$

Formulas for the first and second derivatives are given in the particular case where $x_0 = 0$ and $P(0, y) \equiv 0$. Let $y_i = f_i(0)$.

$$R'(0) = \exp\left(\int_{y_1}^{y_2} \frac{P_x'}{Q}(0, y)dy\right). \tag{5.4}$$

$$\begin{aligned} R''(0) &= R'(0)\Bigg[2\left(f_2'(0)R'(0)\left(\frac{P_x}{Q}\right)(0, y_2) - f_1'(0)\left(\frac{P_x}{Q}\right)(0, y_1)\right) \\ &+ \int_{y_1}^{y_2}\left(\frac{P_x''}{Q}(0, y) - 2\frac{P_x'Q_x'}{Q^2}(0, y)\right)\exp\left(\int_{y_1}^{y} \frac{P_x'}{Q}(0, z)\, dz\right) dy\Bigg]. \end{aligned} \tag{5.5}$$

Proof. We transform (5.1) into the equivalent differential equation

$$\frac{dx}{dy} = \frac{P}{Q}. \tag{5.6}$$

The solution is $x = x(x_0, f_1(x_0), y)$ with initial condition $x(x_0, f_1(x_0), f_1(x_0)) = x_0$. We have that $R(x_0) = x(x_0, f_1(x_0), f_2(R(x_0)))$. Moreover

$$\begin{aligned} \frac{\partial}{\partial y}\frac{\partial x}{\partial x_0} &= \frac{\partial}{\partial x_0}\frac{\partial x}{\partial y} \\ &= \frac{\partial}{\partial x_0}\frac{P(x(x_0, f_1(x_0), y), y)}{Q(x(x_0, f_1(x_0), y), y)} \\ &= \frac{P'_x Q - P Q'_x}{Q^2}\frac{\partial x}{\partial x_0}, \end{aligned} \tag{5.7}$$

from which

$$\frac{\partial x}{\partial x_0} = \exp\left(\int_{f_1(x_0)}^{y} \frac{P'_x Q - P Q'_x}{Q^2}\, dy\right) \tag{5.8}$$

follows. Hence we can rewrite

$$\begin{aligned} \frac{dR}{dx_0}(x_0) = \; & \exp\left(\int_{f_1(x_0)}^{f_2(R(x_0))} \left(\frac{P'_x Q - P Q'_x}{Q^2}\right)\Bigg|_{x = x(x_0, f_1(x_0), y)} dy\right) \\ & \frac{1 - \left(\frac{P}{Q}\right)(x_0, f_1(x_0)) f_1'(x_0)}{1 - \left(\frac{P}{Q}\right)(x_0, f_2(R(x_0))) f_2'(R(x_0))}. \end{aligned} \tag{5.9}$$

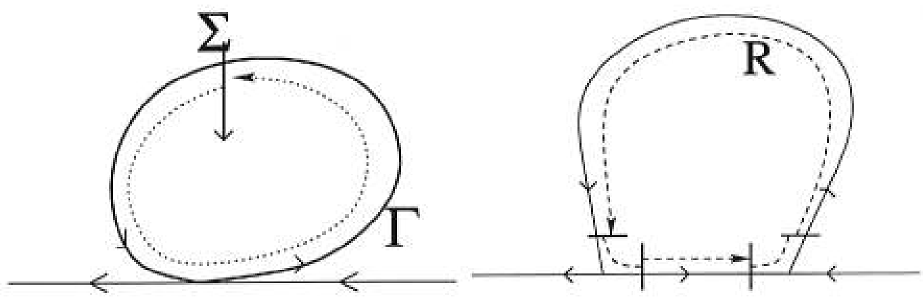
The second derivative of R is most easily calculated from this formula. However the general formula is very long. In the particular case $x_0 = 0$ we get (5.4) and (5.5) for $R'(0)$ and $R''(0)$. □

5.2 Generic property of the hh-graphic

Graphics through a nilpotent saddle point can be of two types: convex or concave. We only consider the convex graphics. Let Γ be the convex hh-graphic of saddle type (see Fig. 5.1(a)). Let Σ' be a section transverse to the connection Γ and parametrized by a regular C^∞ coordinate. We consider the Poincaré first return map

$$P : \Sigma \longrightarrow \Sigma'$$

where $\Sigma \subset \Sigma'$, a neighborhood of $\Gamma \cap \Sigma'$.



(a) Return map (b) Blow-up of the graphic

Figure 5.1: The Poincaré first return map for the hh-graphic of saddle type

Proposition 5.3. *For the convex graphic of saddle type and* $\forall a \in A$, *the Poincaré first return map* $P(z)$ *is at least* C^1 *for* $z \geq 0$ *and*

$$P'(0) = \gamma^* = \exp\left(\int_{-\infty}^{\infty} divX(\Gamma(t))dt\right). \tag{5.10}$$

Proof. We start with the system

$$\begin{cases} \dot{x} &= y + ax^2 \\ \dot{y} &= y(x + \varepsilon_2 x^2 + x^3 h(x)) + y^2 Q(x,y) \end{cases} \tag{5.11}$$

where $h(x)$ and $Q(x,y)$ are C^∞ and $Q(x,y) = O(|(x,y)|^N)$ for N sufficiently large. To study the dynamics near the singularity at $(0,0)$, we make the blow-up (3.6). Let $\bar{y} = 1$ in (3.6), we have

$$\begin{cases} \dot{r} &= \frac{1}{2}r(\bar{x} + rO(|(r,\bar{x})|)) := \overline{P}(r,\bar{x}) \\ \dot{\bar{x}} &= 1 - (\frac{1}{2} - a)\bar{x}^2 + rO(|(r,\bar{x})|) := \overline{Q}(r,x). \end{cases} \tag{5.12}$$

System (5.12) has two singular points P_3 and P_4 and both are hyperbolic saddles (see Fig. 5.1(b)). The eigenvalues at P_3 are $(\frac{\sqrt{2(1-2a)}}{2(1-2a)}, -\sqrt{2(1-2a)})$; the eigenvalues at P_4 are $(-\frac{\sqrt{2(1-2a)}}{2(1-2a)}, \sqrt{2(1-2a)})$. Hence the hyperbolicity ratio at P_3 (resp. P_4) is $\sigma_3(a) = 2(1-2a)$ (resp. $\frac{1}{\sigma_3(a)}$).

Take sections $\overline{\Sigma}_i = \{r_i = r_0\}$($i = 3, 4$), $\overline{\tau}_3 = \{\tilde{x}_3 = -x_0\}$ and $\overline{\tau}_4 = \{\tilde{x}_4 = x_0\}$ in the normal form coordinates in the neighborhood of P_3 and P_4 respectively. For the Dulac maps near P_3 and P_4, we have

$$\begin{aligned} \overline{D}_4(\tilde{x}_4) &= \tilde{x}_4^{\frac{1}{\sigma_3(a)}}[c_4 + \theta_4(\tilde{x}_4)] \\ \overline{D}_3(r_3) &= r_3^{\sigma_3}[c_3 + \theta_3(\tilde{r}_3)] \end{aligned} \tag{5.13}$$

where c_3 and c_4 are positive constants and $c_4 c_3^{\frac{1}{\sigma_3}} = 1$, and $\theta_3, \theta_4 \in (I_0^\infty)$.

Then we can decompose the Poincaré first return map P as

$$P = \overline{R} \circ \overline{D}_3 \circ \overline{T} \circ \overline{D}_4 \tag{5.14}$$

where $\overline{T} : \overline{\tau}_4 \longrightarrow \overline{\tau}_3$ and $\overline{R} : \overline{\Sigma}_3 \longrightarrow \overline{\Sigma}_4$ are two regular transition maps in the normal form coordinates.

Now we use the formula (5.2) of [ALGM] in Prop. 5.1 to calculate the first derivatives of the regular transition maps in (5.14). For the transition map $\overline{T}$ along $r = 0$ for the family (5.12), in the original coordinates $(r, \bar{x})$, the two sections become

$$\begin{aligned} \overline{\tau}_4 &= \{(r, -\tfrac{2}{\sqrt{1-2a}} + x_0 + O(|(r, x_0)|^2)\} \\ \overline{\tau}_3 &= \{(r, \tfrac{2}{\sqrt{1-2a}} - x_0 + O(|(r, x_0)|^2)\} \end{aligned}$$

Note that for the system (5.12), along $r = 0$, $divX|_{r=0} = -(1-2a)\bar{x}$, and $\overline{P}(0, \bar{x}) = 0$, so for $x_0 > 0$ sufficiently small, by Prop. 5.2, we have $\overline{T}'(0) = 1$. Thus we have

$$\overline{T}(r_4) = r_4 + O(r_4^2) \tag{5.15}$$

Therefore, by (5.13) and (5.15), if letting $\tilde{x}_4 = \overline{D}_3 \circ \widetilde{T} \circ \overline{D}_4(\tilde{x}_3)$, then

$$\tilde{x}_4 = \tilde{x}_3 + o(\tilde{x}_3). \tag{5.16}$$

To calculate the map R, as in [DER96], [SP87] and [DRS97], we introduce two auxiliary sections $\widetilde{\Sigma}_i = \{r_i = r_{00}\}$ $(i = 3, 4)$ in the normal form coordinates. Then the map R can be calculated by the decomposition

$$R = R_{40} \circ \overline{R} \circ R_{30} \tag{5.17}$$

where $R_{30}: \overline{\Sigma}_3 \longrightarrow \widetilde{\Sigma}_3$ and $R_{40}: \widetilde{\Sigma}_4 \longrightarrow \overline{\Sigma}_4$ are regular transition maps. Similar to $\overline{T}'(0)$, for R_{30} and R_{40} we have

$$\begin{aligned} R'_{30}(0) &= \left(\tfrac{r_{00}}{r_0}\right)^{1-\sigma_3(a)} \\ R'_{40}(0) &= \left(\tfrac{r_0}{r_{00}}\right)^{1-\sigma_3(a)}. \end{aligned} \tag{5.18}$$

For the transition map $\overline{R}: \widetilde{\Sigma}_3 \longrightarrow \widetilde{\Sigma}_4$, note that in this case, in the original coordinates $(r, \bar{x})$,

$$\begin{aligned} \widetilde{\Sigma}_3 &= \{(r_{00}, \tfrac{2}{\sqrt{1-2a}} + \bar{x} + O(\bar{x}^2) + r_{00}O(|(r, x_0)|)\} \\ \widetilde{\Sigma}_4 &= \{(r_{00}, -\tfrac{2}{\sqrt{1-2a}} - \bar{x} + O(\bar{x}^2) + r_{00}O(|(r, x_0)|)\}. \end{aligned}$$

So again by the formula in Prop. 5.1, we have

$$\overline{R}'(0) = \exp\left(\int_{-T_1}^{T_2} divX(\Gamma(t))dt\right).$$

Note that R is independent of r_{00}, so

$$\begin{aligned} R'(0) &= \lim_{r_{00}\to 0} R'_{40}(0)\ \overline{R}'(0)\ R'_{30}(0) \\ &= \lim_{r_{00}\to 0} \exp\left(\int_{-T_1}^{T_2} divX(\Gamma(t))dt\right) \\ &= \exp\left(\int_{-\infty}^{\infty} divX(\Gamma(t))dt\right). \end{aligned} \tag{5.19}$$

Thus, by (5.17), (5.18) and (5.19), we have

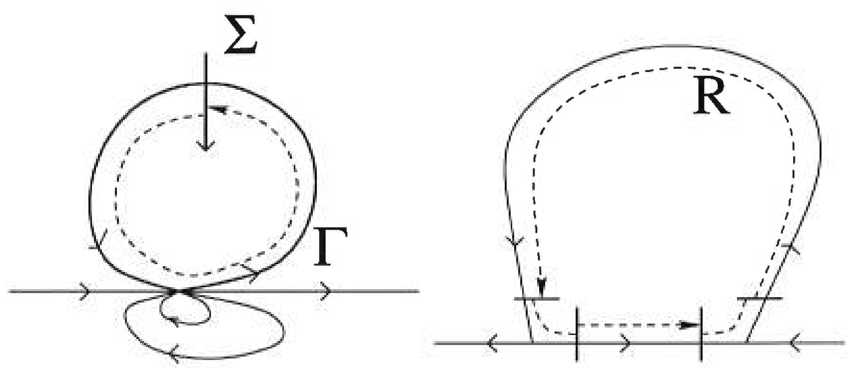
$$R(\tilde{x}_4) = \exp\left(\int_{-\infty}^{\infty} divX(\Gamma(t))dt\right)\tilde{x}_4 + O(\tilde{x}_4^2). \tag{5.20}$$

It follows from (5.14), (5.16) and (5.20) that there holds

$$P(\tilde{x}_3) = \exp\left(\int_{-\infty}^{\infty} divX(\Gamma(t))dt\right)\tilde{x}_3 + o(\tilde{x}_3)$$

thus we proved (5.10). □

Remark 5.4. *Prop.* 5.3 *is true for the hh-graphic of elliptic type(Fig.* 5.2*).*



(a) Return map

(b) Blow-up of the hh-graphic of elliptic type

Figure 5.2: The Poincaré first return map for the hh-graphic of elliptic type

5.3 Main Theorem on the convex graphic of saddle type

For the convex graphic of saddle type, we have

Theorem 5.5. *A convex hh-graphic through a triple nilpotent saddle of codimension 3 has finite cyclicity if the generic hypothesis* $P'(0) \neq 1$ *is satisfied.*

For the proof, by changing the vector field X to $-X$ if necessary, we impose

Hypothesis 5.6. *The convex hh-graphic with a nilpotent saddle is attracting:*

$$[\mathbf{H}]: \quad P'(0) = \gamma^* < 1. \tag{5.21}$$

After the global blow-up in §3.1, for the convex graphic through a triple nilpotent saddle, we get a total of 10 families of convex graphics: Sxhh1, Sxhh2, $\cdots$, Sxhh10 (see Tab. 3.5). For each family Sxhhi ($i = 1, 2, \cdots, 10$), the graphics fall into three groups:

- the upper boundary graphic:

  Sxhhia ($i = 1, 2, \cdots, 10$);

- the intermediate graphics:

  Sxhhib ($i = 1, 2, \cdots, 10$) Sxhh9d and Sxhh10d;

- the lower boundary graphics:

  Sxhhic ($i = 1, 2, \cdots, 10$), Sxhh9e and Sxhh10e.

To prove the finite cyclicity of the convex graphic with a nilpotent saddle, we have to prove that all the graphics listed above have finite cyclicity.

Notation 5.7. *For convenience in the notation, in the following sections and next chapter, let r_0, ρ_0 and y_0 be positive constants, we will always use*

$$\begin{aligned} \Sigma_i &= \{r_i = r_0\}, \quad i = 1,2,3,4 \\ \Pi_i &= \{\rho_i = \rho_0\}, \quad i = 1,2,3,4 \\ \tau_i &= \{\tilde{y}_i = y_0\}, \quad i = 1,2 \\ \tau_i &= \{\tilde{y}_i = -y_0\}, \quad i = 3,4 \end{aligned} \tag{5.22}$$

to denote the sections in normal form coordinates $(r_i, \rho_i, \tilde{y}_i)$ in the neighborhood of the four singular points P_i $(i = 1, 2, 3, 4)$.

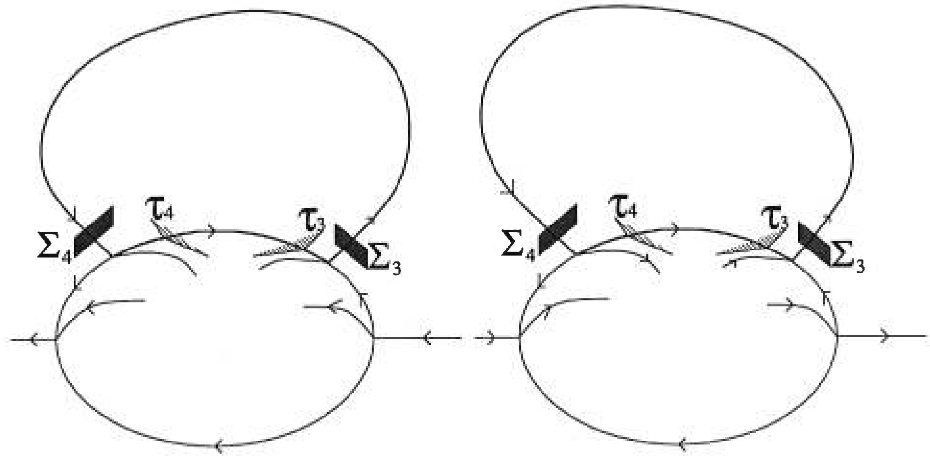
We begin with the upper boundary graphics.

5.4 The upper boundary graphic

Proposition 5.8. *For the convex hh-graphics of saddle type, under Hypothesis* 5.6, *all the upper boundary graphics have cyclicity one.*

Proof. As shown in Fig. 5.3, we study the Poincaré first return map defined on the section Σ_4:

$$P : \Sigma_4 \longrightarrow \Sigma_4.$$



(a) Saddle case (b) Elliptic Case

Figure 5.3: Upper boundary graphics of saddle and elliptic type

We can factorize it as

$$P = R \circ \Theta_3 \circ \overline{T}_{43} \circ \Theta_4^{-1} \tag{5.23}$$

where Θ_4 and Θ_3 are the second type of Dulac maps in the neighborhood of P_4 and P_3 respectively, $\overline{T}_{43}$ and R are the regular transition maps.

At P_3, the eigenvalues are $(1, -1, \sigma_3(a))$, where for $a \in (-\infty, 0)$, $\sigma_3(a) = 2(1-2a) > 0$. By the normal form discussion in Prop. 4.6, depending on $a_0 \notin \mathbb{Q}$ or $a_0 \in \mathbb{Q}$, the vector field near P_3 has the normal form of (4.8) or (4.9) with $\sigma = \sigma_3$. Correspondingly, we use β_i $(i = 1, 2, \cdots, N(k))$ instead of using α_i to make the distinction, especially $\beta_1 = p_3 - \bar{\sigma}_3(a)q_3$.

By Theorem 4.14, the second type Dulac map $\Theta_3 = (\xi_3, \Xi_3) : \tau_3 \longrightarrow \Sigma_3$ has the expression

$$\begin{cases} \xi_3(r_3, \rho_3) & = \nu \\ \Xi_3(r_3, \rho_3) & = \eta_3(\nu, \omega(\frac{r_3}{r_0}, \beta_1)) + (\frac{r_3}{r_0})^{\bar{\sigma}}[y_0 + \theta_3(r_3, \rho_3, \omega(\frac{r_3}{r_0}, -\beta_1))] \end{cases} \tag{5.24}$$

where $\eta_3(\nu, \omega(\frac{r_3}{r_0}, \beta_1)) = \frac{\kappa_3}{r_0^{p_3}} \nu^{p_3} \omega(\frac{r_3}{r_0}, \beta_1)$ and $\theta_3(r_3, \rho_3, \omega(\frac{r_3}{r_0}, -\beta_1))$ satisfies the property (4.30). Due to the symmetry, the Dulac map $\Theta_4 : \tau_4 \longrightarrow \Sigma_4$ has the same form as of Θ_3 in (5.24).

We calculate the transition $\overline{T}_{43} : \tau_4 \longrightarrow \tau_3$ using the polar coordinates $(\bar{x}, \bar{y}) = (r\cos\theta, r^2 \sin\theta)$ in the chart F.R.. Then we have

$$\begin{cases} (1+\sin^2\theta)\dot{r} & = r\cos\theta(a\cos^2\theta + \sin^2\theta + \sin\theta) + O(r^2) \\ (1+\sin^2\theta)\dot{\theta} & = \sin\theta((1-2a)\cos^2\theta - 2\sin\theta) + O(r) \end{cases}$$

or

$$\frac{dr}{d\theta} = -r\frac{\cos\theta(a\cos^2\theta + \sin^2\theta + \sin\theta)}{\sin\theta((1-2a)\cos^2\theta - 2\sin\theta)} + O(r^2)$$

Making the translation $\theta = \bar{\theta} + \frac{\pi}{2}$, then

$$\frac{dr}{d\bar{\theta}} = r\frac{\sin\bar{\theta}(a\sin^2\bar{\theta} + \cos^2\bar{\theta} + \cos\bar{\theta})}{\cos\bar{\theta}((1-2a)\sin^2\bar{\theta} - 2\cos\bar{\theta})} + O(r^2) = f(\bar{\theta})r + O(r^2). \tag{5.25}$$

Note that $f(-\bar{\theta}) = -f(\bar{\theta})$, the two symmetric sections τ_3 and τ_4 correspond to the two symmetric positions $\bar{\theta} = \bar{\theta}_0$ and $\bar{\theta} = -\bar{\theta}_0$. So integrating (5.25) from $\bar{\theta}_0$ to $-\bar{\theta}_0$ gives that for $\nu = 0$

$$r_3 = r_4 \exp(\int_{-\theta_0}^{\theta_0} f(\bar{\theta})d\bar{\theta}) + O(r_4^2) = r_4 + O(r_4^2). \tag{5.26}$$

Let

$$\widehat{T} = \Theta_3 \circ \overline{T}_{43} \circ \Theta_4^{-1}. \tag{5.27}$$

Easily we have $\widehat{T}_1(\nu, \tilde{y}_4) = \nu$. Now we calculate the first derivative of $\widehat{T}_2$. Note that by (5.24), we have

$$\frac{\partial}{\partial r_3}\Xi_3(r_3, \rho_3) = \left(\frac{r_3}{r_0}\right)^{\bar{\sigma}_3 - 1}\left[y_0 + \frac{\beta_1 \kappa_3}{r_0^{\bar{\sigma}_3 - 1}}\rho_3^{p_3} + \hat{\theta}_3(r_3, \rho_3, \omega(\frac{r_3}{r_0}, -\beta_1))\right]. \tag{5.28}$$

For $\Xi_4^{-1}(\nu, \tilde{y}_4)$, we have

$$\frac{\partial}{\partial \tilde{y}_4}\Xi_4^{-1}(\nu, \tilde{y}_4) = \frac{1}{\left(\frac{r_4}{r_0}\right)^{\bar{\sigma}_3 - 1}\left[y_0 + \frac{\beta_1 \kappa_3}{r_0^{\bar{\sigma}_3 - 1}}\rho_4^{p_3} + \hat{\theta}_4(r_4, \rho_4, \omega(\frac{r_4}{r_0}, -\beta_1))\right]}. \tag{5.29}$$

Hence by (5.27), (5.28), (5.26) and (5.29) we have that $\widehat{T}_2$ is at least C^1 and

$$\widehat{T}_2'(0,0) = 1. \tag{5.30}$$

We calculate the transition map R in the chart P.R.3. For its first component, we have $R_1(\nu, \tilde{y}_3) = \nu$. For the second component R_2, as in Prop. 5.3, by using the auxiliary sections and formula of [ALGM] in Prop. 5.2, we obtain

$$R_2'(0,0) = \gamma^*. \tag{5.31}$$

It follows from (5.23), (5.30) and (5.31) that we have

$$\det P(0,0) = \gamma^*.$$

By Hypothesis 5.6, $\gamma^* < 1$. Hence the first return map P has at most 1 fixed point, i.e., $Cycl(Shhia) \leq 1$, $i = 1, 2, \cdots, 10$. □

Remark 5.9. *In the proof we only use the fact* $1 - 2a > 0$. *So for the elliptic case with* $a \in (0, \frac{1}{2})$, *under the assumption in Remark* 5.4, *the same proof gives that the upper boundary hh-graphic of elliptic type has finite cyclicity 1.*

5.5 Intermediate and lower boundary graphics

Let Γ be any intermediate or lower boundary graphic of the 10 families. To study its cyclicity, as shown in Fig. 5.4, take sections Π_3 and Π_4 (as defined in (5.22)) in the normal form coordinates $(r_i, \rho_i, \tilde{y}_i)$ $(i = 3, 4)$. We are going to study the displacement map

$$\begin{aligned} L:\ & \Pi_4 \longrightarrow \Pi_3 \\ & L = R^{-1} - T \end{aligned} \tag{5.32}$$

or the displacement map

$$\begin{aligned} \mathcal{L}:\ & \Pi_3 \longrightarrow \Pi_4 \\ & L = R - T^{-1} \end{aligned} \tag{5.33}$$

where $R : \Pi_3 \longrightarrow \Pi_4$ is the transition map along the regular orbit in the normal form coordinates, and $T : \Pi_4 \longrightarrow \Pi_3$ is the transition map passing through the blown-up singularity. Then by the derivation-division method introduced by Roussarie in [R86], we study the number of small roots of $L = 0$ or $\mathcal{L} = 0$. The maximum number of roots bounds the cyclicity.

We begin with the transition map R. Obviously $R_1(\nu, \bar{y}_3) = \nu$. For the second component $R_2(\nu, \bar{y}_3)$, it is "almost affine"(the two passages near P_3 and P_4 have a "funneling effect" [R98]).

Proposition 5.10. *For any $k \in \mathbb{N}$ and $\forall a_0 \in A = (-\frac{1}{2}, 0)$, there exist $A_0 \subset A$, a neighborhood of a_0 and $\nu_1 > 0$ such that $\forall (a, \bar{\mu}) \in A_0 \times \mathbb{S}^2$ and $\forall \nu \in (0, \nu_1)$, $R_2(\nu, \tilde{y}_1)$ is C^k and*

(1) *If $a_0 \notin \mathbb{Q}$*

$$R_2(\nu, \tilde{y}_3) = m_{340}(\nu)(\tfrac{\nu}{\nu_0})^{-\sigma_3} + (\gamma^* + O(\nu))\tilde{y}_3 + \sum_{j=2}^{k} O(\nu^{j-1})\tilde{y}_3^j + O(\nu^k \tilde{y}_3^{k+1})$$

(2) *If $a_0 \in \mathbb{Q}$*

$$R_2(\nu, \tilde{y}_3) = \gamma_{340}(\nu, \omega(\tfrac{\nu}{\nu_0}, -\beta_1)) + \sum_{i=1}^{k} \gamma_{34i}(\nu, \omega(\frac{\nu}{\nu_0}, -\beta_1))\tilde{y}_3^i + O(\tilde{y}_3^{k+1}) \tag{5.34}$$

where

$$\begin{aligned}
\gamma_{340} &= m_{340}(\nu)\left(\frac{\nu}{\nu_0}\right)^{-\bar{\sigma}_3} + \kappa_3 r_0(\gamma^* - 1)\omega(\frac{\nu}{\nu_0}, -\beta_1) + \kappa_3 O\left(\nu^{\bar{\sigma}_3}\omega^2(\frac{\nu}{\nu_0}, -\beta_1)\right) \\
\gamma_{341} &= \gamma^* + O(\nu^{\bar{p}_3}\omega^{q_3}(\frac{\nu}{\nu_0}, -\beta_1)\ln\frac{\nu}{\nu_0}) \\
\gamma_{34j} &= O\Big(\nu^{\bar{p}_3(1+[\frac{j-2}{q_3}])}\omega^{q_3+1-j-q_3[\frac{j-2}{q_3}]}(\frac{\nu}{\nu_0}, -\beta_1)\ln\frac{\nu}{\nu_0}\Big) \qquad j \geq 2.
\end{aligned}$$

Also $R_2^{-1}(\nu, \tilde{y}_4)$ is C^k and has precisely the same form as R_2.

Proof. We limit ourselves to the second case: $a_0 \in \mathbb{Q}$. Decompose the transition map R as

$$R := \Delta_4^{-1} \circ R_{34} \circ \Delta_3$$

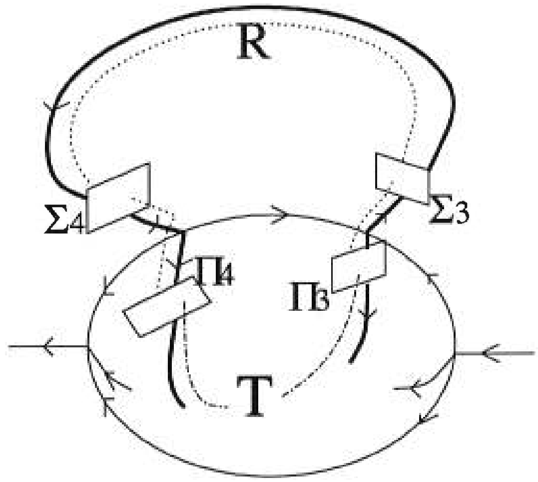


Figure 5.4: Transition map for the intermediate hh-graphics of saddle type

where $\Delta_j : \Pi_j \longrightarrow \Sigma_j$, $(j = 3, 4)$ are the two Dulac maps of the first type in the normal form coordinates near P_3 and P_4 respectively and $R_{34} : \Sigma_3 \longrightarrow \Sigma_4$ is the regular transition map.

(1). The Dulac map Δ_3 and Δ_4:

The systems near P_3 and P_4 have the form (4.8) or (4.9) with $\sigma = \sigma_3(a)$. By Theorem 4.11, we have $(i = 3, 4)$

$$\begin{cases} d_i(\nu, \tilde{y}_i) & = \nu \\ D_i(\nu, \tilde{y}_i) & = \eta_i(\nu, \omega(\frac{\nu}{\nu_0}, \beta_1)) + \left(\frac{\nu}{\nu_0}\right)^{\bar{\sigma}_3} [\tilde{y}_i + \psi_i(\nu, \tilde{y}_i, \omega(\frac{\nu}{\nu_0}, \beta_1))] \end{cases} \tag{5.35}$$

where η_i and ψ_i have the same property in (4.13).

(2). The transition map R_{34}: it is the composition of the regular transition map and two normal form coordinate changes on the section σ_3 and σ_4, so we can write it as

$$\begin{cases} R_{341}(\nu, \tilde{y}_3) & = \nu \\ R_{342}(\nu, \tilde{y}_3) & = m_{341}(\nu) + \sum_{i=1}^{k} m_{34i}(\nu)\tilde{y}_3^i + O(\tilde{y}_3^{k+1}) \end{cases} \tag{5.36}$$

where $m_{340}(0) = 0$ and $m_{341}(0) = \gamma^* + O(\nu)$.

Let

$$\hat{y}_4 = \left(\frac{\nu}{\nu_0}\right)^{-\bar{\sigma}_3}\left[D_4 - \eta_4(\nu, \omega(\frac{\nu}{\nu_0}, \beta_1))\right] \tag{5.37}$$

then by (5.35) and (5.37), we have

$$\hat{y}_4 = \tilde{y}_4 + \psi_4(\nu, \omega(\frac{\nu}{\nu_0}, -\beta_1), \tilde{y}_4). \tag{5.38}$$

Let

$$\tilde{y}_4 = \hat{y}_4 + \overline{\phi}_4(\nu, \omega(\frac{\nu}{\nu_0}, -\beta_1), \tilde{y}_4)$$

be the inverse of (5.38). Then $\overline{\phi}_4$ has the same property as of ψ_4. So if we let $\hat{y}_4 = (\frac{\nu}{\nu_0})^{-\bar{\sigma}_3}\tilde{y}_4$, then for the y−component of Δ_4^{-1}, we have

$$D_4^{-1}(\nu, \tilde{y}_4) = \hat{y}_4 + \overline{\phi}_4(\nu, \omega(\frac{\nu}{\nu_0}, -\beta_1), \tilde{y}_4). \tag{5.39}$$

where $\overline{\phi}_4(\nu, \omega(\frac{\nu}{\nu_0}, -\beta_1), \tilde{y}_4)$ has the same property as of ψ_4.

Hence, for the second component of transition map $\overline{R}$, by (5.35), (5.36) and (5.39), a straightforward calculation gives the result. □

The following proposition will serve to treat the intermediate graphics while the lower boundary graphics will require ad hoc methods in each case.

Proposition 5.11. *Assume that we have a convex hh-graphic* Γ *of saddle or elliptic type shown in Fig.* 5.4. *Let*

$$T : \Pi_4 \longrightarrow \Pi_3$$

be the transition map along the connection in the chart F.R.. Then if T *satisfies one of the following conditions:*

- T *is the identity while the graphic is generic (i.e.,* $\gamma^* < 1$*);*
- $T_2'(0,0)$ *is sufficiently small or* $T_2'(0,0)$ *is sufficiently large;*
- $T_2(0, \tilde{y}_4)$ *is nonlinear of order* n*,*

then Γ *has finite cyclicity.*

Proof. We consider the displacement map L or its inverse defined in (5.32). By Prop. 5.10, the second component $R_2(\nu, \tilde{y}_3)$ of R is almost affine, yielding the results. Γ has cyclicity ≤ 1 in the first two cases and cyclicity $\leq n$ in the third. □

It seems a priori difficult to show that a transition map is nonlinear. For all the cases, we will deal with families of graphics. This allows an interesting observation which we state in the following proposition.

Proposition 5.12. *It is possible to choose normalizing coordinates near P_3 and P_4 such that $\tilde{y}_i(0, \rho_i, y_i)$ $(i = 3, 4)$ is analytic.*

Proof. We modify the normalization process. For both the saddle or elliptic cases, the vector field near P_3 can be written as

$$\begin{cases} \dot{r} &= r \\ \dot{\rho} &= -\rho \\ \dot{y} &= -\sigma_3(a)y + h(a, \bar{\mu}, r, \rho, y) \end{cases} \tag{5.40}$$

where $h(a, \bar{\mu}, r, \rho, y) = o(|r, \rho, y|)$ and for both the saddle $A = (-\frac{1}{2}, 0)$ or the elliptic $A = (0, \frac{1}{2})$ case, we have $\sigma_3(a) = 2(1 - 2a) > 0$.

Let us consider (5.40) for $r = 0$. Then we get

$$\begin{cases} \dot{\rho} &= -\rho \\ \dot{y} &= -\sigma_3(a)y + h(a, \bar{\mu}, 0, \rho, y) \end{cases} \tag{5.41}$$

For the subfamily (5.41), the tuple of eigenvalues $(-1, -\sigma_3(a))$ is in the Poincaré domain, the subfamily has no (resp. one) resonant monomial when $\sigma_3(a_0) \notin \mathbb{N}$ (resp. $\sigma_3(a_0) \in \mathbb{N}$). Hence there exists an analytic map

$$Y = y + \hat{\phi}(\rho, y) \tag{5.42}$$

which brings family (5.41) into the normal form

$$\begin{cases} \dot{\rho} &= -\rho \\ \dot{Y} &= -\sigma_3(a)Y + \kappa_3\rho^{p_3} \end{cases} \tag{5.43}$$

where $p_3 = \sigma_3(a_0)$, and, if $\sigma_3(a_0) \notin \mathbb{N}$, then $\kappa_3 = 0$.

Applying the map (5.42) to the original family (5.40) brings the system to the form

$$\begin{cases} \dot{r} &= r \\ \dot{\rho} &= -\rho \\ \dot{Y} &= -\sigma_3(a)Y + \kappa_3\rho^{p_3} + rH(a, \bar{\mu}, r, \rho, Y). \end{cases} \tag{5.44}$$

For system (5.44), by Prop. 4.6, $\forall(a, \bar{\mu}) \in A \times V$, there exists a C^k map of the form

$$\tilde{y} = Y + r\overline{\phi}(r, \rho, Y) \tag{5.45}$$

which brings system (5.44) into the normal form (4.8) or (4.9).

Combining the transformations (5.41) and (5.45) together, we conclude that $\forall(a, \bar{\mu}) \in A \times V$ and for $\nu > 0$ sufficiently small, a map bringing the original system (5.40) to normal form has the form

$$\tilde{y} = y + \phi(r, \rho, y)$$

with $\phi(r, \rho, y)$ of class C^k and $\phi(0, \rho, y)$ analytic. □

Corollary 5.13. *Assume that* $\forall a_0 \in A$ *and* $\bar{\mu}_0 \in V$ *(*$V \subset \mathbb{S}^2$*), we have a family of graphics of the saddle (convex) or elliptic type which only differ by a segment joining two nodes (Fig. 5.4). Let* Γ *be any intermediate graphic in the family. Then* $\forall(a, \bar{\mu}) \in A \times V$*, the normal form coordinates* $\tilde{y}_3, \tilde{y}_4$ *can be taken so that* $\tilde{y}_i(0, \rho, y_i)$ *is analytic. Take sections* Π_3 *and* Π_4 *in the normal form coordinates in the neighborhood of* P_3 *and* P_4 *respectively. Let* $\Gamma \cap \Pi_4 = (0, \tilde{y}_4^*)$*. Consider the transition map associated with the graphic* Γ

$$\begin{aligned} T:\ \Pi_4 &\longrightarrow \Pi_3 \\ (\nu, \tilde{y}_4) &\mapsto (\nu, T_2(\nu, \tilde{y}_4)). \end{aligned}$$

If $\forall(a,\bar{\mu}) \in A\times V$, $T_2(0,\tilde{y}_4)$ *is nonlinear in the neighborhood of* $\tilde{y}_4^*$, *then its analytic extension in its extension domain* $\mathcal{I}$ *in* $\mathbb{R}$ *is nonlinear at any particular value of* $\tilde{y}_4 \in \mathcal{I}$ *for* $(a,\bar{\mu}) \in A \times V$.

Proof of Theorem 5.5

There are 10 families of convex graphics of saddle type (Table. 3.5). We have proved in Prop. 5.8 that all the upper boundary graphics have finite cyclicity, so we need to prove that in each family all the intermediate and lower boundary graphics have finite cyclicity.

For each family, let Γ be any intermediate graphic and let $T : \Pi_4 \longrightarrow \Pi_3$ be the transition map associated with the graphic Γ in the chart F.R.. Then by Prop. 5.11, to prove the finite cyclicity of Γ, we only need to verify that for $\nu = 0$, the map T or its inverse satisfies one of the three conditions of Prop. 5.11. For the lower boundary graphic, a small adaptation is necessary since T may not be C^k. Thus for each lower boundary graphic, we will study the number of roots for the corresponding displacement map $L = R^{-1} - T$ or $\mathcal{L} : R - T^{-1}$ defined in (5.32) or (5.33). Usually if the criterion that T is nonlinear is used, the starting point is chosen near the lower boundary graphic.

The map R satisfies Prop. 5.10 and R_2 is almost affine. For the transition map T, since $r = 0$ is invariant in the chart F.R., so $T_1(0,\tilde{y}_3) = 0$. We will go over all the 10 families of graphics by considering the second component $T_2(0,\tilde{y}_3)$ or its inverse.

For each family of the graphic Sxhhi ($i = 1, 2, \cdots, 10$), we use $V_i \subset \mathbb{S}^2$ to denote the set of $\bar{\mu}$ in which the family Sxhhi exists.

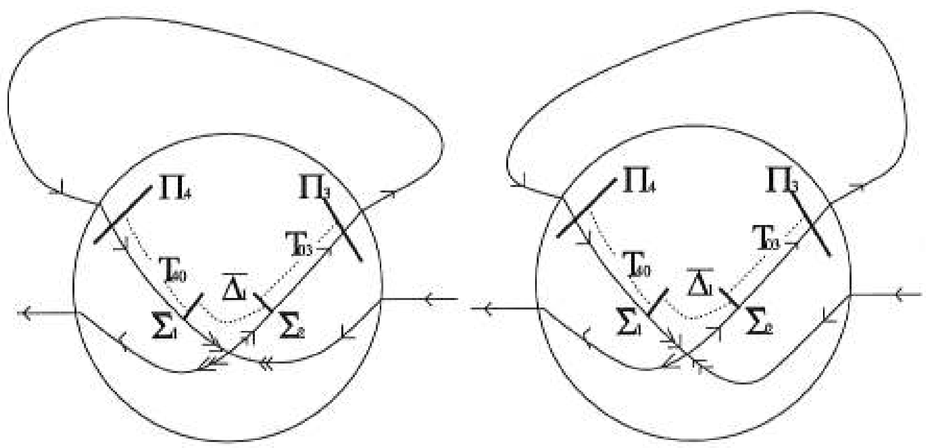
(1). Family Sxhh1

Family Sxhh1 has a lower boundary graphic Sxhh1c which passes through a hyperbolic saddle point in the chart F.R.(Fig. 5.5(a)). Let $\lambda_0(\bar{\mu}_0)$ be the hyperbolicity ratio at this point. Since in the chart F.R., $r = 0$ is invariant, by a linear transformation and C^k normal change of coordinates, we can bring the system in the neighborhood of the saddle point into a normal form.

Take sections $\overline{\Sigma}_1 = \{\tilde{y} = y_0\}$ and $\overline{\Sigma}_2 = \{\tilde{x} = x_0\}$ in the normal form coordinates, let $\overline{\Delta} = (\overline{d}, \overline{D}) : \overline{\Sigma}_1 \longrightarrow \overline{\Sigma}_2$ be the transition map. Then $\overline{d}(0, .) = 0$, and for $r = \nu = 0$, $\overline{D}$ is the Dulac map in the neighborhood of the saddle point. By Prop. 4.2 and Prop. 4.5, for $\nu = 0$, there exists $V_{10} \subset V_1$, a neighborhood of $\bar{\mu}_0$, such that $\forall (a, \bar{\mu}) \in A \times V_{10}$, $\overline{D}$ can be written as

$$\overline{D}(0, \tilde{x}) = \begin{cases} \tilde{x}^{\lambda(\bar{\mu})}(\beta_0 + \overline{\phi}_0(\nu, x)) & \text{if } \lambda_0 \neq 1 \\ \beta_{01}\tilde{x} + \alpha_{01}\tilde{x}\overline{\omega}[1 + \cdots] + \alpha_{02}\tilde{x}^2\overline{\omega}[1 + \cdots] + \cdots & \text{if } \lambda_0 = 1 \end{cases} \tag{5.46}$$

where $\beta_0 > 0$ constant, $\alpha_{01} = \lambda(\bar{\mu}) - \lambda_0$ and $\overline{\omega} = \omega(\tilde{x}, \alpha_{01}(\bar{\mu}))$.



(a) Family Sxhh1 (b) Family Sxhh2

Figure 5.5: Transition map T for the family Sxhh1 and Sxhh2

Let $T_{40} : \Pi_4 \longrightarrow \overline{\Sigma}_1$, $T_{03} : \overline{\Sigma}_2 \longrightarrow \Pi_3$. They are the compositions of normal form coordinate changes and regular transition maps, so $\forall (a, \bar{\mu}) \in A \times V_{10}$, we have

$$\begin{aligned} T_{402}(\nu, \tilde{y}_4) &= m_{400}(\nu) + m_{401}(\nu)\tilde{y}_4 + O(\tilde{y}_4^2) \\ T_{032}(\nu, \tilde{y}) &= m_{030}(\nu) + m_{031}(\nu)\tilde{y} + O(\tilde{y}^2). \end{aligned} \tag{5.47}$$

where $m_{400}(\nu_0) = 0, m_{030}(\nu_0) = 0$, and $m_{401}(0)m_{031}(0) \neq 0$. Then for T, we have

$$T = T_{03} \circ \overline{D} \circ T_{40}. \tag{5.48}$$

We deal with three cases:

(1.1). If $\lambda(\bar{\mu}_0) \neq 1$.

By (5.48), (5.46) and (5.47), a straightforward calculation gives

$$T_2(\nu, \tilde{y}_4) = \tilde{\varepsilon}_1(\nu) + \hat{m}(\nu)(\tilde{\varepsilon}_2(\nu) + \tilde{y}_4)^{\lambda(\bar{\mu})}\Big[1 + \hat{\phi}_0(\nu, \tilde{y}_4)\Big]$$

where $\tilde{\varepsilon}_i(0) = 0$ $(i = 1, 2)$, $\hat{m}(\nu) = m_{401}^{\lambda} m_{031}$. So for the displacement map L defined in (5.32), its second component becomes

$$\begin{aligned} L_2(\nu, \tilde{y}_4) &= \gamma_{340}(\nu, \omega(\tfrac{\nu}{\nu_0}, -\beta_1)) + \Big[\tfrac{1}{\gamma^*} + O(\nu^{\bar{p}_3}\omega^{q_3}(\tfrac{\nu}{\nu_0}, -\beta_1)\ln\tfrac{\nu}{\nu_0})\Big]\tilde{y}_4 + O(\tilde{y}_4^2) \\ &\quad - \Big[\tilde{\varepsilon}_1(\nu) + \hat{m}(\nu)(\tilde{\varepsilon}_2(\nu) + \tilde{y}_4)^{\lambda(\bar{\mu})}\Big(1 + \hat{\phi}_0(\nu, \tilde{y}_4)\Big)\Big]. \end{aligned}$$

Then

$$\begin{aligned} L_2'(\nu, \tilde{y}_4) &= \tfrac{1}{\gamma^*} + O\Big(\nu^{\bar{p}_3}\omega^{q_3}(\tfrac{\nu}{\nu_0}, -\beta_1)\ln\tfrac{\nu}{\nu_0}\Big) + O(\tilde{y}_4) \\ &\quad - \lambda\hat{m}(\nu)(\tilde{\varepsilon}_2(\nu) + \tilde{y}_4)^{\lambda-1}\Big[1 + \hat{\phi}_1(\nu, \tilde{y}_4)\Big]. \end{aligned}$$

So, for $(a, \bar{\mu}) \in A_0 \times \mathrm{V}_{10}$ and for $\nu > 0$ sufficiently small, in both cases $\lambda_0 > 1$ or $\lambda_0 < 1$, $L_2'(\nu, \tilde{y}_4) \neq 0$. By Rolle's theorem, $L(\nu, \tilde{y}_4) = 0$ has at most one small root in the neighborhood of $\tilde{y}_4 = 0$, i.e., $Cycl(Sxhh1c) \leq 1$.

Note that for $\nu = 0$ and $\forall(a, \bar{\mu}) \in A_0 \times V_{10}$, we have

$$T_2(0, \tilde{y}_4) = \hat{m}(\nu)\tilde{y}_4^{\lambda_0} + o(\tilde{y}_4^{\lambda_0})$$

So in both cases $\lambda_0 > 1$ or $\lambda_0 < 1$, $T_2(\nu, \tilde{y}_4)$ is nonlinear in the neighborhood of $\tilde{y}_4 = 0$, by Prop. 5.11, $Cycl(Schh1b)$ is finite.

(1.2). If $\lambda(\bar{\mu}_0) = 1$ and $a \neq -\frac{1}{2}$.

By (5.48), (5.46) and (5.47), for $T_2(0, \tilde{y}_4)$, we have

$$\begin{aligned} T_2(0, \tilde{y}_4) &= m_{410}m_{031}\alpha_{01}\Big[\tilde{y}_4\omega + \cdots\Big] + m_{401}m_{031}\beta_{01}\Big[\tilde{y}_4 + \cdots\Big] \\ &\quad + m_{401}^2 m_{031}\alpha_{02}\Big[\tilde{y}_4^2\omega + \cdots\Big] + O(\tilde{y}_4^2). \end{aligned} \tag{5.49}$$

In this case, we have to calculate the first saddle quantity of the saddle point.

Lemma 5.14. *For the system (3.10), if $a \neq -\frac{1}{2}$ and the saddle point has hyperbolicity ratio equals to 1, then the first saddle quantity equals to*

$$\alpha_{02} = \frac{2a(a-1)(1+2a)^2}{a(4a-1)\bar{\mu}_{30}^2 - (1+2a)^2\bar{\mu}_{20}}\bar{\mu}_{30}. \tag{5.50}$$

Proof. Assume that we have system

$$\begin{cases} \dot{x} &= \quad x + f(x,y) \\ \dot{y} &= -y + g(x,y) \end{cases} \tag{5.51}$$

which has a saddle point with hyperbolicity ratio 1, then by the formula in [JR89], then the first saddle quantity equals to

$$\alpha_{02} = f_{xxy} + g_{xyy} - f_{xx}f_{xy} + g_{xy}g_{yy}. \tag{5.52}$$

Let the saddle point be $(\tilde{x}_0, \tilde{y}_0)$. After translating the singular point to the origin, the system can be written as

$$\begin{cases} \dot{\hat{x}} &= 2a\tilde{x}_0\hat{x} + \hat{y} + 2a\hat{x}^2 \\ \dot{\hat{y}} &= \tilde{y}_0\hat{x} + (\tilde{x}_0 + \bar{\mu}_{30})\hat{y} + \hat{x}\hat{y}. \end{cases} \tag{5.53}$$

Since $\lambda(\bar{\mu}_0) = 1$, which is true if

$$\bar{\mu}_1 = \frac{2a}{1+2a}\bar{\mu}_2\bar{\mu}_3 + \frac{2a^2}{(1+2a)^3}\bar{\mu}^3,$$

then by $(1+2a)\tilde{x}_0 + \bar{\mu}_{30} = 0$, we have

$$\tilde{x}_0 = -\frac{\bar{\mu}_{30}}{1+2a}, \qquad \tilde{y}_0 = -\bar{\mu}_{20} - \frac{a\bar{\mu}_{30}^2}{(1+2a)^2}.$$

After a linear transformation we bring the system at the saddle point to

$$\begin{cases} \dot{u} &= \quad u + f(u,v) \\ \dot{v} &= -v + g(u,v) \end{cases}$$

where

$$\begin{aligned} f(u,v) &= \tfrac{1+2a}{2\Delta_0}\Big[(2a(1-a)\bar{\mu}_{30} + (1+a)\sqrt{\Delta_0})u^2 \\ &\quad +2a((1-a)\bar{\mu}_{30} + \sqrt{\Delta_0})uv + (2a(1-a)\bar{\mu}_{30} + (a-1)\sqrt{\Delta_0})v^2\Big] \\ g(u,v) &= \tfrac{1+2a}{2\Delta_0}\Big[(1-a)(2a\bar{\mu}_{30} + \sqrt{\Delta_0})u^2 \\ &\quad +2a((a-1)\bar{\mu}_{30} + \sqrt{\Delta_0})uv + (2a(a-1)\bar{\mu}_{30} + \sqrt{\Delta_0})v^2\Big] \end{aligned}$$

and $\Delta_0 = a(4a-1)\bar{\mu}_{30}^2 - (1+2a)^2\bar{\mu}_{20} > 0$. So by (5.52) we get the first saddle quantity α_{02} in (5.50). □

End of proof for Sxhh1.

If $\bar{\mu}_{30} = 0$, since $a \neq -\frac{1}{2}$, so $\bar{\mu}_{10} = 0$. Then system (3.10) is symmetric with respect to the $y-$axis, $T_2(0, \tilde{y}_4)$ is the identity. By Prop. 5.11, $Cycl(Sxhh1b) \leq 1$. If $\bar{\mu}_{30} \neq 0$, by (5.50), the first saddle quantity $\alpha_{02} \neq 0$, so $T_2''(0, \tilde{y}_4) \neq 0$, thus T_2 is nonlinear in $\tilde{y}_4$. By Prop. 5.11, we have $Cycl(Sxhh1b)$ is finite.

Now we deal with the lower boundary graphic Sxhh1c. By (5.48), (5.46) and (5.47), we have

$$L_2(\nu, \tilde{y}_4) = \tilde{\varepsilon}_1(\nu) + \hat{\alpha}_{01}[\tilde{y}_4\varpi + \cdots] + \hat{\beta}_{01}\tilde{y}_4 + \hat{\alpha}_{02}[\tilde{y}_4^2\varpi + \cdots] + O(\tilde{y}_4^2) \qquad (5.54)$$

where $\tilde{\varepsilon}_i(0) = 0 (i = 1, 2)$ and $\hat{\alpha}_{02} = *\alpha_{02} \neq 0$.

Similar to the proof in [R86] for the 3−codimension case, by the standard derivation-division method, we can prove that $\forall (a, \bar{\mu}) \in A_0 \times V_{10}$ and $\nu > 0$ sufficiently small, $L_2 = 0$ has at most 3 roots. Thus we get $Cycl(Sxhh1c) \leq 3$.

For the case $\bar{\mu}_{30} = 0$, system (5.53) is symmetric with respect to the $y-$axis, so $m_{401}(0)m_{031}(0) =$ and $\beta_0 = 1$. Thus (5.54) can be further simplified to

$$L_2(\nu, \tilde{y}_4) = \tilde{\varepsilon}_1(\nu) + \hat{\alpha}_{01}[\tilde{y}_4\varpi + \cdots] + \hat{\beta}_{01}\tilde{y}_4 + O(\tilde{y}_4^2\varpi)$$

where $\hat{\beta}_{01}(0) = (\gamma^* - 1) \neq 0$. The derivation-division method ensures that $\forall (a, \bar{\mu}) \in A_0 \times V_{10}$, and for $\nu > 0$ sufficiently small, $L = 0$ has at most two roots which gives $Cycl(Sxhh1c) \leq 2$.

(2). Families Sxhh2 and Sxhh3

For the family Sxhh2, system (3.10) has a semi-hyperbolic saddle on Sxhh2c (Fig. 5.5(b)). Consider the map $\overline{\Delta} = (\overline{d}, \overline{D}) : \overline{\Sigma}_1 \longrightarrow \overline{\Sigma}_2$. In this case for $\nu = 0$, $\overline{D}$ is the stable center transition near the semi-hyperbolic saddle, then by [DRR94],

$\forall i_1, i_2 \in \mathbb{N}$, $\forall (a, \bar{\mu}) \in A_0 \times V_{20}$ and $\nu > 0$ sufficiently small, we have

$$\frac{\partial^{i_1} \overline{D}}{\partial \hat{x}^{i_1}}(\nu, \hat{x}) = O(\hat{x}^{i_2}). \tag{5.55}$$

So for T_2, by (5.48) and (5.55)we have

$$T_2'(0, \tilde{y}_4) \to 0$$

which gives $Cycl(Sxhh2c) \leq 1$ and the nonlinearity of T_2, hence the finite cyclicity of $Cycl(Sxhh2b)$.

By changing $(x, t) \mapsto (-x, -t)$, similar to family Sxhh2, the result hold for the family Sxhh3.

(3). Families Sxhh4 and Sxhh6

For the family Sxhh4, Sxhh5 and Sxhh6, the corresponding lower boundary graphic has a saddle connection $\mathcal{S}_1\mathcal{S}_2$ (Fig. 5.6). At $\mathcal{S}_1$ and $\mathcal{S}_2$, the hyperbolicity ratios are

$$\mathcal{S}_1: \ \lambda_1 = \frac{\bar{\mu}_3 - \sqrt{-\frac{\bar{\mu}_2}{a}}}{2a\sqrt{-\frac{\bar{\mu}_2}{a}}}; \qquad \mathcal{S}_2: \ \lambda_2 = \frac{-2a\sqrt{-\frac{\bar{\mu}_2}{a}}}{\bar{\mu}_3 + \sqrt{-\frac{\bar{\mu}_2}{a}}}.$$

Note that if $\lambda_1\lambda_2 = 1$, then $\bar{\mu}_3 = 0$, this is just the case of family Sxhh5. So we first consider the case $\lambda_1\lambda_2 \neq 1$ when we have families Sxhh4 and Sxhh6. Easy calculations show that the family Sxhh4 exists if and only if

$$V_4 = \{\bar{\mu} \in \mathbb{S}^2 \big| \bar{\mu}_1 = 0, \bar{\mu}_3 > 0, \bar{\mu}_2 > a(a-1)\bar{\mu}_3^2\}.$$

Since system (3.10) in invariant under the transformation

$$(-x, -t, -\bar{\mu}_3) \mapsto (x, \ t, \ \bar{\mu}_3) \tag{5.56}$$

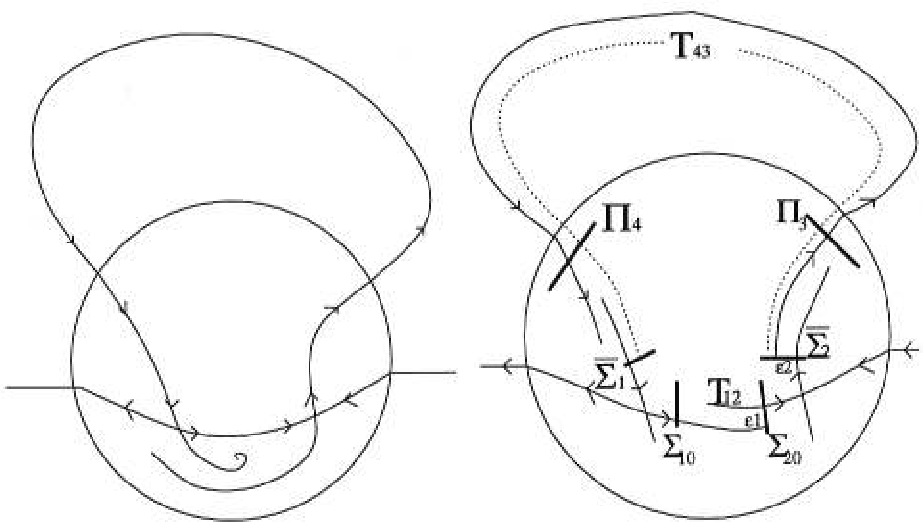
so we only need to study the family Sxhh4 as long as we do not use $\gamma^* \leq 1$. For $\bar{\mu} \in V_4$, we have $\lambda_1\lambda_2 < 1$ and $\lambda_2 < 1$. Let $\bar{\mu}_0 \in V_4$, for the family Sxhh4, we have to consider two cases:

(3.1). $\lambda_1\lambda_2 < 1$, $\lambda_1 \neq 1$ and $\lambda_2 < 1$.

We first consider the lower boundary grahic Sxhh4c. In the chart F.R., in the neighborhood of $\mathcal{S}_1$ and $\mathcal{S}_2$, take sections $\overline{\Sigma}_i = \{y_i = 1\}$ $(i = 1, 2)$, $\overline{\Sigma}_{10} = \{x_1 = 1\}$ and $\overline{\Sigma}_{20} = \{x_2 = -1\}$, and for $i = 1, 2$, let

$$\overline{\Delta}_i = (\overline{d}_i, \overline{D}_i) : \overline{\Sigma}_i \longrightarrow \overline{\Sigma}_{i0}. \tag{5.57}$$

be the transition map in the neighborhood of the saddle $\mathcal{S}_i$. Then, $\overline{d}_i(0, .) = 0$ and for $\nu = 0$, $\overline{D}_1$ and $\overline{D}_2$ are two Dulac maps in the neighborhood of $\mathcal{S}_1$ and $\mathcal{S}_2$ on $r = 0$.



(a) Family Sxhh4 (b) Transition maps

Figure 5.6: Transition maps for the family Sxhh4

Instead of considering the displacement map L, we consider the displacement map

$$\overline{L} : \overline{\Sigma}_1 \longrightarrow \overline{\Sigma}_{20}$$
$$\overline{L} = \overline{T}_{12} \circ \overline{\Delta}_1 - \overline{\Delta}_2^{-1} \circ \overline{T}_{43}$$

where $\overline{T}_{12}$ is a regular transition map and $\overline{T}_{43} = \overline{T}_{32} \circ R \circ \overline{T}_{14}$ is the composition of two regular transition maps and a C^k "almost affine" map R. By [R86] and

[Mou90], there exist V_{40}, a neighborhood of $\bar{\mu}_0$ such that $\forall(a,\bar{\mu}) \in A \times V_{40}$, there exist C^k changes of parameters on each section such that for the components of $\overline{L}$, we have

$$\begin{aligned} &\overline{T}_{122}(\nu,\tilde{y}_1) = \tilde{y}_1 + \tilde{\varepsilon}_1(\nu), \\ &\overline{T}_{432}(\nu,\tilde{x}_1) = \tilde{x}_1 + \tilde{\varepsilon}_2(\nu), \\ &\overline{D}_1(\nu,\tilde{x}_1) = \tilde{x}_1^{\lambda_1}(m_1 + \overline{\phi}_1(\nu,\tilde{x}_1)), \\ &\overline{D}_2^{-1}(\nu,\tilde{x}_2) = \tilde{x}_2^{\lambda_3}(m_2 + \overline{\phi}_1(\nu,\tilde{x}_2)) \end{aligned} \tag{5.58}$$

where $\lambda_3 = \frac{1}{\lambda_2} > 1$ for $\bar{\mu} \in V_4$, and $\tilde{\varepsilon}_i(0) = 0$, $m_i > 0$, $\overline{\phi}_i \in (I_0^\infty)$ $(i = 1, 2)$. Let $x = \tilde{x}_1$, $X = \tilde{x}_1 + \tilde{\varepsilon}_2$. Then for the map $\overline{L}$, we have

$$\overline{L}_2(\nu,x) = x^{\lambda_1}(m_1 + \overline{\phi}_1(\nu,x)) + \tilde{\varepsilon}_1(\nu) - X^{\lambda_3}(m_2 + \overline{\phi}_1(\nu,X)).$$

So

$$\overline{L}_2'(\nu,x) = \lambda_1 x^{\lambda_1 - 1}(m_1 + \hat{\phi}_1(\nu,x)) - \lambda_3 X^{\lambda_3 - 1}(m_2 + \hat{\phi}_1(\nu,X))$$

where $\hat{\phi}_1, \hat{\phi}_2 \in (I_0^\infty)$.

Zeroes of $\overline{L}_2'(\nu,x)$ are the same as zeroes of:

$$\overline{L}_3'(\nu,x) = x(\hat{m}_1 + \hat{\psi}_1(\nu,x)) - \left(\frac{\lambda_3}{\lambda_1}\right)^{\frac{1}{\lambda_1 - 1}} X^{\frac{\lambda_3 - 1}{\lambda_1 - 1}}(\hat{m}_2 + \hat{\psi}_1(\nu,X))$$

where $\hat{m}_1, \hat{m}_2 > 0$ and $\hat{\psi}_1, \hat{\psi}_2 \in (I_0^\infty)$.

Then

$$\overline{L}_3''(\nu,x) = \hat{m}_1 + o(1) - \left[\left(\frac{\lambda_3}{\lambda_1}\right)^{\frac{1}{\lambda_1 - 1}} \frac{\lambda_3 - 1}{\lambda_1 - 1} X^{\frac{\lambda_3 - \lambda_1}{\lambda_1 - 1}}(\hat{m}_3 + \hat{\psi}_3(\nu,X))\right]. \tag{5.59}$$

Since $\lambda_1\lambda_2 < 1$, hence $\lambda_1 < \lambda_3$ thus for $(a,\bar{\mu}) \in A_0 \times \bar{\mu} \in V_{40}$, and for $\nu > 0$ sufficiently samll, $\overline{L}_3''(\nu,x) \neq 0$, yielding a maximum of two zeroes of $\overline{L}$, hence $Cycl(Sxhh4c) \leq 2$.

For the intermediate graphics Sxhh4b near Sxhh4c, as shown in Fig. 5.6, the transition map T defined in Prop. 5.11 associated with the graphic Sxhh4c is the composition of three regular transitions and two Dulac maps. By (5.58), a straightforward calculation gives

$$T_2(0, \tilde{y}_4) = m(0)\tilde{y}_4^{\lambda_1\lambda_2} + o(\tilde{y}_4^{\lambda_1\lambda_2})$$

where $m(0) \neq 0$. So for $\nu = 0$ and $\forall(a, \bar{\mu}) \in A_0 \times V_{40}$, T_2 is nonlinear. Thus by Prop. 5.11 the graphic Sxhh4b has finite cyclicity.

(3.2). $\lambda_1 = 1$, $\lambda_2 < 1$.

Let $\bar{\mu}_{30} \in V_4$, then by $\lambda_1(\bar{\mu}_0) = 1$ we have $\bar{\mu}_{20} = -\frac{a}{(1+2a)^2}\bar{\mu}_{30}^2, \bar{\mu}_{30} > 0$. Therefore the hyperbolic ratio at $\mathcal{S}_2$ is $\lambda_2(\bar{\mu}_0) = -\frac{a}{1+a} \in (0,1)$ (this case can happen only when $a \in (-\frac{1}{2}, 0)$).

This case is in fact similar to the degenerate and non-trivial hyperbolic polycycle with two vertices considered in [Mou94]. As shown in Fig. 5.6, we have to consider the cases $\tilde{\varepsilon}_1 > 0$, $\tilde{\varepsilon}_2 < 0$ and $\tilde{\varepsilon}_1 < 0$, $\tilde{\varepsilon}_2 > 0$ separately. For the other two cases, $Cycl(Sxhh4c) \leq 1$. Here we only give a brief proof for the case $\tilde{\varepsilon}_1 > 0$, $\tilde{\varepsilon}_2 < 0$, details see [Mou94].

Consider the displacement map

$$\widetilde{L} : \overline{\Sigma}_{10} \longrightarrow \overline{\Sigma}_2$$
$$\widetilde{L} = \overline{T}_{43} \circ \overline{\Delta}_1^{-1} - \overline{\Delta}_2 \overline{T}_{12}$$

By using the asymptotic expansion of [R86] for the composition of the Dulac map near $\mathcal{S}_1$ and a regular transition map, for $\widetilde{L}_2(\nu, \tilde{y}_1)$, we have

$$\widetilde{L}_2(\nu, y) = \tilde{\varepsilon}_2 + \alpha_{01}[y\overline{\omega} + \cdots] + +\alpha_{02}[y^2\overline{\omega} + \cdots] + \cdots - Y^{\lambda_2}(m_2 + \overline{\phi}_2(\nu, Y))$$

where $y = \tilde{y}_1$ and $Y = \tilde{y}_1 + \tilde{\varepsilon}_1$; $\alpha_{02}(0) > 0$.

Let $y = t\tilde{\varepsilon}_1$. The study for y small corresponds to $t \in (0, t_0)$. We will prove that there exists t_0 such that for $\bar{\mu} \in V_{40}$ and $\nu > 0$ sufficiently small, $\widetilde{L}_2(\nu, y)$ has at most 2 zeroes $t \in (0, t_0)$. Indeed, as a function of t, $\widetilde{L}_2(\nu, y) = 0$ is equivalent to

$$\begin{aligned} 0 = \ & \tilde{\varepsilon}_2\tilde{\varepsilon}_1^{-\lambda_2} - m_2 + \alpha_{01}\tilde{\varepsilon}_1^{1-\alpha_{01}-\lambda_2}t\overline{\omega}(t)[1 + \cdots] \\ & +(\alpha_{02}\tilde{\varepsilon}_1^{1-\lambda_2} - \lambda_2 m_2 + \alpha_{01}\tilde{\varepsilon}_2^{1-\lambda_2}\overline{\omega}(\tilde{\varepsilon}_1))t[1 + \cdots] + O(t^2) \end{aligned}$$

which has the same form as the expressions considered in [R86]. Note that $m_2\lambda_2 > 0$, so we have the non-degeneracy condition

$$\alpha_{02}\tilde{\varepsilon}_1^{1-\lambda_2} - \lambda_2 m_2 + \alpha_{01}\tilde{\varepsilon}_2^{1-\lambda_2}\overline{\omega}(\tilde{\varepsilon}_1) \neq 0$$

for $\bar{\mu} \in V_{40}$ and $\nu > 0$ sufficiently small.

For the larger zeroes, note that

$$\widetilde{L}_2'(\nu, y) = y^{\lambda_2 - 1}\Big[- \lambda_2(1 + \frac{1}{t})^{\lambda_2 - 1}(m_2 + o(1)) + O(y)\Big]$$

does not vanish for $t > t_0$ and for $\bar{\mu} \in V_{40}$, $\nu > 0$ sufficiently small. Hence $\widetilde{L}_2(\nu, y)$ has at most one zero for $t > t_0$. Therefore we have $Cycl(Sxhh4c) \leq 3$.

For the intermediate graphic Sxhh4b, the corresponding transition map T is the composition of the above two Dulac maps and three regular transition maps, by (5.58), we have that for $\nu = 0$, $T_2(0, \tilde{x}_4)$ is nonlinear hence by Prop. 5.11 we have $Cycl(Sxhh4b)$ is finite.

(4). Family Sxhh5

First note that, family Sxhh5 exists if and only if $\bar{\mu} = (0, 1, 0)$. Then system (3.10) in the chart F.R. is symmetric with a center (Fig. 5.7). For the intermediate graphics, easily we see that for $\nu = 0$, the transition map T is the identity, by Prop. 5.11, graphic Sxhh5b has finite cyclicity.

Now we consider the graphic Sxhh5c. The hyperbolicity ratio λ_1 and λ_2 satisfy $\lambda_1 = -\frac{1}{2a} > 1$, $\lambda_2 = -2a < 1$ and $\lambda_1\lambda_2 = 1$.

By Prop. 4.2, the Dulac maps defined in the neighborhood of the two saddles can be written as

$$\begin{aligned} \overline{D}_1(\nu, x_1) &= x_1^{\lambda_1}\Big(1 + \overline{\phi}_1(\nu, x_1)\Big) \\ \overline{D}_2^{-1}(\nu, x_2) &= x_2^{\frac{1}{\lambda_2}}\Big(1 + \overline{\phi}_2(\nu, x_2)\Big) \end{aligned} \tag{5.60}$$

where $\overline{\phi}_i(\nu, x_i)$ $(i = 1, 2)$ satisfies (I_0^∞) for $(a, \bar{\mu}) \in A_0 \times V_{51}$, $\nu \in (0, \nu_1)$ and x_i sufficiently small.

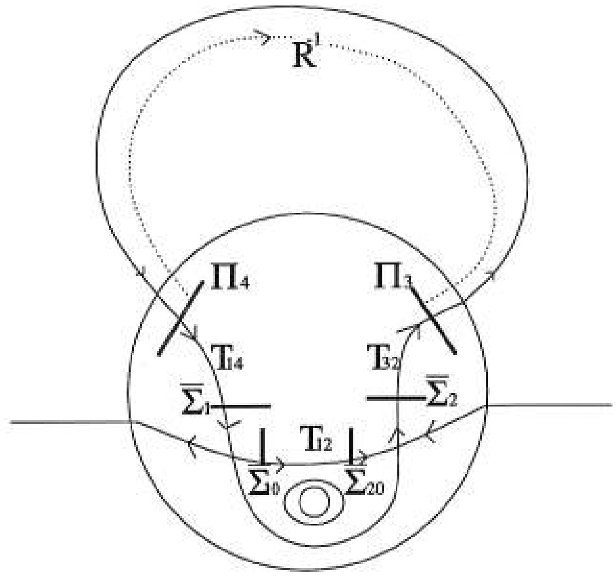


Figure 5.7: Transition map T for the family Sxhh5

Consider the displacement map

$$
\begin{aligned}
L: \ & \overline{\Sigma}_1 \longrightarrow \overline{\Sigma}_{20} \\
& L = \overline{T}_{12} \circ \overline{\Delta}_1 - \overline{\Delta}_2^{-1} \circ \overline{T}_{32} \circ R^{-1} \circ \overline{T}_{14}.
\end{aligned}
\tag{5.61}
$$

where

- $\overline{T}_{12} : \overline{\Sigma}_{10} \longrightarrow \overline{\Sigma}_{20}$,

$$\overline{T}_{122}(\nu, y_1) = m_{120}(\nu) + (1 + O(\nu))y_1 + O(y_1^2).$$

- $\overline{T}_{14} : \overline{\Sigma}_1 \longrightarrow \Pi_4$,

$$\overline{T}_{14}(\nu, x_1) = m_{140}(\nu) + m_{141}(\nu)x_1 + O(x_1^2).$$

- $R^{-1} : \Pi_4 \longrightarrow \Pi_3$, R satisfies Prop. 5.10, and

$$R_2^{-1}(\nu, \tilde{y}_4) = m_{340}(\nu) + (\frac{1}{\gamma^*} + O(\nu))\tilde{y}_4 + O(\tilde{y}_4^2)$$

- $\overline{T}_{32} : \Pi_3 \longrightarrow \overline{\Sigma}_2$,

$$\overline{T}_{322}(\nu, \tilde{y}_3) = m_{320}(\nu) + m_{321}(\nu)\tilde{y}_3 + O(\tilde{y}_3^2).$$

where $m_{321}(0)m_{141}(0) = 1$ because of the symmetry of the system (3.10).

Then a straightforward calculation gives

$$\begin{aligned} L_2(\nu, x_1) &= \overline{\varepsilon}_1(\nu) + x_1^{\lambda_1}(1 + \overline{\phi}_{11}(\nu, x_1)) \\ &\quad -[\overline{\varepsilon}_2(\nu) + \overline{\gamma}^*(\nu)x_1 + O(x_1^2)]^{\frac{1}{\lambda_2}}(1 + \overline{\phi}_{21}(\nu, x_1)) \end{aligned} \tag{5.62}$$

where

$$\begin{aligned} \overline{\varepsilon}_1(\nu) &= m_{120}(\nu) \\ \overline{\varepsilon}_2(\nu) &= m_{320}(\nu) + m_{340}(\nu) + \frac{1}{\gamma^*}m_{140}(\nu) \\ \overline{\gamma}^*(\nu) &= \frac{m_{321}(\nu)m_{141}(\nu)}{\gamma^*} + O(\nu) \end{aligned}$$

with $\overline{\phi}_{11}, \overline{\phi}_{21} \in (I_0^\infty)$. Also $\overline{\gamma}^*(0) = \frac{1}{\gamma^*}$.

By (5.62),

$$\begin{aligned} L_2'(\nu, x_1) &= \lambda_1 x_1^{\lambda_1 - 1}(1 + \overline{\phi}_{12}(\nu, x_1)) \\ &\quad -\frac{1}{\lambda_2}\Big[\overline{\varepsilon}_2(\nu) + \overline{\gamma}^*(\nu)x_1 + O(x_1^2)\Big]^{\frac{1}{\lambda_2}-1}(\overline{\gamma}^*(\nu) + \overline{\phi}_{22}(\nu, x_1)). \end{aligned}$$

where $\overline{\phi}_{12}, \overline{\phi}_{22} \in (I_0^\infty)$.

$L_2'(\nu, x_1)$ has the same number of small roots $x_1 > 0$ as

$$\begin{aligned} L_{21}(\nu, x_1) &= (\lambda_1\lambda_2)^{\frac{\lambda_2}{1-\lambda_2}} x_1^{\frac{(\lambda_1-1)\lambda_2}{1-\lambda_2}}(1 + \overline{\phi}_{13}(\nu, x_1)) \\ &\quad -\Big[\overline{\varepsilon}_2(\nu) - \overline{\gamma}^*(\nu)x_1 + O(x_1^2)\Big](\overline{\gamma}^{*\frac{\lambda_2}{1-\lambda_2}}(\nu) + \overline{\phi}_{23}(\nu, x_1)). \end{aligned} \tag{5.63}$$

Let $\bar{\beta}_1 = \frac{1-\lambda_1\lambda_2}{1-\lambda_2}$. For the term $x_1^{\frac{(\lambda_1-1)\lambda_2}{1-\lambda_2}}$, we make the following development

$$x_1^{\frac{(\lambda_1-1)\lambda_2}{1-\lambda_2}} = x_1^{1-\bar{\beta}_1} = x_1(1 + \bar{\beta}_1\overline{\omega})$$

where $\overline{\omega} = \omega(x_1, \bar{\beta}_1)$.

By (5.63), we then have

$$\begin{aligned} L_{21}'(\nu, x_1) &= (\lambda_1\lambda_2)^{\frac{\lambda_2}{1-\lambda_2}}(1 - \bar{\beta}_1 + \bar{\beta}_1(1 - \bar{\beta}_1)\bar{\omega})(1 + \overline{\phi}_{14}(\nu, x_1)) \\ &\quad -[\overline{\gamma}^{*1+\frac{\lambda_2}{1-\lambda_2}} + O(x_1)](1 + \overline{\phi}_{24}(\nu, x_1)) \end{aligned} \tag{5.64}$$

which has the same number of zeroes as

$$\begin{aligned} L_{22}(\nu, x_1) &= (1-\bar\beta_1+\bar\beta_1(1-\bar\beta_1)\bar\omega) \\ &\quad -[\overline{\gamma}^{*\frac{1}{1-\lambda_2}}+O(x_1)]\frac{(1+\overline{\phi}_{24}(\nu,x_1))}{(\lambda_1\lambda_2)^{\frac{\lambda_2}{1-\lambda_2}}(1+\overline{\phi}_{14}(\nu,x_1))} \\ &= (1-\bar\beta_1+\bar\beta_1(1-\bar\beta_1)\bar\omega)-[\hat{\gamma}^*(\nu)+O(\nu)+O(x_1)](1+\overline{\phi}_{15}(\nu,x_1)) \\ &= 1-\bar\beta_1-\hat{\gamma}^*(\nu)+O(\nu)+\bar\beta_1(1-\bar\beta_1)\bar\omega+O(x_1) \end{aligned} \tag{5.65}$$

where $\hat{\gamma}^*(\nu)=\frac{\overline{\gamma}^{*\frac{1}{1-\lambda_2}}}{(\lambda_1\lambda_2)^{\frac{\lambda_2}{1-\lambda_2}}}$ and $\hat{\gamma}^*(0)=\gamma^*$.

Let $L_{23}=\frac{L_{22}}{\bar\omega}$, then

$$L_{23}=\frac{1-\bar\beta_1-\hat{\gamma}^*(\nu)+O(\nu)}{\overline{\omega}}+\bar\beta-1(1-\bar\beta_1)-\frac{O(x_1)}{\bar\omega}.$$

Derivating L_{23} and letting $L_{24}=\overline{\omega}^2x_1^{1+\bar\beta_1}L_{23}'$, then

$$L_{24}=[-1+\bar\beta_1+\hat{\gamma}^*(\nu)+O(\nu)]+O(x_1).$$

Since $\hat{\gamma}^*(0)=\gamma^*\neq 1$, then $\forall(a,\bar\mu)\in A_0\times V_{51}$ and for $\nu>0$ sufficiently small, we have that L_{24} does not vanish. So $L=0$ has at most 3 roots which gives Cycl(Sxhh5c)≤ 3.

(5). Families Sxhh7 and Sxhh8

As shown in Fig. 5.8(a), there is a saddle point and an attracting saddle node on the lower boundary graphic Sxhh7c. In this case

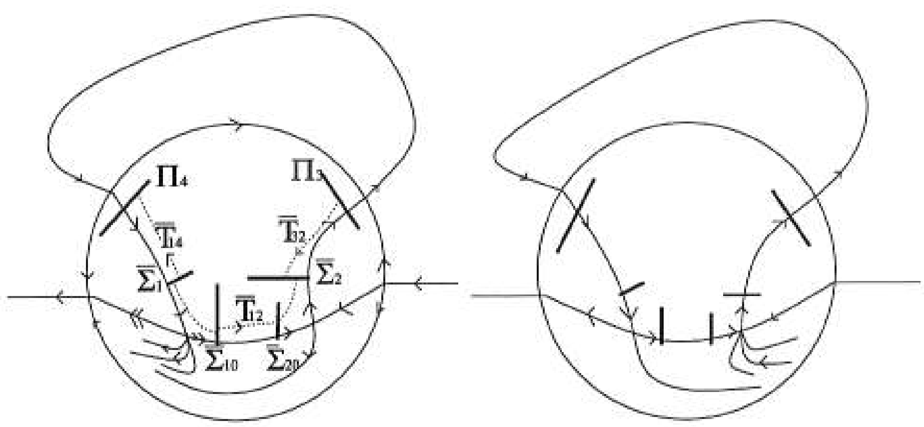
$$V_5=\{\bar\mu\in\mathbb{S}^2\,\Big|\,\bar\mu_1=0,\bar\mu_3=\sqrt{-\frac{\bar\mu_2}{a}},\bar\mu_3>0\}.$$

The hyperbolicity ratio at $\mathcal{S}_2$ becomes $\lambda_2(\bar\mu)=-a<1$.

We first consider the lower boundary graphic Sxhh7c, as shown in Fig. 5.8(a). We study the displacement map

$$\begin{aligned} &\overline{L}:\overline{\Sigma}_{10}\longrightarrow\overline{\Sigma}_2 \\ &\overline{L}=\overline{\Delta}_2\circ\overline{T}_{12}-\widetilde{T}_{12} \end{aligned}$$

where



(a) Family Sxhh7 (b) Family Sxhh8

Figure 5.8: Transition map T for families Sxhh7 and Sxhh8

- $\overline{T}_{12} : \overline{\Sigma}_{10} \longrightarrow \overline{\Sigma}_{20}$ is regular transition map,

$$\overline{T}_{122}(\nu, y_1) = \tilde{\varepsilon}_1 + m_1 y_1 + O(y_1^2) \tag{5.66}$$

where $m_1 \neq 0$.

- $\overline{\Delta}_2 : \overline{\Sigma}_{20} \longrightarrow \overline{\Sigma}_2$, the Dulac map in the neighborhood of $\mathcal{S}_2$ which can be written in the form of (5.46) with $\lambda = \lambda_2 < 1$.

- $\widetilde{T}_{12}$ is the transition map which can be decomposed as

$$\widetilde{T}_{12} = \overline{T}_{32} \circ R^{-1} \circ \overline{T}_{14} \circ \overline{\Delta}_1$$

where $\overline{\Delta}_1 : \overline{\Sigma}_{10} \longrightarrow \overline{\Sigma}_1$ is the transition map with inverse time near the saddle node in the normal form coordinates and it is flat; $\overline{T}_{14} : \overline{\Sigma}_1 \longrightarrow \Pi_4$ and $\overline{T}_{32} : \Pi_3 \longrightarrow \overline{\Sigma}_2$ are regular transition maps and R is almost affine which satisfies Prop. 5.10. All together we have

$$\widetilde{T}_{122}(\nu, y_1) = \tilde{\varepsilon}_2(\nu) + \phi(\nu, y_1) \tag{5.67}$$

with $\phi(\nu, \phi)$ is C^k and flat in y_1.

By (5.66), (5.46) and (5.67), for the second component of the displacement map $\overline{L}$, if we let $Y = \tilde{\varepsilon}_1 + m_1 y_1 + O(y_1^2)$, then we have

$$\overline{L}_2(\nu, y_1) = \tilde{\varepsilon}_2(\nu) + \phi(\nu, y_1) - Y^{\lambda_2}(\beta_0 + \overline{\phi}_0(\nu, Y)).$$

where $\beta_0 \neq 0$ and $\overline{\phi} \in (I_0^\infty)$.

A first derivation of $\overline{L}_2(\nu, y_1)$ gives

$$\overline{L}_2'(\nu, y_1) = \phi'(\nu, y_1) - \lambda_2 m_1 Y^{\lambda_2 - 1}(\beta_0 + \overline{\phi}_1(\nu, Y))$$

which has the same number of small roots $y_1 > 0$ as

$$\overline{L}_{21}(\nu, y_1) = \phi'(\nu, y_1) Y^{1-\lambda_2} - \lambda_2 m_1(\beta_0 + \overline{\phi}_2(\nu, Y).$$

Since $\lambda_2 m_1 \beta_0 \neq 0$ and $\phi(\nu, y_1)$ is flat, so $\overline{L}_{21}(\nu, y_1) \neq 0$ for $(a, \bar{\mu}) \in A_0 \times V_{50}$ and for $\nu > 0$ sufficiently small. Therefore $Cycl(Sxhh7c) \leq 1$.

For the intermediate graphic Sxhh7b, we consider the inverse of T, T^{-1} : $\Pi_3 \longrightarrow \Pi_4$, and decompose it as

$$T^{-1} = \overline{T}_{14} \circ \overline{D}_1 \circ \overline{T}_{21} \circ \overline{D}_2 \circ \overline{T}_{32}$$

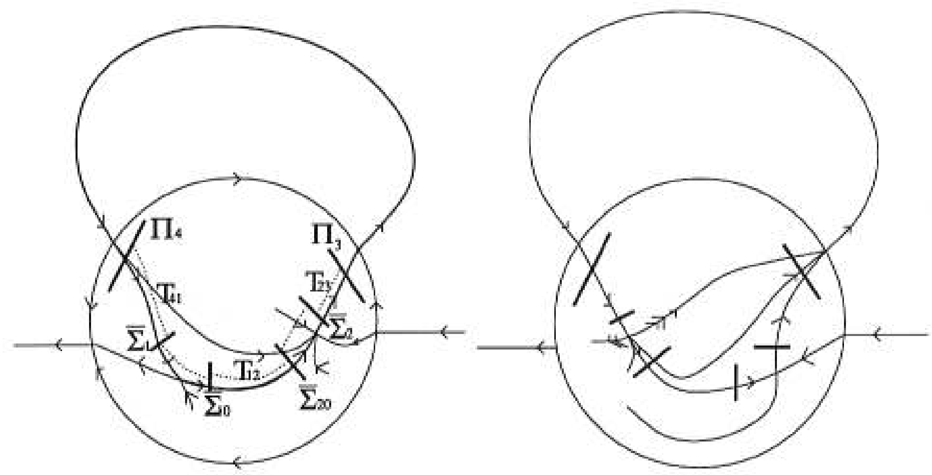
It follows from (5.66), (5.46), (5.67) that we have $\frac{\partial}{\partial \tilde{y}_3} T_2^{-1}(\tilde{y}_3, 0) \to 0$. Again by Prop. 5.11, $Cycl(Sxhh7b)$ is finite.

The finite cyclicity of family Sxhh8 follows from the invariance of system (3.10) under the transformation (5.56) and the finite cyclicity of family Sxhh7.

(6). Families Sxhh9 and Sxhh10

As shown in Fig. 5.9(a), the family Sxhh9 has two subfamilies of graphics: intermediate graphics Sxhh9b and Sxhh9d; two boundary graphics Sxhh9c and Sxhh9e.

First note that the graphic Sxhh9c that passes by an attracting saddle node has the same structure as of the graphic Sxhh2c, so we only need to consider the lower boundary graphic Sxhh9e. As in Fig. 5.9(a), let the hyperbolicity ratio of the saddle point be λ_1. Then for graphic Sxhh9e, we consider two cases:



(a) Family Sxhh9 (b) Family Sxhh10

Figure 5.9: Transition map T for the families Sxhh9 and Sxhh10

(6.1). $\lambda_1 \neq 1$

For graphic Sxhh9e, the corresponding transition map can be factorized as

$$T = \overline{T}_{23} \circ \overline{\Delta}_2 \circ \overline{T}_{12} \circ \overline{\Delta}_1 \circ \overline{T}_{41}. \tag{5.68}$$

where $\overline{\Delta}_1 : \overline{\Sigma}_1 \longrightarrow \overline{\Sigma}_{10}$ is the transition map in the neighborhood of the saddle point, its second component has the form given in (5.46) with $\lambda = \lambda_1$. The map $\overline{\Delta}_2 : \overline{\Sigma}_{20} \longrightarrow \overline{\Sigma}_2$ is the central transition map in normal form coordinates in the neighborhood of the attracting saddle node, its second component satisfies $\overline{D}_2(\nu, y_2) = m_0 y_2$ with $m_0 \to 0$. $\overline{T}_{12}$ and $\overline{T}_{23}$ and $\overline{T}_{41}$ are regular transition maps. A straightforward calculation gives that

$$T_2(\nu, \tilde{y}_4) = \tilde{\varepsilon}_1(\nu) + m_0 Y^{\lambda_1}[1 + \overline{\phi}_1(\nu, Y)] \tag{5.69}$$

where $Y = \tilde{\varepsilon}_2(\nu) + m_3 \tilde{y}_4 + O(\tilde{y}_4^2)$ with $m_3 \neq 0$, m_0 sufficiently small and $\tilde{\varepsilon}_1(0) = \tilde{\varepsilon}_2(0) = 0$, $\overline{\phi}_1 \in (I_0^\infty)$.

Then for the displacement map L defined in (5.32), we have

$$L_2(\nu, \tilde{y}_4) = \tilde{\varepsilon}_1(\nu) + m_0 Y^{\lambda_1}[1 + \overline{\phi}_1(\nu, Y)] - [\gamma_0(\nu) + \gamma_1(\nu)\tilde{y}_4 + O(\tilde{y}_4^2)]$$

where $\gamma_1(0) \neq 0$.

A first derivation gives

$$L_2'(\nu, \tilde{y}_4) = \lambda_1 m_0 Y^{\lambda_1 - 1}[1 + \overline{\phi}_{11}(\nu, Y)] - [\gamma_1(\nu) + O(\tilde{y}_4)].$$

If $\lambda_1 > 1$, $L_2'(\nu, \tilde{y}_4) \neq 0$ which gives $Cycl(Sxhh9e) \leq 1$. For the case $\lambda_1 < 1$, $L_2'(\nu, \tilde{y}_4)$ has the same number of small roots $\tilde{y}_4 > 0$ as

$$L_{21}(\nu, \tilde{y}_4) = \lambda_1 m_0 - \frac{[\gamma_1(\nu) + O(\tilde{y}_4)]Y^{1-\lambda_1}}{1 + \overline{\phi}_{11}(\nu, Y)}.$$

Since

$$L_{21}'(\nu, \tilde{y}_4) = -\frac{[\tilde{\varepsilon}_2(\nu) + m_3\gamma_1(\nu)) + O(\tilde{y}_4)](1 + \overline{\phi}_{11}(\nu, Y)) - (\gamma_1 + O(\tilde{y}_4))Y\frac{\partial \overline{\phi}_{11}}{\partial Y}}{Y^{\lambda_1}(1 + \overline{\phi}_{11}(\nu, Y))^2}.$$

where $\overline{\phi}_{11}, Y\frac{\partial \overline{\phi}_{11}}{\partial Y} \in (I_0^\infty)$, so with $\gamma_1(0) \neq 0$ we have $L_{21}'(\nu, \tilde{y}_4) \neq 0$ which gives $Cycl(Sxhh9e) \leq 2$.

For the intermediate graphics Sxhh9d, note that T_2 can be written as

$$T_2(0, \tilde{y}_4) = *m_0\tilde{y}_4^{\lambda_1} + o(\tilde{y}_4^{\lambda_1})$$

which is nonlinear, hence by Prop. 5.11, $Cycl(Sxhh9d)$ is finite.

(6.2). $\lambda_1 = 1$

In this case, we consider the displacement map

$$\begin{aligned} &\overline{L} : \overline{\Sigma}_1 \longrightarrow \overline{\Sigma}_2 \\ &\overline{L} = \overline{\Delta}_2 \circ \overline{T}_{12} \circ \overline{\Delta}_1 - \overline{T}_{32} \circ R^{-1} \circ \overline{T}_{14}. \end{aligned}$$

Note that the hyperbolicity ratio $\lambda_1 = 1$, so by using the asymptotic expansion in [R86] for the map $\overline{\Delta}_2 \circ \overline{T}_{12} \circ \overline{\Delta}_1$, we have

$$\begin{aligned} \overline{L}_2(\nu, x_1) \; &= m_0\tilde{\varepsilon}_1(\nu) + m_0\Big[\alpha_{01}[x_1\overline{\omega} + \cdots] + \alpha_{02}[x + \cdots] + \cdots\Big] \\ &\quad -[\tilde{\varepsilon}_2(\nu) + m_1x_1 + O(x_1^2)] \\ &= m_0\tilde{\varepsilon}_1(\nu) - \tilde{\varepsilon}_2(\nu) + m_0\alpha_{01}[x_1\overline{\omega} + \cdots] + \overline{m}_1[x + \cdots] + \cdots \end{aligned}$$

where $\alpha_{01} = \lambda_1(\nu) - 1$, $\overline{\omega} = \omega(x_1, \alpha_{01})$ and $m_1, \alpha_{02}(0) \neq 0$. Since $\overline{m_1}(0) = m_1 - m_0\alpha_{02} \neq 0$, the derivation-division method gives that $Cycl(Sxhh9e) \leq 2$.

For the intermediate graphics Sxhh9d, the transition map T along the graphic can be factorized as two regular transition maps and a central transition map, obviously, $T_2(\nu, \tilde{y}_4)$ has a first derivation which can be sufficiently small, thus $Cycl(Sxhh9d)$ is finite.

Therefore, the family Sxhh9 has finite cyclicity.

The finite cyclicity of the family Sxhh10 is similar to that of the family Sxhh9 by reversing time.

Altogether, we obtain that all the generic convex graphics with a nilpotent saddle point of codimension 3 have finite cyclicity, thus completing the proof of Theorem 5.5. □

Chapter 6

Finite cyclicity of graphics with a nilpotent singularity of elliptic type

In this chapter, we study the cyclicity of a graphic through a triple nilpotent elliptic point. There are three types of graphics with a nilpotent elliptic singularity: Epp, Ehp and Ehh. For the pp or hh-graphic, we assume that the graphics are generic. For the hh-graphic, we assume that the nilpotent elliptic point is of codimension 3. Each graphic can be concave or convex, but both cases share identical proofs.

6.1 Finite cyclicity of pp-graphics of elliptic type

In Table. 3.2, we have three families of pp-graphics of elliptic type: Epp1, Epp2 and Epp3. For all the pp-graphics, they do not have a return map.

For the passage near the blown-up shpere, on $r = 0$ in the chart F.R., the system has the form

$$\begin{cases} \dot{\bar{x}} &= \bar{\mu}_2 + \bar{y} + a\bar{x}^2 \\ \dot{\bar{y}} &= \bar{\mu}_1 + (\bar{\mu}_3 + \bar{x})\bar{y}. \end{cases} \tag{3.10}$$

So, if denote

$$V_I = \{\bar{\mu} \in \mathbb{S}^2 \big| \bar{\mu}_1 = 0, \bar{\mu}_2 \geq 0\},$$

then by the result in §3.2, the pp-graphics exist if and only if $\bar{\mu} \in V_I$. Note that if $\bar{\mu}_1 = \bar{\mu}_2 = 0$, an attracting or repelling saddle-node exists on the passage $\bar{y} = 0$ depending on the sign of $\bar{\mu}_3$, so we divide V_I into subsets

$$\begin{aligned} V_{I_1} &= \{\bar{\mu} \in V_I \Big| -\varepsilon_0 \leq \bar{\mu}_3 \leq \varepsilon_0 \} \\ V_{I_2} &= \{\bar{\mu} \in V_I \Big| \quad \tfrac{\varepsilon_0}{2} \leq \bar{\mu}_3 \leq 1 \ \} \\ V_{I_3} &= \{\bar{\mu} \in V_I \Big| -1 \leq \bar{\mu}_3 \leq -\tfrac{\varepsilon_0}{2}\} \end{aligned} \tag{6.1}$$

where $\varepsilon_0 \in (0,1)$ and such that $V_{I_1} \cup V_{I_2} \cup V_{I_3} = V_I$. We will determine ε_0 later.

The following proposition will be important in proving the finite cyclicity of pp and hh graphics of elliptic type.

Proposition 6.1. *Let S_2 be the second component of the transition map S : $\Pi_1 \longrightarrow \Pi_2$ in the normal form coordinates. Then $\forall (a, \bar{\mu}) \in A \times V_{I_1}$ and $\nu > 0$ sufficiently small, we have*

$$\begin{aligned} \tfrac{\partial S_2}{\partial \tilde{y}_1}(0,0) &= \exp\left(\frac{\pi \bar{\mu}_3}{\sqrt{a\bar{\mu}_2}}\right) + O(\rho_0) \\ \tfrac{\partial^2 S_2}{\partial \tilde{y}_1^2}(0,0) &= \frac{1}{a(1-2a)} e^{\frac{\pi\bar{\mu}_3}{\sqrt{a\bar{\mu}_2}}} \left[1 - e^{\frac{\pi\bar{\mu}_3}{\sqrt{a\bar{\mu}_2}}}\right] + O(\rho_0). \end{aligned} \tag{6.2}$$

Proof. The transition map S can be factorized as

$$S = \Psi_2\Big|_{\Pi_2} \circ \Phi_{02}\Big|_{\Pi_2} \circ \overline{S} \circ \Phi_{10}\Big|_{\Pi_1} \circ \Psi_1^{-1}\Big|_{\Pi_1} \tag{6.3}$$

where

(1) Ψ_1 and Ψ_2 are the C^k-coordinate changes normalizing the vector fields (3.8) at P_1 and P_2 respectively, and

$$\Psi_1^{-1}\Big|_{\Pi_1} : \begin{cases} r_1 &= \frac{\nu}{\rho_0} \\ \tilde{y}_1 &= b_{11}\bar{y}_1 + b_{12}\bar{y}_1^2 + O(\bar{y}_1^3) \end{cases}$$

$$\Psi_2\Big|_{\Pi_2} : \begin{cases} r_2 &= \frac{\nu}{\rho_0} \\ \bar{y}_2 &= b_{21}\tilde{y}_2 + b_{22}\tilde{y}_2^2 + O(\tilde{y}_2^3) \end{cases}$$

where b_{1i} and $b_{2i} (i = 1, 2)$ are functions of r_i, ρ_i respectively. On the sections Π_1

and Π_2, we have $\rho_1 = \rho_2 = \rho_0$, so on $r = 0$, we have

$$\begin{aligned} b_{11} &= 1 + \tfrac{\bar{\mu}_3}{a}\rho_0 + O(\rho_0^2) \\ b_{12} &= -\tfrac{1}{a(1-2a)} + O(\rho_0) \\ b_{21} &= 1 + \tfrac{\bar{\mu}_3}{a}\rho_0 + O(\rho_0^2) \\ b_{22} &= \tfrac{1}{a(1-2a)} + O(\rho_0). \end{aligned}$$

(2) Φ_{10} and Φ_{02} are coordinate changes between charts P.R. 1 and F.R, F.R. and P.R. 2 respectively. On the corresponding sections, they are linear:

$$\Phi_{10}\Big|_{\Pi_1} : \begin{cases} \bar{x} &= -\frac{1}{\rho_0} \\ \bar{y} &= \frac{\tilde{y}_1}{\rho_0^2} \end{cases}, \qquad \Phi_{02}\Big|_{\Pi_1} : \begin{cases} r_2 &= \frac{\nu}{\rho_0} \\ \tilde{y}_2 &= \rho_0^2 \bar{y}. \end{cases}$$

(3) The transition map

$$\overline{S} : \{\bar{x} = -x_0\} \longrightarrow \{\bar{x} = x_0\}$$

in the original coordinates $(\bar{x}, \bar{y})$ in the chart F.R., where $x_0 = \frac{1}{\rho_0}$.

In the chart F.R. with the coordinates $(\bar{x}, \bar{y}, r)$, we have system (3.9). Since ν is invariant, so $S_1(\nu, \bar{y}) = \nu$. On $r = 0$, we have system (3.10). For $\bar{\mu} \in V_{I_1}$, system (3.10) has no singular points on the invariant line $\bar{y} = 0$, so $\overline{S}_2$ is a C^k regular transition map and we can write it as

$$\overline{S}_2(\nu, \bar{y}) = m_0(\nu) + m_1(\nu)\bar{y} + m_2(\nu)\bar{y}^2 + O(\bar{y}^3) \tag{6.4}$$

where $m_0(0) = 0$. For the coefficients $m_1(\nu)$ and $m_2(\nu)$, by Prop. 5.2, we have

$$\begin{aligned} m_1(0) &= \exp\Big(\int_{-\bar{x}_0}^{\bar{x}_0} \frac{\bar{x} + \bar{\mu}_3}{a\bar{x}^2 + \bar{\mu}_2} d\bar{x}\Big) = \exp\Big(\frac{2\bar{\mu}_3}{\sqrt{a\bar{\mu}_2}} \arctan \frac{a\bar{x}_0}{\sqrt{a\bar{\mu}_2}}\Big), \\ m_2(0) &= m_1(0)\int_{-\bar{x}_0}^{\bar{x}_0} -\frac{2(\bar{x} + \bar{\mu}_3)}{(a\bar{x}^2 + \bar{\mu}_2)^2} \exp\Big(\int_{-\bar{x}_0}^{\bar{x}} \frac{\bar{x} + \bar{\mu}_3}{a\bar{x}^2 + \bar{\mu}_2} d\bar{x}\Big) d\bar{x} \\ &= m_1(0) I_{12}(\bar{x}_0) \end{aligned}$$

where

$$I_{12}(\bar{x}_0) = \frac{e^{\frac{\bar{\mu}_3}{\sqrt{a\bar{\mu}_2}} \arctan \frac{a\bar{x}_0}{\sqrt{a\bar{\mu}_2}}}}{(a\bar{x}_0^2 + \bar{\mu}_2)^{\frac{1}{2a}}} \int_{-\bar{x}_0}^{\bar{x}_0} \frac{-2(\bar{x} + \bar{\mu}_3) e^{\frac{\bar{\mu}_3}{\sqrt{a\bar{\mu}_2}} \arctan \frac{a\bar{x}}{\sqrt{a\bar{\mu}_2}}}}{(a\bar{x}^2 + \bar{\mu}_2)^{2 - \frac{1}{2a}}} d\bar{x}.$$

By L'Hospital's rule,

$$\begin{aligned}
&\lim_{\bar{x}_0\to\infty} I_{12}(\bar{x}_0)(a\bar{x}_0^2+\bar{\mu}_3)\\
&=\lim_{\bar{x}_0\to\infty}\frac{e^{\frac{\bar{\mu}_3}{\sqrt{a\bar{\mu}_2}}\arctan\frac{a\bar{x}_0}{\sqrt{a\bar{\mu}_2}}}}{(a\bar{x}_0^2+\bar{\mu}_2)^{\frac{1}{2a}-1}}\int_{-\bar{x}_0}^{\bar{x}_0}\frac{-2(\bar{x}+\bar{\mu}_3)e^{\frac{\bar{\mu}_3}{\sqrt{a\bar{\mu}_2}}\arctan\frac{a\bar{x}}{\sqrt{a\bar{\mu}_2}}}}{(a\bar{x}^2+\bar{\mu}_2)^{2-\frac{1}{2a}}}d\bar{x}.\\
&=\lim_{\bar{x}_0\to\infty}\frac{e^{\frac{\pi\bar{\mu}_3}{2\sqrt{a\bar{\mu}_2}}}}{(a\bar{x}_0^2+\bar{\mu}_2)^{\frac{1}{2a}-1}}\int_{-\bar{x}_0}^{\bar{x}_0}\frac{-2(\bar{x}+\bar{\mu}_3)e^{\frac{\bar{\mu}_3}{\sqrt{a\bar{\mu}_2}}\arctan\frac{a\bar{x}}{\sqrt{a\bar{\mu}_2}}}}{(a\bar{x}^2+\bar{\mu}_2)^{2-\frac{1}{2a}}}d\bar{x}\\
&=e^{\frac{\pi\bar{\mu}_3}{2\sqrt{a\bar{\mu}_2}}}\lim_{\bar{x}_0\to\infty}\frac{2\Big[(\bar{x}_0-\bar{\mu}_3)e^{-\frac{\bar{\mu}_3}{\sqrt{a\bar{\mu}_2}}\arctan\frac{a\bar{x}_0}{\sqrt{a\bar{\mu}_2}}}-(\bar{x}_0+\bar{\mu}_3)e^{\frac{\bar{\mu}_3}{\sqrt{a\bar{\mu}_2}}\arctan\frac{a\bar{x}_0}{\sqrt{a\bar{\mu}_2}}}\Big]}{(1-2a)\bar{x}_0}\\
&=\frac{2}{1-2a}\Big(1-e^{\frac{\pi\bar{\mu}_3}{\sqrt{a\bar{\mu}_2}}}\Big).
\end{aligned}$$

So

$$I_{12}(\bar{x}_0)=\frac{2}{a(1-2a)}\Big(1-e^{\frac{\pi\bar{\mu}_3}{\sqrt{a\bar{\mu}_2}}}\Big)\frac{1}{\bar{x}_0^2}+o(\frac{1}{\bar{x}_0^2}).$$

Therefore, for $\bar{x}_0=\frac{1}{\rho_0}$ and $\rho_0>0$ small, we have

$$\begin{aligned}
m_1(0)\ &=e^{\frac{\pi\bar{\mu}_3}{\sqrt{a\bar{\mu}_2}}}+O(\rho_0)\\
m_2(0)\ &=\frac{2}{a(1-2a)}e^{\frac{\pi\bar{\mu}_3}{\sqrt{a\bar{\mu}_2}}}\Big(1-e^{\frac{\pi\bar{\mu}_3}{\sqrt{a\bar{\mu}_2}}}\Big)\rho_0^2+o(\rho_0^2).
\end{aligned}$$

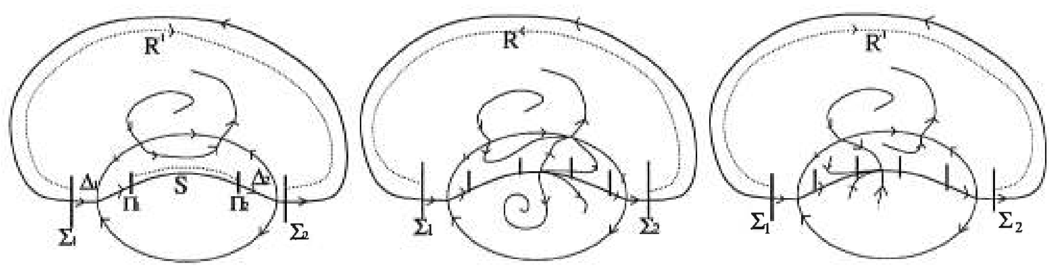
Then by **(1)**, **(2)**, **(3)** and (6.3), we have

$$\begin{aligned}
&\frac{\partial^2 S_2}{\partial\tilde{y}_1^2}(0,0)\\
&=m_1b_{12}b_{21}+\frac{1}{\rho_0^2}b_{11}^2b_{21}m_2+b_{11}^2b_{22}m_1^2+O(\rho_0)\\
&=m_1\Big[-\frac{1}{a(1-2a)}+\frac{2}{a(1-2a)}\Big(1-e^{\frac{\pi\bar{\mu}_3}{\sqrt{a\bar{\mu}_2}}}\Big)+\frac{1}{a(1-2a)}e^{\frac{\pi\bar{\mu}_3}{\sqrt{a\bar{\mu}_2}}}\Big]+O(\rho_0)\\
&=\frac{1}{a(1-2a)}e^{\frac{\pi\bar{\mu}_3}{\sqrt{a\bar{\mu}_2}}}\Big[1-e^{\frac{\pi\bar{\mu}_3}{\sqrt{a\bar{\mu}_2}}}\Big]+O(\rho_0).
\end{aligned}$$

□

Let Γ be any pp-graphic in the family. To prove its finite cyclicity, as shown in Fig. 6.1, we take sections Σ_1 and Σ_2 in normal form coordinates in the neighborhood of P_1 and P_2 respectively. We study the displacement maps

$$L=R^{-1}-T,\qquad\text{or}\qquad\mathcal{L}=R-T^{-1}\tag{6.5}$$



(a) Epp1 (b) Epp2 (c) Epp3

Figure 6.1: Displacement maps for pp-graphics

where $R : \Sigma_2 \longrightarrow \Sigma_1$ is the regular transition map along the regular orbit, and $T : \Sigma_1 \longrightarrow \Sigma_2$ is the transition passing through the blown-up nilpotent elliptic singularity.

For the transition map T, similar to Prop. 5.10, the passage from P_1 to P_2 has the same "funneling effect", i.e., its second component T_2 is almost affine too.

Proposition 6.2. *There exists $\varepsilon_0 > 0$ such that for any $k \in \mathbb{N}$, $a_0 \in (0, \frac{1}{2})$, there exist $A_0 \subset (0, \frac{1}{2})$, a neighborhood of a_0 such that for $\forall(a, \bar{\mu}) \in A_0 \times V_{I_3}$, $T_2'(0,0)$ is sufficiently small; while for $(a, \bar{\mu}) \in A \times V_{I_2}$, $T_2^{-1}(0,0)$ is sufficiently small. For any $(a, \bar{\mu}) \in A_0 \times V_{I_1}$ and $\nu > 0$ sufficiently small, the second component $T_2(\nu, \bar{y})$ of T is C^k, and*

$$\begin{aligned} T_2(\nu, \tilde{y}_1) &= \gamma_{120}(\nu, \omega(\tfrac{\nu}{\nu_0}, -\alpha_0)) + \sum_{i=1}^{k} \gamma_{12i}(\nu, \omega(\frac{\nu}{\nu_0}, -\alpha_0))\tilde{y}_1^i \\ &\quad + O\Big(\nu^{\bar{p}_1(1+[\frac{k-2}{q_1}])}\omega^{q_1+1-k-q_1[\frac{k-2}{q_1}]}(\frac{\nu}{\nu_0}, -\alpha_0)\ln\frac{\nu}{\nu_0}\tilde{y}_1^{k+1}\Big) \end{aligned} \tag{6.6}$$

where

$$\begin{aligned} \gamma_{120} &= m_{120}(\nu)(\tfrac{\nu}{\nu_0})^{-\bar{\sigma}_1} + \kappa_1 r_0^{\bar{p}_1}(1 - m_{121}(\nu))\omega(\tfrac{\nu}{\nu_0}, \alpha_1) + O\Big((\tfrac{\nu}{\nu_0})^{\bar{\sigma}_1}\omega^2(\tfrac{\nu}{\nu_0}, -\alpha_0)\Big) \\ \gamma_{121} &= m_{121}(\nu) + O\Big((\tfrac{\nu}{\nu_0})^{\bar{p}_1}\omega^{q_1}(\tfrac{\nu}{\nu_0}, -\alpha_0)\ln\tfrac{\nu}{\nu_0}\Big) \\ \gamma_{12i} &= O\Big(\nu^{\bar{p}_1(1+[\frac{i-2}{q_1}])}\omega^{q_1+1-i-q_1[\frac{i-2}{q_1}]}(\frac{\nu}{\nu_0}, -\alpha_0)\ln\frac{\nu}{\nu_0}\Big), \quad i \geq 2 \end{aligned}$$

and $m_{121}(0) = \exp(\frac{\pi\bar{\mu}_3}{\sqrt{a\bar{\mu}_2}})$.

Proof. As shown in Fig. 6.1, we decompose the map T as

$$T = \Delta_2^{-1} \circ S \circ \Delta_1 \tag{6.7}$$

where $S : \Pi_1 \longrightarrow \Pi_2$ is the regular transition map defined in Prop. 6.1, Δ_1 and Δ_2 are the first type Dulac maps in the neighborhood of P_1 and P_2. They have the same expression given in (4.12).

For $\bar{\mu} \in V_{I_1}$, the transition map $\overline{S}$ is studied in Prop. 6.1 using (6.3).

For $\bar{\mu}_0 = (0, 0, -1)$, system (3.10) has an attracting saddle-node on the invariant line. Hence we decompose $\overline{S}$ as

$$\overline{S} = T_{02} \circ T_{00} \circ T_{10} \tag{6.8}$$

where

- $T_{10}(\nu, \bar{y}) : \{\bar{x} = -x_0\} \longrightarrow \{\bar{x} = -x_{00}\}$ and $T_{02}(\nu, \bar{y}) : \{\bar{x} = x_{00}\} \longrightarrow \{\bar{x} = x_0\}$ are regular transitions;

- $T_{00}(\nu, \bar{y}) : \{\bar{x} = -x_{00}\} \longrightarrow \{\bar{x} = x_{00}\}$ is the center transition near the saddle node. For its second component $T_{002}(\nu, \bar{y})$, in the plane $r = 0$, by the C^k normal coordinates near the saddle node we have

$$T_{002}(\nu, \bar{y}) = m_{00}(\nu)\bar{y}, \qquad \lim_{\bar{\mu}_2 \to 0+} m_{00}(\nu) = 0.$$

  Therefore there exists $\varepsilon_{01} > 0$ such that for $\bar{\mu} \in V_{I_3}(\varepsilon_{01})$, $T'_{00}(0, 0)$ can be sufficiently small.

  Similarly, for $\bar{\mu}_0 = (0, 0, 1)$, we consider the inverse map $\overline{T}_{12}^{-1}$ by which there exists $\varepsilon_{02} > 0$ such that for $\bar{\mu} \in V_{I_2}(\varepsilon_{02})$, $T_{00}^{-1'}(0, 0)$ can be sufficiently small.

  Let $\varepsilon_0 = \min\{\varepsilon_{01}, \varepsilon_{02}\}$. We use ε_0 to divide V_I into three subcones in (6.1).

So for ε_0 chosen above, for $\bar{\mu} \in V_{I_3}$, $\overline{S}_2$ has the same expression as (6.4), but $\frac{\partial \overline{S}_2}{\partial \bar{y}_1}(0, 0) = m_{121}(0)$ sufficiently small.

Then the same way as in Prop. 5.10, by (4.12), (6.3) and (6.7) a straightforward calculation gives the results. □

Theorem 6.3. *We consider a pp-graphic with a triple nilpotent elliptic point of any codimension. If the second component R_2 of the regular transition map R has its $n-$th derivative nonvanishing, then $Cycl(Epp) \leq n$.*

Proof. There are three types of pp limit periodic sets through a nilpotent elliptic point. To study the cyclicity, we consider the displacement map L or $\mathcal{L}$ defined in (6.5). The transition T satisfies Prop. 6.2. For the regular transition map R, easily, we have $R_1(\nu, \tilde{y}_1) = \nu$; its second component R_2 is C^k, we write it as

$$R_2(\nu, \tilde{y}_1) = \sum_{i=0}^{k} \tilde{\gamma}_i(\nu)\tilde{y}_1^i + o(\tilde{y}_1^k). \tag{6.9}$$

where $\tilde{\gamma}_0(0) = 0$ and $\tilde{\gamma}_1(0) \neq 0$. By assumption, $\forall(a, \bar{\mu}) \in A \times \mathbb{S}^2$, and $\forall \nu \in (0, \nu_1)$ with $\nu_1 > 0$ sufficiently small, we have

$$\frac{\partial^n}{\partial \tilde{y}_1^n} R_2(0,0) = n!\tilde{\gamma}_n(0) \neq 0. \tag{6.10}$$

So for the displacement map L, we have $L_1(\nu, \tilde{y}_1) = 0$, and

$$L_2(\nu, \tilde{y}_1) = \sum_{i=0}^{k} \Big[\gamma_{12i}(\nu, \omega(\frac{\nu}{\nu_0}, -\alpha_0)) - \tilde{\gamma}_i(\nu)\Big]\tilde{y}_1^i + O(\tilde{y}_1^{k+1}). \tag{6.11}$$

For the graphic Epp3, note that $\forall(a, \bar{\mu}) \in A_0 \times V_{I_3}$, $\tilde{\gamma}_1(0) \neq 0$. Also by Prop. 6.2, $\gamma_{121}(\nu, \omega)$ sufficiently small, so we have

$$\frac{\partial L_2}{\partial \tilde{y}_1}(\nu, \tilde{y}_1) = \gamma_{121}(\nu, \omega(\frac{\nu}{\nu_0}, -\alpha_0)) - \tilde{\gamma}_1(\nu) + O(\tilde{y}_1) \neq 0.$$

which gives $Cycl(Epp3) \leq 1$.

For the graphic Epp1, if we choose $k \geq n$, then by (6.6) and (6.10) and (6.11), $\forall(a, \bar{\mu}) \in A_0 \times V_{I_1}$ and $\forall \nu \in (0, \nu_1)$, there holds

$$\begin{aligned} &\tfrac{\partial^n L_2}{\partial \tilde{y}_1^n}(\nu, \tilde{y}_1) \\ &= -n!\tilde{\gamma}_n(\nu) + O\Big(\nu^{\bar{p}_1(1+[\frac{n-2}{q_1}]}\omega^{q_1 1+1-n+q_1[\frac{n-2}{q_1}]}(\tfrac{\nu}{\nu_0}, -\alpha_0)\ln\tfrac{\nu}{\nu_0}\Big) + O(\tilde{y}_1) \\ &\neq 0. \end{aligned}$$

So by Rolle's theorem, for any $(a, \bar{\mu}) \in A_0 \times V_{I_1}$ and $\forall \nu \in (0, \nu_1)$, $L_2(\nu, \tilde{y}_1) = 0$ has at most n small roots in the neighborhood of $\tilde{y}_1 = 0$, i.e., $Cycl(Epp1) \leq n$.

For Epp2, we consider the map $\mathcal{L}$, similarly we get $Cycl(Epp2) \leq 1$. □

Proposition 6.4. *In the Theorem.* 6.3, *for the transition map* R *we assumed that* $R_2^{(n)}(0,0) \neq 0$. *This assumption is intrinsic.*

Proof. In [GR99], there is a detailed discussion on the intrinsic properties of a transition map. By saying that the assumption $R_2^{(n)}(0,0) \neq 0$ is intrinsic we mean that this property does not depend neither on the choices of coordinate changes which bring the system near P_1 and P_2 to normal forms nor on the choice of the sections parallel to the coordinate axes in the normal form coordinates.

Indeed, in the coordinates $(r_1, \rho_1, \tilde{y}_1)$, system near P_1 has the normal form (4.8) or (4.9). The Dulac map $\Delta_1 : \Sigma_1 \longrightarrow \Pi_1$ has the form (4.12). Assume that by an another "nearly-identity" change of coordinates, we bring the system near P_1 into the same normal form with coordinates $(r_1, \rho_1, \tilde{\tilde{y}}_1)$. Let $\widetilde{\Sigma}_1 = \{r_1 = r_{10}\}$ and $\widetilde{\Pi}_1 = \{\rho_1 = \rho_{10}\}$ be two sections parametrized by the new normal form coordinates $\tilde{\tilde{y}}_1$ and $\tilde{\tilde{\Delta}}_1 = (\tilde{\tilde{d}}, \tilde{\tilde{D}}_1)$ be the Dulac map $\Sigma_1 \longrightarrow \Pi_1$ in the new normal form coordinates. Then $\tilde{\tilde{\Delta}}_1$ has the same form as Δ_1 in (4.12), and we should have

$$\tilde{\tilde{\Delta}}_1(\nu, \tilde{\tilde{y}}_1) = \widehat{\Phi}_{11} \circ \Delta_1 \circ \widetilde{\Phi}_{11}(\nu, \tilde{\tilde{y}}_1) \quad \text{or} \quad \tilde{\tilde{\Delta}}_1 \circ \widetilde{\Phi}_{11}^{-1}(\nu, \tilde{y}_1) = \widehat{\Phi}_{11} \circ \Delta_1(\nu, \tilde{y}_1) \tag{6.12}$$

where

$$\begin{aligned} \widetilde{\Phi}_{11}^{-1}(\nu, \tilde{y}_1) &= (\nu, \widetilde{\phi}_{11}^{-1}) : \quad \widetilde{\Sigma}_1 \longrightarrow \Sigma_1 \\ \widehat{\Phi}_{11}(\nu, \tilde{y}_1) &= (\nu, \widehat{\phi}_{11}) : \quad \Pi_1 \longrightarrow \widetilde{\Pi}_1 \end{aligned}$$

are the compositions of coordinate changes and C^k regular transitions respectively. Let

$$\begin{aligned} \widetilde{\phi}_{11}^{-1}(\nu, \tilde{y}_1) &= \tilde{y}_1 + \sum_{j=2}^{k} \widetilde{m}_{11j}(\nu) \tilde{y}_1^j + O(\tilde{y}_1^{k+1}) \\ \widehat{\phi}_{11}(\nu, \tilde{y}_1) &= \tilde{y}_1 + \sum_{j=2}^{k} \widehat{m}_{11j}(\nu) \tilde{y}_1^j + O(\tilde{y}_1^{k+1}). \end{aligned} \tag{6.13}$$

We only consider the most difficult case $a_0 \in \mathbb{Q} \cap A$. Substituting (6.13) and the expressions for Δ_1, $\tilde{\tilde{\Delta}}_1$ into the second equation of (6.12), we have

$$\begin{aligned} &\eta_1(\nu, \omega(\tfrac{\nu}{\nu_0}, -\alpha_1)) + (\tfrac{\nu}{\nu_0})^{\bar{\sigma}_1} \left[\widetilde{\phi}_{11}^{-1}(\nu, \tilde{y}_1) + \phi_1(\nu, \omega(\tfrac{\nu}{\nu_0}), \widetilde{\phi}_{11}^{-1}(\nu, \tilde{y}_1)) \right] \\ &= \eta_1(\nu, \omega(\tfrac{\nu}{\nu_0}, -\alpha_1)) + (\tfrac{\nu}{\nu_0})^{\bar{\sigma}_1} \left[\tilde{y}_1 + \phi_1(\nu, \omega(\tfrac{\nu}{\nu_0}, -\alpha_1), \tilde{y}_1) \right] \\ &\sum_{j=2}^{k} \left[\eta_1(\nu, \omega(\frac{\nu}{\nu_0}, -\alpha_1)) + (\frac{\nu}{\nu_0})^{\bar{\sigma}_1} \Big(\tilde{y}_1 + \phi_1(\nu, \omega(\frac{\nu}{\nu_0} - \alpha_1), \tilde{y}_1) \Big) \right]^j + O(\tilde{y}_1^{k+1}). \end{aligned} \tag{6.14}$$

Equating the coefficient of monomial $\tilde{y}_1^j$ in both sides of (6.14), we get a series of equations about $\widehat{m}_{11j}$ and $\widetilde{m}_{11j}$. Then for $j = 2, 3, \cdots, k$, we have

$$\begin{aligned}
\tilde{y}_1^2: \quad & (\tfrac{\nu}{\nu_0})^{\bar{\sigma}_1} \Big[\widetilde{m}_{112}(\nu) + O\Big((\tfrac{\nu}{\nu_0})^{p_1} \omega^{q_1}(\tfrac{\nu}{\nu_0}, -\alpha_1) \ln \tfrac{\nu}{\nu_0}\Big)\Big] \\
& = \widehat{m}_{112}(\nu)(\tfrac{\nu}{\nu_0})^{2\bar{\sigma}_1} \\
& \quad + O\Big((\tfrac{\nu}{\nu_0})^{\bar{\sigma}_1 + p_1} \omega^{q_1}(\tfrac{\nu}{\nu_0}, -\alpha_1) \ln \tfrac{\nu}{\nu_0}\Big) + O\Big((\tfrac{\nu}{\nu_0})^{2\bar{\sigma}_1} \omega^{q_1}(\tfrac{\nu}{\nu_0}, -\alpha_1)\Big) \qquad (6.15) \\
\tilde{y}_1^j: \quad & (\tfrac{\nu}{\nu_0})^{\bar{\sigma}_1} \Big[\widetilde{m}_{11j}(\nu) + o\Big((\tfrac{\nu}{\nu_0})^{\bar{\sigma}_1}\Big)\Big] \\
& = \widehat{m}_{11j}(\nu)(\tfrac{\nu}{\nu_0})^{j\bar{\sigma}_1} + o\Big((\tfrac{\nu}{\nu_0})^{(j-1)\bar{\sigma}_1}\Big).
\end{aligned}$$

Then by (6.15), for $2 \le j \le k$, we have

$$\widetilde{m}_{11j}(\nu) = (\frac{\nu}{\nu_0})^{\bar{\sigma}_1} \widehat{m}_{11j}(\nu) + o\Big((\frac{\nu}{\nu_0})^{\bar{\sigma}_1}\Big).$$

Therefore we get

$$\widetilde{m}_{11j}(0) = 0, \qquad j = 2, 3, \cdots, k. \qquad (6.16)$$

Let

$$\widetilde{\Phi}_{22}(\nu, \tilde{y}_2) = (\nu, \widetilde{\phi}_{22}) : \widetilde{\Sigma}_2 \longrightarrow \Sigma_2$$

be the corresponding composition of coordinate change and a C^k regular transition map. If we denote

$$\widetilde{\phi}_{22}^{-1}(\nu, \tilde{y}_2) = \tilde{y}_2 + \sum_{j=2}^{k} \widetilde{m}_{22j}(\nu) \tilde{y}_2^j + O(\tilde{y}_2^{k+1}),$$

then similarly to (6.16), we get

$$\widetilde{m}_{22j}(0) = 0, \qquad j = 2, 3, \cdots, k. \qquad (6.17)$$

Let $\tilde{\tilde{R}} : \Sigma_1 \longrightarrow \Sigma_2$ be the transition map in the new normal form coordinates, then we have

$$\tilde{\tilde{R}} = \widetilde{\Phi}_{22}^{-1} \circ R \circ \widetilde{\Phi}_{11}^{-1}. \qquad (6.18)$$

Therefore, by (6.16), (6.17) and (6.18), we have

$$\tilde{\tilde{R}}_2^{(n)}(0, 0) = R_2^{(n)}(0, 0)$$

which is intrinsic. □

Remark 6.5. *In the new normal form coordinates* $(r_i, \rho_i, \tilde{\tilde{y}}_1)$ *(* $i = 1, 2$ *), the second component of the transition map* T *is still almost affine.*

6.2 Finite cyclicity of hp-graphics of elliptic type

Hp-graphic of elliptic type was not mentioned in [KS95] when the authors studied the 3-degenerate polycycles and their "ensembles". For the hp-graphic of elliptic type, we have Ehp1, 3, 4, $\cdots$, 7 and one family Ehp2(a, b, c) in Table. 3.3.

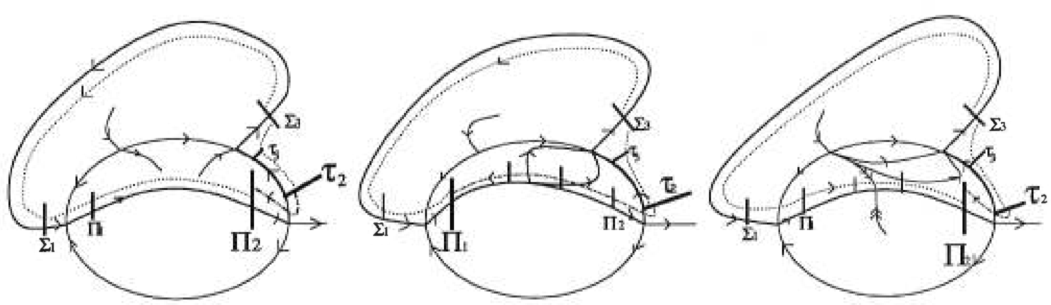
Theorem 6.6. *A hp-graphic with a nilpotent elliptic singularity of any codimension has finite cyclicity provided* **Conjecture 6.8** *given below is true.*

Proof. We consider the concave hp-graphic. By the results in Chap. 3, hp-graphics of elliptic type exist if and only if $\bar{\mu} \in V_I \cup V_{II} \cup V_{III}$ where

$$\begin{aligned} V_{II} &= \{\bar{\mu} \in \mathbb{S}^2 \big| \bar{\mu}_1 = 0, \bar{\mu}_3 \geq \sqrt{\tfrac{-\bar{\mu}_2}{a}}\} \\ V_{III} &= \{\bar{\mu} \in \mathbb{S}^2 \big| \bar{\mu}_1 > 0, a^2\bar{\mu}_3^2 - 9a\bar{\mu}_2\bar{\mu}_3 + \sqrt{a(-3\bar{\mu}_2 + a\bar{\mu}_3^2)^3} \leq \tfrac{27\bar{\mu}_1}{a}\}. \end{aligned}$$

We will study the cyclicity of all the graphics listed in Table. 3.3.

(1). Graphics Ehp1, Ehp2c and Ehp3



(a) Ehp1 (b) Ehp2c (c) Ehp3

Figure 6.2: Displacement maps for graphics Ehp1, Ehp2c and Ehp3

First we consider graphics Ehp1 and Ehp3. As shown in Fig. 6.2, take sections τ_2 and Π_2 (Notation 5.7). We study the displacement map defined on τ_2:

$$\begin{aligned} &L : \tau_2 \longrightarrow \Pi_2 \\ &L = \widetilde{T} - \widehat{T} \end{aligned} \tag{6.19}$$

where $\widetilde{T}$ is the transition map along the graphic, $\widehat{T} = \Theta_2$ is the second type of Dulac map near P_2.

Since the displacement map is defined on section τ_2, so we begin with the parametrization of the section τ_2. On τ_2, the coordinates are (r_2, ρ_2) with $r_2\rho_2 = \nu, \nu > 0$ small and invariant. We want to cover a domain $|r_2| < \varepsilon, |\rho_2| < \varepsilon$, where $\varepsilon > 0$ small. Then $\nu \leq \varepsilon^2$. So $\forall u \in (0,1)$, on the curve $\nu = u\varepsilon^2$, we have $r_2\rho_2 = u\varepsilon^2$, therefore $r_2, \rho_2 \in (u\varepsilon, \varepsilon)$. Let

$$r_2 = \nu^{1-d}, \qquad \rho_2 = \nu^d. \tag{6.20}$$

We then parametrize the section τ_2 using the coordinates $(\nu, d) \in (0, \varepsilon^2) \times \mathcal{I}_\nu$, where $\mathcal{I}_\nu = (\frac{\ln \varepsilon}{\ln \nu}, \frac{\ln u\varepsilon}{\ln \nu}) \subset (0,1)$.

To prove the finite cyclicity of the graphics, we are going to prove that the two functions $\widetilde{T}_2(\nu, d)$ and $\widehat{T}_2(\nu, d)$ have different convexity, i.e., $\widetilde{T}_2''(\nu, d) < 0$ and $\widehat{T}_2(\nu, d) > 0$.

We calculate $\widehat{T}_2''(\nu, d)$ first. Using coordinates (ν, d) on section τ_2, for $\widehat{T} = \Theta_2 = (\xi_2, \Xi_2)$, by Theorem 4.14, we have

$$\begin{cases} \xi_2(\nu, d) & = \nu \\ \Xi_2(\nu, d) & = \eta_2(\nu, \omega(\frac{\nu^d}{\rho_0}, \alpha_1)) + \nu^{\bar{\sigma}_1 d}[l_1 + \theta_2(\nu, \nu^d, \omega(\frac{\nu^d}{\rho_0}, -\alpha_1))] \end{cases} \tag{6.21}$$

where $l_1 = \frac{y_0}{\rho_0^{\bar{\sigma}_1}} > 0$; $\eta_2(\nu, \omega(\frac{\nu^d}{\rho_0}, \alpha_1)) = \frac{\kappa_3}{\rho_0^{p_1}} \nu^{p_1}\ \omega(\frac{\nu^d}{\rho_0}, \alpha_1)$ and $\theta_2(\nu, \nu^d, \omega(\frac{\nu^d}{\rho_0}, -\alpha_1))$ is C^∞. Also $\forall (a, \bar{\mu}) \in A_0 \times \mathbb{S}^2$, for $d \in (0,1)$ and $\nu > 0$ sufficiently small, we have uniformly

$$\begin{aligned} &\theta_2(\nu, \nu^d, \omega(\tfrac{\nu^d}{\rho_0}, -\alpha_1)) = O\Big(\nu^{p_1 d}\omega(\tfrac{\nu^d}{\rho_0}, \alpha_1)\Big) \\ &\tfrac{\partial^i \theta_2}{\partial d^i}(\nu, \nu^d, \omega(\tfrac{\nu^d}{\rho_0}, -\alpha_1)) = O\Big(\nu^{p_1 d}\omega(\tfrac{\nu^d}{\rho_0}, \alpha_1)(\ln \nu)^i\Big), \qquad i \geq 2. \end{aligned} \tag{6.22}$$

By

$$\begin{aligned} \tfrac{\partial}{\partial d}\eta_2(\nu, \omega(\tfrac{\nu^d}{\rho_0}, \alpha_1) &= -\tfrac{\kappa_1}{\rho_0^{p_1 - \alpha_1}} \nu^{p_1}\nu^{-\alpha_1 d} \ln \nu = -\tfrac{\kappa_1}{\rho_0^{\bar{\sigma}_1}} \nu^{p_1 - \alpha_1 d} \ln \nu \\ &= -\tfrac{\kappa_1}{\rho_0^{\bar{\sigma}_1}} \nu^{\bar{\sigma}_1 d} \nu^{p_1 - (\alpha_1 + \bar{\sigma}_1)d} \ln \nu = -\tfrac{\kappa_1}{\rho_0^{\bar{\sigma}_1}} \nu^{\bar{\sigma}_1 d} \nu^{p_1(1-d)} \ln \nu, \end{aligned} \tag{6.23}$$

also note that $\forall d \in (\frac{\ln \varepsilon}{\ln \nu}, \frac{\ln u\varepsilon}{\ln \nu})$, $\nu^{1-d} \in (u\varepsilon, \varepsilon)$ and $\nu^{p_1(1-d)} \in (0, \varepsilon^{p_1})$, so if we derivate $\widehat{T}_2(\nu, d)$ with respect to d twice, then we have

$$\widehat{T}_2''(\nu, d) = \nu^{\bar{\sigma}_1 d}(\ln \nu)^2\Big[\bar{\sigma}_1^2 l_1 + \alpha_1 \frac{\kappa_1}{\rho_0^{\bar{\sigma}_1}} \nu^{p_1(1-d)} + \hat{\theta}_{21}(\nu, \nu^d, \omega(\frac{\nu^d}{\rho_0}, -\alpha_1))\Big]. \tag{6.24}$$

So $\forall(\nu, d) \in (0, \varepsilon^2) \times \mathcal{I}_\nu$ with $\varepsilon > 0$ sufficiently small, there holds $\widehat{T}_2''(\nu, d) > 0$ which means that $\widehat{T}_2(\nu, d)$ is convex.

Next we calculate $\widetilde{T}_2(\nu, d)$ and prove that $\widetilde{T}_2''(\nu, d) < 0$.

For the transition map $\widetilde{T}$, we make the decomposition

$$\widetilde{T} = S \circ \Delta_1 \circ R \circ \Theta_3 \circ V \tag{6.25}$$

where

- Δ_1 is the first type Dulac map near P_1. It satisfies Theorem 4.11 with $\sigma = \sigma_1(a)$,

- Θ_3 is the second type of Dulac map near P_3, it satisfies Theorem 4.14 with $\sigma = \sigma_3(a)$: Using coordinates (ν, r_3) on section τ_3 defined in normal form coordinates by $\{\tilde{y}_3 = -y_0\}$, we have

$$\begin{cases} \xi_3(\nu, r_3) & = \nu \\ \Xi_3(\nu, r_3) & = \eta_3(\nu, \omega(\frac{r_3}{r_0}, \beta_1)) + r_3^{\bar{\sigma}_3}\Big[- l_3 + \theta_3(\nu, r_3, \omega(\frac{r_3}{r_0}, -\beta_1))\Big] \end{cases} \tag{6.26}$$

  where $l_3 = \frac{y_0}{r_0^{\bar{\sigma}_3}} > 0$ a constant, and $\theta_3(\nu, r_3, \omega(\frac{r_3}{r_0}, -\beta_1))$ satisfies a similar property as θ_2 in (6.22).

- $S : \Pi_1 \longrightarrow \Pi_2$ is the transition map defined in Prop. 6.1 with S_2 in (6.4),

- $R : \Sigma_3 \longrightarrow \Sigma_1$, a C^k regular transition map

$$\begin{cases} R_1(\nu, \tilde{y}_3) & = \nu \\ R_2(\nu, \tilde{y}_3) & = m_{310}(\nu) + m_{311}(\nu)\tilde{y}_3 + O(\tilde{y}_3^2), \end{cases} \tag{6.27}$$

  where $m_{310}(0) = 0$ and $m_{311}(0) > 0$,

- $V : \tau_2 \longrightarrow \tau_3$, a C^k regular transition map which can be written as

$$\begin{cases} V_1(r_2, \rho_2) & = r_2[m_{231} + O(|(r_2, \rho_2)|)] \\ V_2(r_2, \rho_2) & = \rho_2[\hat{m}_{231} + O(|(r_2, \rho_2)|)] \end{cases} \tag{6.28}$$

where $m_{231}(0), \hat{m}_{231}(0) > 0$ constants.

Let

$$r_3 = \nu^{1-d}\Big[m_{231} + O(|(\nu^d, \nu^{1-d})|)\Big]. \tag{6.29}$$

Then for the transition map $\widetilde{T}$, by (6.25) and using coordinates (ν, d) on the section τ_2, a straightforward calculation gives

$$\begin{aligned} \widetilde{T}_2(\nu, d) \; &= \delta_{00}(\nu) + \delta_{01}(\nu)\nu^{\bar{\sigma}_1 + p_3}\omega(r_3, -\beta_1)\Big(1 + O(\nu^{p_3}\omega(r_3, -\beta_1))\Big) \\ &\quad + \delta_{11}(\nu)\nu^{\bar{\sigma}_1} r_3^{\bar{\sigma}_3}\Big[1 + \theta_{31}(\nu, r_3, \omega(r_3, -\beta_1))\Big] \end{aligned} \tag{6.30}$$

where

$$\begin{aligned} \delta_{00}(\nu) &= m_0(0) + m_{310}(\nu)(\tfrac{\nu}{\nu_0})^{\bar{\sigma}_1} + O\Big((\tfrac{\nu}{\nu_0})^{\bar{\sigma}_1}\omega(\tfrac{\nu}{\nu_0}, -\alpha_1)\Big) \\ \delta_{01}(\nu) &= m_1(\nu)m_{311}(\nu) \\ \delta_{11}(\nu) &= -\tfrac{l_3 m_1(\nu)m_{311}(\nu)}{\nu_0^{\bar{\sigma}_1}} < 0. \end{aligned} \tag{6.31}$$

Note that if $q_3 = 1$, $p_3 - \beta_1 = \bar{\sigma}_3$ and

$$\begin{aligned} &\nu^{1-d}\nu^{p_3}\nu^{-(1-d)(1+\beta_1)} \\ &= \nu^{p_3 - (1-d)(1+\beta_1) + (1-d)} = \nu^{p_3 - \beta_1(1-d)} \\ &= \nu^{p_3 d + p_3(1-d) - \beta_1(1-d))} = \nu^{p_3 d + (p_3 - \beta_1)(1-d)} = \nu^{p_3 d}\nu^{\bar{\sigma}_3(1-d)}, \end{aligned}$$

a first derivative of $\widetilde{T}_2(\nu, d)$ gives

$$\begin{aligned} &\widetilde{T}_2'(\nu, d) \\ &= -m_{231}\nu^{1-d}\ln\nu(1 + O(\nu^d, \nu^{1-d})) \\ &\quad \Big[\delta_{01}\nu^{\bar{\sigma}_1 + p_3} r_3^{-1-\beta_1}\Big(1 + O(\nu^{p_3}\omega(r_3, -\beta_1))\Big) \\ &\quad + \bar{\sigma}_3\delta_{11}(\nu)\nu^{\bar{\sigma}_1} r_3^{\bar{\sigma}_3 - 1}\Big(1 + \theta_{31}(\nu, r_3, \omega(r_3, -\beta_1)) + \tfrac{1}{\bar{\sigma}_3} r_3 \tfrac{\partial\theta_{31}}{\partial r_3}(\nu, r_3, \omega(r_3, -\beta_1))\Big)\Big] \\ &= -\nu^{\bar{\sigma}_3(1-d)}\nu^{\bar{\sigma}_1}\ln\nu(1 + O(\nu^d, \nu^{1-d})) \\ &\quad \Big[\bar{\sigma}_3\delta_{11}(\nu)m_{231}^{\bar{\sigma}_3} + O(\nu^{p_3 d}) + \theta_{33}(\nu, \nu^d, \omega(m_{231}\tfrac{\nu^{1-d}}{r_0}, -\beta_1))\Big]. \end{aligned} \tag{6.32}$$

where θ_{33} has the same property as of θ_{31}.

Therefore for $\widetilde{T}_2''(\nu, d)$, we have

$$\begin{aligned}\widetilde{T}_2''(\nu, d) &= \nu^{\bar{\sigma}_1+\bar{\sigma}_3(1-d)}(\ln \nu)^2(1+O(\nu^d, \nu^{1-d})) \\ &\quad \left[\bar{\sigma}_3\delta_{11}(\nu)m_{231}^{\bar{\sigma}_3} + O(\nu^{p_3 d}) + \theta_{34}(\nu, \nu^d, \omega(m_{231}\tfrac{\nu^{1-d}}{r_0}, -\beta_1))\right]\end{aligned} \tag{6.33}$$

where $\delta_{11}(\nu) < 0$. So $\forall(\nu, d) \in (0, \varepsilon^2) \times \mathcal{I}_\nu$ with $\varepsilon > 0$ sufficiently small, $\widetilde{T}_2''(\nu, d) < 0$, i.e., $\widetilde{T}_2(\nu, d)$ is concave.

By (6.19), note that $\widehat{T}_2(\nu, d)$ is convex but $\widetilde{T}_2(\nu, d)$ is concave, so $L(\nu, d) = 0$ has at most two small roots for $(\nu, d) \in (0, \varepsilon^2) \times \mathcal{I}_\nu$ with $\varepsilon > 0$ sufficiently small, i.e., $Cycl(Ehp1, Ehp3) \leq 2$.

Now we consider the graphic Ehp2c. There exists a repelling saddle node on the graphic. As shown in Fig. 6.2(b), consider the displacement map

$$\begin{aligned}&L : \tau_2 \longrightarrow \Pi_1 \\ &L = \widetilde{T} - \widehat{T} \\ &\widetilde{T} = \Delta_1 \circ R \circ \Theta_3 \circ V \\ &\widehat{T} = S^{-1} \circ \Theta_2\end{aligned}$$

Similar to the graphic Ehp1 and Ehp3, using coordinates (ν, d) on the section τ_2, then we can prove that $\widehat{T}_2(\nu, d)$ is convex while $\widetilde{T}_2(\nu, d)$ is concave, therefore $L(\nu, d) = 0$ has at most 2 roots which gives $Cycl(Ehp2c) \leq 2$.

Remark 6.7. *For the hp graphics Ehp1, Ehp3 and Ehp2c considered above, we studied the displacement maps defined on the section τ_2 which is transverse to the passage from P_2 to P_3 along the equator. Since $\nu = r_2\rho_2$ is invariant, on τ_2, we have $\rho_2 = \frac{\nu}{r_2}$. So it is the passage from P_2 to P_3 along the equator that forces the two functions $\widetilde{T}_2$ and $\widehat{T}_2$ to have different convexity. Similar phenomenon happens on the passage from P_1 to P_4. Therefore, if a graphic contains one of these two passages and has a structure similar to that of Ehp1, then it has finite cyclicity 2.*

(2). Cyclicity of graphic Ehp2a

Ehp2a is a hp-graphic through a repelling saddle node. As shown in Fig. 6.3(a), let $\overline{\Sigma}_3 = \{\tilde{y} = y_0\}$, $\overline{\Sigma}_1 = \{\tilde{x} = -x_0\}$ be the two sections in the normal form coordinates near the saddle node in the chart F.R.. We consider the displacement map

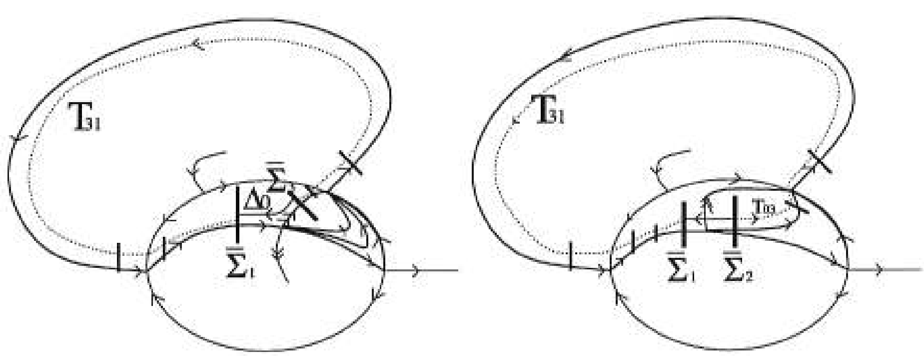
$$\begin{aligned} L: \ & \overline{\Sigma}_3 \longrightarrow \overline{\Sigma}_1 \\ & L := \overline{\Delta}_0 - \overline{T}_{31} \end{aligned} \tag{6.34}$$

where $\overline{\Delta}_0(\nu, \tilde{x}) = (\bar{d}_0, \overline{D}_0) : \overline{\Sigma}_3 \longrightarrow \overline{\Sigma}_1$ is the stable-centre transition near the saddle-node in the normal form coordinates $(\tilde{x}, \tilde{y})$, $\overline{T}_{31}$ is the transition along the flow of the graphic.

For the transition $\overline{\Delta}_0$, obviously $d_0(\nu, \tilde{x}) = \nu$; for $\overline{D}_0(\nu, \tilde{x})$, by [DER96], the graph $\tilde{y} = D_0(\tilde{x}, \nu)$ is a solution of the following differential equation

$$\Omega = F(\tilde{x}, a, \bar{\mu}, \nu)d\tilde{y} - \tilde{y}d\tilde{x} = 0 \tag{6.35}$$

where in this case $F(\tilde{x}, a, \bar{\mu}, \nu) = \bar{\mu}_2 + a\tilde{x}^2 + O(x^3) + O(\nu)$.



(a) Ehp2a (b) Ehp2b

Figure 6.3: Displacement maps for graphics Ehp2a and Ehp2b

For the transition map $\overline{T}_{31}$, we calculate it by making the following decomposition

$$\begin{aligned} \overline{T}_{31}: \ & \overline{\Sigma}_3 \longrightarrow \overline{\Sigma}_1 \\ & (\nu, \ \tilde{x}) \ \longrightarrow (\nu, \overline{T}_{312}) \\ & \overline{T}_{31} = \overline{T}_{10} \circ \Delta_1 \circ R \circ \Delta_3 \circ \overline{T}_{03} \end{aligned} \tag{6.36}$$

where

- $\overline{T}_{03} : \overline{\Sigma}_3 \longrightarrow \Pi_3$, the regular transition map which is the composition of two C^k normal form coordinate changes and a regular transition map from the saddle node to P_3, Altogether, it is C^k, we denote it as

$$\overline{T}_{03} : \begin{cases} \overline{T}_{031} & = \frac{\nu}{\rho_0} \\ \overline{T}_{032} & = m_{030}(\nu) + m_{031}(\nu)\tilde{x} + m_{032}(\nu)\tilde{x}^2 + O(\tilde{x}^3) \end{cases} \tag{6.37}$$

  where $m_{030}(0) = 0$ and $m_{031}(0) \neq 0$.

- $\Delta_3 : \Pi_3 \longrightarrow \Sigma_3$, is the first type Dulac map near P_1, it satisfies Theorem 4.11 with $\sigma = \sigma_3$.

- The regular transition map $R : \Sigma_3 \longrightarrow \Sigma_1$ is defined in (6.27).

- $\Delta_1 : \Sigma_1 \longrightarrow \Pi_1$ is in the first type Dulac map near P_1 and it satisfies Theorem. 4.11 with $\sigma = \sigma_1$.

- $\overline{T}_{10} : \Pi_1 \longrightarrow \overline{\Sigma}_1$, the regular transition map, analogue to $\overline{T}_{03}$, it is C^k and

$$\overline{T}_{10} : \begin{cases} \overline{T}_{101} & = \frac{\nu}{\rho_0} \\ \overline{T}_{102} & = m_{100}(\nu) + m_{101}(\nu)\tilde{y} + m_{102}(\nu)\tilde{y}^2 + O(\tilde{y}^3) \end{cases} \tag{6.38}$$

  where $m_{100}(0) = 0$, $m_{101}(0) \neq 0$.

Then a straightforward calculation from (6.37), (6.27) and (6.38), yielding that for any $k \in \mathbb{N}$, $a_0 \in (0, \frac{1}{2})$, there exist $A_0 \subset (0, \frac{1}{2})$, a neighborhood of a_0 and

$\nu_1 > 0$ such that $\forall(a, \bar{\mu}) \in A_0 \times \mathbb{S}^2$ and $\nu \in (0, \nu_1)$, $T_{312}(\nu, \tilde{y}_3)$ is also C^k and

$$\left\{\begin{array}{ll} \overline{T}_{311}(\nu, \tilde{x}) & = \nu \\ \overline{T}_{312}(\nu, \tilde{x}) & = \gamma_{310}(\nu, \omega(\frac{\nu}{\nu_0}, -\alpha_1), \omega(\frac{\nu}{\nu_0}, -\beta_1)) \\ & \quad + \left(\frac{\nu}{\nu_0}\right)^{\bar{\sigma}_1+\bar{\sigma}_3} [\gamma_{311}(\nu, \omega(\frac{\nu}{\nu_0}, -\alpha_1), \omega(\frac{\nu}{\nu_0}, -\beta_1))\tilde{x} \\ & \quad + \gamma_{312}(\nu, \omega(\frac{\nu}{\nu_0}, -\alpha_1), \omega(\frac{\nu}{\nu_0}, -\beta_1))\tilde{x}^2 + O(\tilde{x}^3)] \end{array}\right. \tag{6.39}$$

where

$$\begin{aligned} \gamma_{310} &= m_{100}(\nu) + m_{101}(\nu)[\kappa_1 r_0^{p_1} (\tfrac{\nu}{\nu_0})^{\bar{\sigma}_1} \omega(\tfrac{\nu}{\nu_0}, -\alpha_1) + O(\tfrac{\nu}{\nu_0})^{\bar{\sigma}_1}))] \\ \gamma_{311} &= m_{101}(\nu) m_{311}(\nu) m_{031}(\nu) + O\Big((\tfrac{\nu}{\nu_0})^{\bar{\sigma}_1} \omega(\tfrac{\nu}{\nu_0}, -\alpha_1), (\tfrac{\nu}{\nu_0})^{\bar{\sigma}_3} \omega(\tfrac{\nu}{\nu_0}, -\beta_1)\Big) \neq 0 \\ \gamma_{312} &= m_{101}(\nu) m_{311}(\nu) m_{032}(\nu) \\ & \quad + O\Big((\tfrac{\nu}{\nu_0})^{\bar{\sigma}_1} \omega(\tfrac{\nu}{\nu_0}, -\alpha_1), (\tfrac{\nu}{\nu_0})^{\bar{\sigma}_3} \omega(\tfrac{\nu}{\nu_0}, -\beta_1), (\tfrac{\nu}{\nu_0})^{\bar{\sigma}_3} \omega^{q_3 - 1}(\tfrac{\nu}{\nu_0}, -\beta_1) \ln \tfrac{\nu}{\nu_0}\Big). \end{aligned}$$

Now let us study the displacement map L. Easily we see $L_1(\nu, \tilde{x}) = 0$. By (6.34), (6.35) and (6.36), the equation $L_2(\nu, \tilde{x}) = 0$ is equivalent to the following system

$$\left\{\begin{array}{ll} \tilde{y} & = \overline{D}_0(\tilde{x}, \nu) \\ \tilde{y} & = \overline{T}_{312}(\nu, \tilde{x}). \end{array}\right. \tag{6.40}$$

Since $\tilde{y} = \overline{D}_0(\tilde{x}, \nu)$ is a connected graph, the generalized Rolle's lemma (Khovanskii procedure) in [K84] shows that the number of solutions of (6.40) is at most 1 plus the number of solutions of the following system

$$\left\{\begin{array}{l} \tilde{y} = \overline{T}_{312}(\nu, \tilde{x}) \\ \Omega \wedge d\overline{T}_{312}(\nu, \tilde{x}) = 0 \end{array}\right.$$

The above system is equivalent to

$$\left\{\begin{array}{l} \tilde{y} = \overline{T}_{312}(\nu, \tilde{x}) \\ \det \begin{pmatrix} \frac{\partial \overline{T}_{312}}{\partial \tilde{x}} & -1 \\ -\tilde{y} & F(\tilde{x}, a, \bar{\mu}, \nu) \end{pmatrix} = 0. \end{array}\right. \tag{6.41}$$

Eliminating $\tilde{y}$ from the equations (6.41), we have an equivalent equation

$$L_{21}(\tilde{x}, \nu) := \overline{T}_{312}(\nu, \tilde{x}) - F(\tilde{x}, a, \bar{\mu}, \nu) \frac{\partial \overline{T}_{312}}{\partial \tilde{x}}(\nu, \tilde{x}). \tag{6.42}$$

By (6.42), derivating with respect to $\tilde{x}$ yields

$$\begin{aligned} L'_{21}(\tilde{x},\nu) &= (1 - F'_x(\tilde{x},a,\bar{\mu},\nu))\frac{\partial \overline{T}_{312}(\nu,\tilde{x})}{\partial \tilde{x}} - F(\tilde{x},a,\bar{\mu},\nu))\frac{\partial^2 \overline{T}_{312}(\nu,\tilde{x})}{\partial \tilde{x}^2} \\ &= \left(\frac{\nu}{\nu_0}\right)^{\bar{\sigma}_1+\bar{\sigma}_3}[(1 - 2a\tilde{x} + O(\tilde{x}^2))(\gamma_{311} + 2\gamma_{312}\tilde{x} + O(\tilde{x}^2)) \\ &- (\hat{\mu}_2 + a\tilde{x}^2 + O(\tilde{x}^3))(2\gamma_{312} + O(\tilde{x}))] \end{aligned}$$

which has the same number of small roots as of

$$L_{22} = \left(\tfrac{\nu}{\nu_0}\right)^{-(\bar{\sigma}_1+\bar{\sigma}_3)} L'_{21}(\nu,\tilde{x}) = \gamma_{311} - 2\gamma_{312}\hat{\mu}_2 + O(\tilde{x}).$$

Hence $\forall (a,\bar{\mu}) \in A_1 \times V_{I_2}$ and $\forall \nu \in (0,\nu_1)$, and for $\hat{\mu}_2 > 0$ sufficiently small, the equation $L_2 = 0$ has at most 2 small zeroes, i.e., $Cycl(Ehp2a) \le 2$.

(3). Cyclicity of graphic Ehp2b

As in Fig. 6.3(b), let $\overline{\Sigma}_1 = \{\tilde{x} = -x_0\}$ and $\overline{\Sigma}_2 = \{\tilde{x} = x_0\}$ be two sections transversal to the graphic Ehp2b, and consider the displacement map

$$\begin{aligned} L:\ \overline{\Sigma}_2 &\longrightarrow \overline{\Sigma}_1 \\ L &:= \overline{\Delta}_0 - \overline{T}_{31} \end{aligned} \tag{6.43}$$

where $\overline{\Delta}_0(\nu,\tilde{x}) = (\overline{d}_0, \overline{D}_0) : \overline{\Sigma}_2 \longrightarrow \overline{\Sigma}_1$ is the centre transition near the saddle-node in the normal form coordinates $(\tilde{x},\tilde{y})$, and

$$\begin{cases} \overline{d}_0(\nu,\tilde{y}) &= \nu \\ \overline{D}_0(\nu,\tilde{y}) &= m(\hat{\mu}_2)\tilde{y} \qquad \lim_{\hat{\mu}_2\to 0} m(\hat{\mu}_2) = 0, \end{cases} \tag{6.44}$$

$\overline{T}_{31}$ is the transition along the flow of the graphic which can be factorized as

$$\begin{aligned} \overline{T}_{31}:\ \overline{\Sigma}_2 &\longrightarrow \overline{\Sigma}_1 \\ (\nu,\ \tilde{x}) &\longrightarrow (\nu, \overline{T}_{312}) \\ \overline{T}_{31} &= \overline{T}_{10} \circ \Delta_1 \circ R \circ \Delta_3 \circ \overline{T}_{03} \end{aligned} \tag{6.45}$$

where except for $\overline{T}_{03}$, the other four components are the same as in (6.36). For $\overline{T}_{03} : \overline{\Sigma}_2 \longrightarrow \Pi_3$, it is a regular transition map in normal form coordinates which can be written as

$$\begin{cases} \overline{T}_{031}(\nu,\tilde{y}) &= \frac{\nu}{\rho_0} \\ \overline{T}_{032}(\nu,\tilde{y}) &= m_{031}(\nu)\tilde{y} + m_{032}(\nu)\tilde{y}^2 + O(\tilde{y}^3). \end{cases}$$

Conjecture 6.8. *We consider the vector field*

$$\begin{cases} \dot{x} &= y + ax^2 \\ \dot{y} &= y(x+1) \end{cases} \tag{6.46}$$

with a saddle node at the origin and a singular point at P at infinity given by $(u,z) = (\frac{1-2a}{2}, 0)$, where $(u,z) = (\frac{y}{x^2}, \frac{1}{x})$ (Fig. 6.4). Let $(\tilde{x}, \tilde{y})$ be normal coordinates near the origin and $(\tilde{u}, \tilde{z})$ be normal coordinates near P. Then the transition map

$$T : \{\tilde{x} = x_0\} \longrightarrow \{\tilde{z} = z_0\}$$

is nonlinear at any point $\tilde{y}_0$ of $\{\tilde{x} = x_0\}$, i.e., $\forall \tilde{y}_0$, there exists $n \geq 2$ such that

$$\frac{d^nT}{d\tilde{y}^n}(\tilde{y}_0) \neq 0. \tag{6.47}$$

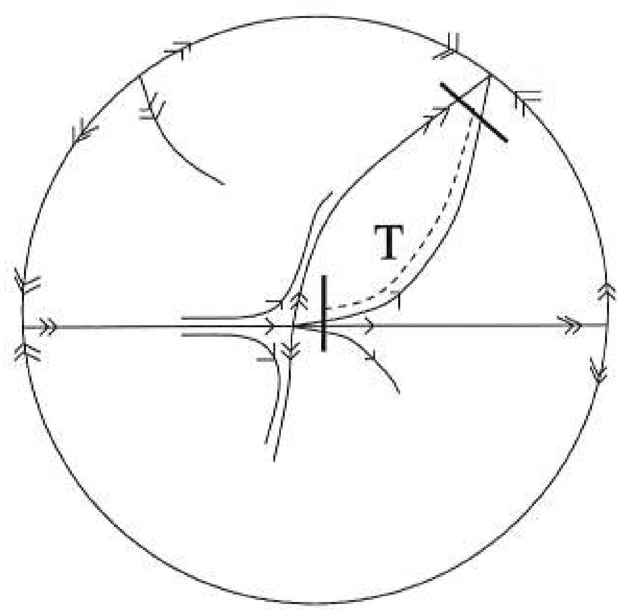


Figure 6.4: The regular transition map T in the Conjecture 6.8

Remark 6.9. *The system* (6.46) *is very simple. To prove the above conjecture, we see two directions:*

- *the first one is an explicit calculation of $\frac{d^2T}{d\tilde{y}^2}$.*

- *the second one is a general argument similar to that in Prop.* 5.12. *In this case there exists no analytic normalizing change of coordinates, but we are in the special case where the system has an analytic center manifold.*

The conjecture implies that there exists $n \geq 2$ such that $\frac{\partial^n T_{032}}{\partial \tilde{y}^n}(0,0) = \tilde{m}_{03n} \neq 0$. Then for the transition map $\overline{T}_{31}$, we have

$$\begin{aligned} \overline{T}_{312}(\nu, \tilde{y}) &= \gamma_{310}(\nu, \omega(\tfrac{\nu}{\nu_0}, -\alpha_1), \omega(\tfrac{\nu}{\nu_0}, -\beta_1)) \\ &+ \left(\tfrac{\nu}{\nu_0}\right)^{\bar{\sigma}_1+\bar{\sigma}_3} \left[\sum_{i=1}^{n} \gamma_{31i}(\nu, \omega(\frac{\nu}{\nu_0}, -\alpha_1), \omega(\frac{\nu}{\nu_0}, -\beta_1))\tilde{y}^i + O(\tilde{y}^{n+1}) \right]. \end{aligned} \tag{6.48}$$

where for $\nu > 0$ sufficiently small, $\gamma_{31n}(0) = *\tilde{m}_{03n} \neq 0$.

Now consider the displacement map $L := \overline{T}_{31} - \Delta_0$. Obviously $L_1(\nu, \tilde{y}) = 0$; for L_2, it follows from (6.43), (6.44) and (6.45) that we have

$$\begin{aligned} L_2(\nu, \tilde{y}) &= -m(\hat{\mu}_2)\tilde{y} + \gamma_{310}(\nu, \omega(\tfrac{\nu}{\nu_0}, -\alpha_1), \omega(\tfrac{\nu}{\nu_0}, -\beta_1) \\ &+ \left(\tfrac{\nu}{\nu_0}\right)^{\bar{\sigma}_1+\bar{\sigma}_3} \left[\sum_{i=1}^{n} \gamma_{31i}(\nu, \omega(\frac{\nu}{\nu_0}, -\alpha_1), \omega(\frac{\nu}{\nu_0}, -\beta_1))\tilde{y}^i + O(\tilde{y}^{n+1}) \right]. \end{aligned}$$

Derivating L_2 with respect to $\tilde{y}$ n times, we have

$$\tilde{L}_n(\nu, \tilde{y}) = \left(\frac{\nu}{\nu_0}\right)^{-(\bar{\sigma}_1+\bar{\sigma}_3)} L_2^{(n)}(\nu, \tilde{y}) = *\tilde{m}_{03n}(0) + O(\nu) + O(\tilde{y}) \neq 0. \tag{6.49}$$

So $L = 0$ has at most n small roots, i.e., $Cycl(Ehp2b) \leq n$.

(4). Cyclicity of Ehp4 , Ehp5

For the graphic Ehp4, as shown in Fig. 6.5(a), there is a saddle point on the connection from P_1 to P_3. In the normal form coordinates $(\tilde{x}, \tilde{y})$ on $r = 0$ in the chart F.R., take sections $\overline{\Sigma}_1 = \{\tilde{x} = -x_0\}$ and $\overline{\Sigma}_3 = \{\tilde{y} = y_0\}$ and consider the transition map

$$\overline{\Delta}_0 = (\overline{d}_0, \overline{D}_0) : \overline{\Sigma}_3 \longrightarrow \overline{\Sigma}_1.$$

Since $r = 0$ is invariant, so $d_0(\nu, \tilde{x}) = \nu$. Let $\lambda_0 = \lambda_{(a, \bar{\mu}_0)}$ be the hyperbolicity ratio at the saddle point. Then $\overline{D}_0(\nu, \tilde{x})$ can be written in the form of (5.46).

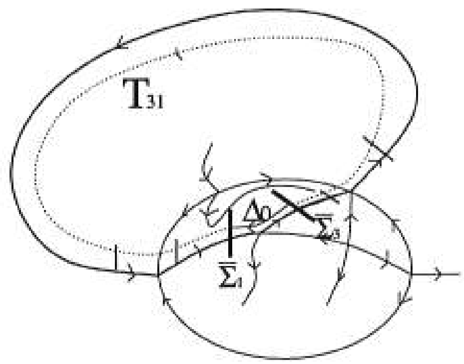


Figure 6.5: Displacement maps for graphic Ehp4

Using the normal formal coordinates on the sections $\overline{\Sigma}_3$ and $\overline{\Sigma}_1$, we consider the displacement map

$$\begin{aligned} L: \ & \overline{\Sigma}_3 \longrightarrow \overline{\Sigma}_1 \\ & (r, \ \tilde{y}) \ \longrightarrow \ (L_1, L_2) \\ & L := \overline{\Delta}_0 - \overline{T}_{31} \end{aligned}$$

where $\overline{T}_{31}$ is the transition map the same as in (6.39). Easily we see $L_1(\nu, \tilde{x}) = 0$. We will prove the finite cyclicity for Ehp4 by considering the displacement map $L_2(\nu, \tilde{x})$ in three cases.

Case (4.1). $\lambda_0 > 1$:

By (6.39) and (5.46) we have

$$\begin{aligned} L_2'(\nu, \tilde{x}) \ &= \tfrac{1}{\lambda}\tilde{x}^{\frac{1}{\lambda}-1}\Big[\beta_0 + \phi_0(\tilde{x}, \nu) + \tilde{x}\tfrac{\partial \phi_0}{\partial \tilde{x}}(\tilde{x}, \nu)\Big] \\ & -(\tfrac{\nu}{\nu_0})^{\bar{\sigma}_1+\bar{\sigma}_3}\Big[\gamma_{311}(\nu, \omega(\nu, -\alpha_1)) + 2\gamma_{312}(\nu, \omega(\nu, -\alpha_1))\tilde{x} + O(\tilde{x}^2)\Big]. \end{aligned} \tag{6.50}$$

Hence

$$\begin{aligned} \tilde{L}_2' \ &= \tilde{x}^{1-\frac{1}{\lambda}} L_2'(\nu, \tilde{x}) \\ &= \frac{1}{\lambda}\beta_0 + \frac{1}{\lambda}(\phi_0 + \tilde{x}\frac{\partial \phi_0}{\partial \tilde{x}}) - \Big(\frac{\nu}{\nu_0}\Big)^{\bar{\sigma}_1+\bar{\sigma}_3}\Big[\gamma_{311}\tilde{x}^{1-\frac{1}{\lambda}} + O\Big(\tilde{x}^{2-\frac{1}{\lambda}}\Big)\Big] \\ &\neq 0. \end{aligned} \tag{6.51}$$

So by (6.51), $L = 0$ has at most 1 small positive root.

Case (4.2). $\lambda_0 < 1$:

By (6.50) and let

$$\begin{aligned}\hat{L}_2(\nu, \tilde{x}) &= \frac{L_2'(\nu, \tilde{x})}{\gamma_{311} + 2\gamma_{312}\tilde{x} + O(\tilde{x}^2)} \\ &= \frac{\frac{1}{\lambda}\tilde{x}^{\frac{1}{\lambda}-1}\Big[\beta_0 + \phi_0(\tilde{x}, \nu) + \tilde{x}\frac{\partial\phi_0}{\partial\tilde{x}}(\tilde{x}, \nu)\Big]}{\gamma_{311} + 2\gamma_{312}\tilde{x} + O(\tilde{x}^2)} - \Big(\frac{\nu}{\nu_0}\Big)^{\bar{\sigma}_1 + \bar{\sigma}_3}\end{aligned}$$

Then for small $\tilde{x}$,

$$\begin{aligned}&\hat{L}_2'(\nu, \tilde{x}) \\ &= \frac{\partial \hat{L}_2(\nu, \tilde{x})}{\partial \tilde{x}} \\ &= \frac{\frac{1}{\lambda}(\frac{1}{\lambda} - 1)\tilde{x}^{\frac{1}{\lambda}-2}(\beta_0 + \hat{\phi}_0)(\gamma_{311} + O(\tilde{x})) - \frac{1}{\lambda}\tilde{x}^{\frac{1}{\lambda}-1}(\beta_0 + \tilde{\phi}_0)(2\gamma_{312} + O(\tilde{x}))}{(\gamma_{311} + O(\tilde{x}))^2} \\ &= \frac{\tilde{x}^{\frac{1}{\lambda}-2}}{\lambda^2(\gamma_{311} + O(\tilde{x}))^2}\Big[(1-\lambda)\beta_0\gamma_{311} + \gamma_{311}\hat{\phi}_0 + O(\tilde{x})\Big] \\ &\neq 0.\end{aligned}$$

Therefore $\hat{L}_2(\nu, \tilde{x}) = 0$ has at most 1 small root and L_2 has at most 2 small positive roots.

Case 3. $\lambda_0 = 1$:

In this case, by (6.39) and (5.46), $L_2 = 0$ is equivalent to

$$\begin{cases} \tilde{y} &= \beta_0\tilde{x} + \alpha_{01}\tilde{x}\omega[1 + \cdots] + \alpha_{02}\tilde{x}^2\omega[1 + \cdots] + \cdots \\ \tilde{y} &= \gamma_{310}(\nu, \omega(\frac{\nu}{\nu_0}, -\alpha_1)\omega(\frac{\nu}{\nu_0}, -\beta_1)) \\ &\quad + \Big(\frac{\nu}{\nu_0}\Big)^{\bar{\sigma}_1 + \bar{\sigma}_3}\Big[\gamma_{311}(\nu, \omega(\frac{\nu}{\nu_0}, -\alpha_1)\omega(\frac{\nu}{\nu_0}, -\beta_1))\tilde{x} \\ &\quad + \gamma_{312}(\nu, \omega(\frac{\nu}{\nu_0}, -\alpha_1)\omega(\frac{\nu}{\nu_0}, -\beta_1))\tilde{x}^2 + O(\tilde{x}^3)\Big]. \end{cases} \tag{6.52}$$

where for the first saddle quantity α_{02}, we have $\alpha_{02} = 2a(2a-1) - 1 \neq 0$.

Indeed, for the vector field at the saddle point, after a translation and a linear transformation, we can bring system in the neighborhood of the saddle into the following form

$$\begin{cases} \dot{u} &= -u + au^2 + (2a-1)uv + (a-1)v^2 \\ \dot{v} &= \quad v + uv + v^2. \end{cases}$$

By Lemma. 5.14, we obtain the first saddle quantity $\alpha_{02}(a, \bar{\mu}) = 2a(2a-1) - 1$ and for $a \in (0, \frac{1}{2})$, $\alpha_{02} \neq 0$. Then from (6.52) and by the standard derivation division method in [R86], we conclude that $L = 0$ has at most 2 small zeroes.

Therefore, for the graphic Ehp4, we have $Cycl(Ehp4) \leq 2$.

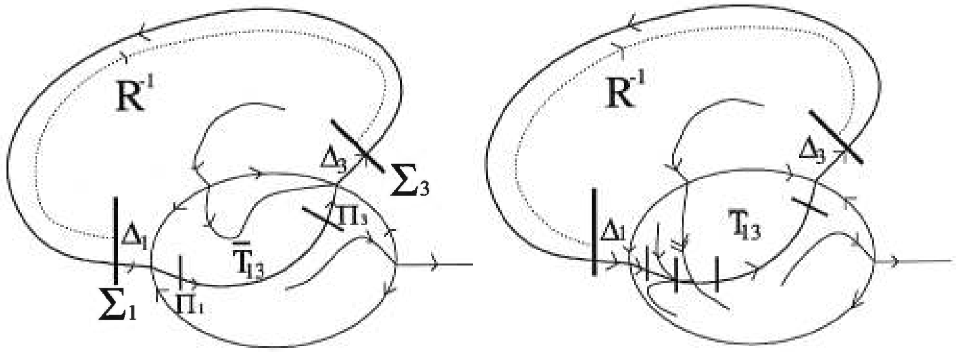
For the l.p.s. Ehp5, since the return map can be written as a composition of regular transition maps and maps with derivatives sufficiently small, we get $Cycl(Ehp5) \leq 1$.

(5). Cyclicity of Ehp6 and Ehp7

For the graphic Ehp6, the passage from P_1 to P_3 is just a regular orbit. Similar to the graphics Ehp1 and Ehp3, as shown in Fig. 6.6, we consider the displacement map

$$\begin{aligned} L : \ & \Sigma_1 \longrightarrow \Sigma_3 \\ & L := R^{-1} - T_{13} \\ & T_{13} := \Delta_3 \circ \overline{T}_{13} \circ \Delta_1 \end{aligned}$$

where $\Delta_1 : \Sigma_1 \longrightarrow \Pi_1$ and $\Delta_3 : \Pi_3 \longrightarrow \Sigma_3$ are the first Dulac maps in the neighborhood of P_1 and P_3, they satisfy Theorem 4.11 with $\sigma = \sigma_1$ and $\sigma = \sigma_3$ respectively; $\overline{T}_{13} : \Pi_1 \longrightarrow \Pi_3$ is the regular C^k transition map. So, for T_{13}, we



(a) graphic Ehp6 (b) graphic Ehp7

Figure 6.6: Displacement maps for graphics Ehp6 and Ehp7

have

$$\begin{cases} T_{131}(\nu, \tilde{y}_1) & = \nu \\ T_{132}(\nu, \tilde{y}_1) & = \gamma_{130}(\nu, \omega(\frac{\nu}{\nu_0}, -\alpha_1), \omega(\frac{\nu}{\nu_0}, -\beta_1)) \\ & \quad + \left(\frac{\nu}{\nu_0}\right)^{\bar{\sigma}_1 + \bar{\sigma}_3} \left[\gamma_{131}(\nu, \omega(\frac{\nu}{\nu_0}, -\alpha_1), \omega(\frac{\nu}{\nu_0}, -\beta_1))\tilde{y}_1 + O\left(\tilde{y}_1^2\right)\right] \end{cases} \tag{6.53}$$

Then by (6.27) and (6.53), it is not difficult to see that $L(\tilde{y}_1, \nu) = 0$ has at most one small zero. Hence $Cycl(Ehp6) \leq 1$.

For the graphic Ehp7, the return map can be written as a composition of regular transition maps and maps with derivatives sufficiently small, so $Cycl(Ehp7) \leq 1$.

Each limit periodic limit has finite cyclicity under extended conjecture, yielding finite cyclicity of the hp graphic. □

6.3 Finite cyclicity of hh-graphics of elliptic type

In this section, we study the 12 families of hh-graphics listed in Table 3.4. We state the main result in §6.3.1 and give a generalized Rolle's Theorem in §6.3.2.

The main theorem is proved in §6.3.3 and §6.3.4.

6.3.1 Main Theorem on the hh-graphics of elliptic type

For the hh-graphic of elliptic type, we have

Theorem 6.10. *An hh-graphic through a triple nilpotent elliptic point of codimension 3 has finite cyclicity if the generic hypothesis $P'(0) \neq 1$ is satisfied.*

For the proof, by changing the family X to $-X$ if necessary, we impose

Hypothesis 6.11. *The hh-graphic with a nilpotent elliptic point is attracting:*

$$[H]: \qquad P'(0) = \gamma^* < 1. \tag{6.54}$$

In Table 3.4, there are 12 families of hh-graphics of elliptic type: Ehhi ($i = 1, 2, \cdots, 12$). By Remark 5.9, all the upper boundary graphics in the 12 families have finite cyclicity 1. So to prove Theorem 6.10, we need to prove that all the lower boundary graphics and intermediate graphics of the 12 families have finite cyclicity. We will finish the proof in two separate sections: in §6.3.3, we prove that all the lower boundary graphic have finite cyclicity, in §6.3.4, we prove that all the intermediate graphics have finite cyclicity.

Before going into the proof of the main theorem, we do some preparations.

6.3.2 Generalized Rolle's Theorem and a transition map

We have Rolle's Theorem to deal with functions of one variable. In proving the finite cyclicity of hh-graphics of elliptic type, we will have to study the number of intersections of two planar curves, hence the following generalization of Rolle's Theorem is useful.

Theorem 6.12. *(Generalized Rolle's Theorem) Let $\mathcal{D} = (x_1, x_2) \times (y_1, y_2)$. Let $F(x, y), G(x, y)$ be two functions continuous on $\overline{\mathcal{D}}$ and smooth in $\mathcal{D}$. Assume that* *in $\mathcal{D}$, $F'_x(x,y)$, $F'_y(x,y) \neq 0$. Consider the system of equations*

$$\begin{cases} F(x,y) = 0 \\ G(x,y) = 0. \end{cases}$$

Denote the number of intersections of $F(x,y) = 0$ and $G(x,y) = 0$ in the region $\mathcal{D}$ by $\#(F,G)$ and let

$$J[F,G](x,y) = F'_y(x,y)G'_x(x,y) - F'_x(x,y)G'_y(x,y).$$

Then

$$\#(F,G) \leq 1 + \#(F, J[F,G]).$$

Proof. First note that if $\forall (x,y) \in \mathcal{D}$, $F(x,y) \neq 0$, then $\#(F,G) = 0$, and the conclusion holds.

Assume that there exists a point $(x_0, y_0) \in \mathcal{D}$ such that $F(x_0, y_0) = 0$. Since $F(x,y)$ is smooth and $F_y(x,y) \neq 0$, by Implicit Function Theorem, there exists $\hat{\varepsilon}_0 > 0$ such that $F(x,y) = 0$ defines a unique smooth curve: $y = f(x)$, in $(x_0 - \hat{\varepsilon}_0, x_0 + \hat{\varepsilon}_0)$. As $F'_y(x,y) \neq 0$, the function $y = f(x)$ can be extended both ways to the boundaries of the region. Let $[x_3, x_4]$ be the maximum interval in which $y = f(x)$ is defined. Then $x_1 \leq x_3 \leq x_4 \leq x_2$.

The curve $y = f(x), x \in [x_3, x_4]$ is the unique branch defined by $F(x,y) = 0$ in the region $\mathcal{D}$. Indeed, if $x_4 < x_2$, since $F'_x(x,y), F'_y(x,y) \neq 0$, so either $F'_x(x,y)F'_y(x,y) > 0$ or $F'_x(x,y)F'_y(x,y) < 0$. In the first case, then for $x \in [x_3, x_4]$, $f'(x) = -\frac{F'_x(x,f(x))}{F'_y(x,f(x))} < 0$, yielding $f(x_4) = y_1$. Therefore, $\forall (x,y) \in (x_4, x_2] \times [y_1, y_2]$ there holds

$$\begin{aligned} F(x,y) &= F(x,y) - F(x_4, y_1) \\ &= [F(x,y) - F(x,y_1)] + [F(x,y_1) - F(x_4,y_1)] \\ &= F'_y(x,\bar{y})(y - y_1) + F'_x(\bar{x}, y_1)(x - x_4) \\ &\neq 0 \end{aligned}$$

where $\bar{x}$ and $\bar{y}$ are between x, x_4 and y, y_1 respectively. The case $F'_x(x,y)F'_y(x,y) < 0$ is similar. So $\forall (x,y) \in (x_4, x_2] \times [y_1, y_2]$, $F(x,y) \neq 0$.

If $x_1 < x_3$, similarly, we can prove that $\forall (x,y) \in [x_1, x_3) \times [y_1, y_2]$, $F(x,y) \neq 0$. So $y = f(x), x \in [x_3, x_4]$ is the unique curve defined by $F(x,y) = 0$ in the region $(x,y) \in [x_1, x_2] \times [y_1, y_2]$.

Let $g(x) = G(x, f(x))$. Then we turn to study the number of roots of $g(x) = 0$ for $x \in [x_3, x_4]$. Since

$$g'(x) = \frac{J[F(x,y), G(x,y)]}{F'_y(x,y)}\Big|_{y=f(x)},$$

by Rolle's Theorem,

$$\begin{aligned}\#(F,G) &\leq 1 + \#(g'(x), 0) \\ &= 1 + \#(J[F,G](x,y), F(x,y)) \\ &= 1 + \#(F, J[F,G]).\end{aligned}$$

□

We will use Theorem. 6.12 for a pair of functions F, G in a region depending on ν.

To study the cyclicity of the family Ehh1, except for the nonlinear transition map S defined in Prop. 6.1, we need the following transition map U to be nonlinear too.

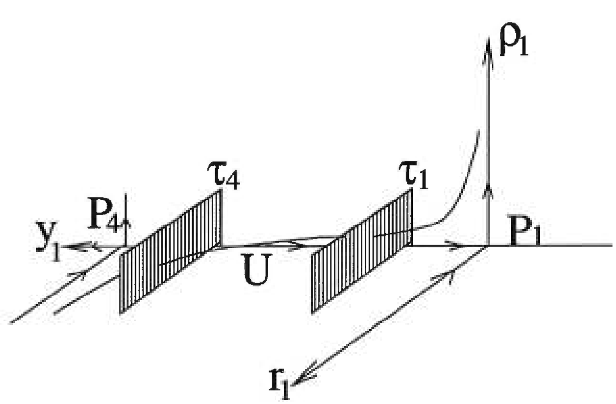


Figure 6.7: The transition map $U : \tau_1 \longrightarrow \tau_4$

Proposition 6.13. *Let* $U = (U_1, U_2) : \tau_1 \longrightarrow \tau_4$ *be the transition map in the normal form coordinates (see Fig. 6.7). If* $a \neq \frac{1}{3}, \frac{1}{4}$, *then*

$$\begin{aligned} U_1(r_1, \rho_1) &= r_1\Big[m_{141} + m_{142}r_1 + m_{143}\rho_1 + O\Big(|(r_1, \rho_1)|^2\Big)\Big] \\ U_2(r_1, \rho_1) &= \rho_1\Big[\hat{m}_{141} + \hat{m}_{142}r_1 + \hat{m}_{143}\rho_1 + O\Big(|(r_1, \rho_1)|^2\Big)\Big] \end{aligned} \tag{6.55}$$

Also $\forall (a, \bar{\mu}) \in A \times \mathbb{S}^2$,

$$\begin{aligned} \tfrac{\partial^2 U_1}{\partial r_1^2}(0,0) &= 2m_{142} = *\varepsilon_2 \\ \tfrac{\partial^2 U_2}{\partial \rho_1^2}(0,0) &= 2\hat{m}_{143} = *\bar{\mu}_3 \end{aligned} \tag{6.56}$$

where $*$ *is a positive constant. Furthermore, if* $m_{142} \neq 0$, *then* $\hat{m}_{142} \neq 0$; *if* $\hat{m}_{143} \neq 0$, *then* $m_{143} \neq 0$.

Proof. The map U is a regular transition map along an invariant line $\{r_1 = 0\} \cap \{\rho_1 = 0\}$. Since $r_1 = 0$ and $\rho_1 = 0$ are invariant, so we can write $U = (U_1, U_2)$ in the form of (6.55) and calculate the derivatives $\frac{\partial U_1}{\partial r_1}(0,0)$, $\frac{\partial^2 U_1}{\partial r_1^2}(0,0)$ in the plane $\rho_1 = 0$, calculate the derivatives $\frac{\partial U_2}{\partial \rho_1}(0,0)$, $\frac{\partial^2 U_2}{\partial \rho_1^2}(0,0)$ in the plane $r_1 = 0$.

We begin with the derivatives with respect to r_1. In the plane $\rho_1 = 0$, the system (4.6) becomes

$$\frac{dr_1}{d\bar{y}_1} = \frac{-(a + \bar{y}_1)r_1}{-(1-2a)\bar{y}_1 + 2\bar{y}_1^2 - \bar{y}_1 r_1(\varepsilon_2 + r_1\bar{h}_1) + r_1\bar{h}_2} = \frac{\overline{P}_1(r_1, \bar{y}_1)}{\overline{Q}_1(r_1, \bar{y}_1)} \tag{6.57}$$

where $\bar{h}_1$ and $\bar{h}_2$ are C^∞ functions and $\bar{h}_2 = \varepsilon_2 a + O(r_1)$.

We are going to do the calculations using system (4.6) in the original coordinates $(r_1, \rho_1, \bar{y}_1)$.

In the neighborhood of P_1, system has the form (4.6). By the normal form change (4.7), system (4.6) is in the normal form (4.9) or (4.8). In the plane $\rho_1 = 0$, if $a \neq \frac{1}{4}, \frac{1}{3}$, the section $\tau_1 = \{\tilde{y}_1 = y_0\}$ becomes

$$\tau_1' : \quad \bar{y}_1 := g_{11}(r_1) = d_{10}(y_0) + d_{11}(y_0)r_1 + O(r_1^2) \tag{6.58}$$

where

$$\begin{aligned} d_{10}(y_0) &= y_0 + O(y_0^2) \\ d_{11}(y_0) &= \varepsilon_2\Big[\tfrac{a}{1-3a} + O(y_0)\Big]. \end{aligned}$$

Similarly, in the coordinates $(r_1, \rho_1, \bar{y}_1)$, the section $\tau_4 = \{\tilde{y}_4 = -y_0\}$ becomes

$$\tau_4' : \quad \bar{y}_1 := g_{14}(r_1) = d_{40}(y_0) + d_{41}(y_0) r_1 + O(r_1^2) \tag{6.59}$$

where

$$\begin{aligned} d_{40}(y_0) &= \tfrac{1-2a}{2} - y_0 + O(y_0^2) \\ d_{41}(y_0) &= \varepsilon_2 \Big[\tfrac{8a(1+4a)}{3-4a} + O(y_0) \Big]. \end{aligned}$$

Then, for $\frac{\partial U_1}{\partial r_1}(0,0)$, by Prop. 5.2, we have

$$\begin{aligned} &\frac{\partial U_1}{\partial r_1}(0,0) \\ &= \frac{1 - \left(\frac{\overline{P}_1}{\overline{Q}_1}\right)(0, g_{11}(0)) g_{11}'(0)}{1 - \left(\frac{\overline{P}_1}{\overline{Q}_1}\right)(0, g_{14}(0)) g_{14}'(0)} \exp\left(\int_{g_{11}(0)}^{g_{14}(0)} \frac{\partial}{\partial r_1} \left(\frac{\overline{P}_1}{\overline{Q}_1} \right) \Big|_{r_1=0} d\bar{y}_1 \right) \\ &= \exp\left(\int_{g_{11}(0)}^{g_{14}(0)} \frac{a + \bar{y}_1}{\bar{y}_1 (1 - 2a - 2\bar{y}_1)} d\bar{y}_1 \right) \\ &= \exp\left(\ln \frac{\bar{y}_1^{\frac{a}{1-2a}}}{(\frac{1-2a}{2} - \bar{y}_1)^{\frac{1}{2(1-2a)}}} \Bigg|_{d_{10}(y_0)}^{d_{40}(y_0)} \right) \\ &= \frac{\left(\frac{1-2a}{2}\right)^{\frac{1+2a}{2(1-2a)}}}{y_0^{\frac{1+2a}{2(1-2a)}}} (1 + O(y_0)) \end{aligned} \tag{6.60}$$

Now we calculate $\frac{\partial^2 U_1}{\partial r_1^2}(0,0)$. By Prop. 5.2, since we are in the case $\overline{P}_1(0, \bar{y}_1) = 0$, so we have

$$\frac{\partial^2 U_1}{\partial r_1^2}(0,0) = \frac{\partial U_1}{\partial r_1}(0,0) \left[PI_1 + PI_2 + PI_3 \right] \tag{6.61}$$

where

$$\begin{aligned} PI_1 &= 2 g_{14}'(0) \frac{\partial U_1}{\partial r_1}(0,0) \left(\frac{\overline{P}_{1r_1}}{\overline{Q}_1} \right) (0, g_{14}(0)) \\ &= \frac{4\varepsilon_2 a (1+4a)}{3-4a} \left(\frac{1-2a}{2} \right)^{\frac{6a-1}{2(1-2a)}} \frac{1 + O(y_0)}{y_0^{\frac{3-2a}{2(1-2a)}}} \\ PI_2 &= -2 g_{11}'(0) \left(\frac{\overline{P}_{1r_1}}{\overline{Q}_1} \right) (0, g_{11}(0)) \\ &= -\frac{\varepsilon_2 a^2 (1 + O(y_0))}{(1-3a)(1-2a) y_0} = \frac{*\varepsilon_2}{y_0} \Big[1 + O(y_0) \Big] \end{aligned} \tag{6.62}$$

and

$$PI_3 = \int_{g_{11}(0)}^{g_{14}(0)} \left[\frac{\overline{P}''_{1r_1}}{\overline{Q}_1}(0,\bar{y}_1) - 2\frac{\overline{P}'_{1r_1}\overline{Q}'_{1r_1}}{\overline{Q}_1^2}(0,\bar{y}_1) \right] \exp\left(\int_{g_{11}(0)}^{u} \frac{\overline{P}'_{1r_1}}{\overline{Q}_1}(0,u)du \right) d\bar{y}_1$$

$$= \int_{g_{11}(0)}^{g_{14}(0)} -\frac{(a+\bar{y}_1)(\bar{y}_1-a)\varepsilon_2}{2\bar{y}_1^2(\frac{1-2a}{2}-\bar{y}_1)^2} \left(\frac{\frac{1-2a}{2}}{\frac{1-2a}{2}-\bar{y}_1} \right)^{\frac{1}{2(1-2a)}} \left(\frac{\bar{y}_1}{y_0} \right)^{\frac{a}{1-2a}} (1+O(y_0))dy_1$$

$$= -\frac{*\varepsilon_2(1+O(y_0))}{y_0^{\frac{a}{1-2a}}} \int_{d_{10}(y_0)}^{d_{40}(y_0)} \frac{(a+\bar{y}_1)(\bar{y}_1-a)}{\bar{y}_1^{\frac{2-5a}{1-2a}}(\frac{1-2a}{2}-\bar{y}_1)^{\frac{5-8a}{2(1-2a)}}} d\bar{y}_1 \tag{6.63}$$

Since

$$\lim_{y_0\to 0} \frac{\displaystyle\int_{d_{10}(y_0)}^{d_{40}(y_0)} \frac{(a+\bar{y}_1)(\bar{y}_1-a)}{\bar{y}_1^{\frac{2-5a}{1-2a}}(\frac{1-2a}{2}-\bar{y}_1)^{\frac{5-8a}{2(1-2a)}}} d\bar{y}_1}{\frac{1}{y_0^{\frac{3-4a}{2(1-2a)}}}}$$

$$= -\lim_{y_0\to 0} \left\{ \frac{2(1-2a)}{4a-3} y_0^{\frac{5-8a}{2(1-2a)}} \left[-\frac{2a(1+4a)}{\left(\frac{1-2a}{2}\right)^{\frac{2-5a}{1-2a}} y_0^{\frac{5-8a}{2(1-2a)}}} - \frac{a^2}{\left(\frac{1-2a}{2}\right)^{\frac{5-8a}{2(1-2a)}} y_0^{\frac{2-5a}{1-2a}}} \right] \right\}$$

$$= -\frac{8a(1+4a)}{3-4a} \left(\frac{2}{1-2a} \right)^{\frac{1-3a}{1-2a}} \lim_{y_0\to 0} \left[1+O(y_0)+O\left(y_0^{\frac{1+2a}{2(1-2a)}} \right) \right]$$
$$= -\frac{8a(1+4a)}{3-4a} \left(\frac{1-2a}{2} \right)^{\frac{3a-1}{1-2a}}$$

So,

$$\int_{d_{10}(y_0)}^{d_{40}(y_0)} \frac{(a+\bar{y}_1)(\bar{y}_1-a)}{\bar{y}_1^{\frac{2-5a}{1-2a}}(\frac{1-2a}{2}-\bar{y}_1)^{\frac{5-8a}{2(1-2a)}}} d\bar{y}_1 = \frac{*\left[1+O(y_0)+O\left(y_0^{\frac{1+2a}{2(1-2a)}} \right) \right]}{y_0^{\frac{3-4a}{2(1-2a)}}} \tag{6.64}$$

Hence, by (6.63) and (6.64), we have

$$PI_3 = \frac{*\varepsilon_2 \left[1+O(y_0)+O\left(y_0^{\frac{1+2a}{2(1-2a)}} \right) \right]}{y_0^{\frac{3-2a}{2(1-2a)}}} \tag{6.65}$$

Therefore, it follows from (6.61), (6.62) and (6.65) that we have

$$\begin{aligned}
&\frac{\partial^2 U_1}{\partial r_1^2}(0,0)\\
&= \varepsilon_2 \frac{\partial U_1}{\partial r_1}(0,0)\left[\frac{4a(1+4a)}{3-4a}\left(\frac{1-2a}{2}\right)^{\frac{6a-1}{2(1-2a)}}\frac{1+O(y_0)}{y_0^{\frac{3-2a}{2(1-2a)}}}\right.\\
&\quad\left.+\frac{*(1+O(y_0))}{y_0}+\frac{*}{y_0^{\frac{3-2a}{2(1-2a)}}}\left[1+O(y_0)+O\left(y_0^{\frac{1+2a}{2(1-2a)}}\right)\right]\right]\\
&= \frac{\partial U_1}{\partial r_1}(0,0)\frac{4\varepsilon_2 a(1+4a)}{3-4a}\left(\frac{1-2a}{2}\right)^{\frac{6a-1}{2(1-2a)}}\frac{1+O\left(y_0^{\frac{1+2a}{2(1-2a)}}\right)+O\left(y_0^{\frac{2a}{2(1-2a)}}\right)}{y_0^{\frac{3-2a}{2(1-2a)}}}
\end{aligned} \tag{6.66}$$

So, if we take $y_0 > 0$ small, then by (6.66), $\frac{\partial^2 U_1}{\partial r_1^2}(0,0) = *\varepsilon_2$.

Now we prove that $\frac{\partial^2 U_2}{\partial \rho_1^2}(0,0) \neq 0$. In the plane $r_1 = 0$, the system (4.6) becomes

$$\frac{d\rho_1}{d\bar{y}_1} = \frac{(a+\bar{y}_1+\bar{\mu}_2\rho_1^2)\rho_1}{-(1-2a)\bar{y}_1+2\bar{y}_1^2+\bar{\mu}_3\rho_1\bar{y}_1+2\bar{\mu}_2\rho_2^2\bar{y}_1+\bar{\mu}_1\rho_1^3} = \frac{\widehat{P}_1(\rho_1,\bar{y}_1)}{\widehat{Q}_1(\rho_1,\bar{y}_1)}. \tag{6.67}$$

We still use system (4.6) in the original coordinates $(r_1, \rho_1, \bar{y}_1)$ to do the calculations.

In the plane $r_1 = 0$, if $a \neq \frac{1}{4}, \frac{1}{3}$, the section $\tau_1 = \{\tilde{y}_1 = y_0\}$ becomes

$$\hat{\tau}_1: \quad \bar{y}_1 := \hat{g}_{11}(r_1) = \hat{d}_{10}(y_0) + \hat{d}_{11}(y_0)\rho_1 + O(\rho_1^2) \tag{6.68}$$

where

$$\begin{aligned}
\hat{d}_{10}(y_0) &= y_0 + O(y_0^2)\\
\hat{d}_{11}(y_0) &= \tfrac{\bar{\mu}_3}{a} + O(y_0).
\end{aligned}$$

Similarly, in the coordinates $(r_1, \rho_1, \bar{y}_1)$, on $r_1 = 0$ the section $\tau_4 = \{\tilde{y}_4 = -y_0\}$ becomes

$$\hat{\tau}_4': \quad \bar{y}_1 = \hat{g}_{14}(\rho_1) = \hat{d}_{40}(y_0) + \hat{d}_{41}(y_0)\rho_1 + O(\rho_1^2) \tag{6.69}$$

where

$$\begin{aligned}
\hat{d}_{40}(y_0) &= \tfrac{1-2a}{2} - y_0 + O(y_0^2)\\
\hat{d}_{41}(y_0) &= -\tfrac{1-2a}{1-4a}\mu_3 + O(y_0).
\end{aligned}$$

Then, for $\frac{\partial U_2}{\partial \rho_1}(0,0)$, by Prop. 5.2, we have

$$\begin{aligned}\frac{\partial U_2}{\partial \rho_1}(0,0) &= \exp\Big(\int_{\hat{g}_{11}(0)}^{\hat{g}_{14}(0)} -\frac{a+\bar{y}_1}{\bar{y}_1(1-2a-2\bar{y}_1)}d\bar{y}_1\Big)\\ &= \exp\left(-\ln\frac{\bar{y}_1^{\frac{a}{1-2a}}}{(\frac{1-2a}{2}-\bar{y}_1)^{\frac{1}{2(1-2a)}}}\Big|_{\hat{d}_{10}(y_0)}^{\hat{d}_{40}(y_0)}\right)\\ &= \frac{y_0^{\frac{1+2a}{2(1-2a)}}(1+O(y_0))}{\left(\frac{1-2a}{2}\right)^{\frac{1+2a}{2(1-2a)}}}.\end{aligned} \tag{6.70}$$

Now we calculate $\frac{\partial^2 U_2}{\partial \rho_1^2}(0,0)$. By Prop. 5.2, note that $\widehat{P}_1(0,\bar{y}_1)=0$, so we have

$$\frac{\partial^2 U_2}{\partial \rho_1^2}(0,0) = \frac{\partial U_2}{\partial \rho_1}(0,0)\left[\widehat{PI}_1+\widehat{PI}_2+\widehat{PI}_3\right] \tag{6.71}$$

where

$$\begin{aligned}\widehat{PI}_1 &= 2\hat{g}_{14}'(0)\frac{\partial U_2}{\partial \rho_1}(0,0)\left(\frac{\widehat{P}_{1\rho_1}}{\widehat{Q}_1}\right)(0,\hat{g}_{14}(0))\frac{\bar{\mu}_3 y_0^{\frac{6a-1}{2(1-2a)}}(1+O(y_0))}{(1-4a)\left(\frac{1-2a}{2}\right)^{\frac{6a-1}{2(1-2a)}}}\\ \widehat{PI}_2 &= -2\hat{g}_{11}'(0)\left(\frac{\widehat{P}_{1\rho_1}}{\widehat{Q}_1}\right)(0,\hat{g}_{11}(0)) = \frac{2\bar{\mu}_3(1+O(y_0))}{(1-2a)y_0}\end{aligned} \tag{6.72}$$

and

$$\begin{aligned}\widehat{PI}_3 &= \int_{\hat{g}_{11}(0)}^{\hat{g}_{14}(0)}\left[\frac{\widehat{P}_{1\rho_1}''}{\widehat{Q}_1}(0,\bar{y}_1)-2\frac{\widehat{P}_{1\rho_1}'\widehat{Q}_{1\rho_1}'}{\widehat{Q}_1^2}(0,\bar{y}_1)\right]\exp\left(\int_{\hat{g}_{11}(0)}^{u}\frac{\widehat{P}_{1\rho_1}'}{\widehat{Q}_1}(0,u)du\right)d\bar{y}_1\\ &= \int_{\hat{g}_{11}(0)}^{\hat{g}_{14}(0)} -\frac{\bar{\mu}_3(a+\bar{y}_1)(1+O(y_0))}{2\bar{y}_1^2(\frac{1-2a}{2}-\bar{y}_1)^2}\left(\frac{\frac{1-2a}{2}-\bar{y}_1}{\frac{1-2a}{2}}\right)^{\frac{1}{2(1-2a)}}\left(\frac{y_0}{\bar{y}_1}\right)^{\frac{a}{1-2a}}d\bar{y}_1\\ &= -\frac{\bar{\mu}_3 y_0^{\frac{a}{1-2a}}(1+O(y_0))}{2(\frac{1-2a}{2})^{\frac{1}{2(1-2a)}}}\int_{\hat{d}_{10}(y_0)}^{\hat{d}_{40}(y_0)}\frac{a+\bar{y}_1}{\bar{y}_1^{\frac{2-3a}{1-2a}}(\frac{1-2a}{2}-\bar{y}_1)^{\frac{3-8a}{2(1-2a)}}}d\bar{y}_1.\end{aligned} \tag{6.73}$$

Since

$$\lim_{y_0\to 0}\frac{\displaystyle\int_{\hat{d}_{10}(y_0)}^{\hat{d}_{40}(y_0)}\frac{a+\bar{y}_1}{\bar{y}_1^{\frac{2-3a}{1-2a}}(\frac{1-2a}{2}-\bar{y}_1)^{\frac{3-8a}{2(1-2a)}}}d\bar{y}_1}{\frac{1}{y_0^{\frac{1-a}{1-2a}}}}$$

$$= \lim_{y_0\to 0}\frac{-\frac{1}{2(\frac{1-2a}{2})^{\frac{2-3a}{1-2a}}y_0^{\frac{3-8a}{2(1-2a)}}}-\frac{a}{(\frac{1-2a}{2})^{\frac{3-8a}{2(1-2a)}}y_0^{\frac{2-3a}{1-2a}}}}{-\frac{1-a}{1-2a}\frac{1}{y_0^{\frac{2-3a}{1-2a}}}} = \frac{a(1-2a)}{(1-a)(\frac{1-2a}{2})^{\frac{3-8a}{2(1-2a)}}}.$$

So,

$$\int_{\hat{d}_{10}(y_0)}^{\hat{d}_{40}(y_0)} \frac{a+\bar{y}_1}{\bar{y}_1^{\frac{2-3a}{1-2a}}\left(\frac{1-2a}{2}-\bar{y}_1\right)^{\frac{3-8a}{2(1-2a)}}} d\bar{y}_1 = \frac{2a\left[1+O\left(y_0^{\frac{1+2a}{2(1-2a)}}\right)\right]}{(1-a)\left(\frac{1-2a}{2}\right)^{\frac{1-4a}{2(1-2a)}} y_0^{\frac{1-a}{1-2a}}}. \tag{6.74}$$

By (6.73) and (6.74), we have

$$\widehat{PI}_3 = \frac{2a\bar{\mu}_3}{(1-2a)y_0}\left[1+O(y_0)+O\left(y_0^{\frac{1+2a}{2(1-2a)}}\right)\right]. \tag{6.75}$$

It follows from (6.71), (6.72) and (6.75) that we have

$$\begin{aligned} &\frac{\partial^2 U_2}{\partial \rho_1^2}(0,0) \\ &= \frac{\partial U_2}{\partial \rho_1}(0,0)\left[\frac{\bar{\mu}_3 y_0^{\frac{6a-1}{2(1-2a)}}(1+O(y_0))}{(1-4a)\left(\frac{1-2a}{2}\right)^{\frac{6a-1}{2(1-2a)}}} + \frac{2\bar{\mu}_3(1+O(y_0))}{(1-2a)y_0}\right. \\ &\qquad \left. + \frac{2a\bar{\mu}_3[1+O(y_0)+O(y_0^{\frac{1+2a}{2(1-2a)}})]}{(1-2a)y_0}\right] \\ &= \frac{(1+2a)\bar{\mu}_3}{(1-2a)y_0}\left[1+O(y_0)+O\left(y_0^{\frac{1+2a}{2(1-2a)}}\right)\right]. \end{aligned} \tag{6.76}$$

So, if we take $y_0 > 0$ small, then by (6.76), $\frac{\partial^2 U_2}{\partial \rho_1^2}(0,0) = *\bar{\mu}_3$.

By the invariance of $\nu = r_1\rho_1 = r_2\rho_2$ and (6.55), we have

$$\begin{aligned} r_2\rho_2 &= r_1\rho_1[m_{141}+m_{142}r_1+m_{143}\rho_1+O(|(r_1,\rho_1)|^2)] \\ &\quad [\hat{m}_{141}+\hat{m}_{142}r_1+\hat{m}_{143}\rho_1+O(|(r_1,\rho_1)|^2)]. \end{aligned} \tag{6.77}$$

Equating the coefficients of terms of r_1 and ρ_1 respectively on both sides of (6.77), we have

$$\begin{cases} m_{141}\hat{m}_{142}+\hat{m}_{141}m_{142}=0 \\ m_{141}\hat{m}_{143}+\hat{m}_{141}m_{143}=0. \end{cases} \tag{6.78}$$

Since $m_{141}\hat{m}_{141} = 1$, so by (6.78) we have that if $m_{142} \neq 0$, then $\hat{m}_{142} \neq 0$; if $\hat{m}_{143} \neq 0$, then $m_{143} \neq 0$. □

Corollary 6.14. *For the transition map $V : \tau_2 \longrightarrow \tau_3$ in the normal form coordinates, if $a \neq \frac{1}{3}, \frac{1}{4}$, then*

$$\begin{aligned} \frac{\partial V_1}{\partial r_2}(0,0) &= \frac{\partial U_1}{\partial r_1}(0,0) \\ \frac{\partial^2 V_1}{\partial r_2^2}(0,0) &= \frac{\partial^2 U_1}{\partial r_1^2}(0,0) \\ \frac{\partial V_2}{\partial \rho_2}(0,0) &= \frac{\partial U_2}{\partial \rho_1}(0,0) \\ \frac{\partial^2 V_2}{\partial \rho_2^2}(0,0) &= \frac{\partial^2 U_2}{\partial \rho_1^2}(0,0). \end{aligned} \tag{6.79}$$

Proof. Just note that in the plane $\rho_2 = 0$, the system is the same as system (6.57) except the term $r_2\bar{h}_1$ will have different sign which does not influence the first and second derivatives. □

6.3.3 Lower boundary hh-graphics of elliptic type

In this section, we will first prove that for a generic hh-graphic through a triple nilpotent elliptic point of codimension 3, all the corresponding 12 lower boundary hh-graphics have finite cyclicity.

Among the 12 lower boundary graphics, Ehh1c, Ehh2e and Ehh3e pass through both passages P_2P_3 and P_1P_4 along the equator (Remark 6.7), and require a special treatment. Indeed in general an explicit formula does not exist for the inverse of the second type Dulac map. We will replace the study of zeroes of the displacement map by the study of the zeroes of a system of two variables using generalized Rolle's Theorem.

To prove the finite cyclicity of the graphic Ehh1, we give the following lemma.

Lemma 6.15. *For the system in the neighborhood of P_3, if $\sigma_3 = \frac{1}{n}$, $n \in \mathbb{N}$, then the first saddle quantity is nonzero for the 2-dimensional system on $\rho = 0$.*

Proof. By (3.8) with $i = 2$, after a translation $y = y_2 - \frac{1-2a}{2}$, system on $\rho = 0$ in the neighborhood of P_4 can be written as

$$\begin{cases} \dot{r} &= r \\ \dot{y} &= -\sigma_3 y - \dfrac{8ay^2}{1+2y} + \varepsilon_2 r + O(r^2) \end{cases} \tag{6.80}$$

where $\sigma_3 = 2(1-2a)$. In the case $\sigma_3 = \frac{1}{n}$, $n \in \mathbb{N}$, we have $a = \frac{2n-1}{4n}$.

By the linear transformation $y = z + \frac{\varepsilon_2}{3-4a} r$, system (6.80) becomes

$$\begin{cases} \dot{r} &= r \\ \dot{z} &= -\dfrac{1}{n} z - \dfrac{8az^2}{1+2z} + \dfrac{\overline{\varepsilon}_2 z(1+z)}{(1+2z)^2} r + O(r^2) \end{cases} \tag{6.81}$$

where $\overline{\varepsilon}_2 = \frac{16a\varepsilon_2}{4a-3}$. For convenience, we still keep a in the higher order terms.

By normal from theory (see for instance [GH83], [IY91]), we will obtain the normal form of (6.80):

$$\begin{cases} \dot{r} &= r \\ \dot{Z} &= -\frac{1}{n} Z + \sum_{i=1}^{k} \beta_{i+1} \left(r Z^n\right)^i Z. \end{cases} \tag{6.82}$$

where β_2, the coefficient of the term rZ^{n+1}, is the first saddle quantity.

In order to obtain the normal form (6.82) from system (6.81), we rewrite system (6.81) as

$$\begin{cases} \dot{r} &= r \\ \dot{z} &= -\frac{1}{n} z - 8a \sum_{i=0}^{\infty} (-2)^i z^{i+2} + \overline{\varepsilon}_2 \Big[z - \sum_{i=0}^{\infty} (-2)^i (i+3) z^{i+2} \Big] r + O(r^2) \end{cases} \tag{6.83}$$

To prove $\beta_2 \neq 0$, we are going to apply the normal form theory to system (6.83). The proof goes in two steps. For any $n \in \mathbb{N}$, we will first kill the terms $rz, rz^2, \cdots, rz^n$. In the second step, we get rid of the nonresonant part $8a \sum_{i=0}^{\infty} (-2)^i z^{i+2}$.

(1). Kill the terms $rz, rz^2, \cdots, rz^n$ successively

(1.1). Kill the term rz first

Let $z = z_1 + rb_1 z_1$. Then by (6.83), we obtain the equation of z_1:

$$\dot{z}_1 = -\frac{1}{n} z_1 + (\bar{\varepsilon}_2 - b_1) r z_1 - 8a \sum_{j=0}^{\infty} (-2)^j z_1^{j+2} - r \sum_{j=0}^{\infty} (-2)^j c_{1j} z_1^{j+2} + O(r^2)$$

where $c_{1j} = 8ab_1(j+1) + \bar{\varepsilon}_2(j+3)$.

Let $b_1 = \bar{\varepsilon}_2$. So we have

$$\dot{z}_1 = -\frac{1}{n} z_1 - 8a \sum_{j=0}^{\infty} (-2)^j z_1^{j+2} - r \sum_{j=0}^{\infty} (-2)^j c_{1j} z_1^{j+2} + O(r^2)$$

where for c_{1j} with $j \in \mathbb{N}$, we have $c_{1j} = \varepsilon_2[8a(j+1) + (j+3)] \neq 0$ and all the coefficients c_{1j} have the same sign as of $\bar{\varepsilon}_2$.

Note that if $n = 1$, the coefficient of the resonant term $r z_1^2$ satisfies $c_{10} \neq 0$. Then the first step stops here.

(1.2). Let $n \geq 2$. Assume that by $n-2$ steps of near-identity transformation of the form $z_{k-1} = z_k + b_k r z_k^k$, $k = 2, \cdots, n-1$, we get rid of the terms $r z_1^2, r z_1^3, \cdots, r z_1^{n-1}$, and obtain the equations of z_n

$$\begin{cases} \dot{r} &= r \\ \dot{z}_n &= -\frac{1}{n} z_n - 8a \sum_{j=0}^{\infty} (-2)^j z_n^{j+2} - r \sum_{j=n-2}^{\infty} (-2)^j c_{nj} z_n^{j+2} + O(r^2). \end{cases} \tag{6.84}$$

where for $j \geq n-2$, $c_{nj} \neq 0$ and they have the same sign as of ε_2.

(1.3). Kill the nonresonant term $r z_n^n$ in (6.84)

Let $z_n = w + b_n r w^n$, then the equation of w becomes

$$\begin{aligned} &(1 + n b_n r w^{n-1}) \dot{w} + b_n r w^n \\ &= -\tfrac{1}{n}(w + b_n r w^n) - 8a \sum_{j=0}^{\infty} (-2)^j (w + b_n r w^n)^{j+2} \\ &- r \sum_{j=n-2}^{\infty} (-2)^j c_{nj} (w + b_n r w^n)^{j+2} + O(r^2) \end{aligned}$$

or

$$\begin{aligned} &(1 + n b_n r w^{n-1}) \dot{w} + b_n r w^n \\ &= -\tfrac{1}{n}(w + b_n r w^n) - 8a \sum_{j=0}^{\infty} (-2)^j [w^{j+2} + (j+2) b_n r w^{j+n+1}] \\ &- r \sum_{j=n-2}^{\infty} (-2)^j c_{nj} w^{j+2} + O(r^2). \end{aligned}$$

Hence w satisfies

$$\begin{aligned}\dot{w} &= -\tfrac{1}{n}w + [-\tfrac{b_n}{n} - (-2)^{n-2}c_{n,n-2}]rw^n \\ &\quad -8a\sum_{j=0}^{\infty}(-2)^j w^{j+2} - c_{n+1,n}rw^{n+1} + rO(w^{n+2}) + O(r^2)\end{aligned} \tag{6.85}$$

where $c_{n+1,n} = -[16ab_n + (-2)^{n-1}c_{n,n-1}]$.

Let $b_n = -nc_{n,n-2}(-2)^{n-2}$, then we get rid of the term rw^n in (6.85) and obtain

$$\begin{cases}\dot{r} &= r \\ \dot{w} &= -\tfrac{1}{n}w - 8a\sum_{j=0}^{\infty}(-2)^j w^{j+2} - c_{n+1,n}rw^{n+1} + rO(w^{n+2}) + O(r^2)\end{cases} \tag{6.86}$$

where

$$\begin{aligned}c_{n+1,n} &= -[16ab_n + (-2)^{n-1}c_{n,n-1}] \\ &= -[-16anc_{n,n-2}(-2)^{n-2} + (-2)^{n-1}c_{n,n-1}] \\ &= (-2)^{n-2}[16anc_{n,n-2} + 2c_{n,n-1}] \\ &\neq 0.\end{aligned}$$

Therefore, we bring system (6.83) into the form

$$\begin{cases}\dot{r} &= r \\ \dot{w} &= -\tfrac{1}{n}w - \tfrac{8w^2}{1+2w} - c_{n+1,n}rw^{n+1} + rO(w^{n+2}) + O(r^2).\end{cases} \tag{6.87}$$

(2). Remove the nonresonant part $-\frac{8w^2}{1+2w}$

By

$$\frac{dw}{-\frac{w}{n} - \frac{8aw^2}{1+2w}} = \frac{dZ}{-\frac{Z}{n}}$$

or equivalently

$$\frac{n(1+2w)dw}{(1+4nw)w} = \frac{ndZ}{Z}$$

we can solve for Z:

$$Z = w(1+4nw)^{\frac{1-2n}{2n}}. \tag{6.88}$$

By (6.88), obviously we have the relation

$$w = Z + O(Z^2). \tag{6.89}$$

So if we make the change of coordinate (6.88), by (6.89), we bring system (6.87) into

$$\begin{cases} \dot{r} &= r \\ \dot{Z} &= -\frac{1}{n}Z - c_{n+1,n} r Z^{n+1} + rO(Z^{n+2}) + O(r^2). \end{cases} \tag{6.90}$$

Note that in removing the higher order nonresonant terms in (6.90), the term $-c_{n+1,n} r Z^{n+1}$ will be invariant. Hence we get that the first saddle quantity $\beta_2 = -c_{n+1,n} \neq 0$. □

Theorem 6.16. *The generic graphic Ehh1c through a nilpotent elliptic point of codimension 3 has finite cyclicity. For the generic graphics Ehh2e and Ehh3e, we have $Cycl(Ehh2e, Ehh3e) \leq 2$.*

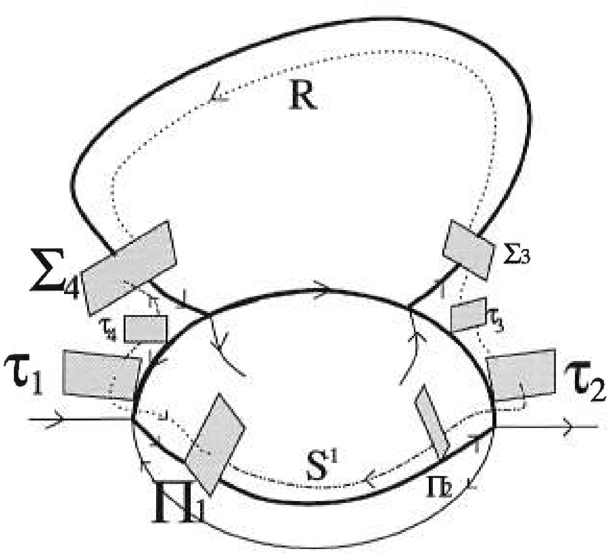


Figure 6.8: The displacement maps defined on τ_1 and τ_2

Proof. For any $(a_0, \bar{\mu}_0) \in A \times V_I$, we have three families of hh graphics with lower boundary graphics Ehh1c, Ehh2e and Ehh3e respectively. As shown in Fig. 6.8, take transversal sections Π_i, Σ_{i+2} $(i = 1, 2)$ and τ_j $(j = 1, 2, 3, 4)$ in the charts P.R. 1, 2, 3 and P.R.4 respectively (sections were defined in 5.7). On the section τ_2, the coordinates are (r_2, ρ_2) with $r_2\rho_2 = \nu, \nu > 0$ small. We want to cover a domain $|r_2| < \varepsilon, |\rho_2| < \varepsilon$, where $\varepsilon > 0$ small. Similar to what we have done in Theorem 6.6 for graphic Ehp1 in (6.20), on τ_2, let

$$r_2 = \nu^{1-d}, \qquad \rho_2 = \nu^d.$$

We then parametrize the section τ_2 using the coordinates $(\nu, d) \in (0, \varepsilon^2) \times \mathcal{I}_\nu$, where $\mathcal{I}_\nu = (\frac{\ln \varepsilon}{\ln \nu}, \frac{\ln u\varepsilon}{\ln \nu}) \subset (0, 1)$ and $u \in (0, 1)$. Similarly, on the section τ_1, let

$$r_1 = \nu^{1-c}, \qquad \rho_1 = \nu^c.$$

We use coordinates $(\nu, c) \in (0, \varepsilon^2) \times \mathcal{I}_\nu$.

To study the cyclicity of the lower boundary graphics, we are going to study the displacement maps defined on the sections τ_1 and τ_2 respectively with images in Π_1 and Σ_4, namely, by using Theorem 6.12 in a region depending on ν with $\nu > 0$ sufficiently small, we will study the number of roots of the system

$$\begin{cases} T_1(\nu, c) = T_2(\nu, d) \\ T_4(\nu, c) = T_3(\nu, d). \end{cases} \tag{6.91}$$

for $(a, \bar{\mu}) \in A_0 \times V_{I_i}$ $(i = 1, 2, 3)$ with $(c, d) \in \mathcal{I}_\nu \times \mathcal{I}_\nu$ and $\nu, \varepsilon > 0$ sufficiently small.

The proof will go in several steps.

(1). Developing the transition maps T_i $(i = 1, 2, 3, 4)$.

(1.1). The transition map T_1. The map $T_1 : \tau_1 \longrightarrow \Pi_1$ is the second type Dulac map near P_1. By Theorem 4.14, for $r = \nu^c$ and $\rho = \nu^{1-c}$, it has an expression similar to (6.21) with $\bar{\sigma}_3 = \bar{\sigma}_1$. Hence

$$\begin{cases} T_{11}(\nu, c) & = \nu \\ T_{12}(\nu, c) & = \eta_1(\nu, \omega(\dfrac{\nu^c}{\rho_0}, \alpha_1)) + \nu^{\bar{\sigma}_1 c}\Big[l_1 + \theta_1(\nu, \nu^c, \omega(\dfrac{\nu^c}{\rho_0}, -\alpha_1))\Big] \end{cases} \tag{6.92}$$

where $l_1 = \frac{y_0}{\rho_0^{\bar{\sigma}_1}} > 0$; $\eta_1(\nu, \omega(\frac{\nu^c}{\rho_0}, \alpha_1)) = \frac{\kappa_1}{\rho_0^{p_1}} \nu^{p_1}\ \omega(\frac{\nu^c}{\rho_0}, \alpha_1)$ and $\theta_1(\nu, \nu^c, \omega(\frac{\nu^c}{\rho_0}, -\alpha_1))$ is C^∞ and satisfies the same properties as of θ_2 in (6.22).

(1.2). Transition map T_4. The transition map

$$T_4 : \ \tau_1 \longrightarrow \Sigma_4$$
$$(\nu, c) \longrightarrow (T_{41}, T_{42})$$

can be factorized as

$$T_4 = \Theta_4 \circ U$$

where

- $U : \tau_1 \longrightarrow \tau_4$ is the regular transition map defined in Prop. 6.13. In the coordinate c on τ_1, the first component U_1 of the map U can be written as

$$\begin{aligned} r_4 &= U_1(\nu^{1-c}, \nu^c) \\ &= \nu^{1-c}\Big[m_{141} + m_{142}(\nu)\nu^{1-c} + m_{143}(\nu)\nu^c + O(\nu^{2(1-c)}, \nu^{2c})\Big] \end{aligned} \tag{6.93}$$

  where by Prop. 6.13, $m_{141}(0) \neq 0$, $m_{142}(0) = *\varepsilon_2$ and $m_{143}(0) = *\bar{\mu}_{30}$.

- $\Theta_4 : \tau_4 \longrightarrow \Sigma_4$, the second type Dulac map near P_4 which satisfies Theorem 4.14 with $\sigma = \sigma_3$. By (6.93), we have

$$r_4 = \nu^{1-c}(m_{141} + O(\nu^c, \nu^{1-c})). \tag{6.94}$$

  Let $m_4 = \frac{m_{141}}{r_0}$. Using (6.94) we have

$$\begin{aligned} \omega(\tfrac{r_4}{r_0}, \beta_1) &= \omega(\frac{m_{14}}{r_0}\nu^{1-c}(1 + O(\nu^c, \nu^{1-c})), \beta_1) \\ &= \omega(m_4\nu^{1-c}, \beta_1) + O(\nu^{c-(1-c)\beta_1}, \nu^{(1-c)(1-\beta_1)}). \end{aligned} \tag{6.95}$$

So by (6.93), (4.29) and (6.95), for the second component of the transition map T_4, we have

$$\begin{cases} T_{42}(\nu, c) & = \eta_4(\nu, \omega(r_4, \beta_1)) \\ & +\nu^{\bar{\sigma}_3(1-c)}\Big[m_{41}(\nu) + m_{42}\nu^{1-c} + m_{43}\nu^c + O(\nu^{2(1-c)}, \nu^{2c}) \\ & +\theta_{41}(\nu, \nu^{1-c}, \omega(m_4\nu^{1-c}, -\beta_1))\Big] \end{cases} \tag{6.96}$$

where

$$m_{41} = \tfrac{y_0}{r_0^{\bar{\sigma}_3}}(m_{141}(\nu))^{\bar{\sigma}_3}$$
$$m_{42}(0) = *m_{142}(0) = K_{142}\varepsilon_2$$
$$m_{43}(0) = *m_{143}(0) = K_{143}\bar{\mu}_{30}$$

where K_{142}, K_{143} are nonzero constants; in $\eta_4(\nu, \omega(\frac{r_4}{r_0}, \beta_1))$ we still keep r_4 as function of c in (6.93); also $\theta_{41}(\nu, \nu^{1-c},\ \omega(m_4\nu^{1-c}, -\beta_1))$ is C^k and satisfies

$$\begin{aligned} \theta_{41} &= O\Big(\nu^{p_3(1-c)}\omega(m_4\nu^{1-c}, \beta_1)\Big) \\ \tfrac{\partial^i\theta_{41}}{\partial c^i} &= O\Big(\nu^{p_3(1-c)}\omega(m_4\nu^{1-c}, \beta_1)(\ln\nu)^i\Big) \qquad i \geq 1. \end{aligned} \tag{6.97}$$

(1.3). Transition map T_2 The transition map

$$\begin{aligned} T_2 : \ & \tau_2 \longrightarrow \Pi_1 \\ & (\nu, d) \longrightarrow (T_{21}, T_{22}) \end{aligned}$$

which can be factored as

$$T_2 = S^{-1} \circ \Theta_2(r_2, \rho_2)$$

where

- $\Theta_2 : \tau_2 \longrightarrow \Pi_2$, the second type of Dulac map near P_2, using the coordinates (ν, d) on the section τ_2, is given in (6.21). $\eta_2(\nu, \omega(\frac{\nu^d}{\rho_0}, \alpha_1)) = \frac{\kappa_1}{\rho_0^{p_1}}\nu^{p_1}\omega(\frac{\nu^d}{\rho_0}, \alpha_1)$. Also $\theta_2(\nu, d, \omega(\frac{\nu^d}{\rho_0}, -\alpha_1))$ is C^∞ and satisfies the same properties as of θ_1 in (6.22).

- $S^{-1} : \Pi_2 \longrightarrow \Pi_1$ is a C^k regular transition map defined in Prop. 6.1. We can write its second component as

$$S_2^{-1}(\nu, \tilde{y}_2) = m_{210}(\nu) + m_{211}(\nu)\tilde{y}_2 + m_{212}(\nu)\tilde{y}^2 + O(\tilde{y}_2^3) \tag{6.98}$$

where $m_{210}(0) = 0$, $m_{211}(0) \neq 0$ and by Prop. 6.1, for Ehh1c, $m_{212}(0) = *\bar{\mu}_{30}$, while for Ehh2e(resp. Ehh3e) $m_{211}(0)$ can be sufficiently small (resp. sufficiently large) .

By (6.98), for the transition map T_2, we have

$$\begin{cases} T_{21}(\nu, d) &= \nu \\ T_{22}(\nu, d) &= m_{20}(\nu, \omega(\frac{\nu^d}{\rho_0}, \alpha_1)) \\ & +\nu^{\bar{\sigma}_1 d}\Big[m_{21}(\nu, \omega(\frac{\nu^d}{\rho_0}, \alpha_1)) + m_{22}(\nu, \omega(\frac{\nu^d}{\rho_0}, \alpha_1))\nu^{\bar{\sigma}_1 d} \\ & +\theta_{211}(\nu, \nu^d, \omega(\frac{\nu^d}{\rho_0}, -\alpha_1))\Big] \end{cases} \tag{6.99}$$

where

$$\begin{aligned} m_{20}(\nu, \omega(\tfrac{\nu^d}{\rho_0}, \alpha_1)) &= m_{210}(\nu) + \kappa_1[\tfrac{m_{211}(\nu)}{\rho_0^{\bar{\sigma}_1}}\nu^{p_1}\omega(\tfrac{\nu^d}{\rho_0}, \alpha_1)) + O(\nu^{2p_1}\omega^2(\tfrac{\nu^d}{\rho_0}, \alpha_1))] \\ m_{21}(\nu, \omega(\tfrac{\nu^d}{\rho_0}, \alpha_1)) &= m_{211}(\nu)l_1 + \kappa_1 O(\nu^{p_1}\omega(\tfrac{\nu^d}{\rho_0}, \alpha_1)) \\ m_{22}(\nu, \omega(\tfrac{\nu^d}{\rho_0}, \alpha_1)) &= m_{212}(\nu)l_1^2 + \kappa_1 O(\nu^{p_1}\omega(\tfrac{\nu^d}{\rho_0}, \alpha_1)), \end{aligned}$$

and θ_{211} is C^k and has the same properties as of θ_1 in (6.22).

(1.4). The transition map T_3 For the transition map $T_3 : \tau_2 \longrightarrow \Sigma_4$, it can be factorized as

$$T_3 = R \circ \Theta_3 \circ V$$

where

- $V : \tau_2 \longrightarrow \tau_3$, the regular transition map defined in Coro. 6.14. Using the coordinates (ν, d) on the section τ_3, we have

$$\begin{aligned} r_3 &= V_1(\nu^{1-d}, \nu^d) \\ &= \nu^{1-d}\Big[\overline{m}_{141} + \overline{m}_{142}(\nu)\nu^{1-d} + \overline{m}_{143}(\nu)\nu^d + O(\nu^{2(1-d)}, \nu^{2d})\Big] \end{aligned} \tag{6.100}$$

  where by Coro. 6.14 we have $\overline{m}_{14i}(0) = m_{14i}(0)$ $(i = 1, 2, 3)$.

- $\Theta_3 : \tau_3 \longrightarrow \sigma_3$, the second type Dulac map near P_3 which satisfies Theorem 4.14 with $\sigma_3(a_0) = 2(1 - 2a_0)$.

- The regular transition map $R : \Sigma_3 \longrightarrow \Sigma_4$ is C^k and can be written as

$$\begin{cases} R_1(\nu, \tilde{y}_3) & = \nu \\ R_2(\nu, \tilde{y}_3) & = m_{340}(\nu) + \sum_{j=1}^{N_2} m_{34j}(\nu)\tilde{y}_3^j + O\left(\tilde{y}_3^{N_2+1}\right) \end{cases} \tag{6.101}$$

where by Hypothesis 6.11, we have $m_{341}(0) < 1$.

So it follows from (6.100), (4.29) and (6.101) that the transition map T_3 satisfies

$$\begin{cases} T_{31}(\nu, d) & = \nu \\ T_{32}(\nu, d) & = m_{30}(\nu, \omega(r_3, \beta_1)) + \nu^{\bar{\sigma}_3(1-d)}\Big[m_{31}(\nu, \omega(m_4\nu^{1-d}, \beta_1)) \\ & +m_{32}(\nu)\nu^{1-d} + m_{33}\nu^d + O(\nu^{2(1-d)}, \nu^{2d}) \\ & +\sum_{j=1}^{N_1} \overline{m}_{34j}\nu^{\bar{\sigma}_3(1-d)j} + O\left(\nu^{(N_1+1)\bar{\sigma}_3(1-d)}\right) + \theta_{31}(\nu, \nu^{1-d}, \omega(m_4\nu^{1-d}, \beta_1))\Big] \end{cases} \tag{6.102}$$

where θ_{31} is C^k and satisfies the properties of θ_{41} in (6.97), also

$$\begin{aligned} m_{30} &= m_{340}(\nu) + \kappa_3\left[\tfrac{m_{341}(\nu)}{r_0^{p_3}}\nu^{p_3}\omega(\tfrac{r_3}{r_0}, \beta_1) + O(\nu^{2p_3}\omega^2(\tfrac{r_3}{r_0}, \beta_1))\right] \\ m_{31} &= \tfrac{\overline{m}_{141}^{\bar{\sigma}_3} m_{341} y_0}{r_0^{\bar{\sigma}_3}} + \kappa_3 O(\nu^{p_3}\omega(m_4\nu^{1-d}, \beta_1)), & m_{31}(0) \neq 0 \\ m_{32} &= \tfrac{\bar{\sigma}_3 \overline{m}_{142}(\nu) y_0}{r_0^{\bar{\sigma}_3}\overline{m}_{141}^{1-\bar{\sigma}_3}} + \kappa_3 O(\nu^{p_3}\omega(m_4\nu^{1-d}, \beta_1)), & m_{32}(0) = *K_{142}\varepsilon_2 \\ m_{33} &= \tfrac{\bar{\sigma}_3 \overline{m}_{143}(\nu) y_0}{r_0^{\bar{\sigma}_3}\overline{m}_{141}^{1-\bar{\sigma}_3}} + \kappa_3 O(\nu^{p_3}\omega(m_4\nu^{1-d}, \beta_1)), & m_{33}(0) = *K_{143}\bar{\mu}_{30} \\ \overline{m}_{341} &= \tfrac{\overline{m}_{141}^{2\bar{\sigma}_3} y_0^2 m_{342}}{r_0^{2\bar{\sigma}_3}} + \kappa_3 O(\nu^{p_3}\omega(m_4\nu^{1-d}, \beta_1)), & \overline{m}_{341}(0) = *m_{342}(0) \end{aligned}$$

and in $m_{30}(\nu, \omega(\frac{r_3}{r_0}, \beta_1))$ we still keep r_3 as the function of d in the expression (6.100).

To get the cyclicity of Ehh1c and Ehh3e, we are going to apply Theorem 6.12 to study $\#(F, G)$ of the following system

$$\begin{cases} F_\nu(c, d) & := T_{12}(\nu, c) - T_{22}(\nu, d) = 0 \\ G_\nu(c, d) & := T_{42}(\nu, c) - T_{32}(\nu, d) = 0 \end{cases} \tag{6.103}$$

for $(a, \bar{\mu}) \in A_0 \times \mathbb{S}^2$ and $(c, d) \in \mathcal{D}_\nu$, where $\mathcal{D}_\nu$ is a square whose size depends on ν. With $\nu = u\varepsilon^2$ and $u \in (0, 1)$, then $\mathcal{D}_\nu = \mathcal{I}_\nu \times \mathcal{I}_\nu$.

(2). Functions $F_\nu(c,d)$ and $G_\nu(c,d)$ satisfy the conditions of Theorem 5.3.2.

For $0<\nu<\varepsilon^2$, $F_\nu(c,d)$ and $G_\nu(c,d)$ are continuous on $\overline{\mathcal{D}}_\nu$ and smooth in $\mathcal{D}_\nu$. Note that $\forall c\in(\frac{\ln\varepsilon}{\ln\nu},\frac{\ln u\varepsilon}{\ln\nu})$, we have $\nu^{1-c}\in(u\varepsilon,\varepsilon)$, hence $\nu^{p_1(1-c)}\in(0,\varepsilon^{p_1})$. So for $0<\nu<\varepsilon^2$ and $\forall(c,d)\in\mathcal{D}_\nu$ with $\varepsilon>0$ sufficiently small, by (6.23), a first derivation gives

$$\begin{aligned}\frac{\partial}{\partial c}F_\nu(c,d) &= T'_{12c}\\ &= -\frac{\kappa_1}{\rho_0^{\bar\sigma_1}}\nu^{\bar\sigma_1 c}\nu^{p_1(1-c)}\ln\nu+\nu^{\bar\sigma_1 c}\ln\nu[\bar\sigma_1 l_1+\theta_1(\nu,\nu^c,\omega(\tfrac{\nu^c}{\rho_0},-\alpha_1))\\ &\quad+\frac{1}{\ln\nu}\frac{\partial}{\partial c}\theta_1(\nu,\nu^c,\omega(\tfrac{\nu^c}{\rho_0},-\alpha_1))]\\ &= \nu^{\bar\sigma_1 c}\ln\nu[\bar\sigma_1 l_1+l_{11}\nu^{p_1(1-c)}+\theta_{11}(\nu,\nu^c,\omega(\tfrac{\nu^c}{\rho_0},\alpha_1))]\\ &\neq 0\end{aligned}\tag{6.104}$$

where $l_{11}=-\frac{\kappa_1}{\rho_0^{\bar\sigma_1}}$, $\theta_{11}(\nu,\nu^c,\omega(\frac{\nu^c}{\rho_0},\alpha_1))$ is C^k and satisfies the properties (6.22). Since for $x>0$ sufficiently small, $(x^{p_1}\omega(x,\alpha_1))'=x^{p_1-1}[\bar\sigma_1\omega(x,\alpha_1)-1]>0$, for θ_{11} with $c\in\mathcal{I}_\nu$, we have the estimation: $\theta_{11}(\nu,\nu^c,\omega(\frac{\nu^c}{\rho_0},\alpha_1))=O\Big(\varepsilon^{p_1}\omega(\varepsilon,\alpha_1)\Big)$.

Similarly, for $0<\nu<\varepsilon^2$ and $\forall(c,d)\in\mathcal{D}_\nu$ with $\varepsilon>0$ sufficiently small, we have

$$\begin{aligned}\frac{\partial}{\partial d}F_\nu(c,d) &= -T'_{22d}\\ &= -\nu^{\bar\sigma_1 d}\ln\nu\Big[\bar\sigma_1 l_2+l_{21}\nu^{p_1(1-d)}+l_{22}(\nu)\nu^{\bar\sigma_1 d}+\theta_{23}(\nu,\nu^d,\omega(\tfrac{\nu^d}{\rho_0},-\alpha_1))\Big]\\ &\neq 0\end{aligned}\tag{6.105}$$

where $l_2(0)=l_1m_{211}(0)\neq0$, $l_{21}=-\frac{\kappa_1 m_{211}}{\rho_0^{\bar\sigma_1}}$, and $l_{22}=2\bar\sigma_1 l_1^2 m_{212}$. Also for Ehh1c, $l_{22}(0)=*m_{212}(0)=*\bar\mu_{30}$. For $\theta_{23}(\nu,\nu^d,\omega(\frac{\nu^d}{\rho_0},-\alpha_1))$, it is C^k and satisfies the properties (6.22).

By (6.104) and (6.105), for $0<\nu<\varepsilon^2$ and $\forall(c,d)\in\mathcal{D}_\nu$ with $\varepsilon>0$ sufficiently small, $F_\nu(c,d)$ and $G_\nu(c,d)$ satisfy the conditions of Theorem 6.12. So we have

$$\#(F,G)\le 1+\#(F,J[F,G]).\tag{6.106}$$

(3). Calculation of $\#(F,J[F,G])$.

To calculate $\#(F,J[F,G])$, we have to calculate $J[F,G]=\frac{\partial F}{\partial c}\frac{\partial G}{\partial d}-\frac{\partial F}{\partial d}\frac{\partial G}{\partial c}$.

Note that for the case $q_3=1$, $\bar\sigma_3+\beta_1=p_3=1$, so

$$\nu\nu^{-\beta_1(1-c)}=\nu^{1-\beta_1(1-c)}=\nu^{(1-\beta_1)(1-c)}\nu^{1-(1-c)}=\nu^c\nu^{\bar\sigma_3(1-c)}.$$

Therefore, for $\eta_4(\nu, \omega(\frac{r_4}{r_0}, \beta_1))$, by (6.93) and similar to (6.23) we have

$$\begin{aligned}
&\tfrac{\partial \eta_4}{\partial c}(\nu, \omega(\tfrac{r_4}{r_0}, \beta_1)) \\
&= \tfrac{\kappa_3}{r_0}\nu \tfrac{\partial}{\partial c}\omega(\tfrac{r_4}{r_0}, \beta_1) \\
&= \tfrac{\kappa_3}{r_0}\nu[-(\tfrac{r_4}{r_0})^{-1-\beta_1}]\tfrac{\partial r_4}{\partial c} \\
&= \kappa_3 m_4^{-\beta_1}\nu\nu^{(1-c)(1-1-\beta_1)}\ln\nu \qquad (6.107) \\
&\quad \Big[1+\overline{m}_{142}(\nu)\nu^{1-c}+\overline{m}_{143}(\nu)\nu^{c}+O(\nu^{2(1-c)},\nu^{2c})\Big]^{-\beta_1} \\
&\quad \Big[1+\tilde{\overline{m}}_{142}(\nu)\nu^{1-c}+\hat{m}_{143}\nu^{c}+O(\nu^{2(1-c)},\nu^{2c})\Big] \\
&= \tfrac{\kappa_3}{m_4^{\bar{\sigma}_1}}\nu^{\bar{\sigma}_3(1-c)}\nu^{c}\ln\nu\Big[1+\hat{\tilde{m}}_{142}(\nu)\nu^{1-c}+\hat{\tilde{m}}_{143}(\nu)\nu^{c}+O(\nu^{2(1-c)},\nu^{2c})\Big].
\end{aligned}$$

So by (6.96), (6.102) and (6.107) and direct derivation, we have

$$\begin{aligned}
\tfrac{\partial}{\partial c}G_\nu(c,d) &= T'_{42c}(\nu,c) \\
&= \nu^{\bar{\sigma}_3(1-c)}\ln\nu\Big[\bar{\sigma}_3 l_4+\bar{l}_{411}\nu^{p_3c}+\bar{l}_{422}(\nu)\nu^{1-c}+\bar{l}_{423}(\nu)\nu^{c} \\
&\quad +O(\nu^{2(1-c)},\nu^{2c})+\theta_{42}(\nu,\nu^{1-c},\omega(m_4\nu^{1-c},\beta_1))\Big] \\
\tfrac{\partial}{\partial d}G_\nu(c,d) &= -T'_{32d}(\nu,d) \\
&= -\nu^{\bar{\sigma}_3(1-d)}\ln\nu\Big[\bar{\sigma}_3 l_3+\bar{l}_{311}\nu^{p_3d}+\bar{l}_{322}(\nu)\nu^{1-d}+\bar{l}_{323}(\nu)\nu^{d}+O(\nu^{2(1-d)},\nu^{2d}) \\
&\quad +\sum_{j=1}^{N_1}\bar{l}_{33j}\nu^{\bar{\sigma}_3(1-d)j}+O\Big(\nu^{(N_1+1)\bar{\sigma}_3(1-d)}\Big) \\
&\quad +\theta_{32}(\nu,\nu^{1-d},\omega(m_4\nu^{1-d},-\beta_1))\Big]
\end{aligned} \qquad (6.108)$$

where $\theta_{42}(\nu,\nu^{1-c},\omega(m_4\nu^{1-c},-\beta_1))$ and $\theta_{32}(\nu,\nu^{1-d},\omega(m_4\nu^{1-d},-\beta_1))$ are C^k and satisfy the properties (6.97); $l_4=\frac{y_0}{r_0^{\bar{\sigma}_1}}m_{141}^{\bar{\sigma}_3}$, $l_3=m_{341}l_4$, and $l_3(0)l_4(0)\neq 0$. Also

$$\begin{aligned}
\bar{l}_{311}(0) &= *\kappa_3 \\
\bar{l}_{411}(0) &= *\kappa_3 \\
\bar{l}_{331}(0) &= *m_{342}(0) \\
\bar{l}_{422}(0) &= *m_{142}(0) = *K_{142}\varepsilon_2 \\
\bar{l}_{423}(0) &= *m_{143}(0) = *K_{143}\bar{\mu}_{30} \\
\bar{l}_{322}(0) &= *m_{341}(0)\bar{l}_{422}(0) = *m_{341}K_{142}\varepsilon_2 \\
\bar{l}_{323}(0) &= *m_{341}(0)\bar{l}_{423}(0) = *m_{341}K_{143}\bar{\mu}_{30}.
\end{aligned}$$

Let

$$G_1(\nu,c,d) := \frac{\bar{\sigma}_3\nu^{\bar{\sigma}_3}}{\bar{\sigma}_1}\frac{J[F,G](\nu,c,d)}{\frac{\partial G}{\partial c}\frac{\partial G}{\partial d}}.$$

It follows from (6.104), (6.105) and (6.108) that

$$G_1(\nu,c,d)=\nu^{(\bar\sigma_1+\bar\sigma_3)c}[1+\bar h_1(\nu,c)]-\nu^{(\bar\sigma_1+\bar\sigma_3)d}\Big[\frac{m_{211}(\nu)}{m_{341}(\nu)}+\bar h_2(\nu,d)\Big].$$

Then for $0<\nu<\varepsilon^2$ and $\forall(c,d)\in\mathcal{D}_\nu$ with $\varepsilon>0$ sufficiently small, the equation $G_1(\nu,c,d)=0$ is equivalent to equation $G_2(\nu,c,d)=0$, where

$$G_2(\nu,c,d):=\nu^c[1+h_1(\nu,c)]-\nu^d[\gamma_1(\nu)+h_2(\nu,d)] \tag{6.109}$$

where $\gamma_1(\nu)=\left(\frac{m_{211}(\nu)}{m_{341}(\nu)}\right)^{\frac{1}{\bar\sigma_1+\bar\sigma_3}}$, and

$$\begin{aligned}h_1(\nu,c)\ &=\bar\gamma_{111}\nu^{p_1(1-c)}+\bar\gamma_{311}\nu^{p_3d}\\ &\quad\bar\gamma_{142}\nu^{1-c}+\bar\gamma_{143}\nu^{c}+O(\nu^{2(1-c)},\nu^{2c})\\ &\quad+H_1(\nu,\nu^c,\nu^{1-c},\omega(\tfrac{\nu^c}{\rho_0},-\alpha_1),\omega(m_4\nu^{1-c},-\beta_1))\end{aligned}$$

$$\begin{aligned}h_2(\nu,d)\ &=\bar\gamma_{211}\nu^{p_1(1-d)}+\bar\gamma_{411}\nu^{p_3d}\\ &\quad+\bar\gamma_{232}\nu^{1-d}+\bar\gamma_{233}\nu^{d}+O(\nu^{2(1-d)},\nu^{2d})\\ &\quad+\sum_{j=1}^{N_1}\bar\gamma_{34j}\nu^{\bar\sigma_3(1-d)j}+O(\nu^{(N_1+1)(1-d)\bar\sigma_3})+\gamma_{22}(\nu)\nu^{\bar\sigma_1 d}\\ &\quad+H_2(\nu,\nu^d,\nu^{1-d},\omega(\tfrac{\nu^d}{\rho_0},-\alpha_1),\omega(m_4\nu^{1-d},-\beta_1))\end{aligned}$$

where

$$\begin{aligned}&\bar\gamma_{111}(0)=*\kappa_1, &&\bar\gamma_{211}(0)=*\kappa_1\\ &\bar\gamma_{311}(0)=*\kappa_3, &&\bar\gamma_{411}(0)=*\kappa_3\\ &\bar\gamma_{22}(0)=*m_{212}(0)\\ &\bar\gamma_{341}(0)=*m_{342}(0)\\ &\bar\gamma_{232}(0)=*\bar\gamma_{142}(0)=*K_{142}\varepsilon_2\\ &\bar\gamma_{233}(0)=*\bar\gamma_{143}(0)=*K_{143}\bar\mu_{30},\end{aligned}$$

also H_1 and H_2 are C^k and

$$\begin{aligned}H_1&=O\Big(\nu^{p_1c}\omega(\tfrac{\nu^c}{\rho_0},\alpha_1),\nu^{p_3(1-c)}\omega(m_4\nu^{1-c},\beta_1)\Big)\\ H_2&=O\Big(\nu^{p_1d}\omega(\tfrac{\nu^d}{\rho_0},\alpha_1),\nu^{p_3(1-d)}\omega(m_4\nu^{1-d},\beta_1)\Big).\end{aligned}$$

Similar to what we did in **(2)** with the functions $F_\nu(c,d)$ and $G_\nu(c,d)$, one can check that for $0<\nu<\varepsilon^2$ and $\forall(c,d)\in\mathcal{D}_\nu$ with $\varepsilon>0$ sufficiently small, $G_2(\nu,c,d)$ and

$F_\nu(c,d)$ satisfy the conditions of Theorem 6.12. Hence we have

$$\begin{aligned} \#(F,G) &\leq \#(F,J[F,G])+1 \\ &= \#(F,G_1)+1 \\ &= \#(F,G_2)+1 \\ &\leq \#(G_2,J[F,G_2])+2. \end{aligned} \tag{6.110}$$

(4). Calculation of $\#(G_2, J[F,G_2])$.

Let

$$G_3(\nu,c,d) := -\bar{\sigma}_1 l_1 \frac{J[F,G_2](\nu,c,d)}{G'_{2c}G'_{2d}}.$$

Then a straightforward calculation gives

$$G_3(\nu,c,d) = \nu^{(\bar{\sigma}_1-1)c}[1+h_{31}(\nu,c)] - \nu^{(\bar{\sigma}_1-1)d}[\gamma_2(\nu)+h_{32}(\nu,d)]. \tag{6.111}$$

where $\gamma_2(\nu) = m_{211}(\nu)\left(\frac{m_{341}(\nu)}{m_{211}(\nu)}\right)^{\frac{1}{\bar{\sigma}_1+\bar{\sigma}_3}}$.

By (6.109), if for $0<\nu<\varepsilon^2$ and $\forall (c,d)\in \mathcal{D}_\nu$ with $\varepsilon>0$ sufficiently small, $G_2(\nu,c,d)\neq 0$, then $\#(G_2,J[F,G_2])=0$ and we already finish the proof. Otherwise, similar to the proof in the Theorem 6.12, $G_2(\nu,c,d)=0$ defines a unique connected curve which satisfies

$$\nu^c = \nu^d \frac{\gamma_1(\nu)+h_2(\nu,d)}{1+h_1(\nu,c)}. \tag{6.112}$$

By iterating the relation (6.112), the unique curve defined by $G_2(\nu,c,d)=0$ can be written as

$$\nu^c = \nu^d[\gamma_1(\nu)+h_0(\nu,d)] \tag{6.113}$$

where

$$\begin{aligned} h_0(\nu,d) &= \bar{\gamma}_{001}\nu^{p_1(1-d)} + \bar{\gamma}_{003}\nu^{p_3 d} \\ &\quad +\bar{\gamma}_{012}\nu^{1-d} + \bar{\gamma}_{013}\nu^{d} + O(\nu^{2(1-d)},\nu^{2d}) \\ &\quad +\sum_{j=1}^{N_1}\bar{\gamma}_{03j}\nu^{\bar{\sigma}_3(1-d)j} + O(\nu^{(N_1+1)(1-d)\bar{\sigma}_3}) \\ &\quad +\bar{\gamma}_{02}(\nu)\nu^{\bar{\sigma}_1 d} + H_0(\nu,\nu^d,\nu^{1-d},\omega(\tfrac{\nu^d}{\rho_0},-\alpha_1),\omega(m_4\nu^{1-d},-\beta_1)) \end{aligned}$$

where

$$\bar{\gamma}_{001}(0) = *\kappa_1, \qquad \bar{\gamma}_{003}(0) = *\kappa_3$$
$$\bar{\gamma}_{02}(0) = *m_{212}(0)$$
$$\bar{\gamma}_{031}(0) = *m_{342}(0)$$
$$\bar{\gamma}_{012}(0) = *m_{142}(0)(m_{211}(0) - m_{341}(0)) = *K_{142}\varepsilon_2$$
$$\bar{\gamma}_{013}(0) = *m_{143}(0)(m_{211}(0) - m_{341}(0)) = *K_{143}\bar{\mu}_{30},$$

also H_0 is C^k and

$$H_0 = O\Big(\nu^{p_1 d}\omega(\frac{\nu^d}{\rho_0}, \alpha_1), \nu^{p_3(1-d)}\omega(m_4\nu^{1-d}, \beta_1)\Big). \tag{6.114}$$

Substitute (6.113) into $G_3(\nu, c, d)$ and let

$$g(\nu, d) = \gamma_1^{1-\bar{\sigma}_1}\nu^{1-\bar{\sigma}_1}G_3(\nu, c, d)\Big|_{(6.113)}.$$

Then a straightforward calculation gives

$$\begin{aligned} g(\nu, d) &= \gamma(\nu) + \delta_{01}\nu^{p_1(1-d)} + \delta_{03}\nu^{p_3 d} \\ &+\delta_2\nu^{1-d} + \delta_3\nu^d + O(\nu^{2(1-d)}, \nu^{2d}) \\ &+\sum_{j=1}^{N_1}\delta_{4j}\nu^{\bar{\sigma}_3(1-d)j} + O(\nu^{(N_3+1)(1-d)\bar{\sigma}_3}) \\ &+\delta_1(\nu)\nu^{\bar{\sigma}_1 d} + H(\nu, \nu^d, \nu^{1-d}, \omega(\tfrac{\nu^d}{\rho_0}, -\alpha_1), \omega(m_4\nu^{1-d}, -\beta_1)) \end{aligned} \tag{6.115}$$

where

$$\gamma(\nu) = 1 - \Big(m_{211}^{\bar{\sigma}_3}(\nu)m_{341}^{\bar{\sigma}_1}(\nu)\Big)^{\frac{1}{\bar{\sigma}_1+\bar{\sigma}_3}}, \tag{6.116}$$

and

$$\begin{aligned} &\delta_{01}(0) = *\kappa_1, \qquad \delta_{03}(0) = *\kappa_3 \\ &\delta_1(0) = *m_{212}(0) = *\bar{\mu}_{30} \\ &\delta_2(0) = *m_{142}(0)(m_{211}(0) - m_{341}(0)) = *K_{142}\varepsilon_2 \\ &\delta_3(0) = *m_{143}(0)(m_{211}(0) - m_{341}(0)) = *K_{143}\bar{\mu}_{30} \\ &\delta_{41}(0) = *m_{342}(0), \end{aligned} \tag{6.117}$$

also H is C^k and satisfies (6.114).

In order to get the cyclicity of Ehh2e, Ehh3e and Ehh1c, we will study the number of roots of the equation $g(\nu, d) = 0$ for $d \in (0, 1)$ and $\nu > 0$ sufficiently small.

(5). Cyclicity of Ehh2e and Ehh3e: Cycl(Ehh2e, Ehh3e)≤ 2.

For the graphic Ehh2e (resp. Ehh3e), since $m_{211}(0)$ can be sufficiently small (resp. large), so $\gamma(0) \to 1$ (resp. sufficiently large), hence for $(a, \bar{\mu}) \in A_0 \times V_{I_2}$ (resp. $(a, \bar{\mu}) \in A_0 \times V_{I_3}$), and $\forall \nu \in (0, \nu_0)$ and $d \in (0,1)$, by (6.115), in both cases we have $g(\nu, d) \neq 0$. Therefore, $\#(G_2, J[F, G_2]) = 0$ and by (6.110), we have $\#(F, G) \leq 2$, i.e., Cycl(Ehh2e, Ehh3e)≤ 2.

(6). Cyclicity of Ehh1c when $m_{211}(0) \leq 1$. (This contains the case $\bar{\mu}_{30} = 0$)

For the graphic Ehh1c, by (6.116), we have

$$\gamma(0) = 1 - \left(m_{211}^{\bar{\sigma}_3}(0) m_{341}^{\bar{\sigma}_1}(0)\right)^{\frac{1}{\bar{\sigma}_1+\bar{\sigma}_3}}. \tag{6.118}$$

By Hypothesis 6.11 we have $m_{341}(0) < 1$, so if $m_{211}(0) \leq 1$, then by (6.118) we have $\gamma(0) \neq 0$, hence Cycl(Ehh1c)≤ 2.

For the graphic Ehh1c with $m_{211}(0) > 1$, we will study the equation $g(\nu, d) = 0$ with $0 < \nu < \varepsilon^2$ and $d \in \mathcal{I}_\nu$ for $a_0 \in (0, \frac{1}{2}) \setminus \mathbb{Q}$ and $a_0 \in (0, \frac{1}{2}) \cap \mathbb{Q}$ in **(7)** and **(8)** respectively.

Note that if $\bar{\mu}_{30} = 0$, then $m_{211}(0) = 1$ which is the case when $\bar{\mu}_{30} = 0$, it is already included in **(6)**; also note that the nilpotent elliptic point is of codimension 3, so $\varepsilon_2 \neq 0$. So in the following two cases, we have

$$\begin{aligned} \delta_2(0) &= *m_{142}(0)[m_{211}(0) - m_{341}(0)] = *K_{142}\varepsilon_2 \neq 0 \\ \delta_3(0) &= *m_{143}(0)[m_{211}(0) - m_{341}(0)] = *K_{143}\bar{\mu}_{30} \neq 0. \end{aligned} \tag{6.119}$$

(7). Cyclicity of Ehh1c when $m_{211}(0) > 1$ ($\bar{\mu}_{30} > 0$): Case $a_0 \in (0, \frac{1}{2}) \setminus \mathbb{Q}$

For $a_0 \in (0, \frac{1}{2}) \setminus \mathbb{Q}$, let $N_1 = [\frac{1}{\bar{\sigma}_3(a_0)}]$, then function $g(\nu, d)$ in (6.115) can be simplified to

$$\begin{aligned} g(\nu, d) \; &= \gamma(\nu) + \delta_1(\nu)\nu^{\bar{\sigma}_1 d} + o(\nu^{\bar{\sigma}_1 d}) \\ &\quad + \delta_2(\nu)\nu^{1-d} + \delta_3(\nu)\nu^d + O(\nu^{2(1-d)}, \nu^{2d}) \\ &\quad + \sum_{j=1}^{[\frac{1}{\bar{\sigma}_3}]} \delta_{4j}\nu^{\bar{\sigma}_3(1-d)j} + O\left(\nu^{([\frac{1}{\bar{\sigma}_3}]+1)\bar{\sigma}_3(1-d)}\right) \end{aligned} \tag{6.120}$$

Let

$$DD: \begin{cases} g_0(\nu,d) &= \frac{1}{\ln\nu}\frac{\partial}{\partial d}g(\nu,d) \\ g_j(\nu,d) &= \frac{\nu^{\bar{\sigma}_3(1-d)j}}{\ln\nu}\frac{\partial}{\partial d}\Big(g_{j-1}(\nu,d)\nu^{-\bar{\sigma}_3(1-d)j}\Big), \qquad j=1,\cdots,[\frac{1}{\bar{\sigma}_3}]. \end{cases} \tag{6.121}$$

Then after $[\frac{1}{\bar{\sigma}_3}]+1$ steps of successive derivation and division in (6.121), we get

$$g_{[\frac{1}{\bar{\sigma}_3}]}(\nu,d) = \bar{\delta}_1(\nu)\nu^{\bar{\sigma}_1 d} + o(\nu^{\bar{\sigma}_1 d}) + \bar{\delta}_2(\nu)\nu^{1-d} + \bar{\delta}_3(\nu)\nu^d + o(\nu^{2(1-d)},\nu^{2d}) \tag{6.122}$$

where $\bar{\delta}_1(0) = *m_{212}(0) = *\bar{\mu}_{30} \neq 0$, and by (6.119), $\bar{\delta}_2(0) = *\delta_2(0) \neq 0$.

We introduce a Lemma.

Lemma 6.17. *Consider the equation*

$$L(\nu,d) = \bar{l}_1(\nu)\nu^{\bar{\sigma}_1 d} + o(\nu^{\bar{\sigma}_1 d}) + \bar{l}_2(\nu)\nu^{1-d} + o(\nu^{1-d}) + \bar{l}_3(\nu)\nu^d + o(\nu^d)$$

for $\nu \in (0,\varepsilon^2)$ *and* $d \in (\frac{\ln\varepsilon}{\ln\nu}, \frac{\ln u\varepsilon}{\ln\nu})$ *with* $u \in (0,1)$ *and* $\varepsilon > 0$ *sufficiently small. If* $\bar{\sigma}_1 > 1$ *and* $\bar{l}_2(0)\bar{l}_3(0) \neq 0$ *or* $\bar{\sigma}_1 < 1$ *and* $\bar{l}_1(0)\bar{l}_2(0) \neq 0$*, then* $L(\nu,d) = 0$ *has at most 1 solution.*

Proof. For the case $\bar{\sigma}_1 > 1$, $L(\nu,d)$ can be simplified to

$$L(\nu,d) = \bar{l}_2(\nu)\nu^{1-d} + o(\nu^{1-d}) + \bar{l}_3(\nu)\nu^d + o(\nu^d)$$

Note that $\bar{l}_2(0)\bar{l}_3(0) \neq 0$, so we have two possibilities:

- if $\bar{l}_2(0)\bar{l}_3(0) > 0$, then $L(\nu,d) \neq 0$;
- if $\bar{l}_2(0)\bar{l}_3(0) < 0$, then $L'_d(\nu,d) \neq 0$, thus $L(\nu,d) = 0$ has at most one solution.

Altogether, $L(\nu,d) = 0$ has at most 1 solution.

The case $\bar{\sigma}_1 < 1$ is similar. □

End of proof of Theorem. 6.16

In this case, note that we have $\bar{\delta}_1(0)\bar{\delta}_2(0)\bar{\delta}_3(0) \neq 0$ and $\bar{\sigma}_1 \neq 1$, so applying Lemma 6.17 to the function in (6.122), we conclude that $g_{[\frac{1}{\bar{\sigma}_3}]}(\nu,d)$ has at most 1 root. So $g(\nu,d) = 0$ has at most $[\frac{1}{\bar{\sigma}_3}]+4$ roots. Hence $Cycl(Ehh1c) \leq [\bar{\sigma}_1] + [\frac{1}{\bar{\sigma}_3}] + 4$. Thus for $a_0 \in (0,\frac{1}{2})/\mathbb{Q}$, the graphic Ehh1c has finite cyclicity.

(8). Cyclicity of Ehh1c when $m_{211}(0) > 1$ ($\bar{\mu}_{30} > 0$): Case $a_0 \in (0,\frac{1}{2}) \cap \mathbb{Q}$

For the graphic Ehh1c, if $a_0 \in (0,\frac{1}{2}) \cap \mathbb{Q}$, then $\sigma_1(a_0), \sigma_3(a_0) \in \mathbb{Q}$. For $i = 1, 3$, let $\sigma_i(a_0) = \frac{p_i}{q_i}, p_i, q_i \in \mathbb{N}$ and $(p_i, q_i) = 1$, thus we have three subcases:

(8.1). Case $a_0 \in (0,\frac{1}{2}) \cap \mathbb{Q} \setminus \{\frac{1}{n+2}, \frac{2n-1}{4n}, n \in \mathbb{N}\}$

In this case, we have $q_1 \geq 2$, $\sigma_1 < p_1$ and $p_3 \geq 2$. Then for $\nu > 0$ sufficiently small, we have $\bar{\sigma}_1 < p_1$ and $[\frac{1}{\bar{\sigma}_3}] < q_3$. Note that in this case $\sigma_1, \frac{1}{\sigma_3} \notin \mathbb{N}$, therefore the equation (6.115) can be reduced to

$$\begin{aligned} g(\nu, d) &= \gamma(\nu) + \delta_1(\nu)\nu^{\bar{\sigma}_1 d} + o(\nu^{\bar{\sigma}_1 d}) + \delta_3(\nu)\nu^d + o(\nu^d) \\ &+\sum_{j=1}^{q_3} \delta_{4j}\nu^{\bar{\sigma}_3(1-d)j} + \delta_2(\nu)\nu^{1-d} + o(\nu^{1-d}) \end{aligned} \tag{6.123}$$

Applying the DD process (6.121) $[\frac{1}{\sigma_3}]$ steps to the function $g(\nu, d)$ in (6.123), we get

$$g_{p_1,q_3}(\nu, d) = \begin{cases} \hat{\delta}_1(\nu)\nu^{\bar{\sigma}_1 d} + o(\nu^{\bar{\sigma}_1 d}) + \hat{\delta}_2(\nu)\nu^{1-d} + o(\nu^{1-d}) & \text{if } \sigma_1 < 1 \\ \hat{\delta}_3(\nu)\nu^d + o(\nu^d) + \hat{\delta}_2(\nu)\nu^{1-d} + o(\nu^{1-d}) & \text{if } \sigma_1 > 1 \end{cases}$$

where $\hat{\delta}_i(0) \neq 0$, $(i = 1, 2, 3)$.

Then by Lemma 6.17 we obtain that $g_{p_1,q_3}(\nu, d) = 0$ has at most 1 solution, then $g(\nu, d) = 0$ has at most $[\frac{1}{\bar{\sigma}_3}] + 4$ solutions, i.e., Ehh1c has finite cyclicity.

(8.2). Case $a_0 = \frac{1}{n+2}, n \in \mathbb{N}, n \neq 1, 2$

This is a particular case of **(8.1)**. In this case, we have $\sigma_1(a_0) = p_1 = n > 1$ and $\sigma_3(a_0) = \frac{2n}{n+2} \in (1, 2)$. Thus for $\sigma_3(a_0) = \frac{p_3}{q_3}$ with $p_3, q_3 \in \mathbb{N}$ and $(p_3, q_3) = 1$, we have $q_3 \geq 2$. Then $g(\nu, d)$ can be reduced to

$$g(\nu, d) = \gamma(\nu) + \delta_2(\nu)\nu^{1-d} + o(\nu^{1-d}) + \delta_3(\nu)\nu^d + o(\nu^d).$$

Similar to the proof of the Lemma 6.17, we obtain that $\gamma(\nu, d)$ has at most 1 solution which gives that $Cycl(Ehh1c) \leq 3$.

(8.3). Case $a_0 = \frac{1}{4}$

In this case, we have $\sigma_1(\frac{1}{4}) = 2$, $p_1 = 2$, $q_1 = 1$ and $\sigma_3(a_0) = 1$, $p_3 = q_3 = 1$.

For the second type of Dulac map near P_3, by Theorem 4.14, we have

$$\theta_4(r_4, \rho_4, \omega(\frac{r_4}{r_0}, -\beta_1) = \beta_2 K_3 r_3 \omega(\frac{r_3}{r_0}, \beta_1) + O\Big(\nu\omega^2(\frac{r_4}{r_0}, -\beta_1)\omega(\frac{r_4}{r_0}, \beta_1)\Big). \tag{6.124}$$

By Lemma 6.15, we have the first saddle quantity at P_3 $\beta_2 \neq 0$.

Then the function in (6.115) has the form

$$\begin{aligned} g(\nu,d) &= \gamma(\nu) + \delta_{11}(\nu)\nu^d + \delta_{12}(\nu)\nu^{2d} \\ &+\delta_1(\nu)\nu^{2d}\omega(\tfrac{\nu^d}{\rho_0},\alpha_1) + O\Big(\nu^2\omega^2(\tfrac{\nu^d}{\rho_0},-\alpha_1)\omega(\tfrac{\nu^d}{\rho_0},\alpha_1)\Big) \\ &+\delta_{21}\nu^{1-d} + \delta_2(\nu)\nu^{1-d}\omega(m_4\nu^{1-d},\beta_1) \\ &+O\Big(\nu\omega^2(m_4\nu^{1-d},-\beta_1)\omega(m_4\nu^{1-d},\beta_1)\Big) \end{aligned} \tag{6.125}$$

where $\delta_{11}(\nu)$ can vanish, $\delta_1(0) = *\alpha_2(\frac{1}{4}) = *\bar{\mu}_{30} \neq 0$, and $\delta_2(0) = *\beta_2 \neq 0$.

By applying the standard division-derivation method to to the function $g(\nu,d)$ in (6.125), we can kill the terms $\gamma(\nu)$ and ν^d. Then the number of roots of $g(\nu,d)=0$ is at most 6 plus the number of roots of

$$\begin{aligned} g_1(\nu,d) &= \hat{\delta}_{12}\nu^{2d} + \hat{\delta}_1(\nu)\nu^{2d}\omega(\tfrac{\nu^d}{\rho_0},\alpha_1) \\ &+O\Big(\nu^2\omega^2(\tfrac{\nu^d}{\rho_0},-\alpha_1)\omega(\tfrac{\nu^d}{\rho_0},\alpha_1)\Big) \\ &+\hat{\delta}_{21}(\nu)\nu^{1-d} + \hat{\delta}_2(\nu)\nu^{1-d}\omega(m_4\nu^{1-d},\beta_1) \\ &+O\Big(\nu\omega^2(m_4\nu^{1-d},-\beta_1)\omega(m_4\nu^{1-d},\beta_1)\Big) \end{aligned} \tag{6.126}$$

where $\hat{\delta}_i(0) = *\delta_i(0) \neq 0$ $(i=1,2)$.

Let $g_2(\nu,d) = \frac{\nu^{2d}}{\ln\nu}\frac{\partial}{\partial d}\Big(\nu^{-2d}g_1(\nu,d)\Big)$, then

$$\begin{aligned} g_2(\nu,d) &= -\rho_0^{\alpha_1}\tilde{\delta}_1\nu^{\bar{\sigma}_1 d} + O\Big(\nu^2\omega^2(\frac{\nu^d}{\rho_0},-\alpha_1)\omega(\frac{\nu^d}{\rho_0},\alpha_1)\Big) \\ &-[(3+\beta_1)\omega(m_14\nu^{1-d},\beta_1)+1]\tilde{\delta}_2(\nu)\nu^{1-d} \\ &+O\Big(\nu\omega^2(m_4\nu^{1-d},-\beta_1)\omega(m_4\nu^{1-d},\beta_1)\Big). \end{aligned}$$

Let $g_3(\nu,d) = \frac{\nu^{1-d}}{\ln\nu}\frac{\partial}{\partial d}\Big(\nu^{-(1-d)}g_2(\nu,d)\Big)$, then

$$\begin{aligned} g_3(\nu,d) &= \bar{\delta}_1\nu^{\bar{\sigma}_1 d} + O\Big(\nu^2\omega^2(\frac{\nu^d}{\rho_0},-\alpha_1)\omega(\frac{\nu^d}{\rho_0},\alpha_1)\Big) \\ &+\bar{\delta}_2(\nu)\nu^{\bar{\sigma}_3 d} + O\Big(\nu\omega^2(m_4\nu^{1-d},-\beta_1)\omega(m_4\nu^{1-d},\beta_1)\Big) \end{aligned}$$

$\bar{\delta}_1(0) = -\rho_0^{\alpha_1}\tilde{\delta}_1(0) \neq 0$, $\bar{\delta}_2(0) = (3+\beta_1)m_4^{-\beta_1}\tilde{\delta}_2(0) \neq 0$.

Again applying Lemma 6.17 to the function $g_3(\nu,d)$ we conclude that $g_3(\nu,d)=0$ has at most 1 root, so for $a_0 = \frac{1}{4}$, $Cycl(Ehh1c) \leq 9$.

(8.4). Case $a_0 = \frac{2n-1}{4n}$, $n \in \mathbb{N}$, $n \neq 1$

In this case, we have $\sigma_1(a_0) = \frac{2}{2n-1} < 1$, $p_1 = 2$, $q_1 = 2n-1$ and $\sigma_3(a_0) = \frac{1}{n}$, $p_3 = 1$, $q_3 = n$.

Since $\sigma_1 < 1$, so this is a case similar to the case **(8.3)**, but simplier. The function in (6.115) has the form

$$\begin{aligned} g(\nu, d) &= \gamma(\nu) + \delta_1(\nu)\nu^{\bar{\sigma}_1 d} + o(\nu^{\bar{\sigma}_1 d}) \\ &+\sum_{j=1}^{n-1}\delta_{2j}\nu^{\bar{\sigma}_3(1-d)j} + \delta_{22}(\nu)\nu^{1-d} \\ &+\delta_2(\nu)\nu^{1-d}\omega(m_4\nu^{1-d}, \beta_1) + O\Big(\nu\omega^2(m_4\nu^{1-d}, -\beta_1)\omega(m_4\nu^{1-d}, \beta_1)\Big) \end{aligned} \tag{6.127}$$

where $\delta_1(0) = *m_{212}(0) = *\bar{\mu}_{30} \neq 0$, and by Lemma 6.15, $\delta_2(0) = *\beta_2 \neq 0$.

After killing the terms $\nu^{\bar{\sigma}_3(1-d)j}$ $(j = 1, 2, \cdots, n-1)$ by the DD process, then similar to the process in **(8.3)**, we obtain that $Cycl(Ehh1c) \leq n + 4$.

(8.5). Case $a_0 = \frac{1}{3}$

Note that in this case, $\sigma_1(\frac{1}{3}) = p_1 = 1$, $\sigma_3(\frac{1}{3}) = \frac{2}{3}$. Then the function in (6.115) has the form

$$\begin{aligned} g(\nu, d) &= \gamma(\nu) + \delta_{11}(\nu)\nu^d + \delta_1(\nu)\nu^d\omega(\tfrac{\nu^d}{\rho_0}, \alpha_1) + O\Big(\nu\omega^2(\tfrac{\nu^d}{\rho_0}, -\alpha_1)\omega(\tfrac{\nu^d}{\rho_0}, \alpha_1)\Big) \\ &+\delta_{21}\nu^{\bar{\sigma}_3(1-d)} + o(\nu^{\bar{\sigma}_3(1-d)}) \end{aligned} \tag{6.128}$$

where $\delta_1(0) = *\alpha_2(\frac{1}{3})$ and $\delta_2(0) = *\beta_2 \neq 0$.

In this case we need to calculate the saddle quantity α_2 for the 2-dimensional system near P_1 on $r = 0$. In this case, system near P_1 on $r = 0$ becomes

$$\begin{cases} \dot{\rho} &= \frac{1}{3}\rho(1 + 3y + 3\bar{\mu}_2\rho 62) \\ \dot{y} &= -\frac{1}{3} + 2y^2 + \bar{\mu}_3\rho y + 2\bar{\mu}_2\rho^2 y + \bar{\mu}_1\rho^3. \end{cases}$$

Then by the formula (5.52) introduced in Lemma 5.14, we have the saddle quantity

$$\alpha_2(\frac{1}{3}, \bar{\mu}_0) = -64\bar{\mu}_{30} \neq 0. \tag{6.129}$$

Then similar to the case **(8.3)**, we obtain the finite cyclicity of Ehh1c.

Altogether we have proved Ehh1c has finite cyclicity. □

Next we study the rest of the lower boundary graphics of Ehh families. First note that for the family Ehh4, it is the same as the family Sxhh1 for the saddle case, so by Theorem 5.5, family Ehh4 has finite cyclicity.

Before finishing the proof for both the lower boundary graphics and intermediate graphics, we make the following remark:

Remark 6.18. *System* (3.10) *is invariant under the transformation*

$$(-t, -x, -\bar{\mu}_1, -\bar{\mu}_3) \mapsto (t, x, \bar{\mu}_1, \bar{\mu}_3) \tag{6.130}$$

so the families Ehh7 and Ehh8 can be obtained from the families Ehh5 and Ehh6, families Ehh11 and Ehh12 can be obtained from the families Ehh9 and Ehh10, we will only need to deal with families Ehh5, Ehh6, Ehh9 and Ehh10 as long as we do not use Hypothesis 6.11*:* $\gamma^* < 1$.

Now we prove that

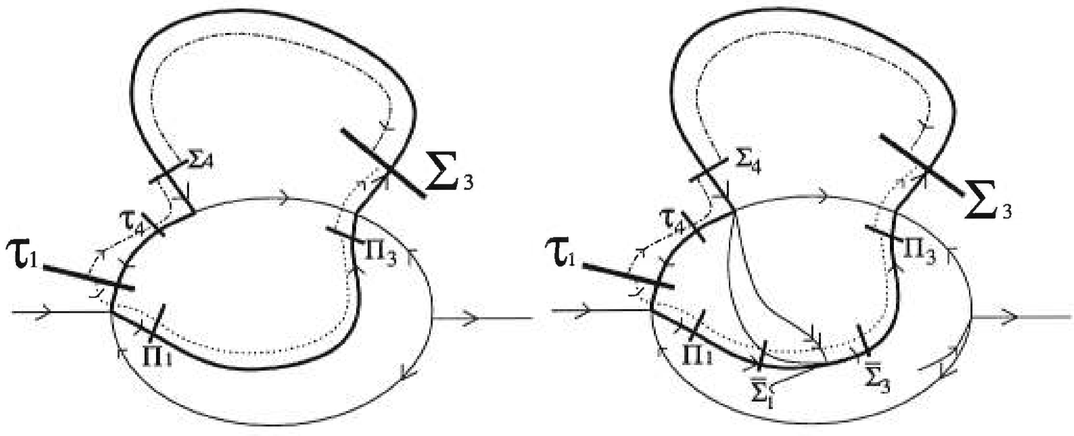
Theorem 6.19. *For the families Ehh5, $\cdots$, Ehh12, all the lower boundary graphics have finite cyclicity.*

Proof. By Remark 6.18, we only need to prove that the lower boundary graphics Ehh5c, Ehh6c, Ehh9c Ehh10e have finite cyclicity. For all these graphics, take sections τ_1 and Σ_3 as defined in Notation 5.7, we are going to study the displacement map defined on the section τ_1:

$$\begin{aligned} L &: \tau_1 \longrightarrow \Sigma_3 \\ L &= \widehat{T} - \widetilde{T} \end{aligned} \tag{6.131}$$

where $\widehat{T}$ is the transition map through the blown-up singularity, $\widetilde{T}$ is the transition map along the regular orbit. Similar to the graphic Ehp1c, on the section τ_1, we will use coordinates (ν, c) with $c \in \mathcal{I}_\nu$.

We begin with the graphics Ehh9c and Ehh10e first.



(a) Graphic Ehh9c

(b) Graphic Ehh10e

Figure 6.9: Lower boundary graphics Ehh9c and Ehh10e: Displacement maps

(1). Lower boundar graphics Ehh9c and Ehh10e

Taking sections τ_4, Σ_4, Σ_3 in the normal form coordinates (as defined in Notation 5.7), the transition map $\widetilde{T}$ can be calculated by the decomposition

$$\widetilde{T} = R^{-1} \circ \theta_4 \circ U$$

where

- $U : \tau_1 \longrightarrow \tau_4$ is the regular transition map defined in Prop. 6.13, it has the expression (6.55),
- $\Theta_4 : \tau_4 \longrightarrow \Sigma_4$ is the second type Dulac map near P_4,
- $R^{-1} : \Sigma_4 \longrightarrow \Sigma_3$ is the inverse of the transition map R defined in (6.101).

Then a straightforward calculation gives

$$\begin{aligned}\widetilde{T}_2(\nu, c) \; &= \widetilde{m}_{130}(\nu) + \widetilde{m}_{131}(\nu)\eta_4(\nu, \omega(r_4, \beta_1)) + O\Big(\nu^{2p_3(1-c)}\omega^2(m_4\nu^{1-c}, \beta_1)\Big) \\ &\quad - m_{13}(\nu)\nu^{\bar{\sigma}_3(1-c)}\Big[1 + \theta_4(\nu, \nu^{1-c}, \omega^2(m_4\nu^{1-c}, \beta_1)\Big]\end{aligned} \tag{6.132}$$

where $r_4 = \nu^{1-c}[m_{141} + O(\nu^c, \nu^{1-c})]$, $m_4 = \frac{m_{141}}{r_0}$ and $m_{13}(0) > 0$.

For the graphic Ehh10e, as shown in Fig. 6.9(b), take sections $\overline{\Sigma}_1$ and $\overline{\Sigma}_3$ in the normal form coordinates in the neighborhood of the saddle-node, then the second component $\overline{D}_0(\nu, \tilde{y})$ of the transition map $\overline{\Delta}_0 : \overline{\Sigma}_1 \longrightarrow \overline{\Sigma}_3$ can be written as

$$\overline{D}_0(\nu, \tilde{y}) = m_{00}(\nu)\tilde{y}$$

where $m_{00}(0)$ is sufficiently small.

Take sections Π_1 and Π_3 as defined in Notation 5.7, then the transition map $\widehat{T}$ can be factorized as

$$\widehat{T} : \tau_1 \longrightarrow \Sigma_3$$
$$\widehat{T} = \Delta_3 \circ \overline{T}_{03} \circ \overline{\Delta}_0 \circ \overline{T}_{10} \circ \Theta_1$$

where

- $\Theta_1 : \tau_1 \longrightarrow \Pi_1$ is the second type Dulac map near P_1 which satisfies Theorem 4.14 with $\sigma = \sigma_1$,
- $\overline{T}_{10} : \Pi_1 \longrightarrow \overline{\Sigma}_1$ and $\overline{T}_{03} : \overline{\Sigma}_3 \longrightarrow \Pi_3$ are C^k regular transition maps, they have the forms of (6.38) and (6.37) respectively,
- $\Delta_3 : \Pi_3 \longrightarrow \Sigma_3$ is the first type Dulac map near P_3 which satisfies Theorem 4.11 with $\sigma = \sigma_3$.

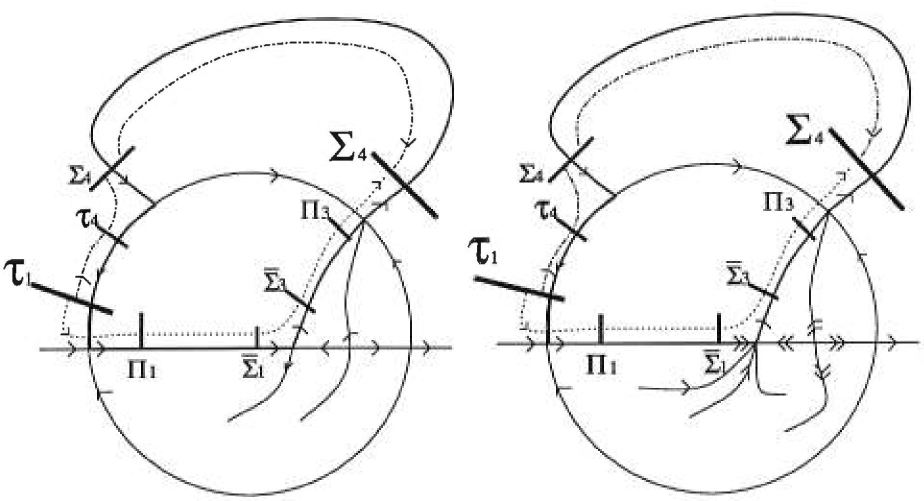
Then a straightforward calculation gives

$$\begin{aligned}\widehat{T}_2 \; &= \widehat{m}_{130}(\nu) + \widetilde{m}_{130}(\nu)\nu^{\bar{\sigma}_3+p_1}\omega(\nu^c, -\alpha_1)\Big(1 + O(\nu^{p_3}\omega(\nu^c, -\alpha_1))\Big) \\ &+ m_{13}(\nu)\nu^{\bar{\sigma}_3+\bar{\sigma}_1 c}\Big[1 + \theta_{13}(\nu, \nu^c, \omega(\nu^c, -\alpha_1))\Big]\end{aligned} \tag{6.133}$$

where $\widehat{m}_{130}(0) = 0$ and $m_{13}(0) = *m_{00}(0) > 0$.

Similar to the case of Ehp1c, it is not difficult to verify that $\forall c \in \mathcal{I}_\nu$ and $\nu > 0$ sufficiently small, there holds $\widetilde{T}_2''(\nu, c) < 0$ and $\widehat{T}_2''(\nu, c) > 0$. Hence the displacement map defined in (6.131) has at most two roots, which gives $Cycl(Ehh10e) \leq 2$.

For the graphic Ehh9c, as shown in Fig. 6.9(a), the map $\widehat{T}$ has a similar expression as in the case of Ehh10e which can be obtained by letting $m_{00}(\nu) = 1$. So it has cyclicity at most 2.



(a) Graphic Ehh5c (b) Graphic Ehh6c

Figure 6.10: Lower boundary graphics Ehh5c and Ehh6c: Displacement maps

(2). Lower boundary graphics Ehh5c and Ehh6c

Since the graphics Ehh5c and Ehh6c pass through a saddle and a saddle node respectively, the transition map $\widehat{T}$ may not be C^2, and the graphics need a special treatment.

Let us see Ehh5c first. As shown in Fig 6.10, in the normal form coordinates in the neighborhood of the saddle point, take sections $\overline{\Sigma}_1 = \{\tilde{x} = -x_0\}$ and $\overline{\Sigma}_3 = \{\tilde{y} = y_0\}$, let $\lambda(\bar{\mu}_0)$ be the hyperbolicity ratio of the saddle point. Then for the transition map

$$\overline{\Delta}_0 = (\bar{d}_0, \overline{D}_0) : \overline{\Sigma}_1 \longrightarrow \Sigma_3$$

its second component $\overline{D}_0(\nu, \tilde{y})$ can be written in the form of 5.46.

Take sections Π_1 and Π_3 as defined in Notation 5.7 in the normal form coordinates, then the transition map $\widehat{T}$ has the following decomposition:

$$\widehat{T} = \Delta_3 \circ \overline{T}_{03} \circ \overline{\Delta}_0 \circ \overline{T}_{10} \circ \Theta_1 \tag{6.134}$$

where

- $\Theta_1 : \tau_1 \longrightarrow \Pi_1$ is the second type Dulac map near P_1 which satisfies Theorem 4.14 with $\sigma = \sigma_1$,
- $\overline{T}_{10} : \Pi_1 \longrightarrow \overline{\Sigma}_1$ and $\overline{T}_{03} : \overline{\Sigma}_3 \longrightarrow \Pi_3$ are C^k regular transition maps they have the same expression as in (6.38) and (6.37) respectively,
- $\Delta_3 : \Pi_3 \longrightarrow \Sigma_3$ is the first type Dulac map near P_3 which satisfies Theorem 4.11 with $\sigma = \sigma_3$.

Since the hyperbolicity ratio λ can be > 1, $= 1$ and < 1, so for graphic Ehh5c we have to deal with three subcases:

(2.1). Case $\lambda(\bar{\mu}_0) > 1$

Let

$$\begin{aligned} \tilde{y}_1 &= \tfrac{\kappa_1}{\rho_0^{p_1}} \nu^{p_1} \omega(\tfrac{\nu^c}{\rho_0}, \alpha_1) + \nu^{\bar{\sigma}_1 c}\Big[l_1 + \theta_1(\nu, \nu^c, \omega(\tfrac{\nu^c}{\rho_0}, -\alpha_1))\Big] \\ \tilde{y}_3 &= m_{030}(\nu) + m_{031}\tilde{y}_1^{\lambda}\Big[\beta_0 + \phi_0(\nu, \tilde{y}_1)\Big] \end{aligned} \tag{6.135}$$

where $l_1 = \frac{y_0}{\rho_0^{\bar{\sigma}_1}} > 0$, $m_{030}(0) = 0$, $m_{031}(0) \neq 0$ and $\phi_0(\nu, \tilde{y}_1) \in (I_0^\infty)$. Then the second component $\widehat{T}_2(\nu, c)$ can be written as

$$\widehat{T}_2(\nu, c) = \kappa_3 \rho_0^{p_3} \omega(\frac{\nu}{\nu_0}, -\beta_3)\Big(\frac{\nu}{\nu_0}\Big)^{\bar{\sigma}_3} + \Big(\frac{\nu}{\nu_0}\Big)^{\bar{\sigma}_3}\Big[\tilde{y}_3 + \phi_3\Big(\nu, \tilde{y}_3, \omega(\frac{\nu}{\nu_0}, -\beta_3)\Big)\Big] \tag{6.136}$$

where $\widetilde{m}_{031}(0) > 0$ and $\tilde{\phi}_{031}$ is C^k and satisfies the property (6.22).

Consider the displacement defined in (6.131). By (6.132) and (6.136), a first derivation of $L_2(\nu, c)$ gives

$$\begin{aligned} L_2'(\nu,c) &= \widehat{T}_2'(\nu,c) - \widetilde{T}_2'(\nu,d) \\ &= \Big(\tfrac{\nu}{\nu_0}\Big)^{\bar{\sigma}_3}\Big[1+\tfrac{\partial\phi_3}{\partial\tilde{y}_3}\Big(\nu,\tilde{y}_3,\omega(\tfrac{\nu}{\nu_0},-\beta_3)\Big)\Big] \\ &\quad \Big[m_{031}(\nu)\lambda\tilde{y}_1^{\lambda_1-1}\Big(1+\phi_{01}(\nu,\tilde{y}_1)\Big)\Big] \\ &\quad \nu^{\bar{\sigma}_1 c}\ln\nu\Big[\bar{\sigma}_1 l_1 + O(\nu^{p_1(1-c)}) + \theta_{11}\Big(\nu,\nu^c,\omega(\tfrac{\nu^c}{\rho_0},-\alpha_1)\Big)\Big] \\ &\quad -\nu^{\bar{\sigma}_3(1-c)}\ln\nu\Big[m_{13}(\nu)\bar{\sigma}_3 + O(\nu^{p_3 c}) + \theta_{41}\Big(\nu,\nu^{1-c},\omega(m_4\nu^{1-c},\beta_3)\Big)\Big] \end{aligned} \tag{6.137}$$

where $\phi_{01} \in (I_0^\infty)$, and θ_{11}, θ_{41} satisfy the property (6.22), also

$$\frac{\partial\phi_3}{\partial\tilde{y}_3} = O\Big(\nu^{p_3}\omega^{q_3}(\frac{\nu}{\nu_0},-\beta_3)\ln\frac{\nu}{\nu_0}\Big).$$

$L_2'(\nu, c)$ has the same number of small roots as of

$$\begin{aligned} L_{21}(\nu,c) &= \frac{\nu^{-\bar{\sigma}_3(1-c)}}{\ln\nu}L_2'(\nu,c) \\ &= -m_{13}(\nu)\bar{\sigma}_3 + O(\nu^{p_3 c}) + \theta_{41}\Big(\nu,\nu^{1-c},\omega(m_4\nu^{1-c},\beta_3)\Big) + O\Big(\tilde{y}_1^{\lambda-1}\Big). \end{aligned} \tag{6.138}$$

Since $m_{13}(0) \neq 0$, so $L_{21}(\nu, c) \neq 0$, thus $L_2(\nu, c) = 0$ has at most one small root, i.e., $Cycl(Ehh5c) \leq 1$.

(2.2). Case $\lambda(\bar{\mu}_0) < 1$

In this case, $L_2'(\nu, c)$ has the same number of small roots as of

$$\begin{aligned} &L_{21}(\nu,c) \\ &= \nu^{\frac{(\bar{\sigma}_1+\bar{\sigma}_3)c}{1-\lambda}}\Big[1+\tfrac{\partial\phi_{31}}{\partial\tilde{y}_3}(\nu,\tilde{y}_3,\omega(\tfrac{\nu}{\nu_0},-\beta_3))\Big] \\ &\quad \Big[(\bar{\sigma}_1 l_1)^{\frac{1}{1-\lambda}} + O(\nu^{p_1(1-c)}) + \theta_{12}(\nu,\nu^c,\omega(\tfrac{\nu^c}{\rho_0},-\alpha_1))\Big] \\ &\quad - \tilde{y}_1(1+\phi_{01}(\nu,\tilde{y}_1))\Big[(m_{13}\bar{\sigma}_3)^{\frac{1}{1-\lambda}} + O(\nu^{p_3 c}) + \theta_{41}(\nu,\nu^{1-c},\omega(m_4\nu^{1-c},\beta_3))\Big] \end{aligned} \tag{6.139}$$

Then

$$
\begin{aligned}
L'_{21}(\nu,c) &= \nu^{\frac{(\bar\sigma_1+\bar\sigma_3)c}{1-\lambda}} \ln\nu \left[1+O\left(\nu^{p_3}\omega^{q_3}(\tfrac{\nu}{\nu_0},-\beta_3)\ln\tfrac{\nu}{\nu_0}\right)\right] \\
&\left[\tfrac{(\bar\sigma_1+\bar\sigma_3)(\bar\sigma_1 l_1)^{\frac{1}{1-\lambda}}}{1-\lambda} + O(\nu^{p_1(1-c)}) + \theta_{13}\left(\nu,\nu^c,\omega(\tfrac{\nu^c}{\rho_0},-\alpha_1)\right)\right] \\
&\\
&-\nu^{\bar\sigma_1 c}\ln\nu\left[l_1\bar\sigma_1 + O(\nu^{p_1(1-c)}) + \theta_{14}(\nu,\nu^c,\omega(\tfrac{\nu^c}{\rho_0},-\alpha_1))\right]\left(1+\phi_{03}(\nu,\tilde y_1)\right) \\
&\left[(m_{13}(\nu)\bar\sigma_3)^{\frac{1}{1-\lambda}} + O(\nu^{p_3 c}) + \theta_{43}(\nu,\nu^{1-c},\omega(m_4\nu^{1-c},\beta_3))\right]
\end{aligned}
\tag{6.140}
$$

which has the same number of small roots as of

$$
L_{22}(\nu,c) = -\frac{\nu^{\bar\sigma_1 c}}{\ln\nu} L'_{21}(\nu,c).
$$

Yet

$$
\begin{aligned}
L_{22}(\nu,c) &= l_1\bar\sigma_1\left(m_{13}(\nu)\bar\sigma_3\right)^{\frac{1}{1-\lambda}}\left[1+\phi_{03}(\nu,\tilde y_1)\right] \\
&\left[1+O(\nu^{p_1(1-c)}) + \theta_{14}\left(\nu,\nu^c,\omega(\tfrac{\nu^c}{\rho_0},-\alpha_1)\right)\right] \\
&\left[1+O(\nu^{p_3 c}) + +\theta_{43}\left(\nu,\nu^{1-c},\omega(m_4\nu^{1-c},\beta_3)\right)\right] + O\left(\nu^{\frac{(\bar\sigma_1\lambda+\bar\sigma_3)c}{1-\lambda}}\right) \\
&\neq 0.
\end{aligned}
\tag{6.141}
$$

Hence, $L_2(\nu,c)=0$ has at most two small roots, therefore $Cycl(Ehh5c)\le 2$.

(2.3). Case $\lambda(\bar\mu_0)=1$

In this case, for the second component $\widehat{T}_2$ of $\widehat{T}$ defined in (6.134), letting

$$
\tilde y = m_{100}(\nu) + m_{101}\nu^{p_1}\omega(\frac{\nu^c}{\rho_0},\alpha_1)) + m_{11}(\nu)\nu^{\bar\sigma_1 c}\left[1+\theta_{11}(\nu,\nu^c,\omega(\frac{\nu^c}{\rho_0},\alpha_1))\right] \tag{6.142}
$$

and using the refinement of Roussarie([R86]) for $\overline{T}_{03}\circ\overline{\Delta}_0$, then a straightforward calculation gives

$$
\begin{aligned}
\widehat{T}_2(\nu,c) &= \alpha_{00}(\nu) + O(\nu^{\bar\sigma_3}\omega(\tfrac{\nu}{\nu_0},-\beta_3)) \\
&+\nu^{\bar\sigma_3}\left[\alpha_{11}(\nu)\tilde y\omega(\tilde y,\alpha_{11}) + \alpha_{22}(\nu)\tilde y + O\left(\nu^{\bar\sigma_3}\tilde y^2\omega(\tilde y,\alpha_{11})\right)\right]
\end{aligned}
\tag{6.143}
$$

where $\alpha_{00}(0) = 0,\ \alpha_{22}(0) \neq 0$.

Then the first derivation of $L_2(\nu, c)$ gives

$$\begin{aligned} L_2'(\nu,c) \ &= \nu^{\bar{\sigma}_3}\Big[\overline{\alpha}_{11}(\nu)\omega(\tilde{y},\alpha_{11}) + \overline{\alpha}_{22}(\nu) + O\Big(\nu^{\bar{\sigma}_3}\tilde{y}\omega(\tilde{y},\alpha_{11})\Big)\Big] \\ &\quad \Big[m_{11}(\nu)\nu^{\bar{\sigma}_1 c}\ln\nu\Big(1 + O(\nu^{p_1(1-c)}) + \theta_{11}(\nu,\nu^c,\omega(\tfrac{\nu^c}{\rho_0},-\alpha_1))\Big)\Big] \\ &\quad -\nu^{\bar{\sigma}_3(1-c)}\ln\nu\Big[m_{13}(\nu)\bar{\sigma}_3 + O(\nu^{p_3 c}) + \theta_{41}(\nu,\nu^{1-c},\omega(m_4\nu^{1-c},\beta_3))\Big] \end{aligned} \tag{6.144}$$

where $\overline{\alpha}_{11}(\nu) = \alpha_{11}(1-\alpha_{11})$ and $\overline{\alpha}_{22} = \alpha_{22} - \alpha_{11}$ with $\overline{\alpha}_{22}(0) \neq 0$.

Denote

$$\begin{aligned} L_{0j}(\nu,c) \ &= 1 + O(\nu^{p_1(1-c)},\nu^{p_3 c}) \\ &\quad +\theta_{1j}(\nu,\nu^c,\omega(\frac{\nu^c}{\rho_0},-\alpha_1)) + \theta_{4j}(\nu,\nu^{1-c},\omega(m_4\nu^{1-c},-\beta_3)) \end{aligned}, \quad j \geq 1.$$

where the θ_{1j} and θ_{4j} will have the similar properties as the θ_{11} and θ_{41} respectively.

Then the equation $L_2'(\nu, c) = 0$ has the same number of small roots as of

$$\begin{aligned} &L_{22}(\nu,c) \\ &= \frac{\nu^{-\bar{\sigma}_3\bar{\sigma}_1 c}L_2'(\nu,c)}{\omega(\tilde{y},\alpha_{11})\ln\nu\Big(1 + O(\nu^{p_1(1-c)}) + \theta_{11}(\nu,\nu^c,\omega(\frac{\nu^c}{\rho_0},-\alpha_1))\Big)} \\ &= m_{11}(\nu)\overline{\alpha}_{11}(\nu) + m_{11}(\nu)\frac{\overline{\alpha}_{22}(\nu)}{\omega(\tilde{y},\alpha_{11})} - \frac{m_{13}(\nu)\nu^{-(\bar{\sigma}_1+\bar{\sigma}_3)c}L_{01}(\nu,c)}{\omega(\tilde{y},\alpha_{11})} \end{aligned} \tag{6.145}$$

The number of roots of the equation $L_{22}(\nu, c) = 0$ is at most one plus the number of roots of

$$\begin{aligned} L_{23}(\nu,c) \ &= \frac{\nu^{(\bar{\sigma}_1+\bar{\sigma}_3)c}\tilde{y}^{1+\alpha_{11}}\omega^2(\tilde{y},\alpha_{11})L_{22}'(\nu,c)}{(\bar{\sigma}_1+\bar{\sigma}_3)m_{13}(\nu)L_{02}(\nu,c)\ln\nu} \\ &= \tilde{y}^{1+\overline{\alpha}_{11}}\omega(\tilde{y},\alpha_{11}) \ - \ \nu^{\bar{\sigma}_1 c}L_{02}(\nu,c) + O\Big(\nu^{(2\bar{\sigma}_1+\bar{\sigma}_3)c}\Big). \end{aligned} \tag{6.146}$$

Let

$$L_{24}(\nu,c) = \frac{L_{23}'(\nu,c)}{\nu^{\bar{\sigma}_1 c}\ln\nu\Big[1 + O(\nu^{p_1(1-c)}) + \theta_{11}(\nu,\nu^c,\omega(\frac{\nu^c}{\rho_0},-\alpha_1))\Big]}$$

Then

$$
\begin{aligned}
L_{24}(\nu,c) &= -m_{11}(\nu)\Big[1+(1+\overline{\alpha}_{11})\tilde{y}^{\overline{\alpha}_{11}}\omega(\tilde{y},\alpha_{11})\Big] \\
&\qquad -\bar{\sigma}_1 L_{03}(\nu,c)+O\Big(\nu^{(2\bar{\sigma}_1+\bar{\sigma}_3)c}\Big) \\
&\neq 0.
\end{aligned}
\tag{6.147}
$$

where the term $\tilde{y}^{\overline{\alpha}_{11}}\omega(\tilde{y},\alpha_{11})$ is positive and sufficiently large.

Therefore, $L_2(\nu,c)=0$ has at most 3 small roots which gives $Cycl(Ehh5c)\leq$ 3.

Now let us study the graphic Ehh6c. In the decomposition of $\widehat{T}$, the second component of the transition map $\overline{\delta}_0=(\overline{d}_0,\overline{D}_0)$ satisfies

$$
\frac{\partial^{i_1}\overline{D}_0}{\partial\tilde{y}^{i_1}}(\nu,\tilde{y})=O\Big(\tilde{y}^{i_2}\Big),\qquad \forall i_1,i_2\in\mathbb{N}. \tag{6.148}
$$

Still letting $\tilde{y}$ be defined as in (6.142), also letting

$$
\tilde{y}_3=m_{030}(\nu)+O\Big(\tilde{y}^{i_2}\Big),\qquad i_2\geq 2
$$

then for $\widehat{T}_2(\nu,c)$, we have (6.134) gives

$$
\widehat{T}_2(\nu,c)=(\frac{\nu}{\nu_0})^{\bar{\sigma}_3}\omega(\frac{\nu}{\nu_0},-\beta_1)+(\frac{\nu}{\nu_0})^{\bar{\sigma}_3}\Big[\tilde{y}_3+\phi_3(\nu,\tilde{y}_3,\omega(\frac{\nu}{\nu_0},-\beta_1))\Big] \tag{6.149}
$$

Then a first derivation of $L_2(\nu,c)$ gives

$$
\begin{aligned}
L_2'(\nu,c) &= \widehat{T}_2'(\nu,c)-\widetilde{T}_2'(\nu,d) \\
&= \Big(\tfrac{\nu}{\nu_0}\Big)^{\bar{\sigma}_3}O\Big(\tilde{y}^{i_2-1}\Big)\Big[1+\tfrac{\partial\phi_3}{\partial\tilde{y}_3}\Big(\nu,\tilde{y}_3,\omega(\tfrac{\nu}{\nu_0},-\beta_3)\Big)\Big] \\
&\quad \nu^{\bar{\sigma}_1 c}\ln\nu\Big[\bar{\sigma}_1 l_1+O(\nu^{p_1(1-c)})+\theta_{11}(\nu,\nu^c,\omega(\tfrac{\nu^c}{\rho_0},-\alpha_1))\Big] \\
&\quad -\nu^{\bar{\sigma}_3(1-c)}\ln\nu\,[m_{13}(\nu)\bar{\sigma}_3+O(\nu^{p_3c})+\theta_{41}(\nu,\nu^{1-c},\omega(m_4\nu^{1-c},\beta_3))]
\end{aligned}
\tag{6.150}
$$

which has the same number of small roots as

$$
\begin{aligned}
L_{21}(\nu,c) &= \frac{L_2'(\nu,c)\nu^{\bar{\sigma}_3(c-1)}}{\ln\nu} \\
&= -\,m_{13}(\nu)\bar{\sigma}_3+O(\nu^{p_3c})+\theta_{41}\Big(\nu,\nu^{1-c},\omega(m_4\nu^{1-c},\beta_3)\Big)+O\Big(\tilde{y}^{i_2-1}\Big) \\
&\neq 0
\end{aligned}
$$

Therefore, $L(\nu,c)=0$ has at most one small root, i.e., $Cycl(Ehh6c)\leq 1$. □

6.3.4 Intermediate graphics of the Ehh families

Now we study the cyclicity of intermediate graphics of the Ehh families.

Theorem 6.20. *Under the generic assumption, all the intermediate hh-graphics of elliptic type of the 12 families Ehh1, Ehh2, $\cdots$, Ehh12 have finite cyclicity.*

Proof. Let Γ be any of the intermediate hh-graphics of elliptic type of the 12 families. Similar to the intermediate concave graphics of saddle type, take sections Π_3 and Π_4 as defined in (5.22) in the normal form coordinates in the neighborhood of P_3 and P_4 respectively. Let

$$\begin{aligned} T:\ \ &\Pi_3 \longrightarrow \Pi_4 \\ &(\nu, \tilde{y}_3) \mapsto (\nu, T_2(\nu, \tilde{y}_3)) \end{aligned}$$

be the transition map similar to the map defined in Prop. 5.11. Then by Prop. 5.11, for each of the intermediate graphic in the 12 families, to prove their finite cyclicity, we only need to verify that for $\nu = 0$, the corresponding transition map T with second component $T_2(0, \tilde{y}_3)$ satisfies one of the conditions listed in Prop. 5.11.

We are going to discuss the transition map $T_2(0, \tilde{y}_3)$ in the chart F.R. on $r = 0$. Recall that in §3.1, we use the quasi-homogeneous compactification (3.15), or

$$z = \frac{1}{\bar{x}}, \qquad u = \frac{\bar{y}}{\bar{x}^2}$$

to study the singularities at infinity on $r = 0$ in the chart F.R.. So by (3.11) we have $(z, u) = (\rho_2, y_2)$, which is precisely the same coordinates we use in the chart P.R.2. Hence in the neighborhood of P_3 and P_4 in chart F.R. on $r = 0$, we still use the coordinates (ρ_2, y_2). By taking $r_3 = 0$ and $r_4 = 0$ in the normal forms in the neighborhood of P_3 and P_4 in Prop. 4.6, we obtain the normal forms in the neighborhoods of P_3 and P_4 in the chart F.R. on $r = 0$. Near P_3,

$$\begin{cases} \dot{\rho}_3 &= -\rho_3 \\ \dot{\tilde{y}}_3 &= -\sigma_3(a)\tilde{y}_3 + \kappa_3 \rho_3^{p_3} \end{cases} \tag{6.151}$$

and in the neighborhood of P_4, we have

$$\begin{cases} \dot{\rho}_4 &= \rho_4 \\ \dot{\tilde{y}}_4 &= \sigma_3(a)\tilde{y}_4 - \kappa_3\rho_4^{p_3} \end{cases} \tag{6.152}$$

where if $a \neq \frac{1}{4}$ then $\kappa_3 = 0$.

Since we are only calculating the map T for $\nu = 0$ and in the chart F.R., $r = \nu$ is invariant, so let $\pi_i = \{\rho_i = \rho_0\}$ $(i = 3, 4)$ be the two line sections in the chart F.R. on $r = 0$ parametrized by $\tilde{y}_i$. Then we are reduced to study the one dimension transition map

$$T_2(0, \tilde{y}_4) : \pi_4 \longrightarrow \pi_3$$

or its inverse. We will verify that for each family, the corresponding map $T_2(0, \tilde{y}_4)$ or its inverse satisfies one of the sufficient conditions listed in Prop. 5.11.

(1). Families Ehh1, Ehh2 and Ehh3

We begin with the family Ehh1. Let Γ be any intermediate graphic of the family Ehh1. Since the systems (6.151) and (6.152) exist globally, so the map T_2 exists globally on π_4 and not only in the neighborhood of $\pi_4 \cap \Gamma$. We are going to prove that $T_2(0, \tilde{y}_4)$ is either the identity or nonlinear. By Prop. 5.12, to prove the nonlinearity of T_2, it suffices to prove that it is nonlinear at certain point on π_3. To do this, as shown in Fig. 6.11, we take line sections $\tau_4 = \{\tilde{y}_4 = -y_0\}$ and $\tau_3 = \{\tilde{y}_3 = -y_0\}$ in the normal form coordinates and the sections τ_3 and τ_4 are chosen such that any intermediate graphic of the family intersects τ_i or π_i inside the neighborhood of P_i $(i = 3, 4)$ respectively. Then over some subinterval of π_4 the map T_2 can be factorized as

$$T_2 = S_3 \circ \widehat{T}_2 \circ S_4^{-1} \tag{6.153}$$

where as shown in Fig. 6.11(a), $S_i : \tau_i \longrightarrow \pi_i$ $(i = 3, 4)$ are the regular transition maps in the normal form coordinates in the neighborhood of P_3 and P_4 respectively, and $\widehat{T}_2 : \tau_4 \longrightarrow \tau_3$ is the transition map which is in particular defined near the lower boundary graphic Ehh1c.

We first calculate S_3 and S_4^{-1}. Due to the easy form of system (6.151), the transition S_3 can be directly calculated by integration, and

$$S_3(0, \rho_3) = \frac{1}{\rho_3^{\sigma_3}}[-C_1 + C_2\kappa_3 \ln \rho_3] \tag{6.154}$$

where C_1 and C_2 are positive constants. Let $v_3 = \frac{1}{\bar{y}_3}$, if we parametrize section π_3 by v_3, then by (6.154), for the map S_3, we have

$$S_3 : v_3 = \frac{\rho_3^{\sigma_3}}{-C_1 + C_2\kappa_3 \ln \rho_3}. \tag{6.155}$$

Note that if we reverse the time in system (6.151) we get system (6.152), so if let $v_4 = \frac{1}{\bar{y}_4}$ and parametrize π_4 by v_4, then we have

$$S_4 : v_4 = \frac{\rho_4^{\sigma_3}}{-C_1 + C_2\kappa_3 \ln \rho_4}. \tag{6.156}$$

By (6.156), the transition map S_4 sends the points on section τ_4 in the positive neighborhood of 0 to the points on the section π_4 at infinity.

Remark 6.21. *Although the normal form is only valid in the neighborhood of P_3 and P_4, the systems* (6.151) *and* (6.152) *exist globally.*

Note that if $\bar{\mu}_{30} = 0$, then $\widehat{T}_2$ is the identity since the system is symmetric. In the case $\bar{\mu}_{30} \neq 0$ we now turn to the calculation of $\widehat{T}_2$. As shown in Fig. 6.11(b), there are two saddles P_1 and P_2 at infinity in the chart F.R. on $r = 0$, so $\widehat{T}_2$ can be calculated by the following decomposition

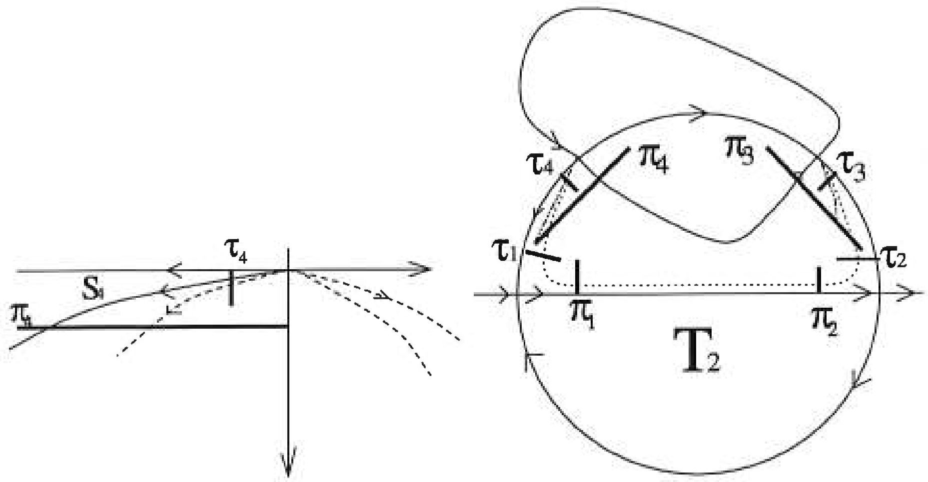
$$\widehat{T}_2 = V_2 \circ \overline{D}_2 \circ S_2 \circ \overline{D}_1 \circ U_2^{-1}. \tag{6.157}$$

Note that $\widehat{T}_2$ is not necessarily valid in the neighborhood of $\Gamma \cap \tau_4$, but we consider it near the end point of the interval of definition of T_2. For the components of $\widehat{T}_2$ in (6.157), we have

- U_2 and V_2 are regular transition maps defined in Prop. 6.13 and Corollary. 6.14. By (6.55) and (6.79), we have

$$\begin{aligned} U_2^{-1}(0, \rho_4) &= m_{41}\rho_4 + m_{42}\rho_4^2 + O(\rho_4^3) \\ V_2(0, \rho_2) &= \frac{1}{m_{41}}\left[\rho_2 - \frac{m_{42}}{m_{41}^2}\rho_2^2 + O(\rho_2^3)\right]. \end{aligned} \tag{6.158}$$

  Also by Prop. 6.13 and Corollary. 6.14, we have $m_{42} \neq 0$ since $\bar{\mu}_{30} \neq 0$.



(a) Regular transition (b) T_2

Figure 6.11: Transition map T for the intermediate graphic of family Ehh1

- $\overline{D}_1$ and $\overline{D}_2$ are Dulac maps in the neighborhood of the infinite singular points P_1 and P_2

$$\overline{D}_1 : \tau_1 \longrightarrow \pi_1$$
$$\overline{D}_2 : \pi_2 \longrightarrow \tau_2$$

and

$$\begin{aligned}
\overline{D}_1(0,\rho_1) &= \begin{cases} \rho_1^{\sigma_1}(\beta_{10} + \phi_{11}(0,\rho_1)) & \text{if } \sigma_1 \neq 1 \\ \beta_{10}\rho_1 + \alpha_1\rho_1\omega_1[1+\cdots] + \alpha_2\rho_1^2\omega_1[1+\cdots] + \cdots & \text{if } \sigma_1 = 1 \end{cases} \\
\overline{D}_2(0,\tilde{y}_2) &= \begin{cases} \tilde{y}_2^{\frac{1}{\sigma_1}}(\bar{\beta}_{10} + \bar{\phi}_{11}(0,\tilde{y}_2)) & \text{if } \sigma_1 \neq 1 \\ \bar{\beta}_{10}\tilde{y}_2 + \alpha_1\tilde{y}_2\omega_2[1+\cdots] + \alpha_2\tilde{y}_2^2\omega_2[1+\cdots] + \cdots & \text{if } \sigma_1 = 1 \end{cases}
\end{aligned} \tag{6.159}$$

where $\omega_1 = \omega(\rho_1, \alpha_1)$ and $\omega_2 = \omega(\tilde{y}_2, \alpha_1)$, ϕ_{11}, $\bar{\phi}_{11}$ satisfy (I_0^∞).

- S_2 is the second component of the transition map S defined in Prop. 6.1 and satisfies (6.2).

It follows from (6.157), (6.158), (6.159) and (6.2) that we have

- Case $\sigma_1 \neq 1$:

$$\widehat{T}_2(0,\rho_4) = m_1\rho_4 + \hat{m}_{42}\rho_4^2 + \hat{m}_2\rho_4^{1+\sigma_1} + \hat{\phi}_1(\rho_4, \omega(\rho_4,\alpha_1)) \tag{6.160}$$

where $\hat{m}_2 = *S_2''(0) = *\bar{\mu}_{30} \neq 0$, and $\hat{m}_{42} = *m_{42} \neq 0$, also $\hat{\phi}_1(\rho_4, \omega(\rho_4,\alpha_1))$ is C^∞ and satisfies I_0^∞.

- Case $\sigma_1 = 1$:

$$\begin{aligned}\widehat{T}_2(0,\rho_4) &= m_1\rho_4 + \alpha_1\tilde{m}_2\rho_4\omega(\rho_4,\alpha_1)[1+\cdots] \\ &\quad +\alpha_2\tilde{m}_1\rho_4^2\omega(\rho_4,\alpha_1)[1+\cdots]+\cdots\end{aligned} \tag{6.161}$$

where $\tilde{m}_1 \neq 0$. For the case $\sigma_1 = 1$ ($a = \frac{1}{3}$), by (6.129), we have $\alpha_2 = *\bar{\mu}_3 \neq 0$.

So for both the cases $\sigma_1 \neq 1$ and $\sigma_1 = 1$, by (6.160) and (6.161), if $\bar{\mu}_3 \neq 0$, $\widehat{T}_2(0,\rho_4)$ is nonlinear in ρ_4.

Now we show that the map $T_2(0,\tilde{y}_4)$ is nonlinear if $\bar{\mu}_{30} \neq 0$. Indeed, by (6.153), we have

$$S_4 = T_2^{-1} \circ S_3 \circ \widehat{T}_2. \tag{6.162}$$

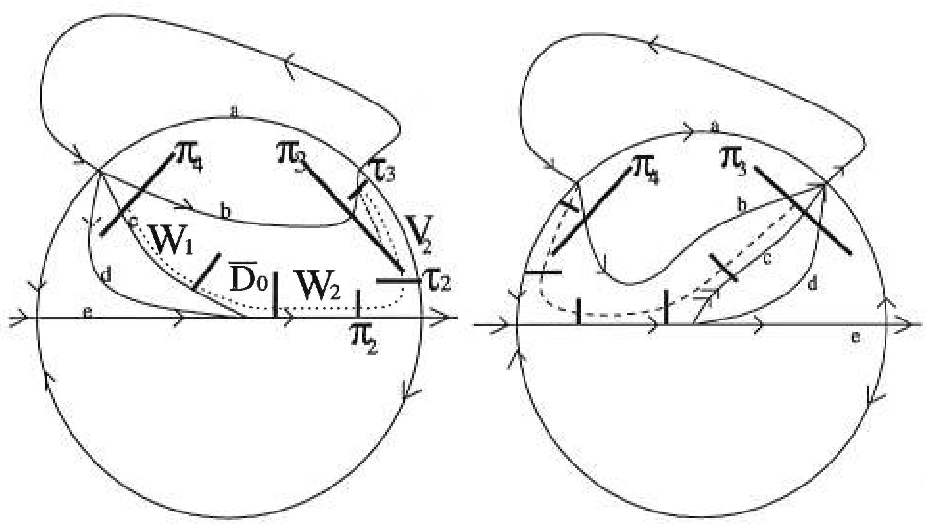
If $T_2(0,\tilde{y}_4)$ is linear in $\tilde{y}_4$, i.e., $T_2(0,\tilde{y}_4) = \hat{b}\tilde{y}_4$ ($\hat{b} \neq 0$), then by (6.162) and (6.155), we should have

$$S_4(0,\rho_4) = \frac{\left(\widehat{T}_2(0,\rho_4)\right)^{\sigma_3}}{\hat{b}\Big[-C_1 + C_2\kappa_3 \ln \widehat{T}_2(0,\rho_4)\Big]}$$

which is a contradiction to (6.156) for all the cases of σ_3 and κ_3.

Therefore, $T_2(0,\tilde{y}_4)$ is either the identity or nonlinear for $\tilde{y}_4 \in \mathbb{R}$. Thus for both cases, the intermediate graphics of the family Ehh1 have finite cyclicity.

Now we consider family Ehh3. As in Fig. 6.12(a), we have a family of intermediate graphics Ehh3b, Ehh3c and Ehh3d. Note that Ehh3c is similar to the graphic Ehh6c while Ehh2d is similar to the graphic Ehh10e. In Theorem 6.19, we have proved Ehh6c and Ehh10e have finite cyclicity. Here instead of starting from section τ_1, we consider the displacement map defined on section τ_2 with image on π_2, we conclude that $Cycl(Ehh3c) \leq 1$ and $Cycl(Ehh3d) \leq 2$. To study the cyclicity of the graphic Ehh3b, we study the transition map T_2 defined on π_4 in the neighborhood of the graphic Ehh3c.



(a) Family Ehh3

(b) Family Ehh2

Figure 6.12: Transition map T for the intermediate graphics of Ehh2 and Ehh3

For Ehh3c, it pass through a attracting saddle node. As shown in Fig. 6.12(a), let $\tilde{\tau}_4 = \{\tilde{y} = y_0\}$ and $\tilde{\tau}_2 = \{\tilde{x} = x_0\}$ be two sections in the neighborhood of the saddle node. Then the corresponding transition map T_2 can be factorized as

$$\begin{aligned} T_2 : \quad & \pi_4 \longrightarrow \pi_3 \\ & T_2 = S_3 \circ V_2 \circ \overline{D}_2 \circ W_2 \circ \overline{D}_0 \circ W_1 \end{aligned} \tag{6.163}$$

where

- $W_1 : \pi_4 \longrightarrow \tilde{\tau}_4$, a C^k regular transition map
- $\overline{D}_0 : \{\tilde{y} = y_0\} \longrightarrow \{\tilde{x} = x_0\}$ is the transition in the neighborhood of the

saddle node in the normal form coordinates. Then $\forall n_1, n_2 \in \mathbb{N}$, we have

$$\frac{\partial^{n_1}\overline{D}_0}{\partial \tilde{x}^{n_1}}(0,\tilde{x}) = O\left(\tilde{x}^{n_2}\right) \tag{6.164}$$

- $W_2 : \tilde{\tau}_2 \longrightarrow \pi_2$ a C^k regular transition map
- $\overline{D}_2$ is the Dulac map in the neighborhood of P_2 and has the form of the (6.159)
- S_3 and V_2 are regular transition maps given in (6.155) and (6.158) respectively.

Note that $\tilde{y}_2 = W_2 \circ \overline{D}_0 \circ W_1(0, \tilde{y}_4)$ as the function of $\tilde{y}_4$, it satisfies (6.164) too, then by (6.163), for T_2, we have $\lim_{\tilde{y}_4 \to 0} T_2(0, \tilde{y}_4) = -\infty$. Hence T_2 maps $(0, \infty)$ to $(-\infty, \infty)$. Since $T_2(0, \tilde{y}_4)$ is analytic and bijective, it has to be nonlinear in $\tilde{y}_4$, therefore it is nonlinear for $\tilde{y}_4 \in \mathbb{R}^+$, thus any intermediate graphic Ehh3b has finite cyclicity.

The finite cyclicity of family Ehh2 follows from Remark 6.18.

(2). Family Ehh4.

For the family Ehh4, the lower boundary graphic passes through a hyperbolic saddle, it has the same structure as the family Sxhh1 of saddle type. The only difference is that the value of a_0 has a different sign, which does not influence the proof. So, the family Ehh4 has finite cyclicity.

(3). Families Ehh9, Ehh10, Ehh11 and Ehh12

As remarked in Remark 6.18, we only need to consider the intermediate graphics for the family Ehh9 and Ehh10. We first consider the family Ehh10 with a attracting saddle-node on its lower boundary(Fig. 6.13(b)). In Theorem 6.19**(1)**, we have proved that Ehh10e has finite cyclicity at most 2. Now we study the transition map $T_2(0, \tilde{y}_4)$ associated with Ehh10d and prove that it is nonlinear. We could have proved directly that Ehh10d has cyclicity ≤ 1, but the proof given here will work with a very small modification for the intermediate graphics of Ehh9.

For the system in the neighborhood of the attracting saddle node, by a C^k normal form coordinate change, we bring the system into normal form. In the normal form coordinates $(\tilde{x}, \tilde{y})$, take sections $\tilde{\tau}_1 = \{\tilde{x} = -x_0\}$ and $\tilde{\tau}_2 = \{\tilde{x} = x_0\}$. Then the central-transition map $\overline{D}_0 : \tilde{\tau}_1 \longrightarrow \tilde{\tau}_2$ satisfies

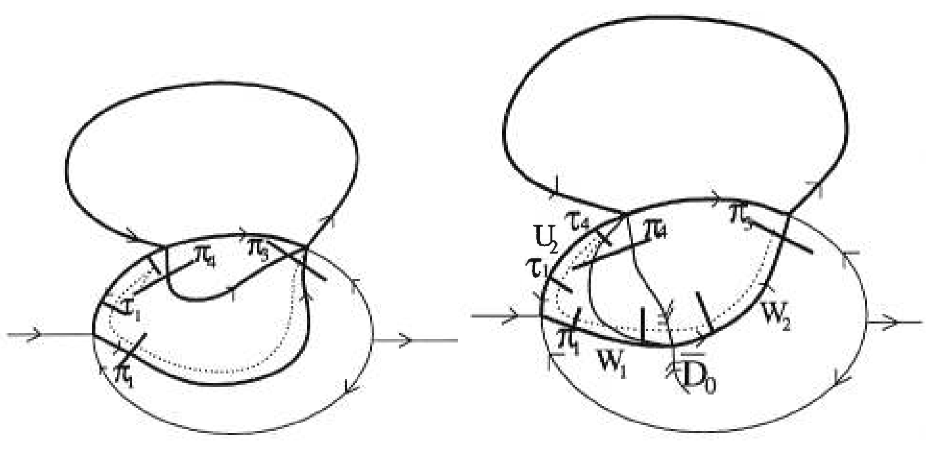
$$\overline{D}_0(0, \tilde{y}) = \overline{m}_0(\bar{\mu})\tilde{y} \tag{6.165}$$

where $\lim_{\bar{\mu}_2 \to 0-} \overline{m}_0(\bar{\mu}) = 0$.

As shown in Fig. 6.13(b), the corresponding transition map T_2 can be factorized as

$$\begin{aligned} T_2 &= \widehat{T}_2 \circ S_4^{-1} \\ \widehat{T}_2 &= W_2 \circ \overline{D}_0 \circ W_1 \circ \overline{D}_1 \circ U_2 \end{aligned} \tag{6.166}$$

where U_2, S_4 are regular transition maps along the lower boundary graphics which are given in (6.156) and (6.158) respectively, $W_1 : \pi_1 \longrightarrow \tilde{\tau}_1$ and $W_2 : \tilde{\tau}_2 \longrightarrow \pi_3$ are regular transition maps, and $\overline{D}_1 : \tau_1 \longrightarrow \pi_1$ is the Dulac in the neighborhood of P_1 in the form of (6.159).



(a) Ehh9b (b) Ehh10d

Figure 6.13: Transition map T for the families Ehh9 and Ehh10

Then a straightforward calculation gives

- Case $\sigma_1 \neq 1$:

$$\widehat{T}_2(0,\rho_4) = \tilde{m}_0(\bar{\mu})\rho_4^{\sigma_1} + o(\rho_4^{\sigma_1}) \tag{6.167}$$

where $\tilde{m}_0 = *\overline{m}_0$.

- Case $\sigma_1 = 1$ $(a = \frac{1}{3})$:

$$\begin{aligned}\widehat{T}_2(0,\rho_4) &= \overline{m}_0(\bar{\mu})\Big[\gamma_1\rho_4 + \alpha_1\rho_4\omega(\rho_4,\alpha_1)[1+\cdots] \\ &\quad +\alpha_2\rho_4^2\omega(\rho_4,\alpha_1)[1+\cdots]+\cdots\Big]\end{aligned} \tag{6.168}$$

where by (6.129), we have the saddle quantity $\alpha_2 = *\bar{\mu}_{30} \neq 0$ since $\bar{\mu}_{30} \neq 0$.

Let $v_4 = \frac{1}{\tilde{y}_4}$, we parametrize section π_4 by v_4 and denote $\widetilde{T}_2(0,v_4) = T_2(0,\tilde{y}_4)$. We claim that the map $\widetilde{T}_2(0,v_4)$ is nonlinear in v_4 in the neighborhood of $v_4 = 0$. Indeed, if not, then it is linear, i.e., $\widetilde{T}_2(0,v_4) = \hat{b}v_4$ $(\hat{b} \neq 0)$, then by (6.166) we have $\widetilde{T}_2 \circ S_4 = \widehat{T}_2$ or depending on σ_1, we have two cases

- Case $\sigma_1 \neq 1$:

$$b\frac{\rho_4^{\sigma_3}}{-C_1 + C_2\ln\rho_4} = \overline{m}_0(\bar{\mu})\gamma_1(\nu)\rho_4^{\sigma_1} + o(\rho_4^{\sigma_1}).$$

- Case $\sigma_1 = 1$:

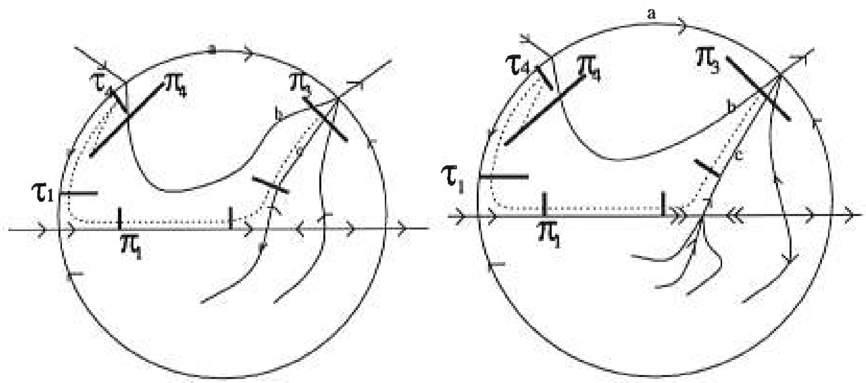
$$\begin{aligned}b\frac{\rho_4^{\sigma_3}}{-C_1+C_2\ln\rho_4} = \;& \gamma_0(\nu) + \overline{m}_0(\bar{\mu})\Big[\gamma_1(\nu)\rho_4 + \alpha_1\rho_4\omega(\rho_4,\alpha_1)[1+\cdots] \\ & +\alpha_2\rho_4^2\omega(\rho_4,\alpha_1)[1+\cdots]+\cdots\Big]\end{aligned}$$

Since $\sigma_1 = \frac{1-2a}{a}$, $\sigma_3 = 2(1-2a)$ and $\forall a \in (0,\frac{1}{2})$, $\sigma_1 \neq \sigma_3$, the above equations are impossible.

Therefore $\widetilde{T}_2(0,v_4)$ is nonlinear in v_4 for v_4 sufficiently small, i.e., $T_2(0,\tilde{y}_4)$ is nonlinear in $\tilde{y}_4$ for $\tilde{y}_4$ sufficiently large. $T_2(0,\tilde{y}_4)$ is analytic, thus $T_2(0,\tilde{y}_4)$ is nonlinear for $\tilde{y}_4 \in \mathbb{R}$. Hence, all the intermediate graphics in the family Ehh10 have finite cyclicity.

Now we turn to the family Ehh9. By Fig. 6.13(b), we see that this family of intermediate graphics can be treated as the graphics of family Ehh10. It suffices to take the central transition map $\overline{S}$ as identity (i.e., $\overline{m}_0 \equiv 1$)

(4). Families Ehh5, Ehh6, Ehh7 and Ehh8



(a) Ehh5b (b) Ehh6b

Figure 6.14: Transition map T for the families Ehh5 and Ehh6

By Remark 6.18, we only need to study families Ehh5 and Ehh6. We first consider family Ehh5. As shown in Fig. 6.14, the lower boundary graphic Ehh5c passes through two saddle points. One is at the infinity P_1, the other lies on the invariant line $\bar{y} = 0$. We have a saddle connection.

In the normal form coordinates $(\tilde{x}, \tilde{y})$ near the finite saddle, take sections $\tilde{\tau}_1 = \{\tilde{x} = -x_0\}$ and $\tilde{\tau}_3 = \{\tilde{y} = y_0\}$, let $\overline{\Delta}_0 : \{\tilde{x} = -x_0\} \longrightarrow \{\tilde{y} = y_0\}$ be the Dulac map, then similar to the family Ehh10, the corresponding transition map T_2 can be factorized as (6.166). Then similar to the case of Ehh 2 we can prove that $\lim_{\tilde{y}_4 \to -\infty} T_2(0, \tilde{y}_4) = 0$ which means the map T_2 maps $(-\infty, \infty)$ to $(0, \infty)$. Since T_2 is bijective and analytic, it has to be nonlinear. Hence all the intermediate graphics Ehh5b have finite cyclicity.

For the family Ehh6, it has a attracting saddle node on the lower boundary graphic. This is similar to the family Ehh2.

Altogether, we have proved that all the intermediate graphics of the Ehh type have finite cyclicity. □

Chapter 7

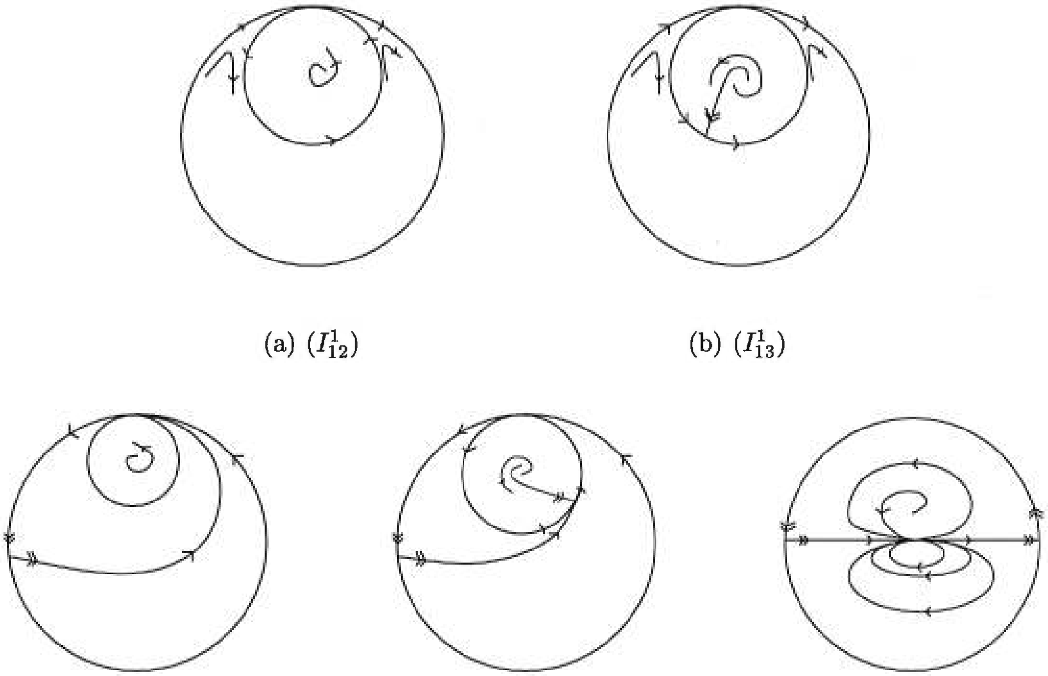
Application of the main theorems to quadratic systems

Hilbert's 16th Problem ([H]) *Finding the maximum number $H(2)$ and relative positions of limit cycles of quadratic vector fields.*

For Hilbert's 16th problem, till now we only know that $H(2) \geq 4$. In [DRR94], Dumortier, Roussarie and Rousseau launched a program aiming at solving the finiteness part of the Hilbert's 16th problem for quadratic vector fields. The paper listed all the 121 limit periodic sets surrounding the origin in a family of quadratic vector fields and reduced the finiteness problem for quadratic systems to the proof that all of these graphics have finite cyclicity. Up to now, about 50 elementary graphics have been proved to have finite cyclicity. To finish the program, it is absolutely essential to be able to prove the finite cyclicity of non-elementary graphics. By the results of this work, we will be able to prove that more than 20 non-elementary graphics have finite cyclicity.

As an application of the main theorems to quadratic systems, in this thesis, we only prove that some of the graphics through a triple nilpotent singular point (listed in Fig. 7.1) have finite cyclicity. More will appear in a forthcoming publication.

Theorem 7.1. *For quadratic systems, the graphics (I^1_{12}), (I^1_{13}), (I^1_{9b}) and (I^1_{11b}) have finite cyclicity if the nilpotent singular point is of codimension 3 (In the proof this condition is calculated explicitly).*



(a) (I_{12}^1) (b) (I_{13}^1)

(c) (I_{9b}^1) (d) (I_{11b}^1) (e) $(H_6^1),(F_{6a}^1)$

Figure 7.1: Some graphics with triple nilpotent singularity for quadratic systems

Proof. By [DRR94], the graphics (I_{12}^1) and (I_{9b}^1) occur in the family

$$\begin{cases} \dot{x} &= \lambda x - y + \varepsilon_1 x^2 \\ \dot{y} &= x + \lambda y + \delta_1 x^2 + \delta_2 xy \end{cases} \tag{7.1}$$

where $0 < \delta_2 < \varepsilon_1$ for (I_{12}^1) (resp. $\varepsilon_1 < \delta_2 < 2\varepsilon_1$ for (I_{9b}^1)), $\delta_1 = \lambda(3\varepsilon_1 - \delta_2) > 0$ and $4\varepsilon_1\lambda^2 - (1+\lambda^2)\delta_2 < 0$. By rescaling we can assume that $\varepsilon_1 = 1$. Then (7.1) becomes

$$\begin{cases} \dot{x} &= \lambda x - y + x^2 := P(x,y) \\ \dot{y} &= x + \lambda y + \lambda(3-\delta_2)x^2 + \delta_2 xy := Q(x,y) \end{cases} \tag{7.2}$$

with $\Delta_0 = (1+\lambda^2)\delta_2 - 4\lambda^2 > 0$ and $\lambda \neq 0$. System (7.2) has an invariant parabola

$$y = (1 - \frac{\delta_2}{2})x^2 - \lambda x - \frac{1}{2}(1+\lambda^2). \tag{7.3}$$

To prove that the graphic (I_{12}^1) (resp. (I_{12}^1)) has finite cyclicity, by Theorem 5.5 and Prop. 5.3 (resp. Theorem 6.10 and Remark 5.4), we only need to prove that, for the quantity γ^* defined in (5.10), there holds $\gamma^* \neq 1$.

Indeed, along the graphic (the invariant parabola (7.3)), we have

$$\begin{aligned}\gamma^* &= \exp\left(\int_{-\infty}^{\infty} div\ X(\gamma(t))\Big|_{(7.3)} dt\right)\\ &= \lim_{x_0\to\infty} \exp\left(\int_{-x_0}^{x_0} \frac{2\lambda+(2+\delta_2)x}{\delta_2 x^2+4\lambda x+1+\lambda^2} dx\right)\\ &= \lim_{x_0\to\infty}\left[\left(\frac{\delta_2 x_0^2+4\lambda x_0+1+\lambda^2}{\delta_2 x_0^2-4\lambda x_0+1+\lambda^2}\right)^{\frac{2+\delta_2}{2\delta_2}} \exp\left(-\frac{8\lambda}{\delta_2\sqrt{\Delta_0}}\arctan\frac{\delta_2 x_0+2\lambda}{\sqrt{\Delta_0}}\right)\right]\\ &= \exp\left(-\frac{4\lambda\pi}{\delta_2\sqrt{\Delta_0}}\right)\\ &\neq 1\end{aligned} \tag{7.4}$$

where $\Delta_0 = (1+\lambda^2)\delta_2 - 4\lambda^2 > 0$ and $\lambda \neq 0$.

So, the graphics (I_{12}^1) and (I_{9b}^1) have finite cyclicity if the nilpotent point has codimension 3. We now calculate the codimension of the nilpotent singularity point.

To study the triple nilpotent singular point on the graphic (at infinity), we introduce the coordinates $(v, z) = (\frac{x}{y}, \frac{1}{y})$. Then we have

$$\begin{cases} \dot v &= -z + (1-\delta_2)v^2 - \lambda(3-\delta_2)v^3 - v^2 z\\ \dot z &= -\delta_2 vz - \lambda z^2 - \lambda(3-\delta_2)v^2 z - vz^2\end{cases} \tag{7.5}$$

Similar to what we did in Theorem 2.3 for quadratic systems, by a near-identity transformation and rescaling, we bring system (7.5) into a "standard form"

$$\begin{cases} \dot v &= w\\ \dot w &= \varepsilon_1 v^3 + w[bv+\varepsilon_2 v^2] + O(v^5) + w^2 O(|(v,w)|^2)\end{cases} \tag{7.6}$$

where $\varepsilon_1 = sgn(\delta_2(\delta_2-1))$ and

$$\begin{aligned} b &= \frac{2-3\delta_2}{\sqrt{|\delta_2(\delta_2-1)|}},\\ \varepsilon_2 &= -\frac{2\lambda(2\delta_2-3)(\delta_2-4)}{\sqrt{|\delta_2(\delta_2-1)|^3}}.\end{aligned}$$

Note that $b = 2\sqrt{2} \iff \delta_2 = 2$. So for $\lambda \neq 0$ and $\delta_2 \in (1,2)$, if $\delta_2 \neq \frac{3}{2}$, the nilpotent elliptic point is of codimension 3, and for $\lambda \neq 0$ and $\delta_2 \in (0,1)$, if $\delta_2 \neq \frac{2}{3}$, the nilpotent saddle point is of codimension 3. Then (I_{12}^1) has finite cyclicity if $\delta_2 \neq \frac{2}{3}$, (I_{9b}^1) has finite cyclicity if $\delta_2 \neq \frac{3}{2}$.

For the graphics (I_{13}^1) and (I_{11b}^1), since $\Delta_0 = 0$ and there exists an attracting saddle node on the invariant parabola, then $\overline{R}'(0)$ is very small. Thus (I_{13}^1) has finite cyclicity if $\delta_2 \neq \frac{2}{3}$; (I_{11b}^1) has finite cyclicity if $\delta_2 \neq \frac{3}{2}$.

□

Theorem 7.2. *$Cycl(H_6^1) \leq 2$, The graphic (F_{6a}^1) has finite cyclicity.*

Proof. By Theorem 2.3, the family unfolding the triple nilpotent singularity for quadratic systems can be written in the form of (2.28) or (2.29). For $0 < c < 2\sqrt{1-a}$, we have hemicycle (H_6^1) and a family of graphics (F_{6a}^1). After the blow-up of the family, take sections Σ_1 and Σ_2 as defined in **Notation 5.7** in the normal form coordinates. To prove that (H_6^1) and (F_{6a}^1) have finite cyclicity, first we study the transition map

$$R : \Sigma_2 \longrightarrow \Sigma_1 \tag{7.7}$$

along the equator. We are going to prove that the second component $R_2(0, \tilde{y}_2)$ is nonlinear in $\tilde{y}_2$.

(1). Normal forms and Dulac maps at infinity

As shown in Fig. 7.2, let P_r be the saddle point at infinity in the direction of the positive $x - axis$. Using coordinates $(z_r, u_r) = (\frac{1}{x}, \frac{y}{x})$, we have

$$\begin{cases} \dot{u}_r &= u_r(1 - a - cu_r + u_r^2 - u_r z_r) \\ \dot{z}_r &= -z_r(a + cu_r - u_r^2 + u_r z_r) \end{cases} \tag{7.8}$$

Hence P_r is a saddle point of hyperbolicity ratio $\sigma_r = \frac{a}{1-a}$. For $a \in (0, \frac{1}{2})$, we have $0 < \sigma_r < 1$. Dividing system (7.8) by $1 - a - cu_r + u_r^2 - u_r z_r$ (positive in the

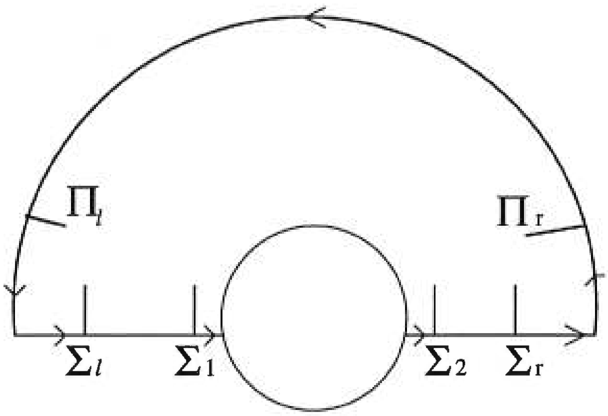


Figure 7.2: The hemicycle (H_6^1)

neighborhood of P_r) and using a coordinate change of the form

$$\begin{cases} u_r &= u_r \\ z_r &= \Psi_r(u_r, Z_r) := d_{r1}(u_r)Z_r + d_{r2}(u_r)Z_r^2 + u_r O(Z_r^3) \end{cases} \tag{7.9}$$

where

$$\begin{aligned} d_{r1}(u_r) &= 1 - \tfrac{c}{(1-a)^2} u_r + O(u_r^2)) \\ d_{r2}(u_r) &= -\tfrac{1}{(1-a)(1-2a)} u_r + O(u_r^2), \end{aligned}$$

then system (7.8) is C^k equivalent to the normalized system

$$\begin{cases} \dot{u}_r &= u_r \\ \dot{Z}_r &= Z_r\Big[-\sigma_r + \sum_{i=1}^{N(k)} \gamma_{ri}(u_r^p Z_r^q)^i \Big] \end{cases} \tag{7.10}$$

where if $a \in (0, 1/2) \setminus Q$, $\gamma_{ri} = 0$; if $a \in (0, 1/2) \bigcap Q$, $\sigma_r = \frac{p}{q}$, $(p, q) = 1, p, q \in \mathbf{N}$.

Take two sections $\Sigma_r = \{Z_r = Z_0\}$ and $\Pi_r = \{u_r = u_0\}$ in the normal form coordinates. By [Mou90], for the Dulac map D_r, we have

$$\begin{aligned} &D_r : \Sigma_r \longrightarrow \Pi_r \\ &D_r(u_r) = u_r^{\sigma_r}(A_r + \psi_r(u_r)) \end{aligned} \tag{7.11}$$

where $A_r > 0$ constant, also $\psi_r \in (I_0^\infty)$.

Similarly, at P_l, the singular point at infinity in the direction of the negative x-axis, if we use coordinates $(z_l, u_l) = (-\frac{1}{x}, -\frac{y}{x})$, then we have

$$\begin{cases} \dot{u}_l &= u_l(a - 1 - cu_l - u_l^2 + u_l z_l) \\ \dot{z}_l &= z_l(a - cu_l - u_l^2 + u_l z_l) \end{cases} \tag{7.12}$$

After dividing by $1 - a + bu_l + u_l^2 - u_l z_l$, and by a coordinate change of the form

$$\begin{cases} u_l &= u_l \\ Z_l &= \Psi_l(u_l, z_l) = d_{l1}(u_l) z_l + d_{l2}(u_l) z_l^2 + u_l O(z_l^3) \end{cases} \tag{7.13}$$

where

$$\begin{aligned} d_{l1}(u_l) &= 1 - \tfrac{c}{(1-a)^2} u_l + O(u_l^2)) \\ d_{l2}(u_l) &= \tfrac{1}{(1-a)(1-2a)} u_l + O(u_l^2), \end{aligned}$$

system (7.12) is C^k equivalent to the normalized system

$$\begin{cases} \dot{u}_l &= -u_l \\ \dot{Z}_l &= Z_l \Big[\sigma_r + \sum_{i=1}^{N(k)} \gamma_{li} (u_l^p Z_l^q)^i \Big] \end{cases} \tag{7.14}$$

where if $a \in (0, 1/2) \setminus Q$, $\gamma_{li} = 0$.

Take two sections $\Pi_l = \{u_l = u_0\}$, $\Sigma_l = \{Z_l = Z_0\}$ in the normal form coordinates. We have the Dulac map

$$\begin{aligned} &D_l : \Pi_l \longrightarrow \Sigma_l \\ &D_l(Z_l) = Z_l^{\sigma_l}(A_l + \psi_l(Z_l)) \end{aligned} \tag{7.15}$$

where $\sigma_l = \frac{1}{\sigma_r} > 1$, $A_l > 0$ constant, also $\psi_l \in (I_0^\infty)$.

(2). Decomposition of the map R_2

For the transition map R defined in (7.7), the second component R_2 can be calculated by the decomposition

$$\begin{aligned} &R_2 : \Sigma_2 \longrightarrow \Sigma_1 \\ &R_2 = T_l \circ D_l \circ R_0 \circ D_r \circ T_r \end{aligned} \tag{7.16}$$

where

- $R_0 : \Pi_r \longrightarrow \Pi_l$ is the transition map in the normal form coordinates along the equator and we can write

$$R_0(Z_r) = \beta_{01} Z_r + \beta_{02} Z_r^2 + O(Z_r^2) \tag{7.17}$$

- $T_r : \Sigma_2 \longrightarrow \Sigma_r$ and $T_l : \Sigma_l \longrightarrow \Sigma_1$ are regular transition maps along the invariant line, they have the forms

$$\begin{aligned} T_r(\tilde{y}) &= m_r \tilde{y} + O(\tilde{y}^2) \\ T_l(\tilde{y}_l) &= m_l \tilde{y}_l + O(\tilde{y}_l^2) \end{aligned} \tag{7.18}$$

Then a straightforward calculation from (7.11), (7.15), (7.16), (7.17) and (7.18) gives

$$R_2(\tilde{y}) = \beta_1 \tilde{y} + \beta_2 \tilde{y}^{1+\sigma_r} + o(\tilde{y}^{1+\sigma_r}) \tag{7.19}$$

where $\beta_1 = m_l m_r (A_r \beta_{01})^{\frac{1}{\sigma_l}} A_l > 0$ constant, $\beta_2 = \frac{1}{\sigma_r} m_l m_r^{\sigma_r} \sigma_l \beta_{01}^{\frac{1}{\sigma_r}-1} \beta_{02} A_l A_r^{1+\frac{1}{\sigma_l}}$. So, in order to prove that $R_2(\tilde{y})$ is nonlinear, it suffice to prove that $\beta_{02} \neq 0$.

(3). Calculation of $R_0''(0)$

(3.1). Decomposition of the map R_0. Now we prove that $\beta_{02} \neq 0$. To do this, we introduce the coordinates $(v, w) = (\frac{x}{y}, \frac{1}{y})$ and make the following decomposition

$$R_0 := \Psi_l \circ \Phi_{l2} \circ R_{00} \circ \Phi_{r2} \circ \Psi_r \tag{7.20}$$

where

$$\begin{aligned} \Psi_r(u_0, Z_r) &= d_{r1}(u_0) Z_r + d_{r2}(u_0) Z_r^2 + u_0 O(Z_r^3) \\ \Psi_l(u_0, z_l) &= d_{l1}(u_0) z_l + d_{l2}(u_0) z_l^2 + u_0 O(z_l^3) \end{aligned} \tag{7.21}$$

are the coordinate changes on $u_r = u_0$ and $u_l = u_0$ respectively and

$$\Phi_r : \begin{cases} v = \frac{1}{u_r} \\ w = \frac{z_r}{u_r} \end{cases}, \qquad \Phi_l : \begin{cases} u_l = -\frac{1}{v} \\ z_l = -\frac{w}{v} \end{cases} \tag{7.22}$$

are the coordinate changes between the charts. The map $R_{00} : \Pi_r \longrightarrow \Pi_l$ is the regular transition map in the coordinates (v, w), and sections become $\Pi_r = \{v = \frac{1}{u_0}\}$ and $\Pi_l = \{v = -\frac{1}{u_0}\}$.

(3.2). Calculation of the transition map R_{00} Using coordinates (v, w), we have system

$$\begin{cases} \dot{v} &= (a-1)v^2 + cv - 1 + w \\ \dot{w} &= -vw. \end{cases} \tag{7.23}$$

Note that the w−axis is invariant, also with $0 < c < 2\sqrt{1-a}$, there holds $\dot{v}\mid_{w=0} = (a-1)v^2 + cv - 1 < 0$. Hence R_{00} is a C^∞ map which can be written as

$$R_{00}(w) = \beta_{001}w + \beta_{002}w^2 + O(w^3). \tag{7.24}$$

By Prop. 5.3, a straightforward calculation gives

$$\begin{aligned} \beta_{001}(u_0) &= \exp\left(\int_{\frac{1}{u_0}}^{-\frac{1}{u_0}} \frac{vdv}{(1-a)v^2 - cv + 1}\right) \\ &= \exp\left(-c_2\left[\arctan\tfrac{2(1-a)-cu_0}{c_1u_0} - \arctan\tfrac{-2(1-a)-cu_0}{c_1u_0}\right]\right)\left[\tfrac{(1-a)+cu_0+u_0^2}{(1-a)-cu_0+u_0^2}\right]^{k_2} \\ &= \exp(-c_2\pi) + O(u_0) \end{aligned} \tag{7.25}$$

and

$$\begin{aligned} \beta_{002}(u_0) &= -\beta_{001}(v_0)\int_{-\frac{1}{u_0}}^{\frac{1}{u_0}} \frac{2v\exp\left(\int_{\frac{1}{u_0}}^{v}\frac{vdv}{(1-a)v^2-cv+1}\right)}{((1-a)v^2-cv+1)^2}dv \\ &= -\frac{\beta_{001}\exp\left(-c_2\arctan\frac{2(1-a)-cu_0}{c_1u_0}\right)}{\frac{1}{u_0^{2k_2}}\left((1-a)-cu_0+u_0^2\right)^{k_2}}\int_{-\frac{1}{u_0}}^{\frac{1}{u_0}}\frac{2v\exp\left(c_2\arctan(\frac{2(1-a)v-c}{c_1})\right)}{((1-a)v^2-cv+1)^{k_1}}dv \\ &= -\frac{2u_0^{2k_2}[1+O(u_0)]\exp(-\frac{3c_2\pi}{2})}{(1-a)^{k_2}}\int_{-\frac{1}{u_0}}^{\frac{1}{u_0}}\frac{2v\exp\left(c_2\arctan(\frac{2(1-a)v-c}{c_1})\right)}{((1-a)v^2-cv+1)^{k_1}}dv \end{aligned} \tag{7.26}$$

where

$$\begin{aligned} k_1 &= \tfrac{3-4a}{2(1-a)} \in (1, \tfrac{3}{2}) \\ k_2 &= \tfrac{1}{2(1-a)} \in (\tfrac{1}{2}, 1) \\ c_1 &= \sqrt{4(1-a)-c^2} \\ c_2 &= \tfrac{c}{(1-a)\sqrt{4(1-a)-c^2}}. \end{aligned}$$

Note that $k_1 \in (1, \frac{3}{2})$, so $2k_1 - 2 \in (0, 1)$. Also

$$\lim_{u_0 \to 0} \frac{\displaystyle\int_{-\frac{1}{u_0}}^{\frac{1}{u_0}} \frac{2v \exp\left(c_2 \arctan(\frac{2(1-a)v-c}{c_1})\right)}{((1-a)v^2 - cv + 1)^{k_1}} dv}{\frac{1}{u_0^{2k_1-2}}} = \frac{\exp(\frac{1}{2}c_2\pi) - \exp(-\frac{1}{2}c_2\pi)}{(2k_1 - 2)(1-a)^{k_1}}.$$

Therefore

$$\int_{-\frac{1}{u_0}}^{\frac{1}{u_0}} \frac{2v \exp\left(c_2 \arctan(\frac{2(1-a)v-c}{c_1})\right)}{((1-a)v^2 - cv + 1)^{k_1}} dv = \frac{[\exp(\frac{1}{2}c_2\pi) - \exp(-\frac{1}{2}c_2\pi)](1 + O(u_0))}{(2k_1 - 2)(1-a)^{k_1} u_0^{2k_1-2}}.$$

Thus, by (7.26), we have

$$\beta_{002}(u_0) = -\frac{\exp(-3c_2\pi)[\exp(c_2\pi) - 1]}{(2k_1 - 2)(1-a)^2} u_0^{2(k_2 - k_1 + 1)}[1 + O(u_0)]. \qquad (7.27)$$

(3.3). R_0 is nonlinear: $\beta_{02} = R_0''(0) \neq 0$

It follows from (7.20), (7.21), (7.22) and (7.24) that

$$R_0(Z_r) = \beta_{01} Z_r + \beta_{02} Z_r^2 + O(Z_r^3)$$

where

$$\begin{aligned}
\beta_{01}(u_0) &= d_{r1}(u_0) d_{l1}(u_0) \beta_{001} \\
&= \left(1 - \tfrac{c}{(1-a)^2} u_0 + O(u_0^2)\right)\left(1 - \tfrac{c}{(1-a)^2} u_0 + O(u_0^2)\right) \\
&\qquad \times \left[\tfrac{(1-a)+cu_0+u_0^2}{(1-a)-cu_0+u_0^2}\right]^{k_2} \exp\left(-\, c_2\left[h_1(\tfrac{1}{u_0}) - h_1(-\tfrac{1}{u_0})\right]\right) \\
&= e^{-c_2\pi} + O(u_0)
\end{aligned}$$

$$\begin{aligned}
\beta_{02}(u_0) &= \beta_{001} d_{l1}(u_0) d_{r2}(u_0) + d_{l2}(u_0) d_{r1}^2(u_0) \beta_{001}^2 + \tfrac{1}{u_0} d_{r1}^2(u_0) d_{l1}(u_0) \beta_{002} \\
&= \beta_{001}\left(1 - \tfrac{c}{(1-a)^2} u_0 + O(u_0^2)\right)\left(-\tfrac{1}{(1-a)(1-2a)} u_0 + O(u_0^2)\right) \\
&\quad + \left(\tfrac{1}{(1-a)(1-2a)} u_0 + O(u_0^2)\right)\left(1 - \tfrac{c}{(1-a)^2} u_0 + O(u_0)\right)^2 \beta_{001}^2 \\
&\quad - \tfrac{1}{u_0}\left(1 - \tfrac{c}{(1-a)^2} u_0 + O(u_0^2)\right)^2 \tfrac{\exp(-3c_2\pi)[\exp(c_2\pi)-1]}{(2k_1-2)(1-a)^2} u_0^{2(k_2-k_1+1)}[1 + O(u_0)] \\
&= -\tfrac{\exp(-3c_2\pi)[\exp(c_2\pi)-1]}{(2k_1-2)(1-a)^2} u_0^{2(k_2-k_1)+1}[1 + O(u_0)] + O(u_0).
\end{aligned}$$

Since $2(k_2 - k_1) + 1 = \frac{3a-1}{1-a} \in (-1, 1)$, so $\beta_{02} \neq 0$. i.e., the second component R_2 of R has the form in (7.19) with $\beta_2 \neq 0$. Note that all steps of the proof of Theorem 6.3 work with this form of the transition in (7.19), yielding $Cycl(H_6^1) \leq 2$. Furthermore, by Prop. 5.12, the graphic (F_{6a}^1) has finite cyclicity. □

Chapter 8

Conclusion

This thesis solves some important problems of finite cyclicity of graphics through a nilpotent singularity of saddle or elliptic type. The methods developed here also allow to prove that the cuspidal loop has finite cyclicity. We also set up some tools which allow to expect to get solutions to several more problems in the near future:

1. Prove Conjecture 6.8 for the hp graphic

   Solving this conjecture by a "general method" would allow to solve several similar conjectures arising in the concave case of the saddle type.

2. Prove that the concave graphic of saddle type has finite cyclicity.

3. Study the finite cyclicity of the convex graphic through a nilpotent of codimension 4 (the case $a = -\frac{1}{2}$, which corresponds to $b = 0$ in (2.4)).

   The case when the saddle connection is fixed is nearly done.

4. Prove that for quadratic systems, many graphics with a triple nilpotent singular point of saddle or elliptic type listed in the paper [DRR94] have finite cyclicity.

   In Ch.7, we have proved several such graphics have finite cyclicity. More can be done by similar calculations.

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