Université de Montréal
Faculté des Arts et des Sciences - Département de Physique

Solitons in Wave Propagation and Spin Systems

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Thèse présentée à la faculté des études supérieures
en vue de l'obtention du grade de
PhilosophiæDoctor (Ph.D.)
en physique

Directeur de Recherche: Professeur Luc Vinet

Juillet 1998

Université de Montréal

Faculté des études supérieures

Cette thèse intitulée:

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Thése acceptée le 25 septembre 1998

0-612-42468-5

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Abstract

This thesis consists of three parts

1) In the first part, a solution of the restricted Hadamard problem is presented. The classical Hadamard problem consists in determining (up to equivalence) all the second order differential operators which satisfy Huygens' Principle in the narrow sense. Physically, such operators describe systems where the diffusion of waves is absent and where signals propagate with maximal velocity. Unlike the original principle of superposition of secondary waves, which holds for all wave propagation phenomena, Huygens' principle in the narrow sense of Hadamard applies only to a very restricted range of wave processes, with sharp signals. We present a new class of Huygens' operators on Minkowski space-time and establish a new link between Huygens' principle and the solitons of the Korteveg-de Vries equation.

2) In the second part, a new class of exactly solvable models in statistical mechanics is presented. We study the connections between the soliton solutions of certain integrable nonlinear equations (hierarchies of equations) and the thermodynamic quantities of one-dimensional Ising models with different types of interactions between spins.The exact solvability of these models can be traced back to this connection. We consider a model linked to soliton solutions of the Korteveg de Vries and of the B-type Kadomtsev-Petiashvili hierarchies. A connection between these Ising chains and random matrix models is considered as well.

3) In the third part, we study solitonic mechanisms of exciton superfluidity. We provide a theoretical explanation of recent experiments on the propagation of excitons in semiconductors. In these experiments, the excitonic transport under the action of a laser pulse has been studied. It turned out that under certain conditions this transport becomes anomalous and the excitons propagate through the crystal in a wave packet without diffusion. We propose a model for this phenomenon which relies on the presence of an exciton-phonon interaction. In this model the exciton propagation is described by soliton solutions of the nonlinear Schrodinger equation. The theory predicts two critical velocities for propagation of the packet which is in agreement with the experimental data.

Résumé

Cette thèse comprend trois parties:

1. Dans la première partie, une étude du principe de Huygens et de ses liens avec des solutions de l'équation de Korteveg-de Vries est effectuée.

Proposé par Christian Huygens en 1690, le principe qui porte son nom offre une explication unifiée d'effets optiques. Plus spécifiquement, l'existence de signaux de haute résolution peut être dérivée du principe de Huygens, comme une conséquence de l'annulation parfaite des ondes secondaires qui suivent le premier front d'ondes.

L'interprétation exacte du principe de Huygens n'a seulement été donnée qu'en 1922 par le mathématicien français J.Hadamard. Selon Hadamard, le principe de Huygens doit être considéré comme un syllogisme qui est basé sur deux prémisses principales. La première prémisse appelée "majeure" est essentiellement un postulat de causalité qui reflète l'hyperbolicité de l'équation.

Au contraire, la prémisse "mineure" fait l'hypothèse d'une propriété spécifique quant à l'absence de diffusion. La prémisse mineure du principe de Huygens a une signification mathématique claire. Elle affirme que la fonction de Green de l'équation de D'Alembert a son support sur la génératrice du cône caractéristique si la dimension de l'espace est impaire, et que la situation est différente pour les dimensions paires.

En général, le principe de Huygens est une propriété analytique rare de l'équation différentielle. Une question naturelle à se poser est alors: "Quelles sont les équations du second ordre pour lesquelles le principe de Huygens est vrai dans son sens mineur". Cette question a été posée par Hadamard. Le problème de la détermination explicite (à équivalence triviale près) de tous les opérateurs possédant la propriété de Huygens est connue sous le nom de problème de Hadamard. Ce problème est difficile à résoudre.

Les premiers exemples non-triviaux d'opérateurs de Huygens ont été trouvés en 1953-1955 par K.L. Stellmacher. Une décennie plus tard, J.E.Lagnese et K.L.Stellmacher ont résolu le problème de Hadamard pour une classe restreinte d'opérateurs dans l'espace de Minkovski avec des coefficients dépendant d'une variable seulement.

La percée suivante dans le contexte de ce problème s'est produite en 1993

quand A.P.Veselov et Yu.Yu.Berest ont donné une généralisation des exemples de Stellmacher. Ils ont lié de nouvelles hiérarchies d'équations de Huygens aux groupes finis de Coxeter.

Plus tard O. Chalych, M. Feigin et A. Veselov ont découvert des configurations "non-Coxetérienne" qui étaient compatibles avec le principe de Huygens.

Dans le premier chapitre de cette thèse, on présente une généralisation de ces exemples quand l'opérateur est la somme d'un D'Alembertien et d'un potentiel à deux variables.

On trouve que chacun de ces potentiels est déterminé par une suite arbitraire et strictement croissante de nombres entiers positifs. C'est le résultat principal du premier chapitre.

2. Dans la deuxième partie de la thèse, on considère quelques types de modèles de la mécanique statistique possédant des solutions exactes.

Les modèles d'Ising sont bien connus en mécanique statistique. Les modèles avec des interactions entre les spins voisins et sans champ magnétique possèdent des solutions exactes en deux dimensions, et en une dimension lorsq'en presénce d' un champ magnétique arbitraire. Ils admettent quelques phénomènes intéressant dont on peut rendre compte à l'aide d'expressions analytiques simples pour la fonction de partition.

D'autre part, les modèles unidimensionels sont intéressants du point de vue de la résolution exacte.

Une nouvelle classe de modèles d'Ising, liées à la fonction τ de certaines hiérarchies intégrables, est introduite dans le deuxième chapitre de la thèse.

On a trouvé que ces modèles correspondent à des solutions self-similaires d'équations nonlinéaires. Leurs spectres sont composés d'un nombre fini de séries géométriques. Cette construction conduit à des chaînes (anti)ferromagnétiques dans un champ magnétique arbitraire. Les interactions d'échange décroissent exponentiellement avec la distance entre les spins. Dans une limite spéciale, on obtient l'interaction rationelle de type Calogero-Moser, qui est reliée au modèle de Kondo.

On présente aussi des modèles de matrices aléatoires liés aux modèles d'Ising et aux solutions solitoniques de hiérarchies intégrables. Ces modèles sont des généralisations discrètes de modèles continus de matrices aléatoires. On obtient des modèles classiques dans la double limite thermodynamique et rationnelle.

Dans cette limite, le nombre de solitons devient infini et le spectre des impulsions continu.

3. Dans la troisième partie, on considère des phénomènes solitoniques dans la propagation d'excitons.

Les excitons sont des excitations bosoniques dans les cristaux et peuvent être engendrés sous l'action de radiations électromagnétiques (e.g. rayons laser).

On donne une explication théorique à des expériences sur la propagation des excitons dans certains semiconducteurs qui ont révélé un transport anormal. En effet, sous des conditions spéciales les excitons forment un soliton qui se propage sans diffusion dans le cristal.

On propose un modèle pour décrire ce phénomène, où la présence d'interactions entre excitons et phonons est importante. Dans ce modèle, la propagation de excitons est décrite par des solutions solitoniques d'un système d'équations non-linéaires. Ce modèle prédit deux vitesses critiques qui sont effectivement observées expérimentalement.

Introduction

The generality of the title of this work reflects diversity of its topics. Although these are quite different and range from the theory of partial differential operators to some condensed matter problems, they still can be collected under a common roof: the theory of solitons.

Four articles form this thesis. The first is devoted to new integrable systems possessing exceptional Huygens' and bispectral properties. In the second and third papers, we introduce a new class of exactly solvable models of statistical physics as well as new random matrix models. The soliton superfluidity of excitons in crystals is studied in the fourth article.

The texts are rather technical and some additional explanations (as well as historical introductions) will be of help.

We start from the study devoted to Huygens' principle and its relation to solutions of the Korteveg-de Vries hierarchy.

Problem of the wave propagation is in the heart of the classical mathematical physics.

The progress in understanding wave phenomena has come with the famous Huygens' principle. Proposed by Christiaan Huygens as early as in 1690, this principle was surprisingly successful in providing a unified explanation for basic optical effects, such as a straight-line propagation, interference and diffraction of light. More specifically, the existence of 'clean-cut' wave signals could also be derived from Huygens' principle, namely, as a result of perfect cancellations of secondary waves which occur behind the leading wave front.

Eventually, Maxwell's electrodynamics brought the physical picture of wave propagation into relation with the theory of linear partial hyperbolic differential equations.

Historically, the first hyperbolic equations studied in detail were the classical wave equations. The one-dimensional equation, which describes small transverse vibrations of an elastic string, was solved by d'Alembert. Later, Poisson and Kirchhoff found the solution to the initial value problem for the wave equation in two and three dimensions. The different mathematical and physical aspects of Huygens' principle arose in their work. However, a precise mathematical interpretation of Huygens' principle was lacking. It was proposed by french mathematican J. Hadamard in his remarkable lectures on Cauchy's problem given at Yale University in 1922. According to Hadamard, Huygens' principle should be treated as a syllogism based on two main premises. The 'major' premise is essentially a causality postulate as applied to any wave phenomena; mathematically, it reflects the hyperbolicity of the 'wave-governing' differential system.

By contrast, the 'minor' premise states a more specific property referred to as the absence of wave diffusion. In a medium, where it holds, the wave carrying an initially localized perturbation does not leave a "trace" behind its fastest front. Clearly, the propagation of waves without diffusion ensures the possibility of transmitting sharp light and sound signals in our world.

Huygens' principle in its 'minor' premise has a clear mathematical meaning. Namely, it amounts to the fact that the Green function of the wave equation

$$\square_{n+1}\Phi(x,t) = \left(\frac{\partial^2}{\partial t^2} - \Delta_n\right)\Phi(x,t) = 0, \quad x \in \mathcal{R}^n \tag{1}$$

with the initial conditions

$$\Phi(t,x)|_{t=0} = 0, \quad \Phi_t(t,x)|_{t=0} = \delta(x)$$

have the following form

$$\Phi(t,x) = \frac{1}{2\pi}\frac{\theta(t^2-|x|^2)}{\sqrt{t^2-|x|^2}}, \quad n=2$$

$$\Phi(t,x) = \frac{1}{2\pi}\delta(t^2-|x|^2), \quad n=3,$$

where δ denotes Dirac and θ Heavyside generalised functions correspondingly.

In other words, the Green function of the wave equation in $n=3$ is supported on the light cone $|x| = \pm t$, i.e. vanishes in the complement to this surface. The same is also true in any *odd* dimension $n \geq 3$, while for even $n = 2, 4, 6, \ldots$ the situation is different.

In general, Huygens' principle is a rare analytic property of a given differential equation inherently related to (the parity of) the space dimension and to the structure of its symmetry group. A natural question: *"For which* [second order normal hyperbolic] *equations is Huygens' principle true in its special* ['minor'] *sense?"* was originally posed by J. Hadamard. The problem of explicit determination (up to a trivial equivalence) of all Huygens operators is known as Hadamard problem. It has turned out to be very hard, and at present, in spite of a good deal of attention, it is still far from being completely solved.

As mentioned above the simplest (sometimes called trivial) examples of Huygens' operators are the ordinary d'Alembertians (1), as well as the operators reducible to (1) by means of elementary transformations, namely, by nonsingular changes of coordinates $x \mapsto f(x)$, and conformal-gauge transformations $L \mapsto \mu(x) \circ L \circ \theta(x)^{-1}$ with some smooth non-vanishing functions $\mu(x)$ and $\theta(x)$.

It has been thought for a long time that these are the only second order normal hyperbolic equations enjoying Huygens' principle in its 'minor' sense. In his classical monograph on mathematical physics R. Courant has attributed this tempting hypothesis to Hadamard.

Strictly speaking, Hadamard's conjecture is valid *only* for real hyperbolic operators with a constant principal symbol in four independent variables, i.e. for wave-type operators in the $(3+1)$-dimensional Minkowski space M^{3+1} (which was justified by M.Mathisson and L.Asgeirsson around 1930), while in higher dimensions it fails to be true.

The first counterexamples that disproved Hadamard's conjecture in higher dimensions $n \geq 5$ were presented by K.L.Stellmacher in 1953-55. More precisely, he proved that if the wave type operator L on Minkovskii space M^{n+1} has the form

$$L := \Box_{n+1} + \left(\frac{\lambda_0}{t^2} - \sum_{i=1}^{n} \frac{\lambda_i}{x_i^2} \right) ,$$

then it satisfies Huygens' principle, if and only if n is odd and $\lambda_k = -m_k(m_k+1)$ with $m_k \in \mathbf{Z}_+$ and $\sum_{k=0}^{n} m_k \leq (n-3)/2$.

In fact, Stellmacher constructed an explicit formula for the Green function of such operators from which validity of Huygens' principle was immediate.

A decade later, J.E.Lagnese took up the Hadamard problem in 7-dimensional Minkovski space. More precisely, he studied the question of Huygens' principle for the wave type operators L of the simplest nontrivial form:

$$L = \Box_{n+1} + u(t) \tag{2}$$

with a real locally analytic potential depending on a single variable only, say $u = u(t)$. He managed to get a remarkable result that such operators satisfy the Huygens principle if and only if the function $u(t)$ has the following form:

$$u(t) = 0 \ ,$$

$$u(t) = -\frac{2}{t^2} \ ,$$

$$u(t) = -\frac{6t(t^3 - 2\gamma)}{(t^3+\gamma)^2} \ ,$$

where γ is an arbitrary constant.

Later, in attempting to understand the 'origin' of these examples, Lagnese and Stellmacher proposed a systematic approach for generating Huygens potentials depending on a single variable. Essentially, they rediscovered the classical factorizaction method known in the theory of one-dimensional Sturm-Liouville (Schrödinger) operators since the work of Darboux and Crum.

They introduced an infinite family (hierarchy) of Huygens' potentials $u = u_k(t)$, $k = 0, 1, \dots$, each u_k depending upon a finite number of (complex) parameters, such that

$$u_k(t) := 2\frac{\partial^2}{\partial t^2} \log P_k \tag{3}$$

where the functions $P_k := P_k(t + \gamma_1, \gamma_2, \dots, \gamma_k)$ are defined as solutions of the following recurrence system of differential equations

$$P'_{k+1}P_{k-1} - P'_{k-1}P_{k+1} = (2k+1)^2 P_k^2 \quad \text{with} \quad P_0 := 1 \ , \ P_1 := t$$

Determined uniquely (up to a normalization) by the last equation, all P_k's turn out to be *polynomials*, so that the corresponding potentials u_k are, in fact, rational functions.

Shortly afterwards Lagnese and Stellmacher show that it is necessary and sufficient for the potentials to have the form (3) in order for the operators (2) be Huygens' type.

In this way, the Hadamard problem, within the restricted class of second order hyperbolic operators (1), had been completely settled. It remains to note that the polynomials P_k and the corresponding potentials $u_k(t)$ were found by J. Burchnall and T. Chaundy as early as in 1929, in connection with the problem of classifying commutative rings of ordinary differential operators. They became widely known much later, due to the work of M. Adler and J. Moser, who established their relation to the KdV equation. The coincidence between the Lagnese-Stellmacher and the Adler-Moser potentials has been noticed by R. Schimming who has also unearthed the earlier paper by Burchnall and Chaundy. Finally, the very same potentials have emerged in the context of the bispectral problem in the work of A. Grünbaum et al.

The next development in the Hadamard problem happened in 1993, when A. P. Veselov and Yu. Yu. Berest gave a sweeping generalization of the original Stellmacher examples by relating new hierarchies of Huygens equations to finite Coxeter groups.

The corresponding wave-type operators may be regarded as a 'hyperbolic' version of the quantum Calogero-Moser Hamiltonians extended to arbitrary root systems. The proof of Huygens' property in that case rests heavily on the machinery of Dunkl operators as well as on preceding studies of algebraic Schrödinger operators (by O. A. Chalykh and A. P. Veselov) associated with Coxeter systems.

To explain the main construction, we note that a Coxeter root system $\Re := \{\alpha\}$ can be defined as a finite set of nonzero pairwise distinct vectors in $\mathbf{R}^n$ invariant under all 'inner' reflections, i.e. the real orthogonal reflections with respect to hyperplanes $(\alpha, x) = 0$ with $\alpha \in \Re$. These 'inner' reflections generate a *finite* subgroup W in $O(n)$ called the Coxeter group.

Given a Coxeter root system $\Re$, we associate with each $\alpha \in \Re$ a non-negative integer number, the multiplicity m_α, so that $m_{w(\alpha)} = m_\alpha$ for all $w \in W$.

Yu. Yu. Berest and A. P. Veselov have shown that if

$$u_m(x) := \sum_{\alpha \in \Re_+} \frac{m_\alpha(m_\alpha + 1)(\alpha, \alpha)}{(\alpha, x)^2}$$

then the differential operator

$$L = \Box_{n+1} + u_m(x)$$

satisfies Huygens' property when dimension of the Minkowski space is big enough:

$$\sum_{\alpha \in \Re_+} m_\alpha \leq (n-3)/2.$$

According to this the original examples of Stellmacher correspond precisely to the simplest Coxeter groups, i.e those of (splited) rank 1, $W \cong \mathbf{Z}_2 \times \ldots \times \mathbf{Z}_2$.

Yu. Yu. Berest and A. P. Veselov have conjectured that the Stellmacher-Lagnese and the Calogero-Moser potentials associated with Coxeter systems completely settle the Hadamard problem in Minkowski spaces. It turned out that this conjecture fails to be true. A new surprising class of examples has been found recently by O. Chalykh, M. Feigin and A. Veselov. The corresponding configurations can be regarded as a *'one-orbit' deformation* of the root system A_n in the case of higher multiplicities $m > 1$.

They found that The wave-type hyperbolic operator

$$L := \Box_{N+1} + \sum_{i<j}^{n} \frac{2m(m+1)}{(x_i - x_j)^2} + \sum_{i=1}^{n} \frac{2(m+1)}{(x_i - \sqrt{m}x_{n+1})^2}$$

in the Minkowski space M^{N+1} satisfies Huygens' principle if N is odd and $N \geq 3 + 2n + mn(n-1)$. Deformations of other root systems were also found.

The existence of 'non-Coxeter' configurations compatible with Huygens' principle motivated further study of Hadamard's problem on Minkowski spaces. Part of these studies have led to the article presented in this work.

Since the Hadamard problem has been completely solved for operators (2) with the potentials u depending on a single variable, the next step was to consider the case of potentials depending on two variables, say $u = u(x_1, x_2)$. A few examples of this form were already known (from the above). However, this appeared to be just the tip of the iceberg. Indeed, the analysis carried out in the work presented here revealed a new large class of Huygens operators associated with *soliton solutions* of the Korteweg-de Vries equation

$$\frac{\partial^3 u}{\partial x^3} - 6u\frac{\partial u}{\partial x} - 4\frac{\partial u}{\partial t} = 0 \tag{4}$$

Again, Huygens' property of the potential u is preserved under flows of the KdV equation (4) (hierarchy) and the link to integrable systems is surprisingly straightforward.

If we introduce the polar coordinates (r, φ) in a fixed 2-dimensional space-like plane E in the Minkowski space $\mathbf{M}^{n+1}$ and consider an arbitrary strictly increasing sequence $(k_j)_{j=1}^m$ of nonnegative integers: $0 \le k_1 < k_2 < \ldots < k_{m-1} < k_m$, with associated to it a set $\{\chi_j(\varphi)\}$ of elementary 2π-periodic functions on $\mathbf{R}^1$:

$$\chi_i(\varphi) := \cos(k_i\,\varphi + \varphi_i)\ , \quad \varphi_i \in \mathbf{R}\ .$$

then the Wronskian of this set

$$W\left[\chi_1, \chi_2, \ldots, \chi_m\right] := \det \begin{pmatrix} \chi_1(\varphi) & \chi_2(\varphi) & \ldots & \chi_m(\varphi) \\ \chi_1'(\varphi) & \chi_2'(\varphi) & \ldots & \chi_m'(\varphi) \\ \vdots & \vdots & \ddots & \vdots \\ \chi_1^{(m-1)}(\varphi) & \chi_2^{(m-1)}(\varphi) & \ldots & \chi_m^{(m-1)}(\varphi) \end{pmatrix}$$

does not vanish indentically since $\chi_j(\varphi)$ are linearly independent.

We define $u(x)$ in terms of cylindrical coordinates in $\mathbf{M}^{n+1}$ with polar components in E :

$$u = u_k(x) := -\frac{2}{r^2}\left(\frac{\partial}{\partial\varphi}\right)^2 \log W\left[\chi_1(\varphi), \chi_2(\varphi), \ldots, \chi_N(\varphi)\right]\ . \tag{5}$$

It is easy to see that in standard Minkowskian coordinates $u(x)$ is a real *rational* function on M^{n+1} with singularities at the zero set of W, and, in particular, it is analytic outside of this set.

Our main result consists in the fact that the wave-type second order hyperbolic operator

$$L_{(k)} := \Box_{n+1} + u_k(x) \tag{6}$$

with the potential (5) associated to an arbitrary strictly monotonic partition (k_j) of length m satisfies Huygens' principle, provided n is odd, and $n \geq 2\,k_m + 3$.

In attempting to give a unified explanation for the examples discussed, Yu. Yu. Berest and myself found the *multidimensional generalization* of the recurrent differential equations of the Adler-Moser (Burchnal-Chaundy) type

$$P_{k+1}\,(\Delta_n P_k) - 2\,\langle \nabla P_k, \nabla P_{k+1}\rangle + P_k\,(\Delta_n P_{k+1}) = 0 \tag{7}$$

It turns out that equation (7) is a necessary condition for operators of the form (6) to possess Huygens' property. Namely, an operator possesses this property in the Minkovskii space of certain dimensions if

$$u_k(x) = -2\Delta_n \log P_k$$

Equation (7) resembles the Hirota bilinear equations for the τ functions of integrable hierarchies.

Working on generalization of the Hirota bilinear relations to higher dimensions, we encountered an observation that lies at the basis of the next topic of this thesis.

Before going into details, let us recall some basic notions from statistical mechanics.

Statistical mechanics describes complex physical systems whose exact states cannot be specified. Instead, macroscopic properties alone may be specified, and the role of the theory is to infer these properties from the microscopic Hamiltonian. Thus, statistical mechanics distinguishes microscopic states from macroscopic ones. A microscopic state is specified by the quantum numbers of all the particles in the system. A macroscopic state is specified by a finite number of macroscopic parameters, which characterize the system from the point of view of observation, such as temperature, magnetization, etc.

The basic idea behind the statistical study of a complex system is that any physical property may be regarded as a statistical average, calculated over a suitable ensemble of microscopic states. The probability that a specific microscopic state is the actual state of the system depends only on its energy and is given by the Boltzman distribution

$$P_i = \frac{1}{Z} e^{-\beta E_i}, \quad \beta = T^{-1}$$

where T is the absolute temperature and Z is the normalization of the distribution called the partition function

$$Z = \sum_i e^{-\beta E_i} \tag{8}$$

The partition function is of central importance in the statistical mechanics since macroscopic quantities are related to derivatives of Z.

In practice, the number of systems for which the partition function can be evaluated exactly is very small. Confronted with the extreme complexity of most realistic systems one relies on simplified models to investigate finite temperature properties. Some of these models are defined in terms of discrete classical variables which live on a lattice of cites.

The best-known and simplest of these discrete models is the Ising model. It consists of a discrete lattice with spin dynamical variables σ_i taking the value $+1$ or -1 at each site. For a lattice with N sites the number of different spin configurations is 2^N and the energy of a spin configuration is

$$E = -\sum_{ij} J_{ij} \sigma_i \sigma_j - h \sum_i \sigma_i \tag{9}$$

The first term in the energy represents the interaction of spins through a ferromagnetic ($J_{ij} > 0$) or antiferromagnetic ($J_{ij} < 0$) coupling. The second term represents the interaction with an external magnetic field.

Ising models are very popular in statistical mechanics. The nearest-neighbor Ising model ($J_{i,j} = J\delta_{i,j+1}$ in one dimension and $J_{i_1,j_1;i_2,j_2} = J_1\delta_{i_1,j_1+1}\delta_{i_2,j_2} + J_2\delta_{i_1,j_1}\delta_{i_2,j_2+1}$ in two dimensions etc.) in arbitrary dimensions were a starting point for the mean field theory of magnetism. The one dimensional Ising model in homogeneous magnetic field is the simplest exactly solvable model of interacting spins. The two-dimensional Ising model with the nearest neighbor interaction is also exactly solvable for zero magnetic field. The latter is particulary well known in statistical physics because its solution deviates in an essential way from the predictions of mean field theory. It provided one of the basic motivations for the scaling hypothesis and for adopting the renormalization group techniques in statistical physics. It shows a number of interesting phenomena reflected in many cases through simple analytical expressions of the partition function and various thermodynamic quantities.

However, one-dimensional models with fast decaying interactions admit as a critical point only the zero temperature. Such models are therefore are primarily interesting from the exact solvability viewpoint as exact derivation of the partition function can be useful for other calculations as well. It is worth mentioning that there are also one-dimensional long-range interaction models with nontrivial phase transitions which are of interest (Kac model). As examples of spin chains with non-nearest neighbor interactions which are solvable we may mention the Haldane-Shastry model and the Inozemtsev model.

A new class of exactly solvable Ising models is introduced in the second chapter of the present thesis. They were discovered while looking for bilinear equations for the τ function of the Huygens operators. It turns out that this particular set of one dimensional Ising models is related to some soliton solutions of integrable nonlinear partial differential equations.

Let us take, for instance, the N soliton solution of the KdV equation (4). It describes propagation of N solitary waves with the moments $k_1, \ldots, k_N$. If we represent the solution of this equation in the Hirota form

$$u = -2\frac{\partial^2}{\partial x^2}\ln\tau, \quad \tau_N = \sum_{\mu=0,1}\exp\Big(\sum_{1\le i<j\le N} A_{ij}\mu_i\mu_j + \sum_{1\le i\le N}\theta_i\mu_i\Big)$$

then the analogy between the last formula and (8), (9) is immediate.

The Hamiltonian of the spin chain (9) contains many degrees of freedom. However, if one imposes the natural constraint of translational invariance (which follows from the fact that all the atoms in the chain are identical and at equal spacing), the exchange interaction loses its arbitrariness.

We found that such models correspond to the self-similar infinite soliton solutions of the Korteweg-de Vries (KdV) equation generated by the Schrödinger equation potentials. Their discrete spectra are composed of a finite number of geometric series. This construction describes antiferromagnetic spin chains in magnetic field. In this case, the interaction is decaying exponentially fast with the distance between spins. The partition function can be calculated exactly for the homogeneous magnetic field and some fixed values of the temperature.

Looking for generalizations of such models we considered different integrable hierarchies and found that the B-type Kadomtsev-Petviashvily (BKP) hierarchy of integrable equations can be related to a wider range of spin phenomena. It turns out that in the BKP case not only antiferromagnetic but also ferromagnetic interaction are permitted. Moreover, in some special limit, one gets the rational Calogero-type interactions $\propto 1/(i-j)^2$. Such interactions are related to the well-known Kondo model, which describes the thermodynamical properties of electron scattering on magnetic impurities. The Kondo problem was among the first models solved with the renormalization group techniques. It is also popular since it can be studied in the framework of conformal field theory.

The solvability of our models implies the existence of a rich group of symmetry. Although renormalization group transformations in this case are not straightforward real or momentum space transformations, this group can be calculated exactly.

It is known on the other hand, that the Ising model can be also viewed as a lattice gas model. The Coulomb gas formalism is used to describe random matrix models. Motivated by this idea, we found that our chains are also related to the random matrix models.

Let us recall that the random matrix method was employed first in the study of complex systems with unknown Hamiltonians. Such systems have a large number of degrees of freedom. As a consequence, the density of levels is high enough and can be described statistically. To describe such systems it is sufficient to consider only the discrete part of the spectrum of the Hamiltonian, so that the Hamiltonian can be approximately reduced to a matrix form. As the interactions in such systems are complex, the matrix elements are unknown. The basic hypothesis of Dyson and Wigner is that the statistical features of such systems can be well described by averaging over ensembles of random matrices, provided the probability distributions are invariant under basic symmetry transformations (e.g. parity, rotation and time-reversal transformations). This approach to the study complex systems was a new kind of statistical physics, in which not only the exact states of a system but also the nature of interactions is unknown.

This approach turns out to be very successful in nuclear physics, as well as in the theory of disordered metals and spin glasses. It is conjectured that the distribution of the zeros of the Riemann ζ function is also described by random matrix models. Furthermore, random matrix models are of great interest in topology and in two dimensional quantum gravity.

In this work we present several new matrix models and show that they are related to the Ising models and the soliton solutions of integrable hierarchies. They are discrete counterparts of the continuous matrix ensembles. In particular, we introduce ensembles of unitary matrices on a circle with probability distributions depending on the classes of matrices. Note that the continuous matrix models correspond to a double thermodynamical and rational limit. In this limit the number of solitons of the corresponding hierarchy goes to infinity while the differences between their momenta vanish.

Our approach turns out to be fruitful since it provides not only a new interpretation but also allows to calculate the partition functions of such systems using the machinery of soliton theory. In concluding the overview of the second chapter, we would like to mention that we have not included results of the studies on the relations between the Ising models and the Huygens operators. These results indicate that such links exist, but they are still obscure.

It is worth mentioning that the τ functions were objects of intensive studies in the context of the theory of correlation functions of two dimensional models at critical temperatures. The question of the relation between these theories and our results is interesting, since they both can be obtained in the framework of the free fermion formalism.

The topic of the last part of this thesis differs from the first two. However, since it involves the theory of solitons, it lies within realm of the present work. Indeed the last part is devoted to the study of the soliton propagation of exciton wave packets in crystals.

We remind the reader that the excitons are boson-like excitations in crystals. The excitons can be created by the electromagnetic radiation (e.g. laser light). Modern experimental techniques allow to extend the lifetime of excitons to such a scale that they can pass macroscopic distances through the crystals.

In recent experiments on excitons an interesting phenomenon has been observed: the excitons created under the action of a laser pulse form a soliton-like wave packet and propagate without diffusion through a crystal when the intensity of the pulse is high enough.

In the third chapter we provide the theoretical explanation of this phenomenon.

Our model is different from other theories presented before. It relies on the fact that the exciton-phonon interaction is crucial in this phenomenon. We show how the Bose-Einstein condensation of excitons occurs due to exciton-phonon interaction in the system.

The phenomenon can be then described by the nonlinear Scrödinger equation coupled to the wave equation. These two components of the model have different covariances: the first one is Galilean invariant while the second one describes the wave propagation and is Lorentz invariant. That is why the exciton-exciton interaction depends on velocity of the packet: it becomes attractive in the propagation direction if the velocity exceeds a critical one. Then the packet propagation is described by the soliton solution of the nonlinear Scrodinger equation. We predict a second critical velocity which is equal to the sound velocity. This is in agreement with the experimental data.

To sum up, the first part of this thesis is devoted to the new integrable systems of partial differential operators possessing Huygens' and bispectral properties; the second part presents new exactly solvable spin models, nd, finally, the third part deals with the soliton propagation of excitons in crystals.

List of articles included in the present thesis

Chapter 1

1.Huygens Principle in Minkowski Spaces and Soliton Solutions of the Korteweg-de Vries Equation, **Y.Berest** and **I.Loutsenko**, Comm.Math.Phys. 190, (1997) 113-132

Chapter 2

1.Self-Similar Potentials and Ising Models, **I.Loutsenko, V.Spiridonov**, JETP Letters 66, (1997) 753-758

2. Spectral self-similarity, one-dimensional Ising models and random matrices, **I.Loutsenko, V.Spiridonov**, preprint of CRM 2551, submitted to Nucl.Phys.B

Chapter 3

1. Critical Velocities in Exciton Superfluidity, **I.Loutsenko, D.Roubtsov**, Phys.Rev.Lett.78,3011 (1997)

Chapter 1

Huygens' Principle, Integrability and Solitons

Huygens' Principle in Minkowski Spaces and Soliton Solutions of the Korteweg-de Vries Equation

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Abstract

A new class of linear second order hyperbolic partial differential operators satisfying Huygens' principle in Minkowski spaces is presented. The construction reveals a direct connection between Huygens' principle and the theory of solitary wave solutions of the Korteweg-de Vries equation.

Mathematics Subject Classification: 35Q51, 35Q53, 35L05, 35L15, 35Q05.

I. Introduction

The present paper deals with the problem of describing all linear second order partial differential operators for which Huygens' principle is valid in the sense of "Hadamard's minor premise". Originally posed by J.Hadamard in his Yale lectures on hyperbolic equations [26], this problem is still far from being completely solved[1].

The simplest examples of Huygens' operators are the ordinary wave operators

$$\Box_{n+1} = \left(\frac{\partial}{\partial x^0}\right)^2 - \left(\frac{\partial}{\partial x^1}\right)^2 - \ldots - \left(\frac{\partial}{\partial x^n}\right)^2 \tag{1}$$

in an odd number $n \geq 3$ of space dimensions and those ones reduced to (1) by means of elementary transformations, i.e. by local nondegenerate changes of coordinates $x \mapsto f(x)$; gauge and conformal transformations of a given operator $\mathcal{L} \mapsto \theta(x) \circ \mathcal{L} \circ \theta(x)^{-1}$, $\mathcal{L} \mapsto \mu(x)\mathcal{L}$ with some locally smooth nonzero functions $\theta(x)$ and $\mu(x)$. These operators are usually called *trivial Huygens' operators*, and the famous "Hadamard's conjecture" claims that all Huygens' operators are trivial.

Such a strong assertion turns out to be valid only for (real) Huygens' operators with a constant principal symbol in $n = 3$ [33]. Stellmacher [40] found the first non-trivial examples of hyperbolic wave-type operators satisfying Huygens' principle, and thereby disproved Hadamard's conjecture in higher dimensional Minkowski spaces. Later Lagnese & Stellmacher [31] extended these examples and even solved [32] Hadamard's problem for a restricted class of hyperbolic operators, namely

$$\mathcal{L} = \Box_{n+1} + u(x^0) \ , \tag{2}$$

where $u\,(x^0)$ is an analytic function (in its domain of definition) depending on a single variable only. It turns out that the potentials $u(z)$ entering into (2) are

[1]Hadamard's problem, or *the problem of diffusion of waves*, has received a good deal of attention and the literature is extensive (see, e.g., [8], [12], [15], [21], [22], [24], [27], [28], [35], and references therein). For a historical account we refer the reader to the articles [19], [25].

rational functions which can be expressed explicitly in terms of some polynomials[2] $\mathcal{P}_k(z)$:

$$u(z) = 2\left(\frac{d}{dz}\right)^2 \log \mathcal{P}_k(z) \ , \quad k = 0, 1, 2, \ldots , \tag{3}$$

the latter being defined via the following differential-recurrence relation:

$$\mathcal{P}'_{k+1}\mathcal{P}_{k-1} - \mathcal{P}'_{k-1}\mathcal{P}_{k+1} = (2k+1)\mathcal{P}_k^2 \ , \quad \mathcal{P}_0 = 1 \ , \ \mathcal{P}_1 = z \ . \tag{4}$$

Since the works of Moser et al. [2], [3] the potentials (3) are known as *rational solutions* of the Korteweg-de Vries equation decreasing at infinity[3].

A wide class of Huygens' operators in Minkowski spaces has been discovered recently by Veselov and one of the authors [9], [10] (see also the review article [8]). These operators can also be presented in a self-adjoint form

$$\mathcal{L} = \Box_{n+1} + u(x) \tag{5}$$

with a locally analytic potential $u(x)$ depending on several variables. More precisely, $u(x)$ belongs to the class of so-called *Calogero-Moser potentials* associated with finite reflection groups (*Coxeter groups*):

$$u(x) = \sum_{\alpha \in \Re_+} \frac{m_\alpha(m_\alpha + 1)(\alpha, \alpha)}{(\alpha, x)^2} \ . \tag{6}$$

In formula (6) $\Re_+ \equiv \Re_+(\mathcal{G})$ stands for a properly chosen and oriented subset of normals to reflection hyperplanes of a Coxeter group $\mathcal{G}$. The group $\mathcal{G}$ acts on M^{n+1} in such a way that the time direction is preserved. The set $\{m_\alpha\}$ is a collection of non-negative integer labels attached to the normals $\alpha \in \Re$ so that $m_{w(\alpha)} = m_\alpha$ for all $w \in \mathcal{G}$. Huygens' principle holds for (5), (6), provided n is odd, and

$$n \geq 3 + 2 \sum_{\alpha \in \Re_+} m_\alpha \ . \tag{7}$$

[2]This remarkable class of polynomials seems to have been found for the first time by Burchnall and Chaundy [11].

[3]The coincidence of such rational solutions of the KdV-hierarchy with the Lagnese-Stellmacher potentials has been observed by Schimming [38], [39].

In the present work we construct a new class of self-adjoint wave-type operators (5) satisfying Huygens' principle in Minkowski spaces. As we will see, this class provides a natural extension of the hierarchy of Huygens' operators associated to Coxeter groups. On the other hand, it turns out to be related in a surprisingly simple and fundamental way to the theory of solitons.

To present the construction we consider a $(n+1)$-dimensional Minkowski space $\mathbf{M}^{n+1} \cong \mathbf{R}^{1,n}$ with the metric signature $(+,-,-,\ldots,-)$ and fixed time direction $\theta \in \mathbf{M}^{n+1}$. We write $\mathbf{Gr}_{\perp}(n+1,2) \subset \mathbf{Gr}(n+1,2)$ for a set of all 2-dimensional space-like linear subspaces in $\mathbf{M}^{n+1}$ orthogonal to θ. Every 2-plane $E \in \mathbf{Gr}_{\perp}(n+1,2)$ is equipped with the usual Euclidean structure induced from $\mathbf{M}^{n+1}$. To define the potential $u(x)$ we fix such a plane E and introduce polar coordinates (r,φ) therein.

Let $(k_i)_{i=1}^{N}$ be a strictly increasing sequence of integer positive numbers[4]: $0 \leq k_1 < k_2 < \ldots < k_{N-1} < k_N$, and let $\{\Psi_i(\varphi)\}$ be a set of 2π-periodic functions on $\mathbf{R}^1$:

$$\Psi_i(\varphi) := \cos(k_i\,\varphi + \varphi_i)\ , \quad \varphi_i \in \mathbf{R}\ , \tag{8}$$

associated to (k_i). The Wronskian of this set

$$\mathcal{W}\,[\Psi_1,\Psi_2,\ldots,\Psi_N] := \det \begin{pmatrix} \Psi_1(\varphi) & \Psi_2(\varphi) & \ldots & \Psi_N(\varphi) \\ \Psi_1'(\varphi) & \Psi_2'(\varphi) & \ldots & \Psi_N'(\varphi) \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_1^{(N-1)}(\varphi) & \Psi_2^{(N-1)}(\varphi) & \ldots & \Psi_N^{(N-1)}(\varphi) \end{pmatrix} \tag{9}$$

does not vanish indentically since $\Psi_i(\varphi)$ are linearly independent.

Let

$$\Xi := \left\{ x \in \mathbf{M}^{n+1} \mid r^{|k|}\,\mathcal{W}\,[\Psi_1,\Psi_2,\ldots,\Psi_N] = 0\ ,\ |k| := \sum_{i=1}^{N} k_i \right\}$$

be an algebraic hypersurface of zeros of the Wronskian (9) in the Minkowski space $\mathbf{M}^{n+1}$, and let $\Omega \subset \mathbf{M}^{n+1} \setminus \Xi$ be an open connected part in its complement.

[4]Using the terminology adopted in the group representation theory we will call such integer monotonic sequences *partitions*.

We define $u(x)$ in terms of cylindrical coordinates in $\mathbf{M}^{n+1}$ with polar components in E :

$$u = u_k(x) := -\frac{2}{r^2}\left(\frac{\partial}{\partial\varphi}\right)^2 \log \mathcal{W}\left[\Psi_1(\varphi), \Psi_2(\varphi), \ldots, \Psi_N(\varphi)\right] \ . \tag{10}$$

It is easy to see that in a standard Minkowskian coordinate chart $u(x)$ is a real *rational* function on $\mathbf{M}^{n+1}$ having its singularities on Ξ. In particular, it is locally analytic in Ω.

Our main result reads as follows.

Theorem. *Let* $\mathbf{M}^{n+1} \cong \mathbf{R}^{1,n}$ *be a Minkowski space, and let*

$$\mathcal{L}_{(k)} := \Box_{n+1} + u_k(x) \tag{11}$$

be a wave-type second order hyperbolic operator with the potential (10) associated to an arbitrary strictly monotonic partition (k_i) *of height* N *:*

$$0 \le k_1 < k_2 < \ldots < k_N \ , \quad k_i \in \mathbf{Z} \ , \quad i = 1, 2, \ldots, N \ .$$

Then operator $\mathcal{L}_{(k)}$ *satisfies Huygens' principle at every point* $\xi \in \Omega$*, provided* n *is odd, and*

$$n \ge 2\,k_N + 3 \ . \tag{12}$$

Remark I. A similar result is also valid if one takes an arbitrary Lorentzian 2-plane $H \in \mathbf{Gr}_{||}(n+1, 2)$ in the Minkowski space $\mathbf{M}^{n+1}$ containing the time-like vector θ. More precisely, in this case the potential $u_k(x)$ associated to the partition (k_i) is introduced in terms of pseudo-polar coordinates (ϱ, ϑ) in H:

$$u_k(x) := -\frac{2}{\varrho^2}\left(\frac{\partial}{\partial\vartheta}\right)^2 \log \mathcal{W}\left[\psi_1, \psi_2, \ldots, \psi_N\right] \ , \tag{13}$$

where $x^0 = \varrho \sinh\vartheta$, and, say, $x^1 = \varrho\cosh\vartheta$. The functions ψ_i involved in (13) are given by

$$\psi_i = \cosh(k_i\,\vartheta + \vartheta_i) \ , \quad \vartheta_i \in \mathbf{R} \ . \tag{14}$$

The theorem formulated above holds when the potential (10) is replaced by (13).

Remark II. The potentials (10) considerably extend the class of Calogero-Moser potentials (6) related to Coxeter groups of rank 2. Indeed, in $\mathbf{R}^2$ any Coxeter group $\mathcal{G}$ is a dihedral group $I_2(q)$, i.e. the group of symmetries of a regular $2q$-polygon. It has one or two conjugacy classes of reflections according as q is odd or even. The corresponding potential (6) can be rewritten in terms of polar coordinates as follows (see [36]):

$$u(r,\varphi) = \frac{m(m+1)q^2}{r^2 \sin^2(q\,\varphi)}\ , \quad \text{when } q \text{ odd}\ ,$$

and

$$u(r,\varphi) = \frac{m(m+1)\,(q/2)^2}{r^2 \sin^2(q/2)\,\varphi} + \frac{m_1(m_1+1)\,(q/2)^2}{r^2 \cos^2(q/2)\,\varphi}\ , \quad \text{when } q \text{ even}\ .$$

It is easy to verify that formula (10) boils down to these forms if we fix $N := m$; $\varphi_i := (-1)^i\pi/2$, $i = 1, 2, \ldots, N$, and choose

$$k := (q,\ 2q,\ 3q,\ \ldots,\ mq)\ ,$$

when q is odd, and

$$k := (\frac{q}{2},\ q,\ \frac{3q}{2},\ \ldots,\ (m-m_1)\frac{q}{2},\ q+(m-m_1)\frac{q}{2},\ 2q+(m-m_1)\frac{q}{2},\ \ldots\ ,\ (m+m_1)\frac{q}{2})\ ,$$

when q is even and $m > m_1$, respectively.

Remark III. Let us set $\varphi_i = 4k_i^3 t + \varphi_{0i}$ and $\vartheta_i = -4k_i^3 t + \vartheta_{0i}$, $i = 1, 2, \ldots, N$; $\varphi_{0i}, \vartheta_{0i} \in \mathbf{R}$. The angular parts of potentials (10), (13), i.e.

$$v(\varphi) = -2\left(\frac{\partial}{\partial\varphi}\right)^2 \log \mathcal{W}\left[\Psi_1(\varphi), \Psi_2(\varphi), \ldots, \Psi_N(\varphi)\right]\ , \tag{15}$$

$$v(\vartheta) = -2\left(\frac{\partial}{\partial\vartheta}\right)^2 \log \mathcal{W}\left[\psi_1(\vartheta), \psi_2(\vartheta), \ldots, \psi_N(\vartheta)\right]\ , \tag{16}$$

are known (see, e.g., [18], [34]) to be respectively singular *periodic* and proper *N-soliton* solutions of the Korteweg-de Vries equation

$$v_t = -v_{\varphi\varphi\varphi} + 6vv_\varphi\ . \tag{17}$$

It is also well-known that N-soliton potentials (16) constitute the whole class of so-called *reflectionless* real potentials for the one-dimensional Schrödinger operator $L = -\partial^2/\partial\vartheta^2 + v(\vartheta)$ (see, e.g., [1]).

In conclusion of this section we put forward the following conjecture[5].

Conjecture. The wave-type operators (11) with potentials of the form (10) give *a complete* solution of Hadamard's problem in Minkowski spaces M^{n+1} within a restricted class of linear second order hyperbolic operators

$$\mathcal{L} = \left(\frac{\partial}{\partial x^0}\right)^2 - \left(\frac{\partial}{\partial x^1}\right)^2 - \left(\frac{\partial}{\partial x^2}\right)^2 - \ldots - \left(\frac{\partial}{\partial x^n}\right)^2 + u(x^1, x^2)$$

with real locally analytic potentials $u = u(x^1, x^2)$ depending on *two* spatial variables and homogeneous of degree (-2): $u(\alpha x^1, \alpha x^2) = \alpha^{-2} u(x^1, x^2)\,,\ \alpha > 0\,.$

Acknowledgements. One of the authors (Yu. B.) is grateful to Prof. A. Veselov (Loughborough University, UK) and his collaborators Dr. O. Chalykh and M. Feigin who have kindly informed him [42] about their recent results showing the existence of new algebraically integrable Schrödinger operators which are not related to Coxeter root systems. In fact, this observation was a motivation for the present work. We would like also to thank Prof. P. Winternitz (CRM, Université de Montréal) for his encouragement and highly stimulating discussions.

The work of Yu. B. was partially supported by the fellowship from Institut des Sciences Mathématiques (Montréal) which is gratefully acknowledged. The second author (I. L.) is grateful to Prof. L. Vinet for his support.

II. Huygens' principle and Hadamard-Riesz expansions

The proof of the theorem stated above rests heavily on the Hadamard theory of Cauchy's problem for linear second order hyperbolic partial differential equations.

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[5]**Note added in the proof.** This conjecture has been proved recently by one of the authors in [6].

Here, we summarize briefly some necessary results from this theory following essentially M. Riesz's approach [37] (see also [21], [24]).

Let $\mathbf{M}^{n+1} \cong \mathbf{R}^{1,n}$ be a Minkowski space, and let Ω be an open connected part in $\mathbf{M}^{n+1}$. We consider a (formally) self-adjoint scalar wave-type operator

$$\mathcal{L} = \Box_{n+1} + u(x) \ , \tag{18}$$

defined in Ω, the scalar field (potential) $u(x)$ being assumed to be in $\mathcal{C}^\infty(\Omega)$. For any $\xi \in \Omega$, we define a cone of isotropic (null) vectors in $\mathbf{M}^{n+1}$ with its vertex at ξ:

$$\gamma(x,\xi) := (x^0 - \xi^0)^2 - (x^1 - \xi^1)^2 - \ldots - (x^n - \xi^n)^2 = 0 \ , \tag{19}$$

and single out the following sets :

$$\begin{aligned} C_\pm(\xi) \ &:= \ \{x \in \mathbf{M}^{n+1} \mid \gamma(x,\xi) = 0 \, , \, \xi^0 \lessgtr x^0\} \ , \\ J_\pm(\xi) \ &:= \ \{x \in \mathbf{M}^{n+1} \mid \gamma(x,\xi) > 0 \, , \, \xi^0 \lessgtr x^0\} \ . \end{aligned} \tag{20}$$

Definition. *A (forward) Riesz kernel* of operator $\mathcal{L}$ is a holomorphic (entire analytic) mapping $\lambda \mapsto \Phi^\Omega_\lambda(x,\xi)$, $\lambda \in \mathbf{C}$, with values in the space of distributions[6] $\mathcal{D}'(\Omega)$, such that for any $\xi \in \Omega$:

$$\begin{aligned} &(i) \quad \operatorname{supp} \Phi^\Omega_\lambda(x,\xi) \subseteq \overline{J_+(\xi)} \ , \\ &(ii) \quad \mathcal{L}\left[\Phi^\Omega_\lambda(x,\xi)\right] = \Phi^\Omega_{\lambda-1}(x,\xi) \ , \\ &(iii) \quad \Phi^\Omega_0(x,\xi) = \delta(x-\xi) \ . \end{aligned} \tag{21}$$

The value of the Riesz kernel $\Phi^\Omega_1(x,\xi) := \Phi_+(x,\xi)$ at $\lambda = 1$ is called *a (forward) fundamental solution* of the operator $\mathcal{L}$:

$$\mathcal{L}[\Phi_+(x,\xi)] = \delta(x-\xi) \ , \quad \operatorname{supp} \ \Phi_+(x,\xi) \subseteq \overline{J_+(\xi)} \ . \tag{22}$$

[6]By a distribution $f \in \mathcal{D}'(\Omega)$ we mean, as usual, a linear continuous form on the space $\mathcal{D}(\Omega)$ of $\mathcal{C}^\infty$-functions with supports compactly imbedded in Ω (cf., e.g., [23]).

Such a solution is known to exist for any $u(x) \in \mathcal{C}^\infty(\Omega)$, and it is uniquely determined.

Definition. The operator $\mathcal{L}$ defined by (18) satisfies *Huygens' principle* in a domain $\Omega_0 \subseteq \Omega$ in $\mathbf{M}^{n+1}$ if

$$\operatorname{supp} \Phi_+(x,\xi) \subseteq \overline{C_+(\xi)} = \partial J_+(\xi) \ . \tag{23}$$

for every point $\xi \in \Omega_0$.

The analytic description of singularities of Riesz kernel distributions (and, in particular, fundamental solutions) for second order hyperbolic differential operators is given in terms of their asymptotic expansions in the vicinity of the characteristic cone by a graded scale of distributions with weaker and weaker singularities. Such "asymptotics in smoothness", usually called *Hadamard-Riesz expansions*, turn out to be very important for testing Huygens' principle for the operators under consideration.

In order to construct an appropriate scale of distributions (*Riesz convolution algebra*) in Minkowski space $\mathbf{M}^{n+1}$ we consider (for a fixed $\xi \in \mathbf{M}^{n+1}$) a holomorphic $\mathcal{D}'$-valued mapping $\mathbf{C} \to \mathcal{D}'(\mathbf{M}^{n+1})$, $\lambda \mapsto R_\lambda(x,\xi)$, such that $R_\lambda(x,\xi)$ is an analytic continuation (in λ) of the following (regular) distribution:

$$\langle R_\lambda(x,\xi), g(x)\rangle = \int\limits_{J_+(\xi)} \frac{\gamma(x,\xi)^{\lambda-\frac{n+1}{2}}}{H_{n+1}(\lambda)}\, g(x)\, dx \ , \quad \operatorname{Re}\lambda > \frac{n-1}{2} \ , \tag{24}$$

where $dx = dx^0 \wedge dx^1 \wedge \ldots \wedge dx^n$ is a volume form in $\mathbf{M}^{n+1}$, $g(x) \in \mathcal{D}(\mathbf{M}^{n+1})$, and $H_{n+1}(\lambda)$ is a constant given by

$$H_{n+1}(\lambda) = 2\pi^{\frac{n-1}{2}} 4^{\lambda-1}\Gamma(\lambda)\Gamma\left(\lambda - (n-1)/2\right) \ . \tag{25}$$

The following properties of this family of distributions are deduced directly from their definition.

For all $\lambda \in \mathbf{C}$ and $\xi \in \mathbf{M}^{n+1}$ we have

$$\operatorname{supp} R_\lambda(x,\xi) \subseteq \overline{J_+(\xi)} \tag{26}$$

$$\Box_{n+1} R_\lambda = R_{\lambda-1} \ , \tag{27}$$

$$R_\lambda * R_\mu = R_{\lambda+\mu} \ , \quad \mu \in \mathbf{C} \ , \tag{28}$$

$$(x-\xi, \partial_x) R_\lambda = (2\lambda - n + 1) R_\lambda \ , \tag{29}$$

$$\gamma^\nu R_\lambda = 4^\nu \, (\lambda)_\nu \, (\lambda - (n-1)/2)_\nu \, R_{\lambda+\nu} \ , \quad \nu \in \mathbf{Z}_{\geq 0} \ , \tag{30}$$

where $(\kappa)_\nu := \Gamma(\kappa+\nu)/\Gamma(\kappa)$ is Pochhammer's symbol, and $\gamma = \gamma(x,\xi)$ is a square of the geodesic distance between x and ξ in $\mathbf{M}^{n+1}$.

In addition, when n is odd, one can prove that

$$R_\lambda(x,\xi) = \frac{1}{2\pi^{\frac{n-1}{2}}} \frac{\delta_+^{(\frac{n-1}{2}-\lambda)}(\gamma)}{4^{\lambda-1}(\lambda-1)!} \quad \text{for} \quad \lambda = 1, 2, \ldots, (n-1)/2 \ , \tag{31}$$

where $\delta_+^{(m)}(\gamma)$ stands for the m-th derivative of Dirac's delta-measure concentrated on the surface of the future-directed characteristic half-cone $\overline{C_+(\xi)}$.

Another important property of Riesz distributions is that

$$R_0(x,\xi) = \delta(x-\xi) \ . \tag{32}$$

Formulas (26), (27), (32) show that $R_\lambda(x,\xi)$ is a Riesz kernel for the ordinary wave operator $\Box_{n+1}$. The property (31) means precisely that in even-dimensional Minkowski spaces $\mathbf{M}^{n+1}$ (n is odd) Huygens' principle holds for sufficiently low powers of the wave operator $\Box^d$, $d \leq (n-1)/2$.

Now we are able to construct the Hadamard-Riesz expansion for the Riesz kernel of a general self-adjoint wave-type operator (18) on $\mathbf{M}^{n+1}$.

First, we have to find a sequence of two-point smooth functions $U_\nu := U_\nu(x,\xi) \in \mathcal{C}^\infty(\Omega \times \Omega)$, $\nu = 0, 1, 2 \ldots$, as a solution of the following *transport equations*:

$$(x-\xi, \partial_x) \, U_\nu(x,\xi) + \nu U_\nu(x,\xi) = -\frac{1}{4} \mathcal{L}\left[U_{\nu-1}(x,\xi)\right] \ , \quad \nu \geq 1 \ . \tag{33}$$

It is well-known (essentially due to [26]) that the differential-recurrence system (33) has a *unique* solution provided each U_ν is required to be bounded in the vicinity of the vertex of the characteristic cone and $U_0(x,\xi)$ is fixed for a normalization, i.e.

$$U_0(x,\xi) \equiv 1 \ , \qquad U_\nu(\xi,\xi) \sim \mathcal{O}(1) \ , \quad \forall \, \nu = 1,2,3,\ldots$$

These functions U_ν are called *Hadamard's coefficients* of the operator $\mathcal{L}$.

In terms of U_ν the required asymptotic expansion can be presented as follows:

$$\Phi_\lambda^\Omega(x,\xi) \sim \sum_{\nu=0}^{\infty} 4^\nu (\lambda)_\nu \, U_\nu(x,\xi) \, R_{\lambda+\nu}(x,\xi) \ . \tag{34}$$

One can prove that for a hyperbolic differential operator $\mathcal{L}$ with locally analytic coefficients the Hadamard-Riesz expansion is locally uniformly convergent. From now on we will restrict our consideration to this case.

For $\lambda = 1$ formula (34) provides an expansion of the fundamental solution of the operator $\mathcal{L}$ in a neighborhood of the vertex $x = \xi$ of the characteristic cone:

$$\Phi_+(x,\xi) = \sum_{\nu=0}^{\infty} 4^\nu \nu! \, U_\nu(x,\xi) \, R_{\nu+1}(x,\xi) \ . \tag{35}$$

When n is even, we have supp $R_{\nu+1}(x,\xi) = \overline{J_+(\xi)}$ for all $\nu = 0,1,2,\ldots$, and therefore Huygens' principle never occurs in odd-dimensional Minkowski spaces M^{2l+1}.

On the other hand, in the case of an odd number of space dimensions $n \geq 3$, we know due to (31) that for $\nu = 0,1,2,\ldots,(n-3)/2$, supp $R_{\nu+1}(x,\xi) = \overline{C_+(\xi)}$. Hence, using (30), we can rewrite the series (35) in following form:

$$\Phi_+(x,\xi) = \frac{1}{2\pi^p} \Big(V(x,\xi) \, \delta_+^{(p-1)}(\gamma) + W(x,\xi) \, \eta_+(\gamma) \Big) \ , \tag{36}$$

where $p := (n-1)/2$, $\eta_+(\gamma)$ is a regular distribution characteristic for the region $J_+(\xi)$:

$$\langle \eta_+(\gamma), g(x) \rangle = \int\limits_{J_+(\xi)} g(x) \, dx \ , \quad g(x) \in \mathcal{D}(\mathrm{M}^{n+1}) \ ,$$

and $V(x,\xi)$, $W(x,\xi)$ are analytic functions in a neighborhood of the vertex $x = \xi$ which admit the following expansions therein:

$$V(x,\xi) = \sum_{\nu=0}^{p-1} \frac{1}{(1-p)\ldots(\nu-p)} \, U_\nu(x,\xi) \, \gamma^\nu \ , \tag{37}$$

$$W(x,\xi)=\sum_{\nu=p}^{\infty}\frac{1}{(\nu-p)!}\,U_\nu(x,\xi)\,\gamma^{\nu-p}\ ,\quad p=\frac{n-1}{2}\ . \tag{38}$$

The function $W(x,\xi)$ is usually called *a logarithmic term* of the fundamental solution[7].

It follows directly from the representation formula (36) that operator $\mathcal{L}$ satisfies Huygens' principle in a neighborhood of the point ξ, if and only if, the logarithmic term $W(x,\xi)$ of its fundamental solution vanishes in this neighborhood identically in x: $W(x,\xi)\equiv 0$.

The function $W(x,\xi)$ is known to be a regular solution of the characteristic Goursat problem for the operator $\mathcal{L}$:

$$\mathcal{L}\,[W(x,\xi)]=0 \tag{39}$$

with a boundary value given on the cone surface $\overline{C_+(\xi)}$. Such a boundary problem has a unique solution, and hence, the necessary and sufficient condition for $\mathcal{L}$ to be Huygens' operator becomes

$$W(x,\xi)\triangleq 0\ , \tag{40}$$

where the symbol $\triangleq$ implies that the equation in hand is satisfied only on $\overline{C_+(\xi)}$. By definition (38), the latter condition is equivalent to the following one

$$U_p(x,\xi)\triangleq 0\ ,\quad p=\frac{n-1}{2}\ . \tag{41}$$

In this way, we arrive at the important criterion for the validity of Huygens' principle in terms of coefficients of the Hadamard-Riesz expansion (34). Equation (41) is essentially due to Hadamard [26]. It will play a central role in the proof of our main theorem.

[7]Such a terminology goes back to Hadamard's book [26], where the function $W(x,\xi)$ is introduced as a coefficient under the logarithmic singularity of an elementary solution (see for details [15], pp. 740–743).

III. Proof of the main theorem

We start with some remarks concerning the properties of the one-dimensional Schrödinger operator

$$L_{(k)} := -\left(\frac{\partial}{\partial\varphi}\right)^2 + v_k(\varphi) \tag{42}$$

with a general periodic soliton potential

$$v_k(\varphi) := -2\left(\frac{\partial}{\partial\varphi}\right)^2 \log \mathcal{W}\left[\Psi_1, \Psi_2, \ldots, \Psi_N\right] \ . \tag{43}$$

Here, as already discussed in the Introduction, $\mathcal{W}\left[\Psi_1, \Psi_2, \ldots, \Psi_N\right]$ stands for a Wronskian of the set of periodic functions on $\mathbf{R}^1$:

$$\Psi_i(\varphi) := \cos(k_i\,\varphi + \varphi_i) \ , \quad \varphi_i \in \mathbf{R} \ , \tag{44}$$

associated to an arbitrary strictly monotonic sequence of real positive numbers ("soliton amplitudes"): $0 \leq k_1 < \ldots < k_{N-1} < k_N$.

It is well-known (see, e.g., [34]) that any such operator $L_{(k)}$ (as well as its proper solitonic counterpart (16)) can be constructed by a successive application of *Darboux-Crum factorization transformations* ([17], [16]) to the Schrödinger operator with the identically zero potential:

$$L_0 := -\left(\frac{\partial}{\partial\varphi}\right)^2 \ . \tag{45}$$

To be precise, let L be a second order ordinary differential operator with a sufficiently smooth potential:

$$L := -\left(\frac{\partial}{\partial\varphi}\right)^2 + v(\varphi) \ . \tag{46}$$

We ask for formal factorizations of the operator

$$L - \lambda\, I = A^* \circ A \ , \tag{47}$$

where I is an identity operator, λ is a (real) constant, and A, A^* are the first order operators adjoint to each other in a formal sense.

According to Frobenius' theorem (see, e.g., [29]), the most general factorization (47) is obtained if we take $\chi(\varphi)$ as a generic element in $\operatorname{Ker}(L - \lambda I) \setminus \{0\}$ and set

$$A := \chi \circ \left(\frac{\partial}{\partial\varphi}\right) \circ \chi^{-1} \quad , \quad A^* := -\chi^{-1} \circ \left(\frac{\partial}{\partial\varphi}\right) \circ \chi \, . \tag{48}$$

Indeed, $A^* \circ A$ is obviously self-adjoint second order operator with the principal part $-\partial^2/\partial\varphi^2$. Hence, it is of the form (46). Moreover, since $A[\chi] = 0$, we have $\chi \in \operatorname{Ker} A^* \circ A$, so that (47) becomes evident.

Note that for every $\lambda \in \mathbf{R}$ we actually get a one-parameter family of factorizations of $L - \lambda I$. This follows from the fact that $\dim \operatorname{Ker}(L - \lambda I) = 2$, whereas $\chi(\varphi)$ and $C\,\chi(\varphi)$ give rise to the same factorization pair (A, A^*).

By definition, the Darboux-Crum transformation maps an operator $L = \lambda I + A^* \circ A$ into the operator

$$\tilde{L} := \lambda I + A \circ A^* \, , \tag{49}$$

in which A and A^* are interchanged. The operator $\tilde{L}$ is also a (formally) self-adjoint second-order differential operator

$$\tilde{L} := -\left(\frac{\partial}{\partial\varphi}\right)^2 + \tilde{v}(\varphi) \, , \tag{50}$$

where $\tilde{v}(\varphi)$ is given explicitly by

$$\tilde{v}(\varphi) = v(\varphi) - 2\left(\frac{\partial}{\partial\varphi}\right)^2 \log \chi(\varphi) \, . \tag{51}$$

The initial operator L and its Darboux-Crum transform $\tilde{L}$ are obviously related to each other via the following intertwining indentities:

$$\tilde{L} \circ A = A \circ L \quad , \quad L \circ A^* = A^* \circ \tilde{L} \, . \tag{52}$$

The Darboux-Crum transformation has a lot of important applications in the spectral theory of Sturm-Liouville operators and related problems of quantum mechanics [30]. In particular, it is used to insert or remove one eigenvalue without changing the rest of the spectrum of a Schrödinger operator (for details see the monograph [34] and references therein).

The explicit construction of the family of operators (42) with periodic soliton potentials (43) is based on the following Crum's lemma:

Lemma ([16]). Let L be a given second order Sturm-Liouville operator (46) with a sufficiently smooth potential, and let $\{\Psi_1, \Psi_2, \ldots, \Psi_N\}$ be its eigenfunctions corresponding to arbitrarily fixed pairwise different eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$, i.e. $\Psi_i \in \mathrm{Ker}(L - \lambda_i I)$, $i = 1, 2, \ldots, N$. Then, for arbitrary $\Psi \in \mathrm{Ker}(L - \lambda I)$, $\lambda \in \mathbf{R}$, the function

$$\chi_N(\varphi) := \frac{\mathcal{W}[\Psi_1, \Psi_2, \ldots, \Psi_N, \Psi]}{\mathcal{W}[\Psi_1, \Psi_2, \ldots, \Psi_N]} \tag{53}$$

satisfies the differential equation

$$\left[-\left(\frac{\partial}{\partial\varphi}\right)^2 + v_N(\varphi) \right] \chi_N(\varphi) = \lambda\, \chi_N(\varphi) \tag{54}$$

with the potential

$$v_N(\varphi) := v(\varphi) - 2\left(\frac{\partial}{\partial\varphi}\right)^2 \log \mathcal{W}[\Psi_1, \Psi_2, \ldots, \Psi_N] \ . \tag{55}$$

Given a sequence of real positive numbers $(k_i)_{i=1}^N$: $0 \leq k_1 < k_2 < \ldots < k_N$, the Darboux-Crum factorization scheme:

$$L_i := A_{i-1} \circ A_{i-1}^* + k_i^2\, I = A_i^* \circ A_i + k_{i+1}^2\, I \ \to \ L_{i+1} := A_i \circ A_i^* + k_{i+1}^2\, I \ , \tag{56}$$

starting from the Schrödinger operator (45) with a zero potential

$$L_0 \equiv -\left(\frac{\partial}{\partial\varphi}\right)^2 = A_0^* \circ A_0 + k_1^2\, I \ ,$$

produces the required operator $L_{(k)} \equiv L_N$ with the general periodic potential (43).

Now we proceed to the proof of our main theorem formulated in the Introduction.

When $N = 0$, the statement of the theorem is evident, since the operator $\mathcal{L}_0$ is just the ordinary wave operator in an odd number n of spatial variables.

Using the Darboux-Crum scheme as outlined above we will carry out the proof by induction in N.

Suppose that the statement of the theorem is valid for all $m = 0, 1, 2, \ldots, N$. Consider an arbitrary *integer* monotonic partition (k_i) of height $N: \ 0 < k_1 < k_2 < \ldots < k_N\,, \ k_i \in \mathbf{Z}$.

By our assumption, the wave-type operator

$$\mathcal{L}_N := \mathcal{L}_{(k)} = \Box_{n+1} + u_k(x)\ , \tag{57}$$

associated to this partition, satisfies Huygens' principle in the $(n+1)$-dimensional Minkowski space $\mathbf{M}^{n+1}$ with n odd, and $n \geq 2\,k_N + 3$. We fix the minimal admissible number of space variables, i.e. $n = 2\,k_N + 3$, and denote

$$p := \frac{n-1}{2} = k_N + 1\ . \tag{58}$$

By construction, the operator $\mathcal{L}_N$ can be written explicitly in terms of suitably chosen cylindrical coordinates in $\mathbf{M}^{n+1}$:

$$\mathcal{L}_N = \Box_{n-1} - \left[\left(\frac{\partial}{\partial r} \right)^2 + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} \left(- \left(\frac{\partial}{\partial \varphi} \right)^2 + v_N(\varphi) \right) \right]\ , \tag{59}$$

where (r, φ) are the polar coordinates in some Euclidean 2-plane E orthogonal to the time direction in $\mathbf{M}^{n+1}$, i.e. $E \in \mathbf{Gr}_{\perp}(n+1, 2)$; $\Box_{n-1}$ is a wave operator in the orthogonal complement $E^{\perp} \cong \mathbf{M}^{n-1}$ of E in $\mathbf{M}^{n+1}$; and $v_N(\varphi)$ is a 2π-periodic potential given by (43).

Let $k := k_{N+1}$ be an arbitrary positive integer such that

$$k > k_N\ . \tag{60}$$

We apply the Darboux-Crum transformation (56) with the spectral parameter k to the angular part of the Laplacian in E. For this we rewrite $\mathcal{L}_N$ in the form

$$\mathcal{L}_N = \Box_{n-1} - \left[\left(\frac{\partial}{\partial r} \right)^2 + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} \left(A_N^* \circ A_N + k^2 \right) \right]\ , \tag{61}$$

and set

$$\mathcal{L}_{N+1} := \Box_{n-1} - \left[\left(\frac{\partial}{\partial r} \right)^2 + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \left(A_N \circ A_N^* + k^2 \right) \right] , \tag{62}$$

where $A_N := A_N(\varphi)$ and $A_N^* := A_N^*(\varphi)$ are the first order ordinary differential operators of the form (48).

According to (52), we have

$$\mathcal{L}_{N+1} \circ A_N = A_N \circ \mathcal{L}_N \quad , \quad \mathcal{L}_N \circ A_N^* = A_N^* \circ \mathcal{L}_{N+1} \, . \tag{63}$$

Let $\Phi_\lambda^N(x,\xi)$ and $\Phi_\lambda^{N+1}(x,\xi)$ be the Riesz kernels of hyperbolic operators $\mathcal{L}_N$ and $\mathcal{L}_{N+1}$ respectively. Then, by virtue of (63) we must have the relation

$$A_N^*(\varphi) \left[\Phi_\lambda^{N+1} \right] - A_N(\phi) \left[\Phi_\lambda^N \right] = 0 \quad \text{for all } \lambda \in \mathbf{C} \, , \tag{64}$$

where $A_N(\phi)$ is the differential operator A_N written in terms of the variable ϕ conjugated to φ. Indeed, if identity (64) were not valid, one could define a holomorphic mapping $\tilde{\Phi}^N : \mathbf{C} \to \mathcal{D}'$, $\lambda \mapsto \tilde{\Phi}_\lambda^N(x,\xi)$, such that

$$\tilde{\Phi}_\lambda^N(x,\xi) := \Phi_\lambda^N(x,\xi) + a \left(A_N^*(\varphi) \left[\Phi_\lambda^{N+1} \right] - A_N(\phi) \left[\Phi_\lambda^N \right] \right) \, . \tag{65}$$

The distribution $\tilde{\Phi}_\lambda^N(x,\xi)$, depending on an arbitrary complex parameter $a \in \mathbf{C}$, would also satisfy all the axioms (21) in the definition of a Riesz kernel for the operator $\mathcal{L}_N$. In this way, we would arrive at the contradiction with the uniqueness of such a kernel.

In particular, when $\lambda = 1$, the identity (64) gives the relation between the fundamental solutions $\Phi_+^N(x,\xi) \equiv \Phi_1^N(x,\xi)$ and $\Phi_+^{N+1}(x,\xi) \equiv \Phi_1^{N+1}(x,\xi)$ of operators $\mathcal{L}_N$ and $\mathcal{L}_{N+1}$. In accordance with (36), we have

$$\Phi_+^N(x,\xi) = \frac{1}{2\pi^p} \left(V_N(x,\xi)\, \delta_+^{(p-1)}(\gamma) + W_N(x,\xi)\, \eta_+(\gamma) \right) \tag{66}$$

and

$$\Phi_+^{N+1}(x,\xi) = \frac{1}{2\pi^p} \left(V_{N+1}(x,\xi)\, \delta_+^{(p-1)}(\gamma) + W_{N+1}(x,\xi)\, \eta_+(\gamma) \right) , \tag{67}$$

where γ is a square of the geodesic distance between the points x and ξ in M^{n+1}. Substituting (66), (67) into (64), we get the relation between the logarithmic terms $W_N(x,\xi)$ and $W_{N+1}(x,\xi)$ of operators $\mathcal{L}_N$ and $\mathcal{L}_{N+1}$

$$A_N^*(\varphi)\ [W_{N+1}(x,\xi)] - A_N(\phi)\ [W_N(x,\xi)] = 0\ . \tag{68}$$

By our assumption, $\mathcal{L}_N$ is a Huygens' operator in M^{n+1}, so that $W_N(x,\xi) \equiv 0$. Hence, equation (68) implies $A_N^*(\varphi)\ [W_{N+1}(x,\xi)] = 0$. On the other hand, as discussed in Sect. II, the logarithmic term $W_{N+1}(x,\xi)$ is a regular solution of the characteristic Goursat problem for $\mathcal{L}_{N+1}$, i.e. in particular,

$$\mathcal{L}_{N+1}\ [W_{N+1}(x,\xi)] = 0\ . \tag{69}$$

Taking into account definition (62) of the operator $\mathcal{L}_{N+1}$, we arrive at the following equation for $W_{N+1}(x,\xi)$:

$$\Box_{n-1} W_{N+1}(x,\xi) = \left(\left(\frac{\partial}{\partial r}\right)^2 + \frac{1}{r}\frac{\partial}{\partial r} - \frac{k^2}{r^2}\right) W_{N+1}(x,\xi)\ . \tag{70}$$

According to (38), the logarithmic term W_{N+1} admits the following expansion

$$W_{N+1}(x,\xi) = \sum_{\nu=p}^{\infty} U_\nu(x,\xi)\,\frac{\gamma^{\nu-p}}{(\nu-p)!}\ ,\quad p = \frac{n-1}{2}\ , \tag{71}$$

where $U_\nu(x,\xi)$ are the Hadamard coefficients of the operator $\mathcal{L}_{N+1}$. Since the potential of the wave-type operator $\mathcal{L}_{N+1}$ depends only on the variables r,φ, its Hadamard coefficients U_ν must depend on the same variables r,φ and their conjugates ρ,ϕ only:

$$U_\nu = U_\nu(r,\varphi,\rho,\phi) \qquad \text{for all} \quad \nu = 0,1,2,\ldots \tag{72}$$

This follows immediately from the uniqueness of solution of Hadamard's transport equations (33).

On the other hand, since

$$\gamma = s^2 - r^2 - \rho^2 + 2r\,\rho\,\cos(\varphi - \phi)\ , \tag{73}$$

where s is a geodesic distance in the space $E^{\perp} \cong \mathrm{M}^{n-1}$ orthogonally complementary to the 2-plane E, we conclude that W_{N+1} is actually a function of five variables: $W_{N+1} = W_{N+1}(s, r, \rho, \varphi, \phi)$. On the space of such functions the wave operator $\Box_{n-1}$ in $E^{\perp}$ acts in the same way as its "radial part", i.e.

$$\Box_{n-1} W_{N+1} = \left(\left(\frac{\partial}{\partial s} \right)^2 + \frac{n-2}{s} \frac{\partial}{\partial s} \right) W_{N+1} \,.$$

Hence, equation (70) becomes

$$\left(\left(\frac{\partial}{\partial r} \right)^2 - \left(\frac{\partial}{\partial s} \right)^2 - \frac{n-2}{s} \frac{\partial}{\partial s} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{k^2}{r^2} \right) W_{N+1} = 0 \,. \tag{74}$$

Now we substitute the expansion (71)

$$W_{N+1} = \sum_{\nu=p}^{\infty} U_\nu(r, \varphi, \rho, \phi) \frac{\gamma^{\nu-p}}{(\nu-p)!} \,, \quad p = \frac{n-1}{2} \,, \tag{75}$$

into the left-hand side of the latter equation and develop the result into the similar power series in γ, taking into account formula (73). After simple calculations we obtain

$$\sum_{\nu=p}^{\infty} \left[\left(U_\nu'' + \frac{1}{r} U_\nu' - \frac{k^2}{r^2} U_\nu \right) - 4\,(r - \rho\,\cos(\varphi - \phi))\; U_{\nu+1}' - \right. \tag{76}$$

$$\left. -2\left(2\,(\nu+1) - \frac{\rho}{r}\,\cos(\varphi-\phi)\right) U_{\nu+1} - 4\,\rho^2\,\sin^2(\varphi-\phi)\,U_{\nu+2} \right] \frac{\gamma^{\nu-p}}{(\nu-p)!} = 0 \,,$$

where the prime means differentiation with respect to r.

Since the functions U_ν do not depend explicitly on γ, equation (76) can be satisfied only if each coefficient under the powers of γ vanishes separately. In this way we arrive at the following differential-recurrence relation for the Hadamard coefficients of the operator $\mathcal{L}_{N+1}$:

$$4\,\rho^2\,\sin^2(\varphi-\phi)\,U_{\nu+2} = \left(U_\nu'' + \frac{1}{r} U_\nu' - \frac{k^2}{r^2} U_\nu \right) +$$

$$+ \frac{2\rho}{r}\,\cos(\varphi-\phi)\,\left(2\,r\,U_{\nu+1}' + U_{\nu+1}\right) - 4\,\left(r\,U_{\nu+1}' + (\nu+1)\,U_{\nu+1}\right) \,, \tag{77}$$

where ν runs from p: $\nu = p,\, p+1,\, p+2, \ldots$

To get a further simplification of equation (77) we notice that all the Hadamard coefficients of the operators under consideration (11), (10) are homogeneous functions of appropriate degrees. More precisely, they have the following specific form

$$U_\nu(r,\varphi,\rho,\phi) = \frac{1}{(r\,\rho)^\nu}\,\sigma_\nu(\varphi,\phi)\ , \qquad \nu = 0,1,2,\dots\ , \tag{78}$$

where $\sigma_\nu(\varphi,\phi) = \sigma_\nu(\phi,\varphi)$ are symmetric 2π-periodic functions depending on the angular variables only.

In order to prove Ansatz (78) we have to go back to the relation (64) between the Riesz kernels of operators $\mathcal{L}_N$ and $\mathcal{L}_{N+1}$:

$$A_N^*(\varphi)\ \left[\Phi_\lambda^{N+1}(x,\xi)\right] - A_N(\phi)\ \left[\Phi_\lambda^N(x,\xi)\right] = 0\ , \quad \lambda \in \mathbf{C}\ , \tag{79}$$

If we substitute the Hadamard-Riesz expansions (34) of the kernels $\Phi_\lambda^N(x,\xi)$ and $\Phi_\lambda^{N+1}(x,\xi)$ into (79) directly and take into account that A_N and its adjoint A_N^* are the first order ordinary differential operators of the following form (cf. (48)):

$$A_N(\varphi) = \frac{\partial}{\partial\varphi} - f_N(\varphi)\ , \quad A_N^*(\varphi) = -\frac{\partial}{\partial\varphi} - f_N(\varphi)\ , \tag{80}$$

where $f_N(\varphi) = (\partial/\partial\varphi)\log\chi_N(\varphi)$, we obtain

$$\sum_{\nu=0}^{\infty} 4^\nu\,(\lambda)_\nu \Big[2r\rho\,\sin(\varphi-\phi)\ \left(U_{\nu+1}^{N+1} - U_{\nu+1}^N\right) -$$

$$- \left(\frac{\partial}{\partial\varphi} + f_N(\varphi)\right)\,U_\nu^{N+1} - \left(\frac{\partial}{\partial\phi} - f_N(\phi)\right)\,U_\nu^N\Big]\,R_{\lambda+\nu} = 0\ , \tag{81}$$

where $U_\nu^N(r,\varphi,\rho,\phi)$ and $U_\nu^{N+1}(r,\varphi,\rho,\phi)$ are the Hadamard coefficients of operators $\mathcal{L}_N$ and $\mathcal{L}_{N+1}$ respectively; $R_\lambda := R_\lambda(x,\xi)$ is the family of Riesz distributions in $\mathbf{M}^{n+1}$.

The same argument as above (see the remark before formula (77)) shows that all the coefficients of the series (81) under the Riesz distributions of different weights must vanish separately. So we arrive at the recurrence relation between the sequences of Hadamard's coefficients of operators $\mathcal{L}_N$ and $\mathcal{L}_{N+1}$:

$$U_{\nu+1}^{N+1} = U_{\nu+1}^N + \frac{1}{2r\rho\,\sin(\varphi-\phi)}\,\left[\left(\frac{\partial}{\partial\varphi} + f_N(\varphi)\right)\,U_\nu^{N+1} + \left(\frac{\partial}{\partial\phi} - f_N(\phi)\right)\,U_\nu^N\right]\ , \tag{82}$$

where $U_0^{N+1} = U_0^N \equiv 1$ and $\nu = 0, 1, 2, \ldots$ Now it is easy to conclude from (82) by induction in N that the Ansatz (78) really holds for Hadamard's coefficients of all wave-type operators (11) with potentials (10).

Returning to equation (77) and substituting (78) therein, we obtain the following three-term recurrence relation for the angular functions $\sigma_\nu(\varphi, \phi)$:

$$4 \sin^2(\varphi - \phi)\, \sigma_{\nu+2} = (\nu^2 - k^2)\, \sigma_\nu - 2(2\nu + 1) \cos(\varphi - \phi)\, \sigma_{\nu+1} \ , \tag{83}$$

where $\nu = p,\ p + 1,\ p + 2, \ldots$.

In order to analyze equation (83) it is convenient to introduce a formal generating function for the quantities $\{\, \sigma_\nu \,\}$:

$$F(t) := \sum_{\nu=p}^{\infty} \sigma_\nu(\varphi, \phi)\, \frac{t^{\nu-p}}{(\nu - p)!} \ . \tag{84}$$

The recurrence relation (83) turns out to be equivalent to the classical hypergeometric differential equation for the function $F(t)$

$$\left(4\,(1 - \omega^2) + 4\omega t - t^2\right) \frac{d^2F}{dt^2} + (2p + 1)\,(2\omega - t)\ \frac{dF}{dt} + (k^2 - p^2)\, F = 0 \ , \tag{85}$$

where $\omega := \cos(\varphi - \phi)$. The general solution to (85) is given in terms of Gauss' hypergeometric series:

$$F(t) = C\, {}_2\mathbf{F}_1(p-k; p+k; p+1/2 \,|\, z) + C_1\, z^{-p+1/2}\, {}_2\mathbf{F}_1(1/2-k; 1/2+k; 3/2-p \,|\, z) \ , \tag{86}$$

where $z := (t - 2\omega + 2)/4$ and ${}_2\mathbf{F}_1$ is defined by

$${}_2\mathbf{F}_1(a; b; c \,|\, z) := \sum_{\mu=0}^{\infty} \frac{(a)_\mu (b)_\mu}{(c)_\mu}\, \frac{z^\mu}{\mu!} \ . \tag{87}$$

As discussed in Sect.II, the Hadamard coefficients $U_\nu(x, \xi)$ must be regular in a neighborhood of the vertex of the characteristic cone $x = \xi$. When $x \to \xi$, we have $\omega \to 1$ and $U_p(\xi, \xi) \propto \sigma_p(\phi, \phi) = F(0)|_{\omega=1}$ is not bounded unless $C_1 = 0$.

In this way, setting $C_1 = 0$ in (86), we obtain

$$\sum_{\nu=p}^{\infty} \sigma_\nu(\varphi, \phi)\, \frac{t^{\nu-p}}{(\nu - p)!} = C\ {}_2\mathbf{F}_1\,(p - k; p + k; p + 1/2 \,|\, (t - 2\omega + 2)/4) \ . \tag{88}$$

Now it remains to recall that by our assumption (60) $k \in \mathbf{Z}$ and $k > k_N$. Since $p = (n-1)/2 = k_N + 1$, we have $k \geq p$. So the hypergeometric series in the right-hand side of equation (88) is truncated. In fact, the generating function (84) is expressed in terms of the classical Jacobi polynomial $\mathbf{P}_{k-p}^{(p-1/2,p+1/2)}(\omega - t/2)$ of degree $k-p$. Hence, $\sigma_{k+1}(\varphi, \phi) \equiv 0$, and the $(k+1)$-th Hadamard coefficient of the operator $\mathcal{L}_{N+1}$ vanishes identically:

$$U_{k+1}(x, \xi) \equiv 0 . \tag{89}$$

According to Hadamard's criterion (41), it means that the operator $\mathcal{L}_{N+1}$ satisfies Huygens' principle in Minkowski space $\mathbf{M}^{n+1}$, if n is odd and

$$n \geq 2\,k + 3 ,$$

Thus, the proof of the theorem is completed.

IV. Concluding remarks and examples

In the present paper we have constructed a new hierarchy of Huygens' operators in higher dimensional Minkowski spaces $\mathbf{M}^{n+1}, n > 3$. However, the problem of complete description of the whole class of such operators for arbitrary n still remains open. As mentioned in the Introduction, the famous Hadamard's conjecture claiming that any Huygens' operator $\mathcal{L}$ can be reduced to the ordinary d'Alembertian $\square_{n+1}$ with the help of trivial transformations is valid only in $\mathbf{M}^{3+1}$. Recently, in the work [4] one of the authors put forward the relevant modification of Hadamard's conjecture for Minkowski spaces of arbitrary dimensions. Here we recall and discuss briefly this statement.

Let Ω be an open set in Minkowski space $\mathbf{M}^{n+1} \cong \mathbf{R}^{n+1}$, and let $\mathcal{F}(\Omega)$ be a ring of partial differential operators defined over the function space $C^\infty(\Omega)$. For a fixed pair of operators $\mathcal{L}_0, \mathcal{L} \in \mathcal{F}(\Omega)$ we introduce the map

$$\mathrm{ad}_{\mathcal{L},\mathcal{L}_0} : \ \mathcal{F}(\Omega) \to \mathcal{F}(\Omega) \ , \quad A \mapsto \mathrm{ad}_{\mathcal{L},\mathcal{L}_0}[A] \ , \tag{90}$$

such that

$$\mathrm{ad}_{\mathcal{L},\mathcal{L}_0}[A] := \mathcal{L} \circ A - A \circ \mathcal{L}_0 \ . \tag{91}$$

Then, given $M \in \mathbf{Z}_{>0}$, the iterated $\mathrm{ad}_{\mathcal{L},\mathcal{L}_0}$-map is determined by

$$\mathrm{ad}^M_{\mathcal{L},\mathcal{L}_0}[A] := \mathrm{ad}_{\mathcal{L},\mathcal{L}_0}\left[\mathrm{ad}_{\mathcal{L},\mathcal{L}_0}\left[\ldots \mathrm{ad}_{\mathcal{L},\mathcal{L}_0}[A]\right]\ldots\right] = \sum_{k=0}^{M}(-1)^k \binom{M}{k} \mathcal{L}^{M-k} \circ A \circ \mathcal{L}_0^k \ . \tag{92}$$

Definition. The operator $\mathcal{L} \in \mathcal{F}(\Omega)$ is called *M-gauge related* to the operator $\mathcal{L}_0 \in \mathcal{F}(\Omega)$, if there exists a smooth function $\theta(x) \in \mathcal{C}^\infty(\Omega)$ non-vanishing in Ω, and an integer positive number $M \in \mathbf{Z}_{>0}$, such that

$$\mathrm{ad}^M_{\mathcal{L},\mathcal{L}_0}\left[\theta(x)\right] \equiv 0 \quad \text{identically in } \mathcal{F}(\Omega) \ . \tag{93}$$

In particular, when $M = 1$, the operators $\mathcal{L}$ and $\mathcal{L}_0$ are connected just by the trivial gauge transformation $\mathcal{L} = \theta(x) \circ \mathcal{L}_0 \circ \theta(x)^{-1}$.

The modified Hadamard's conjecture claims:

Any Huygens' operator $\mathcal{L}$ of the general form

$$\mathcal{L} = \Box_{n+1} + (a(x), \partial) + u(x) \ , \tag{94}$$

in a Minkowski space $\mathbf{M}^{n+1}$ (n is odd, $n \geq 3$) is M-gauge related to the ordinary wave operator $\Box_{n+1}$ in $\mathbf{M}^{n+1}$.

For Huygens' operators associated to the rational solutions of the KdV-equation (2), (3) and to Coxeter groups (5), (6) this conjecture has been proved in [4] and [7]. In these cases the required identities (93) are the following

$$\mathrm{ad}^{M_k+1}_{\mathcal{L}_k,\mathcal{L}_0}[\mathcal{P}_k(x^0)] = 0 \ , \quad M_k := \frac{k(k+1)}{2} \ , \tag{95}$$

where $\mathcal{L}_k$ is given by (2) with the potential (3) for $k = 0, 1, 2, \ldots$ and

$$\mathrm{ad}^{M_m+1}_{\mathcal{L}_m,\mathcal{L}_0}[\pi_m(x)] = 0 \ , \quad M_m := \sum_{\alpha \in \Re_+} m_\alpha \ , \tag{96}$$

where $\mathcal{L}_m$ is defined by (5), (6) and $\pi_m(x) := \prod_{\alpha\in\Re_+}(\alpha, x)^{m_\alpha}$.

It is remarkable that for the operators constructed in the present work the modified Hadamard's conjecture is also verified. More precisely, for a given wave-type operator

$$\mathcal{L}_{(k)} = \Box_{n+1} - \frac{2}{r^2}\left(\frac{\partial}{\partial\varphi}\right)^2 \log \mathcal{W}\left[\Psi_1(\varphi), \Psi_2(\varphi), \ldots, \Psi_N(\varphi)\right] \ , \qquad (97)$$

associated to a positive integer partition $(k_i): 0 \le k_1 < k_2 < \ldots < k_N$, we have the identity

$$\mathrm{ad}^{|k|+1}_{\mathcal{L}_{(k)},\mathcal{L}_0}[\Theta_{(k)}(x)] = 0 \ , \qquad (98)$$

where $\Theta_{(k)}(x) := r^{|k|}\,\mathcal{W}[\Psi_1, \Psi_2, \ldots, \Psi_N]$ and $|k| := \sum_{i=1}^N k_i$ is a weight of the partition (k_i) .

We are not going to prove (98) in the present paper. A more detailed discussion of this identity and associated algebraic structures will be the subject of our subsequent work. Here, we only mention that such type identities naturally appear [4]–[5] in connection with a classification of overcomplete commutative rings of partial differential operators [13], [14], [41], and with the bispectral problem [20].

We conclude the paper with several concrete examples illustrating our main theorem.

1. As a first example we consider the dihedral group $I_2(q)$, $q \in \mathbf{Z}_{>0}$, acting on the Euclidean plane $E \cong \mathbf{R}^2 \subset \mathbf{Gr}_\perp(n+1, 2)$ and fix the simplest partition $k = (q)$ and the phase $\varphi = \pi/2$. According to Remark II, in this case our theorem gives the wave-type operator with the Calogero-Moser potential related to the Coxeter group $I_2(q)$ with $m = 1$:

$$\mathcal{L}_{(k)} = \Box_{n+1} + \frac{2\,q^2}{r^2 \sin^2(q\,\varphi)} \ .$$

This operator satisfies Huygens' principle in $\mathbf{M}^{n+1}$ if n is odd and $n \ge 2\,q + 3$. The Hadamard coefficients of $\mathcal{L}_{(k)}$ can be presented in a simple closed form in terms of polar coordinates on E:

$$U_0 = 1 \ ,$$

$$U_\nu = \frac{1}{(2r\rho)^\nu} \frac{T_q^{(\nu)}(\cos(\varphi - \phi))}{\sin(q\,\varphi)\sin(q\,\phi)}\,, \quad \nu \geq 1\,,$$

where $T_q(z) := \cos(q \arccos(z))$, $z \in [-1,1]$, is the q-th Chebyshev polynomial, and $T_q^{(\nu)}(z)$ is its derivative of order ν with respect to z. These formulas are easily obtained with the help of recurrence relation (82).

2. Now we fix $N = 2$, $k_1 = 2$, $k_2 = 3$ and $\varphi_1 = \pi/2$, $\varphi_2 = 0$. The corresponding wave-type operator

$$\mathcal{L}_{(k)} = \Box_{n+1} + \frac{10\ (x_1^2 + x_2^2)\,(15\,x_2^2 - x_1^2)}{(5x_2^2 + x_1^2)^2\ x_1^2}\,,$$

satisfies Huygens' principle for odd $n \geq 9$. The nonzero Hadamard coefficients of this operator are given explicitly by the formulas:

$$U_0 = 1,$$

$$U_1 = \frac{40\,x_2\xi_1\xi_2x_1 + 15\,{\xi_1}^2{x_2}^2 + 75\,{\xi_2}^2{x_2}^2 + 15\,{\xi_2}^2{x_1}^2 - 5\,{\xi_1}^2{x_1}^2}{2\,\xi_1x_1\,(5\,{x_2}^2 + {x_1}^2)\left(5\,{\xi_2}^2 + {\xi_1}^2\right)},$$

$$U_2 = \frac{120\,x_2\xi_1\xi_2x_1 + 15\,{\xi_2}^2{x_1}^2 - 5\,{\xi_1}^2{x_1}^2 + 15\,{\xi_1}^2{x_2}^2 + 75\,{\xi_2}^2{x_2}^2}{4\,{\xi_1}^2{x_1}^2\,(5\,{x_2}^2 + {x_1}^2)\left(5\,{\xi_2}^2 + {\xi_1}^2\right)},$$

$$U_3 = -\frac{15\,x_2\xi_2}{{\xi_1}^2{x_1}^2\,(5\,{x_2}^2 + {x_1}^2)\left(5\,{\xi_2}^2 + {\xi_1}^2\right)}\,.$$

3. Now we take $N = 3$, the partition $k = (1,\,3,\,4)$, and the phases $\varphi_1 = \varphi_2 = \varphi_3 = \pi/2$. The corresponding operator

$$\mathcal{L}_{(k)} = \Box_{n+1} + \frac{12\ (49x_1^4 + 28x_1^2x_2^2 - x_2^4)}{x_2^2\ (7x_1^2 + x_2^2)^2},$$

is a Huygens operator in $\mathbf{M}^{n+1}$ when n is odd and $n \geq 11$. The nonzero Hadamard's coefficients are

$$U_0 = 1,$$

$$U_1 = \frac{-21\,{\xi_2}^2{x_1}^2 - 42\,x_2\xi_1\xi_2x_1 - 21\,{\xi_1}^2{x_2}^2 + 3\,{\xi_2}^2{x_2}^2 - 147\,{\xi_1}^2{x_1}^2}{\xi_2x_2\,(7\,{x_1}^2 + {x_2}^2)\left(7\,{\xi_1}^2 + {\xi_2}^2\right)},$$

$$U_2 = \frac{735\,{\xi_1}^2{x_1}^2 + 504\,x_2\xi_1\xi_2x_1 + 105\,{\xi_1}^2{x_2}^2 - 21\,{\xi_2}^2{x_2}^2 + 105\,{\xi_2}^2{x_1}^2}{4\,{\xi_2}^2{x_2}^2\,(7\,{x_1}^2 + {x_2}^2)\left(7\,{\xi_1}^2 + {\xi_2}^2\right)},$$

$$U_3 = \frac{-1260\, x_2\xi_1\xi_2 x_1 + 21\, {\xi_2}^2{x_2}^2 - 105\, {\xi_1}^2{x_2}^2 - 105\, {\xi_2}^2{x_1}^2 - 735\, {\xi_1}^2{x_1}^2}{8\, {\xi_2}^3{x_2}^3\,(7\, {x_1}^2 + {x_2}^2)\left(7\, {\xi_1}^2 + {\xi_2}^2\right)},$$

$$U_4 = \frac{315\, x_1\xi_1}{4\, {\xi_2}^3{x_2}^3\,(7\, {x_1}^2 + {x_2}^2)\left(7\, {\xi_1}^2 + {\xi_2}^2\right)}\,.$$

4. The last example illustrates Remark I following the theorem (see Introduction). In this case we consider the operator (11) with the potential (13) associated with the proper N-soliton solution of the KdV equation. We take $N = 2$ and fix $k_1 = 1$, $k_2 = 2$. The real phases are chosen as follows $\vartheta_1 = \operatorname{arctanh}(1/2)$, $\vartheta_2 = \operatorname{arctanh}(1/4)$. The corresponding operator $\mathcal{L}_{(k)}$ reads

$$\mathcal{L}_{(k)} = \Box_{n+1} + \frac{2\,(2x_0 - 3x_1)\,(3x_1^3 - 6x_0x_1^2 + 4x_1x_0^2 + 8x_0^3)}{x_1^2\,(4x_0^2 - 2x_0x_1 - x_1^2)^2}\,.$$

According to the theorem, it is huygensian provided n is odd and $n \geq 7$. The nonzero Hadamard coefficients are given by the following formulas:

$$U_0 = 1,$$

$$U_1 = \frac{4\xi_0^2x_1^2 + 9\xi_1^2x_1^2 - 16\xi_0^2x_0^2 + 8\xi_0^2x_0x_1}{2x_1\,(4x_0^2 - 2x_0x_1 - x_1^2)\,\xi_1\,(4\xi_0^2 - 2\xi_0\xi_1 - \xi_1^2)} +$$
$$+ \frac{8\xi_0\xi_1x_0^2 - 12\xi_1^2x_0x_1 + 4\xi_1^2x_0^2 - 12\xi_0\xi_1x_1^2 + 16\xi_0\xi_1x_0x_1}{2x_1\,(4x_0^2 - 2x_0x_1 - x_1^2)\,\xi_1\,(4\xi_0^2 - 2\xi_0\xi_1 - \xi_1^2)}\,,$$

$$U_2 = -\frac{5\,(2\,\xi_0 - \xi_1)\,(2\,x_0 - x_1)}{4\,x_1\,(4\,{x_0}^2 - 2\,x_0x_1 - {x_1}^2)\,\xi_1\left(4\,{\xi_0}^2 - 2\,\xi_0\xi_1 - {\xi_1}^2\right)}\,.$$

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Chapter 2

Ising models, random matrices and solitons

Self similar potentials and Ising models

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30 October 1997

Abstract

A new link between soliton solutions of integrable nonlinear equations and one-dimensional Ising models is established. Translational invariance of the spin lattice associated with the KdV equation is related to self-similar potentials of the Schrödinger equation. This gives antiferromagnets with exponentially decaying interaction between the spins. Partition function is calculated exactly for a homogeneous magnetic field and two discrete values of the temperature.

PACS 03.65.-w, 05.50.+q

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The one-dimensional Schrödinger equation

$$L\psi(x) \equiv -\psi_{xx}(x) + u(x)\psi(x) = \lambda\psi(x) \tag{1}$$

lies in the foundations of quantum mechanics and theory of solitons. The class of potentials $u(x)$, for which the spectrum and eigenfunctions of the operator L are known in the closed form, is of a particular interest. It includes simple potentials tied to the Gauss hypergeometric function (for a review, see [1]), finite-gap potentials, and the potentials whose discrete spectra consist of a number of arithmetic or geometric progressions (see [2, 3] and references therein). The latter potentials appear after a self-similar reduction of the factorization chain or the chain of Darboux transformations. In this note we discuss relation of the self-similar potentials to one-dimensional Ising type spin chain models. Below we use the language of the soliton theory described, e.g., in [4, 5].

It is well known that if the potential $u(x,t)$ and the wave function $\psi(x,t)$ in (1) depend on 'time' t in such a way that

$$\psi_t(x,t) = B\psi(x,t), \qquad B \equiv -4\partial_x^3 + 6u(x,t)\partial_x + 3u_x(x,t), \tag{2}$$

then the compatibility condition of (1) and (2), $L_t = [B, L]$, is equivalent to the Korteweg – de Vries (KdV) equation $u_t + u_{xxx} - 6uu_x = 0$. The N-soliton solution of this equation can be represented in the form $u(x,t) = -2\partial_x^2 \ln \tau_N(x,t)$, where $\tau_N = \det C$ is the determinant of the matrix

$$C_{ij} = \delta_{ij} + \frac{2\sqrt{k_i k_j}}{k_i + k_j} e^{(\theta_i + \theta_j)/2}, \qquad \theta_i = k_i x - k_i^3 t + \theta_i^{(0)}. \tag{3}$$

Here k_i are the amplitudes of solitons related to the bound state energies of (1), $\lambda_i = -k_i^2/4$, and $\theta_i^{(0)}/k_i$ are the zero time phases. The ordering $0 < k_N < \ldots < k_1$ is assumed. Equivalently, this τ-function can be rewritten in the form [4, 5]:

$$\tau_N = \sum_{\mu_i = 0,1} \exp\Big(\sum_{1 \le i < j \le N} A_{ij}\mu_i\mu_j + \sum_{1 \le i \le N} \theta_i \mu_i \Big), \tag{4}$$

where the phase shifts A_{ij} are determined by the formula

$$e^{A_{ij}} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}. \tag{5}$$

There are generalizations of the expressions (3)-(5) such that the corresponding $u(x,t,\ldots)$ satisfy higher order members of the KdV-hierarchy, sin-Gordon, Kadomtsev – Petviashvili (KP), Toda, and some other integrable equations [4].

We start from the observation that the expression (4) has nice interpretation within the statistical mechanics. Namely, for $\theta_i = \theta^{(0)} = \text{const}$ it defines the grand partition function of the lattice gas model [6]. In this case μ_i play the role of filling factors of the lattice sites by repulsing molecules, $\theta^{(0)}$ is proportional to the chemical potential, and A_{ij} are proportional to the interaction energy between the i-th and j-th molecules.

Simultaneously, (4) is closely related to the partition function of the one-dimensional Ising model [6]:

$$Z_N = \sum_{\sigma_i=\pm 1} e^{-\beta E}, \qquad E = \sum_{i<j} J_{ij}\sigma_i\sigma_j - \sum_{1\le i\le N} H_i\sigma_i, \tag{6}$$

where N is the number of spins $\sigma_i = \pm 1$, $J_{ij} = J_{ji}$ is the coupling between i-th and j-th spins, H_i is the external magnetic field, $\beta = 1/kT$ is the inverse temperature. Indeed, let us introduce into (4) the spin variables via the substitution $\mu_i = (\sigma_i + 1)/2$. After some simple calculations one finds

$$\tau_N = e^{\Phi} Z_N, \qquad \Phi = \tfrac{1}{4}\sum_{i<j} A_{ij} + \tfrac{1}{2}\sum_{1\le j\le N}\theta_j, \tag{7}$$

provided

$$A_{ij} = -4\beta J_{ij}, \qquad \theta_i = 2\beta(H_i + \sum_{1\le j\ne i\le N} J_{ij}). \tag{8}$$

As a result, one arrives at an interesting fact: from a given N-soliton τ-function of the KdV equation (4), one recovers the partition function of the N-spin Ising model (7). The τ-function is defined only up to a gauge factor $\exp(ax+b)$, and (7) fits this freedom. Therefore one may identify the N-soliton τ-function itself with (6) for the specific exchange interaction (5). This fact alone does not help much in the evaluation of Z_N. However, the recursive way of building N-soliton potentials with the help of Darboux transformations or the factorization method appears to be quite useful. Let us provide the representation of Z_N following from the Wronskian form of τ_N [7, 8]

$$Z_N = \frac{2^{N(N+1)/2}\, W_N}{\prod_{i<j}(k_i^2 - k_j^2)^{1/2}}, \qquad W_N = \det\left(\frac{d^{i-1}\Psi_j}{dx^{i-1}}\right), \tag{9}$$

where $\Psi_{2j-1} = \operatorname{ch}\beta H_{N-2j+2}$, $\Psi_{2j} = \operatorname{sh}\beta H_{N-2j+1}$. Dependence of H_j on the soliton parameters is read from (8).

The factorization method transforms a given potential $u_j(x) = f_j^2(x) - f_{jx}(x) + \lambda_j$ with some discrete spectrum to the potential $u_{j+1}(x) = u_j(x) + 2f_{jx}(x)$ containing an additional (the lowest) bound state with the prescribed energy λ_j. Within the Ising models context, this corresponds to the extension of the lattice by one more site. Then the infinite-soliton potentials correspond to the thermodynamic limit $N \to \infty$. Characterization of general τ_N at $N \to \infty$ is a challenging problem, but for the specific choice of parameters $k_i, \theta_i^{(0)}$ this function can be analyzed to some extent through the basic infinite chain of equations [1]

$$(f_j(x) + f_{j+1}(x))_x + f_j^2(x) - f_{j+1}^2(x) = \rho_j \equiv \lambda_{j+1} - \lambda_j, \quad j \in \mathbf{Z}. \tag{10}$$

In general both τ_N and Z_N diverge in the limit $N \to \infty$. If the corresponding solutions of (10) are finite, then the divergences gather into the gauge factor.

A key observation of the present work is that the simplest physical constraints imposed upon the form of spin interactions J_{ij} of the infinite Ising chain select the potentials with the discrete spectrum composed from a number of geometric progressions. First, let us demand that all the spins are situated on equal distance from each other and that they are identical, i.e. that there is a translational invariance, $A_{i+1,j+1} = A_{ij}$. This means that the intensities of interaction A_{ij} depend only on the distance between the sites $|i-j|$, $A_{ij} = A(|i-j|)$. Such a natural constraint has the *unique* solution

$$k_i = k_1 q^{i-1}, \qquad q = e^{-2\alpha}, \qquad A_{ij} = 2\ln|\operatorname{th}\alpha(i-j)|, \tag{11}$$

where $\alpha > 0$ is an arbitrary constant. For finite N this spectrum corresponds to reflectionless potentials with the eigenvalues condensing near $\lambda = 0$. For $q > 1$, one should write $k_i = k_1 q^{-i+1}$ for correct ordering of k_i. (The exponentially growing spectrum is formally obtained for purely imaginary k_1 and $q > 1$, but the corresponding potential contains singularities.) In the $N \to \infty$ limit, one gets an infinite soliton potential with the discrete spectrum $\lambda_j = -k_1^2 q^{2(j-1)}/4$ describing a specific semi-infinite spin chain (j takes only positive values). As $q^j \to 0$ for $j \to \infty$, the magnetic field is decaying exponentially from the edge of the lattice. The limit $x \to \infty$ corresponds to the growing depth of penetration of the magnetic field inside the bulk. Note that one can analyze boundary effects by working with a difference of the free energy at two fixed values of the magnetic field.

Since $0 < |\operatorname{th} \alpha(i-j)| < 1$, one has $J_{ij} > 0$, i.e. an antiferromagnetic interaction (the spins are not aligned in the ground state). It has nice physical characteristics — its intensity falls exponentially fast with the distance between the sites. It is well known that the one-dimensional systems with finite range interactions do not have phase transitions at non-zero temperature. There is a model with the exponential interaction $J_{ij} = -\gamma |J_0| e^{-\gamma|i-j|}$ solved in the limit $\gamma \to 0$ by M. Kac [9]. This limit corresponds to the very weak but long-range interaction and shows a phase transition with the Van der Waals equation of state.

There should be some relation of our model to the Kac one, but it is not clear whether there exists a direct connection. A similar molecular approximation limit is reached in our case if $\alpha \to 0$. Formally $A_{ij} \propto J_{ij}/kT$ diverge in this limit. If we renormalize interaction constants $J_{ij}^{ren} = J_{ij}(q^{-1} - q)$ and the temperature $T_{ren} = T(q^{-1} - q)$, then the maximal interaction energy of a single i-th spin (determined by the summation of J_{ij}^{ren} over j) will be finite for $\alpha \to 0$ (or $q \to 1$). Therefore the limit $q \to 1$ corresponds to the *long range interaction* model *at low temperature.* Note that one should rescale simultaneously the magnetic field $H = h/(q^{-1} - q)$ to imitate the change of the temperature.

The particular form of the renormalization factor $q^{-1} - q$ was chosen in order in the limit $q \to 0$ to recover the interaction $J_{ij}^{ren} \propto \delta_{i+1,j}$. If one takes h as a real magnetic field then one gets the nearest neighbor interaction Ising model at high temperature. If the magnetic field is not rescaled then the $q \to 0$ limit corresponds to the completely non-interacting spins. Thus our formalism allows to analyze partition function upon two dimensional planes in the space of variables (T, H, q). Unfortunately, for fixed q the temperature is fixed as well and we may normalize the "KdV temperature" to $kT = 1$.

The discrete spectrum does not characterize completely even the reflectionless potentials — one has to fix the phases θ_i. Only for the special choice of these parameters one arrives at the self-similar potentials. E.g., the simplest case is determined by the condition that the scaling of x and t by q and q^3 respectively is equivalent to removing of one soliton. Formally this corresponds to the constraint $\theta_i(qx, q^3t) = \theta_{i+1}(x, t)$ assuming the choice $\theta_i^{(0)} = \theta^{(0)} =$const. However, τ_N, Z_N and Φ in (7) are diverging for $N \to \infty$ and a more careful analysis is thus called for. Note that the shift of H_i in (8) remains finite and it becomes a fixed constant for $i \to \infty$. This means that in the thermodynamic limit the zero chemical potential in the lattice gas partition function corresponds to a fixed nonzero magnetic field in the Ising model, and, vice versa, zero magnetic field matches with a prescribed value of the chemical potential.

Let us consider now the "M-color" Ising model for which the chain is formed by the embedded sublattices when the blocks of M spins with different distances between them are periodically repeated. Within each of this block the distances between spins are not equal, so that the interaction constants between the first M sites are given by arbitrary (random) numbers. Equivalently, one may think that in the equal distance lattice points one has different magnetic moment particles, i.e. some kind of ferrimagnetic interaction. Such physical constraints are bound to the condition $A_{i+M,j+M} = A_{ij}$, which leads to the constraint upon the soliton energies of the form $k_{j+M} = qk_j$, generalizing the previous case. For a specific choice of the phases $\theta_{i+M}^{(0)} = \theta_i^{(0)}$ one arrives at the general self-similar potentials for which one has $\theta_i(qx, q^3t) = \theta_{i+M}(x, t)$. The rigorous definition of these potentials for fixed time is given by the constraints [2]

$$f_{j+M}(x) = q f_j(qx), \qquad \rho_{j+M} = q^2 \rho_j \tag{12}$$

imposed upon the chain (10). The system of mixed differential and q-difference equations arising after this reduction describes q-deformations of the Painlevé transcendents and their higher order analogs. For $M = 1$ one has a q-harmonic oscillator model, for $M = 2$ a system with the $su_q(1,1)$ symmetry algebra, etc.

Using the Wronskian representation (9), we calculated exactly the free energy per site f_I in the thermodynamic limit $Z_N \to e^{-\beta N f_I}$, $N \to \infty$, for a homogeneous magnetic field and arbitrary M. In the simplest $M = 1$ case one has

$$-\beta f_I(H) = \ln \frac{2(q^4;q^4)_\infty \operatorname{ch} \beta H}{(q^2;q^2)_\infty^{1/2}} + \frac{1}{2\pi} \int_0^\pi d\nu \ln \left(|\rho(\nu)|^2 - q \operatorname{th}^2 \beta H \right), \tag{13}$$

where $(a;q)_\infty = \prod_{j=0}^\infty (1 - aq^j)$ and

$$|\rho(\nu)|^2 = \frac{(q^2 e^{2i\nu};q^4)_\infty^2 (q^2 e^{-2i\nu};q^4)_\infty^2}{4 \sin^2 \nu \; (q^4 e^{2i\nu};q^4)_\infty^2 (q^4 e^{-2i\nu};q^4)_\infty^2} = q \frac{\theta_4^2(\nu,q^2)}{\theta_1^2(\nu,q^2)}. \tag{14}$$

The Jacobi θ-functions are defined in the standard way [10]. The density function $\rho(\nu)$ has integrable singularities near the $\nu = 0, \pi$ points. Note that it satisfies a curious identity such that the second term in (13) vanishes for $H = 0$.

Dependence of the magnetization $m(H) = -\partial_H f_I(H)$ on H looks as follows

$$m(H) = \left(1 - \frac{1}{\pi} \int_0^\pi \frac{\theta_1^2(\nu,q^2) d\nu}{\theta_4^2(\nu,q^2) \operatorname{ch}^2 \beta H - \theta_1^2(\nu,q^2) \operatorname{sh}^2 \beta H} \right) \operatorname{th} \beta H. \tag{15}$$

We substitute into this expression $\beta H = h/(q^{-1} - q)$ and plot $m(h)$ in Figure by the dashed lines for $q = 0.1$ (the lower curve) and $q = 0.8$. We would like to note that it is not clear how to solve the considered Ising model with the help of the traditional Bethe anzats and transfer matrix methods [6].

As was mentioned, a drawback of the given construction is that the KdV-generated partition function has a fixed temperature for fixed α. In order to obtain the full thermodynamical description it is necessary to extend the formalism and replace A_{ij} (11) at least by nA_{ij}, where n is a positive integer. The KdV temperature is thus normalized to $\beta = n = 1$ (for $n > 1$ one has to renormalize the magnetic field $H_i \to nH_i$ in order to imitate the effect of the temperature lowering). This means that we need to look for an integrable model with the phase shifts given by the powers of (5). Then one may hope to recover the partition function with arbitrary values of the temperature $\propto 1/n$ by an analytic continuation.

The phase shifts A_{ij} for a given Hirota polynomial $P(x_1, x_3, \ldots)$, determining a particular integrable evolution equation, can be represented in the form [5]

$$e^{A_{ij}} = -\frac{P(k_1 - k_2, k_2^3 - k_1^3, \ldots, (-1)^\ell(k_1^{2\ell+1} - k_2^{2\ell+1}))}{P(k_1 + k_2, -k_2^3 - k_1^3, \ldots, (-1)^\ell(k_1^{2\ell+1} + k_2^{2\ell+1}))}, \tag{16}$$

where ℓ is the number of variables in P. We have looked for integrable systems with the prescribed phase shifts, substituting homogeneous (with the account of weights of the variables) polynomials with undefined coefficients into (16). Then, we applied an additional test to select the polynomials admitting N-soliton solutions. It turns out that the taken conditions are very restrictive, and the only solution we were able to find is the hierarchy which starts from

$$P(x_1, x_3, x_5) = 16x_1^6 + 20x_1^3x_3 + 9x_1x_5 - 5x_3^2 \tag{17}$$

corresponding to $n = 2$, i.e. to the temperature $kT = 1/2$ normalized to the KdV case. The polynomial (17) corresponds to the B-type KP (BKP) equation [11]: its original form is $x_1^6 - 5x_1^3x_3 - 5x_3^2 + 9x_1x_5$ which is obtained from (17) up to a common numerical factor 16 after the substitutions $x_1 \to x_1, x_3 \to -4x_3, x_5 \to 16x_5$.

Using the Pfaffian representation of the N-soliton solutions of the BKP equation [11], we calculated exactly the partition function in the thermodynamical limit $N \to \infty$ for a homogeneous magnetic field and arbitrary M. For $M = 1$ one has

$$-\beta f_I(H) = \frac{1}{2\pi}\int_0^\pi d\nu \ln 2 \left| \frac{(q;q)_\infty^2}{(-q;q)_\infty^2} \operatorname{ch} 4\beta H + \frac{\partial_\nu \theta_1(\nu, q^{1/2})}{\theta_2(\nu, q^{1/2})} \right|, \tag{18}$$

where ∂_ν means the derivative with respect to the variable ν and $\theta_2(\nu, q^{1/2})$ is another Jacobi θ-function [10]. The dependence of magnetization on H is

$$m(H) = \left(1 - \frac{1}{\pi}\int_0^\pi d\nu \left(1 + \frac{(q;q)_\infty^2 \theta_2(\nu, q^{1/2}) \operatorname{ch} 4\beta H}{(-q;q)_\infty^2 \partial_\nu \theta_1(\nu, q^{1/2})}\right)^{-1}\right) \operatorname{th} 4\beta H. \qquad (19)$$

For $q \to 0$ one gets the simple answer $m(H) = \operatorname{th} 2\beta H$.

We substitute into (19) $\beta H = h/(q^{-1} - q)$ and plot $m(h)$ in the Figure by the solid lines for $q = 0.1$ (the lower curve) and $q = 0.8$. From the comparison of the magnetization curves one can see that with the lowering of temperature, which corresponds both to the transition from $n = 1$ to $n = 2$ and to the growing of q, $m(h)$ becomes more steep. This may be interpreted as a trend towards formation of a staircase-like fractal function that should take place at zero temperature according to the arguments of [12]. Formation of the platos for $m(h)$ at low temperatures can be easily checked analytically for the nearest neighbor interaction Ising antiferromagnet.

The attempts to find integrable systems with $n > 2$ have failed for Hirota polynomials of up to 20-th degree. Probably one has to pass from the scalar Lax pairs to the matrix ones in order to imitate other values of the discrete temperature. The lattice of temperatures itself resembles a discrete variable unifying different hierarchies of integrable systems into one class.

A relation between the two-dimensional nearest neighbor interaction Ising model and the sinh-Gordon hierarchy was discussed in [13]. In particular, the corresponding N-soliton solution τ-function, $N \to \infty$, was shown to be the generating function of correlation functions. It should be noted that our identification of the one-dimensional Ising model partition function with τ-functions of some integrable equations is different from the constructions considered in [13] and earlier related works. However, it is expected that the self-similar potentials (or q-analogs of the Painlevé transcendents) are related to some correlation functions in the corresponding setting as well. A hint on this comes from the fact that the supersymmetric quantum mechanical representation of the factorization method is related to the Lax pair of the sinh-Gordon equation.

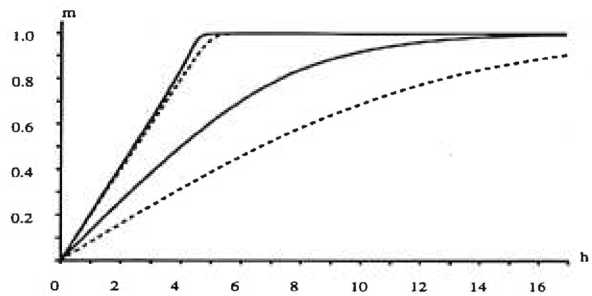

Figure 1: Dependence of the magnetization $m(h)$ for the renormalized magnetic field $\beta H = h/(q^{-1} - q)$ for the KdV case $n = 1$ (dashed lines) and for the BKP case $n = 2$ (solid lines). The lower curves correspond to $q = 0.1$ and the upper ones to $q = 0.8$.

The authors are indebted to Yu. Berest, V. Inozemtsev, T. Shiota, C. Tracy for stimulating discussions. This work is supported in part by NSERC (Canada), RFBR (Russia) grant 97-01-01041 and by INTAS 96-700.

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Spectral self-similarity, one-dimensional Ising chains and random matrices

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Abstract

Partition functions of the one-dimensional Ising chains with specific long distance interaction between N spins are connected to N-soliton tau-functions of the Korteweg-de Vries (KdV) and B-type Kadomtsev-Petviashvili (BKP) integrable equations. The condition of translational invariance of the spin lattice selects infinite soliton solutions with soliton amplitudes forming a number of geometric series. The KdV equation generates a spin chain with the exponentially decaying antiferromagnetic interaction. The BKP case is more rich. It comprises both ferromagnets and antiferromagnets and, as a limiting case, includes an asymptotic interaction $\propto 1/(i-j)^2$. The corresponding partition functions are calculated exactly for a homogeneous magnetic field and some fixed values of the temperature. A connection between these Ising chains and random matrix models is considered as well.

1 Introduction

Ising chains are very popular in statistical mechanics [1]. They show a number of interesting phenomena detected in many cases on the basis of simple analytical expressions for partition function and various thermodynamic quantities. It is well known that the 2D Ising model with the nearest-neighbor interaction between spins is solvable for zero magnetic field and there is a phase transition at non-zero temperature. Its 1D partner is solvable for arbitrary homogeneous magnetic field. However, 1D models with fast decreasing interactions admit as a critical point only the zero temperature. Therefore such models are primarily interesting from the exact solvability viewpoint. Any exact derivation of the partition function can be useful for other calculations as well. It is worth to mention that there are also one-dimensional long-range interaction models with nontrivial phase transitions which are interesting on their own [2]. As some particular examples of spin chains with the non-nearest neighbor exchange which are solvable, we mention the Haldane-Shastry model [3, 4] and the Inozemtsev model [5].

Recently we have described [6] an interesting relation between the particular set of 1D Ising chains and integrable nonlinear partial differential equations. Namely, it was shown that self-similar infinite soliton solutions of the Korteweg-de Vries (KdV) equation describe an antiferromagnetic spin chain in a magnetic field. The corresponding soliton solutions are generated by the Schrödinger equation potentials whose discrete spectra are composed from a finite number of geometric series [7]. A characteristic property of the arising Ising chains is that the interaction is decaying exponentially fast with the distance between spins. The partition function was calculated exactly for a homogeneous magnetic field and two fixed values of the temperature. Quantization of the temperature, induced by the condition of solvability of the model, is a curious point of this scheme. In a sense, this is a price one has to pay in order to have solvable model with complicated form of the exchange interaction.

In this paper we give a detailed account of the results of [6] and extend the approach to more general integrable equations and Ising chains. We discuss also a relation of the emerging Ising chains to the random matrix theory. The paper is organized as follows. In the next section, we present the details of calculations for the simplest translationally invariant Ising chain generated by the KdV equation. In section 3, the chains with translational invariance with respect to the shifts by arbitrary number of sites M are discussed. The corresponding results allow us to estimate magnetizations of various spin sublattices for the homogeneous magnetic field. In section 4, a more complicated set of Ising chains based upon the B-type Kadomtsev-Petviashvili (BKP) equation is analyzed. In this case, not only the antiferromagnetic but the ferromagnetic exchange is permitted as well. In some special limit one gets an exchange which becomes rational (of the Calogero-type) at large distances $\propto 1/(i-j)^2$. In section 5 we consider the possibilities of changing the discrete temperature with the help of particular reductions of M-periodic systems to the $M = 1$ case. In section 6 we describe a connection of our Ising chains to the random matrix models. In particular, we show that the old Gaudin model [8] is related to the tau-function of the KP equation. The last section contains a brief discussion of open problems and possibilities for further extension of the derived results. Throughout the paper we use the language of the soliton theory, some basic notions to be employed can be found, e.g., in [9, 10, 11].

2 Integrable equations and 1D Ising chains

The most popular eigenvalue problem in physics is determined by the 1D Schrödinger equation

$$L\psi(x) \equiv -\psi_{xx}(x) + u(x)\psi(x) = \lambda\psi(x), \tag{1}$$

where the subscript x denotes the derivative with respect to the space coordinate x. Physical system is determined completely after providing some boundary conditions fixing the Hilbert space structures. It is convenient to assume that $x \in R$ and $\psi(x) \in L^2(R)$. In general, one can consider (1) without any boundary conditions and take x, λ as complex variables $x, \lambda \in C$.

The Schrödinger operators L, whose discrete spectra are parametrized in terms of simple, say, elementary functions, are of particular interest. The corresponding eigenfunctions $\psi(x)$ may be quite complicated, especially in the presence of the continuous spectrum. Sometimes they are expressed as quadratures of the functions entering the potential $u(x)$, but more often they define some transcendental special functions. In the latter case, it is supposed that the key properties of these functions, like the asymptotics, are derivable (at least in principle) by known means. This class of "exactly solvable" potentials is wide enough and, actually, does not have sharp bounds. It includes elementary functions potentials related in a simple way to the Gauss hypergeometric function ${}_2F_1$ which were considered at the early stages of development of quantum mechanics. Development of the theory of solitons and integrable systems brought to light new classes of such potentials. The most complete theory is built for the finite-gap potentials associated with the finite-genus Riemann surfaces formed by the spectral parameter. As other examples we mention the potentials whose discrete spectra consist of a number of arithmetic or geometric progressions. The latter cases appear from q-periodic (self-similar) reductions of the chain of Darboux transformations. A partial review of the properties of these systems and relevant literature can be found in [7, 12]. One of their characteristic features is that quantum algebras play the role of spectrum generating algebras for the corresponding Schrödinger operators. In the present paper we exploit a relation of these self-simliar potentials to the infinite soliton systems and specific 1D Ising spin chains.

The connection between the Schrödinger equation and the KdV equation is well known. Let the potential $u = u(x,t)$ and the wave functions $\psi = \psi(x,t)$ in (1) depend on some continuous parameter t and let the evolution in t be determined by the equation

$$\psi_t(x,t) = B\psi(x,t), \qquad B \equiv -4\partial_x^3 + 6u(x,t)\partial_x + 3u_x(x,t), \tag{2}$$

where the subscript t denotes the partial derivative with respect to t. The compatibility condition of (1) and (2) has the following operator form

$$L_t = [B, L].$$

For the particular choice of the operators L and B we have taken, it yields the KdV equation

$$u_t + u_{xxx} - 6uu_x = 0.$$

The single traveling wave (soliton) solution of this equation was known for a long time. The inverse scattering method has revealed interesting properties of the general N-soliton solution which can be represented in the form

$$u(x,t) = -2\partial_x^2 \ln \tau_N(x,t), \tag{3}$$

where τ_N is the determinant of a $N \times N$ matrix C,

$$\tau_N = \det C, \qquad C_{ij} = \delta_{ij} + \frac{2\sqrt{k_i k_j}}{k_i + k_j} e^{(\theta_i + \theta_j)/2}, \tag{4}$$

$$\theta_i = k_i x - k_i^3 t + \theta_i^{(0)}, \qquad i, j = 1, 2, \ldots, N.$$

This expression contains a number of free variables $k_i, \theta_i^{(0)}$ parametrizing properties of individual solitons. So, k_i describes the amplitude of i-th soliton. It is related to i-th bound state energy of (1) in a simple way, $\lambda_i = -k_i^2/4$. From the scattering kinematics it is seen that $\theta_i^{(0)}/k_i$ are the zero time phases of solitons and k_i^2 are their velocities. It is convenient to fix the ordering $0 < k_N < \ldots < k_1$ for later technical considerations.

The potentials obtained after a number of Darboux transformations or their discrete analogs have the determinant representations [14, 15, 16, 17, 18]. The first two references concern the three term recurrence relation for orthogonal polynomials which can be considered as a finite-difference analog of (1). The potentials obtained in this way comprise reflectionless potentials with N bound states determining N-soliton systems.

The following representation of τ_N (4) was widely discussed in the literature (see, e.g., [9, 10, 11] and references therein):

$$\tau_N = \sum_{\mu_i=0,1} \exp\left(\sum_{1\le i<j\le N} A_{ij}\mu_i\mu_j + \sum_{i=1}^{N} \theta_i\mu_i \right), \tag{5}$$

where the phase shifts A_{ij} are expressed via k_i in the following way

$$e^{A_{ij}} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}. \tag{6}$$

As observed in [6] the expression (5) defines the grand partition function of the lattice gas model for $\theta_i = \theta^{(0)} = const.$ Indeed, a simple comparison with [1] shows that μ_i are the filling factors of the lattice sites by molecules and $\theta^{(0)}$ is proportional to the chemical potential. The phase shifts A_{ij} are interpreted as the constants proportional to the interaction energies of i-th and j-th molecules.

It is not difficult to see that τ_N (5) is related to the partition function of a 1D Ising chain:

$$Z_N = \sum_{\sigma_i=\pm 1} e^{-\beta E}, \qquad \beta = \frac{1}{kT}, \tag{7}$$

$$E = \sum_{1\le i<j\le N} J_{ij}\sigma_i\sigma_j - \sum_{i=1}^{N} H_i\sigma_i,$$

where N is the number of spins $\sigma_i = \pm 1$ and J_{ij} are the exchange constants between i-th and j-th spins. The variables H_i describe an external magnetic field. The notations for the temperature T, the Boltzmann's constant k and the inverse temperature β are standard. In order to see this connection one can introduce into (5) spin variables via the substitution $\mu_i = (\sigma_i + 1)/2$. Simple calculations yield

$$\tau_N = e^{\Phi} Z_N, \qquad \Phi = \frac{1}{4}\sum_{i<j} A_{ij} + \frac{1}{2}\sum_{j=1}^{N} \theta_j, \tag{8}$$

with the following identifications

$$\beta J_{ij} = -\frac{1}{4}\, A_{ij}, \qquad \beta H_i = \frac{1}{2}\theta_i + \frac{1}{4} \sum_{j=1, i\neq j}^{N} A_{ij}. \tag{9}$$

So, the N-soliton τ-function of the KdV equation (5) determines partition function of a 1D N-spin Ising chain (8). Since the τ-function is defined only up to a multiplicative factor $\exp(ax+b)$, one may identify τ_N with the partition function itself (for fixed exchange constants), but we shall not do it.

A similar relation with Ising chains is valid for the whole Kadomtsev-Petviashvili (KP) hierarchy and many other differential and difference nonlinear integrable equations. For passing to the corresponding tau-functions it is necessary to change simply the phase shifts A_{ij} and the phases θ_i [9, 10]. In particular, for the KP hierarchy one has

$$A_{ij} = \ln \frac{(a_i - a_j)(b_i - b_j)}{(a_i + b_j)(b_i + a_j)}, \tag{10}$$

$$\theta_i = \theta_i^{(0)} + \sum_{n=1}^{\infty} (a_i^n - (-b_i)^n) x_n,$$

where a_i, b_i are the soliton parameters and x_n are the hierarchy "times" with $x_1 = x, x_2 = y, x_3 = -4t$ being the standard (2+1)-coordinates of the KP equation. The choice $a_i = b_i = k_i/2$ reduces KP to the KdV equation. For the B-type KP (BKP) equation the phase shifts and soliton phases are (see, e.g., [13])

$$A_{ij} = \ln \frac{(a_i - a_j)(b_i - b_j)(a_i - b_j)(b_i - a_j)}{(a_i + a_j)(b_i + b_j)(a_i + b_j)(b_i + a_j)}, \tag{11}$$

$$\theta_i = \theta_i^{(0)} + \sum_{n=1}^{\infty} \left(a_i^{2n-1} + b_i^{2n-1}\right) x_{2n-1},$$

i.e. the even "times" are absent. We limit our consideration to the KdV and BKP equations only.

3 The simplest Ising chain induced by the KdV equation

Whether the correspondence established in the previous section helps in the evaluation of the partition functions Z_N? The answer is positive and in [6] we have presented the results of the calculation of Z_N at $N \to \infty$ for the KdV and BKP

N-soliton solutions with the specific choice of the soliton parameters. In the KdV case we used the Darboux transformations techniques for the Schrödinger equation providing the following Wronskian representation of Z_N [16]:

$$Z_N = \frac{2^{N(N+1)/2}\, W_N}{\prod_{i<j}(k_i^2 - k_j^2)^{1/2}}, \qquad W_N = \det\left(\frac{d^{i-1}\Psi_j}{dx^{i-1}}\right), \tag{12}$$

where

$$\Psi_{2j-1} = \cosh \beta H_{N-2j+2}, \qquad \Psi_{2j} = \sinh \beta H_{N-2j+1}.$$

Soliton parameters enter the definition of the magnetic field H_j as it is prescribed in (9).

Before going into the technical details of calculations let us mention that the Darboux transformations allow one to make simple spectral surgeries upon a given potential — to remove or to add one bound state, or to build another potential with the same spectrum. In the context of the theory of solitons this corresponds to the addition or removal of a soliton. Within the context of the Ising chains, this results in the extension or shortening of the lattice by one site. The thermodynamic limit $N \to \infty$ is of the prime interest in statistical mechanics. In order to find it one has to be able to work with the infinite-soliton potentials, but the description of general τ_N at $N \to \infty$ is a hard problem. Below we are calculating only the diverging factor of τ_∞ for a special choice of k_i and $\theta_i^{(0)}$ which is sufficient for the determination of the free energy per site of a large class of Ising chains.

In this approach, one represents first a given potential $u_j(x)$ in the form

$$u_j(x) = f_j^2(x) - f_{jx}(x) + \lambda_j,$$

where λ_j is a free parameter. Then, under a particular choice of λ_j and $f_j(x)$, the Schrödinger operator with the potential

$$u_{j+1}(x) = u_j(x) + 2f_{jx}(x) = f_j^2(x) + f_{jx}(x) + \lambda_j$$

contains an additional bound state with the eigenvalue λ_j as the lowest discrete spectrum point. The problem of searching for solvable potentials is thus equivalent to the search of the simplest possible solutions of the infinite chain of equations

$$(f_j(x) + f_{j+1}(x))_x + f_j^2(x) - f_{j+1}^2(x) = \rho_j \equiv \lambda_{j+1} - \lambda_j, \quad j \in \mathbf{Z}, \tag{13}$$

where both functions $f_j(x)$ and λ_j are considered as unknown objects. Evidently this is an underdetermined problem and there is too much freedom. The constraints imposed upon the solutions of (13), namely, the requirement of the presence of additional symmetries provides a guide in the solution of this problem. A wide class of potentials pretending to the title "exactly solvable" is defined by the following self-similarity reduction [7]

$$f_{j+M}(x) = q^M f_j(q^M x), \qquad \rho_{j+M} = q^{2M} \rho_j \tag{14}$$

imposed upon (13). This anzats is based upon two symmetries of (13) mapping the space of its solutions onto itself — the discrete translation $j \to j + N$ and the scaling $x \to qx, f_j(x) \to q f_j(qx), \lambda_j \to q^2 \lambda_j$. As a result of (14), one arrives at a system of N coupled nonlinear differential-q-difference equations whose solutions include infinite soliton potentials with discrete spectra composed from a number of geometric series. It is convenient to work with the factor q^M in (14), then the $M = 1$ system satisfies the anzats for arbitrary M without renormalization of q.

Algebraically, symmetries of these potentials are described by some q-deformed algebras, which for $M = 1$ correspond to a q-harmonic oscillator, for $M = 2$ one gets a system with the $su_q(1,1)$ symmetry algebra, etc. Analytically, one arrives at complicated transcendents which include q-analogs of some of the Painlevé functions. Let q be a primitive root of unity, $q^K = 1$. Then for odd K and some cases of even K one finds the finite-gap potentials with additional (quasi-) crystallographic symmetries [12].

Consider now the partition function Z_N for arbitrary k_i and homogeneous magnetic field $H_i = H = const.$ The latter condition corresponds to the rather trivial limit of the KdV solutions when all KdV hierarchy "times", including x and t, are equal to zero and the constants $\theta_i^{(0)}$ take some prescribed values.

It is convenient to work with even number of solitons $N = 2p$. Then the Wronskian W_N looks as follows

$$W_{2p} = \frac{1}{2^{p(2p-1)}} \begin{vmatrix} \cosh\beta H & \sinh\beta H & \dots & \cosh\beta H & \sinh\beta H \\ k_{2p}\sinh\beta H & k_{2p-1}\cosh\beta H & \dots & k_2\sinh\beta H & k_1\cosh\beta H \\ k_{2p}^2\cosh\beta H & k_{2p-1}^2\sinh\beta H & \dots & k_2^2\cosh\beta H & k_1^2\sinh\beta H \\ \vdots & & & & \\ k_{2p}^{2p-1}\sinh\beta H & k_{2p-1}^{2p-1}\cosh\beta H & \dots & k_2^{2p-1}\sinh\beta H & k_1^{2p-1}\cosh\beta H \end{vmatrix}.$$

After a simple permutation of rows and columns this determinant can be brought to the form

$$W_N = (-1)^p \frac{\cosh^{2p}\beta H}{2^{p(2p-1)}} \begin{vmatrix} az & b \\ c & dz \end{vmatrix}, \tag{15}$$

where $z = \tanh\beta H$ and

$$a = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ k_1^2 & k_3^2 & \dots & k_{2p-3}^2 & k_{2p-1}^2 \\ \vdots & & & & \\ k_1^{2p-2} & k_2^{2p-2} & \dots & k_{2p-3}^{2p-2} & k_{2p-1}^{2p-2} \end{pmatrix}, b = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ k_2^2 & k_4^2 & \dots & k_{2p-2}^2 & k_{2p}^2 \\ \vdots & & & & \\ k_2^{2p-2} & k_4^{2p-2} & \dots & k_{2p-2}^{2p-2} & k_{2p}^{2p-2} \end{pmatrix},$$

$$c = \begin{pmatrix} k_1 & k_3 & \dots & k_{2p-3} & k_{2p-1} \\ k_1^3 & k_3^3 & \dots & k_{2p-3}^3 & k_{2p-1}^3 \\ \vdots & & & & \\ k_1^{2p-1} & k_2^{2p-1} & \dots & k_{2p-3}^{2p-1} & k_{2p-1}^{2p-1} \end{pmatrix}, d = \begin{pmatrix} k_2 & k_4 & \dots & k_{2p-2} & k_{2p} \\ k_2^3 & k_4^3 & \dots & k_{2p-2}^3 & k_{2p}^3 \\ \vdots & & & & \\ k_2^{2p-1} & k_4^{2p-1} & \dots & k_{2p-2}^{2p-1} & k_{2p}^{2p-1} \end{pmatrix}.$$

Expanding the determinant into series over z one derives a compact polynomial expression for the partition function Z_N. Introducing the products

$$F(t_1, \dots, t_K) = \prod_{i<j}(t_i^2 - t_j^2), \qquad G(t_1, \dots, t_K) = \left(\prod_{s=1}^{K} t_s\right)\prod_{i<j}(t_i^2 - t_j^2),$$

one can write

$$W_N(z) = \frac{\cosh^N\beta H}{2^{N(N-1)/2}} \sum_{\ell=0}^{[N/2]} (-z^2)^\ell \sum_{i_1<\dots<i_\ell}^{[(N+1)/2]} \sum_{j_1<\dots<j_\ell}^{[N/2]} F_{i_1\dots i_\ell}^{j_1\dots j_\ell}\, G_{j_1\dots j_\ell}^{i_1\dots i_\ell}, \tag{16}$$

$$F^{j_1 \ldots j_\ell}_{i_1 \ldots i_\ell} = F(k_1, k_3, \ldots, k_{2i_1-3}, k_{2j_1}, k_{2i_1+1}, \ldots, k_{2i_\ell-3}, k_{2j_\ell}, k_{2i_\ell+1}, \ldots, k_{2[(N+1)/2]-1}),$$

$$G^{i_1 \ldots i_\ell}_{j_1 \ldots j_\ell} = G(k_2, k_4, \ldots, k_{2j_1-2}, k_{2i_1-1}, k_{2j_1+2}, \ldots, k_{2j_\ell-2}, k_{2i_\ell-1}, k_{2j_\ell+2}, \ldots, k_{2[N/2]}).$$

This representation is actually valid for both even and odd N. The fact that W_N should depend only on the powers of z^2 follows from the obvious parity symmetry $Z_N(-H) = Z_N(H)$. The polynomial representation is not satisfactory since it is difficult to analyze the $N \to \infty$ limit. A more effective would be the representation of Z_N in the product form $Z_{2p} \propto \prod_{i=1}^{p}(z^2 - z_i^2)$, where z_i are zeros of the partition function (evidently they should be complex).

In order to understand the structure of z_i one should diagonalize matrices a and d in (15) by simple multiplication of rows by appropriate factors and subsequent subtraction of them from each other in a 'triangulation' manner. After some work one can derive the following formula

$$W_{2p} = \frac{(-1)^p \cosh^{2p} \beta H}{2^{p(2p-1)}} \prod_{s=1}^{p} k_{2s} \prod_{1 \le i < j \le p} (k_{2i-1}^2 - k_{2j-1}^2)(k_{2i}^2 - k_{2j}^2) \begin{vmatrix} zI & \alpha \\ \beta & zI \end{vmatrix}, \quad (17)$$

where I is the unit matrix and two other $p \times p$ matrices α and β have the following form

$$\alpha_{ij} = \frac{k_{2i-1}^2}{k_{2j}} \prod_{\substack{m=1 \\ m \ne i}}^{p} \left(k_{2m-1}^2 - k_{2j}^2\right) \prod_{\substack{l=1 \\ l \ne j}}^{p} \left(k_{2l}^2 - k_{2j}^2\right)^{-1},$$

$$\beta_{ij} = \frac{1}{k_{2j-1}} \prod_{\substack{m=1 \\ m \ne i}}^{p} \left(k_{2m}^2 - k_{2j-1}^2\right) \prod_{\substack{l=1 \\ l \ne j}}^{p} \left(k_{2l-1}^2 - k_{2j-1}^2\right)^{-1}. \quad (18)$$

Consider the eigenvalue problem for the matrix

$$D = \begin{pmatrix} zI & \alpha \\ \beta & zI \end{pmatrix},$$

where the matrices α and β are not degenerate. The eigenvectors Ψ of D, $D\Psi = \lambda\Psi$, are

$$\Psi = \begin{pmatrix} \phi \\ \xi \end{pmatrix}, \quad \begin{pmatrix} zI & \alpha \\ \beta & zI \end{pmatrix} \begin{pmatrix} \phi \\ \xi \end{pmatrix} = \lambda \begin{pmatrix} \phi \\ \xi \end{pmatrix}.$$

It follows from the last equation that

$$\alpha\beta\phi = (\lambda - z)^2\phi, \quad \beta\alpha\xi = (\lambda - z)^2\xi. \tag{19}$$

Denote as z_i^2 and ϕ_i, $i = 1, \dots, p$, eigenvalues and eigenvectors of the matrix $\alpha\beta$. Now it is obvious that the vectors $(\phi_i, \pm(\beta\phi)_i/z_i)^T$ are eigenvectors of D with the eigenvalues $z \pm z_i$. Therefore

$$\det D = \begin{vmatrix} zI & \alpha \\ \beta & zI \end{vmatrix} = \lambda_1\lambda_2 \dots \lambda_{2p} = \prod_{j=1}^{p}(z^2 - z_j^2).$$

Calculation of the eigenvalues of the matrix $\alpha\beta$ (18) for arbitrary choice of parameters k_i is a difficult task. In the models considered in [6] and below the situation is tremendously simplified in the thermodynamic limit $N \to \infty$ due to the self-similarity restrictions imposed upon k_i.

The crucial observation of our previous work [6] has related the condition of translational invariance of the phase shifts A_{ij} for the KdV equation (or of the spin exchange constants J_{ij} of the corresponding infinite Ising chain) to self-similar potentials mentioned above. The simplest translational symmetry consists in the invariance of system with respect to the shift by the shortest distance between the spins $j \to j + 1$ which means that $J_{i+1,j+1} = J_{ij}$. Thus the intensities J_{ij} and, so, A_{ij} depend only on the distance between the sites $|i - j|$, $A_{ij} = A(|i - j|)$. This natural physical condition fixes all k_i in terms of k_1 and a parameter q in a unique way:

$$k_i = k_1 q^{i-1}, \qquad q = e^{-2\alpha}, \qquad A_{ij} = 2\ln|\tanh\alpha(i - j)|. \tag{20}$$

The models with bigger periods of translational invariance $M > 1$, when $J_{i+M,j+M} = J_{ij}$, will be considered in the next section.

From the ordering of k_i's we have chosen, it follows that $\alpha > 0$. Strictly speaking there is no translational invariance due to the boundary effects since $1 \le j \le N$. An infinite chain emerges in the thermodynamic limit $N \to \infty$. In this case one gets an infinite soliton potential with the discrete spectrum

$\lambda_j = -k_1^2 q^{2(j-1)}/4$. All other variables of the solutions of hierarchies of integrable equations (the coordinate x and higher "times" x_n) are interpreted as parameters of the magnetic field H_i. Since $q^j \to 0$ for $j \to \infty$, the $x, t, \ldots$ dependent part of H_i is decaying exponentially fast from the $j = 1$ edge of the lattice. Therefore their influence upon the partition function is negligible in the thermodynamic limit, i.e. only the constants $\theta_i^{(0)}$ are relevant. Below we are considering the partition function for the homogeneous or M-periodic magnetic fields, $H_{i+M} = H_i$.

The KdV equation generates only an antiferromagnetic Ising chain. Indeed, one has $0 < |\tanh \alpha(i-j)| < 1$ and $J_{ij} = -A_{ij}/4 > 0$. Physically, the exchange between spins we have obtained is more natural than the nearest-neighbor one — it has the long distance character but the intensity falls off exponentially fast. The phase transition takes place in such systems only at the zero temperature.

In the interesting limit $\alpha \to 0$ or $q \to 1$ the constants $A_{ij} \propto J_{ij}/kT$ are diverging. In order to have finite energy of interaction of a single spin with other ones, it is necessary to renormalize the constants $J_{ij}^{ren} = J_{ij}(q^{-1} - q)$ and the temperature $kT_{ren} = kT(q^{-1} - q)$. Then the summation of J_{ij}^{ren} over j is finite for $q \to 1$. As a result, in the limit $q \to 1$ one actually gets the long range nonlocal interaction model with low effective temperature. Let us remark that in order to imitate the change of the temperature one should renormalize simultaneously the magnetic field $H = H^{ren}/(q^{-1} - q)$.

Due to the specific choice of the renormalization factor $q^{-1} - q$, the limit $q \to 0$ gives $J_{ij}^{ren} \propto \delta_{i+1,j}$. Evidently, for finite H^{ren} one gets the high temperature nearest neighbor interaction model. If H is kept finite then the $q \to 0$ limit describes the non-interacting spins. Thus our formalism does not provide full description of the partition function. It captures only the behavior of the system in a two-dimensional subspace of the variables (T, H, q). Since for fixed q the temperature is fixed as well, we normalize the "KdV temperature" to $\beta^{-1} = kT = 1$.

It is not possible to recover potentials uniquely from their spectrum. For example, in order to fix the reflectionless potential it is necessary to specify the variables k_i and the phases $\theta_i^{(0)}$. Self-similar potentials defined by the reduction (14) correspond to the case when the scaling of x and t by q^M and q^{3M} respectively deletes M solitons. This means that $\theta_i(q^M x, q^{3M} t) = \theta_{i+M}(x,t)$, which in turn imposes the constraints $k_{i+M} = q^M k_i$, $\theta_{i+M}^{(0)} = \theta_i^{(0)}$. The same constraints are derived from the spin chain periodicity condition $J_{i+M,j+M} = J_{ij}$. For $M = 1$ one has homogeneous phases $\theta_i^{(0)} = \theta^{(0)} = const.$

Substituting (20) into (17) we get the following expression for the partition function,

$$Z_{2p} = \frac{2^{2p}\prod_{i=1}^{p-1}(q^4;q^4)_i^2 \cosh^{2p}\beta H}{\prod_{i=1}^{2p-1}(q^2;q^2)_i^{1/2}} (-q)^p \begin{vmatrix} zI & \alpha \\ \beta & zI \end{vmatrix}, \tag{21}$$

where

$$\alpha_{ij} = \frac{k_1 q^{-1}}{1-q^{4j-4i+2}} \frac{(q^2;q^4)_j (q^2;q^4)_{p-j}}{(q^4;q^4)_{j-1}(q^4;q^4)_{p-j}}, \quad i,j = 1,2,\ldots,p, \tag{22}$$

$$\beta_{ij} = \frac{k_1^{-1}}{1-q^{4i-4j+2}} \frac{(q^2;q^4)_{j-1}(q^2;q^4)_{p-j+1}}{(q^4;q^4)_{j-1}(q^4;q^4)_{p-j}} \tag{23}$$

and the standard notation for the q-shifted factorial $(a;q)_n$ is

$$(a;q)_0 = 1, \quad (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

Let us fix another numeration of the indices of the matrices. Assuming that p is even (for simplicity), we can shift $j \to j + p/2$ and write

$$\alpha_{ij} = \frac{k_1 q^{-1}}{1-q^{4j-4i+2}} \frac{(q^2;q^4)_{p/2+j}(q^2;q^4)_{p/2-j}}{(q^4;q^4)_{p/2+j-1}(q^4;q^4)_{p/2-j}}, \quad i,j = -p/2+1,\ldots,p/2,$$

with a similar change of indices for β_{ij}. Now we take the thermodynamic limit $p \to \infty$ and get

$$\alpha_{ij} = \frac{\gamma k_1 q^{-1}}{1-q^{4j-4i+2}}, \qquad \beta_{ij} = \frac{\gamma k_1^{-1}}{1-q^{4i-4j+2}}, \quad i,j \in Z,$$

where

$$\gamma = \frac{(q^2;q^4)_\infty^2}{(q^4;q^4)_\infty^2}.$$

The indices i and j keep finite values in this limiting procedure, i.e. we consider "the middle part" of the infinite spin chain. Note that the magnetic field corresponding to the $\theta_i^{(0)} = const$ choice is not homogeneous for $x = t = 0$, since the shifts in (9) depend on i. However, in the thermodynamic limit $N \to \infty$ the difference is negligible and the magnetic field differs from $\theta^{(0)}/2$ only by a finite constant

$$\beta H_i = \frac{1}{2}\theta^{(0)} + \ln \frac{(q;q)_\infty}{(-q;q)_\infty}, \tag{24}$$

where the index i is assumed to lie in "the middle part" of the chain.

As a consequence of the spectral self-similarity hidden in the problem, the matrix elements α_{ij}, β_{ij} depend only on the differences $i-j$, i.e. we get the Toeplitz matrices: $\alpha_{ij} = \alpha(i-j), \beta_{ij} = \beta(i-j)$. It is known that such matrices are digonalized by the discrete Fourier transformation [19].

For a given $K \times K$ matrix β_{nm} the discrete Fourier transformation is defined as follows

$$\rho_{\mu\nu} = \frac{1}{K}\sum_{m,n=0}^{K-1} e^{i(\mu n - \nu m)}\beta_{nm},$$

$$\mu = \frac{2\pi\mu'}{K}, \qquad \nu = \frac{2\pi\nu'}{K}, \qquad \mu', \nu' = 0, 1, \ldots, K-1.$$

For the Toeplitz matrix one has

$$\rho_{\mu\nu} = \frac{1}{K}\sum_{m,n=0}^{K-1} e^{i(\mu n - \nu m)}\beta(n-m) = \frac{1}{K}\sum_{m=0}^{K-1} e^{i(\mu-\nu)m} \sum_{s=-m}^{K-1-m} e^{is\mu}\beta(s) \underset{K\to\infty}{=} \delta_{\mu\nu}\rho(\mu),$$

where $\delta_{\mu\nu} = 1$ if $\mu = \nu$ and zero otherwise. The density function $\rho(\nu)$ is determined by the following bilateral series

$$\rho(\nu) = \sum_{s=-\infty}^{\infty} e^{is\nu}\beta(s).$$

The matrix $\beta(n-m)$ becomes thus diagonal in the $K \to \infty$ limit after a simple Fourier transformation. Because of this property the equations (19) are diagonalized too and yield $z^2(\nu) = q^{-1}|\rho(\nu)|^2$, where $|\rho(\nu)|^2$ is the diagonal part of the Fourier transform of the $p \times p$ (i.e. $K = p$) matrix $q\alpha\beta$ and $z(\nu)$ are continuous analogs of the partition function zeros z_i. We write $|\rho|^2$ because the density functions of the matrices $q\alpha/k_1$ and $k_1\beta$ are conjugated to one another.

Before taking the limit $p \to \infty$, the logarithm of the partition function is represented as a finite sum over eigenvalues z_i,

$$\ln Z_{2p} \propto \sum_{i=1}^{p} \ln(z_i^2 - z^2).$$

In the limit $p \to \infty$ the step $\Delta = 2\pi/p$ of the variable ν characterizing the "distance" between eigenvalues goes to zero. Therefore the sum over the index i should be replaced by an integral over the varibale $\nu = 2\pi i/p$,

$$\sum_{i=1}^{p} \ln\left(z_i^2 - z^2\right) \underset{p\to\infty}{\longrightarrow} \frac{p}{2\pi}\int_0^{2\pi} \ln\left(q^{-1}|\rho(\nu)|^2 - z^2\right) \mathrm{d}\nu.$$

As a result, after tracing the normalization factors, we obtain

$$(-q)^p \begin{vmatrix} zI & \alpha \\ \beta & zI \end{vmatrix} \underset{p\to\infty}{=} \exp\left[\frac{p}{2\pi}\int_0^{2\pi} d\nu \ln(|\rho(\nu)|^2 - q\tanh^2\beta H)\right],$$

where the density function $\rho(\nu)$ is

$$\rho(\nu) = \gamma \sum_{k=-\infty}^{\infty} \frac{e^{i\nu k - \epsilon k}}{1 - q^{4k+2}} = \frac{(q^2 e^{i\nu-\epsilon}; q^4)_\infty (q^2 e^{-i\nu+\epsilon}; q^4)_\infty}{(e^{i\nu-\epsilon}; q^4)_\infty (q^4 e^{-i\nu+\epsilon}; q^4)_\infty}. \qquad (25)$$

Here the dumping factor depending on a small parameter $\epsilon > 0$ was introduced in order to guarantee the absolute convergency of the infinite sum.

The formula (25) was obtained with the help of the ${}_1\psi_1$ Ramanujan's bilateral basic hypergeometric series sum [22]:

$$\sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} z^n = \frac{(q, b/a, az, q/az; q)_\infty}{(b, q/a, z, b/az; q)_\infty},$$

where the compact notations

$$(a_1, a_2, \ldots, a_n; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \ldots (a_n; q)_\infty$$

are used. Setting $b = qa$, one finds the formula

$$\sum_{n=-\infty}^{\infty} \frac{z^n}{1 - aq^n} = \frac{(q, q, az, q/az; q)_\infty}{(a, q/a, z, q/z; q)_\infty},$$

which is repeatedly used in our considerations.

Now we can remove the regularization, i.e. take the limit $\epsilon \to 0$ in (25). It is seen that in this limit $\rho(\nu)$ becomes singular only near the points $\nu = 0, 2\pi$. But these singularities are integrable and, so, harmless.

Gathering all together we get $Z_{2p} \to \exp(-2p\beta f_I)$, where the free energy per site f_I has the form

$$-\beta f_I(q,H) = \ln \frac{2(q^4;q^4)_\infty \cosh \beta H}{(q^2;q^2)_\infty^{1/2}} + \frac{1}{4\pi}\int_0^{2\pi} d\nu \ln(|\rho(\nu)|^2 - q\tanh^2 \beta H), \quad (26)$$

$$|\rho(\nu)|^2 = \frac{(q^2e^{i\nu};q^4)_\infty^2 (q^2e^{-i\nu};q^4)_\infty^2}{(q^4e^{i\nu};q^4)_\infty^2 (q^4e^{-i\nu};q^4)_\infty^2} \frac{1}{4\sin^2(\nu/2)} = q\frac{\theta_4^2(\nu/2,q^2)}{\theta_1^2(\nu/2,q^2)}.$$

Here $\theta_{1,4}(y,q)$ are the standard Jacobi theta-functions of the argument y and base q (our base is q^2)

$$\begin{aligned}\theta_4(\nu,q^2) &= 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{2n^2}\cos(2n\nu)\\ &= (q^4;q^4)_\infty (q^2e^{2i\nu};q^4)_\infty (q^2e^{-2i\nu};q^4)_\infty,\end{aligned}$$

and

$$\begin{aligned}\theta_1(\nu,q^2) &= 2\sum_{n=0}^{\infty}(-1)^n q^{2(n+1/2)^2}\sin((2n+1)\nu)\\ &= 2q^{1/2}\sin\nu\ (q^4;q^4)_\infty (q^4e^{2i\nu};q^4)_\infty (q^4e^{-2i\nu};q^4)_\infty.\end{aligned}$$

The relations between the series and product representations of elliptic functions are called the triple product Jacobi identities [22].

From the polynomial representation (16) it is easy to find the partition function Z_N at zero magnetic field:

$$Z_{2p+1} = 2^{2p+1}(q^4;q^4)_p \prod_{j=1}^{2p}(q^2;q^2)_j^{-1/2}\prod_{\ell=1}^{p-1}(q^4;q^4)_\ell^2, \quad (27)$$

$$Z_{2p} = 2^{2p}\prod_{j=1}^{2p-1}(q^2;q^2)_j^{-1/2}\prod_{\ell=1}^{p-1}(q^4;q^4)_\ell^2. \quad (28)$$

Therefore for $N \to \infty$ one has

$$-\beta f_I = \ln \frac{2(q^4;q^4)_\infty}{(q^2;q^2)_\infty^{1/2}}.$$

Comparing with the $H \neq 0$ derivation we see that the following identity should take place

$$\int_0^{2\pi} d\nu \ln |\rho(\nu)|^2 = 0, \tag{29}$$

which can be easily proved. First, one has to split the interval of integration into half. Then, shift the variable $\nu \to \nu + \pi$ in the integral over the interval $[\pi, 2\pi]$. This will change $\sin(\nu/2)$ into $\cos(\nu/2)$ and $e^{i\nu}$ into $-e^{i\nu}$. Adding two integrals together one gets the integral equal to $1/2$ of the initial one with q replaced by q^2. One may iterate this procedure to infinity and see that the limit is equal to zero.

Taking the derivative of $f_I(H)$ with respect to the magnetic field we find the total magnetization of the lattice:

$$m(H) = -\partial_H f_I = \lim_{p\to\infty} \frac{1}{2p}\sum_{i=1}^{2p}\langle\sigma_i\rangle$$

$$= \tanh \beta H \left(1 - \frac{1}{\pi}\int_0^\pi \frac{\theta_1^2(\nu,q^2)d\nu}{\theta_4^2(\nu,q^2)\cosh^2\beta H - \theta_1^2(\nu,q^2)\sinh^2\beta H}\right). \tag{30}$$

In order to imitate the change of the temperature, we substitute $\beta H = h/(q^{-1}-q)$ into this formula and plot $m(h)$ in Fig. 1 by the dashed lines for $q = 0.1$ (the lower curve) and $q = 0.8$ (the upper one). The curves have a simple shape whose qualitative properties are predicted by the general considerations of 1D systems with the fast decaying interaction.

One can check that the magnetic susceptibility for $H = 0$ derived with the help of two different formulae (16) and (26) coincide with each other. For instance, one can differentiate (30) with respect to H and set $H = 0$:

$$\chi(H=0) = \beta^{-1}\partial_H m(H=0) = 1 - \frac{1}{\pi}\int_0^\pi \frac{\theta_1^2(\nu,q^2)}{\theta_4^2(\nu,q^2)}d\nu.$$

Converting the ratio of θ-functions into the series form via the Ramanujan ${}_1\psi_1$-series, taking its square and calculating the integral over the emerging double series termwise, we get

$$\chi(H=0) = 1 - 2q\frac{(q^2;q^4)_\infty^4}{(q^4;q^4)_\infty^4}\sum_{n=0}^{\infty}\frac{q^{4n}}{(1-q^{4n+2})^2}.$$

This series can be expressed in terms of the Jacobi elliptic functions.

4 The M-periodic Ising chains

Let us consider now the M-periodic Ising chains which have the translational invariance with respect to the shifts by M sites. Such a chain consists of M embedded sublattices, or, in other words, of the periodic blocks of M spins with different distances between them. Another interpretation refers to ferrimagnetics, when the distances between the lattice points are equal but one has different magnetic moments at sites. For any interpretation the exchange constants between the neighboring M spins are given by arbitrary numbers decreasing with the distance between sites. These physical conditions generate the constraints $J_{i+M,j+M} = J_{ij}$, which, in turn, demand that the soliton parameters satisfy the simple condition $k_{j+M} = q^M k_j$ (here we take M-th power of q just for convenience). As was mentioned already, for $\theta_{i+M}^{(0)} = \theta_i^{(0)}$ one arrives at the general self-similar infinite soliton potentials (14). In the thermodynamic limit this condition is equivalent to the periodicity of the magnetic field $H_{i+M} = H_i$ which we assume from now on.

Calculation of Z_N for the $M > 1$ self-similar potentials is simplified after setting $N = 2Mp$ and taking the $p \to \infty$ limit. Indeed, the Fourier transformation of a given $K \times K, K = Mp$, matrix $\beta_{i+M,j+M} = \beta_{ij}$ is diagonal again in the limit $p \to \infty$. In order to see this it is necessary to split the matrix indices $n = Mk + l, k = 0, \dots, p-1, l = 0, \dots, M-1$ in the calculation of the transformation:

$$\rho_{\mu\nu} = \frac{1}{K}\sum_{n,m=0}^{K-1} e^{i(\mu n - \nu m)}\beta_{nm} = \frac{1}{Mp}\sum_{k,k'=0}^{p-1}\sum_{l,l'=0}^{M-1} e^{iM(\mu k - \nu k') + i(\mu l - \nu l')}\beta_{M(k-k')+l,l'}$$

$$= \frac{1}{Mp}\sum_{k=0}^{p-1} e^{iM(\mu-\nu)k}\sum_{s=-k}^{p-1-k} e^{i\mu M s}\sum_{l,l'=0}^{M-1} e^{i(\mu l - \nu l')}\beta_{Ms+l,l'} \underset{p\to\infty}{=} \delta_{\mu\nu}\rho(\nu),$$

where

$$\rho(\nu) = \frac{1}{M} \sum_{s=-\infty}^{\infty} e^{i\nu M} \sum_{l,l'=0}^{M-1} e^{i\nu(l-l')} \beta_{Ms+l,l'}. \tag{31}$$

As an example, for $M = 2$ one gets

$$\rho(\nu) = \frac{1}{2} \sum_{k=-\infty}^{\infty} e^{2i\nu k} \left(\beta_{2k,0} + \beta_{2k,1} e^{-i\nu} + \beta_{2k+1,0} e^{i\nu} + \beta_{2k+1,1}\right). \tag{32}$$

It is not easy to apply the formula (31) to calculation of the partition function of M-periodic Ising chains in the Wronskian representation. The condition $H_{i+M} = H_i$ calls for a complicated split of the determinant into a $M \times M$ block structure responsible for the M-periodicity of the system. In this respect, the gramian form of the KdV τ-function is much more convenient.

After some heuristic analysis of the gramian representation of the p-soliton solution of the KP equation we have obtained the following compact form of W_N for even number of solitons, $N = 2p$:

$$W_{2p}\left(\cosh\beta H_1, \ldots, \sinh\beta H_{2p}\right) = \frac{(-1)^{p(p-1)/2}}{2^{2p^2}} \left(\prod_{i=1}^{p} k_{2i-1}^{-1} \prod_{j=1}^{p} (k_{2i-1}^2 - k_{2j}^2)\right) \det D, \tag{33}$$

where the $p \times p$ matrix D has the form

$$D_{ij} = \frac{\cosh\beta(H_{2i-1} + H_{2j})}{1 + k_{2j}/k_{2i-1}} + \frac{\cosh\beta(H_{2i-1} - H_{2j})}{1 - k_{2j}/k_{2i-1}}.$$

Indeed, the KP p-soliton τ-function has the following determinant representation [23, 13]

$$\tau_p = \det D, \quad D_{ij} = C_{ij} + \int^x \phi_i(x_n)\phi_j'(x_n)dx_1, \quad i,j = 1,2\ldots,p,$$

where ϕ_i, ϕ_i' are functions of $x_1 = x, x_2 = y, \ldots$ satisfying the following auxiliary differential equations

$$\frac{\partial \phi_i}{\partial x_n} = \frac{\partial^n \phi_i}{\partial x^n}, \qquad \frac{\partial \phi_i'}{\partial x_n} = (-1)^{n-1} \frac{\partial^n \phi_i'}{\partial x^n},$$

and C_{ij} are some constants. The space of permitted functions ϕ_i, ϕ'_i is quite large. The following non-trivial choice

$$\phi_i = \sinh \beta H_{2i-1}, \qquad \phi'_i = \cosh \beta H_{2i}, \qquad C_{ij} = 0,$$

where $\beta H_i = (\theta_i^{(0)} + (a_i + b_i)x)/2 + \dots$ with the subsequent reduction $a_i = b_i = k_i/2$ appears to determine the $2p$-soliton solution of the KdV equation. So, $\tau_p^{KP} \propto W_{2p}^{KdV}$ with the proportionality constant determined from the analysis of the leading $H_i = H \to \infty$ terms in (33).

In the thermodynamical limit one can apply the formula (31) to (33) directly (without the auxiliary block diagonalizations) after imposing the constraints $k_{i+M} = q^M k_i$, $H_{i+M} = H_i$. Actually, for $M = 2$ one gets the determinant of a Toeplitz matrix without the use of the formula (32). This is easily seen after the substitution of the relations

$$k_{2i-1} = k_1 q^{2(i-1)}, \quad k_{2i} = k_2 q^{2(i-1)}, \quad H_{2i-1} = H_1, \quad H_{2i} = H_2,$$

which gives

$$Z_{2p} = 2^{2p} \det \widetilde{D} \prod_{i=1}^{p} \frac{(r^2; q^4)_i^{1/2} (q^4/r^2; q^4)_{i-1}^{1/2}}{(q^4, q^4)_{i-1}},$$

$$\widetilde{D}_{ij} = \frac{\cosh \beta H_1 \cosh \beta H_2 - r q^{2(j-i)} \sinh \beta H_1 \sinh \beta H_2}{1 - r^2 q^{4(j-i)}},$$

where we have denoted

$$r = k_2/k_1, \qquad q^2 < r < 1.$$

The Fourier transformation diagonalizes the matrix $\widetilde{D}$ in the limit $p \to \infty$ and allows us to find its determinant. Using the regularization procedure described earlier and applying the ${}_1\psi_1$ summation formula, we find the free energy per site

$$-\beta f_I = \ln \frac{2(q^4; q^4)_\infty^{1/2}}{(r^2, q^4/r^2; q^4)_\infty^{1/4}} + \frac{1}{4\pi} \int_0^{2\pi} d\nu \ln |\rho(\nu)|, \tag{34}$$

where

$$\rho(\nu) = \frac{(r^2 z, q^4 r^{-2} z^{-1}, q^2 z, q^2 z^{-1}; q^4)_\infty}{(r^2 q^2 z, q^2 r^{-2} z^{-1}, z, q^4 z^{-1}; q^4)_\infty} \cosh \beta H_1 \cosh \beta H_2$$

$$-r\sinh\beta H_1\sinh\beta H_2, \qquad z = e^{-i\nu}. \tag{35}$$

In the derivation of this formula some identities similar to (29) have been used.

It can be seen that for $k_2 = qk_1, H_1 = H_2 = H$ the derived result coincides with the one for $M = 1$ system (26). We get thus one more independent test of the expression for the free energy per site for the KdV equation case.

Let us put $k_2 = k_1 q$, take the derivative of $\ln Z_{2p}$ with respect to H_1, and set $H_1 = H_2 = H$. This gives the odd index sublattice magnetization

$$m_1(H) = -2\partial_{H_1} f_I(H_1, H_2)|_{H_1=H_2} = \lim_{p\to\infty}\frac{1}{p}\sum_{i=1}^{p}\langle\sigma_{2i-1}\rangle$$

in the simplest $M = 1$ model. From the derived form of the free energy per site it is clearly seen that $m_1(H) = m(H)$, i.e. there is no sublattice magnetization for $H \to 0$. This means that the system is above the critical temperature and there is no long-range ordering for sublattices. Actually, for $k_i = k_1 q^{i-1}$ the density function $\rho(\nu)$ of arbitrary M-periodic system is analytic in $H_1, \ldots, H_M$ and invariant with respect to the permutation of these variables. This means that the magnetization of any periodic spin sublattice of the $M = 1$ chain is equivalent to the total magnetization.

5 Ising chains induced by the BKP tau-function

A drawback of the construction described in the previous sections is that the partition function associated with the KdV-equation has a fixed temperature for fixed q. For a more complete description of thermodynamical quantities one may try to replace (20) by $A_{ij} = 2n\ln|\tanh\alpha(i-j)|$, where n is a positive integer playing the role of the inverse temperature, $\beta = n$. The KdV case is normalized to $n = 1$, for $n > 1$ it is necessary to renormalize the magnetic field $H_i \to nH_i$ for imitation of the temperature lowering. If there are integrable models with the phase shifts of the following form

$$e^{A_{ij}} = \frac{(k_i - k_j)^{2n}}{(k_i + k_j)^{2n}}, \tag{36}$$

for some infinite sequence of integers n, then one may hope to recover the partition function for arbitrary variations of the inverse temperature β by an analytic continuation.

Our attempts to find integrable equations with such phase shifts succeeded only partially. Namely, one example was found [6] for $n = 2$ corresponding to the reduced form of the N-soliton solution of the BKP equation. The canonical form of the BKP equation is [13]

$$\frac{\partial}{\partial x_1}\left(9\frac{\partial u}{\partial x_5} - 5\frac{\partial^3 u}{\partial x_3 \partial x_1^2} + \frac{\partial^5 u}{\partial x_1^5} - 30\frac{\partial u}{\partial x_3}\frac{\partial u}{\partial x_1} + 30\frac{\partial u}{\partial x_1}\frac{\partial^3 u}{\partial x_1^3} + 60\left(\frac{\partial u}{\partial x_1}\right)^3\right) - 5\frac{\partial^2 u}{\partial x_3^2} = 0. \tag{37}$$

The unconstrained tau-function of this integrable equation (or of the whole BKP hierarchy) is related to the partition function of the essentially more general Ising chain than we have considered in the previous sections. The general exchange constants have now the following form

$$\beta J_{ij} = -\frac{1}{4}A_{ij}, \quad e^{A_{ij}} = \frac{(a_i - a_j)(b_i - b_j)(a_i - b_j)(b_i - a_j)}{(a_i + a_j)(b_i + b_j)(a_i + b_j)(b_i + a_j)} \tag{38}$$

with the energy and the partition function given by (7). For $a_i = b_i = k_i/2$ these A_{ij} are reduced to (36) for $n = 2$.

It is known that the N-soliton tau-function of the BKP hierarchy can be expresed through the pfaffian of $2N \times 2N$ antisymmetric matrix D, with the matrix elements [13]

$$D_{ij} = C_{ij} + \int^x \left(\frac{\partial f_i}{\partial x_1} f_j - \frac{\partial f_j}{\partial x_1} f_i\right) \mathrm{d}x_1,$$

where C_{ij} are some constants and the functions f_i depend only on odd hierarchy "times" x_{2m-1} and satisfy the following auxiliary linear equations

$$\frac{\partial f_i}{\partial x_{2m-1}} = \frac{\partial^{2m-1} f_i}{\partial x^{2m-1}}.$$

One particular choice of f_i was taken in [13], and a simple example of building 2-soliton solution was described via the pfaffian of a 4×4 matrix.

However, for an even number of solitons $N = 2p$ it is possible to simplify the situation and get a "folded" determinant representation when the $2p$ soliton solution of the BKP equation comes from the pfaffian of a $2p \times 2p$ matrix. Indeed, if we take $C_{ij} = 0$ and set

$$f_i = g_i \exp\left(h_i + b_i x_1 + b_i^3 x_3 + \dots\right) + g_i^{-1} \exp\left(-h_i - a_i x_1 - a_i^3 x_3 - \dots\right)$$

then the auxiliary linear equations are satisfied. Integrating over x_1 and setting all x_m equal to zero, we get the needed determinant formula for the tau-function.

As a result, after establishing the correspondence between $2p$-soliton tau-function and the partition function of the Ising chain of $2p$ spins with the exchange constants given by (38) and the magnetic field $\beta H_i \equiv h_i$, one finds

$$Z_{2p} = \left(\prod_{1 \le i < j \le 2p} \frac{(a_i - a_j)(b_i - b_j)(a_i + b_j)(b_i + a_j)}{(a_i + a_j)(b_i + b_j)(a_i - b_j)(b_i - a_j)} \right)^{-1/4} \sqrt{\det G}, \qquad (39)$$

where

$$G_{ij} = g_i g_j \frac{b_i - b_j}{b_i + b_j} e^{\beta(H_i + H_j)} + g_i g_j^{-1} \frac{b_i + a_j}{b_i - a_j} e^{\beta(H_i - H_j)}$$
$$+ g_i^{-1} g_j \frac{a_i + b_j}{a_i - b_j} e^{\beta(-H_i + H_j)} + g_i^{-1} g_j^{-1} \frac{a_i - a_j}{a_i + a_j} e^{\beta(-H_i - H_j)},$$

and

$$g_i = \left(\prod_{\substack{j=1 \\ j \neq i}}^{2p} \frac{(a_i - a_j)(b_i + b_j)(b_i - a_j)(a_i + b_j)}{(a_i + a_j)(b_i - b_j)(b_i + a_j)(a_i - b_j)} \right)^{1/4}. \qquad (40)$$

The condition of simple translational invariance of the spin lattice, $J_{ij} = J(i - j)$, yields the following spectral self-similarity constraints

$$a_i = q^{i-1}, \quad b_i = b q^{i-1}, \quad q = e^{-2\alpha}, \qquad (41)$$

where for simplicity we normalize $a_1 = 1$ and assume that $\alpha > 0$. This gives the following form of the exchange constants

$$\beta J_{ij} = -\frac{1}{4} \ln \frac{\tanh^2 \alpha(i-j) - (b-1)^2/(b+1)^2}{\coth^2 \alpha(i-j) - (b-1)^2/(b+1)^2}.$$

Because of the $b \to b^{-1}$ symmetry we restrict complex values of b to the unit disk $|b| \leq 1$. Now it is seen that for

$$-1 < b < -q$$

one has the ferromagnetic interaction, $J_{ij} < 0$, which was not possible in the KdV case. For two other regions

$$q < b \leq 1 \qquad \text{and} \qquad b = e^{i\phi}, \quad \phi \neq \pi,$$

one has the antiferromagnetic interaction, $J_{ij} > 0$. For other choices of $b \neq -1$ one gets unphysical complex exchange constants J_{ij} which we are not interested in.

For physical values of b, the thermodynamic limit $p \to \infty$ leads to the simplification, $g_i \to 1$, so that

$$Z_{2p} = \frac{(-q, -q, bq, q/b; q)_\infty^{p/2}}{(q, q, -bq, -q/b; q)_\infty^{p/2}} \sqrt{\det G}, \tag{42}$$

where the matrix G has the following form

$$G_{ij} = 2\frac{1 - q^{j-i}}{1 + q^{j-i}} \cosh \beta(H_i + H_j) + \frac{b + q^{j-i}}{b - q^{j-i}} e^{\beta(H_i - H_j)} + \frac{1 + bq^{j-i}}{1 - bq^{j-i}} e^{\beta(-H_i + H_j)}. \tag{43}$$

Taking the Fourier transformation and applying the Ramanujan ${}_1\psi_1$ sum, one finds the free energy per site for the homogeneous magnetic field $H_i = H$ in the following form

$$-\beta f_I(H) = \frac{1}{4} \ln \frac{(q, q, bq, q/b; q)_\infty}{(-q, -q, -bq, -q/b; q)_\infty} + \frac{1}{4\pi} \int_0^{2\pi} d\nu \ln |2\rho(\nu)|,$$

where

$$\rho(\nu) = \cosh 2\beta H + \frac{(-q; q)_\infty^2}{(-e^{i\nu}, -qe^{-i\nu}; q)_\infty} \left(\frac{(b^{-1}e^{i\nu}, qbe^{-i\nu}; q)_\infty}{(b^{-1}, qb; q)_\infty} + \frac{(be^{i\nu}, qb^{-1}e^{-i\nu}; q)_\infty}{(b, qb^{-1}; q)_\infty} \right). \tag{44}$$

Taking the derivative with respect to H we find the magnetization

$$m(H) = \tanh 2\beta H \left(1 - \frac{1}{\pi} \int_0^{\pi} \frac{d\nu}{1 + d(\nu) \cosh 2\beta H} \right), \tag{45}$$

where

$$d(\nu) = \frac{(qb, q/b; q)_\infty (b^{-1/2} - b^{1/2}) \theta_2(\nu, q^{1/2})}{(-q; q)_\infty^2 2\mathrm{Im}\ \theta_1(\nu - (i/2)\ln b, q^{1/2})}$$

for $q < b \leq 1$,

$$d(\nu) = \frac{(qb, q/b; q)_\infty (|b|^{-1/2} + |b|^{1/2}) \theta_2(\nu, q^{1/2})}{(-q; q)_\infty^2 2\mathrm{Re}\ \theta_2(\nu - (i/2)\ln |b|, q^{1/2})}$$

for $-1 < b < -q$, and

$$d(\nu) = \frac{(qe^{i\phi}, qe^{-i\phi}; q)_\infty 2\sin(\phi/2)\ \theta_2(\nu, q^{1/2})}{(-q; q)_\infty^2 [\theta_1(\nu + \phi/2, q^{1/2}) - \theta_1(\nu - \phi/2, q^{1/2})]}$$

for $b = e^{i\phi}$. In these formulae another Jacobi θ-function $\theta_2(\nu, q^{1/2})$ is appearing

$$\theta_2(\nu, q^{1/2}) = 2\sum_{n=0}^{\infty} q^{(n+1/2)^2/2} \cos{(2n+1)\nu}$$

$$= 2q^{1/8} \cos\nu\ (q; q)_\infty (-qe^{2i\nu}; q)_\infty (-qe^{-2i\nu}; q)_\infty.$$

In the limit $b \to 1$ we find partition function of the $M = 1$ Ising chain inspired by the KdV-equation for another value of the discrete temperature $\beta = n = 2$ (it is necessary to renormalize magnetic field $H \to 2H$ for this interpretation)

$$-\beta f_I(H) = \frac{1}{2\pi} \int_0^\pi d\nu \ln 2 \left| \frac{(q; q)_\infty^2}{(-q; q)_\infty^2} \cosh 4\beta H + \frac{\partial_\nu \theta_1(\nu, q^{1/2})}{\theta_2(\nu, q^{1/2})} \right|, \tag{46}$$

where ∂_ν is the derivative with respect to ν. This gives the magnetization formula

$$m(H) = \tanh 4\beta H \left(1 - \frac{1}{\pi} \int_0^\pi d\nu \left(1 + \frac{(q; q)_\infty^2 \theta_2(\nu, q^{1/2}) \cosh 4\beta H}{(-q; q)_\infty^2 \partial_\nu \theta_1(\nu, q^{1/2})} \right)^{-1} \right), \tag{47}$$

which was derived in [6] with the help of a slightly different procedure. For $q \to 0$ one has $m(H) = \tanh 2H/kT$.

We substitute into (47) $\beta H = h/(q^{-1} - q)$ and plot $m(h)$ in Fig. 1 by the solid lines for $q = 0.1$ (the lower curve) and $q = 0.8$ (the upper one). From the comparison of curves one can see that for lower temperatures (i.e. for the higher values of n or q) the magnetization $m(h)$ becomes steeper. According the arguments of [21] a staircase-like fractal function may emerge in the $q \to 1$ limit.

Some steps towards its formation can be traced for Ising antiferromagnets with nonzero interactions between few neighboring spins.

Let us illustrate the influence of the parameter b upon the magnetization shape for the general BKP case. For this we substitute $\beta H = 2h/(q^{-1} - q)$ into (45) and plot in Fig. 2 dependence of $m(h)$ on h for $q = 0.5$ and the following three choices of the parameter b: 1) $b = 0.8$ (the solid curve), 2) $b = -0.8$ (the dashed curve), 3) $b = i$ (the dash-dotted curve). The only qualitative difference between them is that for the ferromagnetic interaction the magnetization values are higher than in the antiferromagnetic cases, other differences being inessential at the taken temperature.

In the parametrization $B = -e^{-2\alpha\eta}$ the exchange constants have the form

$$\beta J(j) = -\frac{1}{2} \ln|\tanh \alpha j| + \frac{1}{4} \ln|\tanh \alpha(j+\eta)| + \frac{1}{4} \ln|\tanh \alpha(j-\eta)|.$$

Real values of η correspond to the ferromagnetic interaction, for the purely imaginary choice $\eta = i|\eta|$ we get the antiferromagnetic chain. For $\eta \to 0$ and fixed α, one finds an Ising model with the interaction

$$\beta J(j) = -\frac{\alpha^2\eta^2}{4} \left(\frac{1}{\cosh^2 \alpha j} + \frac{1}{\sinh^2 \alpha j} \right),$$

with high values of the effective temperature $\beta \propto |\eta|^2$. Let us remark that in a similar limit of the Ising model associated with the KP equation one has $J(j) \propto 1/\sinh^2 \alpha j$.

An interesting model emerges in the limit $\alpha \to 0$, η finite, since the exchange constants acquire the Calogero type behavior $J(j) \propto 1/j^2$ as $j \to \infty$. Indeed, in this limit one has

$$\beta J(j) = \frac{1}{4} \ln \left(1 - \left(\frac{\eta}{j} \right)^2 \right) \underset{j \to \infty}{\longrightarrow} -\frac{\eta^2}{4j^2} < 0 \tag{48}$$

for real η, and

$$\beta J(j) = \frac{1}{4} \ln \left(1 + \left(\frac{|\eta|}{j} \right)^2 \right) \underset{j \to \infty}{\longrightarrow} \frac{|\eta|^2}{4j^2} > 0$$

for imaginary η. The rational limit of our chain describes thus the Kondo model for the effective temperature $T \propto 1/|\eta|^2$ since the Ising chains with the asymptotic inverse square interaction correspond to this system [20]. The range of the parameter η^2 is restricted in the ferromagnetic case to $\eta^2 < 1$ (this corresponds to $-1 < b < -q$), then the argument of the logarithm in (48) is positive. However, the $\alpha \to 0$ limit in (44) is delicate since the original bilateral series form of the density function $\rho(\nu)$ diverges which interferes with our regularization procedure.

Considering the spin chain as consisting of two sublattices (we suppose that in the ground state of the antiferromagnetic phase spins are aligned in the Neel order), one can choose the sublattice magnetization as the order parameter. This definition of the order parameter also fits the ferromagnetic case, since the mean magnetization of sublattice coincides with the total one. Using the pfaffian representation (39) one can find the partition function for arbitrary M-periodic chains determined by the conditions $J_{i+M,j+M} = J_{ij}$ or, equivalently, $a_{i+M} = q^M a_i, b_{i+M} = q^M b_i$. However, the arguments similar to the ones presented in the previous section show that for $q < 1$ the system is above the critical temperature and all sublattice magnetizations do not differ (in the proper normalizations) from the total one.

6 On the discrete temperature renormalization

In this section we discuss possibilities to connect the M-periodic chains for fixed temperature T to the $M = 1$ chain at the lower temperature T/μ^2, where $0 < \mu \leq M$.

Consider an Ising chain with Mp spins. Let the exchange constants satisfy the constraints $J_{i+M,j+M} = J_{ij}$ and the magnetic field be homogeneous, $H_i = H = const$. Then the energy function can be split into three parts

$$E = \frac{1}{2}\sum_{\substack{i,j=1\\i\neq j}}^{Mp} J_{ij}\sigma_i\sigma_j - H\sum_{j=1}^{Mp}\sigma_j = \sum_{k=0}^{p-1} E_k$$

$$+\frac{1}{2}\sum_{\substack{k,k'=0\\k\neq k'}}^{p-1}\sum_{l,l'=1}^{M} J_{Mk+l,Mk'+l'}\sigma_{Mk+l}\sigma_{Mk'+l'} - H\sum_{k=0}^{p-1}\sum_{l=1}^{M}\sigma_{Mk+l}, \tag{49}$$

where E_k is the energy of interaction of spins in the k-th block

$$E_k = \frac{1}{2}\sum_{\substack{l,l'=1\\l\neq l'}}^{M} J_{ll'}\sigma_{Mk+l}\sigma_{Mk+l'}$$

and the second term describes the interblock interations. In the representation of E_k we used the translational invariance of J_{ij}.

Suppose now that the interaction within the blocks is much stronger than between the blocks

$$\frac{|J_{ij}|}{|J_{Mk+l,l'}|} \to \infty, \quad i,j = 1,\ldots,M, \quad k \neq 0.$$

For instance, in the BKP case such a situation is achieved if the parameters $a_l \approx a_{l'}, b_l \approx b_{l'}$ for $l, l' = 1, \ldots, M$ and q^M is not close to 1. Then, the exchange constants for spins of two different blocks are approximately equal to each other $J_{Mk+l,Mk'+l'} \approx J_{Mk,Mk'}$. Moreover, the internal interaction energies of blocks diverge so that the dominant contribution into the partition function is given by the ground states of the blocks. Clearly the ground states of different blocks may differ from each other only by the inversion of all spins in these blocks, which means that the block energies E_k coincide, $E_k = E_0$. As a result, the following approximate representation of the partition function takes place

$$Z_{Mp} = e^{-p\beta E_0}\sum_{\substack{ground\ states\\of\ blocks}} e^{-\beta\widetilde{E}},$$

where

$$\widetilde{E} = \frac{1}{2}\sum_{\substack{k,k'=1\\k\neq k'}}^{p-1} J_{Mk,Mk'}\sum_{l=1}^{M}\sigma_{Mk+l}\sum_{l'=1}^{M}\sigma_{Mk'+l'} - H\sum_{k=0}^{p-1}\sum_{l=1}^{M}\sigma_{Mk+l}.$$

Let us denote

$$\sum_{l=1}^{M}\sigma_{Mk+l} = \mu\sigma_k^{eff}, \qquad \sigma_k^{eff} = \pm 1.$$

In the ground state of ferromagnets all spins are aligned so that $\mu = M$. For antiferromagnets the situation is more complicated. For weak magnetic fields the second term in E_{eff} does not influence the structure of ground state and one can expect that $\mu = 0$ for even M and $\mu = 1$ for odd M due to the Neel ordering of spins. For large magnetic fields, when H is of the order of magnitute of the exchange constants for spins inside the blocks, the ground states become nontrivial [21]. Therefore, various choices $0 \leq \mu \leq M$ are possible for antiferromagnets.

Thus, the heuristic considerations given above suggest that

$$Z_{Mp} = e^{-p\beta E_0} \sum_{\sigma_k^{eff}=\pm 1} e^{-\mu^2 \beta E_{eff}},$$

$$E_{eff} = \sum_{k<k'} J(M(k-k'))\sigma_k^{eff}\sigma_{k'}^{eff} - \frac{H}{\mu}\sum_{k=0}^{p-1}\sigma_k^{eff},$$

where $J(M(k-k')) = J_{Mk,Mk'}$ are the exchange constants of the simplest $M = 1$ translationally invariant spin chain. In the thermodynamic limit this relation connects the free energy per site of the M-periodic chains, $M > 1$, to the one of the $M = 1$ case for renormalized discrete temperature T/μ^2 and renormalized magnetic field H/μ:

$$M f_I^{(M)}(q^M, T, H) - E_0 \to f_I^{(1)}(q^M, T/\mu^2, H/\mu),$$

where we assume that $\mu \neq 0$.

Unfortunately, this trick with the renormalization of the discrete temperature does not work for our Ising chains in its simplest realizations. The first problem appears from the fact that if for some fixed i and j one takes the limit $k_i \to k_j$ in the KdV case or $a_i \to a_j$ (or $b_i \to b_j$) for the BKP equation, then $J_{ij} \to +\infty$, i.e. at short distances the interactions are repulsive (antiferromagnetic). This orders the neighboring spins in the opposite directions so that μ is small. If one puts $a_i \approx -b_{i+1}$ in the BKP case and keeps a_i as free parameters, then $J_{i,i+1} \to -\infty$ and the spins are parallel. However, in this situation the condition $J_{Mk+l,Mk'+l'} \approx J_{Mk,Mk'}$ for $k \neq k'$ is not satisfied and the effective renormalization of the interaction constants by the M^2 factor does not take place.

7 Connections with the random matrix theory

Let us recall briefly some basic notions from the theory of random matrices. The random matrix method was first employed in the study of complex systems with a large number of degrees of freedom. As a consequence of complexity, the density of levels is high enough for sufficiently high excitation energies and can be described statistically. Since one is interested in the discrete part of the spectrum, the hamiltonian may be approximately reduced to a matrix form. Because interactions in such systems are complicated, entries of the hamiltonian matrix are unknown. The basic hyphothesis by Wigner and Dyson was that statistical characteristics of such systems can be described by averaging over ensembles of random matrices, provided the probability distributions are invariant under basic symmetry transformations such as parity, rotation and time-reversal transformations.

Historically, the Gaussian (or linear) ensembles were considered first [24]. It was found that if H is a real symmetric, Hermitian or self-dual Hermitian random matrix with the statistically independent elements H_{ik}, then the probability density $P(H)$ of the matrix elements to lie in a unit volume is proportional to

$$\exp\left(-a \operatorname{tr} H^2 + b \operatorname{tr} H + c\right),$$

and the measure is invariant under orthogonal, unitary or symplectic transformations. The statistical independence of hamiltonian entries is a rather artificial hyphotesis. If one abandons this requirement restricting the freedom too strongly, then the invariant probability density of linear ensembles is

$$e^{Q(\operatorname{tr} H)}, \tag{50}$$

where Q is a polynomial of degree higher than 2.

In order to overcome some drawbacks of the noncompact Gaussian ensembles, Dyson suggested (circular) ensembles of unitary random matrices with the eigenvalues $e^{i\phi_j}$, such that the local distribution of the phases ϕ_i is isomorphic to the distribution of eigenvalues of the hamiltonian. The condition of invariance of the measure under all unitary authomorphisms defines uniquely the probability density. For example, let S be a $n \times n$ unitary matrix with the eigenvalues $\epsilon_j = e^{i\phi_j}$, $j = 1, \ldots, n$. Then there exists a unique (up to a normalization factor) measure which is invariant under the transformations

$$S \to USW, \tag{51}$$

where U, W are arbitrary unitary matrices [24].

An arbitrary unitary matrix S can be represented in the following form

$$S = U^{-1}EU, \tag{52}$$

where the matrix E is diagonal with the diagonal elements equal to ϵ_j and U is a unitary matrix. In this representation, the probability measure invariant under the transformation (51) is

$$\Omega(dS) = \prod_{i<j} |\epsilon_i - \epsilon_j|^2 \, d\phi_1 \ldots d\phi_n \omega(dU). \tag{53}$$

Since U does not depend on ϵ_j, the part of the measure related to U, $\omega(dU)$ can be integrated out and the eigenvalue distribution becomes

$$P d\phi_1 \ldots d\phi_n \propto \prod_{i<j} |\epsilon_i - \epsilon_j|^2 d\phi_1 \ldots d\phi_n.$$

The hypothesis of equal probability for transformed matrices (51) reflects total ignorance of all details of interactions, except of the symmetries of S. This is not completely justified and can be weakened to the condition of equal probability under general unitary transformations, but not under arbitrary left or right translations which are allowed in (51). In this case, the invariant measure is not uniquely defined

$$\mu(dS) = F(S)\Omega(dS), \tag{54}$$

where the weight function $F(S)$ is invariant with respect to (52), i.e. it depends only on the eigenvalues of S. After integrating out of the $\omega(dU)$ part of the measure, the eigenvalue distribution becomes

$$Pd\phi_1 \dots d\phi_n = f(\phi_1, \dots, \phi_n) \prod_{i<j} |\epsilon_i - \epsilon_j|^2 d\phi_1 \dots d\phi_n,$$

where the function f is symmetric under permutation of its arguments.

One of such ensembles has been considered by Gaudin long time ago [8]. In his model

$$F(S) = \det |1 - zA(S)|^{-1} = \prod_{i,k} (1 - z\epsilon_i \epsilon_k^{-1})^{-1},$$

where $A(S)$ is a $n^2 \times n^2$ matrix of ajoint representation of the Lie algebra G_n generating the group of unitary matrices U_n. For a fixed $S \in U_n$ one has

$$A(S) : x \to SxS^{-1}, \qquad x \in G_n,$$

and the eigenvalues of $A(S)$ are $\epsilon_j \epsilon_k^{-1}$, $j, k = 1, \dots, n$. As a result, one gets the following probability law for the eigenvalues

$$Pd\phi_1 \dots d\phi_n \propto \prod_{i<j} \left| \frac{\epsilon_i - \epsilon_j}{\epsilon_i - z\epsilon_j} \right|^2 d\phi_1 \dots d\phi_n, \tag{55}$$

which interpolates between the distribution of the Dyson unitary ensemble ($z = 0$) and the uniform distribution ($z = 1$). This model can be considered also as a circular Coulomb gas with the partition function

$$Z_n \propto \int_0^{2\pi} \dots \int_0^{2\pi} d\phi_1 \dots d\phi_n \exp \left(-\beta \sum_{i<j} V(\phi_i - \phi_j) \right),$$

where β is the inverse temperature and the interaction energy is

$$\beta V(\phi_i - \phi_j) = \ln \left(1 + \frac{\sinh^2 \gamma}{\sin^2((\phi_i - \phi_j)/2)} \right), \quad z = e^{-2\gamma}. \tag{56}$$

Note that the value of β is fixed.

Let us show that the grand partition function of the model (55), (56) can be recovered in the specific infinite soliton (thermodynamic) limit of the τ-function of the KP hierarchy. The corresponding finite soliton solutions provide thus a particular discretization of the model. In the Coulomb gas interpretation this discretization describes a lattice gas on the circle with equal spacing between the sites. Such a model is equivalent to the Ising spin chain on the circle in homogeneous magnetic field, the exchange constants being given by (56). One can recover also the generalized Gaussian unitary ensembles with the probability density (50) in a rational limit of the KP or BKP equations, so that the coefficients of the polynomial $Q(\mathrm{tr}\ H)$ in (50) are defined by linear combinations of the hierarchy "times". The latter case corresponds to the Coulomb gas on the line in a charged background.

The BKP equation soliton phase shifts suggest a unitary matrix model with the following probability law

$$Pd\phi_1 \dots d\phi_n \propto \prod_{i<j} \left| \frac{\epsilon_i - \epsilon_j}{\epsilon_i + \epsilon_j} \right|^2 \left| \frac{\epsilon_i + z\epsilon_j}{\epsilon_i - z\epsilon_j} \right|^2 d\phi_1 \dots d\phi_n, \tag{57}$$

which corresponds to the weight function

$$F(S) = \det \frac{1 + zA(S)}{(1 - zA(S))(1 + A(S))}. \tag{58}$$

However, because $\epsilon_i + \epsilon_j = 0$ for $\phi_j = \phi_i + \pi$, which creates singularities, one has to restrict the region of variables ϕ_j to $[0, \pi]$.

Consider a space of unitary $n \times n$ matrices with the eigenvalues equal to N-th roots of unity. The angles ϕ_j take now the discrete values $\phi_j = 2\pi m_j/N, m_j = 1, \dots, N$. This space is preserved under the unitary transformations (52). The measure becomes semidiscrete: it is continuous in the "dummy" variables $\omega(dU)$ and one takes the sums over ϕ_j instead of the integrals. The "classical" continuous model is recovered for $N \to \infty$, ϕ_j fixed:

$$\left(\frac{2\pi}{N}\right)^n \sum_{m_1=1}^{N} \cdots \sum_{m_n=1}^{N} \underset{N\to\infty}{\longrightarrow} \int_0^{2\pi} d\phi_1 \dots \int_0^{2\pi} d\phi_n. \tag{59}$$

Consider the distribution (55). For matrices of dimension n the partition function is

$$Z_n(N, z) = \left(\frac{2\pi}{N}\right)^n \sum_{m_1=1}^{N} \cdots \sum_{m_n=1}^{N} \prod_{1\le i<j\le n} \left| \frac{\epsilon_i - \epsilon_j}{\epsilon_i/\sqrt{z} - \sqrt{z}\epsilon_j} \right|^2,$$

$$\epsilon_j = \exp\frac{2i\pi m_j}{N}, \qquad z = e^{-2\gamma}.$$

The grand canonical ensemble partition function corresponding to this distribution can be written in the following form

$$Z(z,\theta) = \sum_{n=0}^{N}\frac{Z_n(N,z)e^{\theta n}}{n!} = \sum_{\mu_m=0,1}\exp\left(\sum_{1\le m<k\le N} A_{mk}\mu_m\mu_k + (\theta+\eta)\sum_{m=1}^{N}\mu_m\right), \tag{60}$$

where $\eta = \ln(2\pi/N)$, θ denotes the chemical potential and (cf. with (56))

$$A_{mk} = \ln\frac{\sin^2(\pi(m-k)/N)}{\sin^2(\pi(m-k)/N)+\sinh^2\gamma} = \ln\frac{(a_m-a_k)(b_m-b_k)}{(a_m+b_k)(b_m+a_k)},$$

is the KP phase shift with the following identification of parameters

$$a_m = e^{2i\pi m/N}, \qquad b_m = -za_m, \quad m = 1,2,\ldots,N. \tag{61}$$

From the comparison of (5), (10) with (60) it follows, that the grand partition function of this matrix model is nothing else than the N-soliton τ-function at zero KP hierarchy "times". In the thermodynamical limit $N\to\infty$ the relation (59) takes place and we get the Gaudin matrix model.

The grand partition function of the matrix model (57) inspired by the BKP equation is also given by (60) with

$$A_{mk} = \ln\frac{(a_m-a_k)(b_m-b_k)(a_m-b_k)(b_m-a_k)}{(a_m+a_k)(b_m+b_k)(a_m+b_k)(b_m+a_k)}.$$

In order to escape singularities the parameters a_m, b_m have to be restricted now. The choice (61) is permissible only for odd N, which makes the $N\to\infty$ limit ill defined. One can replace also m/N in (61) by $m/2N$, i.e. to consider only the half-circle.

As was discussed in the Section 2, these lattice gas models are equivalent to the Ising model after the replacement $\mu_m = (\sigma_m+1)/2$ and the identification of the phase shifts and the exchange constants $\beta J_{mk} = -A_{mk}/4$. The relation between the magnetic field and the chemical potential is

$$\beta H_m = \frac{1}{2}(\theta+\eta) + \frac{1}{4}\sum_{k=1,k\neq m}^{N} A_{mk}.$$

Evidently, the exchange constants are rotationally invariant, $J_{mk} = J(m-k)$:

$$\beta J(m) = -\frac{1}{4} \ln \frac{1 + \sinh^2 \gamma / \cos^2(\pi m/N)}{1 + \sinh^2 \gamma / \sin^2(\pi m/N)}.$$

In order to calculate the partition function one can employ the determinant (pfaffian) representation (39) from the section 5.

In a conclusion of this section, let us make few remarks on the hermitian random matrices. The linear unitary ensembles are ensembles of such matrices with the distribution function invariant under any unitary transformation. If we abandon again the hyphotesis of the statistical independence of the matrix elements, we arrive at the distributions with the measure of the form (54) and the degree of the polynomial Q in the exponent (50) may be arbitrarily large. Now, if we choose soliton amplitudes as in (41) with real $q = \exp(r/N)$, $r = const$, and take the KP or BKP hierarchy "times" to be different from zero we get discrete models with nontrivial $Q(\mathrm{tr}\, H)$ in (50).

After taking the thermodynamical limits $N \to \infty$, or $q \to 1$, we get the distributions of the form (50). The partition function can be represented in the determinant form, but the fourier transformed matrices seem to be not diagonal because of the nonzero hierarchy "times" and calculation of the partition function becomes more complicated. We postpone the detailed analysis of the Gaudin and BKP versions of the circular ensembles as well as other ensembles of random matrices and their relations to the Ising chains to a later publication.

8 Conclusions

In this work we were considering only the 1D Ising chains with some non-nearest neighbor exchange between spins. We were able to calculate partition functions exactly for translationally invariant spin chains with an arbitrary period M in a periodic magnetic field of the same period M. An interesting problem which we did not address is the calculation of the correlation functions for spin operators.

It would be appropriate to try to extend our considerations to other nonlinear integrable equations admitting N-soliton solutions. It is worth to consider the possibilities to build higher dimensional solvable lattice models on the basis of the results of this work. Vice versa, it would be appropriate to analyze what kind of nonlinear partial differential equations could be associated to known exactly solvable lattice models [1].

A curious point concerning the quantization of the temperature requires a deeper understanding. It is well known [24] that orthogonal, unitary and symplectic ensembles of random matrices correspond to particular quantized values of the inverse temperature β within the Coulomb gas model interpretation. However, the corresponding partition function is calculable exactly for arbitrary β, being determined by the Selberg integral. A similar result for the considered Ising chains would be of great interest.

As further extenstions of the results of this work, one can consider the M-periodic Ising chains emerging from the KP and BKP equations and analyze the corresponding matrix models. Purely discrete nonlinear integrable equations like the discrete-time and space Toda, KP and other equations worth of analysis from the Ising chains viewpoint as well. However, the corresponding spectral self-similarity [25] does not lead to the translationally invariant spin lattices. As a result, the partition functions are not related to the determinats of Toeplitz matrices and our methods seem not to apply in this cases in a direct fashion.

9 Acknowledgments

The authors are indebted to Yu. Berest, R. Hirota, V. Inozemtsev, K. Kajiwara, A. Orlov, A. Zhedanov for attention to the work and to T. Shiota, C. Tracy for stimulating discussions at its early stages. The results of this paper were reported at the third workshop "Symmetries and integrability of difference equations" (Sabaudia, Italy, 16-22 May, 1998). This work is supported in part by

NSERC (Canada), RFBR (Russia) grant No. 97-01-01041 and by INTAS-96-0700.

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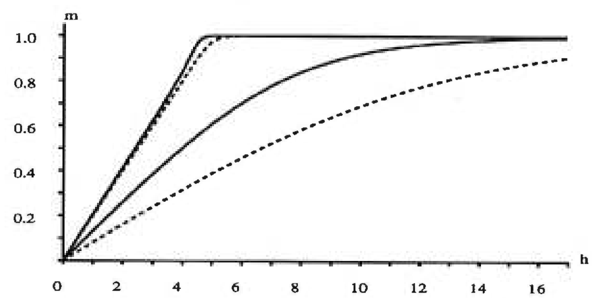

Figure 1: Dependence of the magnetization $m(h)$ on the renormalized magnetic field $\beta H = h/(q^{-1} - q)$ for the KdV equation inspired Ising model. The dashed lines correspond to the discrete temperature $\beta = 1$ and the solid lines to second value $\beta = 2$ appearing from the BKP equation at $a_i = b_i$. The lower curves correspond to the $q = 0.1$ values of the basic parameter and the upper ones to $q = 0.8$.

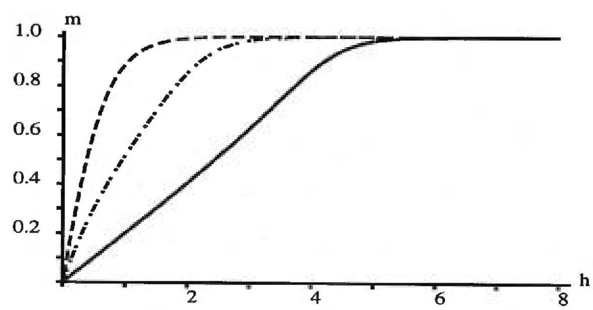

Figure 2: Magnetization $m(h)$ for the general Ising model generated by the BKP equation. In order to match with the KdV case the magnetic field is renormalized to $\beta H = 2h/(q^{-1} - q)$. For all three curves $q = 0.5$. The solid line corresponds to the antiferromagnet with $b = 0.8$, the dashed line — to the ferromagnet with $b = -0.8$, and the dashed-dotted line is for the curious antiferromagnetic case $b = i$.

Chapter 3

Solitons and exciton superfluidity

Critical Velocities in Exciton Superfluidity

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Abstract

The presence of exciton phonon interactions is shown to play a key role in the exciton superfluidity. It turns out that there are essentially two critical velocities in the theory. Within the range of these velocities the condensate can exist only as a bright soliton. The excitation spectrum and differential equations for the wave function of this condensate are derived.

PACS numbers: 05.30.Jp, 63.20.Ls, 64.60.Ht, 67.40.Db

The problem of critical velocities in the theory of superfluidity arose a long time ago when the experiments with the liquid He showed a substantial discrepancy with quantum-mechanical predictions. Later, the effect was analyzed and its phenomenological description was given (e.g. see [1]). The fact that the liquid He could not be treated as a weakly non-ideal Bose gas was believed to be the main reason for inconsistency of microscopic theory with experimental data.

For a long time He has been the only substance where the superfluidity can be observed. The recent experiments with the dilute gas of excitons [2], [3] provide new possibilities for studying different types of superfluidity.

In this series of experiments the Cu_2O crystall was irradiated with laser light pulses of several ns duration. At low intensities of the laser beam (low concentration of excitons) the system revealed a typical diffusive behavior of exciton gas. Once the intensity of the beam exceeds some value, the majority of particles move together in the packet. Their common propagation velocity is close to the longitudinal sound velocity, and the packet evolves as a bright soliton.

Some alternative explanations of the phenomena are known. One of them [2] implies that the bright soliton is a one-dimensional traveling wave which satisfies the Gross-Pitaevsky (nonlinear Schrödinger) equation [4] for the Bose-condensate wave function $\Psi(\mathbf{x}, t)$

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi + \nu\Psi^*\Psi^2 \tag{1}$$

with attractive potential of exciton-exciton interaction $\nu < 0$.

A quantitative treatement given in [5] provides an iterative solution for the Heisenberg equation with the use of perturbational methods. In this picture the second order interactions, neglected in the Bogoliubov approximation, contribute to the negative value of ν. However, the influence of exciton-phonon interactions on the dynamics of the condensed excitons is not treated [5].

Another interpretation is based on a classical model [6] where the normal exciton gas is pushed towards the interior of a sample by the phonon wind emanating from the surface. Such an explanation seems to be in discrepancy with the experiment, because the signal observed is one order of magnitude longer than the excitation pulse duration [3].

In this study we give an alternative and, in our opinion, more intrinsic interpretation of these phenomena. We argue that it is a propagation of a superfluid *exciton-phonon condensate* which is observed experimentally. The presence of exciton-phonon interactions is crucial for a "soliton-like superfluidity". This interaction plays a key role when the propagation velocity approaches the longitudional sound velocity.

We start with the Hamiltonian of the exciton-phonon system

$$H = H_{\text{ex}} + H_{\text{ph}} + H_{\text{int}},$$

$$H_{\text{ex}} = -\frac{\hbar^2}{2m}\int \hat{\Psi}^*(\mathbf{x})\Delta\hat{\Psi}(\mathbf{x})d\mathbf{x} + \frac{1}{2}\int \hat{\Psi}^*(\mathbf{x})\hat{\Psi}^*(\mathbf{y})\,\nu(\mathbf{x}-\mathbf{y})\,\hat{\Psi}(\mathbf{x})\hat{\Psi}(\mathbf{y})d\mathbf{x}d\mathbf{y},$$

$$H_{\text{ph}} = \int \left\{\frac{1}{2\rho}\hat{\boldsymbol{\pi}}(\mathbf{x})^2 + \frac{c^2\rho}{2}\left(\nabla\,\hat{\mathbf{u}}(\mathbf{x})\right)^2\right\} d\mathbf{x},$$

$$H_{\text{int}} = \int \sigma(x-y)\hat{\Psi}^*(\mathbf{x})\hat{\Psi}(\mathbf{x})\left(\nabla\,\hat{\mathbf{u}}(\mathbf{y})\right) d\mathbf{x}d\mathbf{y}, \tag{2}$$

where $\hat{\Psi}$ and $\hat{\mathbf{u}}$ are the operators of the exciton and phonon fields correspondingly, c is the longitudional sound velocity and ρ denotes the mass density of the crystal. The field variables obey the following commutation relations

$$\left[\hat{\Psi}(\mathbf{x}), \hat{\Psi}^*(\mathbf{y})\right] = \hbar\delta(\mathbf{x}-\mathbf{y}), \quad [\hat{\pi}_i(\mathbf{x}), \hat{u}_j(\mathbf{y})] = -i\hbar\delta_{ij}\delta(\mathbf{x}-\mathbf{y}),\ i,j = 1,2,3.$$

In (2) we omit the terms with the transverse sound velocity, since the interaction of excitons with transverse sound waves is much weaker than with the longitudional ones.

It is convenient to change the reference system when we consider a uniform motion of the Bose gas. The transition to the reference system moving uniformly with the velocity $\mathbf{v} = (v, 0, 0)$ is immediate. In new coordinates the classical field equations become:

$$\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\Delta + \frac{mv^2}{2} - \int \nu(\mathbf{x}-\mathbf{y})|\psi(\mathbf{y},t)|^2 dy^3\right)\psi(\mathbf{x},t)$$

$$= \psi(\mathbf{x},t) \int \sigma(\mathbf{x}-\mathbf{y})\, (\nabla \mathbf{u}(\mathbf{y},t))\, \mathrm{d}\mathbf{y} \tag{3}$$

$$\left(\frac{\partial^2}{\partial t^2} - 2v\frac{\partial^2}{\partial t \partial x_1} + v^2\frac{\partial^2}{\partial x_1^2} - c^2\Delta\right) \mathbf{u}(\mathbf{x},t) = \frac{1}{\rho}\nabla \int \sigma(\mathbf{x}-\mathbf{y})|\psi(\mathbf{y},t)|^2 \mathrm{d}\mathbf{y}, \tag{4}$$

where $\psi(\mathbf{x},t) = \Psi(x_1+vt, x_2, x_3, t)\exp\left(-imvx_1/\hbar\right)$.

The l.h.s. of Equation (3) is Galileian invariant, while the l.h.s. of (4) is Lorentz invariant. As a result, the system (3), (4) is neither Galileian nor Lorentz invariant. As we will see later, it is due to this noninvariance that the effective potential of exciton-exciton interactions depends on velocity.

Let us consider slowly varying solutions of the system (3), (4). In this (long wavelength) limit one can replace $\nu(\mathbf{x})$ and $\sigma(\mathbf{x})$ by $\nu_0\delta(\mathbf{x})$ and $\sigma_0\delta(\mathbf{x})$, where $\nu_0\,(>0)$ and σ_0 denote the zero-mode Fourier components of the corresponding potentials.

Solving (4), one can express the bounded at infinity time-independent solution $\mathbf{u}(x)$ in terms of $\psi(\mathbf{x})$. The effective potential of the exciton-exciton interaction is obtained after substituting this expression into (3). The phonon field makes this potential long-range, anisotropic and v-dependent. The potential becomes asymptotically attractive along the $\mathbf{v}$-direction and asymptotically repulsive in directions perpendicular to $\mathbf{v}$. It follows that stability of the corresponding solutions $\psi = \phi(x_1)\exp(-i\omega_0 t)$, $u_i = \delta_{i1}q(x_1)$ is preserved under the one-dimensional reduction of the system (3), (4). The functions $\phi(x_1)$, $q(x_1)$ obey the following equations

$$\left(\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_1^2} - \lambda\right)\phi(x_1) = \left(\nu_0 - \frac{\sigma_0^2}{(c^2-v^2)\rho}\right)\phi(x_1)^3, \quad \lambda = -\hbar\omega_0 - \frac{mv^2}{2} + C\sigma_0, \tag{5}$$

$$\frac{\partial q(x_1)}{\partial x_1} = C - \frac{\sigma_0\phi(x_1)^2}{(c^2-v^2)\rho}, \tag{6}$$

where the integration constant C is fixed by the condition $q \to \text{const}$ as $|x_1| \to \infty$. In the last equations ϕ assumed to be real. This choice does not change the result but simplifies our calculations.

It follows from (5) that the effective potential becomes attractive when v exceeds the critical velocity

$$v_0 = \sqrt{c^2 - (\sigma_0^2/\nu_0\rho)}. \tag{7}$$

But if v exceeds the sound velocity c, the potential becomes repulsive again. As for the solution varying in the direction $\mathbf{n} = (n_1, n_2, n_3)$, $\psi = f(\mathbf{n}\,\mathbf{x})\exp(-i\omega_0 t)$, $\mathbf{u} = \mathbf{n}q(\mathbf{n}\,\mathbf{x})$, the critical velocity is $v_0(\mathbf{n}) = v_0/\cos(\theta), |\cos(\theta)| > v_0/c$, where θ is the angle between $\mathbf{n}$ and $\mathbf{v}$.

When v is less than the critical velocity (7), equations (5),(5) have the following stable stationary solutions

$$\text{(i)} \quad \phi = \phi_0 = \sqrt{N/V} = \text{const}, \quad \mathbf{u} = \text{const}, \quad \text{and}$$

$$\text{(ii)} \quad \phi = \phi_0 \tanh\left(\beta\phi_0(x_1 - a)\right), \frac{\partial q(x_1)}{\partial x_1} = -\frac{\sigma_0\phi_0^2}{(c^2 - v^2)\rho}\cosh^{-2}\left(\beta\phi_0(x_1 - a)\right), \tag{8}$$

$$\beta = \sqrt{\frac{m\nu}{\hbar^2}\frac{|v_0^2 - v^2|}{|c^2 - v^2|}}, \lambda = \left(\frac{\sigma_0^2}{(c^2 - v^2)\rho} - \nu_0\right)\phi_0^2 = \nu_0\phi_0^2\frac{v^2 - v_0^2}{c^2 - v^2}, C = \frac{\sigma_0\phi_0^2}{(c^2 - v^2)\rho}.$$

In (8,i) N and V stand for the number of particles in the condensate and the volume of the system.

When v exceeds v_0, we have only one stable stationary solution

$$\phi = \phi_0\cosh^{-1}\left(\beta\phi_0(x_1 - a)\right), \quad \frac{\partial q(x_1)}{\partial x_1} = -\frac{\sigma_0\phi_0^2}{(c^2 - v^2)\rho}\cosh^{-2}\left(\beta\phi_0(x_1 - a)\right),$$

$$\lambda = \frac{\phi_0^2}{2}\left(\frac{\sigma_0^2}{(c^2 - v^2)\rho} - \nu_0\right) = \frac{\nu_0\phi_0^2}{2}\frac{v^2 - v_0^2}{c^2 - v^2}, \quad C = 0. \tag{9}$$

To find the excitation spectrum of the system we expand the field operators near the proper classical solutions:

$$\hat{\psi}(\mathbf{x}, t) = \left(\phi(x_1) + \hat{\chi}(\mathbf{x}, t)\right)e^{-i\omega_0 t}, \quad \hat{u}_i(\mathbf{x}, t) = \delta_{i1}q(x_1) + \hat{\eta}_i(\mathbf{x}, t).$$

The Hamiltonian of the system can be written as follows

$$H = H_0 + \hbar H_2 + \ldots, \tag{10}$$

where $H_0 = H(\phi e^{-i\omega_0 t}, q)$ stands for the classical part of H. It is important that H_2 is bilinear in $\hat{\chi}(\mathbf{x}, t)$, $\hat{\eta}(\mathbf{x}, t)$, whereas the linear terms are absent in (10) (since the classical fields satisfy the stationary equations (5),(6)). From now on we are working in quasiclassical approximation and neglecting the terms of power greater than one (in $\hbar$).

The quasiclassical Hamiltonian (10) is reduced to the normal form

$$H_2 = \sum_i \omega_i \hat{b}_i^* \hat{b}_i + \text{const}, \quad \left[\hat{b}_i, \hat{b}_j^*\right] = \delta_{ij}, \left[\hat{b}_i, \hat{b}_j\right] = 0. \tag{11}$$

Indeed, since H_2 is a bilinear function of $\hat{\chi}$, $\hat{\eta}$, the equations of motion are linear in field operators. They coincide with the corresponding classical equations (i.e. equations (3),(4) linearized around $\psi(\mathbf{x}, t) = \phi(x_1)\exp(-i\omega_0 t)$, $u_i(\mathbf{x}, t) = \delta_{i1} q(x_1)$):

$$\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\Delta - \lambda + C\sigma_0 + \left\{ \frac{\sigma_0^2}{(c^2 - v^2)\rho} - 2\nu_0 \right\} \phi(x)^2 \right) \chi$$

$$-\nu_0 \phi(x)^2 \chi^* - \sigma_0 \phi(x) (\nabla\, \boldsymbol{\eta}) = 0, \tag{12}$$

$$\left(c^2\Delta - v^2\frac{\partial^2}{\partial x_1^2} + 2v\frac{\partial^2}{\partial t \partial x_1} - \frac{\partial^2}{\partial t^2} \right) \boldsymbol{\eta} + \frac{\sigma_0}{\rho} \nabla \left(\phi(x)(\chi + \chi^*) \right) = 0. \tag{13}$$

The quantities ω_i in (11) are characteristic frequencies of the system (12), (13).

Let us consider the homogeneous Bose gas moving uniformly with velocity $v < v_0$. The condensate wave function is given by (8,i). The differential equations (12),(13) have constant coefficients so that the characteristic frequencies $\omega(\mathbf{k})$ are determined as roots of the following characteristic polynomial

$$(\Omega^2 - c^2 k^2) \left[\hbar^2 (\Omega + v k_1)^2 - \frac{\hbar^2 k^2}{2m} \left(\frac{\hbar^2 k^2}{2m} + \left\{ 2\nu_0 - \frac{\sigma_0^2}{(c^2 - v^2)\rho} \right\} \phi_0^2 \right) \right] -$$

$$\frac{\hbar^2 k^2}{2m} \frac{\sigma_0^2 \phi_0^2 k^2}{\rho} = 0, \tag{14}$$

where $\Omega = \omega(\mathbf{k}) - v k_1$ are the excitation frequencies in the crystal reference frame. In the limit $\sigma_0 \to 0$ one gets the Bogoliubov [7] spectrum $\hbar\omega(k) = \sqrt{\frac{\hbar^2 k^2}{2m} \left(\frac{\hbar^2 k^2}{2m} + 2\nu_0 \phi_0^2 \right)}$ for the exciton gas as well as the free phonon spectrum

$\Omega = ck$. When we switch on an exciton-phonon interaction, the spectrum $\omega(\mathbf{k})$ becomes $\mathbf{v}$-dependent.

The quantization near the translationally noninvariant classical solution (8,ii) in the region $v < v_0$ yields the same continuous spectrum $\omega(\mathbf{k})$. The only new feature is that there appears a bounded state at $\omega = 0$ in the $\mathbf{v}$-direction. This fact has a simple explanation: the family of the solutions (8,ii) contains an arbitrary translation parameter a, which, in fact, is a collective coordinate. Differentiation of (8,ii) with respect to a gives then necessary time independent solution of (12),(13). This bounded state does not affect the quasiclassical excitation spectrum and contributes only to highest approximations (e.g. see [8]).

If the velocity v exceeds (7), the characteristic polynomial (14) has complex roots and there is no stable constant solutions. The condensate (i.e. classical) wave function turns into the (bright) soliton (9) of the one-dimensional nonlinear Schrödinger equation (5). This solution decreases exponentially. This allows us to obtain the continuous spectrum from asymptotics of (12), (13). We have

$$\hbar\omega(\mathbf{k}) = \lambda + \frac{\hbar^2 k^2}{2m}$$

for the exciton branch of the model, and

$$\omega(\mathbf{k}) = ck + vk_1$$

for the phonon branch. As in the previous case we get a bounded state at zero energy. We skip the question of existence of other bound states, since it is not essential for our purposes.

The spectrum now has a *gap* in the exciton branch which is equal to λ. In a sense, the situation is similar to the BCS theory: the exciton-phonon interaction makes the effective exciton-exciton potential attractive, and the excitation spectrum acquires a gap.

The transition to the ballistic regime is accompanied by the symmetry breakdown: a new condensate wave function (9) is no more translationally invariant.

However, it contains a free translation parameter. We can interprete this as a phase transition of the second order.

The value ϕ_0 is readily computed from the normalization condition $\int \phi(\mathbf{x})^2 d\mathbf{x} = N$, and λ is then obtained from (9)

$$\lambda = \frac{m\nu_0^2 N^2}{8\hbar^2 S^2}\left(\frac{v^2 - v_0^2}{c^2 - v^2}\right)^2 . \tag{15}$$

In (15) S denotes the packet cross-section in x_2x_3-plane. When v approaches the longitudional sound velocity c, the gap magnitude increases and soliton becomes more stable. The soliton energy can be estimated from (2)

$$E = N\left\{\frac{m\nu_0^2 N^2}{24\hbar^2 S^2}\frac{(v^2 - v_0^2)}{(c^2 - v^2)^3}\left(v^4 + 3v^2c^2 + v_0^2c^2 - 5v_0^2v^2\right) + \frac{mv^2}{2}\right\} + \ldots .$$

It follows from the last formula that $E \to \infty$ as $v \to c$. Roughly speaking, the soliton effective mass tends to infinity when its speed approaches the longitudional sound velocity. Then its motion is less subjected to the external forces.

The onset of ballistical regime is determined by the condition $v > v_0$. It is easy to see that the solution (9) is the most stable in the class of one-dimensional traveling waves moving uniformly with given $v(> v_0)$ and N. We argue that (9) is also the most stable solution in the class of all solutions with given $v(> v_0)$ and N, because the effective exciton-exciton potential is attractive in $\mathbf{v}$-direction and repulsive in the perpendecular directions. We would like to stress that effective one-dimensional solutions of three-dimensional nonlinear Schrödinger equations (1) with attractive potentials do not have the similar properties. In particular, the stability of such solutions is doubtful [9].

In the present work we have discussed the properties of the system at zero temperature. The extension of our results to finite temperatures seems to be a more difficult problem.

We hope that the similar approach (involving solitonic mechanisms) can be applied to the solution of the general problem of critical velocities in the superfluidity of liquid helium.

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Conclusion

I would like to mention some questions which have not been discussed in the thesis as well as some open problems.

We discussed the relation between soliton solutions of the KdV equation and Huygens' principle in the first part of the work. It is worth mentioning that there are relations between Huygens' principle and integrability. According to A.P. Veselov, the Schrödinger counterparts of Huygens' operators (i.e. Laplacian plus a Huygens' potential) possess the property of algebraic integrability. Recall, that a Schrödinger operator in an n dimensional space is integrable, if there exist a ring generated by n pairwise commuting differential operators with algebraically independent constant highest symbols.

On the one hand, the Schrödinger operator is called algebraically integrable, if there exists at least one more operator, which commutes with all the ring, such that its highest symbol takes different values at the intersection of the zeros of the highest symbols of other ring generators. Thus the Huygens systems describe the overcomplete rings of commuting differential operators.

On the other hand, according to J.L. Burchnall and T.W. Chaundy, the problem of the classification of the rings of commuting differential operators is related to the bispectral problem. The bispectral problem in one dimension has been considered by J.J. Duistermat and F.A. Grünbaum The Huygens operators provide examples of bispectral systems in higher dimensions.

The problem of describing all bispectral operators in arbitrary dimensions is very hard. Howevwr, it is beleived that this problem can be solved in two dimensions. This is for future studies.

Another open question is the problem of the generalization of the examples of O.A. Chalych and A.P. Veselov in dimensions higher than two. In this respect, the multidimensional generalizations of the Burchnall-Chaundy equations could be of help.

In the investigations that led to the second part of the thesis, we faced the question about links between the Huygens systems and the Ising models related to the integrable hierarchies. Performing computer experiments, we observed that the partition functions of the Ising systems at particular values of the homogeneous magnetic field are also the τ functions of the Huygens operators. Although such examples exist for all types of regular Weyl root systems, we neither found their generalizations to higher multiplicities nor explained the existing examples.

Note that in the present work we considered only the Ising models in one dimension with some special interaction between spins. It would be appropriate to consider the correspondent models in higher dimensions.

It would be interesting to extend our considerations to other integrable equations admitting soliton solutions.It would be worthy analyzing what kind of equations could be associated with other known solvable models.

The study of relations between the random matrix models and the Ising would also diserve a deeper analysis.

Acknowledgements

This thesis has been completed under direction of professor Luc Vinet. I am grateful to him for his patience, help and good advice. I would like to thank my collaborators Yuri Berest, Vyacheslav Spiridonov and Danila Roubtsov and also Alexey Zhedanov and Alexandr Orlov for helpful discussions.

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