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Multiplicative functions with small partial sums and an estimate of Linnik revisited

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Résumé

Cette thèse se compose de deux projets. Le premier concerne la structure des fonctions multiplicatives dont les moyennes sont petites. En particulier, dans ce projet, nous établissons le comportement moyen des valeurs $f(p)$ de f aux nombres premiers pour des fonctions f multiplicatives appropriées lorsque leurs sommes partielles $\sum_{n\leqslant x} f(n)$ sont plus petites que leur borne supérieure triviale par un facteur d'une puissance de $\log x$. Ce résultat poursuit un travail antérieur de Koukoulopoulos et Soundararajan [12] et il est construit sur des idées provenant du traitement plus soigné de Koukoulopoulos [9] sur le cas special des fonctions multiplicatives bornées.

Le deuxième projet de la thèse est inspiré par un analogue d'une estimation que Linnik a déduit dans sa tentative de prouver son célèbre théorème concernant la taille du plus petit nombre premier d'une progression arithmétique. Cette estimation fournit une formule asymptotique fortement uniforme pour les sommes de la fonction de von Mangoldt Λ sur les progressions arithmétiques. Dans la littérature, ses preuves existantes utilisent des informations non triviales sur les zéros des fonctions L de Dirichlet $L(\cdot,\chi)$ et le but du deuxième projet est de présenter une approche différente, plus élémentaire qui récupère cette estimation en évitant la "langue" de ces zéros. Pour le développement de cette méthode alternative, nous utilisons des idées qui apparaissent dans le grand crible prétentieux (pretentious large sieve) de Granville, Harper et Soundararajan [6]. De plus, comme dans le cas du premier projet, nous empruntons également des idées du travail de Koukoulopoulos [9] sur la structure des fonctions multiplicatives bornées à petites moyennes.

Mots clés: Théorie analytique des nombres, théorie prétentieuse des nombres, fonction multiplicative, sommes partielles, sommes partielles sur nombres premiers, méthode de Landau-Selberg-Delange, théorème inverse, théorème de Linnik, zéro de Siegel, caractère exceptionnel

Abstract

This thesis consists of two projects. The first one is concerned with the structure of multiplicative functions whose averages are small. In particular, in this project, we establish the average behaviour of the prime values $f(p)$ for suitable multiplicative functions f when their partial sums $\sum_{n\leqslant x} f(n)$ admit logarithmic cancellations over their trivial upper bound. This result extends previous related work of Koukoulopoulos and Soundararajan [**12**] and it is built upon ideas coming from the more careful treatment of Koukoulopoulos [**9**] on the special case of bounded multiplicative functions.

The second project of the dissertation is inspired by an analogue of an estimate that Linnik deduced in his attempt to prove his celebrated theorem regarding the size of the smallest prime number of an arithmetic progression. This estimate provides a strongly uniform asymptotic formula for the sums of the von Mangoldt function Λ on arithmetic progressions. In the literature, its existing proofs involve non-trivial information about the zeroes of Dirichlet L-functions $L(\cdot,\chi)$ and the purpose of the second project is to present a different, more elementary approach which recovers this estimate by avoiding the "language" of those zeroes. For the development of this alternative method, we make use of ideas that appear in the pretentious large sieve of Granville, Harper and Soundararajan [**6**]. Moreover, as in the case of the first project, we also borrow insights from the work of Koukoulopoulos [**9**] on the structure of bounded multiplicative functions with small averages.

Keywords: Analytic number theory, pretentious number theory, multiplicative function, partial sums, partial sums over primes, Landau-Selberg-Delange method, converse theorem, Linnik's theorem, Siegel zero, exceptional character

To the memory of my grandparents,
Afroditi and Zisis.

Contents

Notation and definitions

▶ General standard notation and definitions

First we provide a list of some standard notation and definitions that will be used throughout the dissertation.

- For a set $\mathcal{S}$, the symbol $\#\mathcal{S}$ denotes its cardinality.
- In this thesis, the number 0 is not included in the set $\mathbb{N}$, that is $\mathbb{N} = \{1,2,3,\ldots\}$.
- Following a classical notational convention of analytic number theory, the symbol log will be denoting the natural logarithm ln.
- Following another notational convention of analytic number theory, the letter s always denotes a complex number and we write $s = \sigma + it$, where $\sigma = \mathrm{Re}(s)$ and $t = \mathrm{Im}(s)$.
- We denote the set of non-positive integers by $\mathbb{Z}_{\leqslant 0}$.
- Throughout the text, the letter p always denotes a prime number.
- If $a, b \in \mathbb{N}$, we denote their greatest common divisor by (a,b).
- For $(m,n) = 1$, we set $\overline{n}$ to denote the inverse of n modulo m, that is $n\overline{n} \equiv 1 \pmod{m}$.
- In this thesis, $(\mathbb{Z}/q\mathbb{Z})^* = \{1 \leqslant n \leqslant q : (n,q) = 1\}$.
- The notation $p^\nu \| n$ means that $p^\nu \mid n$ but $p^{\nu+1} \nmid n$.
- We denote the smallest prime factor of an integer $n > 1$ by $P^-(n)$. For $n = 1$, we define $P^-(1) = +\infty$.
- A function $f : \mathbb{N} \to \mathbb{C}$ is called *arithmetic*.
- An arithmetic function f for which $f(1) = 1$ and $f(mn) = f(m)f(n)$ whenever $(m,n) = 1$ is called *multiplicative*.
- For any two arithmetic functions f and g, we write $f * g$ for their *Dirichlet convolution* which is defined as $(f * g)(n) := \sum_{ab=n} f(a)g(b)$ for all $n \in \mathbb{N}$.
- For an arithmetic function f with $f(1) \neq 0$, the function f^{-1} will denote the *Dirichlet inverse* of f and it is the unique arithmetic function such that $(f * f^{-1})(n) = (f^{-1} * f)(n) = 1$ for all $n \in \mathbb{N}$.
- For $m \in \mathbb{N}$, the symbol τ_m denotes the *m-fold divisor function* defined as $\tau_m(n) = \sum_{d_1 \cdots d_m = n} 1$ for all $n \in \mathbb{N}$. When $m = 2$, we write $\tau_2 = \tau$ and this function, called *the divisor function*, counts the number of divisors of a positive integer.

- In our text, the arithmetic functions $\mu, \Lambda, \phi, \omega$ and Ω also appear. The *Möbius function* μ is defined as

$$\mu(n) = \begin{cases} (-1)^{\omega(n)} & \text{if } n \text{ is square-free and} \\ 0 & \text{otherwise.} \end{cases}$$

  The *von Mangoldt function* Λ is defined as

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{ for a prime } p \text{ and } m \in \mathbb{N} \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

  The *Euler totient function* ϕ is given as $\phi(n) = \#(\mathbb{Z}/n\mathbb{Z})^*$. Lastly, the functions ω and Ω are defined through the relations $\omega(n) = \sum_{p|n} 1$ and $\Omega(n) = \sum_{p^a \| n} a$.
- The Greek letter χ always denotes a Dirichlet character (see Definition 1.3.1) modulo some positive integer. The symbol χ_0 will alway denote a principal character (see Definition 1.3.2).
- An arithmetic function that is bounded by a divisor function τ_m is called *divisor-bounded*.
- Let $X \subseteq \mathbb{R}$. For $x \in X$, the three expressions $f(x) \ll g(x), f(x) = O(g(x))$ and $g(x) \gg f(x)$ mean that there exists some constant $C > 0$ such that $|f(x)| \leqslant Cg(x)$ for all $x \in X$. If the constant C is not absolute and depends on several parameters, then these parameters are sometimes included as subscripts at the symbols $\ll$, $\gg$ and O.
- The notation $f(x) \asymp g(x)$ indicates that $f(x) \ll g(x) \ll f(x)$.
- We write $f(x) \sim g(x)$ if $\lim_{x\to\infty} f(x)/g(x) = 1$.
- The *Riemann ζ function* will appear some times in the text. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, it is defined as $\zeta(s) = \sum_{n\geqslant 1} n^{-s}$.
- We will denote the Dirichlet series of an arithmetic function f at $s \in \mathbb{C}$ by $L(s,f) = \sum_{n\geqslant 1} f(n)n^{-s}$, provided that the series of the right-hand side converges.
- For $m \in \mathbb{N}$ and $a \in \mathbb{C}$, we write $\binom{a}{m} = \frac{1}{m!} \prod_{m}^{j=1} (a - j + 1)$.

▶ Special notation

For easy reference, we collect here some non-standard notation which appears in the text.

- Given a multiplicative function f, the symbol Λ_f denotes the unique arithmetic function defined through the relation $f \cdot \log = \Lambda_f * f$.

- Let D be a positive integer and let $A > 0$ be a real number. The following two classes of multiplicative functions occur in many places of the text.

$$\mathcal{F}(D) := \{f : \mathbb{N} \to \mathbb{C}, f \text{ multiplicative}, |\Lambda_f| \leqslant D \cdot \Lambda\}$$
$$\mathcal{F}(D,A) := \Big\{f \in \mathcal{F}(D),\ \sum_{n \leqslant x} f(n) \ll \frac{x}{(\log x)^A} \text{ for all } x \geqslant 2\Big\}$$

- For $t \in \mathbb{R}$, we set $V_t := \exp\{100(\log(3 + |t|))^{2/3}(\log\log(3 + |t|))^{1/3}\}$.

List of abbreviations and acronyms

GRH	Generalized Riemann Hypothesis
PNT	Prime Number Theorem

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Introduction

0.1. Multiplicative functions with small partial sums

In analytic number theory there are several results concerning the mean values of multiplicative functions. One of them is the Landau-Selberg-Delange method [3, 14, 15, 22], a powerful tool which provides asymptotics for the partial sums of arithmetic functions f whose Dirichlet series can be written as $\zeta^v \cdot G$, where $v \in \mathbb{C}, \zeta$ is the Riemann zeta function and G satisfies suitable regularity conditions. In the case where f is multiplicative and bounded by the D-fold divisor function τ_D for some fixed positive integer D, Granville and Koukoulopoulos [7] weakened the assumptions of the Landau-Selberg-Delange method and showed that if $f(p) \approx v$ on average, in the sense that

$$\sum_{p \leqslant x} f(p) \log p = vx + O\left(\frac{x}{(\log x)^{D+1-\mathrm{Re}(v)}}\right) \tag{0.1.1}$$

for all $x \geqslant 2$, then

$$\sum_{n \leqslant x} f(n) = \frac{c(f,v)}{\Gamma(v)} x(\log x)^{v-1} + O_{D,v}\left(x(\log x)^{\mathrm{Re}(v)-2}(\log\log x)^{\mathbb{1}_{v=D}}\right), \tag{0.1.2}$$

where Γ is the Gamma function and

$$c(f,v) = \prod_p \left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \cdots\right)\left(1 - \frac{1}{p}\right)^v.$$

Typically, the Euler product $c(f,v)$ is non-zero unless f exhibits a specific behaviour on small primes and their powers. For instance, if $f(2^k) = -1$ for all $k \in \mathbb{N}$, then

$$\sum_{k \geqslant 1} \frac{f(2^k)}{2^k} = -1,$$

which implies that $c(f,v) = 0$. We put such special examples of functions f aside and assume that $c(f,v) \neq 0$. In this case, notice that if $v \notin \mathbb{Z}_{\leqslant 0}$, then the "main term" of (0.1.2) does not vanish and it thus determines the asymptotic behaviour of the partial sums of f. More precisely, it is then true that

$$\left|\sum_{n\leqslant x} f(n)\right| \sim \left|\frac{c(f,v)}{\Gamma(v)}\right| \frac{x}{(\log x)^{1-\mathrm{Re}(v)}} \quad \text{as} \quad x\to\infty. \tag{0.1.3}$$

Since $|f(p)| \leqslant \tau_D(p) = D$, the prime number theorem (Theorem 1.2.3) and condition (0.1.1) imply $|v| \leqslant D$, and so (0.1.3) yields

$$\left|\sum_{n\leqslant x} f(n)\right| \gg \frac{x}{(\log x)^{D+1}}.$$

Hence, if $c(f,v) \neq 0$, then by further assuming that there exists a real number $A > D+1$ such that

$$\sum_{n\leqslant x} f(n) \ll \frac{x}{(\log x)^{A}} \quad \text{for all} \quad x \geqslant 2, \tag{0.1.4}$$

we force v to lie in $\{0, -1, \ldots, -D\}$. We thus see that if $f(p) \approx v$ on average, then the only way that f can satisfy (0.1.4) is when $v \in \{0, -1, \ldots, -D\}$. Now, one might start wondering whether it is possible to draw conclusions about the average size of $f(p)$ when only (0.1.4) holds, that is without (0.1.1).

In 2013, Koukoulopoulos [9] initially addressed this question for $D = 1$ and $A > 2$. Seven years later, in 2020, in joint work with Soundararajan [12], they worked on the case where D is any fixed positive integer and $A > D+2$. Specifically, they proved the following result.

Theorem K-S. *Fix $D \in \mathbb{N}$ and a real number $A > D + 2$. Let also f be a multiplicative function such that $|\Lambda_f| \leqslant D \cdot \Lambda$, where Λ_f is the unique arithmetic function defined through the relation $f \cdot \log = \Lambda_f * f$ and Λ is the von Mangoldt function. Assume further that*

$$\sum_{n\leqslant w} f(n) \ll \frac{w}{(\log w)^{A}} \quad \textit{for all} \quad w \geqslant 2.$$

Then there exists a multiset Γ of m real numbers with $m \leqslant D$ and such that

$$\left|\sum_{p\leqslant x}\Big(f(p) + \sum_{\gamma\in\Gamma} p^{i\gamma}\Big)\log p\right| \leqslant C_1 \cdot \frac{x}{\sqrt{\log x}} + C_2 \cdot \frac{x}{\sqrt{T}}$$

for all $x, T \geqslant 2$, where C_1 is a constant depending only on f and T, and C_2 is an absolute constant.

In this theorem, the stronger condition $A > D + 2$ is considered, but Koukoulopoulos and Soundararajan also dealt with the full range $A > D + 1$ in the special case where the continuous extension of the Dirichlet series of f has a single root of multiplicity D on the line $\mathrm{Re}(s) = 1$ [12, p. 12-13].

Theorem K-S II. *Fix a natural number D and a real number $A > D+1$. Let also f be a multiplicative function as the one appearing in the statement of Theorem K-S. If the Dirichlet series $L(\cdot,f)$ of f has a zero $1+i\gamma$ of multiplicity D, then*

$$\sum_{p\leqslant x} |f(p) + Dp^{i\gamma}| \log p \ll_f \frac{x}{(\log x)^{(A-1-D)/2}}.$$

In [21], we extended Theorem K-S in the full range $A > D+1$ without making any assumptions about the zeroes of the Dirichlet series of f, as was done in Theorem K-S II. The extension of Theorem K-S is the main object of study for the first project of the present dissertation. In particular, we will prove the following.

Theorem 1. *Fix a positive integer D and a real number $A > D+1$. If f is a multiplicative function satisfying the conditions of Theorem K-S, then there exists a multiset Γ of m real numbers with $m \leqslant D$ and such that*

$$\Big|\sum_{p\leqslant x}\Big(f(p) + \sum_{\gamma\in\Gamma} p^{i\gamma}\Big)\log p\Big| \leqslant O_{f,T}\left(\frac{x(\log\log x)^{D+m}}{(\log x)^{\min\{1,A-D-1\}/2}}\right) + O_\Gamma\left(\frac{x(\log T)^{D+m}}{\sqrt{T}}\right), \tag{0.1.5}$$

for all $x \geqslant 3$ and any $T \geqslant 2$. The implied constants depend also on D and A.

It turns out that, in both Theorems K-S and 1, the multiset Γ consists of the ordinates of the zeroes of the Dirichlet series of f on $\mathrm{Re}(s) = 1$ and this generalizes what Koukoulopoulos [9] also proved in the case $D = 1$. Moreover, the implied constant in the first big-Oh term at the right-hand side of (0.1.5) could be explicitly given as the maximum of the logarithmic derivative of a Dirichlet series related to f. For the sake of a simpler statement, we chose not to make this dependence explicit here, but it can be easily deduced from the proof of Theorem 1 in Section 3.3.

Now, we can divide both sides of (0.1.5) by x, take the $\limsup$ as x tends to infinity and then let $T \to \infty$. This way, we see clearly that $f(p) \approx -\sum_{\gamma\in\Gamma} p^{i\gamma}$ on average.

Corollary 1. *Under the assumptions of Theorem 1, there exists a multiset Γ of at most D real numbers such that*

$$\lim_{x\to+\infty} \frac{1}{x}\sum_{p\leqslant x}\Big(f(p) + \sum_{\gamma\in\Gamma} p^{i\gamma}\Big)\log p = 0.$$

Classical results in the study of the mean values of multiplicative functions, like Wirsing's theorem [27, 28] and the Landau-Selberg-Delange method [3, 14, 15, 22], require information about the prime values of a multiplicative function and they give us back information about its averages. Theorem 1 does the converse, as does *Koukoulopoulos' converse theorem* [9] from 2013. It uses the fact that the averages of an appropriate multiplicative function are small and it returns information about its structure on primes. Thus, it may be seen as a partial converse to these results.

Remark 1. If the hypothesis $A > D+1$ is replaced by the inequality $A < D+1$, then the prediction of Theorem 1 is false and we can verify this with a counterexample.

Let $\kappa \in \mathbb{R} \setminus \mathbb{Q}$ be such that $A-1 < \kappa < D$ and consider the arithmetic function $\tau_{-\kappa}$ whose values are given by the coefficients of the Dirichlet series of $\zeta^{-\kappa}$, namely

$$\zeta(s)^{-\kappa} = \sum_{n \geqslant 1} \frac{\tau_{-\kappa}(n)}{n^s} \quad \text{for} \quad \mathrm{Re}(s) > 1.$$

As usual, ζ is the Riemann zeta function.

By writting $\zeta(s) = \prod_p (1-p^{-s})^{-1}$ (this is the Euler product of ζ, see [1, p. 231]) and multiplying the Taylor expansions of the terms $(1-p^{-s})^{-\kappa}$, we deduce that $\tau_{-\kappa}$ is multiplicative and that its values on prime powers p^m are

$$\tau_{-\kappa}(p^m) = \binom{m-\kappa-1}{m}.$$

For the D-fold divisor function τ_D, it is known that (see 1.4.1)

$$\tau_D(p^m) = \binom{m+D-1}{m},$$

and so a simple application of the triangle inequality implies that $|\tau_{-\kappa}| \leqslant \tau_D$. Moreover, $\tau_{-\kappa}(p) = -\kappa$ and the prime number theorem (Theorem 1.2.3) guarantees that $\tau_{-\kappa}$ satisfies (0.1.1) with $v = -\kappa$. Therefore, the Koukoulopoulos and Granville variant of the Landau-Selberg-Delange method [7, Theorem 1] leads to the estimate

$$\sum_{n \leqslant x} \tau_{-\kappa}(n) \ll_\kappa x/(\log x)^{\kappa+1} \leqslant x/(\log x)^A \quad \text{for all} \quad x \geqslant 2. \tag{0.1.6}$$

For a multiplicative function f, it is easy to verify that the function Λ_f defined through the relation $f \cdot \log = \Lambda_f * f$ is given by the coefficients of the Dirichlet series of $-(L'/L)(\cdot,f)$, where $L(\cdot,f)$ is the Dirichlet series of f. Hence, $\Lambda_{\tau_{-\kappa}} = -\kappa \cdot \Lambda$, as $-(\zeta'/\zeta)(s) = \sum_{n \geqslant 1} \Lambda(n) n^{-s}$ for $\mathrm{Re}(s) > 1$. Therefore, apart from (0.1.6), we also have that $|\Lambda_{\tau_{-\kappa}}| = \kappa \cdot \Lambda \leqslant D \cdot \Lambda$, and this means that $\tau_{-\kappa}$ would meet the conditions of Theorem 1 with $A < D+1$. Then, there would exist diestinct real numbers $\gamma_1, \ldots, \gamma_k$ and positive integers $m_1, \ldots, m_k$ such that

$$\sum_{p \leqslant x} (m_1 p^{i\gamma_1} + \cdots + m_k p^{i\gamma_k} - \kappa) \log p = o(x).$$

The existence of the numbers $\gamma_1, \ldots, \gamma_k$ is guaranteed by the prime number theorem (Theorem 1.2.3). Indeed, if no such real numbers existed, then we would have $-\kappa x + o(x) =$

$-\kappa\sum_{p\leqslant x}\log p = o(x)$, which is absurd, since $\kappa \neq 0$, as it is an irrational number. Now, for $t \in \mathbb{R}$, the prime number theorem and partial summation (Lemma 1.1.1) give

$$\sum_{p\leqslant x} p^{it}\log p = \frac{x^{1+it}}{1+it} + O\Big(\frac{x}{\log x}\Big). \tag{0.1.7}$$

Consequently, with the change of variables $x = e^u$, we have that

$$\sum_{j=1}^{k} c_j e\Big(u\frac{\gamma_j}{2\pi}\Big) = \kappa + o(1), \tag{0.1.8}$$

where $c_j = m_j/(1+i\gamma_j)$ and $e(w) := e^{2\pi i w}$ for $w \in \mathbb{R}$.

We now show that (0.1.8) leads to a contradiction. Let us first consider the case where all the γ_j's are non-zero. Without loss of generality, we suppose that $\gamma_1,\ldots,\gamma_\ell$ form a maximal linearly independent subset of $\{\gamma_1,\ldots,\gamma_k\}$ over $\mathbb{Q}$. Then γ_j is a $\mathbb{Q}$-linear combination of $\gamma_1,\ldots,\gamma_\ell$ for $j > \ell$. Of course, we can assume that these linear combinations are $\mathbb{Z}$-linear. Indeed, if m is the least common multiple of the denominators from all the $\mathbb{Q}$-linear combinations, then we can switch from the set of γ_j's to that of the γ_j/m's without affecting our arguments.

Now, let $\varepsilon \in (0,1)$ and $\theta_1,\ldots,\theta_\ell$ be arbitrary real numbers. If $\|\cdot\|$ denotes the distance from the closest integer, then by *Dirichlet's simultaneous approximation theorem* there exist integers $q_n \geqslant 1$ such that $\|q_n\frac{\gamma_j}{2\pi}\| < \varepsilon/2^{n+1}$ for all $n \in \mathbb{N}$ and for all $j \in \{1,\ldots,\ell\}$. So, letting $a_n = \sum_{i=1}^n q_i$, we conclude that $\|a_n\frac{\gamma_j}{2\pi}\| \leqslant \sum_{i=1}^n \|q_i\frac{\gamma_j}{2\pi}\| < \sum_{i\geqslant 1}\varepsilon/2^{i+1} = \varepsilon/2$, for all $n \in \mathbb{N}$ and for all $j \in \{1,\ldots,\ell\}$. Moreover, since $\gamma_1,\ldots,\gamma_\ell$ are linearly independent over $\mathbb{Q}$, *Kronecker's approximation theorem* implies that there exists a real number $y > 0$ such that $\|y\frac{\gamma_j}{2\pi} - \theta_j\| < \varepsilon/2$ for all $j \in \{1,\ldots,\ell\}$. We consider the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ with $\alpha_n = a_n + y$ for all $n \in \mathbb{N}$. This sequence diverges to infinity, because a_n is a sum of positive integers. Furthemore, for any $n \in \mathbb{N}$ and $j \in \{1,\ldots,\ell\}$, we have that $\|\alpha_n\frac{\gamma_j}{2\pi} - \theta_j\| \leqslant \|y\frac{\gamma_j}{2\pi} - \theta_j\| + \|a_n\frac{\gamma_j}{2\pi}\| < \varepsilon/2 + \varepsilon/2 = \varepsilon$, and so $e(\alpha_n\frac{\gamma_j}{2\pi}) = e(\theta_j) + O(\varepsilon)$ by Taylor's theorem. We now take $u = \alpha_n$ in (0.1.8), let $n \to \infty$ first and $\varepsilon \to 0$ after. This way we deduce that there exist distinct linear functions $L_j : \mathbb{R}^\ell \to \mathbb{R}$ with integer coefficients such that

$$\sum_{j=1}^{\ell} c_j e(\theta_j) + \sum_{j=\ell+1}^{k} c_j e(L_j(\theta_1,\ldots,\theta_\ell)) = \kappa,$$

for arbitrary real numbers $\theta_1,\ldots,\theta_\ell$. But, $\{e(n_1x_1 + \ldots + n_\ell x_\ell) : n_1,\ldots,n_\ell \in \mathbb{Z}, x_1,\ldots,x_n \in [0,1]\}$ is an orthononormal basis of $L^2([0,1]^\ell)$, and so $\kappa = 0$, which is a contradiction.

Now, if one of the γ_j's was zero, say γ_1, we can follow the above argument with the rest of the γ_j's and with $\kappa* = \kappa - m_1$ in place of κ. In this case we reach a contradiction, because we show that $\kappa* = 0$. This implies that $\kappa = m_1 \in \mathbb{N}$, also a contradiction, as κ is irrational.

Remark 2. Note that, like in Theorem K-S, in Theorem 1 one can restrict the sum $\sum_{\gamma\in\Gamma} p^{i\gamma}$ to the ordinates $|\gamma| \leqslant T$. Indeed, for $|\gamma| > T$, by the formula (0.1.7), we have that

$$\sum_{p\leqslant x} p^{i\gamma} \log p = \frac{x^{1+i\gamma}}{1+i\gamma} + O\Big(\frac{x}{\log x}\Big) \ll \frac{x}{T} + \frac{x}{\log x}.$$

We prove Theorem 1 in Chapter 3. In the first section of this chapter, Section 3.1, we explain why the condition $A > D+2$ of Theorem K-S is essential for the proof of Koukoulopoulos and Soundararajan to work. Then we describe the different method that we will use to avoid the technical details which lead to the requirement of the stronger assumption $A > D+2$. We basically develop an approach which involves lower derivatives of $L(\cdot,f)$, as the presence of higher derivatives in the work of Koukoulopoulos and Soundararajan is responsible for the range $A > D+2$. The approach is based on ideas and techniques from the work of Koukoulopoulos [9] in the case $D = 1$ and $A > 2$.

0.2. Primes in arithmetic progressions: An estimate of Linnik revisited

Prime numbers have been fascinating the human mind for thousands of years. A careful glance at the list of the first primes gives the impression that there is some irregularity, some randomness in the way they spawn among the other positive integers. As we go through the list of the primes, it becomes apparent that it is not very easy to predict where the next prime is going to land, if it does at all. At this point, one might ask if prime numbers will keep appearing forever. The answer to this question is known since the time of Euclid who proved that there are infinitely many primes. Now, let us take this question one step further. What can be said about the number of primes in other sets of integers? For example, we could ask if there exist infinitely many primes in an arithmetic progression or if there are infinitely many primes of the form n^2+1. Some of these questions remain unanswered till to this day. For example, it is still unknown whether there are infinitely many primes of the form n^2+1. In fact, this is a famous open problem which was first introduced at the 1912 International Mathematics Conference by Landau and it is usually referred to as Landau's fourth problem. As for the number of primes in a given arithmetic progression $\{qn+a, n\in\mathbb{N}\}$ with a,q coprime, almost 2 millenia after Euclid, in 1837, Dirichlet [4] proved that the number of primes in the arithmetic progression $a \pmod q$ is infinite. Consequently, if a and q are two coprime positive integers, then there is a prime which occurs first in the arithmetic progression $a \pmod q$. Let us denote this prime by $p(q,a)$.

An interesting problem regarding $p(q,a)$ asks how far it will make its appearance in the arithmetic progression $a \pmod q$ and the goal is to look for bounds on $p(q,a)$ which depend on the modulus q. Such bounds can be obtained by examining when the sums

$$\theta(x;q,a) := \sum_{\substack{p \leqslant x \\ p \equiv a \,(\mathrm{mod}\, q)}} \log p$$

become positive. Indeed, since $\log p > 0$ for all primes p, these sums are positive when they contain at least one term which will correspond to a prime $p \equiv a \,(\mathrm{mod}\, q)$. Consequently, if $\theta(x;q,a) > 0$ for $x = h(q)$, where h is some function of q, then $p(q,a) \leqslant h(q)$.

If $q \leqslant (\log x)^A$ for some $A > 0$, the Siegel-Walfisz theorem [19, Corollary 11.21, p. 382] states that there is an absolute constant $c > 0$ such that

$$\theta(x;q,a) = \frac{x}{\phi(q)} + O\Big(x \exp\big\{-c\sqrt{\log x}\big\}\Big),$$

where ϕ is the Euler totient function and the implied constant is ineffective. In this case, for $\varepsilon > 0$, it follows that $\theta(x;q,a) > 0$ for $x \asymp \exp(q^\varepsilon)$, and so the above method implies that

$$p(q,a) \ll_\varepsilon \exp(q^\varepsilon).$$

However, under the Grand Riemann Hypothesis (GRH), it can be shown that [26]

$$p(q,a) \ll (\phi(q)\log q)^2.$$

Moreover, the implied constant can be taken to be 1, as was shown in 2015 by Lamzouri, Li and Soundararajan [13].

The unconditional bound provided by the Siegel-Walfisz theorem is exponential in q and this is far from that of GRH or even a large power of q. So, in 1944, it came as a big surprise when Linnik [16, 17] unconditionally proved that there exist positive universal constants C and L such that

$$p(q,a) \leqslant Cq^L$$

for any choice of coprime positive integers a and q. Furthermore, the constants C and L are effectively computable. One can be led to this breakthrough of Linnik, now known as *Linnik's theorem*, by the following estimates [8, Proposition 18.5, p. 441].

Assume that $x \geqslant q^2$. It is known that at most one of the functions $L(\cdot,\chi)$ has a real zero β in its classical zero-free region. This zero is called a *Siegel zero*, and if it does not exist, then

$$\sum_{\substack{n \leqslant x \\ n \equiv a \,(\mathrm{mod}\, q)}} \Lambda(n) = \frac{x}{\phi(q)} + O\left(\frac{x^{1-c_0/\log(2q)}}{\phi(q)} + \frac{x\log q}{q\phi(q)}\right),$$

where Λ is the von Mangoldt function. However, if the Siegel zero β exists, then

$$\sum_{\substack{n\leqslant x\\ n\equiv a\,(\mathrm{mod}\,q)}} \Lambda(n) = \frac{x}{\phi(q)} - \frac{\chi_1(a)x^\beta}{\phi(q)\beta} + O\left(\frac{x^{1-c_0'|\log(2(1-\beta)\log(2q))|/\log(2q)}}{\phi(q)} + \frac{x\log q}{q\phi(q)}\right),$$

where χ_1 is the exceptional character mod q for which $L(\beta,\chi_1)=0$. In these two estimates, the positive absolute constants c_0 and c_0' are effectively computable.

If $\sqrt{\log x} \ll \log q$, then at the cost of slightly weakening the error term in the case where a Siegel zero exists, we can compactly rewrite the aforementioned estimates as

$$\sum_{\substack{n\leqslant x\\ n\equiv a\,(\mathrm{mod}\,q)}} \Lambda(n) = \frac{x}{\phi(q)} + \frac{\chi^*(a)}{\phi(q)}\sum_{n\leqslant x}\Lambda(n)\chi^*(n) + O\left(\frac{x^{1-c_1/\log(2q)}}{\phi(q)} + \frac{xe^{-c_2\sqrt{\log x}}}{\phi(q)}\right), \qquad (0.2.1)$$

where c_1, c_2 are two positive absolute constants and χ^* is the potential exceptional character. Indeed, by adjusting the classical estimation of $\sum_{n\leqslant x}\Lambda(n)\chi^*(n)$ [19, Theorem 11.16, p. 378] to the range $\sqrt{\log x} \ll \log q$, we deduce that there is a constant $C>0$ such that [19, Exercise 2, p. 382]

$$\sum_{n\leqslant x}\Lambda(n)\chi^*(n) \ll x^{1-C/\log(2q)}$$

if $L(\cdot,\chi^*)$ does not have a Siegel zero, and

$$\sum_{n\leqslant x}\Lambda(n)\chi^*(n) = -\frac{x^\beta}{\beta} + O\left(x^{1-C/\log(2q)}\right)$$

when β is the Siegel zero of $L(\cdot,\chi^*)$. Throughout the text, we will be frequently referring to (0.2.1) as *Linnik's estimate* (since it is related to his celebrated theorem).

Linnik's work on the bound of $p(q,a)$ was later simplified, as was done by Bombieri in [2], but the new proofs, including Linnik's original proof, relied in one form or another on the following three ingredients [8, Principles 1,2 and 3, p. 428]:

- The classical zero-free regions of $L(\cdot,\chi)$ for the characters $\chi(\mathrm{mod}\ q)$;
- A log-free zero-density estimate which is a strong bound on the total number of zeroes of all $L(\cdot,\chi)$ in the rectangles $\{s\in\mathbb{C} : \alpha \leqslant \sigma \leqslant 1, |t| \leqslant T\}$ for $\alpha\in[1/2,1]$ and $T \geqslant 1$;
- The exceptional zero repulsion, also known as the Deuring-Heilbronn phenomenon, stating that it is possible to enlarge the classical zero-free region when it contains a Siegel zero.

Proofs that make use of these three principles can be found in modern treatments, like the one which is presented by Iwaniec and Kowalski in their book [8, Chapter 18].

In the last years, some more elementary proofs that avoid these tools have appeared. For example, in 2002, such a proof was developed by Elliot in [5]. Later, in 2016, Granville, Harper and Soundararajan [6] studied the distribution of multiplicative functions on arithmetic progressions using the so called pretentious methods and as a consequence of their general results, they were able to show a weaker form of (0.2.1) [6, Corollary 1.12]. In turn, this estimate served as the stepping stone for another proof of Linnik's theorem which circumvented the combination of the three aforementioned principles. Another largely elementary proof of Linnik's theorem is presented in [11, Chapter 27] and a basic element of the proof is a flexible variant of (0.2.1) [11, Theorem 27.2, p. 289] where every prime is weighted with $1/p$ instead of $\log p$. Even though the recent alternative approaches recover Linnik's theorem by more elementary means, they do not provide a simpler proof of Linnik's estimate (0.2.1).

This motivated the second project of this thesis, where we infer Linnik's estimate (0.2.1) by adopting a rather elementary approach that we developed in [20]. In particular, we will show the following theorem which is the main result of the currently described project. Note that it slightly improves Linnik's estimate with a Korobov-Vinogradov-type term in place of $x\exp\{-c_2\log x\}$.

Theorem 2. *Let $q \geqslant 1$ be an integer and consider a real number $x \geqslant q^2$. For any character χ mod q, we set*

$$L_q(1,\chi) = \prod_{p>q}(1-\chi(p)/p)^{-1}.$$

We also define $\mathcal{R}_q$ as the set of real, non-principal characters mod q and we take a character ψ such that $L_q(1,\psi) = \min_{\chi\in\mathcal{R}_q} L_q(1,\chi)$. Then, for any $a \in (\mathbb{Z}/q\mathbb{Z})^*$, we have that

$$\sum_{\substack{n\leqslant x\\ n\equiv a \,(\mathrm{mod}\, q)}} \Lambda(n) = \frac{x}{\phi(q)} + \frac{\psi(a)}{\phi(q)}\sum_{n\leqslant x}\psi(n)\Lambda(n) + O\left(\frac{x^{1-C_1/\log(2q)}}{\phi(q)} + \frac{xe^{-C_2(\log x)^{3/5}(\log\log x)^{-3/5}}}{\phi(q)}\right),$$

where C_1 and C_2 are two positive absolute constants.

We may bound the sum $\sum_{n\leqslant x}\psi(n)\Lambda(n)$ in Theorem 2 by referring to Theorem 1.6(a) of [9]. Doing so, it follows that there exist positive constants c' and c'' such that

$$\sum_{n\leqslant x}\psi(n)\Lambda(n) \ll x^{1-c' L_q(1,\psi)/\log(2q)} + xe^{-c''\sqrt{\log x}}, \tag{0.2.2}$$

for all $x \geqslant q^2$. The same theorem also provides information about the size of the quantity $L_q(1,\psi)$ that is involved in the bound above. It is known that there exists a constant $\delta \in (0,1)$ such that $L(\cdot,\psi)$ has at most one zero β in $[1-\delta/\log q, 1)$. If such a zero does not exist, we set $\beta = 1 - \delta/\log q$. In either case, Theorem 1.6(a) of [9] shows that $L_q(1,\psi) \asymp (1-\beta)\log q$. Therefore, we arrive at the following consequence of Theorem 2.

Corollary 2. *Under the considerations and assumptions of Theorem 2, we have that*

$$\sum_{\substack{n\leqslant x\\ n\equiv a\,(\mathrm{mod}\,q)}} \Lambda(n) = \frac{x}{\phi(q)} + O\left(\frac{x^{1-\alpha_2 L_q(1,\psi)/\log(2q)}}{\phi(q)} + \frac{xe^{-\alpha_3\sqrt{\log x}}}{\phi(q)}\right),$$

for some positive constants α_1, α_2 and α_3. Moreover, there exists a constant $\delta \in (0,1)$ such that $L(\cdot,\psi)$ has at most one zero β in $[1-\delta/\log q, 1)$. If such a zero does not exist, we put $\beta = 1-\delta/\log q$ and in any case, $L_q(1,\psi) \asymp (1-\beta)\log q$.

Remark. Note that the term $\phi(q)^{-1}x^{1-C_1/\log(2q)}$ is absorbed by the first fraction in the big-Oh term of Corollary 2, because $L_q(1,\psi) \asymp (1-\beta)\log q \leqslant \delta$.

Remark. The term $x^{1-c' L_q(1,\psi)/\log(2q)}$ in estimate (0.2.2) leads to the first fraction of the big-Oh term in Corollary 2. The size of this fraction might be comparable in size to $x/\phi(q)$. Indeed, there might exist a sequence $\{q_j\}_{j\in\mathbb{N}}$ of moduli q that correspond to zeroes

$$\beta_j > 1 - \frac{\delta_j}{\log q_j},$$

where $\{\delta_j\}_{j\in\mathbb{N}}$ is a sequence of real numbers converging to 0. This means that upon using the fact that $L_{q_j}(1,\psi) \asymp (1-\beta_j)\log q_j < \delta_j$, for $x = {q_j}^A$, no matter how large the exponent A is, the aforementioned fraction is always of size ${q_j}^A/\phi(q_j)$ up to some factors which are positive absolute constants.

Remark. Even though Theorem 2 provides a refinement of Linnik's estimate, our result in not the best to date. In fact, the best error term to date is due to Thorner and Zaman. Their recent work [25, Corollary 1.4] includes a slightly stronger error term where $(\log\log x)^{-3/5}$ is replaced by $(\log\log x)^{-1/5}$. One difference between our work and theirs is the set of methods that are used. Their arguments fall in the realm of the classical approaches, as they are based on non-trivial information about the zeroes of the functions $L(\cdot,\chi)$. Our techniques are more elementary and avoid a "heavy" involvement of those zeroes. Besides that, they achieve an error term which is almost as strong as that of Thorner and Zaman, and in future work we hope to modify our methods to match their error term completely. Another difference between our work and that of Thorner and Zaman lies in the term $\sum_{n\leqslant x}\psi(n)\Lambda(n)$ when ψ is the exceptional character. In order to stay in the lines of the pretentious methods, we only bounded this sum by using Theorem 1.6(a) of [9]. Thorner and Zaman used the explicit formula for it and showed that in the range $x \geqslant q^{12}$, the sums $\sum_{n\leqslant x, n\equiv a\,(\mathrm{mod}\,q)}\Lambda(n)$

are asymptotic to $\lambda x/\phi(q)$, where $\lambda = 1 - \chi_1(a)x^{\beta-1}/\beta$ and β is the corresponding Siegel zero.

We prove Theorem 2 in Chapter 4. Its proof borrows ideas from the pretentious large sieve that Granville, Harper and Soundararjan developed in [6]. It is also inspired by techniques of Koukoulopoulos from his work on bounded multiplicative functions with small partial sums [9]. These are techniques that we use in a similar fashion for the first project.

0.3. Organization of the thesis

Chapters 1 and 2 include all the preliminary results required for the proofs of Theorems 1 and 2. The objective of Chapters 3 and 4 are the proofs of the main theorems of this dissertation. Chapter 3 has three sections. In the first one, we ouline the proof of Theorem 1. The second one is centered around the proof of a single result, that of Proposition 3.2.6. The reason for this is that the estimate of Proposition 3.2.6 will be the main input in the first step of the proof of Theorem 1. In the last section of Chapter 3, Section 3.3, we finally give the proof of Theorem 1. Chapter 4 is basically the proof of Theorem 2 which is broken down into Lemmas 4.2.1, 4.2.2 and 4.2.4. In the beginning of the chapter we also included a short section where we explain the main ideas that lead to the proof of Theorem 2 through these lemmas.

Chapter 1

Background material

In this chapter, we have categorized standard number theory knowledge that is used throughout the text.

1.1. Summation formulas

Partial summation [1, Theorem 4.2, p. 77] is one of the most basic and widely used tools of analytic number theory. Its power lies on the fact that it can provide useful, non-trivial information for sums of the form $\sum_{n\leqslant x} a_n f(n)$ when f is smooth and the partial sums of the complex numbers a_n are well undertood.

Lemma 1.1.1 (Partial summation)**.** *Let $x \geqslant y$ be two positive real numbers. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of complex numbers and define its summatory function as $A(u) = \sum_{n\leqslant u} a_n$ for $u > 0$. If the complex valued function f is continuously differentiable on an open interval containing $(y,x]$, then*

$$\sum_{y<n\leqslant x} a_n f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)\mathrm{d}t.$$

Partial summation can be easily proven using the Riemann-Stieltjes integral. As a matter of fact, sometimes, instead of using partial summation right away, it is more convenient to turn our sum of interest into a Riemann-Stieltjes integral, because we can then estimate the sums by deploying properties of the Riemann-Stieltjes integral, like integration by parts. We will do this in the proof of Lemma 2.1.3.

Apart from partial summation, another useful summation formula is the so called *hyperbola method* [1, Theorem 3.17, p. 69]. It originates from a lattice-counting idea that Dirichlet used in order to estimate the sums $\sum_{n\leqslant x} \tau(n)$ of the divisor function τ up to an error term $O(\sqrt{x})$. In fact, Dirichlet arrived at his estimate by symmetricaly counting the lattice points under the hyperbola $ab = x$ of the (a,b) cartesian plane. This also explains the name of the method. The hyperbola method allows us to estimate sums of Dirichlet convolutions $f * g$ when there is information about the partial sums of f and g.

Lemma 1.1.2 (Dirichlet's hyperbola method)**.** *Let f and g be two arithmetic functions. If $F(u) = \sum_{n \leqslant u} f(n)$ and $G(u) = \sum_{n \leqslant u} g(n)$ for $u > 0$, then, for $x, y > 0$, we have*

$$\sum_{n \leqslant x} (f * g)(n) = \sum_{n \leqslant y} f(n) G\left(\frac{x}{n}\right) + \sum_{n \leqslant x/y} g(n) F\left(\frac{x}{n}\right) - F(y) G\left(\frac{x}{y}\right).$$

1.2. Prime number results

In this section, we record some classical theorems about prime numbers. We begin with four estimates of Mertens [11, Theorem 3.4, p. 40]. The first two are pretty much the same, but written in a different form.

Theorem 1.2.1. *If $\gamma = 1 - \int_1^\infty \{t\} t^{-2} \mathrm{d}t$ is the Euler-Mascheroni constant, then, for $x \geqslant 2$, we have*

(a) $\displaystyle\sum_{n \leqslant x} \frac{\Lambda(n)}{n} = \log x + O(1);$

(b) $\displaystyle\sum_{p \leqslant x} \frac{\log p}{p} = \log x + O(1);$

(c) $\displaystyle\sum_{p \leqslant x} \frac{1}{p} = \log\log x + c + O(1/\log x),$ *where c is a constant*;

(d) $\displaystyle\prod_{p \leqslant x} \left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x}(1 + O(1/\log x)).$

We now define the prime counting function π and Chebyshev's functions θ and ψ. For $x \geqslant 1$, they are given as $\pi(x) = \#\{p \leqslant x\}, \theta(x) = \sum_{p \leqslant x} \log p$ and $\psi(x) = \sum_{n \leqslant x} \Lambda(n)$.

Theorem 1.2.2. *For $x \geqslant 2$, we have that*

(a) $\pi(x) \asymp x/\log x;$

(b) $\psi(x) \asymp x;$

(c) $\psi(x) = \theta(x) + O(\sqrt{x}).$

Proof. For part (a), see [19, Corollary 2.6, p. 49], and for part (b), see [19, Theorem 2.4, p. 46]. Finally, for part (c), the reader is advised to see [19, Corollary 2.5, p. 49]. ☐

The first two parts of Theorem 1.2.2 are known as *Chebyshev's estimates.* However, the *Prime Number Theorem* (PNT) provides strongest distributional information about the primes than these estimates. We state PNT with the best error term to date which is due to Korobov and Vinogradov [8, Corollary 8.30, p. 227].

Theorem 1.2.3 (Prime number theorem)**.** *If $x \geqslant 2$, then there exists an absolute constant $c > 0$ such that*

$$\psi(x) = x + O(x \exp\{-c(\log x)^{3/5}(\log\log x)^{-1/5}\}).$$

This form of PNT was also proved by Koukoulopoulos [10] by more elementary means. So, it may be viewed as a number-theoretic result that can be deduced by simpler methods, the so called *pretentious* methods.

We close the current section with a variant of the Brun-Titchmarsch inequality [11, Theorem 20.1, p. 206] about prime numbers in short arithmetic progressions.

Lemma 1.2.4. *Let $q \geqslant 1$ be an integer. For $a \in (\mathbb{Z}/q\mathbb{Z})^*$ and $x \geqslant y \geqslant 2q\sqrt{x}$, we have that*

$$\sum_{\substack{x-y<n\leqslant x \\ n\equiv a \,(\mathrm{mod}\, q)}} \Lambda(n) \ll \frac{y}{\phi(q)}.$$

Proof. Note that $\Lambda(n) \leqslant \log x$ for $n \leqslant x$. Hence,

$$\sum_{\substack{x-y<n\leqslant x \\ n\equiv a \,(\mathrm{mod}\, q)}} \Lambda(n) \leqslant \log x \sum_{\substack{x-y<p\leqslant x \\ p\equiv a \,(\mathrm{mod}\, q)}} 1 + \sum_{\substack{p^k\leqslant x \\ k\geqslant 2}} \log p. \tag{1.2.1}$$

In virtue of Theorem 1.2.2(c), we have that

$$\sum_{\substack{p^k\leqslant x \\ k\geqslant 2}} \log p = \psi(x) - \theta(x) \ll \sqrt{x} \ll \frac{y}{\phi(q)}. \tag{1.2.2}$$

Now, since $x \geqslant 2q\sqrt{x}$, it follows that $x \geqslant 4q^2$, and so we have that $y \geqslant 2q\sqrt{x} \geqslant \max\{4q^2, 2\sqrt{x}\}$. Thus, $\log(y/q) \geqslant (\log y)/2$ and $\log y \asymp \log x$. Consequently, the Brun-Titchmarsh inequality [11, Theorem 20.1, p. 206] yields

$$\sum_{\substack{x-y<p\leqslant x \\ p\equiv a \,(\mathrm{mod}\, q)}} 1 \ll \frac{y}{\phi(q)\log(y/q)} \ll \frac{y}{\phi(q)\log x}. \tag{1.2.3}$$

We insert (1.2.2) and (1.2.3) in (1.2.1) and this finishes the proof. ☐

1.3. Dirichlet characters

This section is a brief discussion on the notion of *Dirichlet characters*. First we see their definition and then we state one of their basic properties as well as a useful lemma.

Definition 1.3.1. Let $q \geqslant 1$ be an integer. We call a function $\chi : \mathbb{Z} \to \{z \in \mathbb{C} : |z| \leqslant 1\}$ a *Dirichlet character* modulo q if

(a) χ is q-periodic, that is, $\chi(n+q)=\chi(n)$ for all $n\in\mathbb{Z}$;
(b) $\chi(n)=0$ if and only if $(n,q)=1$;
(c) χ is completely multiplicative, meaning that $\chi(mn)=\chi(m)\chi(n)$ for all $m,n\in\mathbb{Z}$.

There are $\phi(q)$ characters modulo q [1, Theorem 6.15, p. 138] and one of them is called *principal*:

Definition 1.3.2. Let $q\geqslant 1$ be an integer. The Dirichlet character χ_0 for which $\chi_0(n)=1$ for all $(n,q)=1$ is called the *principal character* modulo q.

An important property of the Dirichlet characters which detects integers in arithmetic progressions is the following *orthogonality relation* [1, Theorem 6.16, p. 140].

Theorem 1.3.1 (Orthogonality relation)**.** *Let $q\in\mathbb{N}$. For $a\in(\mathbb{Z}/q\mathbb{Z})^*$ and $n\in\mathbb{Z}$, we have*

$$\frac{1}{\phi(q)}\sum_{\chi\,(\mathrm{mod}\,q)}\chi(n)\overline{\chi(a)}=\mathbb{1}_{n\equiv a\,(\mathrm{mod}\,q)}.$$

We now prove a lemma which will be needed multiple times in the proof of Theorem 2.

Lemma 1.3.2 (Parseval's identity)**.** *Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence of complex numbers. If the series $\sum_{n\geqslant 1}\alpha_n\chi(n)$ converges for all $\chi\,(\mathrm{mod}\,q)$, then*

$$\sum_{\chi\,(\mathrm{mod}\,q)}\left|\sum_{n\geqslant 1}\alpha_n\chi(n)\right|^2=\phi(q)\sum_{a\in(\mathbb{Z}/q\mathbb{Z})^*}\left|\sum_{n\equiv a\,(\mathrm{mod}\,q)}\alpha_n\right|^2.$$

Proof. Let N be a positive integer. For any $a\in(\mathbb{Z}/q\mathbb{Z})^*$, Theorem 1.3.1 implies that

$$\sum_{\substack{n\leqslant N\\ n\equiv a\,(\mathrm{mod}\,q)}}\alpha_n=\frac{\overline{\chi(a)}}{\phi(q)}\sum_{n\leqslant N}\alpha_n\chi(n).$$

Consequently, since the series $\sum_{n\geqslant 1}\alpha_n\chi(n)$ converges for all $\chi\,(\mathrm{mod}\,q)$, it follows that the series $\sum_{n\equiv a\,(\mathrm{mod}\,q)}\alpha_n$ converges too.

Now, applying Theorem 1.3.1, we conclude that

$$\begin{aligned}\sum_{\chi\,(\mathrm{mod}\,q)}\left|\sum_{n\leqslant N}\alpha_n\chi(n)\right|^2 &= \sum_{n_1,n_2\leqslant N}\alpha_{n_1}\overline{\alpha_{n_2}}\sum_{\chi\,(\mathrm{mod}\,q)}\chi(n_1)\overline{\chi(n_2)}\\ &= \phi(q)\sum_{\substack{n_1,n_2\leqslant N\\ n_1\equiv n_2\,(\mathrm{mod}\,q)}}\alpha_{n_1}\overline{\alpha_{n_2}}\\ &= \phi(q)\sum_{a\in(\mathbb{Z}/q\mathbb{Z})^*}\Bigg(\sum_{\substack{n_1\leqslant N\\ n_1\equiv a\,(\mathrm{mod}\,q)}}\alpha_{n_1}\Bigg)\overline{\Bigg(\sum_{\substack{n_2\leqslant N\\ n_2\equiv a\,(\mathrm{mod}\,q)}}\alpha_{n_2}\Bigg)}\end{aligned}$$

$$= \phi(q) \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^*} \Bigg| \sum_{\substack{n \leqslant N \\ n \equiv a \,(\mathrm{mod}\, q)}} \alpha_n \Bigg|^2 .$$

We let $N \to \infty$ at the first and last line and this finishes the proof of the lemma. ☐

1.4. Multiplicative functions

A characteristic example of a multiplicative function is the *m-fold divisor function* τ_m, where $m \in \mathbb{N}$. It is defined as $\tau_m(n) = \sum_{d_1 \cdots d_m = n} 1$ for all $n \in \mathbb{N}$, For $n = 2$, we write $\tau_2 = \tau$ and this is *the divisor function* which counts the number of divisors of a positive integer. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the Dirichlet series of τ_m is ζ^m, where ζ is the Riemann zeta function. The values of τ_m on prime powers are (by a reasoning similar to the one that we developed in Remark 1)

$$\tau_m(p^k) = \binom{m+k-1}{k}. \tag{1.4.1}$$

Lemma 1.4.1. *If $m \in \mathbb{N}$, then for all $n \in \mathbb{N}$, we have*

(a) $m^{\omega(n)} \leqslant \tau_m(n) \leqslant m^{\Omega(n)}$;

(b) $\tau_m(n) \ll_{\varepsilon,m} n^{\varepsilon}$ *for any* $\varepsilon > 0$

Proof. *(a)* Since the functions m^{ω}, m^{Ω} and τ_m are positive and multiplicative, it suffices to show that $m \leqslant \tau_m(p^k) \leqslant m^k$ for any prime p and every integer $k \geqslant 1$. Since $\tau_m(p) = m$ from (1.4.1), the inequalities are trivially true when $k = 1$. Now, let us assume that $k \geqslant 2$. Then, note that $m + k - j \geqslant 1 + k - j$ for $j \in \{1, \cdots, k-1\}$. Hence, using (1.4.1), we deduce that

$$\tau_m(p^k) = m \prod_{1 \leqslant j \leqslant k-1} \frac{m+k-j}{1+k-j} \geqslant m.$$

It now remains to show that $\tau_m(p^k) \leqslant m^k$. Since

$$\frac{m+k-j}{1+k-j} = 1 + \frac{m-1}{1+k-j},$$

the fractions of the left-hand side are increasing in j, and so, for $j \leqslant k$, the largest fraction corresponds to $j = k$. Therefore, we obtain that $(m+k-j)/(1+k-j) \leqslant m$, and combining this with (1.4.1) gives

$$\tau_m(p^k) = \prod_{1 \leqslant j \leqslant k} \frac{m+k-j}{1+k-j} \leqslant m^k.$$

This completes the proof for the first part of the lemma.

(b) Let $\varepsilon > 0$. We prove this part by induction on m. The base case is [24, Cororllary 1.1, p. 81] and we will use it with $\varepsilon/2$. Now, assume that $\tau_{m-1}(n) \ll_{\varepsilon,m} n^{\varepsilon/2}$. Upon noting that $\tau_m = \tau_{m-1} * 1$, we deduce that $\tau_m(n) \leqslant \tau(n) \max_{d|n}(\tau_{m-1}(d))$ for all $n \in \mathbb{N}$. So,

$$\tau_m(n) \ll_{\varepsilon,m} \tau(n) \max_{d|n} d^{\varepsilon/2} \ll_{\varepsilon,m} n^{\varepsilon/2} \cdot n^{\varepsilon/2} = n^{\varepsilon},$$

and this concludes the proof of part (b) too. □

For $m = 2$, a stronger inequality than that of Lemma 1.4.1(b) follows from the maximal order of $\log \tau$ [24, Theorem 2, p. 82] which implies that $\max_{n \leqslant x} \log \tau(n) \ll \log x / \log\log x$ for $x \geqslant 2$. If we put this bound together with the inequality $2^{\omega} \leqslant \tau$ (from Lemma 1.4.1(a)), then we infer the following lemma.

Lemma 1.4.2. *For $x \geqslant 2$, we have $\omega(n) \ll \log x / \log\log x$ for all positive integers $n \leqslant x$.*

Asymptotics for the partial sums of τ_m are also known [11, Theorem 7.4, p. 76].

Theorem 1.4.3. *Let $x \geqslant 1$ and let $m \in \mathbb{N}$. There exists a real number $\eta = \eta(m) \in (0,1)$ such that*

$$\sum_{n \leqslant x} \tau_m(n) = x \sum_{i=0}^{m-1} \alpha_{i,m} (\log x)^i + O(x^{1-\eta}),$$

where $\alpha_{i,m}$ are real numbers that depend at most on i and m for $i \in \{0, \cdots, m-1\}$, and $a_{m-1,m} = 1/(m-1)!$.

Apart from all the above results about the divisors functions, let us also recall a classical estimate regarding the Euler totient function ϕ [24, Theorem 4, p. 84].

Lemma 1.4.4. *For a positive integer $n \geqslant 2$, it is true that $\phi(n) \gg n / \log\log n$.*

In the case of the divisor functions, we have asymptotic formulas for their partial sums. In general, asymptotic formulas constitute a very precise piece of information and sometimes weaker information, such as that of an upper bound, is sufficient. The next theorem [11, Theorem 14.2] allows us to bound the partial sums of a non-negative, divisor bounded multiplicative function f rather easily, by using only its prime values $f(p)$.

Theorem 1.4.5. *If f is a multiplicative function such that $0 \leqslant f \leqslant \tau_k$ for some positive integer k, then*

$$\sum_{n \leqslant x} f(n) \ll_k x \exp\Big\{ \sum_{p \leqslant x} \frac{f(p) - 1}{p} \Big\}.$$

Example 1.4.1. For $y \geqslant 1$, if we take $f = \mathbb{1}_{P^-(\cdot)>y}$ in Theorem 1.4.5, then by referring to Theorem 1.2.1(c), we obtain the estimate

$$\Phi(x,y) := \#\{n \leqslant x : P^-(n) > y\} \ll \frac{x}{\log y}. \tag{1.4.2}$$

A stronger version of Theorem 1.4.5, where the partial sums are shorter and are taken over an arithmetic progression, was proven by Shiu [23].

Theorem 1.4.6. *Fix $m \in \mathbb{N}$ and $\varepsilon > 0$. Given any choice of $q \in \mathbb{N}, a \in (\mathbb{Z}/q\mathbb{Z})^*$, real numbers $x \geqslant y \geqslant 1$ with $y/q \geqslant x^\varepsilon$ and a multiplicative function f such that $0 \leqslant f \leqslant \tau_m$, we have that*

$$\sum_{\substack{x-y<n\leqslant x \\ n\equiv a \pmod q}} f(n) \ll_{m,\varepsilon} \frac{y}{q} \exp\left\{\sum_{\substack{p\leqslant x \\ p\nmid q}} \frac{f(p)-1}{p}\right\}.$$

1.5. Sieve theory: Two fundamental lemmas

Sieve theory is a field of number theory that deals with the size of sets of integers whose prime factors avoid a prescribed set of primes. One of the most important results in this field is the next theorem which is also known as the fundamental lemma of sieve theory. A proof of this theorem [11, Theorem 18.11, p. 190] may be found in [11, p. 196].

Theorem 1.5.1 (The Fundamental Lemma of Sieve Theory)**.** *Let $\mathcal{A} = \{a_n\}_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers with $\sum_{n\geqslant 1} a_n < \infty$ and let $\mathcal{P}$ be a set of primes. Let also $y \geqslant 1$ be a real number and set $P(y) := \prod_{p\leqslant y,\ p\in\mathcal{P}} p$. We define*

$$S(\mathcal{A},\mathcal{P},y) := \sum_{(n,P(y))=1} a_n,$$

and for $d \mid P(y)$, we put

$$A_d := \sum_{d\mid n} a_n.$$

If there exists a non-negative multiplicative function v, some real number X, remainder terms r_d and positive constants κ and C such that $v(p) < p$ for all $p \mid P(y)$,

$$A_d = X \cdot \frac{v(d)}{d} + r_d \quad \textit{for all} \quad d \mid P(y) \quad \textit{and}$$

$$\prod_{\substack{p\in\mathcal{P} \\ w_1<p\leqslant w_2}} \left(1 - \frac{v(p)}{p}\right)^{-1} \leqslant \left(\frac{\log w_2}{\log w_1}\right)^\kappa \left(1 + \frac{C}{\log w_1}\right) \quad \textit{for} \quad 2 \leqslant w_1 \leqslant w_2 < y,$$

then, for every real real number $u \geqslant 1$, we have that

$$S(\mathcal{A},\mathcal{P},y) = X \prod_{p|P(y)} \left(1 - \frac{v(p)}{p}\right) (1 + O_{\kappa,C}(u^{-u/2})) + O\Bigg(\sum_{\substack{d|P(y)\\ d\leqslant y^u}} |r_d|\Bigg).$$

The last result [11, Exercise 19.4(b), p. 204] of the section may be seen as a variant of another classical form of the fundamental lemma of sieve theory [11, Theorem 19.1, p. 195]. It is a variant that contains logarithmic weights.

Lemma 1.5.2. *Let k, κ and C be positive constants and let also $z \geqslant 2, \mathcal{P} \subseteq \{p \leqslant z\}, P(y) = \prod_{p\in\mathcal{P},p\leqslant y} p$ for all $y \leqslant z$ and $u \geqslant u_\kappa := 1 + 2/(e^{0.53/\kappa} - 1)$. Suppose further that ν is a multiplicative function such that $0 \leqslant \nu(p) < \min\{k,p\}$ for all $p \in \mathcal{P}$ and*

$$\prod_{\substack{p\in\mathcal{P},\\ w_1<p\leqslant w_2}} \left(1 - \frac{\nu(p)}{p}\right)^{-1} \leqslant \left(\frac{\log w_2}{\log w_1}\right)^{\kappa} \left(1 + \frac{C}{\log w_1}\right) \quad \text{for} \quad 2 \leqslant w_1 \leqslant w_2.$$

There exist two arithmetic functions λ^- and λ^+ such that

- $\lambda^\pm(1) = 1, |\lambda^\pm| \leqslant 1,$
- $\operatorname{supp}(\lambda^\pm) \subseteq \{d \mid \prod_{p\in\mathcal{P}} p : d \leqslant z^u\},$
- $(1 * \lambda^-)(n) \leqslant \mathbb{1}_{(\cdot,P(z))=1}(n) \leqslant (1 * \lambda^+)(n)$ *for all $n \in \mathbb{N}$ and*

$$\sum_{d|\prod_{p\in\mathcal{P}} p} \frac{\lambda^\pm(d)\nu(d)(\log d)^r}{d} = \sum_{d|\prod_{p\in\mathcal{P}} p} \frac{\mu(d)\nu(d)(\log d)^r}{d} + O_{r,k}\Bigg((\log z)^r u^{-u/2} \prod_{p\in\mathcal{P}} \left(1 - \frac{\nu(p)}{p}\right)\Bigg),$$

for every $r \in \mathbb{N}$.

Proof. First we show the following claim and then we use it to prove Lemma 1.5.2.

Claim 1. Let ν^* be a multiplicative function such that $\nu^*(p) = 1$ for $p \,|\, P(2ke)$ and $\nu^*(p) = \nu(p)p^{1/\log z}$ when $p \in \mathcal{P} \cap (2ke, +\infty)$. Then

$$\left|\sum_{d|P(y)} \frac{\mu(d)\nu(md)(\log(md))^r}{d}\right| \ll_{r,k} (\log z)^r \nu^*(m) \prod_{p|P(y)} \left(1 - \frac{\nu^*(p)}{p}\right), \tag{1.5.1}$$

for all $y \in [2,z]$ and for all $m \mid \prod_{p\in\mathcal{P},p\geqslant y} p$.

Proof of Claim 1. Let $y \in [2,z]$. For any $s \in \mathbb{C}$ with $|s| \leqslant 1/\log z$, we may use Theorem 1.2.1(b) to easily show that

$$\left|\prod_{p\in\mathcal{P}\cap(2ke,y)} \left(1 - \frac{\nu(p)}{p^{1-s}}\right)\right| \asymp_k \prod_{p|P(y)} \left(1 - \frac{\nu(p)}{p}\right), \tag{1.5.2}$$

upon noticing that $p^s = 1 + O(\log p/\log z)$ and that $|\nu(p)/p^{1-s}| \leqslant 1/2$ for $p \in [2ke, z] \cap \mathcal{P}$.

We now consider the function $g : \mathbb{C} \to \mathbb{C}$ with

$$g(s) = m^s \prod_{p|P(y)} \left(1 - \frac{\nu(p)}{p^{1-s}}\right) = \sum_{d|P(y)} \frac{\mu(d)\nu(d)(md)^s}{d}, \quad s \in \mathbb{C}.$$

For $r \in \mathbb{N}$ and $m \mid \prod_{p \in \mathcal{P}, p \geqslant y} p$, Cauchy's residue theorem implies that

$$\begin{aligned}\left|\sum_{d|P(y)} \frac{\mu(d)\nu(md)(\log(md))^r}{d}\right| &= \nu(m)|g^{(r)}(0)| = \frac{r! \cdot \nu(m)}{2\pi} \cdot \left|\int_{|s|=\frac{1}{\log z}} \frac{g(s)}{s^{r+1}} \mathrm{d}s\right| \\ &\ll_r (\log z)^r \nu(m) m^{\frac{1}{\log z}} \max_{|s|=1/\log z} \left|\prod_{p|P(y)} \left(1 - \frac{\nu(p)}{p^{1-s}}\right)\right| \\ &\ll_{r,k} (\log z)^r \nu(m) m^{\frac{1}{\log z}} \prod_{p|P(y)} \left(1 - \frac{\nu(p)}{p}\right). \end{aligned} \tag{1.5.3}$$

For the last estimate we first noted that $|\prod_{p|P(2ke)}(1 - \nu(p)p^{s-1})| \leqslant \prod_{p|P(2ke)}(1 + kp^{\mathrm{Re}(s)-1}) \leqslant \prod_{p|P(2ke)}(1 + kp^{1/\log z - 1}) \leqslant \prod_{p \leqslant 2ke}(1 + kp^{1/\log 2 - 1}) =: c_3(k)$ and then we used (1.5.2).

Applying (1.5.2) with $s = 1/\log z$, it follows that

$$\prod_{p|P(y)} \left(1 - \frac{\nu^*(p)}{p}\right) = \prod_{p|P(2ke)} \left(1 - \frac{1}{p}\right) \prod_{p \in \mathcal{P} \cap (2ke, y)} \left(1 - \frac{\nu^*(p)}{p}\right) \asymp_k \prod_{p|P(y)} \left(1 - \frac{\nu(p)}{p}\right). \tag{1.5.4}$$

The term $\prod_{p|P(2ke)}(1 - 1/p)$ was absorbed in the implied constants, because $c_4(k) := \prod_{p \leqslant 2ke}(1 - 1/p) \leqslant \prod_{p|P(2ke)}(1 - 1/p) < 1$. We make use of (1.5.4), and so (1.5.3) becomes

$$\left|\sum_{d|P(y)} \frac{\mu(d)\nu(md)(\log(md))^r}{d}\right| \ll_{r,k} (\log z)^r \nu(m) m^{\frac{1}{\log z}} \prod_{p|P(y)} \left(1 - \frac{\nu^*(p)}{p}\right). \tag{1.5.5}$$

At this point, we write $m = m_1 m_2$, where $m_1 = \prod_{p|m, p \leqslant 2ke} p$ and $m_2 = \prod_{p|m, p > 2ke} p$. For the positive integer m_1 we have

$$0 \leqslant \nu(m_1) \leqslant \prod_{p|m, p \leqslant 2ke} \nu(p) \leqslant k^{2ke} \quad \text{and} \quad m_1^{\frac{1}{\log z}} \leqslant m_1^{\frac{1}{\log 2}} \leqslant (2ke)^{\frac{2ke}{\log 2}}.$$

Thus, $\nu(m) m^{1/\log z} \ll_k \nu(m_2) m_2^{1/\log z} = \nu^*(m)$, and combining this inequality with (1.5.5), we arrive at the estimate (1.5.1). So, the claim has now been proven.

Since $p \leqslant z$ for any prime $p \in \mathcal{P}$, if $p \in \mathcal{P} \cap (2ke, +\infty)$, then $\nu^*(p) \leqslant ke < p/2$. On the other hand, if $p \in \mathcal{P}$ and $p \leqslant 2ke$, then $\nu^*(p) = 1 < p/2$ as well. So, $0 \leqslant \nu^*(p) < p/2$ and ν^* is bounded on the primes $p \in \mathcal{P}$. Now we use the inequalities $0 \leqslant \nu^*(p) < p/2$ for all $p \in \mathcal{P}$

and Claim 1 to prove that

$$\sum_{d|\prod_{p\in\mathcal{P}} p} \frac{\lambda^{\pm}(d)\nu(d)(\log d)^r}{d} = \sum_{d|\prod_{p\in\mathcal{P}} p} \frac{\mu(d)\nu(d)(\log d)^r}{d} + O_{r,k}\left((\log z)^r u^{-u/2} \prod_{p\in\mathcal{P}} \left(1 - \frac{\nu^*(p)}{p}\right)\right),$$

where $\lambda^{\pm}$ are arithmetic functions satisfying the conditions mentioned in the statement of the lemma. The proof of the lemma will then be complete, since one may replace ν^* by ν in the big-Oh term by using (1.5.4) with $y = z$.

First, let us put $f = \nu \cdot \log^r$ for simplicity. The construction of the functions $\lambda^{\pm}$ is described in [11, Chapter 19]. The proof of the last aforementioned equality is along the same lines as the proof of Theorem 19.1 in [11]. One only has to make minor notational changes and rename the y_j's to z_j's and redefine $V(y)$ as

$$V_f(y) = \sum_{d|P(y)} \frac{\mu(d)f(d)}{d}.$$

They also have to redefine V_n as

$$V_{n,f} = \sum_{\substack{z_n < p_n < \ldots < p_1 \leqslant z \\ p_1,\ldots,p_n \in \mathcal{P} \\ p_i \leqslant z_i \ (i<n, i\equiv n (\mathrm{mod}\, 2))}} \frac{1}{p_1 \cdots p_n} \sum_{d|P(p_n)} \frac{\mu(d)f(p_1\cdots p_m d)}{d}.$$

Then, in an analogous way as in [11, Theorem 19.1], we have that

$$V_f(z) - \sum_{d|\prod_{p\in\mathcal{P}} p} \frac{\lambda^+(d)f(d)}{d} = -\sum_{j>J} V_{2j-1,f}, \tag{1.5.6}$$

for the integer J which is defined in [11, Chapter 19, p. 195, 197]. Furthermore, since $z_n < p_n$, using (4.2.3), we deduce that

$$\begin{aligned}
|V_n| &\ll_{r,k} (\log z)^r \sum_{\substack{z_n < p_n < \ldots < p_1 \leqslant z \\ p_1,\ldots,p_n \in \mathcal{P}}} \frac{\nu^*(p_1)\cdots\nu^*(p_n)}{p_1\cdots p_n} \prod_{p|P(p_n)} \left(1 - \frac{\nu^*(p)}{p}\right) \\
&\ll_{r,k} (\log z)^r \prod_{p|P(z_n)} \left(1 - \frac{\nu^*(p)}{p}\right) \sum_{\substack{z_n < p_n < \ldots < p_1 \leqslant z \\ p_1,\ldots,p_n \in \mathcal{P}}} \frac{\nu^*(p_1)\cdots\nu^*(p_n)}{p_1\cdots p_n} \\
&\ll_{r,k} \frac{(\log z)^r}{n!} \cdot \prod_{p|P(z_n)} \left(1 - \frac{\nu^*(p)}{p}\right) \cdot \left(\sum_{p\in\mathcal{P}\cap(z_n,z]} \frac{\nu^*(p)}{p}\right)^n.
\end{aligned} \tag{1.5.7}$$

In the last step we applied the Erdős trick to drop the ordering condition from the sum of the second line. Now we can simply use (1.5.6) and (1.5.7) and follow the proof of Theorem

19.1 of [11] to show that

$$\sum_{d|\prod_{p\in\mathcal{P}}p}\frac{\lambda^{\pm}(d)\nu(d)(\log d)^r}{d}=\sum_{d|\prod_{p\in\mathcal{P}}p}\frac{\mu(d)\nu(d)(\log d)^r}{d}+O_{r,k}\left((\log z)^r u^{-u/2}\prod_{p\in\mathcal{P}}\left(1-\frac{\nu^*(p)}{p}\right)\right).$$

The proof for the function λ^- is similar. Its only difference is the use of the identity

$$V_f(z)-\sum_{d|\prod_{p\in\mathcal{P}}p}\frac{\lambda^+(d)f(d)}{d}=\sum_{j>J}V_{2j,f}$$

in place of (1.5.6). ☐

Chapter 2

Auxiliary results

In this chapter, we state and prove some results which will be essential for the proofs of Theorems 1 and 2.

2.1. Asymptotics of sifted sums

This section includes estimates on sifted sums of several arithmetic functions. The first two are useful for the first project of the thesis.

Lemma 2.1.1. *Let $D \in \mathbb{N}$ and set $k_D = \prod_{p \leqslant D^3} p$. There exists a real number $c = c(D) \in$ (0,1) such that*

$$\sum_{\substack{n \leqslant x \\ (n,k_D)=1}} D^{\Omega(n)} = x \sum_{i=0}^{D-1} a_{i,D} (\log x)^i + O(x^{1-c}) \tag{2.1.1}$$

for some constants $a_{i,D}$ that depend at most on i and D.

Proof. We write $\mathbb{1}_{(\cdot,k_D)=1} \cdot D^{\Omega} = \tau_D * h$, where $h = \tau_{-D} * (\mathbb{1}_{(\cdot,k_D)=1} \cdot D^{\Omega})$ and τ_{-D} is given by the coefficients of the Dirichlet series of $\zeta(s)^{-D}$ when $\mathrm{Re}(s) > 1$. The values of τ_{-D} can be easily computed by the argument that we saw in the beginning of Remark 1. Now, if $p \leqslant D^3$, then $h(p^\nu) = \tau_{-D}(p^\nu)$. Hence, by (1.4.1),

$$\prod_{p \leqslant D^3} \left(\sum_{j \geqslant 0} \frac{h(p^j)}{p^{js}} \right) = \prod_{p \leqslant D^3} \left(\sum_{j \geqslant 0} \binom{j-D-1}{j} \frac{1}{p^{js}} \right) = \prod_{p \leqslant D^3} (1-p^{-s})^D. \tag{2.1.2}$$

The product of the leftmost side is an entire function of s as a finite product of entire functions. Furthermore, each of the series in the product is absolutely convergent for $\sigma > 0$, because $|p^{-s}| = p^{-\sigma} < 1$ and the radius of convergence of the power series of $x \mapsto (1-x)^D$ equals 1. Therefore, the finite product of the leftmost side of (2.1.2) is absolutely convergent for $\sigma > 0$. If $p > D^3$, then note that $h(p) = 0$ and

$$h(p^\nu) = D^\nu \sum_{\kappa=0}^{\nu} D^{-\kappa} \binom{\kappa - D - 1}{\kappa} = D^\nu \Big(1 - \frac{1}{D}\Big)^D + O(1) \ll D^\nu \quad \text{for} \quad \nu \geqslant 2.$$

The second equality may be justified by using the Taylor expansion of order ν for the function $x \mapsto (1-x)^D$ when $|x| < 1$. Consequently, for $p > D^3$ and $\sigma > 5/6$, we have that $\sum_{j \geqslant 1} |h(p^j) p^{-js}| \ll \sum_{j \geqslant 2} (D/p^\sigma)^j \leqslant \sum_{j \geqslant 2} p^{-j(\sigma - 1/3)} \ll p^{-2(\sigma-1/3)}$. Since $2(\sigma - 1/3) > 1$, when $\sigma > 5/6$, we conclude that the product

$$\prod_{p > D^3} \Big(\sum_{j \geqslant 0} \frac{h(p^j)}{p^{js}} \Big)$$

is absolutely convergent in the half-plane $\mathrm{Re}(s) > 5/6$ and it thus defines an analytic function of s in this region. Combining all the above conclusions, we infer that the Euler product

$$H(s) := \prod_{p} \Big(\sum_{j \geqslant 0} \frac{h(p^j)}{p^{js}} \Big)$$

is absolutely convergent and an analytic function of s when $\sigma > 5/6$. So, the same hold for its Dirichlet series $H(s) = \sum_{n \geqslant 1} h(n) n^{-s}$.

Now,

$$\sum_{\substack{n \leqslant x \\ (n, k_D) = 1}} D^{\Omega(n)} = \sum_{a \leqslant \sqrt{x}} \tau_D(a) \sum_{\sqrt{x} < b \leqslant x/a} h(b) + \sum_{b \leqslant \sqrt{x}} h(b) \sum_{a \leqslant x/b} \tau_D(a), \tag{2.1.3}$$

and let S_1 and S_2 be the left and right sums of the right-hand side, respectively. Let also $G(s) = \sum_{n \geqslant 1} |h(n)| n^{-s}$ for $\sigma > 5/6$.

For S_1, we apply Rankin's trick twice and obtain

$$\begin{aligned} |S_1| \;\leqslant\; & \sum_{a \leqslant \sqrt{x}} \tau_D(a) \sum_{\sqrt{x} < b \leqslant x/a} \Big(\frac{x}{ab}\Big)^{7/8} |h(b)| \leqslant x^{7/8}\, G\Big(\frac{7}{8}\Big) \sum_{a \leqslant \sqrt{x}} \frac{\tau_D(a)}{a^{7/8}} \\ \leqslant\; & x^{7/8}\, G\Big(\frac{7}{8}\Big) \sum_{a \leqslant \sqrt{x}} \Big(\frac{\sqrt{x}}{a}\Big)^{1/6} \frac{\tau_D(a)}{a^{7/8}} \leqslant G\Big(\frac{7}{8}\Big) \zeta\Big(\frac{25}{24}\Big)^D x^{23/24}, \end{aligned} \tag{2.1.4}$$

where at the end we used the Dirichlet series of τ_D which is ζ^D.

We continue by bounding S_2. In this case, we make use of the asymptotic formula of Theorem 1.4.3 and get that

$$S_2 = x \sum_{i=0}^{D-1} \alpha_{i,D} \sum_{\ell=0}^{i} \binom{i}{\ell} (\log x)^\ell \sum_{b \leqslant \sqrt{x}} \frac{h(b)(-\log b)^{i-\ell}}{b} + O\Big(x^{1-\eta} \sum_{b \leqslant \sqrt{x}} \frac{|h(b)|}{b^{1-\eta}} \Big). \tag{2.1.5}$$

We have that

$$\Big|\sum_{b>\sqrt{x}} \frac{h(b)(-\log b)^{i-\ell}}{b}\Big| \leqslant \sum_{b>\sqrt{x}} \frac{|h(b)|(\log b)^{D-1}}{b} \ll_D \sum_{b>\sqrt{x}} \frac{|h(b)|}{b^{23/24}}$$
$$\leqslant \sum_{b>\sqrt{x}} \left(\frac{b}{\sqrt{x}}\right)^{1/24} \frac{|h(b)|}{b^{23/24}} \leqslant G\Big(\frac{11}{12}\Big) x^{-1/48},$$

and so

$$\sum_{b\leqslant\sqrt{x}} \frac{h(b)(-\log b)^{i-\ell}}{b} = H^{(i-\ell)}(1) + O(x^{-1/48}). \tag{2.1.6}$$

If we define $\vartheta := 1 - 11/(12\eta) < 1$, then one last suitable application of Rankin's trick gives

$$x^{1-\eta} \sum_{b\leqslant\sqrt{x}} \frac{|h(b)|}{b^{1-\eta}} \leqslant x^{1-\eta} \sum_{b\leqslant\sqrt{x}} \left(\frac{\sqrt{x}}{b}\right)^{\eta\cdot\vartheta} \frac{|h(b)|}{b^{1-\eta}} = x^{1-\eta(1-\vartheta/2)} \sum_{b\leqslant\sqrt{x}} \frac{|h(b)|}{b^{11/12}} \leqslant G\Big(\frac{11}{12}\Big) x^{1-\eta/2}.$$

We insert (2.1.6) and (2.1.7) into (2.1.5) and we combine the result with (2.1.3) and (2.1.4). This completes the proof of the lemma. □

Lemma 2.1.2. *Let $t \in \mathbb{R}$ and set $V_t := \exp\{100(\log(3+|t|))^{2/3}(\log\log(3+|t|))^{1/3}\}$. For $x \geqslant z \geqslant V_t$, we have that*

$$\sum_{\substack{n\leqslant x\\ P^-(n)>z}} n^{it} = \frac{x^{1+it}}{1+it} \prod_{p\leqslant z}\left(1-\frac{1}{p}\right) + O\left(\frac{x^{1-1/(30\log z)}}{\log z}\right).$$

Proof. See [9, Lemma 3.1]. □

Lemma 2.1.3. *Given a $q \in \mathbb{N}$, let χ be a Dirichlet character modulo q. For $j \in \mathbb{N}\cup\{0\}$ and any real numbers $x \geqslant y \geqslant (10q)^{100}$, we have*

$$\sum_{\substack{n\leqslant x\\ P^-(n)>y}} \chi(n)(\log n)^j = \mathbb{1}_{\chi=\chi_0} \cdot \left(\int_y^x (\log t)^j \mathrm{d}t\right) \prod_{p\leqslant y}\left(1-\frac{1}{p}\right) + O\left(\frac{(\log x)^j x^{1-\kappa/\log y}}{\log y}\right),$$

where χ_0 is the principal character modulo q and $\kappa > 0$ is an absolute constant.

Proof. For any $w \geqslant y$, application of [11, Lemma 22.2, p. 224] with $t = 0$ gives

$$\sum_{\substack{n\leqslant w\\ P^-(n)>y}} \chi(n) = \mathbb{1}_{\chi=\chi_0} \cdot w \prod_{p\leqslant y}\left(1-\frac{1}{p}\right) + R_y(w), \tag{2.1.7}$$

where $R_y(w) \ll (\log y)^{-1} w^{1-\kappa/\log y}$ for some constant $\kappa > 0$. Using the Riemann-Stieltjes integral, we have

$$\sum_{\substack{n \leqslant x \\ P^-(n)>y}} \chi(n)(\log n)^j = \int_y^x (\log t)^j \mathrm{d}\Big(\sum_{\substack{n \leqslant t \\ P^-(n)>y}} \chi(n) \Big) \tag{2.1.8}$$
$$= \mathbb{1}_{\chi=\chi_0} \cdot \left(\int_y^x (\log t)^j \mathrm{d}t \right) \prod_{p \leqslant y} \left(1 - \frac{1}{p}\right) + \int_y^x (\log t)^j \mathrm{d}R_y(t).$$

Now, if $x \geqslant y^A$ for some large $A > 0$, so that $y \leqslant x^{1-\kappa/\log 2} \leqslant x^{1-\kappa/\log y}$, then a simple integration by parts implies that

$$\int_y^x (\log t)^j \mathrm{d}R_y(t) \ll \frac{(\log x)^j x^{1-\kappa/\log y}}{\log y} + \frac{j}{\log y} \int_y^x (\log t)^{j-1} t^{-\kappa/\log y} \mathrm{d}t.$$

But, since

$$j \int_y^x (\log t)^{j-1} t^{-\kappa/\log y} \mathrm{d}t \leqslant j x^{1-\kappa/\log y} \int_1^x \frac{(\log t)^{j-1}}{t} \mathrm{d}t = (\log x)^j x^{1-\kappa/\log y},$$

we deduce that

$$\int_y^x (\log t)^j \mathrm{d}R_y(t) \ll \frac{(\log x)^j x^{1-\kappa/\log y}}{\log y}. \tag{2.1.9}$$

We insert (2.1.9) in (2.1.8) and complete the proof of the lemma when $x \geqslant y^A$. It only remains to establish the lemma in the case $y \leqslant x < y^A$. In this case, $\log x/\log y \asymp 1$, and so applying Example 1.4.1, we have that

$$\Bigg| \sum_{\substack{n \leqslant x \\ P^-(n)>y}} \chi(n)(\log n)^j \Bigg| \leqslant (\log x)^j \sum_{\substack{n \leqslant x \\ P^-(n)>y}} 1 \ll \frac{x(\log x)^j}{\log y} \ll \frac{(\log x)^j x^{1-\kappa/\log y}}{\log y}.$$

This means that the lemma does hold in the range $y \leqslant x < y^A$ as well and this finishes the proof. □

We close the section by establishing an asymptotic for the sifted partial sums of $(\log n)^j$ on arithmetic progressions. The following lemma, as well as Lemma 2.1.3, are only required for the proof of Theorem 2.

Lemma 2.1.4. *Let j be a non-negative integer. For any $q \in \mathbb{N}, a \in (\mathbb{Z}/q\mathbb{Z})^*$ and real numbers $x \geqslant y \geqslant 2q$, there exists a constant $\lambda > 0$ such that*

$$\sum_{\substack{n \leqslant x, P^-(n)>y \\ n \equiv a \,(\mathrm{mod}\, q)}} (\log n)^j = \frac{1}{\phi(q)} \left(\int_y^x (\log t)^j \mathrm{d}t \right) \prod_{p \leqslant y} \left(1 - \frac{1}{p}\right) + O\left(\frac{(\log x)^j x^{1-\lambda/\log y}}{\phi(q) \log y} \right). \tag{2.1.10}$$

Proof. Let $A > 0$ be a sufficiently large real number. We are going to prove the lemma separately in the ranges $x \geqslant y^A$ and $y \leqslant x < y^A$.

First, we start with the case $x \geqslant y^A$. In this case, we will apply Theorem 1.5.1 with $\mathcal{P} = \{p \leqslant y, p \nmid q\}$ and $\mathcal{A} = \{a_n\}_{n\in\mathbb{N}}$, where

$$a_n = \begin{cases} (\log n)^j, & \text{when } n \leqslant x \text{ and } n \equiv a \pmod q \\ 0, & \text{otherwise.} \end{cases}$$

Observe that

$$S(\mathcal{A},\mathcal{P},y) = \sum_{\substack{n\leqslant x, P^-(n)>y \\ n\equiv a \pmod q}} (\log n)^j.$$

Consequently, with the chosen sets $\mathcal{A}$ and $\mathcal{P}$, an application of Theorem 1.5.1 will provide an asymptotic formula for the sums of interest at the right-hand side.

Now, let d be a positive integer dividing $\prod_{p\leqslant y,\, p\nmid q} p$. Because of the chinese remainder theorem, the system of linear congruences $n \equiv a \pmod q, n \equiv 0 \pmod d$ is equivalent to $n \equiv a^* \pmod{qd}$ for some $a^* \in (\mathbb{Z}/(qd)\mathbb{Z})$. Since $\sum_{n\leqslant x, n\equiv a^* \pmod{qd}} 1 = x/(qd) + O(1)$, partial summation implies that

$$\begin{aligned} \mathcal{A}_d = \sum_{\substack{n\leqslant x, d\mid n \\ n\equiv a \pmod q}} (\log n)^j = \sum_{\substack{n\leqslant x \\ n\equiv a^* \pmod{qd}}} (\log n)^j &= \frac{1}{qd}\left\{x(\log x)^j - j\int_1^x (\log t)^{j-1}\mathrm{d}t\right\} + O((\log x)^j) \\ &= \frac{1}{qd}\int_1^x (\log t)^j \mathrm{d}t + O((\log x)^j). \end{aligned} \tag{2.1.11}$$

Therefore, following the notation of Theorem 1.5.1, we have $X = xq^{-1}\int_1^x (\log t)^j \mathrm{d}t$, $v(d) = 1$ and $r_d = (\log x)^j$ for all $d \mid \prod_{p\leqslant y,\, p\nmid q} p$. Then Theorem 1.2.1(d) implies that we may choose $\kappa = 1$ and some large $C > 0$ in Theorem 1.5.1.

For $u = e\log x/(A\log y) \geqslant e$, note that

$$\begin{aligned} X\prod_{p\leqslant y\, p\nmid q}\left(1 - \frac{v(p)}{p}\right)u^{-u/2} &< \frac{x(\log x)^j}{q}\prod_{p\leqslant y}\left(1-\frac{1}{p}\right)\prod_{p\mid q}\left(1-\frac{1}{p}\right)^{-1}\exp\left\{-\frac{e\log x}{2A\log y}\right\} \\ &\leqslant \frac{(\log x)^j x^{1-\lambda/\log y}}{\phi(q)}\prod_{p\leqslant y}\left(1-\frac{1}{p}\right) \\ &\ll \frac{(\log x)^j x^{1-\lambda/\log y}}{\phi(q)\log y} \end{aligned} \tag{2.1.12}$$

for some $\lambda \in (0, e(2A)^{-1}]$. At the last step we applied one of Mertens' estimates (Theorem 1.2.1(d)). Furthermore, with $P(y) = \prod_{p\leqslant y,\, p\nmid q} p$ and the choice of u that we made, we have that

$$\sum_{\substack{d|P(y)\\ d\leqslant y^u}} |r_d| \leqslant \sum_{d\leqslant y^u} r_d \leqslant x^{e/A}(\log x)^j \ll x^{2e/A}(\log x)^{j-1}$$
$$< \frac{(\log x)^j x^{(2e+1)/A}}{\phi(q)\log y} \leqslant \frac{(\log x)^j x^{1-\lambda/\log y}}{\phi(q)\log y}, \tag{2.1.13}$$

where we passed from the first line to the second by using the inequality $x^{1/A} \geqslant y \geqslant 2q > \phi(q)$. The last step follows from the fact that A is sufficiently large.

Combination of (2.1.12) and (2.1.13) with Theorem 1.5.1 yields

$$\sum_{\substack{n\leqslant x, P^-(n)>y\\ n\equiv a \,(\mathrm{mod}\, q)}} (\log n)^j = \frac{1}{\phi(q)}\left(\int_1^x (\log t)^j \mathrm{d}t\right)\prod_{p\leqslant y}\left(1-\frac{1}{p}\right) + O\left(\frac{(\log x)^j x^{1-\lambda/\log y}}{\phi(q)\log y}\right). \tag{2.1.14}$$

But, $\prod_{p\leqslant y}(1-1/p) \ll (\log y)^{-1}$ by Theorem 1.2.1(d), and

$$\int_1^y (\log t)^j \mathrm{d}t < y(\log x)^j \ll (\log x)^j x^{1-\lambda/\log y},$$

since $x \geqslant y^A$ with A sufficiently large. Hence, relation (2.1.14) implies (2.1.10) in the range $x \geqslant y^A$.

When $y \leqslant x < y^A$, a similar argument as the one that we developed at the end of the proof of Lemma 2.1.3 shows that

$$\sum_{\substack{n\leqslant x, P^-(n)>y\\ n\equiv a \,(\mathrm{mod}\, q)}} (\log n)^j \ll \frac{(\log x)^j x^{1-\lambda/\log y}}{\phi(q)\log y}$$

and the proof of the lemma is complete. □

2.2. Bounds for sifted $L-$Dirichlet series

Let us consider a real number $y \geqslant 1$ and a positive integer q. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, we define the y-rough Dirichlet series of a Dirichlet character χ modulo q as

$$L_y(s,\chi) := \sum_{\substack{n\geqslant 1\\ P^-(n)>y}} \frac{\chi(n)}{n^s} = L(s,\chi)\prod_{p\leqslant y}(1-\chi(p)/p^s). \tag{2.2.1}$$

In (2.2.1), the series is absolutely convergent and its rightmost side implies the meromorphical continuation of $L_y(\cdot,\chi)$ on the whole complex plane (with one pole at 1 only in the case of the principal character χ_0).

The present section constitutes a collection of some existing upper and lower bounds for the values of a y-rough Dirichlet series. We will use all these bounds in the proof of Theorem 2. The theorems listed below are stated without their proofs, as they are already part of the literature. We only provide a reference for each one of them. To simplify their statements, we will use the notation

$$V_t := \exp\{100(\log(3+|t|))^{2/3}(\log\log(3+|t|))^{1/3}\}, \quad (t \in \mathbb{R})$$

which was introduced in Lemma 2.1.2.

We start with the following theorem [10, Lemma 4.1] concerning an upper bound for the derivatives of $L(\cdot,\chi)$.

Theorem 2.2.1. *Let q be a positive integer and χ be a non-principal character modulo q. Let also $j \in \mathbb{N}$ and $s = \sigma + it$ with $\sigma > 1$ and $t \in \mathbb{R}$. For $y \geqslant qV_t$, we have that*

$$|L_y^{(j)}(s,\chi)| \ll j!(C\log y)^j,$$

where $C > 0$ is an absolute constant.

The next result [10, Lemma 4.2] equips us with lower bounds for $L_y(s,\chi)$.

Theorem 2.2.2. *Fix a positive integer q and let χ be a character modulo q. Let $s = \sigma + it$ with $\sigma > 1$ and $t \in \mathbb{R}$ and consider the real number $y \geqslant qV_t$.*

(a) If χ is not real, then $|L_y(s,\chi)| \gg 1$.

(b) If χ is real and non-principal, then $|L_y(s,\chi)| \gg L_y(1,\chi)$.

The last result of this section is a theorem [11, Lemma 27.5, p. 291] that deals with the size of $L_q(\sigma,\chi)$ for $\sigma \geqslant 1$ when $\chi \neq \psi$, where ψ is defined as in the statement of Theorem 2.

Theorem 2.2.3. *Let q be a positive integer and let $\mathcal{R}_q$ be the set of real, non-principal characters modulo q. If we take a character ψ such that $L_q(1,\psi) = \min_{\chi\in\mathcal{R}_q} L_q(1,\chi)$ and $\mathcal{C}_q := \{\chi \,(\operatorname{mod} q) : \chi \neq \chi_0, \psi\}$, then $|L_y(\sigma,\chi)| \asymp 1$ for all $\chi \in \mathcal{C}_q, y \geqslant q$ and $\sigma \geqslant 1$.*

2.3. Mean value theorems

We open this section with a mean value theorem of Montgomery [18, Thoerem 3, p. 131].

Lemma 2.3.1. *Let $A(s) = \sum_{n\geqslant 1} a_n n^{-s}$ and $B(s) = \sum_{n\geqslant 1} b_n n^{-s}$ be two Dirichlet series which converge for $\operatorname{Re}(s) > 1$. If $|a_n| \leqslant b_n$ for all $n \in \mathbb{N}$, then*

$$\int_{-T}^{T} |A(\sigma+it)|^2 \mathrm{d}t \leqslant 3\int_{-T}^{T} |B(\sigma+it)|^2 \mathrm{d}t,$$

for any $\sigma > 1$ and any $T \geqslant 0$.

Now, we prove a mean value theorem for the derivatives of the Dirichlet series

$$\sum_{\substack{n\equiv a \,(\mathrm{mod}\, q) \\ P^-(n)>y}} \frac{\Lambda(n)}{n^s}.$$

Lemma 2.3.2. *Consider an integer $q \geqslant 1$ and a real number $y \geqslant 16q^2$. For $j \in \mathbb{N}\cup\{0\}, T \geqslant 1, \sigma \in (1,2)$ and $a \in (\mathbb{Z}/q\mathbb{Z})^*$, we have*

$$\int_{|t|>T} \Bigg| \sum_{\substack{n\equiv a \,(\mathrm{mod}\, q) \\ P^-(n)>y}} \frac{\Lambda(n)(\log n)^j}{n^{\sigma+it}} \Bigg|^2 \frac{\mathrm{d}t}{t^2} \ll \frac{(\log 4)^j (2j)!}{\phi(q)^2(\sigma-1)^{2j+1}T}.$$

Proof. First, for $k \in \mathbb{N}\cup\{0\}$, because of Lemma 2.3.1, we observe that

$$\begin{aligned}\int_{k-1/2}^{k+1/2} \Bigg| \sum_{\substack{n\equiv a \,(\mathrm{mod}\, q) \\ P^-(n)>y}} \frac{\Lambda(n)(\log n)^j}{n^{\sigma+it}} \Bigg|^2 \mathrm{d}t &= \int_{-1/2}^{1/2} \Bigg| \sum_{\substack{n\equiv a \,(\mathrm{mod}\, q) \\ P^-(n)>y}} \frac{\Lambda(n)(\log n)^j n^{-ik}}{n^{\sigma+it}} \Bigg|^2 \mathrm{d}t \\ &\leqslant 3\int_{-1/2}^{1/2} \Bigg| \sum_{\substack{n\equiv a \,(\mathrm{mod}\, q) \\ P^-(n)>y}} \frac{\Lambda(n)(\log n)^j}{n^{\sigma+it}} \Bigg|^2 \mathrm{d}t.\end{aligned}$$

Consequently,

$$\begin{aligned}\int_{|t|>T} \Bigg| \sum_{\substack{n\equiv a \,(\mathrm{mod}\, q) \\ P^-(n)>y}} \frac{\Lambda(n)(\log n)^j}{n^{\sigma+it}} \Bigg|^2 \frac{\mathrm{d}t}{t^2} &\leqslant \sum_{|k|>T-1/2} \int_{k-1/2}^{k+1/2} \Bigg| \sum_{\substack{n\equiv a \,(\mathrm{mod}\, q) \\ P^-(n)>y}} \frac{\Lambda(n)(\log n)^j}{n^{\sigma+it}} \Bigg|^2 \frac{\mathrm{d}t}{t^2} \\ &\leqslant 4 \sum_{|k|>T/2} \frac{1}{k^2} \int_{k-1/2}^{k+1/2} \Bigg| \sum_{\substack{n\equiv a \,(\mathrm{mod}\, q) \\ P^-(n)>y}} \frac{\Lambda(n)(\log n)^j}{n^{\sigma+it}} \Bigg|^2 \mathrm{d}t \\ &\ll \Bigg(\sum_{k>T/2} \frac{1}{k^2} \Bigg) \cdot \int_{-1/2}^{1/2} \Bigg| \sum_{\substack{n\equiv a \,(\mathrm{mod}\, q) \\ P^-(n)>y}} \frac{\Lambda(n)(\log n)^j}{n^{\sigma+it}} \Bigg|^2 \mathrm{d}t \\ &\ll \frac{1}{T} \int_{-1/2}^{1/2} \Bigg| \sum_{\substack{n\equiv a \,(\mathrm{mod}\, q) \\ P^-(n)>y}} \frac{\Lambda(n)(\log n)^j}{n^{\sigma+it}} \Bigg|^2 \mathrm{d}t. \qquad (2.3.1)\end{aligned}$$

Now, we focus on estimating the integral at the last line of (2.3.1). We will do this by adopting the rather standard technique which is used for proving similar mean value theorems. We consider the function $\Phi : \mathbb{R} \to \mathbb{R}$ given by the formula $\Phi(x) = (2\pi \sin(x/4))^2 x^{-2}$ for all $x \in \mathbb{R}^*$ and $\Phi(0) = \pi^2/4$. Notice that $\Phi(x) \geqslant 1$ on $[-1/2,1/2]$. So, if $\widehat{\Phi}$ is the Fourier transform of Φ, then

$$\int_{-1/2}^{1/2}\Bigg|\sum_{\substack{n\equiv a\,(\mathrm{mod}\,q)\\ P^-(n)>y}}\frac{\Lambda(n)(\log n)^j}{n^{\sigma+it}}\Bigg|^2\mathrm{d}t\leqslant\int_{\mathbb{R}}\Bigg|\sum_{\substack{n\equiv a\,(\mathrm{mod}\,q)\\ P^-(n)>y}}\frac{\Lambda(n)(\log n)^j}{n^{\sigma+it}}\Bigg|^2\Phi(t)\mathrm{d}t \tag{2.3.2}$$
$$=\sum_{\substack{m\equiv a\,(\mathrm{mod}\,q)\\ n\equiv a\,(\mathrm{mod}\,q)\\ P^-(m),P^-(n)>y}}\frac{\Lambda(n)\Lambda(n)(\log m)^j(\log n)^j}{m^\sigma n^\sigma}\widehat{\Phi}(\log(m/n)),$$

where we arrived at the last line by expanding the square and by interchanging the order of summation and integration. The Fourier transform $\widehat{\Phi}$ is an even function, because Φ is also even. Therefore, we may bound the last line of (2.3.2) by twice the sum

$$\sum_{\substack{m\equiv a\,(\mathrm{mod}\,q)\\ n\leqslant m,n\equiv a\,(\mathrm{mod}\,q)\\ P^-(m),P^-(n)>y}}\frac{\Lambda(n)\Lambda(n)(\log m)^j(\log n)^j}{m^\sigma n^\sigma}\widehat{\Phi}(\log(m/n)). \tag{2.3.3}$$

The Fourier transform $\widehat{\Phi}$ is continuous and compactly supported on $[-1/2,1/2]$. Moreover, for $n\leqslant m$, we have that $|n-m|\leqslant m\log(m/n)$. Therefore, the sum (2.3.3) is smaller than or equal to

$$\sum_{\substack{m\equiv a\,(\mathrm{mod}\,q)\\ P^-(m)>y}}\frac{\Lambda(m)(\log m)^j}{m^\sigma}\sum_{\substack{n\leqslant m,|n-m|\leqslant m/2\\ n\equiv a\,(\mathrm{mod}\,q)}}\frac{\Lambda(n)(\log n)^j}{n^\sigma}$$
$$\ll\sum_{\substack{m\equiv a\,(\mathrm{mod}\,q)\\ P^-(m)>y}}\frac{\Lambda(m)(\log m)^{2j}}{m^{2\sigma}}\sum_{\substack{m/2\leqslant n\leqslant m\\ n\equiv a\,(\mathrm{mod}\,q)}}\Lambda(n)$$
$$\ll\frac{1}{\phi(q)}\sum_{\substack{m\equiv a\,(\mathrm{mod}\,q)\\ P^-(m)>y}}\frac{\Lambda(m)(\log m)^{2j}}{m^{2\sigma-1}}.$$

For the last estimate we made use of Lemma 1.2.4. Its application was allowed, because $\Lambda(1)=0$, which means that the condition $P^-(m)>y$ implies $m>y\geqslant 16q^2$, which in turn gives that $m/2\geqslant 2q\sqrt{m}$. According to all the above, relation (2.3.2) becomes

$$\int_{-1/2}^{1/2}\Bigg|\sum_{\substack{n\equiv a\,(\mathrm{mod}\,q)\\ P^-(n)>y}}\frac{\Lambda(n)(\log n)^j}{n^{\sigma+it}}\Bigg|^2\mathrm{d}t\ll\frac{1}{\phi(q)}\sum_{\substack{m\equiv a\,(\mathrm{mod}\,q)\\ P^-(m)>y}}\frac{\Lambda(m)(\log m)^{2j}}{m^{2\sigma-1}}. \tag{2.3.4}$$

We continue by bounding the sum of the right-hand side of (2.3.4). By decomposing this sum in dyadic intervals, we get that

$$\begin{aligned}\sum_{\substack{m\equiv a\,(\mathrm{mod}\,q)\\ P^-(m)>y}}\frac{\Lambda(m)(\log m)^{2j}}{m^{2\sigma-1}} &\leqslant (\log 4)^j\sum_{r>\log y/\log 2}\frac{r^{2j}}{2^{r(2\sigma-1)}}\sum_{\substack{2^r\leqslant m<2^{r+1}\\ m\equiv a\,(\mathrm{mod}\,q)}}\Lambda(m)\\ &\ll \frac{(\log 4)^j}{\phi(q)}\sum_{r\geqslant 0}\frac{r^{2j}}{4^{r(\sigma-1)}},\end{aligned}$$

where we applied Lemma 1.2.4 for the last step, because $2^r > y \geqslant 16q^2$, which implies that $2^r > 2q\sqrt{2^{r+1}}$. Using the $2j$-th derivative of the geometric series, it follows that

$$\sum_{r\geqslant 0}\frac{r^{2j}}{4^{r(\sigma-1)}} \leqslant \sum_{r\geqslant 0}\frac{(r+2j)!}{4^{r(\sigma-1)}r!} = \frac{(2j)!}{(1-4^{1-\sigma})^{2j+1}} \ll \frac{(2j)!}{(\sigma-1)^{2j+1}},$$

as $1-4^{1-\sigma} \gg \sigma-1$ for $\sigma\in(1,2)$ by the mean value theorem. Thus,

$$\sum_{\substack{m\equiv a\,(\mathrm{mod}\,q)\\ P^-(m)>y}}\frac{\Lambda(m)(\log m)^{2j}}{m^{2\sigma-1}} \ll \frac{(\log 4)^j(2j)!}{\phi(q)(\sigma-1)^{2j+1}}. \tag{2.3.5}$$

We now combine (2.3.1), (2.3.4) and (2.3.5) and conclude the proof of the lemma. □

Chapter 3

Prime values of multiplicative functions with small averages

In this chapter, we prove Theorem 1. First, we give an overview of its proof in Section 3.1. This overview makes the chapter easier to follow, as it illuminates the important parts of the proof as well as the ideas behind them. In Section 3.2, we establish a set of preparatory results which lead to Proposition 3.2.6, a proposition which is necessary in the beginning of the proof of Theorem 1. Finally, we reach our end goal by concluding the proof of Theorem 1 in Section 3.3.

3.1. Outline of the proof

In this section we sketch and motivate the proof of Theorem 1. But, before doing so, we give some necessary notation.

First, as in the statement of Theorem 1, for a multiplicative function g, we will be using the notation Λ_g for the unique arithmetic function defined through the relation

$$g \cdot \log = \Lambda_g * g.$$

The function Λ_g is supported on prime powers and in particular, $\Lambda_g(p) = g(p)\log p$ on the primes. It is also true that $\Lambda_{g*h} = \Lambda_g + \Lambda_h$ for any two multiplicative functions g and h. These classical properties can be proved by looking at the formal Dirichlet series of Λ_g and Λ_h.

We now introduce two classes of multiplicative functions and make a few comments about the first one. Given an integer $D \in \mathbb{N}$ and a real number $A > 0$, we define the sets

$$\mathcal{F}(D) := \{f : \mathbb{N} \to \mathbb{C}, f \text{ multiplicative}, |\Lambda_f| \leqslant D \cdot \Lambda\},$$
$$\mathcal{F}(D,A) := \left\{ f \in \mathcal{F}(D),\ \sum_{n \leqslant x} f(n) \ll \frac{x}{(\log x)^A} \text{ for all } x \geqslant 2 \right\}.$$

Note that for a function $f \in \mathcal{F}(D)$, we have that

$$-\frac{L'}{L}(s,f) = \sum_{n \geqslant 1} \frac{\Lambda_f(n)}{n^s} \quad \text{for} \quad \mathrm{Re}(s) > 1$$

and that the series convergences absolutely when $\mathrm{Re}(s) > 1$. Indeed, if $g(n)$ are the coefficients of the Dirichlet series of $-(L'/L)(\cdot,f)$, then, since $L'(\cdot,f) = (L'/L)(\cdot,f) \cdot L(\cdot,f)$, we conclude that $f \cdot \log = g * f$, by comparing coefficients. However, $f(1) = 1 \neq 0$, because f is multiplicative, and so by applying its Dirichlet inverse on both sides of $f \cdot \log = g * f$, it follows that $g = (f \cdot \log) * f^{-1}$. From the definition of Λ_f, the same reasoning shows that $\Lambda_f = (f \cdot \log) * f^{-1}$. Hence, $\Lambda_f = g$ and the claim is proven.

The class $\mathcal{F}(D)$ includes many important number-theoretic functions, like the Möbius function μ and the generalized divisor functions τ_k for $k \leqslant D$. There are technical reasons that make the class $\mathcal{F}(D)$ very convenient to work with. For example, if $f \in \mathcal{F}(D)$, then $f^{-1} \in \mathcal{F}(D)$, where f^{-1} is the Dirichlet inverse of f, namely, the inverse of f with respect to the Dirichlet convolution. Furthermore, if $f \in \mathcal{F}(D)$, then both f and f^{-1} satisfy the inequalities

$$|f| \leqslant \tau_D, \quad |f^{-1}| \leqslant \tau_D.$$

These two last results are proved in [12, Lemma 2.2].

Having now defined the classes $\mathcal{F}(D)$ and $\mathcal{F}(D,A)$, we continue by describing the ideas behind the proof of Theorem 1. For a multiset $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$, we let

$$\tau_\Gamma(n) = \sum_{d_1 \cdots d_m = n} d_1^{i\gamma_1} \cdots d_m^{i\gamma_m} \quad \text{for} \quad n \in \mathbb{N}.$$

Then, for a function $f \in \mathcal{F}(D,A)$, we consider the multiplicative function $f_\Gamma = f * \tau_\Gamma$. On the primes p, the values of the function f_Γ are

$$f_\Gamma(p) = f(p) + \sum_{\gamma \in \Gamma} p^{i\gamma}.$$

From now on, we focus on establishing a bound for the sums $\sum_{p \leqslant x} f_\Gamma(p) \log p$ for some suitable multiset Γ. For $\gamma \in \Gamma$ and $\mathrm{Re}(s) > 1$ we have that [1, p. 236]

$$-\frac{\zeta'}{\zeta}(s - i\gamma) = \sum_{n \geqslant 1} \frac{\Lambda(n) n^{i\gamma}}{n^s},$$

which implies that

$$\Lambda_{f_\Gamma}(n) = \Lambda_f(n) + \sum_{\gamma \in \Gamma} \Lambda(n) n^{i\gamma}. \tag{3.1.1}$$

Therefore, $|\Lambda_{f_\Gamma}| \leqslant (D+m)\cdot\Lambda$, as $f \in \mathcal{F}(D)$. Moreover, since Λ_{f_Γ} is supported on prime powers, Theorem 1.2.2(c) implies

$$\Bigg|\sum_{\substack{p^k\leqslant x\\ k\geqslant 2}} \Lambda_f(p^k)\Bigg| \leqslant (D+m)\sum_{\substack{p^k\leqslant x\\ k\geqslant 2}}\Lambda(n) = (D+m)(\psi(x)-\theta(x)) \ll_{m,D} \sqrt{x},$$

and so

$$\sum_{p\leqslant x} f_\Gamma(p)\log p = \sum_{n\leqslant x}\Lambda_{f_\Gamma}(n) + O_{m,D}(\sqrt{x}). \tag{3.1.2}$$

Now our goal now is to estimate the sums of the right-hand side.

In [12], Koukoulopoulos and Soundararajan bounded the sums $\sum_{n\leqslant x}\Lambda_{f_\Gamma}(n)$ by applying a smoothed version of Perron's formula [12, relation (6.1), p. 14] to the Dirichlet series of $-(L'/L)'(\cdot,f_\Gamma)$. This approach works when the logarithmic derivative $(L'/L)(\cdot,f_\Gamma)$ does not attain very large values near the vertical line $\mathrm{Re}(s)=1$, that is when $L(\cdot,f_\Gamma)$ does not vanish. For this reason, Koukoulopoulos and Soundararajan chose Γ to be the multiset of all those real numbers γ for which $1+i\gamma$ is a zero of the potential continuous extension of $L(\cdot,f)$ on $\mathrm{Re}(s)\geqslant 1$. Then, since $f_\Gamma = f * \tau_\Gamma$, we see that

$$L(s,f_\Gamma) = L(s,f)\prod_{\gamma\in\Gamma}\zeta(s-i\gamma), \tag{3.1.3}$$

and so $L(s,f_\Gamma)\neq 0$ for $\mathrm{Re}(s)\geqslant 1$, because the zeroes of $L(\cdot,f)$ are cancelled by the poles of the ζ factors of (3.1.3). Now that the choice of Γ is determined, the method of Koukoulopoulos and Soundararajan requires upper and lower bounds for $L(\cdot,f_\Gamma)$ and its two first derivatives. In this direction, they proved the following proposition [12, Proposition 2.4] which also provides information about the size of the multiset Γ.

Proposition 3.1.1. *Let f be a function of the class $\mathcal{F}(D,A)$ with $A > D+1$.*

(a) The series $L^{(j)}(s,f)$ with $0\leqslant j < A-1$ all converge uniformly in compact subsets of the region $\mathrm{Re}(s)\geqslant 1$.

(b) Counted with multiplicity, $L(s,f)$ has at most D zeroes on the line $\mathrm{Re}(s)=1$.

(c) Let Γ denote the (possibly empty) multiset of ordinates γ of zeroes $1+i\gamma$ of $L(s,f)$. Let $\widetilde{\Gamma}$ denote a (multi-)subset of Γ and let $m_{\widetilde{\Gamma}}$ denote the largest multiplicity of an element in $\widetilde{\Gamma}$. The Dirichlet series

$$L(s,f_{\widetilde{\Gamma}}) = L(s,f)\prod_{\gamma\in\widetilde{\Gamma}}\zeta(s-i\gamma)$$

and the series of derivatives $L^{(j)}(s,f_{\widetilde{\Gamma}})$ for $1\leqslant j < A-m_{\widetilde{\Gamma}}-1$ all converge uniformly in compact subsets of the region $\mathrm{Re}(s)\geqslant 1$.

If $A - m_\Gamma - 1 > 2$, then Proposition 3.1.1(c) implies the continuity of $L(\cdot, f_\Gamma)$ as well as the continuity of its two first derivatives. So, we have that the functions $L^{(j)}(\cdot, f_\Gamma)$ for $j \in \{0,1,2\}$ are bounded in the compact subsets of the half-plane $\mathrm{Re}(s) \geqslant 1$ when $A - m_\Gamma - 1 > 2$. Now, since Theorem K-S II is in the spirit of Theorem 1 in the case where there exists a single zero of multiplicity D, we can assume that $m_\Gamma \leqslant D - 1$. However, with this extra assumption, the quantity $A - m_\Gamma - 1$ might be as small as $A - D$. Hence, in order to guarantee the continuity of $L^{(j)}(\cdot, f_\Gamma)$ for $j = 0,1,2$, Koukoulopoulos and Soundararajan assumed that $A - D > 2$. On the other hand, if they had used a variant of Perron's formula for $(L'/L)(\cdot, f_\Gamma)$, so that they avoided the presence of the second derivative $L''(\cdot, f_\Gamma)$, the resulting bound at the end of their arguments would have been off by one factor of $\log x$, making it trivial. All this explains why Koukoulopoulos and Soundararajan needed the stricter condition $A > D + 2$.

In the present chapter, we circumvent the use of higher derivatives which are responsible for the condition $A > D + 2$. We do so when $m_\Gamma \leqslant D - 1$ by resorting to the recursiveness of the mean values of multiplicative functions. By this, we mean the identity

$$\sum_{n \leqslant x} f_\Gamma(n) \log n = \sum_{n \leqslant x} \Lambda_{f_\Gamma}(n) \sum_{d \leqslant x/n} f_\Gamma(d). \tag{3.1.4}$$

The idea that is about to be described is inspired by the work of Koukoulopoulos [9]. By establishing an estimate for the sums $\sum_{n \leqslant x} f_\Gamma(n)$, we can apply partial summation (Lemma 1.1.1) to bound the sums of the left-hand side in (3.1.4). One might then make use of Dirichlet's hyperbola method (Lemma 1.1.2) to obtain an estimate for

$$\sum_{n \leqslant \sqrt{x}} f_\Gamma(n) \sum_{d \leqslant x/n} \Lambda_{f_\Gamma}(d).$$

The summand corresponding to $n = 1$ is $\sum_{d \leqslant x} \Lambda_f(d)$, which is the sum that we are trying to bound. The problem is that the next term equals $f(2) \sum_{d \leqslant x/2} \Lambda_f(d)$ and this term is expected to have roughly the same size as the "main term" $\sum_{d \leqslant x} \Lambda_f(d)$. In order for this obstacle to be overcome, sieve methods come into play. By combining sieves with the hyperbola method (Lemma 1.1.2) and the "recursiveness" of averages for the function $f_\Gamma \cdot \mathbb{1}_{P^-(\cdot) > z}$, we aim to estimate

$$\sum_{\substack{n \leqslant \sqrt{x} \\ P^-(n) > z}} f_\Gamma(n) \sum_{d \leqslant x/n} \Lambda_{f_\Gamma}(d).$$

The summand for $n = 1$ is again the sum $\sum_{d \leqslant x} \Lambda_f(d)$, but this time all the summands for $n \in (1, z]$ vanish. So, the problem that occurred before is now resolved. In addition, the summands with $n > z$ are supported on a set of density $\ll 1/\log z$.

With this approach we will bound $\sum_{d\leqslant x}\Lambda_{f_\Gamma}(d)$ in terms of its integrals

$$\int_1^x \frac{|\sum_{d\leqslant t}\Lambda_{f_\Gamma}(d)|}{t^2}\mathrm{d}t,$$

much like in the case of a multiplicative function. This is the same idea which is used in the original proof of Halász's theorem [24, Section 4.3, p. 335-347]. Then, arguing along the lines of the latter allows us to complete the proof of Theorem 1. More precisely, we estimate the above integrals in the following way. After an application of the Cauchy-Schwarz inequality, we apply Parseval's theorem to bound them by an integral involving the logarithmic derivative of $L(\cdot,f_\Gamma)$. We continue by splitting this integral in the ranges $|t|\leqslant T$ and $|t|>T$. In the first range we bound trivially by using the continuity of $L(\cdot,f_\Gamma)$ and $L'(\cdot,f_\Gamma)$, provided by Proposition 3.1.1(c), and in the second one we apply Lemma 2.3.1.

3.2. Estimates of sifted partial sums

Let D be a positive integer and consider a real number $A>D+1$. Let also $f\in\mathcal{F}(D,A)$ be a multiplicative function. The ultimate goal of this section is to establish a good bound for the sifted sums $\sum_{n\leqslant x,P^-(n)>z}f_\Gamma(n)$, where Γ is the multiset of the ordinates γ of the zeroes $1+i\gamma$ of $L(\cdot,f)$. Such bounds are necessary, because they will serve as the first basic ingredient going into the proof of Theorem 1. To estimate the aforementioned sums of interest, we first estimate the simpler sums $\sum_{n\leqslant x,P^-(n)>z}f(n)$. But, in order to bound these, we need to understand the size of sums of the form $\sum_{n\leqslant x,(n,d)=1}f(n)$ for all $d\leqslant x$. Lemma 3.2.3 provides a bound for these last sums. For its proof, we will make use of the next two lemmas. The first one may be found in [1, Theorem 2.22].

Lemma 3.2.1. *Let $F:(0,+\infty)\to\mathbb{C}$ and $G:(0,+\infty)\to\mathbb{C}$ be two complex-valued functions such that $F(x)=G(x)=0$ for $x\in(0,1)$. Let h be an arithmetic function which has an inverse h^{-1} under Dirichlet. If*

$$G(x)=\sum_{n\leqslant x}h(n)F(x/n),$$

then

$$F(x)=\sum_{n\leqslant x}h^{-1}(n)G(x/n).$$

Proposition 3.2.2. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers and let k be a positive integer. For $y\geqslant 0$, we have that*

$$\#\Big\{(\nu_1,\ldots,\nu_k)\in\mathbb{N}^k:\sum_{j=1}^k a_j\nu_j\leqslant y\Big\}\leqslant\frac{\left(y+\sum_{j=1}^k a_j\right)^k}{k!\prod_{j=1}^k a_j}.$$

Proof. See [24, Theorem 3, p. 363]. □

For the sake of notational simplicity, in the proof of the following lemma, as well as for the rest of the chapter, any dependence of the implied constants on D and A will be suppressed.

Lemma 3.2.3. *Fix a natural number D and a real number $A > 0$. If $x \geqslant 3$ and $f \in \mathcal{F}(D,A)$, then*

$$\sum_{\substack{n \leqslant x \\ (n,d)=1}} f(n) \ll \frac{x}{(\log x)^A} \left(\frac{d}{\phi(d)}\right)^D \quad \text{for} \quad d \leqslant x. \tag{3.2.1}$$

Proof. For any n, there is a unique way to write it as $n = am$, where all prime divisors of a divide d and $(m,d) = 1$. Consequently, we find that

$$S(x) := \sum_{n \leqslant x} f(n) = \sum_{n \leqslant x} h(n) \sum_{\substack{m \leqslant x/n \\ (m,d)=1}} f(m),$$

where h is the multiplicative function with $h(p^\nu) = f(p^\nu)\mathbb{1}_{p|d}$ for $\nu \in \mathbb{N}$. By applying Lemma 3.2.1, we get that

$$\sum_{\substack{n \leqslant x \\ (n,d)=1}} f(n) = \sum_{n \leqslant x} h^{-1}(n) S(x/n), \tag{3.2.2}$$

where h^{-1} denotes the Dirichlet inverse of h. Note that $h \in \mathcal{F}(D)$. Therefore, $h^{-1} \in \mathcal{F}(D)$ too. Furthermore, one observes that $h^{-1}(p^\nu) = f^{-1}(p^\nu)\mathbb{1}_{p|d}$ for any $\nu \in \mathbb{N}$.

We now split the sum of the right-hand side of (3.2.2) into the two parts

$$T_1 = \sum_{n \leqslant \sqrt{x}} h^{-1}(n) S(x/n) \quad \text{and} \quad T_2 = \sum_{\sqrt{x} < n \leqslant x} h^{-1}(n) S(x/n).$$

We begin with the estimation of T_1. Since $h^{-1} \in \mathcal{F}(D)$, it is true that $|h^{-1}| \leqslant \tau_D$, and so

$$\begin{aligned} |T_1| \leqslant \sum_{\substack{n \leqslant \sqrt{x} \\ p|n \Rightarrow p|d}} \tau_D(n) |S(x/n)| &\ll \frac{x}{(\log x)^A} \sum_{\substack{n \leqslant \sqrt{x} \\ p|n \Rightarrow p|d}} \frac{\tau_D(n)}{n} \\ &\leqslant \frac{x}{(\log x)^A} \prod_{p|d} \left(1 + \frac{\tau_D(p)}{p} + \frac{\tau_D(p^2)}{p^2} + \dots\right) \\ &= \frac{x}{(\log x)^A} \prod_{p|d} \left(\sum_{j \geqslant 0} \binom{D+j-1}{j} \frac{1}{p^j}\right) = \frac{x}{(\log x)^A} \left(\frac{d}{\phi(d)}\right)^D. \end{aligned} \tag{3.2.3}$$

For the passage from the second to the third line we used (1.4.1) and then, at the last step, we calculated the series by applying the Maclaurin expansion of $x \mapsto (1-x)^{-D}$ for $x \in (0,1)$

We continue by bounding the sum T_2. Since $\tau_D(n) \ll n^{1/4} \leqslant x^{1/4}$ for $n \leqslant x$ by Lemma 1.4.1(b), we have

$$|T_2| \leqslant \sum_{\substack{\sqrt{x}<n\leqslant x \\ p|n\Rightarrow p|d}} \tau_D(n)|S(x/n)| \ll x \sum_{\substack{\sqrt{x}<n\leqslant x \\ p|n\Rightarrow p|d}} \frac{\tau_D(n)}{n} < \sqrt{x} \sum_{\substack{\sqrt{x}<n\leqslant x \\ p|n\Rightarrow p|d}} \tau_D(n) \ll x^{3/4} \sum_{\substack{n\leqslant x \\ p|n\Rightarrow p|d}} 1.$$

Proposition 3.2.2 implies that

$$\sum_{\substack{n\leqslant x \\ p|n\Rightarrow p|d}} 1 \leqslant \omega(d) \cdot \frac{(\log x + \log d)^{\omega(d)}}{\omega(d)! \left(\prod_{p|d} \log p\right)} < 4^{\omega(d)} \frac{(\log x)^{\omega(d)}}{\log 2 \cdot \omega(d)!} \ll 32^{\omega(d)} x^{1/8},$$

and so $T_2 \ll 32^{\omega(d)} x^{7/8}$. Because of Lemma 1.4.2, we have that $\omega(d) \ll \log x / \log\log x$ for $d \leqslant x$. Therefore, there exists a constant $C > 0$ such that

$$T_2 \ll x^{\frac{7}{8}+\frac{C}{\log\log x}} \ll \frac{x}{(\log x)^A} \left(\frac{d}{\phi(d)}\right)^D. \tag{3.2.4}$$

Combination of (3.2.2), (3.2.3) and (3.2.4) completes the proof of the lemma. □

Now that Lemma 3.2.3 is proven, we combine it with Lemma 1.5.2 to establish an upper bound for the sifted partial sums $\sum_{n\leqslant x, P^-(n)>z} f(n)$ of a function $f \in \mathcal{F}(D,A)$, where $D \in \mathbb{N}$ and $A > 0$.

Proposition 3.2.4. *Fix a natural number D and a positive real number A. If $x \geqslant 3$ and $f \in \mathcal{F}(D,A)$, then there exists a constant $\alpha = \alpha(D) \in (0,1)$ such that*

$$\sum_{\substack{n\leqslant x \\ P^-(n)>z}} f(n) \ll \frac{x(\log z)^D}{(\log x)^A} + \frac{x^{1-\alpha/\log z}}{\log z}$$

for all $z \in [2,x]$.

Proof. Let $C = \min\{1/16, c/2\}$, where c is the constant appearing in Lemma 2.1.1. First we show that the estimate holds trivially when $z > x^{C/(4D+1)}$. Indeed, in this case $\log x/\log z \asymp 1$, and so using Theorem 1.2.1(c) and Theorem 1.4.5 with the divisor-bounded, multiplicative function $\tau_D \cdot \mathbb{1}_{P^-(\cdot)>z}$, we conclude that

$$\left|\sum_{\substack{n\leqslant x \\ P^-(n)>z}} f(n)\right| \leqslant \sum_{\substack{n\leqslant x \\ P^-(n)>z}} \tau_D(n) \ll \frac{x}{\log x}\left(\frac{\log x}{\log z}\right)^D \ll \frac{x^{1-C/(2\log z)}}{\log z}.$$

Now it remains to prove the proposition when $z \leqslant x^{C/(4D+1)}$. Assuming that x is large enough in terms of D, when $z \leqslant D^3$, we can use Lemma 3.2.3. For $z > D^3$, the condition

$P^-(n) > z$ implies that $(n,k_D) = 1$, where $k_D = \prod_{p\leqslant D^3} p$. So, using the arithmetic functions λ^- and λ^+ of Lemma 1.5.2 with $u = C\log x/\log z$, we write

$$\sum_{\substack{n\leqslant x\\ P^-(n)>z}} f(n) = \sum_{\substack{n\leqslant x\\ (n,k_D)=1}} (\lambda^+ * 1)(n)f(n) + O\Big(\sum_{\substack{n\leqslant x\\ (n,k_D)=1}} (\lambda^+ * 1 - \lambda^- * 1)(n)|f(n)| \Big). \tag{3.2.5}$$

According to Lemma 1.4.1(a), we have $|f(n)| \leqslant \tau_D(n) \leqslant D^{\Omega(n)}$ for all $n \in \mathbb{N}$, and it then follows that

$$\sum_{\substack{n\leqslant x\\ (n,k_D)=1}} (\lambda^+ * 1 - \lambda^- * 1)(n)|f(n)| \leqslant \sum_{(d,k_D)=1} (\lambda^+(d) - \lambda^-(d))D^{\Omega(d)} \sum_{\substack{m\leqslant x/d\\ (m,k_D)=1}} D^{\Omega(m)}. \tag{3.2.6}$$

We insert the formula of Lemma 2.1.1 in the right-hand side of (3.2.6) and use the binomial theorem to expand the resulting powers $(\log(x/d))^\nu = (\log x - \log d)^\nu$ with $\nu \leqslant D-1$. Since $|\lambda^\pm| \leqslant 1$, the contribution coming from the error term of (2.1.1) is

$$\ll x^{1-c} \sum_{d\leqslant x^C} D^{\Omega(d)} \ll x^{1-c+C}(\log x)^{D-1} \leqslant x^{1-c/2}(\log x)^{D-1} \ll \frac{x(\log z)^D}{(\log x)^A},$$

where we bounded the sums $\sum_{d\leqslant x^C} D^{\Omega(n)}$ with an application of Theorems 1.4.5 and 1.2.1(c). The summands coming from the main term of (2.1.1) contain expressions of the form

$$\sum_d \frac{(\lambda^+(d) - \lambda^-(d))D^{\Omega(d)}\mathbb{1}_{(d,k_D)=1}(\log d)^j}{d}$$

for $j \in \{0,\dots,D-1\}$. Each one of these expressions is multiplied by a logarithmic factor $(\log x)^\ell$ with $\ell + j \leqslant D-1$. Since Lemma 1.5.2 implies that

$$\sum_{(d,k_D)=1} \frac{(\lambda^+(d) - \lambda^-(d))D^{\Omega(d)}(\log d)^j}{d} \ll \frac{x^{-\frac{C}{\log z}}}{(\log z)^{D-j}} \quad \text{for} \quad j \in \{0,\dots,D-1\},$$

we finally infer that

$$\sum_{n\leqslant x} (\lambda^+ * 1 - \lambda^- * 1)(n)|f(n)| \ll \frac{x^{1-\frac{C}{\log z}}(\log x)^{D-1}}{(\log z)^D} \ll \frac{x^{1-\frac{C}{2\log z}}}{\log z}.$$

So, if we define $\alpha := C/2$, relation (3.2.5) becomes

$$\sum_{\substack{n\leqslant x\\ P^-(n)>z}} f(n) = \sum_{\substack{n\leqslant x\\ (n,k_D)=1}} (\lambda^+ * 1)(n)f(n) + O\left(\frac{x(\log z)^D}{(\log x)^A} + \frac{x^{1-\alpha/\log z}}{\log z} \right). \tag{3.2.7}$$

We now turn to the estimation of the "main term" of (3.2.7). For every two natural numbers m,d, there is a unique way to write $md = m'd'$ where $(m',d') = 1$, $d \mid d'$ and all the prime factors of d' divide d. Then

$$\sum_{\substack{n\leqslant x\\(n,k_D)=1}}(\lambda^+ * 1)(n)f(n) = \sum_{\substack{md\leqslant x\\(md,k_D)=1}}\lambda^+(d)f(md) = \sum_{(d,k_D)=1}\lambda^+(d)\sum_{\substack{d'\leqslant x, d|d'\\p|d\Leftrightarrow p|d'\\(k_D,d')=1}}f(d')\sum_{\substack{m'\leqslant x/d'\\(dk_D,m')=1}}f(m').$$

We divide the inner sum on d' into two sums S_1 and S_2. In S_1 we are summing over the range $d' \leqslant \sqrt{x}$. In the sum S_2 we have $\sqrt{x} < d' \leqslant x$. We apply Lemma 3.2.3 with x large enough to the sums

$$\sum_{\substack{m'\leqslant x/d'\\(m',dk_D)=1}}f(m').$$

Then, the sum on d, coming from S_1, is

$$\ll \frac{x}{(\log x)^A}\sum_d\left(\frac{d}{\phi(d)}\right)^D\sum_{\substack{d',\, d|d'\\p|d\Leftrightarrow p|d'}}\frac{\tau_D(d')}{d'} \tag{3.2.8}$$

and the sum on d, coming from S_2, is

$$\ll \sqrt{x}\sum_d\left(\frac{d}{\phi(d)}\right)^D\sum_{\substack{d'\leqslant x\\p|d\Leftrightarrow p|d'}}\tau_D(d'). \tag{3.2.9}$$

So, in order for the proof to be completed, we need to show that the quantities of (3.2.8) and (3.2.9) are $\ll x(\log z)^D/(\log x)^A$.

First, by replicating the passage from the second to the third line of (3.2.3), we have that

$$\sum_{\substack{d',\, d|d'\\p|d\Leftrightarrow p|d'}}\frac{\tau_D(d')}{d'} \leqslant \prod_{p|d}\left(\sum_{j\geqslant 1}\frac{\tau_D(p^j)}{p^j}\right) = \prod_{p|d}\left(\left(1-\frac{1}{p}\right)^{-D}-1\right) \leqslant \frac{D^{\omega(d)}}{d}\left(\frac{d}{\phi(d)}\right)^{D+1},$$

where the last estimate follows from the inequality $(1-1/p)^{-D} - 1 \leqslant Dp^{-1}(1-1/p)^{-D-1}$, which may be obtained by applying the Mean Value Theorem to the function $t \mapsto t^{-D}$. Therefore, upon using Mertens' third estimate (Theorem 1.2.1(c)), the expression of (3.2.8) is bounded by

$$\frac{x}{(\log x)^A}\sum_{d|\prod_{p\leqslant z}p}\frac{D^{\omega(d)}}{d}\left(\frac{d}{\phi(d)}\right)^{2D+1} \leqslant \frac{x}{(\log x)^A}\prod_{p\leqslant z}\left(1+\frac{D}{p}\left(1+\frac{1}{p-1}\right)^{2D+1}\right)$$

$$
\begin{aligned}
&= \frac{x}{(\log x)^A} \prod_{p \leqslant z} \left(1 + \frac{D}{p} + O\left(\frac{1}{p^2}\right)\right) \\
&\ll \frac{x}{(\log x)^A} \exp\left(D \sum_{p \leqslant z} \frac{1}{p}\right) \\
&\ll \frac{x(\log z)^D}{(\log x)^A}.
\end{aligned}
$$

We now continue with the estimation of the expression of (3.2.9). As in the proof of Lemma 3.2.3, Proposition 3.2.2 implies that

$$\sum_{\substack{d' \\ p|d \Leftrightarrow p|d'}} 1 < 4^{\omega(d)} \frac{(\log x)^{\omega(d)}}{\log 2 \cdot \omega(d)!} \ll 16^{\omega(d)} x^{1/4}.$$

We also have the inequalities $d/\phi(d) \ll \log\log d \leqslant \log\log x$ (Lemma 1.4.4) and $\tau_D(d) \ll d^{1/8} \leqslant x^{1/8}$ for $d \leqslant x$ (Lemma 1.4.1(b)). Consequently, the expression of (3.2.9) is

$$\ll x^{7/8}(\log\log x)^D \sum_{d \leqslant x^C} 16^{\omega(d)} \ll x^{15/16}(\log x)^{15}(\log\log x)^D \ll \frac{x(\log z)^D}{(\log x)^A}$$

and the proposition is finally proved. □

The next lemma is a rather technical result and is useful for the proof of Proposition 3.2.6. It concerns a bound for the tails of a Dirichlet series and its proof is an easy application of partial summation.

Lemma 3.2.5. *Fix a natural number D and two real numbers $\varepsilon > 0$ and $z \geqslant 2$. Let Γ be a mutliset of m elements, counting the multiplicities. Let also f be an arithmetic function and suppose that there exist some $\delta \in (0,1)$ and some $A \geqslant m + 1 + \varepsilon$ such that*

$$\sum_{\substack{n \leqslant x \\ P^-(n) > z}} f_\Gamma(n) \ll \frac{x(\log z)^{D-m}}{(\log x)^{A-m}} + \frac{x^{1-\delta/\log z}}{\log z} \quad \textit{whenever} \quad x \geqslant z, \tag{3.2.10}$$

where $f_\Gamma = f * \tau_\Gamma$. For $N \geqslant \max\{3,z\}$ and $s = \sigma + it$ with $\sigma \in [1,2]$ and $t \in \mathbb{R}$, we have

$$\sum_{\substack{n > N \\ P^-(n) > z}} \frac{f_\Gamma(n)}{n^s} \ll_{\varepsilon,\delta} (1 + |t|) N^{1-\sigma} \left(\frac{(\log z)^{D-m}}{(\log N)^{A-m-1}} + N^{-\frac{\delta}{\log z}}\right).$$

Proof. Let $M > N$. Then, partial summation implies that

$$\sum_{\substack{N < n \leqslant M \\ P^-(n) > z}} \frac{f_\Gamma(n)}{n^s} = \Bigg(\sum_{\substack{n \leqslant x \\ P^-(n) > z}} f_\Gamma(n)\Bigg) x^{-s}\Bigg|_{x=N}^{M} + s\int_N^M \Bigg(\sum_{\substack{n \leqslant y \\ P^-(n) > z}} f_\Gamma(n)\Bigg) \frac{\mathrm{d}y}{y^{s+1}}. \tag{3.2.11}$$

Using the hypothesis (3.2.10) twice, once with $x=N$ and once with $x=M$, we conclude that

$$\Big(\sum_{\substack{n\leqslant x\\ P^-(n)>z}} f_\Gamma(n)\Big)x^{-s}\Big|_{x=N}^{M} \ll \frac{(\log z)^{D-m}}{(\log N)^{A-m}N^{\sigma-1}}+\frac{N^{-\frac{\delta}{\log z}}}{N^{\sigma-1}(\log z)}. \tag{3.2.12}$$

We now focus on the integral of the right-hand side of (3.2.11). The hypothesis (3.2.10) yields

$$\int_N^M \Big(\sum_{\substack{n\leqslant y\\ P^-(n)>z}} f_\Gamma(n)\Big)\frac{\mathrm{d}y}{y^{s+1}} \ll (1+|t|)\Big(\int_N^M \frac{(\log z)^{D-m}}{(\log y)^{A-m}y^\sigma}\mathrm{d}y+\frac{1}{\log z}\int_N^M \frac{\mathrm{d}y}{y^{\frac{\delta}{\log z}+\sigma}}\Big).$$

We also have that

$$\int_N^M \frac{(\log z)^{D-m}}{(\log y)^{A-m}y^\sigma}\mathrm{d}y \leqslant \frac{(\log z)^{D-m}}{N^{\sigma-1}}\int_N^\infty \frac{\mathrm{d}y}{(\log y)^{A-m}y} \ll_\varepsilon \frac{(\log z)^{D-m}}{(\log N)^{A-m-1}N^{\sigma-1}}.$$

Furthermore,

$$\int_N^M \frac{\mathrm{d}y}{y^{\frac{\delta}{\log z}+\sigma}} \leqslant \int_N^\infty \frac{\mathrm{d}y}{y^{\frac{\delta}{\log z}+\sigma}} = \frac{N^{-\frac{\delta}{\log z}}}{N^{\sigma-1}(\sigma+\delta/\log z-1)} \leqslant \delta^{-1}(\log z)N^{1-\sigma-\frac{\delta}{\log z}}.$$

Therefore, we infer that

$$\int_N^M \Big(\sum_{\substack{n\leqslant y\\ P^-(n)>z}} f_\Gamma(n)\Big)\frac{\mathrm{d}y}{y^{s+1}} \ll_{\varepsilon,\delta} (1+|t|)N^{1-\sigma}\left(\frac{(\log z)^{D-m}}{(\log N)^{A-m-1}}+N^{-\frac{\delta}{\log z}}\right). \tag{3.2.13}$$

We insert (3.2.12) and (3.2.13) in (3.2.11) and obtain the desired inequality. □

We now close Section 3.2 by reaching its final goal, namely an estimate for the sums $\sum_{n\leqslant x, P^-(n)>z} f_\Gamma(n)$, where Γ is the multiset that we have defined in the beginning of the section.

Proposition 3.2.6. *Suppose that $D\in\mathbb{N}$ and that $A>D+1$. Let also $f\in\mathcal{F}(D,A)$ and $\widetilde{\Gamma}$ be a (multi-)subset of the multiset Γ of the ordinates of the zeroes of $L(\cdot,f)$ on $\mathrm{Re}(s)=1$. Using the definition of V_t for $t\in\mathbb{R}$ before Lemma 2.1.2, if $x\geqslant z\geqslant V_{\widetilde{\Gamma}} := \max_{\gamma\in\widetilde{\Gamma}} V_\gamma$, then there exists a real number $\kappa=\kappa(D)\in(0,1)$ such that*

$$\sum_{\substack{n\leqslant x\\ P^-(n)>z}} f_{\widetilde{\Gamma}}(n) \ll_{\widetilde{\Gamma}} \frac{x(\log z)^{D-m}}{(\log x)^{A-m}}+\frac{x^{1-\kappa/\log z}}{\log z},$$

where m is the number of elements of $\widetilde{\Gamma}$ with the multiplicities being counted.

Proof. We perform induction on the number of elements m of a multisubset of Γ. The proposition holds when $m=0$ because of Proposition 3.2.4. For $m \geqslant 1$, we assume that the proposition is true for any multisubset of Γ with $m-1$ elements. We will show that it remains true for a multisubset $\widetilde{\Gamma}$ of m elements.

When $\sqrt{x} < z$, then $\log x \asymp \log z$ and we may argue as in the beginning of the proof of Proposition 3.2.4 to show that

$$\sum_{\substack{n \leqslant x \\ P^-(n)>z}} f_{\widetilde{\Gamma}}(n) \ll_{\widetilde{\Gamma},\varepsilon} \frac{x^{1-\varepsilon/\log z}}{\log z},$$

for any $\varepsilon > 0$.

If $\sqrt{x} \geqslant z$, let γ be an element of $\widetilde{\Gamma}$ and write $\widetilde{\Gamma} = \Gamma' \cup \{\gamma\}$. We have that the cardinality of Γ' is $m-1$ and that $f_{\widetilde{\Gamma}}(n) = \sum_{ab=n} f_{\Gamma'}(a) b^{i\gamma}$ for all $n \in \mathbb{N}$. So,

$$\sum_{\substack{n \leqslant x \\ P^-(n)>z}} f_{\widetilde{\Gamma}}(n) = \sum_{\substack{a \leqslant \sqrt{x} \\ P^-(a)>z}} f_{\Gamma'}(a) \sum_{\substack{b \leqslant x/a \\ P^-(b)>z}} b^{i\gamma} + \sum_{\substack{b \leqslant \sqrt{x} \\ P^-(b)>z}} b^{i\gamma} \sum_{\substack{\sqrt{x}<a \leqslant x/b \\ P^-(a)>z}} f_{\Gamma'}(a) := S_1 + S_2, \tag{3.2.14}$$

say. Since $\sqrt{x} \geqslant z$, Lemma 2.1.2 gives that

$$S_1 = \frac{x^{1+i\gamma}}{1+i\gamma} \prod_{p \leqslant z} \left(1 - \frac{1}{p}\right) \sum_{\substack{a \leqslant \sqrt{x} \\ P^-(a)>z}} \frac{f_{\Gamma'}(a)}{a^{1+i\gamma}} + O\Bigg(\frac{x^{1-\frac{1}{30 \log z}}}{\log z} \sum_{\substack{a \leqslant \sqrt{x} \\ P^-(a)>z}} \frac{\tau_{D+m-1}(a)}{a^{1-\frac{1}{30 \log z}}} \Bigg). \tag{3.2.15}$$

First we bound the sum in the big-Oh term. From Theorem 1.2.1(d), it follows that

$$\begin{aligned} \sum_{\substack{a \leqslant \sqrt{x} \\ P^-(a)>z}} \frac{\tau_{D+m-1}(a)}{a^{1-\frac{1}{30 \log z}}} &\leqslant x^{\frac{1}{60 \log z}} \sum_{\substack{a \leqslant \sqrt{x} \\ P^-(a)>z}} \frac{\tau_{D+m-1}(a)}{a} \leqslant x^{\frac{1}{60 \log z}} \prod_{z<p \leqslant \sqrt{x}} \Bigg(\sum_{j \geqslant 0} \frac{\tau_{D+m-1}(p^j)}{p^j} \Bigg) \\ &= x^{\frac{1}{60 \log z}} \prod_{z<p \leqslant \sqrt{x}} \left(1 - \frac{1}{p}\right)^{-(D+m-1)} \ll x^{\frac{1}{60 \log z}} \left(\frac{\log x}{\log z} \right)^{D+m-1} \ll_{\widetilde{\Gamma}} x^{\frac{1}{40 \log z}}. \end{aligned} \tag{3.2.16}$$

We continue by bounding the sum outside the big-Oh term on the right-hand side of (3.2.15). Since we have that $V_{\Gamma'} \leqslant V_{\widetilde{\Gamma}} \leqslant z$ and that Γ' contains $m-1$ elements, the inductive hypothesis implies that for all $w \geqslant z$ it is true that

$$\sum_{\substack{a \leqslant w \\ P^-(a)>z}} f_{\Gamma'}(a) \ll_{\Gamma'} \frac{w\,(\log z)^{D-m+1}}{(\log w)^{A-m+1}} + \frac{w^{1-\kappa_0/\log z}}{\log z}, \tag{3.2.17}$$

where $\kappa_0 = \kappa_0(D)$ is some real number of (0,1). We have that $m \leqslant D$ because of part (b) of Proposition 3.1.1. So, $A > (m-1)+2$. Therefore, we can use (3.2.17) to apply Lemma

3.2.5 with $\varepsilon = 1, \delta = \kappa_0, N = \sqrt{x}$ and $\sigma = 1$ and deduce that

$$\sum_{\substack{a>\sqrt{x} \\ P^-(a)>z}} \frac{f_{\Gamma'}(a)}{a^{1+i\gamma}} \ll_{\widetilde{\Gamma}} \frac{(\log z)^{D-m+1}}{(\log x)^{A-m}} + x^{-\kappa_0/(2\log z)}.$$

Since γ is an element of $\widetilde{\Gamma}$, the complex number $1+i\gamma$ is a zero of $L(\cdot,f)$. In addition, the multiplicity of γ in Γ' is smaller than its multiplicity in Γ, because $\widetilde{\Gamma} = \Gamma' \cup \{\gamma\}$ and $\widetilde{\Gamma}$ is a multisubset of Γ. Therefore, $L(1+i\gamma,f_{\Gamma'}) = 0$, and so

$$\begin{aligned} \sum_{\substack{a\leqslant\sqrt{x} \\ P^-(a)>z}} \frac{f_{\Gamma'}(a)}{a^{1+i\gamma}} &= L(1+i\gamma,f_{\Gamma'}) \cdot \prod_{p\leqslant z} \left(\sum_{j\geqslant 0} \frac{f_{\Gamma'}(p^j)}{p^{j(1+i\gamma)}} \right)^{-1} - \sum_{\substack{a>\sqrt{x} \\ P^-(a)>z}} \frac{f_{\Gamma'}(a)}{a^{1+i\gamma}} \\ &= - \sum_{\substack{a>\sqrt{x} \\ P^-(a)>z}} \frac{f_{\Gamma'}(a)}{a^{1+i\gamma}} \ll_{\widetilde{\Gamma}} \frac{(\log z)^{D-m+1}}{(\log x)^{A-m}} + x^{-\kappa_0/(2\log z)}. \end{aligned} \tag{3.2.18}$$

Now, combination of (3.2.15) with Theorem 1.2.1(d) and the estimates (3.2.16), (3.2.18), gives

$$S_1 \ll_{\widetilde{\Gamma}} \frac{x(\log z)^{D-m}}{(\log x)^{A-m}} + \frac{x^{1-\kappa_1/\log z}}{\log z},$$

for some $\kappa_1 = \kappa_1(D) \in (0,1)$.

It only remains to estimate

$$S_2 = \sum_{\substack{b\leqslant\sqrt{x} \\ P^-(b)>z}} b^{i\gamma} \sum_{\substack{\sqrt{x}<a\leqslant x/b \\ P^-(a)>z}} f_{\Gamma'}(a).$$

In fact, we are going to show that S_2 satisfies the same bound as S_1 and then the proof of the proposition will be complete. In the innermost sum of S_2, we have that $x/b \geqslant \sqrt{x} \geqslant z$. So, using the inductive hypothesis once for $\sqrt{x}$ and once for x/b, we get that

$$S_2 \ll_{\Gamma'} \frac{x(\log z)^{D-m+1}}{(\log x)^{A-m+1}} \sum_{\substack{b\leqslant\sqrt{x} \\ P^-(b)>z}} \frac{1}{b} + \frac{x^{1-\kappa_0/\log z}}{\log z} \sum_{\substack{b\leqslant\sqrt{x} \\ P^-(b)>z}} \frac{1}{b^{1-\frac{\kappa_0}{\log z}}}. \tag{3.2.19}$$

For $u \geqslant z$, from Emample 1.4.2, it is known that $\#\{n \leqslant u : P^-(n) > z\} \ll u/\log z$. This estimate and partial summation (Lemma 1.1.1) imply that $\sum_{b\leqslant\sqrt{x},P^-(b)>z} 1/b \ll \log x/\log z$. Then,

$$\sum_{\substack{b\leqslant\sqrt{x} \\ P^-(b)>z}} \frac{1}{b^{1-\frac{\kappa_0}{\log z}}} \leqslant x^{\kappa_0/(2\log z)} \sum_{\substack{b\leqslant\sqrt{x} \\ P^-(b)>z}} \frac{1}{b} \ll x^{\kappa_0/(2\log z)} \frac{\log x}{\log z} \ll x^{2\kappa_0/(3\log z)}.$$

So, finally, the estimate (3.2.19) becomes

$$S_2 \ll_{\widetilde{\Gamma}} \frac{x(\log z)^{D-m}}{(\log x)^{A-m}} + \frac{x^{1-\kappa_0/(3\log z)}}{\log z}$$

and the proof is finished with $\kappa := \min\{\kappa_0/3,\kappa_1\}$. □

3.3. Proof of Theorem 1

We now move on to the proof of Theorem 1. First, we introduce some auxiliary notation. If g is an arithmetic function and $x \geqslant 2$ is a real number, we define

$$S(x,g) = \sum_{n \leqslant x} g(n).$$

Moreover, for $z \geqslant 1$, we set $g_z(n) = g(n)$ when $P^-(n) > z$ and $g_z(n) = 0$ otherwise.

The multiset Γ in the statement of the Theorem 1 consists of the ordinates of the zeroes of $L(\cdot,f)$ on the vertical line $\sigma = 1$. If $L(\cdot,f)$ has a single root $1+i\gamma$ of multiplicity D, then Theorem 1 follows directly from Theorem K-S II. So, according to Proposition 3.1.1(b), for the rest of this proof we assume that the largest multiplicity of the roots of $L(\cdot,f)$ on $\mathrm{Re}(s) = 1$ is at most $D-1$. Let m be the number of elements of Γ, with the multiplicities being counted. In Proposition 3.2.6 we take $\widetilde{\Gamma} = \Gamma$ and

$$z = x^{1/(L\log u)} \text{ with } L = 8(\min\{\kappa, \log\log 3\cdot(A-D-1)\})^{-1} \text{ and}$$
$$u = \min\{(\log x)^{A-D-1},\sqrt{T}\}.$$

We further assume that x is large enough in terms of Γ. Indeed, for bounded x, Theorem 1 holds trivially by adjusting the implied constant in its statement. Then, for $w \in [x^{1/4},x] \subseteq [z^{1/4},x]$, we have

$$\sum_{\substack{n\leqslant w\\ P^-(n)>z}} f_\Gamma(n) \ll_\Gamma \frac{w}{u\log x}. \tag{3.3.1}$$

Consequently,

$$\sum_{\substack{n\leqslant x\\ P^-(n)>z}} f_\Gamma(n)\log n = O(x^{1/3}) + \sum_{\substack{x^{1/4}<n\leqslant x\\ P^-(n)>z}} f_\Gamma(n)\log n \ll_\Gamma \frac{x}{u},$$

where we used the inequality $|f_\Gamma| \leqslant \tau_{D+m}$ to control the summands with $n \leqslant x^{1/4}$, whereas we used partial summation and (3.3.1) for the summands with $n > x^{1/4}$. Now, combining this bound with Dirichlet's hyperbola method (Lemma 1.1.2) applied to the convolution

$(f_\Gamma)_z * \Lambda_{(f_\Gamma)_z}$, we get

$$\sum_{\substack{n\leqslant x^{1/4}\\ P^-(n)>z}} f_\Gamma(n)S(x/n,\Lambda_{(f_\Gamma)_z}) = S(x^{1/4},(f_\Gamma)_z)S(x^{3/4},\Lambda_{(f_\Gamma)_z}) - \sum_{n\leqslant x^{3/4}} \Lambda_{(f_\Gamma)_z}(n)S(x/n,(f_\Gamma)_z) + O_\Gamma\left(\frac{x}{u}\right). \tag{3.3.2}$$

Note that $\Lambda_{(f_\Gamma)_z} = (\Lambda_{f_\Gamma})_z$. So, since $|\Lambda_{f_\Gamma}| \leqslant (D+m)\cdot\Lambda$, we deduce that $|S(x^{3/4},\Lambda_{(f_\Gamma)_z})| \leqslant \sum_{n\leqslant x^{3/4}} |\Lambda_{f_\Gamma}(n)|\mathbb{1}_{P^-(n)>z} \leqslant \sum_{n\leqslant x^{3/4}} |\Lambda_{f_\Gamma}(n)| \leqslant (D+m)\sum_{n\leqslant x^{3/4}} \Lambda(n) \ll x^{3/4}$, where the last inequality follows from Theorem 1.2.2(b). Hence, from (3.3.1) and (3.3.2), we conclude that

$$\sum_{\substack{n\leqslant x^{1/4}\\ P^-(n)>z}} f_\Gamma(n)S(x/n,\Lambda_{(f_\Gamma)_z}) \ll_\Gamma \frac{x}{u} + \frac{x}{u\log x}\sum_{\substack{n\leqslant x^{3/4}\\ P^-(n)>z}} \frac{|\Lambda_{f_\Gamma}(n)|}{n}.$$

Making use of Theorem 1.2.1(a), we get that the sum of the right-hand side is smaller than or equal to $\sum_{n\leqslant x^{3/4}} |\Lambda_{f_\Gamma}(n)|/n \leqslant (D+m)\sum_{n\leqslant x^{3/4}} \Lambda(n)/n \ll \log x$. So,

$$\sum_{\substack{n\leqslant x^{1/4}\\ P^-(n)>z}} f_\Gamma(n)S(x/n,\Lambda_{(f_\Gamma)_z}) \ll_\Gamma \frac{x}{u}.$$

In addition, for $v \geqslant 1$ we have that

$$|S(v,\Lambda_{(f_\Gamma)_z}) - S(v,\Lambda_{f_\Gamma})| \leqslant \sum_{\substack{p\leqslant z\\ p^\nu\leqslant v}} |\Lambda_{f_\Gamma}(p^\nu)| \leqslant (D+m)\sum_{\substack{p\leqslant z\\ p^\nu\leqslant v}} \Lambda(p^\nu) \ll \log v\sum_{p\leqslant z} 1 \ll z\log v.$$

Consequently, since $|f_\Gamma| \leqslant \tau_{D+m}$ and $\sum_{n\leqslant y} \tau_{D+m}(n)/n \ll (\log y)^{D+m}$ for any $y \geqslant 1$ (this can be proved with partial summation and Theorem 1.4.3), it follows that

$$\begin{aligned} S(x,\Lambda_{f_\Gamma}) + \sum_{\substack{1<n\leqslant x^{1/4}\\ P^-(n)>z}} f_\Gamma(n)S(x/n,\Lambda_{f_\Gamma}) &= \sum_{\substack{n\leqslant x^{1/4}\\ P^-(n)>z}} f_\Gamma(n)S(x/n,\Lambda_{f_\Gamma}) \\ &= \sum_{\substack{n\leqslant x^{1/4}\\ P^-(n)>z}} f_\Gamma(n)S(x/n,\Lambda_{(f_\Gamma)_z}) + O(zx^{1/4}(\log x)^{D+m}) \ll_\Gamma \frac{x}{u}. \end{aligned} \tag{3.3.3}$$

For the last estimate we used the fact that $z \leqslant x^{1/8}$, an inequality which follows from the choice of z that we made. Lemma 1.2.4 guarantees that

$$S(x/n,\Lambda_{f_\Gamma}) - S(t/n,\Lambda_{f_\Gamma}) \ll \frac{x}{nu}$$

for $t \in [x - x/u,x]$ and $n \leqslant x^{1/4}$. Using again the inequality $|f_\Gamma| \leqslant \tau_{D+m}$, we have

$$\sum_{\substack{n\leqslant x^{1/4}\\ P^-(n)>z}} \frac{|f_z(n)|}{n} \leqslant \sum_{\substack{n\leqslant x^{1/4}\\ P^-(n)>z}} \frac{\tau_{D+m}(n)}{n} \leqslant \prod_{z<p\leqslant x^{1/4}} \left(1 + \frac{\tau_{D+m}(p)}{p} + \frac{\tau_{D+m}(p^2)}{p^2} + \dots\right)$$
$$= \prod_{z<p\leqslant x^{1/4}} \left(1 - \frac{1}{p}\right)^{-(D+m)} \ll (\log u)^{D+m}.$$

We used (1.4.1) and the Taylor expansion of $x \mapsto (1-x)^{-D}$ to go from the first line to the second, whereas the last estimate follows from an application of Theorem 1.2.1(d). Thus, if we set $\Delta = x/u$, relation (3.3.3) becomes

$$\begin{aligned}
S(x,\Lambda_{f_\Gamma}) &= -\frac{1}{\Delta} \sum_{\substack{1<n\leqslant x^{1/4}\\ P^-(n)>z}} f_\Gamma(n) \int_{x-\Delta}^{x} S(t/n,\Lambda_{f_\Gamma})\mathrm{d}t + O_\Gamma\left(\frac{x(\log u)^{D+m}}{u}\right) \\
&= -\frac{1}{\Delta} \sum_{\substack{1<n\leqslant x^{1/4}\\ P^-(n)>z}} f_\Gamma(n)n \int_{\frac{x-\Delta}{n}}^{\frac{x}{n}} S(t,\Lambda_{f_\Gamma})\mathrm{d}t + O_\Gamma\left(\frac{x(\log u)^{D+m}}{u}\right) \\
&= -\frac{1}{\Delta} \int_{\frac{x-\Delta}{x^{1/4}}}^{\frac{x}{z}} S(t,\Lambda_{f_\Gamma}) \left(\sum_{\substack{(x-\Delta)/t<n\leqslant x/t\\ P^-(n)>z}} f_\Gamma(n)n \right)\mathrm{d}t + O_\Gamma\left(\frac{x(\log u)^{D+m}}{u}\right).
\end{aligned}$$

Using Theorem 1.2.1(c) and Theorem 1.4.6 applied to the non-negative, divisor-bounded, multiplicative function $\tau_{D+m} \cdot \mathbb{1}_{P^-(\cdot)>z}$, we obtain

$$\left| \sum_{\substack{(x-\Delta)/t<n\leqslant x/t\\ P^-(n)>z}} f_\Gamma(n)n \right| \leqslant \frac{x}{t} \sum_{\substack{(x-\Delta)/t<n\leqslant x/t\\ P^-(n)>z}} \tau_{D+m}(n) \ll \frac{x\Delta(\log u)^{D+m}}{t^2 \log x},$$

for $t \leqslant x/z$ and $\Delta = x/u$. Thus, we arrive at the estimate

$$S(x,\Lambda_{f_\Gamma}) \ll \frac{x(\log u)^{D+m}}{\log x}\int_1^x \frac{|S(t,\Lambda_{f_\Gamma})|}{t^2}\mathrm{d}t + O_\Gamma\left(\frac{x(\log u)^{D+m}}{u}\right). \tag{3.3.4}$$

We continue by bounding the integral of the right-hand side and we start with the Cauchy-Schwarz inequality which implies that

$$\begin{aligned}
\int_1^x \frac{|S(t,\Lambda_{f_\Gamma})|}{t^2}\mathrm{d}t &\leqslant \left(\log x \int_1^x \frac{|S(t,\Lambda_{f_\Gamma})|^2}{t^3}\mathrm{d}t\right)^{1/2} \\
&\ll \left(\log x \int_1^{+\infty} \frac{|S(t,\Lambda_{f_\Gamma})|^2}{t^{3+2/\log x}}\mathrm{d}t\right)^{1/2}.
\end{aligned} \tag{3.3.5}$$

Partial summation and a suitable change of variables give

$$s\int_0^{+\infty} S(e^u,\Lambda_{f_\Gamma})e^{-u\sigma}e^{-iut}\mathrm{d}u = -\frac{L'}{L}(s,f_\Gamma) \quad \text{when} \quad \sigma > 1.$$

So, for $c = 1 + 1/\log x$, Parseval's theorem allows us to write

$$\int_1^{+\infty} \frac{|S(t,\Lambda_{f_\Gamma})|^2}{t^{3+2/\log x}}\mathrm{d}t = \int_0^{+\infty} |S(e^u,\Lambda_{f_\Gamma})|^2 e^{-2(1+1/\log x)u}\mathrm{d}u = \frac{1}{2\pi}\int_{\mathbb{R}} \left|\frac{L'}{L}(c+it,f_\Gamma)\right|^2 \frac{\mathrm{d}t}{c^2+t^2}$$

and (3.3.5) implies that

$$\int_1^x \frac{|S(t,\Lambda_{f_\Gamma})|}{t^2}\mathrm{d}t \ll \sqrt{\log x}\left(\int_{\mathbb{R}} \left|\frac{L'}{L}(c+it,f_\Gamma)\right|^2 \frac{\mathrm{d}t}{c^2+t^2}\right)^{1/2}. \tag{3.3.6}$$

As was explained in Section 2, we proceed to splitting the integral into two parts. In the first part we integrate over the interval $[-T,T]$, whereas the range of integration of the second integral consists of the large values $|t| > T$. In the beginning of the proof we assumed that the largest multiplicity of the zeros of $L(\cdot,f)$ is at most $D-1$. So, using Proposition 3.1.1 (c) and the non-vanishing of $L(s,f_\Gamma)$ on the vertical line $\sigma = 1$, we conclude that

$$\int_{|t|\leqslant T} \left|\frac{L'}{L}(c+it,f_\Gamma)\right|^2 \frac{\mathrm{d}t}{c^2+t^2} \ll \max_{|t|\leqslant T,\,\sigma\in[1,2]} \left|\frac{L'}{L}(\sigma+it,f_\Gamma)\right|^2 =: C_f(T). \tag{3.3.7}$$

The first part has been estimated and it remains to bound the second one. Since $|\Lambda_{f_\Gamma}(n)n^{-ik}| \leqslant (D+m)\Lambda(n)$ for any two positive integers n and k and $\zeta(s) \asymp (s-1)^{-1}$ in a fixed region around 1 for the Riemann ζ function, Lemma 2.3.2 yields that

$$\begin{aligned}
\int_{|t|>T} \left|\frac{L'}{L}(c+it,f_\Gamma)\right|^2 \frac{\mathrm{d}t}{c^2+t^2} &\leqslant \sum_{|k|>T-1/2} \int_{k-1/2}^{k+1/2} \left|\frac{L'}{L}(c+it,f_\Gamma)\right|^2 \frac{\mathrm{d}t}{c^2+t^2} \\
&\leqslant 4\sum_{|k|>T/2} \frac{1}{k^2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{L'}{L}(c+i(t+k),f_\Gamma)\right|^2 \mathrm{d}t \\
&\ll \left(\sum_{k>T/2}\frac{1}{k^2}\right)\cdot\int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{\zeta'}{\zeta}(c+it)\right|^2 \mathrm{d}t \\
&\asymp \frac{1}{T}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\mathrm{d}t}{\frac{1}{(\log x)^2}+t^2} \leqslant \frac{\log x}{T}\int_{\mathbb{R}} \frac{\mathrm{d}\alpha}{1+\alpha^2} \ll \frac{\log x}{T}.
\end{aligned} \tag{3.3.8}$$

The logarithmic derivative of the Riemann ζ function appeared at the passage from the second to the third line, because it is the Dirichlet series of the von Mangoldt function Λ for

$\mathrm{Re}(s) > 1$ (see [1, p. 236]). Combining (3.3.7) and (3.3.8), we get that

$$\int_{\mathbb{R}} \left| \frac{L'}{L}(c+it,f_\Gamma) \right|^2 \frac{\mathrm{d}t}{c^2+t^2} \ll C_f(T) + \frac{\log x}{T}.$$

We now insert this estimate into (3.3.6) to find that

$$\int_1^x \frac{|S(t,\Lambda_{f_\Gamma})|}{t^2} \mathrm{d}t \ll \sqrt{C_f(T)\log x} + \frac{\log x}{\sqrt{T}}.$$

Together with (3.3.4), this implies that

$$S(x,\Lambda_{f_\Gamma}) \leqslant O_{f,T}\left(\frac{x(\log u)^{D+m}}{\sqrt{\log x}}\right) + O_\Gamma\left(\frac{x(\log u)^{D+m}}{\min\{u,\sqrt{T}\}}\right).$$

Recalling the estimate (3.1.2) and the fact that we chose $u = \min\{(\log x)^{A-D-1},\sqrt{T}\}$, the proof of Theorem 1 readily follows.

Chapter 4

Linnik's estimate - An alternative proof

4.1. Plan of the proof

In this chapter we establish the refined form of Linnik's estimate (0.2.1) stated in Theorem 2. Here we present an overview of the main steps that we are going to develop.

- **Step 1: The sifting condition.** For $q \in \mathbb{N}, x \geqslant q^2$ and $a \in (\mathbb{Z}/q\mathbb{Z})^*$, the principal goal is to estimate the quantity

$$\sum_{\substack{n\leqslant x\\ n\equiv a\,(\mathrm{mod}\,q)}} \Lambda(n) - \frac{x}{\phi(q)} - \frac{\psi(a)}{\phi(q)}\sum_{n\leqslant x}\psi(n)\Lambda(n),$$

where ψ is a Dirichlet character modulo q defined as in the statement of Theorem 2. Since Λ is supported on prime powers, we note that imposing the condition $P^-(n) > y$ on the sums above is not very wasteful when y is appropriately smaller than x. Indeed, using Theorem 1.2.2(a), we have

$$\Bigg|\sum_{\substack{n\leqslant x\\ n\equiv a\,(\mathrm{mod}\,q)}} \Lambda(n) - \sum_{\substack{n\leqslant x\\ n\equiv a\,(\mathrm{mod}\,q)\\ P^-(n)>y}} \Lambda(n)\Bigg| \leqslant \sum_{\substack{p^\nu\leqslant x\\ p\leqslant y}} \Lambda(p^\nu) \leqslant \pi(y)\log x \ll \frac{y\log x}{\log y} \tag{4.1.1}$$

and the same estimate holds for the differences of the analogous sums of $\psi\cdot\Lambda$. Consequently, the sifting condition $P^-(n) > y$ is harmless, as it excludes sums of negligible contribution. However, it can also be beneficial, because in later steps it can save a $\log y$ from the obtained bounds. For this reason, in Lemma 4.2.1, we turn our focus to the estimation of the expressions

$$\sum_{\substack{n\leqslant x\\ n\equiv a\,(\mathrm{mod}\,q)\\ P^-(n)>y}} \Lambda(n) - \frac{x}{\phi(q)} - \frac{\psi(a)}{\phi(q)}\sum_{\substack{n\leqslant x\\ P^-(n)>y}}\psi(n)\Lambda(n). \tag{4.1.2}$$

- **Step 2: Inspiration from Halász's theorem.** As in the proof of Theorem 1, inspired by the classical proof of Halász's theorem [24, Section 4.3, p. 335-347], we use the recursiveness of the mean values of multiplicative functions (that is, the identity (3.1.4)) with sieve estimates in order to bound the quantities (4.1.2) by averages of themselves. Then, after a few more technical steps, this leads to an estimate involving the sums of integrals

$$\sum_{\substack{\chi\,(\mathrm{mod}\,q)\\ \chi\neq\chi_0,\psi}} \int_{\frac{\sqrt{x}}{2}}^{x} \Bigg| \sum_{\substack{\ell\leqslant t\\ P^-(\ell)>y}} \Lambda(\ell)\chi(\ell)\Bigg|^2 \frac{\mathrm{d}t}{t^3}. \tag{4.1.3}$$

- **Step 3: Logarithmic weights.** At the analogous point in the proof of Theorem 1, we applied Parseval's identity to convert the integrals of the summatory functions into L^2 norms of Dirichlet series. Here we delay the application of Parseval's theorem and choose to switch from the sums of integrals (4.1.3) to

$$\sum_{\substack{\chi\,(\mathrm{mod}\,q)\\ \chi\neq\chi_0,\psi}} \int_{\frac{\sqrt[4]{x}}{2}}^{x} \Bigg| \sum_{\substack{\ell\leqslant t\\ P^-(\ell)>y}} \Lambda(\ell)\chi(\ell)(\log \ell)^k\Bigg|^2 \frac{\mathrm{d}t}{t^{3+2/\log x}}.$$

  We basically do this with partial summation (Lemma 1.1.1) and the motivating factor for this change is that we can now apply Parseval's theorem to the new integrals and then optimize k to obtain a better, non-trivial bound.

- **Step 4: Parseval's theorem.** We apply Parseval's theorem and bound the last sums of integrals by

$$\sum_{\substack{\chi\,(\mathrm{mod}\,q)\\ \chi\neq\chi_0,\psi}} \int_{\mathbb{R}} \Bigg| \left(\frac{L_y'}{L_y}\right)^{(k)}(c+it,\chi)\Bigg|^2 \frac{\mathrm{d}t}{c^2+t^2}, \tag{4.1.4}$$

  where $c = 1 + 1/(\log x)$. For some T, suitably chosen in terms of x, we treat these integrals separately in the ranges $|t| > T$ and $|t| \leqslant T$.

- **Step 5: Large values of t.** To assess the contribution of the integrals over the interval $|t| > T$, we apply Lemmas 1.3.2 and 2.3.2.

- **Step 6: Small values of t.** For the range $|t| \leqslant T$ of the smaller values of t, we use Lemma 4.2.3 and the bounds of Section 2.2 on $L_y(\cdot,\chi)$. This handles the presence of the peculiar derivatives of $(L'/L)(\cdot,\chi)$ and it eventually reduces the estimation of (4.1.4) to that of

$$\sum_{\substack{\chi\,(\mathrm{mod}\,q)\\ \chi\neq\chi_0,\psi}} \int_{\mathbb{R}} |L_y^{(j)}(c+it,\chi)|^2 \frac{\mathrm{d}t}{c^2+t^2}.$$

We then apply Parseval's theorem again and complete the estimation by applying sieves to the resulting sums.

- **Step 7. Optimization.** For the last part of the proof we collect all our bounds and choose k appropriately in terms of x and y in order to optimize our estimation. This will basically complete the proof of Theorem 2.

4.2. The proof of Theorem 2

The objective of this section is to prove Theorem 2. In this direction, we prove Lemmas 4.2.1, 4.2.2 and 4.2.4. Then, the proof of Theorem 2 follows by putting these lemmas together.

Lemma 4.2.1. *Assume that q is a positive integer and let x and y be two real numbers such that $\sqrt{x} \geqslant y \geqslant (10q)^{100}$. If κ and λ are the constants from the statements of Lemmas 2.1.3 and 2.1.4, respectively, then for $\delta \in (0, \min\{(\log 2)/8, \kappa/6, \lambda/6\})$, we put $D = x^{1-\delta/(\log y)}$. Moreover, if ψ is the Dirichlet character from the statement of Theorem 2, we define*

$$\Delta(u,z;r,b) := \sum_{\substack{\ell \leqslant u \\ \ell \equiv b \,(\mathrm{mod}\, r) \\ P^-(\ell) > z}} \Lambda(\ell) - \frac{1}{\phi(r)} \sum_{\substack{\ell \leqslant u \\ P^-(\ell) > z}} \Lambda(\ell)\chi_0(\ell) - \frac{\psi(b)}{\phi(r)} \sum_{\substack{\ell \leqslant u \\ P^-(\ell) > z}} \Lambda(\ell)\psi(\ell),$$

for $u \geqslant z \geqslant 1, r \in \mathbb{N}$ and $b \in (\mathbb{Z}/r\mathbb{Z})^*$. Then, for any $k \in \mathbb{N}$, we have

$$\Delta(x,y;q,a) = -\frac{1}{D} \sum_{\substack{1 < m \leqslant \sqrt{x} \\ P^-(m) > y}} \int_{x-D}^{x} \Delta\Big(\frac{t}{m}, y; q, a\overline{m}\Big) \mathrm{d}t + O_\delta\bigg(\frac{x^{1-\delta/(2\log y)}}{\phi(q)}\bigg).$$

Proof. Since $\log = \Lambda * 1$, by taking the logarithm of the unique prime factorization of n, we have that

$$\sum_{\substack{n \leqslant x \\ n \equiv a \,(\mathrm{mod}\, q) \\ P^-(n) > y}} \log n = \sum_{\substack{m \leqslant \sqrt{x} \\ P^-(m) > y}} \sum_{\substack{\ell \leqslant \frac{x}{m} \\ \ell \equiv a\overline{m} \,(\mathrm{mod}\, q) \\ P^-(\ell) > y}} \Lambda(\ell) + \sum_{\substack{\ell \leqslant \sqrt{x} \\ P^-(n) > y}} \Lambda(\ell) \sum_{\substack{\sqrt{x} < m \leqslant \frac{x}{\ell} \\ m \equiv a\overline{\ell} \,(\mathrm{mod}\, q) \\ P^-(m) > y}} 1. \tag{4.2.1}$$

For $\chi \in \{\chi_0, \psi\}$, it is true that $\log \cdot \chi = (\Lambda \cdot \chi) * \chi$, which directly follows from the previous convolution identity and the complete multiplicativity of χ. Therefore, we similarly have

$$\sum_{\substack{n \leqslant x \\ P^-(n) > y}} \chi(n) \log n = \sum_{\substack{m \leqslant \sqrt{x} \\ P^-(m) > y}} \chi(m) \sum_{\substack{\ell \leqslant \frac{x}{m} \\ P^-(\ell) > y}} \Lambda(\ell)\chi(\ell) + \sum_{\substack{\ell \leqslant \sqrt{x} \\ P^-(n) > y}} \Lambda(\ell)\chi(\ell) \sum_{\substack{\sqrt{x} < m \leqslant \frac{x}{\ell} \\ P^-(m) > y}} \chi(m). \tag{4.2.2}$$

We now use Lemma 2.1.3 for the sums $\sum_{n \leqslant x,\, P^-(n) > y} \chi_0(n) \log n$ and $\sum_{n \leqslant x,\, P^-(n) > y} \psi(n) \log n$. We also use Lemma 2.1.4 for the sum $\sum_{\substack{n \leqslant x,\, P^-(n) > y \\ n \equiv a \,(\mathrm{mod}\, q)}} \log n$. Combination of the obtained formulas yields

$$\sum_{\substack{n\leqslant x\\ n\equiv a\,(\mathrm{mod}\,q)\\ P^-(n)>y}} \log n - \frac{1}{\phi(q)}\sum_{\substack{n\leqslant x\\ P^-(n)>y}} \chi_0(n)\log n - \frac{\psi(a)}{\phi(q)}\sum_{\substack{n\leqslant x\\ P^-(n)>y}} \psi(n)\log n \ll \frac{x^{1-c_1/\log y}}{\phi(q)}, \tag{4.2.3}$$

where $c_1 = \min\{\kappa,\lambda\}/2$. Now, for simplicity, we put

$$\Delta^*(u,y;q,b) := \sum_{\substack{m\leqslant u\\ m\equiv b\,(\mathrm{mod}\,q)\\ P^-(m)>y}} 1 - \frac{1}{\phi(q)}\sum_{\substack{m\leqslant u\\ P^-(m)>y}} \chi_0(m) - \frac{\psi(b)}{\phi(q)}\sum_{\substack{m\leqslant u\\ P^-(m)>y}} \psi(m).$$

We take (4.2.2) once with $\chi = \chi_0$ and once with $\chi = \psi$ and then we add the two relations term by term. Then we subtract the resulting relation from (4.2.1). This leads to

$$\begin{aligned}\sum_{\substack{m\leqslant\sqrt{x}\\ P^-(m)>y}} \Delta\Big(\frac{x}{m},y;q,a\overline{m}\Big) &= -\sum_{\substack{\ell\leqslant\sqrt{x}\\ P^-(\ell)>y}} \Lambda(\ell)\Delta^*\Big(\frac{x}{\ell},y;q,a\overline{\ell}\Big)\\ &\quad + \sum_{\substack{\ell\leqslant\sqrt{x}\\ P^-(\ell)>y}} \Lambda(\ell)\Delta^*(\sqrt{x},y;q,a\overline{\ell}) + O\bigg(\frac{x^{1-c_1/\log y}}{\phi(q)}\bigg),\end{aligned}$$

where the big-Oh term comes from the contribution of the left-hand side of (4.2.3). Since $x/\ell \geqslant \sqrt{x} \geqslant y$ for $\ell \leqslant \sqrt{x}$, we can apply Lemmas 2.1.3 and 2.1.4 with $j = 0$ to bound the three sums in the definitions of $\Delta^*(x/\ell,y;q,a\overline{\ell})$ and $\Delta^*(\sqrt{x},y;q,a\overline{\ell})$. Doing so yields

$$\begin{aligned}\sum_{\substack{\ell\leqslant\sqrt{x}\\ P^-(\ell)>y}} \Lambda(\ell)\Big\{\Delta^*\Big(\frac{x}{\ell},y;q,a\overline{\ell}\Big) - \Delta^*(\sqrt{x},y;q,a\overline{\ell})\Big\} &\ll \frac{x^{1-c_1/\log y}}{\phi(q)\log y}\sum_{\substack{\ell\leqslant\sqrt{x}\\ P^-(\ell)>y}} \frac{\Lambda(\ell)}{\ell^{1-c_1/\log y}}\\ &\ll \frac{x^{1-c_1/(2\log y)}}{\phi(q)\log y}\sum_{\ell\leqslant\sqrt{x}} \frac{\Lambda(\ell)}{\ell}\\ &\ll \frac{x^{1-c_1/(2\log y)}\log x}{\phi(q)\log y} \ll \frac{x^{1-c_1/(3\log y)}}{\phi(q)},\end{aligned}$$

where we applied Theorem 1.2.1(a) to bound the sums $\sum_{\ell\leqslant\sqrt{x}}\Lambda(\ell)/\ell$. Hence,

$$\Delta(x,y;q,a) = -\sum_{\substack{1<m\leqslant\sqrt{x}\\ P^-(m)>y}} \Delta\Big(\frac{x}{m},y;q,a\overline{m}\Big) + O\bigg(\frac{x^{1-c_2/\log y}}{\phi(q)}\bigg), \tag{4.2.4}$$

where $c_2 = c_1/3$.

Since $x \geqslant (10q)^{200} > (2q)^8$, we have that $D/m \geqslant 2q\sqrt{x/m}$ when $m \leqslant \sqrt{x}$. Thus, for $t \in [x-D,x]$, an application of Lemma 1.2.4 gives

$$\sum_{\substack{\frac{t}{m}<\ell\leqslant\frac{x}{m} \\ \ell\equiv a\overline{m}\,(\mathrm{mod}\,q) \\ P^-(\ell)>y}} \Lambda(\ell) \leqslant \sum_{\substack{\frac{x-D}{m}<\ell\leqslant\frac{x}{m} \\ \ell\equiv a\overline{m}\,(\mathrm{mod}\,q)}} \Lambda(\ell) \ll \frac{D}{m\phi(q)}.$$

Similarly,

$$\sum_{\substack{\frac{t}{m}<\ell\leqslant\frac{x}{m} \\ P^-(\ell)>y}} \Lambda(\ell)\chi_0(\ell) \ll \frac{D}{m} \qquad \text{and} \qquad \sum_{\substack{\frac{t}{m}<\ell\leqslant\frac{x}{m} \\ P^-(\ell)>y}} \Lambda(\ell)\psi(\ell) \ll \frac{D}{m},$$

by bounding the characters trivially before making use of Lemma 1.2.4. With these estimates, we deduce that

$$\begin{aligned}\sum_{\substack{1<m\leqslant\sqrt{x} \\ P^-(m)>y}} \Delta\Big(\frac{x}{m},y;q,a\overline{m}\Big) &= \frac{1}{D}\sum_{\substack{1<m\leqslant\sqrt{x} \\ P^-(m)>y}} \int_{x-D}^{x} \Delta\Big(\frac{t}{m},y;q,a\overline{m}\Big)\mathrm{d}t \\ &\quad + O\Bigg(\frac{D}{\phi(q)}\sum_{\substack{m\leqslant\sqrt{x} \\ P^-(m)>y}} \frac{1}{m}\Bigg). \end{aligned} \tag{4.2.5}$$

But according to Theorem 1.2.1(d),

$$\sum_{\substack{m\leqslant\sqrt{x} \\ P^-(m)>y}} \frac{1}{m} \leqslant \prod_{y<p\leqslant\sqrt{x}} \Big(1-\frac{1}{p}\Big)^{-1} \ll \frac{\log x}{\log y},$$

and so (4.2.4) and (4.2.5) lead to the desired result since $x^{\delta/(2\log y)} \gg_\delta \log x/(\log y)$. □

Lemma 4.2.2. *Consider a positive integer q and a real number $\delta > 0$. Moreover, let x and y be two real numbers such that $x^\delta \geqslant y \geqslant 4q^2$. If ψ is defined as in Theorem 2 and $L_y(s,\chi) = \sum_{P^-(n)>y}\chi(n)n^{-s}$ for the character χ mod q when $\mathrm{Re}(s) > 1$, then, for $k \in \mathbb{N}$, we have*

$$\begin{aligned}\frac{1}{D}\sum_{\substack{1<m\leqslant\sqrt{x} \\ P^-(m)>y}} \int_{x-D}^{x} \Delta\Big(\frac{t}{m},y;q,a\overline{m}\Big)\mathrm{d}t &\ll \frac{x^{1/2+\delta/(\log y)}}{\log y}\sqrt{\frac{\log x}{\phi(q)}} \\ &+ \frac{M^k x^{1+\delta/(\log y)}(\log x)^{1/2-k}}{\phi(q)\log y}\Bigg(\sum_{\substack{\chi\,(\mathrm{mod}\,q) \\ \chi\neq\chi_0,\psi}} \int_{\mathbb{R}} \bigg|\Big(\frac{L_y'}{L_y}\Big)^{(k)}(c+it,\chi)\bigg|^2 \frac{\mathrm{d}t}{c^2+t^2}\Bigg)^{1/2},\end{aligned}$$

where $D = x^{1-\delta/(\log y)}$, $c = 1 + 1/(\log x)$ and $M > 0$ is some sufficiently large constant.

Proof. By referring to the orthogonality of the characters modulo q (Theorem 1.3.1), it follows that

$$\Delta\Big(\frac{t}{m},y;q,a\overline{m}\Big)=\frac{1}{\phi(q)}\sum_{\substack{\chi\,(\mathrm{mod}\,q)\\ \chi\neq\chi_0,\psi}}\overline{\chi}(a)\chi(m)\sum_{\substack{\ell\leqslant\frac{t}{m}\\ P^-(\ell)>y}}\Lambda(\ell)\chi(\ell).$$

Therefore,

$$\begin{aligned}
&\frac{1}{D}\Bigg|\sum_{\substack{1<m\leqslant\sqrt{x}\\ P^-(m)>y}}\int_{x-D}^{x}\Delta\Big(\frac{t}{m},y;q,a\overline{m}\Big)\mathrm{d}t\Bigg|\\
&\leqslant\frac{1}{D\phi(q)}\sum_{\substack{\chi\,(\mathrm{mod}\,q)\\ \chi\neq\chi_0,\psi}}\Bigg|\sum_{\substack{1<m\leqslant\sqrt{x}\\ P^-(m)>y}}\chi(m)\int_{x-D}^{x}\Bigg(\sum_{\substack{\ell\leqslant\frac{t}{m}\\ P^-(\ell)>y}}\Lambda(\ell)\chi(\ell)\Bigg)\mathrm{d}t\Bigg|\\
&=\frac{1}{D\phi(q)}\sum_{\substack{\chi\,(\mathrm{mod}\,q)\\ \chi\neq\chi_0,\psi}}\Bigg|\sum_{\substack{1<m\leqslant\sqrt{x}\\ P^-(m)>y}}\chi(m)m\int_{\frac{x-D}{m}}^{\frac{x}{m}}\Bigg(\sum_{\substack{\ell\leqslant t\\ P^-(\ell)>y}}\Lambda(\ell)\chi(\ell)\Bigg)\mathrm{d}t\Bigg|\\
&=\frac{1}{D\phi(q)}\sum_{\substack{\chi\,(\mathrm{mod}\,q)\\ \chi\neq\chi_0,\psi}}\Bigg|\sum_{\substack{1<m\leqslant\sqrt{x}\\ P^-(m)>y}}\chi(m)m\int_{\frac{x-D}{\sqrt{x}}}^{\frac{x}{y}}\mathbb{1}_{\left(\frac{x-D}{m},\frac{x}{m}\right]}(t)\cdot\Bigg(\sum_{\substack{\ell\leqslant t\\ P^-(\ell)>y}}\Lambda(\ell)\chi(\ell)\Bigg)\mathrm{d}t\Bigg|\\
&=\frac{1}{D\phi(q)}\sum_{\substack{\chi\,(\mathrm{mod}\,q)\\ \chi\neq\chi_0,\psi}}\Bigg|\int_{\frac{x-D}{\sqrt{x}}}^{\frac{x}{y}}\Bigg(\sum_{\substack{\ell\leqslant t\\ P^-(\ell)>y}}\Lambda(\ell)\chi(\ell)\Bigg)\Bigg(\sum_{\substack{\frac{x-D}{t}<m\leqslant\frac{x}{t}\\ P^-(m)>y}}\chi(m)m\Bigg)\mathrm{d}t\Bigg|.
\end{aligned}$$

We move the absolute value inside the integral and then use the Cauchy-Schwarz inequality twice to obtain

$$\begin{aligned}
&\frac{1}{D}\Bigg|\sum_{\substack{1<m\leqslant\sqrt{x}\\ P^-(m)>y}}\int_{x-D}^{x}\Delta\Big(\frac{t}{m},y;q,a\overline{m}\Big)\mathrm{d}t\Bigg|\\
&\leqslant\frac{1}{D\phi(q)}\int_{\frac{x-D}{\sqrt{x}}}^{\frac{x}{y}}\Bigg(\sum_{\substack{\chi\,(\mathrm{mod}\,q)\\ \chi\neq\chi_0,\psi}}\Bigg|\sum_{\substack{\ell\leqslant t\\ P^-(\ell)>y}}\Lambda(\ell)\chi(\ell)\Bigg|^2\Bigg)^{1/2}\Bigg(\sum_{\substack{\chi\,(\mathrm{mod}\,q)\\ \chi\neq\chi_0,\psi}}\Bigg|\sum_{\substack{\frac{x-D}{t}<m\leqslant\frac{x}{t}\\ P^-(m)>y}}\chi(m)m\Bigg|^2\Bigg)^{1/2}\mathrm{d}t\\
&\leqslant\frac{1}{D\phi(q)}\Bigg(\sum_{\substack{\chi\,(\mathrm{mod}\,q)\\ \chi\neq\chi_0,\psi}}\int_{\frac{x-D}{\sqrt{x}}}^{\frac{x}{y}}\Bigg|\sum_{\substack{\ell\leqslant t\\ P^-(\ell)>y}}\Lambda(\ell)\chi(\ell)\Bigg|^2\frac{\mathrm{d}t}{t^3}\Bigg)^{1/2}\\
&\times\Bigg(\int_{\frac{x-D}{\sqrt{x}}}^{\frac{x}{y}}\sum_{\chi\,(\mathrm{mod}\,q)}\Bigg|\sum_{\substack{\frac{x-D}{t}<m\leqslant\frac{x}{t}\\ P^-(m)>y}}\chi(m)m\Bigg|^2t^3\mathrm{d}t\Bigg)^{1/2}.
\end{aligned}\tag{4.2.6}$$

However, due to Lemma 1.3.2, we have that

$$\sum_{\chi\,(\mathrm{mod}\,q)}\Bigg|\sum_{\substack{\frac{x-D}{t}<m\leqslant\frac{x}{t}\\ P^-(m)>y}}\chi(m)m\Bigg|^2=\phi(q)\sum_{b\in(\mathbb{Z}/q\mathbb{Z})^*}\Bigg(\sum_{\substack{\frac{x-D}{t}<m\leqslant\frac{x}{t}\\ m\equiv b\,(\mathrm{mod}\,q)\\ P^-(m)>y}}m\Bigg)^2$$

$$\leqslant\frac{\phi(q)x^2}{t^2}\sum_{b\in(\mathbb{Z}/q\mathbb{Z})^*}\Bigg(\sum_{\substack{\frac{x}{2t}<m\leqslant\frac{x}{t}\\ m\equiv b\,(\mathrm{mod}\,q)\\ P^-(m)>y}}1\Bigg)^2\ll\frac{x^4}{t^4(\log y)^2}.$$

Note that we enlarged the range of summation over m because $x^\delta\geqslant y$. At the final step, we applied Theorem 1.4.6 to each sum of the last line under the condition $y\geqslant 4q^2$. So,

$$\int_{\frac{x-D}{\sqrt{x}}}^{\frac{x}{y}}\sum_{\chi\,(\mathrm{mod}\,q)}\Bigg|\sum_{\substack{\frac{x-D}{t}<m\leqslant\frac{x}{t}\\ P^-(m)>y}}\chi(m)m\Bigg|^2t^3\mathrm{d}t\ll\frac{x^4}{(\log y)^2}\int_1^x\frac{\mathrm{d}t}{t}=\frac{x^4\log x}{(\log y)^2}.$$

This means that (4.2.6) leads to the estimate

$$\frac{1}{D}\sum_{\substack{1<m\leqslant\sqrt{x}\\ P^-(m)>y}}\int_{x-D}^{x}\Delta\Big(\frac{t}{m},y;q,a\overline{m}\Big)\mathrm{d}t\ll\frac{x^{1+\delta/(\log y)}\sqrt{\log x}}{\phi(q)\log y}$$

$$\times\Bigg(\sum_{\substack{\chi\,(\mathrm{mod}\,q)\\ \chi\neq\chi_0,\psi}}\int_{\frac{\sqrt{x}}{2}}^{x}\Bigg|\sum_{\substack{\ell\leqslant t\\ P^-(\ell)>y}}\Lambda(\ell)\chi(\ell)\Bigg|^2\frac{\mathrm{d}t}{t^3}\Bigg)^{1/2}. \tag{4.2.7}$$

Now, we continue by bounding the sum of integrals at the second line of (4.2.7). For $t\in[\sqrt{x}/2,x]$, by Theorem 1.2.2(b), we have that

$$\sum_{\substack{\ell\leqslant t\\ P^-(\ell)>y}}\Lambda(\ell)\chi(\ell)=O(\sqrt{t})+\int_{\sqrt{t}}^{t}(\log u)^{-k}\mathrm{d}\Bigg(\sum_{\substack{\ell\leqslant u\\ P^-(\ell)>y}}\Lambda(\ell)\chi(\ell)(\log\ell)^k\Bigg)$$

$$=O(\sqrt{t})+(\log t)^{-k}\sum_{\substack{\ell\leqslant t\\ P^-(\ell)>y}}\Lambda(\ell)\chi(\ell)(\log\ell)^k$$

$$+\int_{\sqrt{t}}^{t}\sum_{\substack{\ell\leqslant u\\ P^-(\ell)>y}}\Lambda(\ell)\chi(\ell)(\log\ell)^k\frac{\mathrm{d}u}{u(\log u)^{k+1}}$$

$$\ll\sqrt{t}+M^k(\log x)^{-k}\Bigg|\sum_{\substack{\ell\leqslant t\\ P^-(\ell)>y}}\Lambda(\ell)\chi(\ell)(\log\ell)^k\Bigg|$$

$$+M^k(\log x)^{-(k+1)}\int_{\sqrt{t}}^{t}\Bigg|\sum_{\substack{\ell\leqslant u\\ P^-(\ell)>y}}\Lambda(\ell)\chi(\ell)(\log\ell)^k\Bigg|\frac{\mathrm{d}u}{u},$$

where M is some sufficiently large positive constant. Because of the basic inequality $3(\alpha^2+\beta^2+\gamma^2) \geqslant (\alpha+\beta+\gamma)^2$ for all $\alpha,\beta,\gamma\in\mathbb{R}$, we deduce that

$$\Big|\sum_{\substack{\ell\leqslant t\\ P^-(\ell)>y}}\Lambda(\ell)\chi(\ell)\Big|^2 \ll t + M^{2k}(\log x)^{-2k}\Big|\sum_{\substack{\ell\leqslant t\\ P^-(\ell)>y}}\Lambda(\ell)\chi(\ell)(\log\ell)^k\Big|^2 + M^{2k}(\log x)^{-2(k+1)}\bigg(\int_{\sqrt{t}}^{t}\Big|\sum_{\substack{\ell\leqslant u\\ P^-(\ell)>y}}\Lambda(\ell)\chi(\ell)(\log\ell)^k\Big|\frac{\mathrm{d}u}{u}\bigg)^2.$$

But, upon noticing that

$$\bigg(\int_{\sqrt{t}}^{t}\Big|\sum_{\substack{\ell\leqslant u\\ P^-(\ell)>y}}\Lambda(\ell)\chi(\ell)(\log\ell)^k\Big|\frac{\mathrm{d}u}{u}\bigg)^2 \leqslant \frac{\log t}{2}\int_{\sqrt{t}}^{t}\Big|\sum_{\substack{\ell\leqslant u\\ P^-(\ell)>y}}\Lambda(\ell)\chi(\ell)(\log\ell)^k\Big|^2\frac{\mathrm{d}u}{u},$$

by the Cauchy-Schwarz inequality, we conclude that

$$\begin{aligned}&\bigg(\sum_{\substack{\chi\,(\mathrm{mod}\,q)\\ \chi\neq\chi_0,\psi}}\int_{\frac{\sqrt{x}}{2}}^{x}\Big|\sum_{\substack{\ell\leqslant t\\ P^-(\ell)>y}}\Lambda(\ell)\chi(\ell)\Big|^2\frac{\mathrm{d}t}{t^3}\bigg)^{1/2}\\ &\ll\sqrt{\frac{\phi(q)}{x}} + M^k(\log x)^{-k}\bigg(\sum_{\substack{\chi\,(\mathrm{mod}\,q)\\ \chi\neq\chi_0,\psi}}\int_{\frac{\sqrt{x}}{2}}^{x}\Big|\sum_{\substack{\ell\leqslant t\\ P^-(\ell)>y}}\Lambda(\ell)\chi(\ell)(\log\ell)^k\Big|^2\frac{\mathrm{d}t}{t^3}\bigg)^{1/2}\\ &+M^k(\log x)^{-k}\bigg(\sum_{\substack{\chi\,(\mathrm{mod}\,q)\\ \chi\neq\chi_0,\psi}}\int_{\frac{\sqrt{x}}{2}}^{x}\bigg(\int_{\sqrt{t}}^{t}\Big|\sum_{\substack{\ell\leqslant u\\ P^-(\ell)>y}}\Lambda(\ell)\chi(\ell)(\log\ell)^k\Big|^2\frac{\mathrm{d}u}{u}\bigg)\frac{\mathrm{d}t}{t^3}\bigg)^{1/2}.\end{aligned}$$

We use Fubini's theorem to interchange the order of integration, and so the double integral above equals

$$\int_{\frac{\sqrt[4]{x}}{\sqrt{2}}}^{x}\Big|\sum_{\substack{\ell\leqslant u\\ P^-(\ell)>y}}\Lambda(\ell)\chi(\ell)(\log\ell)^k\Big|^2\bigg(\int_{u}^{u^2}\frac{\mathrm{d}t}{t^3}\bigg)\frac{\mathrm{d}u}{u} \leqslant \int_{\frac{\sqrt[4]{x}}{2}}^{x}\Big|\sum_{\substack{\ell\leqslant u\\ P^-(\ell)>y}}\Lambda(\ell)\chi(\ell)(\log\ell)^k\Big|^2\frac{\mathrm{d}u}{u^3}.$$

Hence,

$$\begin{aligned}&\bigg(\sum_{\substack{\chi\,(\mathrm{mod}\,q)\\ \chi\neq\chi_0,\psi}}\int_{\frac{\sqrt{x}}{2}}^{x}\Big|\sum_{\substack{\ell\leqslant t\\ P^-(\ell)>y}}\Lambda(\ell)\chi(\ell)\Big|^2\frac{\mathrm{d}t}{t^3}\bigg)^{1/2}\\ &\ll\sqrt{\frac{\phi(q)}{x}} + M^k(\log x)^{-k}\bigg(\sum_{\substack{\chi\,(\mathrm{mod}\,q)\\ \chi\neq\chi_0,\psi}}\int_{\frac{\sqrt[4]{x}}{2}}^{x}\Big|\sum_{\substack{\ell\leqslant t\\ P^-(\ell)>y}}\Lambda(\ell)\chi(\ell)(\log\ell)^k\Big|^2\frac{\mathrm{d}t}{t^3}\bigg)^{1/2}\end{aligned}$$

$$\ll \sqrt{\frac{\phi(q)}{x}} + M^k(\log x)^{-k}\Bigg(\sum_{\substack{\chi \,(\mathrm{mod}\, q)\\ \chi\neq\chi_0,\psi}} \int_{\frac{\sqrt[4]{x}}{2}}^{x} \Bigg|\sum_{\substack{\ell\leqslant t\\ P^-(\ell)>y}} \Lambda(\ell)\chi(\ell)(\log \ell)^k\Bigg|^2 \frac{\mathrm{d}t}{t^{3+2/\log x}}\Bigg)^{1/2}.$$

Since Parseval's theorem for Dirichlet series, as was applied in the proof of Theorem 1, guarantees that

$$\int_1^{\infty} \Bigg|\sum_{\substack{\ell\leqslant u\\ P^-(\ell)>y}} \Lambda(\ell)\chi(\ell)(\log \ell)^k\Bigg|^2 \frac{\mathrm{d}u}{u^{3+2/\log x}} = \frac{1}{2\pi}\int_{\mathbb{R}} \Bigg|\bigg(\frac{L_y'}{L_y}\bigg)^{(k)}(c+it,\chi)\Bigg|^2 \frac{\mathrm{d}t}{c^2+t^2},$$

with $c = 1 + 1/(\log x)$, we arrive at

$$\begin{aligned}&\Bigg(\sum_{\substack{\chi \,(\mathrm{mod}\, q)\\ \chi\neq\chi_0,\psi}} \int_{\frac{\sqrt{x}}{2}}^{x} \Bigg|\sum_{\substack{\ell\leqslant t\\ P^-(\ell)>y}} \Lambda(\ell)\chi(\ell)\Bigg|^2 \frac{\mathrm{d}t}{t^3}\Bigg)^{1/2}\\ &\ll \sqrt{\frac{\phi(q)}{x}} + M^k(\log x)^{-k}\Bigg(\sum_{\substack{\chi \,(\mathrm{mod}\, q)\\ \chi\neq\chi_0,\psi}} \int_{\mathbb{R}} \Bigg|\bigg(\frac{L_y'}{L_y}\bigg)^{(k)}(c+it,\chi)\Bigg|^2 \frac{\mathrm{d}t}{c^2+t^2}\Bigg)^{1/2}. \qquad (4.2.8)\end{aligned}$$

We finish the proof of the lemma by combining (4.2.8) with (4.2.7). □

The following result is used in the proof of Lemma 4.2.4. A proof of it may be found in [9, Lemma 9.1].

Lemma 4.2.3. *Let $k \in \mathbb{N}$, S be an open set of $\mathbb{C}$, $s \in S$ and $F : S \to \mathbb{C}$ be a function which is differentiable k times at s. We further assume that $F(s) \neq 0$ and we set*

$$K = \max_{1\leqslant j\leqslant k}\left\{\frac{1}{j!}\left|\frac{F^{(j)}}{F}(s)\right|\right\}^{1/j} \quad \text{and} \quad L = \max_{1\leqslant j\leqslant k}\left\{\frac{1}{j!}\left|\bigg(\frac{F'}{F}\bigg)^{(j-1)}(s)\right|\right\}^{1/j}.$$

Then $K/2 \leqslant L \leqslant 2K$.

Lemma 4.2.4. *Let q be a positive integer and consider the three real numbers $x, T \geqslant 1$ and $y \geqslant (10q)^{100}V_T$ (recall the definition of V_t for $t \in \mathbb{R}$ from Lemma 2.1.2 or from p. 15). If ψ is as in the statement of Theorem 2 and $L_y(s,\chi) = \sum_{P^-(n)>y}\chi(n)n^{-s}$ is the y-rough Dirichlet series of the character χ mod q for $\mathrm{Re}(s) > 1$, then, for $k \in \mathbb{N}$, we have*

$$\sum_{\substack{\chi \,(\mathrm{mod}\, q)\\ \chi\neq\chi_0,\psi}} \int_{\mathbb{R}} \Bigg|\bigg(\frac{L_y'}{L_y}\bigg)^{(k)}(c+it,\chi)\Bigg|^2 \frac{\mathrm{d}t}{c^2+t^2} \ll (4C)^{2k}((k+1)!)^2(\log y)^{2k+1} + \frac{(c_1k)^{2k}(\log x)^{2k+1}}{T},$$

where $c_1 > 0$ is a constant, $C > 0$ is the constant from Theorem 2.2.1 and $c = 1 + 1/(\log x)$.

Proof. We are going to estimate the integrals by splitting them into two parts. In the first parts, we will be integrating over $|t| \leqslant T$. We will bound these parts by mainly using the results of Section 2.2. For the remaining parts, where we integrate over the range $|t| > T$, we will use Lemma 2.3.2. We start with the integrals over $|t| > T$ first.

By referring to Lemma 1.3.2, it is true that

$$\sum_{\chi \,(\mathrm{mod}\, q)} \left| \left(\frac{L_y'}{L_y}\right)^{(k)} (c+it,\chi) \right|^2 = \phi(q) \sum_{b \in (\mathbb{Z}/q\mathbb{Z})^*} \left| \sum_{\substack{n \equiv b \,(\mathrm{mod}\, q) \\ P^-(n) > y}} \frac{\Lambda(n)(\log n)^k}{n^{c+it}} \right|^2 .$$

So, now one can use Lemma 2.3.2 to infer that

$$\begin{aligned} &\sum_{\substack{\chi \,(\mathrm{mod}\, q) \\ \chi \neq \chi_0, \psi}} \int_{|t|>T} \left| \left(\frac{L_y'}{L_y}\right)^{(k)} (c+it,\chi) \right|^2 \frac{\mathrm{d}t}{c^2+t^2} \\ &\leqslant \phi(q) \sum_{b \in (\mathbb{Z}/q\mathbb{Z})^*} \int_{|t|>T} \left| \sum_{\substack{n \equiv b \,(\mathrm{mod}\, q) \\ P^-(n) > y}} \frac{\Lambda(n)(\log n)^k}{n^{c+it}} \right|^2 \frac{\mathrm{d}t}{t^2} \ll \frac{(c_1 k)^{2k} (\log x)^{2k+1}}{T}, \end{aligned} \tag{4.2.9}$$

where $c_1 = 2\sqrt{\log 4}$. Our treatment for the integrals corresponding to the large values of t is complete and we turn our focus to the integrals whose range of integration is $|t| \leqslant T$. For a character $\chi \notin \{\chi_0,\psi\}$, using Lemma 4.2.3, it follows that

$$\begin{aligned} &\int_{|t| \leqslant T} \left| \left(\frac{L_y'}{L_y}\right)^{(k)} (c+it,\chi) \right|^2 \frac{\mathrm{d}t}{c^2+t^2} \\ &\leqslant 4^k ((k+1)!)^2 \sum_{j=1}^{k+1} (j!)^{-\frac{2(k+1)}{j}} \int_{|t| \leqslant T} \left| \frac{L_y^{(j)}}{L_y} (c+it,\chi) \right|^{\frac{2(k+1)}{j}} \frac{\mathrm{d}t}{c^2+t^2}. \end{aligned} \tag{4.2.10}$$

Since $\chi \notin \{\chi_0,\psi\}$ and $y \geqslant (10q)^{100} V_T > qV_t$ when $|t| \leqslant T$, a proper combination of Theorems 2.2.2 and 2.2.3 implies that $|L_y^{-1}(c+it,\chi)| \ll 1$. Moreover, $|L_y^{(j)}(c+it,\chi)| \ll j!(C\log y)^j$ for all $j \in \{1,\dots,k+1\}$, as can be seen from Theorem 2.2.1. Hence, (4.2.10) gives

$$\begin{aligned} &\sum_{\substack{\chi \,(\mathrm{mod}\, q) \\ \chi \neq \chi_0, \psi}} \int_{|t| \leqslant T} \left| \left(\frac{L_y'}{L_y}\right)^{(k)} (c+it,\chi) \right|^2 \frac{\mathrm{d}t}{c^2+t^2} \\ &\ll (2C)^{2k} ((k+1)!)^2 \sum_{j=1}^{k+1} \frac{(\log y)^{2(k+1-j)}}{(j!)^2} \sum_{\substack{\chi \,(\mathrm{mod}\, q) \\ \chi \neq \chi_0, \psi}} \int_{\mathbb{R}} |L_y^{(j)}(c+it,\chi)|^2 \frac{\mathrm{d}t}{c^2+t^2}. \end{aligned} \tag{4.2.11}$$

From an application of Parseval's theorem, it follows that

$$\sum_{\substack{\chi\,(\mathrm{mod}\,q)\\ \chi\neq\chi_0,\psi}}\int_{\mathbb{R}}|L_y^{(j)}(c+it,\chi)|^2\frac{\mathrm{d}t}{c^2+t^2}\leqslant\int_y^{\infty}\sum_{\substack{\chi\,(\mathrm{mod}\,q)\\ \chi\neq\chi_0}}\Big|\sum_{\substack{n\leqslant u\\ P^-(n)>y}}\chi(n)(\log n)^j\Big|^2\frac{\mathrm{d}u}{u^3}. \tag{4.2.12}$$

For $b\in(\mathbb{Z}/q\mathbb{Z})^*$ and $u\geqslant y$, Lemmas 2.1.3 and 2.1.4 yield that

$$\sum_{\substack{n\leqslant u\\ n\equiv b\,(\mathrm{mod}\,q)\\ P^-(n)>y}}(\log n)^j-\frac{1}{\phi(q)}\sum_{\substack{n\leqslant u\\ P^-(n)>y}}\chi_0(n)(\log n)^j\ll\frac{(\log u)^j u^{1-c_2/\log y}}{\phi(q)\log y},. \tag{4.2.13}$$

where $c_2=\min\{\kappa,\lambda\}$. Since $\int_y^v(\log t)^j\mathrm{d}t\leqslant v(\log v)^j$ for $v\geqslant y$, these lemmas also imply that

$$\sum_{\substack{n\leqslant u\\ n\equiv b\,(\mathrm{mod}\,q)\\ P^-(n)>y}}(\log n)^j+\frac{1}{\phi(q)}\sum_{\substack{n\leqslant u\\ P^-(n)>y}}\chi_0(n)(\log n)^j\ll\frac{u(\log u)^j}{\phi(q)\log y}. \tag{4.2.14}$$

We combine (4.2.13) and (4.2.14) with the elementary identity $w^2-z^2=(w-z)(w+z)$ and infer that

$$\begin{aligned}&\sum_{\substack{\chi\,(\mathrm{mod}\,q)\\ \chi\neq\chi_0}}\Big|\sum_{\substack{n\leqslant u\\ P^-(n)>y}}\chi(n)(\log n)^j\Big|^2\\ &=\phi(q)\sum_{b\in(\mathbb{Z}/q\mathbb{Z})^*}\Bigg\{\Bigg(\sum_{\substack{n\leqslant u\\ n\equiv b\,(\mathrm{mod}\,q)\\ P^-(n)>y}}(\log n)^j\Bigg)^2-\Bigg(\frac{1}{\phi(q)}\sum_{\substack{n\leqslant u\\ P^-(n)>y}}\chi_0(n)(\log n)^j\Bigg)^2\Bigg\}\\ &\ll\frac{(\log u)^{2j}u^{2-c_2/\log y}}{(\log y)^2},\end{aligned}$$

for all $j\in\{1,\ldots,k+1\}$. The first step is justified with an application of Lemma 1.3.2. We now plug this estimate into (4.2.12) and obtain

$$\begin{aligned}\sum_{\substack{\chi\,(\mathrm{mod}\,q)\\ \chi\neq\chi_0,\psi}}\int_{\mathbb{R}}|L_y^{(j)}(c+it,\chi)|^2\frac{\mathrm{d}t}{c^2+t^2}&\ll\frac{1}{(\log y)^2}\int_y^{\infty}(\log u)^{2j}u^{-1-c_2/\log y}\mathrm{d}u\\ &=c_2^{-2j-1}(\log y)^{2j-1}\Gamma(2j+1)\leqslant c_2^{-3}(2j)!(\log y)^{2j-1},\end{aligned}$$

for $j\in\{1,\ldots,k+1\}$ and Γ being the Gamma funtion. With this last bound, (4.2.11) turns into

$$\sum_{\substack{\chi\,(\mathrm{mod}\,q)\\ \chi\neq\chi_0,\psi}}\int_{|t|\leqslant T}\left|\left(\frac{L_y'}{L_y}\right)^{(k)}(c+it,\chi)\right|^2\frac{\mathrm{d}t}{c^2+t^2}\ll(2C)^{2k}((k+1)!)^2(\log y)^{2k+1}\sum_{j=1}^{k+1}\binom{2j}{j}.$$

But, $\binom{2j}{j} \leqslant \sum_{\ell=0}^{j} \binom{2j}{\ell} = 4^j$, and so

$$\sum_{j=1}^{k+1} \binom{2j}{j} \leqslant 4^{k+1} \sum_{j=0}^{k} 4^{-j} < 4^{k+!} \sum_{j \geqslant 0} 4^{-j} \ll 4^k.$$

Therefore, we complete the estimation of the integrals over $|t| \leqslant T$ by arriving at the bound

$$\sum_{\substack{\chi \,(\mathrm{mod}\, q) \\ \chi \neq \chi_0, \psi}} \int_{|t| \leqslant T} \left| \left(\frac{L_y'}{L_y} \right)^{(k)} (c+it,\chi) \right|^2 \frac{\mathrm{d}t}{c^2+t^2} \ll (4C)^{2k} ((k+1)!)^2 (\log y)^{2k+1}. \tag{4.2.15}$$

Putting (4.2.9) and (4.2.15) together, we conclude the proof of the lemma. □

• **Endgame - Proof of Theorem 2:** Now that the proofs of Lemmas 4.2.1, 4.2.2 and 4.2.4 are complete, we are combining them to prove Theorem 2.

First, we may assume that $x \geqslant q^A$ for some sufficiently large real number $A > 0$. Indeed, if $q^2 \leqslant x < q^A$, then $\log x / \log q \asymp 1$ and the theorem follows from a trivial application of Lemma 1.2.4. Now, let k be a positive integer that we will choose later and set

$$y = (10q)^{100} V_T \quad \text{with} \quad T = \exp\{2B(\log x)^{3/5} (\log\log x)^{2/5}\},$$

where $V_T = \exp\{100(\log(T+3))^{2/3}(\log\log(T+3))^{1/3}\}$ and $B > 0$ is some large constant.

With these choices of y and T, applications of Lemmas 4.2.2 and 4.2.4 give

$$\frac{1}{D} \sum_{\substack{1<m\leqslant\sqrt{x} \\ P^-(m)>y}} \int_{x-D}^{x} \Delta\Big(\frac{t}{m},y;q,a\overline{m}\Big) \mathrm{d}t \ll_\delta \frac{x^{1+2\delta/(\log y)}}{\phi(q)} \left\{ \left(\frac{M'\ell \log y}{\log x} \right)^{\ell} + \frac{(c_1 M \ell)^\ell}{\sqrt{T}} \right\}, \tag{4.2.16}$$

where $\delta > 0$ is sufficiently small, $\ell = k+1$ and $M' = 4MC$. Note that we omitted the term

$$\frac{x^{1/2+\delta/(\log y)}}{\log y} \sqrt{\frac{\log x}{\phi(q)}},$$

since we are working with a sufficiently small $\delta > 0$ and a $x \geqslant q^A$ for some sufficiently large A. Now, we insert the estimate (4.2.16) into Lemma 4.2.1 and obtain that

$$\Delta(x,y;q,a) \ll_\delta \frac{x^{1+2\delta/(\log y)}}{\phi(q)} \left\{ \left(\frac{M'\ell \log y}{\log x} \right)^{\ell} + \frac{(c_1 M\ell)^\ell}{\sqrt{T}} \right\} + \frac{x^{1-\delta/(2\log y)}}{\phi(q)}. \tag{4.2.17}$$

Since M can be sufficiently large, we can choose δ sufficiently small as $\delta = 1/(5eM')$. Then, for

$$\ell = \left\lfloor \frac{\log x}{eM' \log y} \right\rfloor,$$

it follows that

$$\left(\frac{M'\ell\log y}{\log x}\right)^{\ell} \leqslant \frac{\log x}{M'\log y}x^{-1/(eM'\log y)} \ll x^{-1/(2eM'\log y)},$$

and so, with the above choice of δ, we deduce that

$$\frac{x^{1+2\delta/(\log y)}}{\phi(q)}\left(\frac{M'\ell\log y}{\log x}\right)^{\ell} \ll \frac{x^{1-\delta/(2\log y)}}{\phi(q)}. \tag{4.2.18}$$

We now observe that $\log V_T \asymp (\log x)^{2/5}(\log\log x)^{3/5}$, which implies that $\ell < \log x/(\log y) < \log x/(\log V_T) \ll (\log x)^{3/5}(\log\log x)^{-3/5}$. Moreover, $\log \ell < \log\log x$. Consequently, there exist some positive constants c_2 and c_3 such that

$$\begin{aligned} x^{\frac{2\delta}{\log y}}\frac{(c_3M_0\ell)^{\ell}}{\sqrt{T}} &= \exp\left\{\frac{2\delta\log x}{\log y} + \ell\log\ell + \log(c_3M_0)\ell - B(\log x)^{3/5}(\log\log x)^{2/5}\right\} \\ &\leqslant \exp\{c_2(\log x)^{3/5}(\log\log x)^{1-3/5} - B(\log x)^{3/5}(\log\log x)^{2/5}\} \\ &\leqslant \exp\{-c_3(\log x)^{3/5}(\log\log x)^{2/5}\}, \end{aligned} \tag{4.2.19}$$

because the constant B in the definition of T is sufficiently large.

With the selection of ℓ that we made, we have the estimates (4.2.18) and (4.2.19) and then (4.2.17) becomes

$$\Delta(x,y;q,a) \ll \frac{x^{1-\delta/(2\log y)}}{\phi(q)} + \frac{xe^{-c_3(\log x)^{3/5}(\log\log x)^{2/5}}}{\phi(q)} \tag{4.2.20}$$

$$\ll \frac{x^{1-c_4/(\log(2q))}}{\phi(q)} + \frac{xe^{-c_4(\log x)^{3/5}(\log\log x)^{-3/5}}}{\phi(q)}, \tag{4.2.21}$$

for some $c_4 > 0$. The first term of the second line is for the range where $\log(10q) \geqslant \log V_T$, whereas the second term covers the range $\log(10q) \leqslant \log V_T$. The proof of the theorem is almost complete. It only remains to observe that $\Delta(x,y;q,a)$ does not differ much from

$$\sum_{\substack{n\leqslant x\\ n\equiv a\,(\mathrm{mod}\,q)}} \Lambda(n) - \frac{x}{\phi(q)} - \frac{\psi(a)}{\phi(q)}\sum_{n\leqslant x}\Lambda(n)\psi(n).$$

First, we have the estimate (4.1.1) which can be rewritten as

$$\sum_{\substack{n\leqslant x\\ n\equiv a\,(\mathrm{mod}\,q)\\ P^-(n)>y}} \Lambda(n) = \sum_{\substack{n\leqslant x\\ n\equiv a\,(\mathrm{mod}\,q)}} \Lambda(n) + O\left(\frac{y\log x}{\log y}\right).$$

Similarly,

$$\sum_{\substack{n\leqslant x\\ P^-(n)>y}} \Lambda(n)\chi(n) = \sum_{n\leqslant x} \Lambda(n)\chi(n) + O\left(\frac{y\log x}{\log y}\right)$$

for $\chi \in \{\chi_0,\psi\}$. Now, all the prime factors of q in its prime factorization are greater than or equal to 2, which implies that $2^{\omega(q)} \leqslant q$. Using this inequality, we deduce that

$$\sum_{\substack{n\leqslant x\\ (n,q)>1}} \Lambda(n) \leqslant \sum_{\substack{p^k\leqslant x\\ p\mid q}} \log p \leqslant \sum_{p\mid q} \log p \sum_{k\leqslant \log x/\log p} 1 \leqslant \omega(q)\log x \ll (\log x)(\log q),$$

and so an application of Theorem 1.2.3 (the prime number theorem) yields

$$\sum_{n\leqslant x} \Lambda(n)\chi_0(n) = x + O(x\exp\{-c_5(\log x)^{3/5}(\log\log x)^{-1/5}\}),$$

for some absolute constant $c_5 > 0$. So, finally, when $x \geqslant q^A$ for a large $A > 0$, with the y that we have chosen, we conclude that

$$\sum_{\substack{n\leqslant x\\ n\equiv a\,(\mathrm{mod}\,q)}} \Lambda(n) - \frac{x}{\phi(q)} - \frac{\psi(a)}{\phi(q)}\sum_{n\leqslant x} \Lambda(n)\psi(n) - \Delta(x,y;q,a) \ll \frac{xe^{-c_5(\log x)^{3/5}(\log\log x)^{-1/5}}}{\phi(q)}.$$

In virtue of (4.2.20), the theorem has now been proven.

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