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Rigid and Strongly Rigid Relations on Small Domains

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Les relations fortement rigides jouent un rôle important dans l'étude de la complexité des problèmes de satisfaction de contraintes (CSPs) (Feder et Vardi [22], Schaefer [9], Jeavons [23], Bulatov, Jeavons et Krokhin [24], Larose et Tesson [34], Larose [31], Barto et Kozik [36], et Bulatov, Jeavons et Krokhin [27]) qui font l'objet de recherches intenses à la fois en intelligence artificielle et en recherche opérationnelle (Russell et Norvig [19]). Une relation n-aire ρ sur un ensemble U est rigide si elle n'admet aucun automorphisme non-trivial ; elle est fortement rigide si elle n'est préservée que par les projections. De plus ρ est dite projective si les seules opérations idempotentes qui la préservent sont les projections.

Rosenberg (1973) a caracterisé toutes les relations fortement rigides sur un ensemble à deux éléments, et a construit une relation binaire fortement rigide sur tout ensemble de plus de deux éléments. Larose et Tardif (2001) ont étudié les graphes projectifs et fortement rigides, et ont construit de grandes familles de graphes fortement rigides. Łuczak et Nešetřil (2004) ont démontré une conjecture de Larose and Tardif qui prévoyait que la plupart des graphes avec suffisamment de sommets sont projectifs, et ont caractérisé tous les graphes homogènes qui sont projectifs. Łuczak et Nešetřil (2006) ont ensuite confirmé une conjecture de Rosenberg qui prédisait que la plupart des relations sur un ensemble suffisamment grand sont fortement rigides.

Le premier résultat principal de cette thése est une caractérisation des relations fortement rigides sur un ensemble d'au moins 3 éléments, résolvant ainsi un problème ouvert de Rosenberg (Rosenberg [7], Problème 6 de [13]). Ensuite nous montrons qu'à isomorphisme près, il n'existe que 4 relations binaires rigides sur un ensemble à trois éléments, parmi lesquelles deux seulement sont fortement rigides. De plus, nous déterminons, à isomorphisme près, les 40 relations binaires rigides sur un univers à quatre éléments, et montrons que 25 d'entre elles sont fortement rigides (Exemple 5.4 et Exemple 6.1 dans Sun [41]). Nous généralisons une de ces relations pour construire une nouvelle relation binaire fortement rigide sur tout ensemble d'au moins 4 éléments (Sun [43]), et décrivons de plus une relation ternaire fortement rigide sur tout ensemble fini avec au moins 2 éléments et conjecturons une relation k-aire fortement rigide sur tout domaine fini (Sun [42]).

Mots clés : Algèbre universelle, théorie des clones, polymorphismes, relations rigides et fortement rigides.

ABSTRACT

Strongly rigid relations play an important role in the study of the complexity of Constraint Satisfaction Problems (CSPs) (Feder and Vardi [22], Schaefer [9], Jeavons [23], Bulatov, Jeavons and Krokhin [24], Larose and Tesson [34], Larose [31], Barto and Kozik [36], and Bulatov, Jeavons and Krokhin [27]) which are the subject of intense research in both artificial intelligence and operations research (Russell and Norvig [19]). An n-ary relation ρ on a set U is strongly rigid if it is preserved only by trivial operations. It is projective if the only idempotent operations in $Pol\rho$ are projections.

Rosenberg (1973) characterized all strongly rigid relations on a set with two elements and found a strongly rigid binary relation on every domain U of at least 3 elements. Larose and Tardif (2001) studied the projective and strongly rigid graphs, and constructed large families of strongly rigid graphs. Łuczak and Nešetřil (2004) settled in the affirmative a conjecture of Larose and Tardif that most graphs on a large set are projective, and characterized all homogenous graphs that are projective. Łuczak and Nešetřil (2006) confirmed a conjecture of Rosenberg that most relations on a big set are strongly rigid.

In this thesis we characterize all strongly rigid relations on a set with at least three elements to answer an open question by Rosenberg (1973) (Rosenberg [7], Problem 6 in Rosenberg [13]). We classify the binary relations on the 3-element domain and demonstrate that there are merely 4 pairwise nonisomorphic rigid binary relations on the same domain (among them 2 are pairwise nonisomorphic strongly rigid), and we classify the binary relations on the 4-element domain and show that there are merely 40 pairwise nonisomorphic rigid binary relations on the same domain (among them 25 are pairwise nonisomorphic strongly rigid) (Example 5.4 and Example 6.1 in Sun [41]). We extend a strongly rigid relation on a 4-element domain to any finite domain (Sun [43]). Finally, we give a strongly rigid ternary relation on any finite domain and conjecture a strongly rigid k-ary relation on any finite domain (Sun [42]).

Keywords: Universal algebra, clone theory, polymorphisms, rigid relations and

strongly rigid relations.

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CHAPTER 1

INTRODUCTION

1.1 Introduction

In 1941, Post [1] first gave a full description of the lattice of clones on $\{0,1\}$. Since then, researchers have been trying to develop their understanding on clones in more generic terms. In 1965, Vopěnka [4] et al. showed the existence of a rigid binary relation on any set. In 1973, Rosenberg [7] initiated the study of strongly rigid relations (and relational structures) which was later developed by Poizat [8], Grabowski and Pöschel [16, 21], Fearnley [18], Larose and Tardif [25], and Lenz [28]. In 1983, Rosenberg characterized the minimal clones in his celebrated work [13]. Minimal clones and trivial clones are interesting in the study of the complexity of Constraint Satisfaction Problems (CSPs) (Feder and Vardi [22], Schaefer [9], Jeavons [23], Bulatov, Jeavons and Krokhin [24], Larose and Tesson [34], Larose [31], Barto and Kozik [36], and Bulatov, Jeavons and Krokhin [27]) which are the subject of intense research in both artificial intelligence and operations research (Russell and Norvig [19]). In our work, we concentrate mainly on rigid relations and strongly rigid relations (trivial clones).

In 1973, Rosenberg [7] characterized all strongly rigid relations on a set with two elements, and constructed a strongly rigid binary relation on any domain (see Fig. 1.1). He also proposed two open questions about strongly rigid relations:

1. How to characterize all strongly rigid relations on a set with more than two elements (see also Problem 6 in Rosenberg [13])?
2. He conjectured that for a finite set with k elements the number of $s(n)$ of n-ary strongly rigid relations satisfies

$$\lim_{n\to\infty} 2^{-k^n} s(n) = 1,$$

i.e., for a big n almost all n-ary relations are strongly rigid.

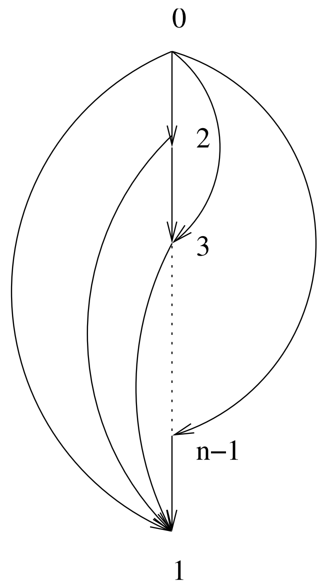


Figure 1.1 – A strongly rigid binary relation on a finite domain by I. Rosenberg

The second question was confirmed only recently by Łuczak and Nešetřil [32] using a probabilistic approach (see also Kazda [39]). Note also that Łuczak and Nešetřil [29] used a similar probabilistic approach to settle in the affirmative a conjecture of Larose and Tardif [25] that most graphs on a large set are projective. In Chapter 2, we will give a characterization of all strongly rigid relations on a set with at least three elements. This answers the first open question.

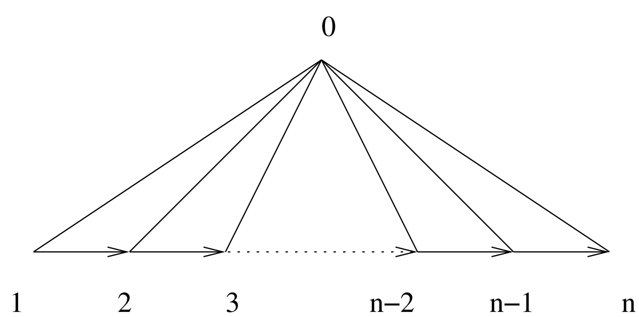


Figure 1.2 – A strongly rigid binary relation on a finite domain by A. Fearnley

The computation of small examples could prove useful for researchers (Csákány [11], Szczepara [17], Berman and Burris [20], Barto and Stanovský [37], Machida and Rosenberg [38], and Jovanović [40]). Fearnley [18] showed that for a 3-element domain, up to relational isomorphism, there are only two strongly rigid binary relations. In Chapter 3, an algorithmic approach will be applied to classify the binary relations on a domain with 3 elements, and it turns out that there are only 4 pairwise nonisomorphic rigid binary relations (see Table 1.1 which accounts for all binary relations on a 3-element domain), 2 of them are pairwise nonisomorphic strongly rigid binary relations. In Chapter 4, a similar approach will be applied to classify the binary relations on a domain with 4 elements, and it turns out that there are only 40 pairwise nonisomorphic rigid binary relations (see Table 1.2 which accounts for all binary relations on a 4-element domain), 25 of them are pairwise nonisomorphic strongly rigid binary relations (see Fig.1.4).

Fearnley [18] presented a family of strongly rigid binary relations (see Fig. 1.2), Larose and Tardif [25] studied the projective and strongly rigid graphs and constructed large families of strongly rigid graphs. In Chapter 5, as an extension of one result in Chapter 4, we will present a new strongly rigid binary relation on a finite domain. In Chapter 6, we propose a strongly rigid ternary relation on a finite domain and conjecture a strongly rigid k-ary ($k > 3$) relation on a finite domain. In Chapter 7, we suggest some further research ideas.

Proposition	# of Relations	# of Relations up to isomorphism	Property
Proposition 3.1.6	$(2^3-1)2^6$		Not rigid
Fact 3.2.1	40		Not rigid
Proposition 3.4.1	12	2	rigid but not strongly rigid
Proposition 3.5.1	12	2	Strongly rigid
Total:	2^{3^2}	4	

Table 1.1 – Classification of binary relations on a 3-element domain

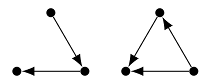


Figure 1.3 – The two strongly rigid binary relations on a 3-element domain (up to isomorphism)

Proposition	# of Relations	# of Relations up to isomorphism	Property
Proposition 4.1.1	$(2^4-1)2^{12}$		Not rigid
Fact 4.1.2	2644		Not rigid
Proposition 4.2.1 (The source-sink rule)	612	15	Rigid but not strongly rigid
Proposition 4.3.1 (Fig.1.4)	840	25	Strongly rigid
Total:	2^{4^2}	40	

Table 1.2 – Classification of binary relations on a 4-element domain

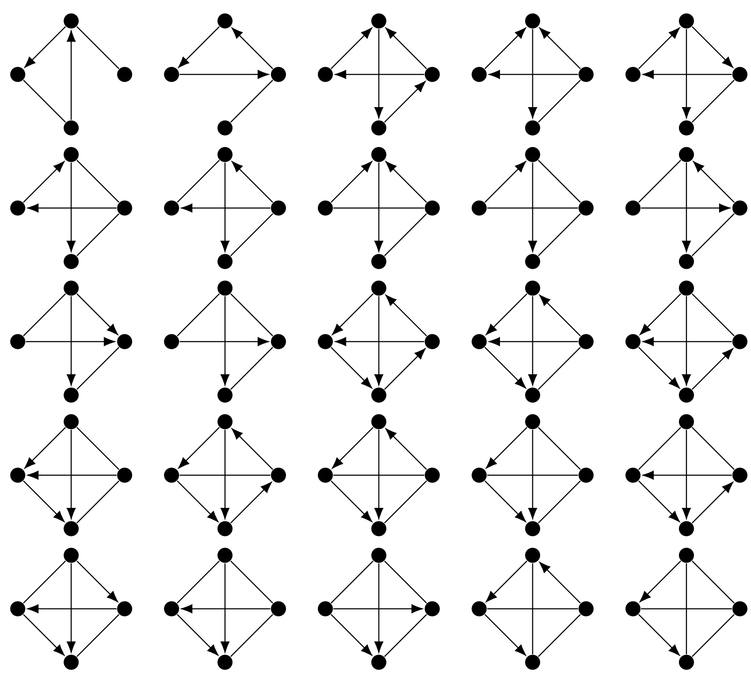


Figure 1.4 – The twenty-five strongly rigid binary relations on a 4-element domain (up to isomorphism)

CHAPTER 2

CHARACTERIZATION OF STRONGLY RIGID RELATIONS

2.1 Preliminaries

As a general reference for the basic definitions and terminology in this field, the reader is referred to Á. Szendrei's *Clones in Universal Algebra* [14] or D. Lau's *Function Algebras on Finite Sets* [33].

Let U be a non-empty universe, and let n be a positive integer. A map $f : U^n \to U$ is called an *n-ary operation* (or *function*) on U, the set of all n-ary operations on U is denoted by $\mathcal{O}_U^{(n)}$, and $\mathcal{O}_U = \bigcup_{n=1}^{\infty} \mathcal{O}_U^{(n)}$. For a positive integer h, a subset ρ of U^h is an *h-ary relation* on U. In the following, $:\approx$ defines an operation while $\approx$ means that both sides are equal with all variables universally quantified.

Definition 2.1.1. *For $1 \le i \le n$, the i-*th *n-*ary projection *e_i^n is defined by setting $e_i^n(x_1, ..., x_n) :\approx x_i$ for all $x_1, ..., x_n \in U$*

Definition 2.1.2. *Let $f \in \mathcal{O}_U^{(n)}$, and let ρ be an h-ary relation on U. The operation f* preserves *ρ if for all $(a_{1,i}, a_{2,i}, \ldots, a_{h,i}) \in \rho$ $(i = 1, \ldots, n)$,*

$$(f(a_{1,1}, a_{1,2}, \ldots, a_{1,n}), f(a_{2,1}, a_{2,2}, \ldots, a_{2,n}), \ldots, f(a_{h,1}, a_{h,2}, \ldots, a_{h,n})) \in \rho.$$

Definition 2.1.3. *An operation $f \in \mathcal{O}_U^{(n)}$ is* idempotent *if $f(x, ..., x) \approx x$.*

Definition 2.1.4. *Let f be an n-ary operation and $g_1, ..., g_n$ be k-ary operations on U. A k-ary operation $f[g_1, ..., g_n]$ on U, called the* composition *(or* superposition *or* substitution*), is defined as follows:*

$$f[g_1, ..., g_n](x_1, ..., x_k) :\approx f(g_1(x_1, ..., x_k), ..., g_n(x_1, ..., x_k)).$$

Definition 2.1.5. *A subset of $\mathcal{O}_U$ is said to be a* clone *on U if it contains the projections and is closed under composition.*

Evidently every projection preserves every relation in Γ on U where Γ is a set of relations. We denote by $Pol\Gamma$ the set of all operations on U preserving every relation in Γ. If $\Gamma = \{\rho\}$ we write $Pol\rho$ instead of $Pol\{\rho\}$ the set of all operations on U preserving ρ, and $Pol^{(n)}\rho$ instead of $Pol^{(n)}\{\rho\}$ the set of all n-ary operations on U preserving ρ. It is well known and easy to verify that $Pol\rho$ is a clone on U.

Definition 2.1.6. *A relation ρ on U is* rigid *if every unary operation on U preserving ρ is a projection. i.e., $Pol^{(1)}\rho = \{e_1^1\}$.*

Definition 2.1.7. *A relation ρ on U is* projective (idempotent trivial) *if every idempotent operation on U preserving ρ is a projection.*

Definition 2.1.8. *A relation ρ on U is* strongly rigid *if every operation on U preserving ρ is a projection, i.e., $Pol\rho = \{e_i^n : 1 \leq i \leq n < \omega\}$. In other words, $Pol\rho$ is the smallest clone, the clone of all projections, also known as the trivial clone.*

Definition 2.1.9. *A clone is said to be a minimal clone if it has the trivial clone as its only proper subclone. Nontrivial operations of minimal arity in a minimal clone are called minimal operations.*

Definition 2.1.10. *A clone is said to be a maximal clone if it is a coatom in the lattice formed by the set of all clones with respect to inclusion.*

For a general introduction to minimal clones, please see Csákány [30].

Lemma 2.1.11. *A relation ρ is (strongly) rigid if and only if ρ^{-1} is (strongly) rigid, where $\rho^{-1} = \{(x,y) : (y,x) \in \rho\}$.*

Proof. This follows from $Pol\,\rho = Pol\,\rho^{-1}$ (which is well known and can be verified directly). $\square$

2.2 Characterization of strongly rigid relations

We begin by defining the following operations on $\{0,1\}$:

1) The unary $\mathbf{0}$ and $\mathbf{1}$ constant operations given by $\mathbf{0}(x) :\approx 0, \mathbf{1}(x) :\approx 1$.

2) The unary negation operation $\neg$ defined by $\neg 0 = 1$ and $\neg 1 = 0$.

3) The binary conjunction operation $\wedge$ (conjunction) defined by $x \wedge y = 1$ if and only if $x = y = 1$.

4) The binary disjunction operation $\vee$ (disjunction) defined as $x \vee y = 1$ if and only if $x = 1$ or $y = 1$.

5) The ternary Boolean majority operation $m(x, y, z) = 1$ if and only if at least two of x, y, z are 1.

6) The ternary Boolean minority operation $p(x, y, z) = x + y + z$ (mod 2).

Theorem 2.2.1. *(Post [1]) Every minimal operation on a two element set is among one of the following:*

1) The unary negation operation,

2) The binary conjunction,

3) The binary disjunction operation,

4) The ternary Boolean majority operation,

5) The ternary Boolean minority operation.

Using Post's results, Rosenberg characterized all strongly rigid relations on a set with two elements.

Theorem 2.2.2. *(Rosenberg [7]) A relation ρ on $\{0, 1\}$ is strongly rigid if and only if ρ satisfies the following:*

1) Polρ contains neither unary constant operation **0** *or* **1***,*

2) Polρ does not contain the negation operation,

3) Polρ does not contain the binary conjunction operation,

4) Polρ does not contain the binary disjunction operation,

5) Polρ contains no ternary Boolean majority operation,

6) Polρ contains no ternary Boolean minority operation.

Definition 2.2.3. *A ternary operation f on U is a* Mal'tsev operation *if*

$$f(x,x,y) \approx y \approx f(y,x,x).$$

Definition 2.2.4. *A ternary operation f on U is a* majority operation *if*

$$f(x,x,y) \approx f(x,y,x) \approx f(y,x,x) \approx x.$$

Definition 2.2.5. *A ternary operation f on U is a* minority operation *if*

$$f(x,x,y) \approx f(x,y,x) \approx f(y,x,x) \approx y.$$

Definition 2.2.6. *For $n \geq 3$ and $1 \leq i \leq n$, an n-ary operation f on U is a* semiprojection on its i-*th* variable *if*

$$f(x_1, ..., x_n) \approx x_i$$

whenever $x_1, ..., x_n \in U$ are not pairwise distinct.

The following construction will be useful.

Definition 2.2.7. *Let $f \in \mathcal{O}_U^{(n)}$, $n \geq 3$. For all $1 \leq i < j \leq n$* the operation *f_{ij} is defined by*

$$f_{ij}(x_1, ..., x_{n-1}) :\approx f(x_1, ..., x_{j-1}, x_i, x_j, ..., x_{n-1})$$

(e.g., for $n = 5$, $f_{13}(x_1, x_2, x_3, x_4) = f(x_1, x_2, x_1, x_3, x_4)$)

Lemma 2.2.8. *(Swierczkowski [3]) If $n > 3$ and $f \in \mathcal{O}_U^{(n)}$ is such that for all $1 \leq i < j \leq n$ the operation f_{ij} is a projection, then f is a semiprojection.*

Definition 2.2.9. *We say that a set of relations Σ on U is pp-definable (positively primitively definable) from a set of relations P if each relation in Σ can be defined by a first order formula of the form*

$$\exists \cdots [\bigwedge_i atomic_i]$$

where each atomic$_i$ is of the form $(x_1,...,x_k) \in \Theta$ *where* $\Theta \in P$ *or equality* $(x = y)$*. That is, P pp-defines* Σ *if each relation in* Σ *can be defined by a first order formula which only uses relations in P, the equality relation, conjunction and existential quantification.*

The following lemma can be easily verified.

Lemma 2.2.10. *Let P,* Σ *and* Γ *be sets of relations on* U*.*

(1) If Σ *is pp-definable from P, then* $PolP \subseteq Pol\Sigma$*.*

(2) If Σ *is pp-definable from P, and* Γ *is pp-definable from* Σ*, then* Γ *is pp-definable from P.*

The clone of polymorphisms controls pp-definability in the sense of the following classical result.

Theorem 2.2.11. *(Bodnarčuk, Kalužhnin, Kotov, Romov [6]; Geiger [5]). Let* Γ *and* Σ *be sets of relations on* U*. Then* Σ *is pp-definable from* Γ *if and only if* $Pol\Gamma \subseteq Pol\Sigma$*.*

Let ρ be a binary relation on $U = \{0, ..., k\}$. A subset V of U is *derived* from ρ if $f(v_1, ..., v_n) \in V$ for all $f \in Pol\,\rho$ and all $v_1, ..., v_n \in V$. It is known that V is derived from ρ if and only if there exist $l \geq 1$ and a binary relation Γ on $\{1, ..., l\}$ such that

$$V = \{\phi(1) : \phi \in Hom(\Gamma, \rho)\}$$

(where $Hom(\Gamma, \rho)$ is the set of relational homomorphisms from Γ into ρ; In other words, the mappings $f : \{1, ..., l\} \to U$ such that $(x,y) \in \Gamma$ implies $(f(x), f(y)) \in \rho$). Denote by $D_i(\rho)$ the set of all i-element subsets of V derived from ρ.

We are now ready to present a well-known universal algebraic result about minimal operations, namely Rosenberg's Classification Theorem [13], in the form it appears in Proposition 1.12, [14].

Theorem 2.2.12. *(Rosenberg's Classification Theorem (RCT)). Every minimal operation is of one of the following types:*

1) a unary operation that is not the identity, *2) a binary idempotent operation that is not a projection,*

3) a ternary majority operation,

4) a ternary minority operation,

5) a k-ary semiprojection ($k > 2$) which is not a projection.

Motivated by the relationship between Theorem 2.2.1 and Theorem 2.2.2, and in spirit of Theorem 2.2.12 (RCT), we present a characterization of strongly rigid relations on a set with at least three elements.

Theorem 2.2.13. *(Theorem 3,16 [41]) (*Master Theorem*) Let U be finite, $|U| = n+1$, $n \geq 2$. A relation ρ on U is strongly rigid if and only if ρ satisfies:*

1) $\{u\} \in D_1(\rho)$ for every $u \in U$, (in other words, ρ is rigid)

2) there exists $\{a, b\} \in D_2(\rho)$ such that

(i) e_1^2 is the only idempotent $f \in Pol^{(2)}\rho$ satisfying $f(a, b) = a$, and

(ii) no idempotent $f \in Pol^{(2)}\rho$ satisfies $f(a, b) = f(b, a)$,

3) $Pol\,\rho$ contains no Mal'tsev operation,

4) there exists a ternary relation σ_3 on U such that

(i) $\forall a,\ b,\ c\ \in U$ with $b, c \neq a$, either $(a, b, c) \in \sigma_3$ or $(a, c, b) \in \sigma_3$, and

(ii) $\forall f \in Pol^{(3)}\rho$ with $f(x, x, y) \approx x \approx f(x, y, x)$
we have $(a, b, c) \in \sigma_3 \implies f(a, b, c) = a$,

5) $\forall i = 4, ..., n+1$, there exists an i-ary relation σ_i on U such that

(a) for each $a_1 \in U$, every $(i-1)$-element subset A of $U \setminus \{a_1\}$ can be written as $\{a_2, ..., a_i\}$ so that $(a_1, ..., a_i) \in \sigma_i$, and

(b) if $f \in Pol^{(i)}\rho$ is a semiprojection on its first variable, then $(c_1, ..., c_i) \in \sigma_i \implies f(c_1, ..., c_i) = c_1$.

Proof. ($\Rightarrow$)

1) is immediate.

2) Take any $\{a,b\} \subset U$ with $a \neq b$. Since $Pol\rho$ is the trivial clone, $\{a,b\} \in D_2(\rho)$. Let $f \in Pol^{(2)}$ be idempotent. Then $f \in \{e_1^2, e_2^2\}$. If $f(a,b) = a$, then $f = e_1^2$; and in both cases, $f(a,b) \neq f(b,a)$.

3) Obvious since no projection is a Mal'tsev operation.

4) Take $\sigma_3 = U^3$; Then (i) and (ii) are immediate.

5) Similarly, take $\sigma_i = U^i$. Then (a) and (b) are immediate.

$(\Leftarrow)$

Suppose that $Pol\rho$ is not trivial.

By RCT, $Pol\rho$ contains either a unary operation that is not the identity, a binary idempotent operation that is not a projection, a ternary majority operation, a ternary minority operation, or a k-ary semiprojection ($k > 2$) which is not a projection.

Let f be such an operation. By 1), f is at least binary. If f is a binary idempotent operation and $\{a,b\}$ is the set from 2); If $f(a,b) = b$, then $f(b,a) = a$. Let $g(x,y) = f(y,x)$. Then $g \in Pol^{(2)}\rho$ is idempotent and $g(a,b) = a$. So $g = e_1^2$ and hence $f = e_2^2$. Otherwise $f(a,b) = a$ so $f = e_1^2$. Thus f is at least 3-ary. f cannot be a Mal'tsev operation, hence it is not a minority operation; If it is a majority operation, then it satisfies $f(x,x,y) \approx x \approx f(x,y,x)$. Let $a,b \in U, a \neq b$. Then by 4), $(a,b,b) \in \rho_3$ so $f(a,b,b) = a$, a contradiction. Thus f must be a semiprojection. Without loss of generality, we can assume that it is on the first variable. In particular, $f(x,x,y) \approx x \approx f(x,y.x)$. Let $(a,b,c) \in U^3$, a,b,c all distinct. If $(a,b.c) \in \rho_3$, then $f(a,b,c) = a$. Otherwise, $(a,c,b) \in \rho_3$. Let $g(x,y,z) :\approx f(x,y,z)$. Clearly g is also a semiprojection on the first variable, so by 4) again, $f(a,b,c) = g(a,c,b) = a$. In any case, $f = e_1^3$.

So now we assume that f is a semiprojection, without loss of generality, on the first variable, say f has arity i. If $i > |U|$, then by the pigeonhole principle $f = e_1^i$. Thus assume that $i \leq |U|$. Let $(a_1, ..., a_i) \in U^i$ with $|\{a_1, ..., a_i\}| = i$. There exists $(b_1, ..., b_i) \in \rho_i$ such that $a_1 = b_1$ and $\{a_2, ..., a_i\} = \{b_2, ..., b_i\}$. There exists a suitable permutation τ of $\{2, ..., i\}$ such that $g(x_1, x_2, ..., x_i) :\approx f(x_1, x_{\tau_2}, ..., x_{\tau_i})$ satisfies $g(b_1, ..., b_i) = f(a_1, ..., a_i)$. It is clear that g is also a semiprojection on the first variable, $g \in Pol\rho$. Thus by 5), we have $f(a_1, ..., a_i) = g(b_1, ..., b_i) = a_1$, so $f = e_j^i$. Thus by RCT, $Pol\rho$ must be trivial. $\square$

CHAPTER 3

RIGID BINARY RELATIONS ON A 3-ELEMENT DOMAIN

In this chapter, we present some sufficient conditions to determine if a relation is not rigid or not strongly rigid, then we will describe all rigid and strongly rigid binary relations on a 3-element domain.

3.1 Loop, overlap and interchange rules for non-rigidity

Let ρ be a binary relation on a fixed universe U where $|U| \geq 3$. In the following definitions, we will treat ρ as a directed graph. Elements of U are called *vertices*. For $u, v \in U$, we write uv for the ordered pair (u, v).

Definition 3.1.1. *We say that two vertices u and v are* nonadjacent *if neither $uv \in \rho$ nor $vu \in \rho$.*

Definition 3.1.2. *We say that two vertices u and v are* connected *by an edge if both $uv \in \rho$ and $vu \in \rho$.*

Definition 3.1.3. *Let u, v be distinct nonadjacent vertices in ρ. We say that u* dominates *v, in symbols $v \preceq u$, if for every vertex w distinct from u and v, we have that $vw \in \rho \Rightarrow uw \in \rho$ and $wv \in \rho \Rightarrow wu \in \rho$.*

Definition 3.1.4. *Let u, v be two vertices connected by an edge in ρ. We say that u and v are* equivalent *if for every vertex w distinct from u and v, we have that $vw \in \rho$ if and only if $uw \in \rho$ and $wv \in \rho$ if and only if $wu \in \rho$.*

Definition 3.1.5. *A relation ρ is* irreflexive *if $uu \notin \rho$ for all $u \in U$.*

Lemma 3.1.6. *(1) (Loop rule) A rigid relation is irreflexive. In other words, if there exists a vertex u such that $uu \in \rho$, then ρ is not rigid.*

*(2) (*Overlap rule) *If there exist vertices u and v such that u dominates v, then ρ is not rigid.*

*(3) (*Interchange rule) *If there exist two equivalent vertices u and v, then ρ is not rigid.*

Proof. (1) Define a unary operation f on U by setting $f(x) = u$ for every $x \in U$. Then $f \in Pol\,\rho$ but f is not trivial.

(2) Define a unary operation f on U by setting $f(v) = u$ and $f(x) = x$ otherwise. Then $f \in Pol\,\rho$ but f is not trivial.

(3) Define a unary operation f on U by setting $f(v) = u$, $f(u) = v$ and $f(x) = x$ otherwise. Then $f \in Pol\,\rho$ but f is not trivial.

□

Denote by S_U the set of all permutations of U. For $p \in S_U$ set $p(\rho) := \{p(a)p(b) : ab \in \rho\}$. We say that binary relations ρ and σ on U are isomorphic, in symbols $\rho \approx \sigma$, if $\sigma = p(\rho)$ for some $p \in S_U$. Two binary relations ρ and σ on U are *equivalent*, in symbols $\rho \sim \sigma$, if $\sigma \approx \rho$ or $\sigma \approx \rho^{-1}$. *Set* $eqv(\rho) := |\{\sigma : \sigma \sim \rho\}|$.

3.2 Nonrigid binary relations on a 3-element domain

Fact 3.2.1. *There are exactly* 40 *non-rigid irreflexive binary relations on* $\{0,1,2\}$.

This is done by the overlap rule, the interchange rule and direct verification.

3.3 Source-Sink rule

Definition 3.3.1. *Let v be a vertex in ρ. The* outdegree *of v, denoted by $deg^+(v)$, is defined to be the cardinality of the set $\{u \in U : vu \in \rho\}$. The* indegree *of v, denoted by $deg^-(v)$, is defined to be the cardinality of the set $\{u \in U : uv \in \rho\}$. A source (respectively a sink) is a vertex of indegree (respectively outdegree) zero.*

We show that a digraph with a source and a sink is not strongly rigid.

Lemma 3.3.2. *If there are distinct vertices u and v such that v is a sink, and for all $w \in U$,*

$$wu \in \rho \implies wv \in \rho, \quad (*)$$

then ρ is not strongly rigid.

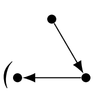
Proof. Define a binary operation f on U by setting $f(u,v) = v$ and $f(x,y) = x$ otherwise. We verify that f preserves ρ. Suppose $ac, bd \in \rho$. If neither ab nor cd is uv, then $f(ab)f(cd) = ac \in \rho$. Otherwise, since v is a sink, we have $ab \neq uv$ and $cd = uv$, so $f(ab)f(cd) = av$; now $au = ac \in \rho$ implies $av \in \rho$ and we are done. □

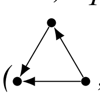
We have the following as a corollary of the above theorem because the premise of (*) is void.

Lemma 3.3.3. (Source-Sink Rule) *If ρ contains both a source and a sink, then ρ is not strongly rigid.*

3.4 Rigid but not strongly rigid binary relations on a 3-element domain

Proposition 3.4.1. *Let $U = \{0, 1, 2\}$, then the following binary relations on U are rigid but not strongly rigid.*

(1) $\rho = \{12, 20\}$ *(**, $eqv(\rho) = 6$ and $\rho \sim \rho^{-1}$),*

(2) $\rho = \{10, 20, 21\}$ *(**, $eqv(\rho) = 6$ and $\rho \sim \rho^{-1}$).*

Proof. We will prove the first case. The other is similar. We verify that ρ is rigid by using the following pp-definitions:

(1) $\alpha = \{x : ux, xv \in \rho \; for\, some\, u, v \in U\} = \{2\}$

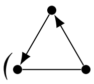
(2) $\beta = \{x : xu \in \rho \; for\, some\, u \in \alpha\} = \{x : x2 \in \rho\} = \{1\}$

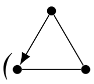
(3) $\gamma = \{x : ux \in \rho \; for\, some\, u \in \alpha\} = \{0\}$

But ρ is not strongly rigid by Lemma 3.3.3 $(u = 1, v = 0)$. □

3.5 Strongly rigid binary relations on a 3-element domain

Proposition 3.5.1. *Let $U = \{0,1,2\}$. Then the following binary relations on U are strongly rigid:*

(1) $\rho = \{02, 10, 20, 21\}$ *(**, $eqv(\rho) = 6$ and $\rho \sim \rho^{-1}$),*

(2) $\rho = \{02, 10, 12, 20, 21\}$ *(**, $eqv(\rho) = 6$ and $\rho \sim \rho^{-1}$).*

Proof. We shall use the following notation: for a subset $S \subseteq U$, let

$$S^{\rightarrow} := \{x \in U : \exists y \in S, y \rightarrow x\}$$

and symmetrically

$$^{\rightarrow}S := \{x \in U : \exists y \in S, x \rightarrow y\}.$$

Notice that if S is pp-definable, so are these two sets.

(1) This was first proved in Theorem 2 [7] for the case $n = 3$; We give a direct proof to illustrate the use of the Master Theorem.

We use the Master Theorem.

1) Observe that the complement of every singleton is pp-definable:

- $\alpha = \{x : \exists y \in U \textit{ such that } x \leftrightarrow y\} = \{0, 2\}$;
- $\beta = \{2\}^{\rightarrow} = \{0, 1\}$;
- $\gamma =^{\rightarrow} \{0\} = \{1, 2\}$.

We have

- $\delta = \alpha \cap \beta = \{0\}$,
- $\varepsilon = \alpha \cap \gamma = \{2\}$,
- $\zeta = \beta \cap \gamma = \{1\}$.

Therefore, ρ is rigid.

2) We've shown in 1) that $\{0, 2\} \in D_2(\rho)$. Since $02 \rightarrow 20$, it follows that $\forall f \in Pol^{(2)}\rho$, $f(0, 2) \neq f(2, 0)$. Fix $f \in Pol^{(2)}\rho$, let $[xy] = f(x, y)$. Suppose $[02] = 0$. Then as above $[20] = 2$. Thus

- $0 = [02] \rightarrow [21] \Rightarrow [21] = 2$;
- $[01] \rightarrow [20] = 2 \Rightarrow [01] = 0$;

- $2 = [20] \to [12] \to [01] = 0 \Rightarrow [12] = 1$;
- $2 = [21] \to [10] \to [02] = 0 \Rightarrow [10] = 1$.

Therefore we have proved in this case that f is a projection.
3) Suppose to the contrary that there exists an operation $f \in Pol^{(3)}\rho$ such that $f(x,x,y) \approx y \approx f(y,x,x)$. As $122 \to 001$, we obtain the contradiction $1 = f(1,2,2) \to f(0,0,1) = 1$.
4) Suppose $f \in Pol^{(3)}\rho$ such that $f(x,x,y) \approx x \approx f(x,y,x)$. Then $g(x,y) :\approx f(y,x,x) \in Pol^{(2)}\rho$ is a projection, hence f is either a majority operation or a semiprojection.

(a) If it is a majority operation, write $[xyz]$ for $f(x,y,z)$, we have

- $2 = [220] \to [012] \to [200] = 0 \Rightarrow [012] = 1$;
- $2 = [202] \to [120] \to [002] = 0 \Rightarrow [120] = 1$.

But $1 = [120] \to [012] = 1$, a contradiction.

(b) If it is a semiprojection, say on the first variable, we have

- $2 = [220] \to [102] \to [020] = 0 \Rightarrow [102] = 1$;
- $2 = [220] \to [012] \to [201] \to [020] = 0 \Rightarrow [012] = 0$ and $[201] = 2$;
- $1 = [102] \to [021] \Rightarrow [021] = 0$;
- $0 = [021] \to [210] \Rightarrow [210] = 2$;
- $2 = [202] \to [120] \to [012] = 0 \Rightarrow [120] = 1$.

Thus in any case, $[xxy] = x = [xyx]$ implying f is a projection. Hence we may take $\sigma_3 = U^3$.

(2) By Theorem 1 [18] for the case $n = 3$. □

Alternative Proof. I. Let $\rho_1 := \{(0,1),(1,2),(2,0),(2,1)\}$ (We switched the symbols 0 and 1 in the relation given in (1)). Let $\alpha := \{(0,1),(1,2),(2,0)\}$ and let $\leq$ denote the natural order relation

$$\{(0,0),(0,1),(0,2),(1,1),(1,2),(2,2)\}$$

on U. It is well known that both $Pol\,\alpha$ and $Pol \leq$ are maximal clones on U([2], [10]); It is shown in Proposition 3.1 of Czédi et al. [26] that the intersection $Pol\,\alpha \cap Pol \leq$ is the clone of all projections on U. Now by direct verification we have

$$\alpha = \{(x,y) : \exists v_3 \in U\, with\, (x,y) \wedge (y,v_3) \wedge (v_3,y) \in \rho_1\}.$$

On the other hand the linear order $\leq$ is exactly

$$\{(x,y) : \exists u_2,u_3,u_4,u_5 \in U\, with\, (x,u_2) \wedge (u_2,u_3) \wedge (u_3,u_4) \wedge (u_4,u_5) \wedge (x,u_5) \wedge (u_3,y) \in \rho_1\}.$$

Thus by Theorem 2.2.11 we have $Pol\rho_1 \subseteq Pol\alpha$ and $Pol\rho_1 \subseteq Pol \leq$, i.e., $Pol\rho_1 \subseteq (Pol\alpha \cap Pol \leq)$, proving that $Pol\rho_1$ consists of projections only.

II. Let $\rho_2 := \{(0,1),(0,2),(1,2),(2,0),(2,1)\}$. Let

$$\beta := \{(x,y) : \exists u_3,u_4,u_5 \in U\, with\, (x,y) \wedge (x,u_3) \wedge (u_3,u_4) \wedge (u_4,y) \wedge (y,u_5) \wedge (u_4,u_5) \in \rho_2\}.$$

One can verify that $\beta = \rho_1$. Again, by Theorem 2.2.11, we have $Pol\rho_2 \subseteq Pol\beta = Pol\rho_1$, and since $Pol\rho_1$ is trivial, the relation ρ_2 is strongly rigid. $\square$

There are $2^{9-3} = 64$ different irreflexive binary relations on $\{0,1,2\}$ and they are all accounted for in the above results. Thus, by combining Propositions 3.1.6, 3.2.1, 3.4.1 3.5.1, we have the classification of binary relations on a 3-element domain listed in Table 1.1.

CHAPTER 4

RIGID BINARY RELATIONS ON A 4-ELEMENT DOMAIN

In this chapter, we will classify all binary relations on a 4-element domain by means of the loop, overlap, interchange and in-out rules introduced in the previous chapter.

4.1 Nonrigid binary relations on a 4-element domain

Proposition 4.1.1. *There are 61440 non-irreflexive relations on a* 4*-element domain.*

Proof. We know that there are $2^{4^2} - 2^{2^4-4} = 2^{16} - 2^{12} = 61440$ non-irreflexive relations on a 4-element domain. $\square$

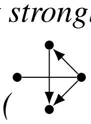
Since a relation with a loop is not rigid, now we can only consider irreflexive relations. There are $2^{16-4} = 2^{12} = 4096$ of these.

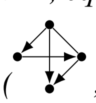
Fact 4.1.2. *There are exactly* 2644 *non-rigid irreflexive binary relations on* $\{0,1,2,3\}$.

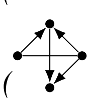
This is done by the overlap rule, the interchange rule and direct verification.

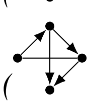
4.2 Rigid but not strongly rigid binary relations on a 4-element domain

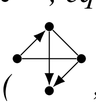
Proposition 4.2.1. *(Example 5.4 [41]) Let* $U = \{0, 1, 2, 3\}$*, then the following binary relations on* U *are rigid but not strongly rigid.*

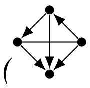
(1) $\rho = \{13, 20, 30, 31, 32\}$ *(*  *,* $eqv(\rho) = 48$ *and* $\rho \not\sim \rho^{-1}$*),*

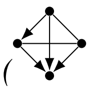
(2) $\rho = \{13, 20, 21, 23, 30, 32\}$ *(*  *,* $eqv(\rho) = 48$ *and* $\rho \not\sim \rho^{-1}$*),*

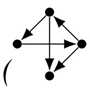
(3) $\rho = \{12, 13, 20, 30, 31, 32\}$ *(*  *,* $eqv(\rho) = 48$ *and* $\rho \not\sim \rho^{-1}$*),*

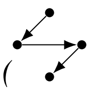
(4) $\rho = \{12, 13, 20, 23, 30, 31\}$ *(*  *,* $eqv(\rho) = 48$ *and* $\rho \not\sim \rho^{-1}$*),*

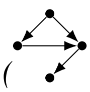
(5) $\rho = \{12, 13, 20, 23, 30, 31, 32\}$ *(*  *,* $eqv(\rho) = 48$ *and* $\rho \not\sim \rho^{-1}$*),*

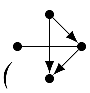
(6) $\rho = \{10, 13, 20, 21, 30, 31, 32\}$ *(*  *, eqv*$(\rho) = 48$ *and* $\rho \not\sim \rho^{-1}$*),*

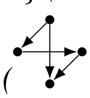
(7) $\rho = \{10, 13, 20, 21, 23, 30, 31, 32\}$ *(*  *, eqv*$(\rho) = 48$ *and* $\rho \not\sim \rho^{-1}$*),*

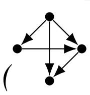
(8) $\rho = \{13, 20, 21, 30, 32\}$ *(*  *, eqv*$(\rho) = 48$ *and* $\rho \not\sim \rho^{-1}$*),*

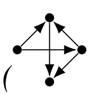
(9) $\rho = \{13, 21, 30\}$ *(*  *, eqv*$(\rho) = 24$ *and* $\rho \sim \rho^{-1}$*),*

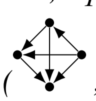
(10) $\rho = \{13, 21, 23, 30\}$ *(*  *, eqv*$(\rho) = 48$ *and* $\rho \not\sim \rho^{-1}$*),*

(11) $\rho = \{13, 20, 23, 30, 31\}$ *(*  *, eqv*$(\rho) = 24$ *and* $\rho \sim \rho^{-1}$*),*

(12) $\rho = \{13, 20, 21, 30\}$ *(*  *, eqv*$(\rho) = 24$ *and* $\rho \sim \rho^{-1}$*),*

(13) $\rho = \{13, 20, 21, 23, 30\}$ *(*  *, eqv*$(\rho) = 48$ *and* $\rho \not\sim \rho^{-1}$*),*

(14) $\rho = \{12, 13, 20, 30, 32\}$ *(*  *, eqv*$(\rho) = 24$ *and* $\rho \sim \rho^{-1}$*),*

(15) $\rho = \{10, 20, 21, 30, 31, 32\}$ *(*  *, eqv*$(\rho) = 24$ *and* $\rho \sim \rho^{-1}$*).*

Proof. (1) We verify that ρ is rigid. Let $f \in Pol^{(1)}\rho$.

(i) Let $\alpha = \{x : xu, ux, uv, uw, vw \in \rho \text{ for some } u, v, w \in U\}$. We first prove that $\alpha = \{1\}$. In fact, for every $x \in \alpha$, from $xu, ux \in \rho$, we know that $x, u \in \{1, 3\}$, but x can not be 3. Otherwise $u = 1$, then $1v, 1w, vw \in \rho$ implying that $v = 3, w = 3$ and $vw \in \rho$, a contradiction. Therefore $x = 1$ and $\alpha = \{1\}$. It follows that $f(1) = 1$. In fact, $1 \in \alpha$ implies that there are some $u, v, w \in U$ such that $1u, u1, uv, uw, vw \in \rho$. As $f \in Pol^{(1)}\rho$, we have

$$f(1)f(u), f(u)f(1), f(u)f(v), f(u)f(w), f(v)f(w) \in \rho,$$

and thus $f(1) \in \alpha$. i.e. $f(1) = 1$.

(ii) Let $\beta = \{x : ux \in \rho \text{ for some } u \in \alpha\}$. Then $\beta = \{x : 1x \in \rho\} = \{3\}$. Thus $f(3) = 3$.

(iii) Let $\gamma = \{x : ux, uw, xw \in \rho \textit{ for some } u \in \beta \textit{ and some } w \in U\}$, then $\gamma = \{2\}$. Thus $f(2) = 2$.

(iv) Let $\delta = \{x : ux \in \rho \textit{ for some } u \in \gamma\}$, then $\delta = \{0\}$. Thus $f(0) = 0$.

By (i)-(iv), ρ is rigid. But ρ is not strongly rigid by Lemma 3.3.2 ($u = 2, v = 0$).

Similarly we can prove other cases.

(2) We verify that ρ is rigid:

(i) $\alpha = \{x : ux, vx, uv \in \rho \textit{ for some } u, v \in U\} = \{0\}$,

(ii) $\beta = \{x : xu, vu, xv \in \rho \textit{ for some } u \in \alpha \textit{ and some } v \in U\} = \{3\}$,

(iii) $\gamma = \{x : xu, vx, vu \in \rho \textit{ for some } u \in \alpha \textit{ and some } v \in \beta\} = \{2\}$,

(iv) $\delta = \{x : ux, xv, vu \in \rho \textit{ for some } u \in \gamma \textit{ and some } v \in \beta\} = \{0\}$.

But ρ is not strongly rigid by Lemma 3.3.2 ($u = 2, v = 0$).

(3) We verify that ρ is rigid:

(i) $\alpha = \{x : vx, ux, uv, vu \in \rho \textit{ for some } u, v \in U\} = \{0\}$,

(ii) $\beta = \{x : vx, xu, uv, vu \in \rho \textit{ for some } u, v \in U\} = \{1\}$,

(iii) $\gamma = \{x : xu, vx \in \rho \textit{ for some } u \in \alpha \textit{ and some } v \in \beta\} = \{3\}$,

(iv) $\delta = \{x : xu, ux \in \rho \textit{ for some } u \in \gamma\} = \{2\}$.

But ρ is not strongly rigid by Lemma 3.3.2 ($u = 1, v = 0$).

(4) We verify that ρ is rigid:

(i) $\alpha = \{x : vx, ux, uv, vu \in \rho \textit{ for some } u, v \in U\} = \{2\}$,

(ii) $\beta = \{x : ux \in \rho \textit{ for some } u \in \alpha\} = \{x : 2x \in \rho\} = \{0\}$,

(iii) $\gamma = \{x : xu, xv, uv \in \rho \textit{ for some } u \in \alpha \textit{ and some } v \in \beta\} = \{3\}$,

(iv) $\delta = \{x : xu, ux \in \rho \textit{ for some } u \in \gamma\} = \{1\}$.

But ρ is not strongly rigid by Lemma 3.3.2 ($u = 1, v = 0$).

(5) We verify that ρ is rigid:

(i) $\alpha = \{x : vx, xu, uv, vu \in \rho \textit{ for some } u, v \in U\} = \{2\}$,

(ii) $\beta = \{x : xu \in \rho \textit{ for some } u \in \alpha\} = \{x : x2 \in \rho\} = \{1\}$,

(iii) $\gamma = \{x : xu, ux \in \rho \textit{ for some } u \in \beta\} = \{3\}$,

(iv) $\delta = \{x : ux, vx, uv \in \rho \textit{ for some } u \in \alpha \textit{ and some } v \in \gamma\} = \{0\}$.

But ρ is not strongly rigid by Lemma 3.3.2 ($u = 2, v = 0$).

(6) We verify that ρ is rigid:

(i) $\alpha = \{x : xu, ux, xv, vx, uv \in \rho \textit{ for some } u, v \in U\} = \{3\}$,

(ii) $\beta = \{x : ux, xu, uv, vu, xv \in \rho \textit{ for some } u \in \alpha \textit{ and some } v \in U\} = \{1\}$,

(iii) $\gamma = \{x : xu, ux, uv, vu, vx \in \rho \textit{ for some } u \in \alpha \textit{ and some } v \in \beta\} = \{2\}$,

(iv) $\delta = \{x : ux, vx, uv, vu \in \rho \textit{ for some } u \in \alpha \textit{ and some } v \in \gamma\} = \{0\}$.

But ρ is not strongly rigid by Lemma 3.3.2 ($u = 1, v = 0$).

(7) We verify that ρ is rigid:

(i) $\alpha = \{x : ux, vx, uv, vu \in \rho \textit{ for some } u, v \in U\} = \{0\}$,

(ii) $\beta = \{x : ux, xv, uv, vu \in \rho \textit{ for some } u, v \in U\} = \{x : 1x \in \rho\} = \{2\}$,

(iii) $\gamma = \{x : xu, vx, vu \in \rho \textit{ for some } u \in \alpha \textit{ and some } v \in \beta\} = \{1\}$,

(iv) $\delta = \{x : xu, ux \in \rho \textit{ for some } u \in \gamma\} = \{3\}$.

But ρ is not strongly rigid by Lemma 3.3.2 ($u = 2, v = 0$).

(8) We verify that ρ is rigid:

(i) $\alpha = \{x : xu, ux, xv, vx, uv \in \rho \textit{ for some } u, v \in \rho\} = \{3\}$,

(ii) $\beta = \{x : ux, xu, vx, uv, vu \in \rho \textit{ for some } u \in \alpha \textit{ and some } v \in U\} = \{1\}$

(iii) $\gamma = \{x : ux, xu, xv \in \rho \textit{ for some } u \in \alpha \textit{ and some } v \in \beta\} = \{2\}$,

(iv) $\delta = \{x : ux, vx, wx \textit{ for some } u \in \alpha, v \in \beta \text{ and } w \in \gamma\} = \{0\}$.

But ρ is not strongly rigid by Lemma 3.3.2 ($u = 1, v = 0$).

(9) We verify that ρ is rigid:

(i) $\alpha = \{x : xu, uv, vw \in \rho \textit{ for some } u, v, w \in U\} = \{2\}$

(ii) $\beta = \{x : ux \in \rho \textit{ for some } u \in \alpha\} = \{x : 2x \in \rho\} = \{1\}$,

(iii) $\gamma = \{x : ux \in \rho \textit{ for some } u \in \beta\} = \{x : 1x \in \rho\} = \{3\}$,

(iv) $\delta = \{x : ux \in \rho \textit{ for some } u \in \gamma\} = \{x : 3x \in \rho\} = \{0\}$.

But ρ is not strongly rigid by Lemma 3.3.3 ($u = 2, v = 0$).

(10) We verify that ρ is rigid:

(i) $\alpha = \{x : xu, xv, uv \in \rho \text{ for some } u, v \in U\} = \{2\}$,

(ii) $\beta = \{x : ux, uv, vx \in \rho \text{ for some } u \in \alpha\} = \{3\}$,

(iii) $\gamma = \{x : ux, xv \in \rho \text{ for some } u \in \alpha \text{ and some } v \in \beta\} = \{1\}$,

(iv) $\delta = \{x : ux \in \rho \text{ for some } u \in \beta\} = \{0\}$.

But ρ is not strongly rigid by Lemma 3.3.3 ($u = 2, v = 0$).

(11) We verify that ρ is rigid:

(i) $\alpha = \{x : ux, vx, uv \in \rho \text{ for some } u, v \in U\} = \{0\}$,

(ii) $\beta = \{x : xu, xv, vx \in \rho \text{ for some } u \in \alpha \text{ and some } v \in U\} = \{3\}$,

(iii) $\gamma = \{x : ux, xu \in \rho \text{ for some } u \in \beta\} = \{1\}$,

(iv) $\delta = \{x : xu, xv \in \rho \text{ for some } u \in \alpha \text{ and some } v \in \beta\} = \{2\}$.

But ρ is not strongly rigid by Lemma 3.3.3 ($u = 2, v = 0$).

(12) We verify that ρ is rigid:

(i) $\alpha = \{x : xu, uv, uw \in \rho \text{ for some } u, v, w \in U\} = \{2\}$,

(ii) $\beta = \{x : ux, xv \in \rho \text{ for some } u \in \alpha \text{ and some } v \in U\} = \{1\}$,

(iii) $\gamma = \{x : ux \in \rho \text{ for some } u \in \beta\} = \{3\}$,

(iv) $\delta = \{x : ux \in \rho \text{ for some } u \in \gamma\} = \{0\}$.

But ρ is not strongly rigid by Lemma 3.3.3 ($u = 2, v = 0$).

(13) We verify that ρ is rigid:

(i) $\alpha = \{x : ux, vx, uv \in \rho \text{ for some } u, v \in U\} = \{0\}$,

(ii) $\beta = \{x : xu, xv, uv \in \rho \text{ for some } u \in \alpha \text{ and some } v \in U\} = \{2\}$,

(iii) $\gamma = \{x : ux, vx \in \rho \text{ for some } u \in \alpha \text{ and some } v \in \beta\} = \{3\}$,

(iv) $\delta = \{x : ux, xv \in \rho \text{ for some } u \in \beta \text{ and some } v \in \gamma\} = \{1\}$.

But ρ is not strongly rigid by Lemma 3.3.3 ($u = 2, v = 0$).

(14) We verify that ρ is rigid:

(i) $\alpha = \{x : ux, vx, uv \in \rho \text{ for some } u, v \in U\} = \{0\}$,

(ii) $\beta = \{x : xu, xv, uv \in \rho \text{ for some } u, v \in U\} = \{1\}$,

(iii) $\gamma = \{x : ux, xv, uv \in \rho \textit{ for some } u \in \beta\} = \{3\}$,

(iv) $\delta = \{x : ux, xv \in \rho \textit{ for some } u \in \gamma \textit{ and some } v \in \alpha\} = \{2\}$.

But ρ is not strongly rigid by Lemma 3.3.3 ($u = 1, v = 0$).

(15) We verify that ρ is rigid:

(i) $\alpha = \{x : ux, wx, uw \in \rho \textit{ for some } u, v, w \in U\} = \{0\}$

(ii) $\beta = \{x : xu, xv, vu \in \rho \textit{ for some } u \in \alpha \textit{ and some } v \in U\} = \{3\}$,

(iii) $\gamma = \{x : xv, uw, wx \in \rho \textit{ for some } u, w \in \beta, v \in \alpha\} = \{1\}$,

(iv) $\delta = \{x : ux, xv \in \rho \textit{ for some } u \in \beta \textit{ and some } v \in \gamma\} = \{2\}$.

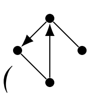
But ρ is not strongly rigid by Lemma 3.3.3 ($u = 3, v = 0$).

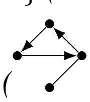
□

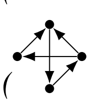
4.3 Strongly rigid binary relations on a 4-element domain

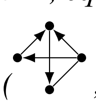
As an application of the Master Theorem, we present the following 25 strongly rigid binary relations on a 4-element domain.

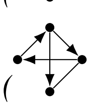
Proposition 4.3.1. *(Example 6.1 [41]) Let* $U = \{0, 1, 2, 3\}$*, then the following are strongly rigid.*

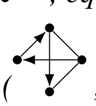
(1) $\rho = \{01, 02, 10, 21, 23, 32\}$ ( , $eqv(\rho) = 24$ *and* $\rho \sim \rho^{-1}$),

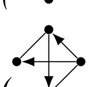
(2) $\rho = \{03, 13, 21, 30, 32\}$ ( , $eqv(\rho) = 24$ *and* $\rho \sim \rho^{-1}$),

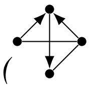
(3) $\rho = \{03, 12, 20, 31, 32\}$ ( , $eqv(\rho) = 24$ *and* $\rho \sim \rho^{-1}$),

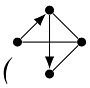
(4) $\rho = \{03, 12, 20, 30, 31, 32\}$ ( , $eqv(\rho) = 48$ *and* $\rho \not\sim \rho^{-1}$),

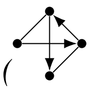
(5) $\rho = \{03, 12, 20, 23, 30, 31\}$ ( , $eqv(\rho) = 48$ *and* $\rho \not\sim \rho^{-1}$),

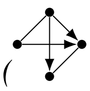
(6) $\rho = \{03, 12, 20, 23, 30, 31, 32\}$ ( , $eqv(\rho) = 48$ *and* $\rho \not\sim \rho^{-1}$),

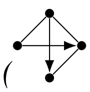
(7) $\rho = \{03, 12, 20, 21, 30, 31, 32\}$ ( , $eqv(\rho) = 48$ *and* $\rho \not\sim \rho^{-1}$),

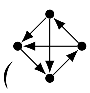
(8) $\rho = \{03, 12, 13, 20, 30, 31, 32\}$ ( , $eqv(\rho) = 48$ and $\rho \not\sim \rho^{-1}$),

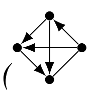
(9) $\rho = \{03, 12, 13, 20, 23, 30, 31, 32\}$ ( , $eqv(\rho) = 24$ and $\rho \sim \rho^{-1}$),

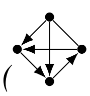
(10) $\rho = \{03, 12, 13, 20, 21, 30, 32\}$ ( , $eqv(\rho) = 24$ and $\rho \sim \rho^{-1}$),

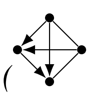
(11) $\rho = \{03, 12, 13, 20, 21, 23, 30\}$ ( , $eqv(\rho) = 24$ and $\rho \sim \rho^{-1}$),

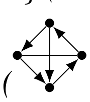
(12) $\rho = \{03, 12, 13, 20, 21, 23, 30, 32\}$ ( , $eqv(\rho) = 24$ and $\rho \sim \rho^{-1}$),

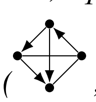
(13) $\rho = \{03, 10, 20, 21, 31, 32\}$ ( , $eqv(\rho) = 24$ and $\rho \sim \rho^{-1}$),

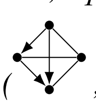
(14) $\rho = \{03, 10, 20, 21, 30, 31, 32\}$ ( , $eqv(\rho) = 24$ and $\rho \sim \rho^{-1}$),

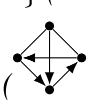
(15) $\rho = \{03, 10, 20, 21, 23, 31, 32\}$ ( , $eqv(\rho) = 48$ and $\rho \not\sim \rho^{-1}$),

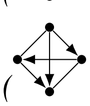
(16) $\rho = \{03, 10, 20, 21, 23, 30, 31, 32\}$ ( , $eqv(\rho) = 48$ and $\rho \not\sim \rho^{-1}$),

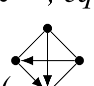
(17) $\rho = \{03, 10, 13, 20, 21, 31, 32\}$ ( , $eqv(\rho) = 24$ and $\rho \sim \rho^{-1}$),

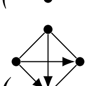
(18) $\rho = \{03, 10, 13, 20, 21, 30, 31, 32\}$ ( , $eqv(\rho) = 48$ and $\rho \not\sim \rho^{-1}$),

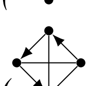
(19) $\rho = \{03, 10, 13, 20, 21, 23, 30, 31, 32\}$ ( , $eqv(\rho) = 24$ and $\rho \sim \rho^{-1}$),

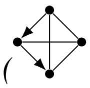
(20) $\rho = \{03, 10, 12, 20, 21, 23, 31, 32\}$ ( , $eqv(\rho) = 48$ and $\rho \not\sim \rho^{-1}$),

(21) $\rho = \{03, 10, 12, 20, 21, 23, 30, 31\}$ ( , $eqv(\rho) = 24$ and $\rho \sim \rho^{-1}$),

(22) $\rho = \{03, 10, 12, 20, 21, 23, 30, 31, 32\}$ ( , $eqv(\rho) = 48$ and $\rho \not\sim \rho^{-1}$),

(23) $\rho = \{03, 10, 12, 13, 20, 21, 23, 30, 32\}$ ( , $eqv(\rho) = 24$ and $\rho \sim \rho^{-1}$),

(24) $\rho = \{02, 03, 10, 13, 20, 21, 30, 31, 32\}$ ( , $eqv(\rho) = 24$ and $\rho \sim \rho^{-1}$),

(25) $\rho = \{02, 03, 10, 13, 20, 21, 23, 30, 31, 32\}$ *(*  *, $eqv(\rho) = 24$ and $\rho \sim \rho^{-1}$).*

(see Fig.1.4)

Proof. (9) and (14) are special cases of known results:

(9) By Theorem 1 [18] for $n = 4$,

(14) By Theorem 2 [7] for $n = 4$.

We will prove case (1); the rest are similar.

First we verify 1) in the Master Theorem. Let $S = \{0, 1, 2\}$. Notice that the restriction of ρ to S is strongly rigid by Proposition 3.5.1(1), and hence any $f \in Pol\rho$ restricted to S is a projection and every subset of S is pp-definable; we also have

- $\{2\}^{\rightarrow} = \{1, 3\}$,
- ${}^{\rightarrow}\{2\} = \{0, 3\}$,
- ${}^{\rightarrow}\{2\} \cap \{2\}^{\rightarrow} = \{3\}$,
- $\{1, 2\}^{\rightarrow} = \{0, 1, 3\}$,
- $\alpha =^{\rightarrow} \{1, 2\} = \{0, 2, 3\}$,
- $\beta = \{0, 2\}^{\rightarrow} = \{1, 2, 3\}$,
- $\alpha \cap \beta = \{2, 3\}$.

From the above, we know $f(A) \subseteq A$ for $A \in 2^U$ and for all $f \in Pol^{(1)}\rho$.

Next we verify 2) in the Master Theorem. Let $f \in Pol^{(2)}\rho$ be idempotent. Write [xy] for $f(x, y)$. Set $E = \{xy \in U^2 : [xy] = x\}$. We know that f restricted to S is a projection. Without loss of generality, we can assume that $f|_S$ is the first projection. Thus, $01, 10, 02, 20, 12, 21 \in E$. As $[13] \in \{1, 3\}$ and $[13] \rightarrow [02] = 0$ we have $13 \in E$, and $0 = [02] \rightarrow [23] \in \{2, 3\}$ implies $23 \in E$, and then $2 = [23] \leftrightarrow [32]$ yields $32 \in E$. Now $2 = [22] \rightarrow [31] \rightarrow [20] = 2$ leads to $31 \in E$. Next $1 = [12] \rightarrow [03] \in \{0, 3\}$ leads to $03 \in E$. $2 = [21] \leftrightarrow [30] \in \{0, 3\}$ implies $30 \in E$. Therefore we have proved in this case f is the first projection.

Now we verify 3) in the Master Theorem. Suppose to the contrary that there exists an operation $f \in Pol^{(3)}\rho$ such that $f(x,x,y) \approx y \approx f(x,y,x)$. As $[202] \to [113]$, we obtain the contradiction $0 = [202] \to [113] = 3$.

Further we verify 4) in the Master Theorem. Set $\sigma_3 = \{013, 021, 023, 102, 103, 123, 201, 203, 213, 301, 302, 312\}$. Let f satisfy the hypotheses of 4)(ii). Set $E := \{xyz \in U^3 : f(x,y,z) = x$ for every $f \in Pol^{(3)}\rho$ satisfying $4)(ii)\}$. Set $E := \{xyz \in U^3 : f(x,y,z) = x\}$. We have $xxy, xyx \in E$ for all $x, y \in U$. We know that f restricted to S is a projection. Without loss of generality, we can assume that $f|_S$ is the first projection. Thus, $021, 102, 201 \in E$.

Notice that $2 = [212] \to [301] \to [210] = 2$ leads to $301 \in E$. From $2 = [210] \to [302] \to [221] = 2$, we obtain $302 \in E$. Now $2 = [220] \to [312] \to [201] = 2$ implies $312 \in E$, and $3 = [302] \leftrightarrow [213]$ gives $213 \in E$. Since $3 = [312] \leftrightarrow [203]$, we have $203 \in E$. $1 = [102] \to [013] \to [202] = 2$ leads to $013 \in E$. Now $2 = [212] \to [103] \to [012] = 0$ implies $103 \in E$. As $1 = [102] \to [023]$, we get $023 \in E$. Now, $2 = [202] \to [123] \to [032] = 0$ shows $123 \in E$. As $0 = [022] \leftrightarrow [133]$, we have $133 \in E$. Notice that $2 = [211] \leftrightarrow [300]$ yields $300 \in E$. Now $0 = [022] \to [233] \in \{2,3\}$ implies $233 \in E$. Next $2 = [233] \leftrightarrow [322]$ shows $322 \in E$. As $1 = [122] \leftrightarrow [033]$, we know $033 \in E$, and $2 = [200] \leftrightarrow [311]$ gives $311 \in E$. Consequently we have proved $xyy \in E$ and thus $xyz \in E$ whenever $x, y, z \in U$. i.e., every ternary operation $f \in Pol\rho$ is a projection.

Finally we verify 5) in the Master Theorem. Set $\sigma_4 = \{0123, 1023, 2013, 3102\}$. Let f satisfy the hypotheses of 5)(ii). Set $E := \{xyzw \in U^4 : f(x,y,z,w) = x$ for every $f \in Pol^{(4)}\rho$ satisfying $5)(ii)\}$. We abbreviate $f(x,y,z,w)$ by $[xyzw]$.

Notice that $[0123] \leftrightarrow [1032] \to [0223] = 0$ implies $1023, 0123 \in E$. Finally $[3102] \leftrightarrow [2013] \to [3202] = 3$ gives $2013, 3102 \in E$.

Therefore, ρ is a strongly rigid binary relation.

$\square$

By combining Propositions 4.1.1, 4.1.2, 4.2.1 and 4.3.1, we have a classification of binary relations on a 4-element domain listed in Table 1.2.

CHAPTER 5

A STRONGLY RIGID BINARY RELATION

In this chapter, we construct a new strongly rigid binary relation on a finite domain.

5.1 A new strongly rigid binary relation on a finite domain

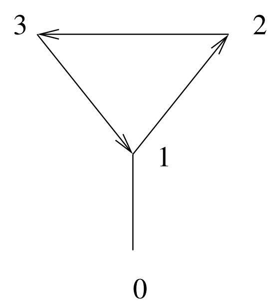


Figure 5.1 – A strongly rigid binary relation on a 4-element domain

Consider the binary relation $\rho := \{01, 10, 12, 23, 31\}$ (see Fig. 5.1). Notice that ρ is a directed 3-cycle on the set $\{1,2,3\}$ together with the undirected edge $\{01, 10\}$. This relation is equivalent to the relation in (2) of Proposition 4.3.1 and therefore is strongly rigid. We use the pattern in ρ to construct a strongly rigid binary relation on any finite domain with $2n$ elements where $n \geq 2$.

Theorem 5.1.1. *Let k be a natural number with $k \geq 3$. Consider the $(k+2)$-element set $U = \{0, 1, ..., k, k+1\}$ with the following binary relation ρ (see Fig. 5.2):*

$$\rho = \{(0,1), (1,2), ..., (k-1,k), (k,0)\} \cup \{(k+1,0), (0,k+1)\},$$

then

(a) ρ *is not rigid if k is odd;*

(b) ρ *is strongly rigid if k is even.*

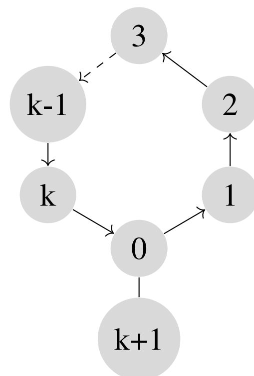


Figure 5.2 – A strongly rigid binary relation on a finite domain

The series of claims in the proof of the theorem follows the description of types of minimal clones in Theorem 2.2.12. For brevity, a ρ-walk is a series $\alpha = u_1 \to u_2 \to \cdots \to u_s$ (abbreviated as $\alpha = u_1u_2\ldots u_s$) of elements in U such that $(u_i, u_{i+1}) \in \rho$ for all i $(1 \leq i \leq s-1)$; α is a ρ-walk of length $(s-1)$. If $\alpha_1, \ldots, \alpha_n$ are ρ-walks with the same length ℓ and f is an n-ary operation that preserves ρ then we can define $f(\alpha_1, \ldots, \alpha_n)$ in a natural way to get another ρ-walk with length ℓ. If α and β are ρ-walks, then the product of α and β is the concatenation of α and β, denoted by $\alpha\beta$; For a ρ-walk α and a natural number n, α^n will denote the n-fold product $\alpha \ldots \alpha$.

Throughout the rest of this chapter, we set $\gamma = 01\ldots k$, $\eta = 0(k+1)$ and $\delta = (k+1)0$.

The following properties of ρ-walks, for even k, are immediate consequences of the structure of ρ.

Lemma 5.1.2. *Let* $k = 2n$ *be even,* $n > 1$*. Then*

(1) the only ρ*-walks of even length from* 0 *to* 0 *are of the form* $\eta^i 0$ *or* $0\delta^i$ *where* $i \geq 1$*;*

(2) for any ρ*-walk* $u_1u_2\ldots u_s$ *of length less than or equal to k, if* $u_s = s$*, then* $u_1u_2\ldots u_s = 12\ldots s$*;*

(3) there is no ρ-walk from s to s of odd length less than $k+1$ where $s \in U$;

(4) the only ρ-walks of odd length $l \geq k+1$ from 0 to 0 are of the form $\eta^i(\gamma 0)\delta^j$ where $i, j \geq 0$; in particular, the only ρ-walk of length $(k+1)$ from 0 to 0 is $\gamma 0$.

Proof of Theorem 5.1.1. (a) If k is odd then consider the unary operation f on U defined by the rule

$$f : U \to U, s \mapsto \begin{cases} k+1 & \text{if } s = k+1 \text{ or } s \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Then f preserves ρ and f is not the identity operation, which proves that ρ is not rigid.

(b) Let k be even, say $k = 2n$ ($n \geq 2$ integer). To prove that ρ is strongly rigid, it clearly suffices by Theorem 2.2.12 to show that the digraph is rigid, has no non-binary polymorphisms, no minority nor majority polymorphisms, nor any semiprojections of arity > 2 other than projections.

Claim 1. If f is an operation in $\mathrm{Pol}^{(1)}\rho$ then f is the identity operation on U.

Proof. Since $\rho \cap \rho^{-1} = \{(k+1,0),(0,k+1)\}$ and f preserves ρ then $f(\{0,k+1\}) = \{0,k+1\}$ holds. We first prove that $f(0) = 0$ and $f(k+1) = k+1$. Suppose, for a contradiction, that $f(0) = k+1, f(k+1) = 0$. Since $\alpha = \gamma 0$ is a ρ-walk of length $(k+2)$, $f(\alpha) = \delta \ldots \eta$ is a ρ-walk, as well, that contains a ρ-walk of odd length $(k-1)$ from 0 to 0, which contradicts Lemma 5.1.2(3).

Now we prove that for $i \in \{1,2,\ldots,k\}$ we have $f(i) = i$. It is enough to prove that $f(1) = 1$ holds. Since $f(0) = 0$ by the preceding argument, we get that either $f(1) = 1$ or $f(1) = k+1$. If $f(1) = k+1$, then $k+1 = f(1) \to f(2)$ and thus $f(2) = 0$. Hence, $0 = f(2) \to \ldots \to f(k) \to f(0) = 0$ is a ρ-walk of odd length $(k-1)$ from 0 to 0, which contradicts Lemma 5.1.2(3). Therefore, $f(1) = 1$. It follows from $1 = f(1) \to f(2) \to \cdots \to f(k)$ that $f(i) = i$ holds for every $i \in \{2,\ldots,k\}$, which concludes the proof. $\square$

Claim 2. Let $f \in \mathrm{Pol}^{(2)}\rho$. Then f is a projection.

Proof. The unary operation $f(x,x)$ belongs to Pol ρ, and so, $f(x,x)$ is the identity operation by Claim 1. Consider the table below

Table 5.1: $k+3$ steps from 0 to 0 in the row $f(x,y)$

x	0	1	2	3	$\cdots$	k	0	$k+1$	0
y	0	$k+1$	0	1	$\cdots$	$k-2$	$k-1$	k	0
$f(x,y)$	0	$*$	$*$	$*$	$\cdots$	$*$	$*$	$*$	0

By Lemma 5.1.2(4), there are only two ρ-walks from 0 to 0 consisting of $k+3$ steps, either $f|_H = e_1^2|_H$ or $f = e_2^2|_H$, where $H = \{(1,k+1),(2,0),(3,1),\ldots,(k,k-2),(0,k-1),(k+1,k)\}$. Changing the order of variables, we may assume that $f|_H = e_1^2|_H$. We will prove that $f = e_1^2$ everywhere.

Claim 2.1. $f(x,y) = x$ holds for every pair (x,y) in which 0 or $k+1$ appears.

Claim 2.1.1. $f|_J = e_1^2|_J$ where $J = \{(k+1,0),(0,k+1)\}$.

Consider the ρ-walks $\alpha = \gamma\eta^{n+1}0$ and $\beta = \eta^{n+1}\gamma 0$ of length $(2k+3)$. Set $\varepsilon = f(\alpha,\beta)$. Then $\varepsilon_1\varepsilon_2\varepsilon_3 = 012$ and $\varepsilon_{2k+2}\varepsilon_{2k+3}\varepsilon_{2k+4} = 0(k+1)0$ by Table 5.1, and so, $\varepsilon_1\varepsilon_2\ldots\varepsilon_{k+2} = \gamma 0$, which proves that $f(0,k+1) = \varepsilon_{k+2} = 0$. Since, $\zeta = \varepsilon_{k+2}\ldots\varepsilon_{2k+2}$ is a ρ-walk of length k from 0 to 0 we have that $\zeta = \eta^k 0$ by Lemma 5.1.2(1), hence, $f(k+1,0) = \varepsilon_{k+3} = k+1$, and $f(0,k+1) = 0$ as $f(0,k+1) \leftrightarrow f(k+1,0) = k+1$.

Claim 2.1.2. $f|_J = e_1^2|_J$ where $J = \{(k+1,i),(i,k+1),(0,i+1),(i+1,0) : i$ is an odd integer with $1 \leq i \leq k\}$.

Consider the ρ-walks $\alpha = \eta^{n+1}$ and $\beta = \gamma 0$ of length $(k+1)$. First set $\tau = f(\beta,\alpha)$. By Claim 2.1.1 we have $f(0,k+1) = 0$ so τ is a ρ-walk of length $(k+1)$ from 0 to 0. By Lemma 5.1.2(4), $\tau = \gamma 0$. It follows that $f(i,k+1) = i$ and $f(i+1,0) = i+1$

for all odd i between 0 and k. Now set $\varepsilon = f(\alpha,\beta)$. Then $\varepsilon_1 = 0$ and $\varepsilon_{k+1} = 0$ as $\varepsilon_{k+1} \to \varepsilon_{k+2} = f(k+1,0) = k+1$. Since, $\zeta = \varepsilon_1\varepsilon_2\ldots\varepsilon_{k+1}$ is a ρ-walk of length k from 0 to 0 we have that $\zeta = \eta^k 0$ by Lemma 5.1.2(1), hence, $f(k+1,i) = \varepsilon_{i+1} = k+1$ where i is an odd integer with $1 \le i \le k$. Therefore, $f(0,i+1) = 0$ for i odd in $\{0,\ldots,k\}$. Similarly, we can show that $f(i,k+1) = i$ where i is an odd integer with $1 \le i \le k$.

Claim 2.1.3. $f|_J = e_1^2|_J$ where $J = \{(k+1,i),(i,k+1),(0,i+1),(i+1,0) : i$ is an even integer with $1 \le i \le k\}$.

Consider the ρ-walks $\alpha = \delta^{n+1}$ and $\beta = \gamma 0$ of length $((k+1))$. First set $\tau = f(\beta,\alpha)$. By Claim 2.1.1 we have $f(0,k+1) = 0$ so τ is a ρ-walk of length $(k+1)$ from 0 to 0. By Lemma 5.1.2(4), $\tau = \gamma 0$. It follows that $f(i,k+1) = i$ and $f(i+1,0) = i+1$ for all odd i between 0 and k. Now set $\varepsilon = f(\alpha,\beta)$. As $k+1 = f(k+1,0) = \varepsilon_1 \to \varepsilon_2$, we have $\varepsilon_2 = 0$, and $\varepsilon_{k+2} = f(0,0) = 0$. Since, $\zeta = \varepsilon_2\varepsilon_3\ldots\varepsilon_{k+2}$ is a ρ-walk of length k from 0 to 0 we have that $\zeta = 0\delta^k$ by Lemma 5.1.2(1), hence, $f(k+1,i) = k+1$ where i is an even integer with $1 \le i \le k$. Similarly, we can show that $f(i,k+1) = i$ where i is an even integer with $1 \le i \le k$.

Claim 2.2. $f(i,j) = i$ holds for $i,j \in \{1,2,\ldots,k\}$.

By changing the order of ρ-walks in the following arguments, without loss of generality, we may assume that $i < j$.

Claim 2.2.1. $f|_J = e_1^2|_J$ where $J = \{(i,i+1) : 0 \le i < k\}$.

Consider the ρ-walks $\alpha = (k+1)\gamma 0$ and $\beta = \gamma\eta$ of length $(k+2)$. Set $\varepsilon = f(\alpha,\beta)$. As $k+1 = f(k+1,0) = \varepsilon_1 \to \varepsilon_2$, we have $\varepsilon_2 = 0$, and $\varepsilon_{k+2} = f(0,k+1) = 0$. Since, $\zeta = \varepsilon_2\varepsilon_3\ldots\varepsilon_{k+2}$ is a ρ-walk of length $(k+1)$ from 0 to 0 we have that $\zeta = \gamma$ by Lemma 5.1.2(4), hence, $f(i,i+1) = i$.

Claim 2.2.2. $f|_J = e_1^2|_J$ where $J = \{(i,j) : 0 \le i < k$ and $i+1 < j\}$.

Consider the ρ-walks of length $(k-j+i+2)$:

$$\alpha = k+1 \to 0 \to 1 \to \cdots \to k-j+i+1,$$
$$\beta = j-i-1 \to j-i \to \cdots \to k \to 0,$$

and set $\varepsilon = f(\alpha,\beta)$. We have $\varepsilon_{k-j+i+2} = f(k-j+i+1,0) = k-j+i+1$. Notice that $\varepsilon_{i+1} \to \cdots \to \varepsilon_{k-j+i+1} \to \varepsilon_{k-j+i+2} = k-j+i+1$, by Lemma 5.1.2(2), we get $f(i,j) = \varepsilon_{i+1} = i$. Therefore, $f = e_1^2$ everywhere. □

Claims 3 and 4 follow immediately from Theorem 3.1 in Barto et al. [35] but we still give a self-contained proof.

Claim 3. Let $f \in \mathrm{Pol}^{(3)}\rho$. Then f is not a minority operation.

Proof. Suppose by contradiction that, without loss of generality, $f(x,x,y) = y$. Consider the ρ-walks $\alpha = 0\delta^n 1$, $\beta = \gamma 0$ and $\theta = \delta^{n+1}$ of length $(k+1)$. Set $\varepsilon = f(\alpha,\beta,\theta)$. Then $\varepsilon_1 = f(0,0,k+1) = k+1$ which implies $\varepsilon_2 = 0$, and $\varepsilon_{k+2} = f(1,0,0) = 1$ which implies $\varepsilon_{k+1} = 0$, and so, $\varepsilon_2 \ldots \varepsilon_{k+1}$ is a ρ-walk of odd length from 0 to 0 and less than $k+1$ steps, a contradiction by Lemma 5.1.2(3). □

Claim 4. Let $f \in \mathrm{Pol}^{(3)}\rho$. Then f can not be a majority operation.

Proof. Suppose, by contradiction, that f is a majority operation. Consider the ρ-walks $\alpha = 0\delta^n(k+1)$, $\beta = \delta^{n+1}$ and $\theta = \gamma 0$ of length $(k+1)$. Set $\varepsilon = f(\alpha,\beta,\theta)$. Then $\varepsilon_1 = f(0,k+1,0) = 0$ and $\varepsilon_{k+2} = f(k+1,0,0) = 0$, and so, $\varepsilon_1 \ldots \varepsilon_{k+2}$ is a ρ-walk from 0 to 0 of length $(k+1)$ and therefore $f(k+1,0,1) = 1$ by Lemma 5.1.2(4). Consider the ρ-walks $\alpha' = 0\delta'^{n+1}$, $\beta' = (k+1)\gamma' 0$ and $\theta' = \gamma'\eta$ of length $(k+2)$, where $\gamma' = \gamma$, $\delta' = \delta$. Set $\varepsilon' = f(\alpha',\beta',\theta')$. Then ε' is a ρ-walk of even length from 0 to 0 but $\varepsilon'_2 = f(k+1,0,1) = 1$ contradicting Lemma 5.1.2(1). □

Claim 5. Let $f \in \mathrm{Pol}^{(3)}\rho$ be a semiprojection. Then f is a projection.

Proof. By Claim 2, the binary operations $f(x,y,y), f(y,x,y)$ and $f(y,y,x)$ are projections. So, there are the following three possibilities for f because it can be neither a minority operation for more than one position nor a majority operation:

1) for all $x,y \in U$, $f(x,y,y) = x, f(y,x,y) = y, f(y,y,x) = y$, hence, $f|_H = e_1^3|_H$,
2) for all $x,y \in U$, $f(x,y,y) = y, f(y,x,y) = x, f(y,y,x) = y$, hence, $f|_H = e_2^3|_H$,
3) for all $x,y \in U$, $f(x,y,y) = y, f(y,x,y) = y, f(y,y,x) = x$, hence, $f|_H = e_3^3|_H$,

where $H = \{(x,y,y),(y,x,y),(y,y,x) : x,y \in U\}$. By reordering the variables of f we may assume that $f|_H = e_1^3|_H$. Now we show that the equality $f = e_1^3$ holds on all other triples.

Claim 5.1. The values of f and e_1 coincide on the set of all triples in which both 0 and $k+1$ appear.

Claim 5.1.1. $f|_J = e_1^3|_J$ where $J = \{(i,k+1,0),(i+1,0,k+1) : i$ is even with $1 \leq i \leq k\}$.

Consider the ρ-walks $\alpha = \gamma 0$, $\beta = \delta^{n+1}$ and $\theta = 0\delta^n(k+1)$ of length $(k+1)$. Set $\varepsilon = f(\alpha,\beta,\theta)$. Then $\varepsilon_1 = f(0,k+1,0) = 0$ and $\varepsilon_{k+2} = f(0,0,k+1) = 0$. Since, $\zeta = \varepsilon_1\varepsilon_2\ldots\varepsilon_{k+2}$ is a ρ-walk of length $(k+1)$ from 0 to 0 we have $\zeta = \gamma 0$ by Lemma 5.1.2(4), hence, $f(i,k+1,0) = i$ and $f(i+1,0,k+1) = i+1$.

Claim 5.1.2. $f|_J = e_1^3|_J$ where $J = \{(i,k+1,0),(i+1,0,k+1) : i$ is odd with $1 \leq i \leq k\}$.

Consider the ρ-walks $\alpha = \gamma 0$, $\theta = 0\delta^n(k+1)$ and $\beta = \delta^{n+1}$ of length $(k+1)$. Set $\varepsilon = f(\alpha,\theta,\beta)$. Then $\varepsilon_1 = f(0,0,k+1) = 0$ and $\varepsilon_{k+2} = f(0,k+1,0) = 0$. Since the ρ-walk ζ coincides with ε we have $\zeta = \gamma 0$ by Lemma 5.1.2(4), hence, $f(i,k+1,0) = i$, and $f(i+1,0,k+1) = i+1$.

Claim 5.1.3. $f|_J = e_1^3|_J$ where $J = \{(k+1,i,0),(0,i+1,k+1) : i$ is even with $1 \leq i \leq k\}$.

Consider the ρ-walks $\alpha = \delta^{n+1}$, $\beta = \gamma 0$ and $\theta = 0\delta^n(k+1)$ of length $(k+1)$. Set $\varepsilon = f(\alpha,\beta,\theta)$. Then $\varepsilon_1 = f(k+1,0,0) = k+1$ and $\varepsilon_{k+2} = f(0,0,k+1) = 0$. Since, $\zeta = \varepsilon_2 \ldots \varepsilon_{k+2}$ is a ρ-walk of length k from 0 to 0 we have $\zeta = 0\delta^n$ by Lemma 5.1.2(1), hence, $f(k+1,i,0) = k+1$ and $f(0,i+1,k+1) = 0$.

Claim 5.1.4. $f|_J = e_1^3|_J$ where $J = \{(k+1,i,0),(0,i+1,k+1) : i$ is odd with $1 \leq i \leq k\}$.

Consider the ρ-walks $\theta = 0\delta^n$(k+1), $\beta = \gamma 0$ and $\alpha = \delta^{n+1}$ of length $(k+1)$. Set $\varepsilon = f(\theta,\beta,\alpha)$. Then $\varepsilon_1 = f(0,0,k+1) = 0$ and $\varepsilon_{k+2} = f(k+1,0,0) = k+1$ which implies $\varepsilon_{k+1} = 0$. Since, $\zeta = \varepsilon_1\varepsilon_2 \ldots \varepsilon_{k+1}$ is a ρ-walk of length k from 0 to 0 we have $\zeta = 0\delta^n$ by Lemma 5.1.2(1), hence, $f(k+1,i,0) = k+1$ and $f(0,i+1,k+1) = 0$.

Claim 5.1.5. $f|_J = e_1^3|_J$ where $J = \{(k+1,0,i),(0,k+1,i+1) : i$ is even with $1 \leq i \leq k\}$.

Consider the ρ-walks $\alpha = \delta^{n+1}$, $\beta = 0\delta^n(k+1)$ and $\theta = \gamma 0$ of length $(k+1)$. Set $\varepsilon = f(\alpha,\beta,\theta)$. Then $\varepsilon_1 = f(k+1,0,0) = k+1$ and $\varepsilon_{k+2} = f(0,k+1,0) = 0$. Since, $\zeta = \varepsilon_2 \ldots \varepsilon_{k+2}$ is a ρ-walk of length k from 0 to 0 we have $\zeta = 0\delta^n$ by Lemma 5.1.2(1), hence, $f(k+1,0,i) = k+1$ and $f(0,k+1,i+1) = 0$.

Claim 5.1.6. $f|_J = e_1^3|_J$ where $J = \{(k+1,0,i),(0,k+1,i+1) : i$ is odd with $1 \leq i \leq k\}$.

Consider the ρ-walks $\alpha = 0\delta^n$(k+1), $\beta = \delta^{n+1}$ and $\theta = \gamma 0$ of length $(k+1)$. Set $\varepsilon = f(\alpha,\beta,\theta)$. Then $\varepsilon_1 = f(0,k+1,0) = 0$ and $\varepsilon_{k+2} = f(k+1,0,0) = k+1$ which implies $\varepsilon_{k+1} = 0$. Since, $\zeta = \varepsilon_1\varepsilon_2 \ldots \varepsilon_{k+1}$ is a ρ-walk of length k from 0 to 0 we have $\zeta = 0\delta^n$ by Lemma 5.1.2(1), hence, $f(k+1,0,i) = k+1$, and $f(0,k+1,i+1) = 0$.

The values of f and e_1^3 coincide on the set of all triples $(a,b,c) \in U^{'} \times \{0,k+1\} \times U^{'}$, where $U^{'} = U \setminus \{0,k+1\}$. Let i and j be distinct elements of $U^{'}$. We may assume that $i < j$ because the proof can be easily modified by exchanging ρ-walks.

Claim 5.2. The values of f and e_1^3 coincide on the set of all triples in which 0 or $k+1$ appears.

Claim 5.2.1. $f|_J = e_1^3|_J$ where $J = \{(i,0,j),(i,k+1,j) : i,j \in U', i < j\}$.

Consider the ρ-walks $\alpha = \gamma 0$, $\beta = \delta^{n+1}$ and

$$\theta = j-i \to \cdots \to k \to 0 \to 1 \to \cdots \to j-i$$

of length $(k+1)$. Set $\varepsilon = f(\alpha,\beta,\theta)$. Then $\varepsilon_1 = f(0,k+1,j-i) = 0$ and $\varepsilon_{k+2} = f(0,0,*) = 0$. Since $\zeta = \varepsilon$ is a ρ-walk of length $(k+1)$ from 0 to 0, we obtain that $f(i,s,j) = i$, where s is 0 or $k+1$ depending on the parity of i. By using $\beta = \eta^{n+1}$ instead, we obtain by the same argument that $f(i,s',j) = i$ where $\{s,s'\} = \{0,k+1\}$.

Claim 5.2.2. $f|_J = e_1^3|_J$ where $J = \{(i,j,0),(i,j,k+1) : i,j \in U', i < j\}$.

Consider the ρ-walks $\alpha = \gamma 0$,

$$\theta = j-i \to \cdots \to k \to 0 \to 1 \to \cdots \to j-i$$

and $\beta = \delta^{n+1}$ of length $(k+1)$. Set $\varepsilon = f(\alpha,\theta,\beta)$. Then $\varepsilon_1 = f(0,j-i,k+1) = 0$ and $\varepsilon_{k+2} = f(0,*,0) = 0$. Since $\zeta = \varepsilon_1\varepsilon_2\ldots\varepsilon_{k+2}$ is a ρ-walk of length $(k+1)$ from 0 to 0, we obtain that $f(i,j,s) = i$, where s is 0 or $k+1$ depending on the parity of i. By using $\theta = \eta^{n+1}$ instead, we obtain by the same argument that $f(i,j,s') = i$ where $\{s,s'\} = \{0,k+1\}$.

Claim 5.2.3. $f|_J = e_1^3|_J$ where $J = \{(0,i,j),(k+1,i,j) : i,j \in U', i < j\}$.

Consider the ρ-walks $\beta = \delta^{n+1}$, $\alpha = \gamma 0$ and

$$\theta = j-i \to \cdots \to k \to 0 \to 1 \to \cdots \to j-i$$

of length $(k+1)$. Set $\varepsilon = f(\beta,\alpha,\theta)$. Then $\varepsilon_1 = f(k+1,0,j-i) = k+1$ and $\varepsilon_{k+2} = f(0,0,*) = 0$. Since $\zeta = \varepsilon_2\varepsilon_2\ldots\varepsilon_{k+2}$ is a ρ-walk of length k from 0 to 0, we obtain that $f(s,i,j) = s$, where s is 0 or $k+1$ depending on the parity of i. By using $\alpha = \eta^{n+1}$ instead, we obtain by the same argument that $f(s',i,j) = s'$ where $\{s,s'\} = \{0,k+1\}$.

Finally, if a triple consists of distinct elements $i,j,l \in U'$ then, by reordering the ρ-walks in the argument, we may assume that $i < j < l$. Consider the ρ-walks of length $(k-l+i+1)$

$$\begin{aligned}
\alpha &= 0 \to 1 \to \cdots \to i \to \cdots \to k-l+i+1, \\
\beta &= j-i \to \cdots \to j \to \cdots \to k-l+j+1, \\
\theta &= l-i \to \cdots \to l \to \cdots \to 0,
\end{aligned}$$

and set $\varepsilon = f(\alpha,\beta,\theta)$. Then $\varepsilon_1 = f(0,j-i,l-i) = 0$ and $\varepsilon_{k-l+i+2} = f(k-l+i+1,*,0) = k-l+i+1$. By Lemma 5.1.2(2), we have $\varepsilon_1\varepsilon_2\ldots\varepsilon_{k-l+i+1} = 1,2,\cdots,k-l+i+1$. Hence $f(i,j,l) = i$.

$\square$

Claim 6. Let $f \in \mathrm{Pol}^{(h)}\rho$ be a semiprojection, $h > 3$. Then f is a projection.

Proof. Without loss of generality, we can assume that f is a semiprojection onto the first variable. We will show that f is the first projection.

Claim 6.1. We first prove that $f(0,x_2,\ldots,x_h) = 0$ and $f(k+1,x_2,\ldots,x_h) = k+1$ for all $x_i \in U$, $2 \leq i \leq h$.

As $|\{x_2, x_3, \ldots, x_h\}| \geq 3$, there exist two entries x_i and x_j with the same parity. Without loss of generality we may reorder variables and assume that $x_3 > x_2$ and $x_3 - x_2$ is even.

Claim 6.1.1. $f(0, x_2, \ldots, x_h) = 0$ for all $x_i \leq k$, $2 \leq i \leq h$.

Since f is a semiprojection we can assume that $x_2 \geq 1$, hence, $k - x_2 + 1 \leq k$. We have that $f(k - x_2 + 1, 0, 0, \ldots, *) = k - x_2 + 1$ as f is semiprojection onto the first variable. Consider the ρ-walks of length $(k - x_2 + 1)$:

$$\alpha_1 = 0 \to 1 \to \cdots \to k - x_2 + 1,$$
$$\alpha_2 = x_2 \to x_2 + 1 \to \cdots \to k \to 0,$$
$$\alpha_3 = x_3 \to x_3 + 1 \to \cdots \to k \to 0 \to \delta^{(x_3 - x_2)/2},$$

and let $\alpha_4 = \cdots = \alpha_h = \alpha_1$. Set $\varepsilon = f(\alpha_1, \alpha_2, \cdots, \alpha_h)$. We have $\varepsilon_2 \ldots \varepsilon_{k - x_2 + 1}$ is a ρ-walk of length $(k - x_2 + 1) \leq k$ and

$$\varepsilon_{k - x_2 + 2} = f(k - x_2 + 1, 0, 0, \ldots) = k - x_2 + 1.$$

Applying Lemma 5.1.2(2), the statement follows.

Claim 6.1.2. $f(k + 1, x_2, \ldots, x_h) = k + 1$ for all $x_i \in U$, $2 \leq i \leq h$.

Since f is a semiprojection we can safely assume that all $x_i \leq k$, $2 \leq i \leq h$. For each i, $2 \leq i \leq h$, there exist $y_i \leq k$ and $z_i \leq k$ such that $y_i \to x_i \to z_i$. Then by Claim 6.1.1, we have

$$0 = f(0, y_2, \cdots, y_h) \to f(k + 1, x_2, \cdots, x_h) \to f(0, z_2, \cdots, z_h) = 0$$

which implies $f(k + 1, x_2, \ldots, x_h) = k + 1$.

Claim 6.1.3. $f(0,x_2,\ldots,x_h)=0$ for all $x_i \in U$, $2 \leq i \leq h$.

Since $f(0,x_2,\ldots,x_h) \to f(k+1,x_2\dot{+}1,\ldots,x_h\dot{+}1)=k+1$, where $\dot{+}$ is the addition modulo $(k+2)$, we have $f(0,x_2,\ldots,x_h)=0$.

If there were an element $j \in U^{'}$ for which $f(j,x_2,\ldots,x_h)=k+1$ would hold, then the ρ-walks

$$\alpha_1 = j(j+1)\ldots k01\ldots(j-1)j;$$
$$\alpha_l = \gamma 0(2 \leq l \leq h)$$

of length $(k+1)$ would yield the ρ-walk $\beta = f(\alpha_1,\ldots,\alpha_h)$. Since $\beta_1 = \beta_{k+2} = k+1$, we have $\beta_2 = \beta_{k+1} = 0$. Therefore, $\beta_2 \ldots \beta_{k+1}$ would be a ρ-walk of length $(k-1)$ from 0 to 0, which contradicts Lemma 5.1.2(3). Therefore, for $j \in \{1,2,\ldots,k\}$, we have $f(j,x_2,\ldots,x_{i+1}) \neq k+1$.

Consider the ρ-walks

$$\alpha_1 = \gamma 0;$$
$$\alpha_l = \alpha_1(2 \leq l \leq i+1).$$

Set $\beta = f(\alpha_1,\ldots,\alpha_{i+1})$. Then $\beta_1 = f(0,\ldots,0)=0$, and so, $\beta_j = f(j-1,\ldots,j-1) = j-1$ holds by (1) for every $j(2 \leq j \leq k+1)$.

$$0 = f(0,*,*,\ldots,*) = \beta_1 \to \beta_2 \to \cdots \to \beta_{k+1},$$

and we have $\beta_j = j-1$ $(2 \leq j \leq k+1)$. In other words, $f(j,*,*,\ldots,*) = j$ $(j \in U^{'})$. Thus, $f(x,x_2,\ldots,x_{h+1}) = x$ holds for every element $x \in U$. Hence, f is the first projection. Hence, f is the first projection, which concludes the proof of Theorem 5.1.1.

$\square$

CHAPTER 6

A STRONGLY RIGID TERNARY RELATION

In this chapter, we present a strongly rigid ternary relation on a finite domain.

6.1 A strongly rigid ternary relation on a finite domain

A relation ρ is strongly C-rigid if every operation on U preserving ρ is a projection or a constant function. It is shown in H. Länger and R. Pöschel [12] that for $n \geq 3$ the following ternary relation

$$\{(a,a,a) : 1 \leq a \leq n\} \cup \{(a,a+1,a+2) : 1 \leq a \leq n-2\}$$

on $U = \{1,...,n\}$ is strongly C-rigid.

As an extension of Theorem 1 [18], we give a strongly rigid ternary relation on a finite domain.

Theorem 6.1.1. *(Theorem 2.1 [42]) Let $3 < n < \omega$, let $U = \{i : 0 \leq i < n\}$ and let $U^* := U \setminus \{0\}$ and $U^o := \{u \in U : u+2 \in U\}$. The ternary relation*

$$\rho := \{(0,a,a) : a \in U^*\} \cup \{(a,a,0) : a \in U^*\}$$
$$\cup \{(0,0,a) : a \in U^*\} \cup \{(a,0,0) : a \in U^*\} \cup \{(a,a+1,a+2) : a \in U^o\}$$

on U is strongly rigid.

Proof. Let $h \in Pol\rho$ be unary.

Claim 1 : $h = e_1^1$; *i.e.* $h(x) = x$ *for all* $x \in U$.

Proof. Set $a_i := h(i)$ for all $i \in U$. First we prove

Fact 1. $a_i := i$ *for* $i = 0, ..., 3$.

Proof. Since $(0,1,1),(0,2,2),(0,3,3),(0,1,2),(1,2,3) \in \rho$, clearly

$$\begin{aligned} b_0 &:= (a_0, a_1, a_1), \\ b_1 &:= (a_0, a_2, a_2), \\ b_2 &:= (a_0, a_3, a_3), \\ b_3 &:= (a_0, a_1, a_2), \\ b_4 &:= (a_1, a_2, a_3) \end{aligned}$$

all belong to ρ. First $a_0 = 0$ since otherwise from b_0, b_1 and b_2 we obtain $a_1 = a_2 = a_3 = 0$ which contradicts $b_4 \in \rho$. From $b_0, b_1, b_2 \in \rho$ we see that $a_1, a_2, a_3 \in U^*$. Now $a_1 \neq a_2$ since otherwise $b_4 \in \rho$ would yield $a_3 = 0 \notin U^*$. Next $a_1 = 1$ due to $(0, a_1, a_2) = b_3 \in \rho$ and $a_1 \neq a_2$. Finally $(0, 1, a_2) = b_3 \in \rho$ and $a_2 \neq 1$ shows $a_2 = 2$ and $(1, 2, a_3) = b_4 \in \rho$ proves $a_3 = 3$. □

Let $3 \leq i < n$ be such that $i+1 < n$ and $a_j = j$ for all $j = 0, ..., i$. From $(i-1, i, a_{i+1}) = (a_{i-1}, a_i, a_{i+1}) \in \rho$ we obtain $a_{i+1} = i+1$. This concludes the induction step and proves the claim. □

By Claim 1 we have that $f(i, ..., i) = i$ for each $f \in Pol\rho$ and $i \in U$.

Claim 2 : *Let* $a \in U^*$. Then (i) $Pol\rho \subseteq Pol\ \{0, a\}$, (ii) $Pol\rho \subseteq Pol\ \{0,1,2\}$, (iii) $Pol\rho \subseteq Pol\ U^*$ *and* (iv) $Pol\rho \subseteq Pol\ \{(0,a),(a,0)\}$.

Proof. For $a \in U^*$ set $\sigma_a := \{x : (0, x, a) \in \rho\}$. It is easy to see that $\sigma_2 = \{0,1,2\}$ and $\sigma_a = \{0, a\}$ for $a \neq 2$. Together with the fact that $Pol\rho \subseteq Pol\ \{0\} \cap Pol\ \{a\}$ (Claim 1) we obtain (i) for $a \neq 2$ and (ii). To prove (i) for $a = 2$ set $\tau := \{x \in \{0,1,2\} : (x, 2, u), (2, u, 0) \in \rho$ for some $u \in U\}$. Here $\{0,2\} \subseteq \tau$ because for $x \in \{0,2\}$ we can choose $u = 2 - x$. Next $1 \notin \tau$ because were $1 \in \tau$ then $(1, 2, u) \in \rho$ would imply $u = 3$ and $(2,3,0) \in \rho$, a contradiction. Thus $\tau = \{0,2\}$.

(iii) Clearly $U^* = \{x : (0,x,x) \in \rho\}$.

(iv) Set $\lambda := \{(x,y) \in \{0,a\}^2 : (x,y,y) \in \rho\}$. It is easy to see that $\lambda = \{(0,a),(a,0)\}$. □

We denote by g the restriction of f to $\{0,1\}$. By Claim 2 (i) (for a = 1), we have $g : \{0,1\}^m \to \{0,1\}$, so g is a Boolean function. Moreover, by Claim 2 (iv) the function f preserves $\{(0,1),(1,0)\}$ and so g is a selfdual Boolean function.

We say that g is *monotone* (or *order-preserving*) if $x_1 \leq y_1, x_2 \leq y_2, ..., x_m \leq y_m$ imply

$$g(x_1,x_2,...,x_m) \leq g(y_1,y_2,...,y_m),$$

where $\leq$ is the usual order on $\{0,1\}$.

Claim 3: *$Pol\rho \subseteq Pol$ $\{(0,0),(0,1),(1,1)\}$. i.e., the restriction of every $f \in Pol\rho$* to $\{0,1\}$ *is a monotone Boolean function.*

Proof. Set

$$\lambda = \{(x,y) \in \{0,1\}^2 : (x,y,u),(y,x,v),(w,y,x),(t,x,y),(v,t,z),(w,u,z) \in \rho$$

$$for\ some\ u,v,w,t,z \in U\}.$$

It suffices to show that $\lambda = \{(0,0),(0,1),(1,1)\}$. To show $\supseteq$ it suffices to choose the values u,v,w,t,z as shown in Table 1.

x	y	u	v	w	t	z
0	0	2	1	1	2	3
0	1	2	0	1	0	3
1	1	0	0	0	0	1

Table 1

To prove $\subseteq$ suppose to the contrary that $(1,0) \in \lambda$ and let u,v,w,t,z be the corresponding values. Then $u = 0$, $v \in \{1,2\}$, $w = 0$, $t = 1$. Next $(0,0,z) = (w,u,z) \in \rho$

shows $z \in U^*$. Finally $(v,t,z) \in (\{1,2\} \times \{1\} \times U^*) \cap \rho = \emptyset$, a contradiction. $\square$

Claim 4: *The restriction of every $f \in Pol\rho$ to $\{0,1\}$ is a projection.*

Proof. Let $f \in Pol\rho$ and suppose that g, the restriction of f to $\{0,1\}$, is not a projection.

From the Claims 2 and 3, we know that the Boolean function g is monotone and self-dual. By applying the arguments of Claim 5 in [18] to the functions g and f, we can find a ternary function $k \in Pol\rho$ such that $k|_{\{0,1\}} = m$, where m is the majority function defined by

$$m(x,y,z) = \begin{cases} 0 & \text{if x+y+z} \leq 1 \\ 1 & \text{otherwise} \end{cases}.$$

Fact 1. Let $b \in U^*$. Then $k(b,0,0) = k(0,b,0) = k(0,0,b) = 0$ and $k(b,b,0) = k(b,0,b) = k(0,b,b) = b$.

Proof. Since $(0,1,1),(b,0,0) \in \rho$, we have

$$\begin{aligned}(k(b,0,0),1,1) &= (k(b,0,0),m(0,1,1),m(0,1,1)) \\ &= (k(b,0,0),k(0,1,1),k(0,1,1)) \in \rho.\end{aligned}$$

Notice that from $k(b,0,0) \in \{0,b\}$, we get $k(b,0,0) = 0$. By symmetry, we have $k(0,b,0) = 0$, $k(0,0,b) = 0$ and the other equality follows the first one by Claim 2 (iv). $\square$

Since $(b,0,0),(0,1,1),(0,0,b) \in \rho$, it follows that

$$(0,0,k(0,1,b)) = (k(b,0,0),k(0,1,0),k(0,1,b)) \in \rho,$$

hence $k(0,1,b) \in U^*$. By symmetry, $k(1,0,b) \in U^*$. Notice that $(0,0,1),(1,1,0)$, $(1,2,3) \in \rho$, and so $(1,k(0,1,2),k(1,0,3)) = (k(0,1,1),k(0,1,2),k(1,0,3)) \in \rho$. As $k(0,1,2)$, $k(0,1,3) \in U^*$, this yields $k(0,1,2) = 2$. Finally, since $(0,0,3)$, $(1,2,3)$, $(2,0,0) \in \rho$, we obtain $(2,0,3) = (k(0,1,2),k(0,2,0),k(3,3,0)) \in \rho$.

This is a contradiction. Therefore, g is a projection. $\square$

Claim 5: *Every $f \in Pol\rho$ is a projection.*

Proof. By Claim 4, we already know that $f|_{\{0,1\}}$ is a projection, i.e.,without loss of generality we may assume that it is the 1st projection, i.e., that

$$f(a_1,...,a_m) = a_1$$

for all $a_1,...,a_m \in \{0,1\}$. By induction on $k = 2,...,n$, we now show that

$$f(b_1,...,b_m) = b_1$$

for all $b_1,...,b_m \in \underline{k} := \{0,...,k-1\}$.

The statement is true for the case $k = 2$. Suppose that the statement is true for some $2 \leq k < n$. Let $x_1, x_2, ..., x_m \in \underline{k+1}$.

For $i \in \{1,...,m\}$, define a_i, b_i and d_i as follows

$$a_i := \begin{cases} 1 & \text{if } \mathrm{x}_i = 0 \\ 0 & \text{otherwise} \end{cases}, \; b_i = \begin{cases} 1 - x_i & \text{if } \mathrm{x}_i \in \{0,1\} \\ x_i - 2 & \text{otherwise} \end{cases}, \; d_i = \begin{cases} 1 & \text{if } \mathrm{x}_i \in \{0,1\} \\ x_i - 1 & \text{otherwise} \end{cases}.$$

Notice that $a_1,...,a_m, b_1,...,b_m, d_1,...,d_m \in \underline{k}$ and so by the induction hypothesis $f(a_1,...,a_m) = a_1, f(b_1,...,b_m) = b_1$ and $f(d_1,...,d_m) = d_1$.

Next for every $i = 1,2,...,m$, clearly $(a_i, x_i, x_i) \in \rho$ and so

$$(a_1, f(x_1,...,x_m), f(x_1,...,x_m)) = (f(a_1,...,a_m), f(x_1,...,x_m), f(x_1,...,x_m)) \in \rho. \quad (1)$$

Note that $a_1 \in \{0,1\}$. We consider the two cases separately.

Case 1: $a_1 = 1$.

By the definition of a_1, clearly $x_1 = 0$, and so by (1) and the definition of ρ we have $f(x_1,...,x_m) = 0 = x_1$.

Case 2: $a_1 = 0$.

By the definition of a_1, clearly $x_1 \neq 0$ and from (1) also $f(x_1,...,x_m) \neq 0$.

Let $x_1 > 2$. Then $b_1 = x_1 - 2$. Moreover $(b_i, d_i, x_i) \in \rho$, and thus we have $(x_1 - 2, x_1 - 1, f(x_1, ..., x_m)) = (b_1, d_1, f(x_1, ..., x_m)) = (f(b_1, ..., b_m), f(d_1, ..., d_m), f(x_1, ..., x_m)) \in \rho$. Hence $f(x_1, ..., x_m) = x_1$.

Now, it remains to consider the two cases (A) $x_1 = 1$, (B) $x_1 = 2$.

(A) Let $x_1 = 1$. We need to show that $f(x_1, ..., x_m) = 1$.

For $i \in \{1, ..., m\}$, we define c_i as follows

$$c_i = \begin{cases} 2 & \text{if } \mathrm{x}_i = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Notice that $c_1 = 2$ as $x_1 = 1$. Since $c_1, ..., c_m \in \{0, 2\}$, we know by Claim 2 (i) ,

$$f(c_1, ..., c_m) \in \{0, 2\}.$$

Note that $\frac{1}{2}c_1 = 1$ and $\frac{1}{2}c_1, ..., \frac{1}{2}c_m \in \{0, 1\}$. As $f|_{\{0,1\}}$ is the 1st projection, it implies

$$f(\frac{1}{2}c_1, ..., \frac{1}{2}c_m) = \frac{1}{2}c_1 = 1.$$

Observe that for $i = 1, ..., m$, clearly $(2 - c_i, \frac{1}{2}c_i, \frac{1}{2}c_i) \in \{(2, 0, 0), (0, 1, 1)\} \subseteq \rho$, and in view of Claim 2 (iv) (for a = 2) we have

$$(2 - f(c_1, ..., c_m), 1, 1) = (f(2 - c_1, ..., 2 - c_m), f(\frac{1}{2}c_1, ..., \frac{1}{2}c_m), f(\frac{1}{2}c_1, ..., \frac{1}{2}c_m)) \in \rho.$$

It follows that $2 - f(c_1, ..., c_m) = 0$ and thus $f(c_1, ..., c_m) = 2$.

For $i \in \{1, ..., m\}$, define e_i by

$$e_i = \begin{cases} 3 & \text{if } \mathrm{x}_i \in \{0, 1\} \\ 0 & \text{otherwise} \end{cases}.$$

Notice that $e_1 = 3$ as $x_1 = 1$. Since $e_1, ..., e_m \in \{0,3\}$, we know by Claim 2 (i) ,

$$f(e_1, ..., e_m) \in \{0,3\}.$$

Note that $\frac{1}{3}e_1 = 1$ and $\frac{1}{3}e_1, ..., \frac{1}{3}e_m \in \{0,1\}$. As $f|_{\{0,1\}}$ is the 1st projection, it implies

$$f(\frac{1}{3}e_1, ..., \frac{1}{3}e_m) = \frac{1}{3}e_1 = 1.$$

Observe that for $i = 1, ..., m$, clearly $(3 - e_i, \frac{1}{3}e_i, \frac{1}{3}e_i) \in \{(3,0,0), (0,1,1)\} \subseteq \rho$, and in view of Claim 2 (iv) (for a = 3) we have

$$(3 - f(e_1, ..., e_m), 1, 1) = (f(3 - e_1, ..., 3 - e_m), f(\frac{1}{3}e_1, ..., \frac{1}{3}e_m), f(\frac{1}{3}e_1, ..., \frac{1}{3}e_m)) \in \rho.$$

It follows that $3 - f(e_1, ..., e_m) = 0$ and thus $f(e_1, ..., e_m) = 3$.

By the definition of c_i and e_i, we know $(x_i, c_i, e_i) \in \rho$, thus

$$(f(x_1, ..., x_m), 2, 3) = (f(x_1, ..., x_m), f(c_1, ..., c_m), f(e_1, ..., e_m)) \in \rho$$

and we obtain $f(x_1, ..., x_m) = 1 = x_1$.

(B) Let $x_1 = 2$. We need to show that $f(x_1, ..., x_m) = 2$.

For $i \in \{1, ..., m\}$, define g_i by

$$g_i = \begin{cases} 3 & \text{if } \mathrm{x}_i = 2 \\ 0 & \text{otherwise} \end{cases}.$$

Notice that $g_1 = 3$ as $x_1 = 2$. Since $g_1, ..., g_m \in \{0,3\}$, we know by Claim 2 (i) ,

$$f(g_1, ..., g_m) \in \{0,3\}.$$

Note that $\frac{1}{3}g_1 = 1$ and $\frac{1}{3}g_1, ..., \frac{1}{3}g_m \in \{0,1\}$. As $f|_{\{0,1\}}$ is the 1st projection, it implies

$$f(\frac{1}{3}g_1, ..., \frac{1}{3}g_m) = \frac{1}{3}g_1 = 1.$$

Observe that for $i = 1, ..., m$, clearly $(3 - g_i, \frac{1}{3}g_i, \frac{1}{3}g_i) \in \{(3,0,0),(0,1,1)\} \subseteq \rho$, and in view of Claim 2 (iv) (for a = 3) we have

$$(3 - f(g_1, ..., g_m), 1, 1) = (f(3 - g_1, ..., 3 - g_m), f(\frac{1}{3}g_1, ..., \frac{1}{3}g_m), f(\frac{1}{3}g_1, ..., \frac{1}{3}g_m)) \in \rho.$$

It follows that $3 - f(g_1, ..., g_m) = 0$ and thus $f(g_1, ..., g_m) = 3$.

For $i \in \{1, ..., m\}$, define h_i by

$$h_i = \begin{cases} 4 & \text{if } \mathrm{x}_i \in \{0,2\} \\ 0 & \text{otherwise} \end{cases} .$$

Notice that $h_1 = 4$ as $x_1 = 2$. Since $h_1, ..., h_m \in \{0,4\}$, we know by Claim 2 (i) ,

$$f(h_1, ..., h_m) \in \{0,4\}.$$

Note that $\frac{1}{4}h_1 = 1$ and $\frac{1}{4}h_1, ..., \frac{1}{4}h_m \in \{0,1\}$. As $f|_{\{0,1\}}$ is the 1st projection, it implies

$$f(\frac{1}{4}h_1, ..., \frac{1}{4}h_m) = \frac{1}{4}h_1 = 1.$$

Observe that for $i = 1, ..., m$, clearly $(4 - h_i, \frac{1}{4}h_i, \frac{1}{4}h_i) \in \{(4,0,0),(0,1,1)\} \subseteq \rho$, and in view of Claim 2 (iv) (for a = 4) we have

$$(4 - f(h_1, ..., h_m), 1, 1) = (f(4 - h_1, ..., 4 - h_m), f(\frac{1}{4}h_1, ..., \frac{1}{4}h_m), f(\frac{1}{4}h_1, ..., \frac{1}{4}h_m)) \in \rho.$$

It follows that $4 - f(h_1, ..., h_m) = 0$ and thus $f(h_1, ..., h_m) = 4$.

By the definition of g_i and h_i, we know $(x_i, g_i, h_i) \in \rho$, thus

$$(f(x_1,\ldots,x_m),3,4) = (f(x_1,\ldots,x_m), f(g_1,\ldots,g_m), f(h_1,\ldots,h_m)) \in \rho$$

and we obtain $f(x_1,\ldots,x_m) = 2 = x_1$. In all the cases, $f(x_1,\ldots,x_m) = x_1$. This concludes the induction. Thus f is a projection.

□

Therefore, $Pol\rho$ only contains the projections.

□

Conjecture 6.1.2. *(Conjecture 2.3 [42]) Let* $3 \le k < n < \omega$*, let* $U = \{i : 0 \le i < n\}$ *and let* $U^* := U \setminus \{0\}$ *and* $U^o := \{u \in U : u+k-1 \in U\}$*. The k-ary relation*

$$\rho := \{(0,a,\ldots,a) : a \in U^*\} \cup \{(a,\ldots,a,0) : a \in U^*\}$$
$$\cup \{(0,\ldots,0,a) : a \in U^*\} \cup \{(a,0,\ldots,0) : a \in U^*\} \cup \{(a,a+1,\ldots,a+k-1) : a \in U^o\}$$

on U *is strongly rigid.*

CHAPTER 7

CONCLUSION AND FUTURE WORK

7.1 Conclusion and future work

We characterized all strongly rigid relations on a set with more than two elements. The problem to characterize all strongly rigid relations on an infinite domain remains open.

We proposed several simple lemmas to exclude non-rigid binary relations which can be turned into computer programs. These lemmas are quite effective not only in a 4-element domain but also in a k-element domain when $k \geq 5$. Some techniques used in recent developments in Barto and Stanovský [37] and Jovanović [40] can also help in this aspect. Applying these rules to a 5-element domain, we could obtain a result as in Table 7.1. It would be interesting to find out all strongly rigid binary relations out of the potential list on a 5-element domain.

Proposition	# of Relations	# of Relations up to isomorphism	Property
Lemma 3.1.6 (The loop rule)	$(2^5-1)2^{20}$		Not rigid
Lemma 3.1.6 (The overlap rule)	544756		Not rigid
Lemma 3.1.6 (The interchange rule)	21336		Not rigid
Nontrivial automorphism	51444		Not rigid
Lemma 3.3.3 (The source-sink rule)	119520	520	Rigid but not strongly rigid
	311520	1425	Possible strongly rigid
Total:	2^{5^2}		

Table 7.1 – Classification of binary relations on a 5-element domain

The classification of binary relations on 4 elements with a trivial clone is a useful framework and rich source of examples for understanding trivial clones in general, such as trivial clones on binary relations, and for constructing other families of strongly rigid relations. It would be interesting to investigate the classification problem for similar types of relations, e.g., rigid ternary relations, idempotent trivial relations, and strongly C-rigid binary and ternary relations on small domains.

Nous avons caractérisé les relations fortement rigides sur un ensemble fini de plus de deux éléments. L'analogue sur un univers infini demeure ouvert.

Nous avons établi plusieurs lemmes simples qui permettent d'exclure les relations binaires non-rigides de façon algorithmique. Ces lemmes sont efficaces non seulement sur un domaine à 4 éléments mais pour tout univers fini. Certaines techniques récentes de Barto et Stanovský [37] et Jovanović [40] peuvent aussi s'avérer utiles dans ce contexte. Si on applique ces lemmes à un domaine avec 5 éléments, on obtiendra un résultat de la forme présentée dans la Table 7.1. Il serait intéressant de déterminer quelles sont les relations binaires fortement rigides parmi cette liste de cas potentiels.

La caractérisation des relations binaires sur 4 éléments dont le clone est trivial offre une riche source d'exemples pour comprendre les relations fortement rigides dans le cas général, et pourra s'avérer utile pour construire d'autres familles de telles relations. Il est à espérer que ce travail permettra d'avancer la classification de relations avec des propriétés semblables, telles les relations ternaires rigides, les relations idempotentes-triviales, et les relations C-rigides sur de petits univers.

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