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Asymptotiques spectrales et géométrie des nombres

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Sommaire

Dans cette thèse, nous étudions le spectre du laplacien ainsi que celui d'autres opérateurs qui lui sont associés. Sur une variété riemannienne compacte M fermée, ou possédant un bord et munie de conditions frontières auto-adjointes, le laplacien a un spectre réel, discret

$$\lambda_1(M) \leq \lambda_2(M) \leq \ldots \nearrow \infty$$

ne s'accumulant qu'à l'infini, où les $\lambda_j(M)$ sont les nombres réels pour lesquels il existe une solution non-triviale à l'équation

$$\Delta\varphi + \lambda\varphi = 0.$$

Nous nous sommes particulièrement intéressé au comportement asymptotique de la fonction de compte

$$N(\lambda; M) := \#\left\{\lambda_j(M) < \lambda\right\}.$$

Hermann Weyl a démontré en 1911 [**80**] ce qui s'appelle aujourd'hui la loi de Weyl,

$$N(\lambda; M) \sim \frac{\omega_d}{(2\pi)^d} \mathrm{Vol}(M) \lambda^{d/2},$$

où ω_d est le volume de la boune unité en dimension d. Nous cherchons à déterminer la taille de

$$R(\lambda; M) := N(\lambda; M) - \frac{\omega_d}{(2\pi)^d} \mathrm{Vol}(M) \lambda^{d/2}.$$

Dans les contextes que nous avons étudiés, nous avons traduit ce problème dans les termes de la géométrie des nombres, *i.e.* l'étude de l'interaction entre les points de réseaux, par exemple $\mathbb{Z}^d$, et les ensembles convexes. Dans le premier chapitre, nous décrivons précisément les problèmes à l'étude ainsi que les liens qu'ils possèdent avec la géométrie des nombres, et décrivons plus en détails les principales techniques utilisées.

Le second chapitre, intitulé *On a generalised Gauss circle problem and integrated density of states* [**54**], est le fruit d'une collaboration avec Leonid Parnovski. Nous y étudions le spectre du laplacien sur un produit d'un tore plat et de l'espace euclidien. Dans ce cas le spectre n'est pas discret mais nous étudions une quantité, la densité intégrée des états, qui remplit le rôle de la fonction de compte des valeurs propres et qui suit elle-même une loi de Weyl. Nous obtenons des bornes supérieures et inférieures sur $R(\lambda)$ dans ce contexte, qui dépendent des dimensions relatives du tore et de l'espace euclidien. Nous obtenons que lorsque la dimension du tore est strictement inférieure à celle de l'espace euclidien, nos bornes inférieures et supérieures sont du même ordre polynomial. Nous obtenons aussi un développement asymptotique jusqu'à l'ordre constant pour la densité d'états intégrée de l'opérateur de Schrödinger magnétique avec potentiel constant.

Le troisième chapitre, intitulé *The Steklov spectrum of cuboids* [**26**] provient d'une collaboration avec Alexandre Girouard, Iosif Polterovich et Alessandro Savo. Nous y étudions le problème aux valeurs propres de Steklov sur des cuboïdes en toute dimension. Cet opérateur a été peu étudié sur des domaines dont la frontière n'est pas lisse et nous utilisons le cuboïde comme premier modèle d'un tel cas. Le spectre reste discret et ne s'accumule qu'à l'infini, nous obtenons une loi de Weyl à deux termes ainsi qu'une inégalité isopérimétrique pour la première valeur propre non triviale. Finalement, nous y obtenons aussi que certaines suites de fonctions propres se concentrent asymptotiquement sur des ensembles de mesure nulle, un comportement qu'on appelle la cicatrisation.

Dans le dernier chapitre, intitulé *Eigenvalue optimisation on flat tori and lattice points in anisotropically expanding domains* [**53**], nous étudions le spectre du laplacien sur des tores plats. Nous obtenons des bornes pour $R(\lambda; M)$ dépendant du rayon d'injectivité. Nous utilisons ensuite ces bornes pour démontrer que toute suite de tores plats $\mathbb{T}_k$ maximisant la k-ième valeur propre du laplacien doit dégénérer lorsque la dimension est inférieure à 10. Pour ce faire, nous avons ramené le problème à celui de compter les points de $\mathbb{Z}^d$ dans un domaine qui croît de façon anisotrope, généralisant des résultats obtenus par Yuri Kordyukov et Andrei Yakovlev [**49**].

Mots-clefs : Géométrie spectrale ; asymptotique spectrale ; géométrie des nombres ; densité d'états ; problème de Steklov ; loi de Weyl ; optimisation asymptotique

Summary

In this thesis, we study the spectrum of the Laplacian and of other related operators. When defined on either a closed compact Riemannian manifold, or a manifold with boundary and self-adjoint boundary conditions, the Laplacian Δ has a real and discrete spectrum

$$\lambda_1(M) \le \lambda_2(M) \le \ldots \nearrow \infty$$

accumulating only at ∞. The numbers $\lambda_j(M)$ are those for which there is a non-trivial solution to the equation

$$\Delta\varphi + \lambda\varphi = 0.$$

We are more specifically interested in the asymptotic behaviour of the counting function

$$N(\lambda; M) := \#\left\{\lambda_j(M) < \lambda\right\}.$$

Hermann Weyl has shown in 1911 [**80**] what is now known as Weyl's law,

$$N(\lambda; M) \sim \frac{\omega_d}{(2\pi)^d}\mathrm{Vol}(M)\lambda^{d/2} \qquad \text{as } \lambda \to \infty,$$

where ω_d is the volume of the unit ball in dimension d. We want to determine the size of

$$R(\lambda; M) := N(\lambda; M) - \frac{\omega_d}{(2\pi)^d}\mathrm{Vol}(M)\lambda^{d/2}.$$

In the context at hand, we have translated this problem in terms of the geometry of numbers, the study of the interaction between lattice points, *e.g.* $\mathbb{Z}^d$ and convex sets. In the first chapter, we make a precise description of the problems studied and how they can be linked to the geometry of numbers. Furthermore, we describe in more detail the main techniques that we have used.

The second chapter, titled *On a generalised Gauss circle problem and integrated density of states* [**54**], has been written in collaboration with Leonid Parnovski. There, we study the spectrum of the Laplacian on a product of a flat torus and Euclidean space. In this case, the spectrum is not discrete. However, we study the integrated density of states, which takes the role of the eigenvalue counting function and also satisfies Weyl's law. We obtain upper and lower bounds on $R(\lambda)$ in this context, which depend on the relative dimensions of the flat torus and Euclidean space. When the dimension of the torus is strictly smaller than that of the Euclidean space the upper and lower bound share the same polynomial order. We also obtain an asymptotic expansion up to constant order for the integrated density of states of a magnetic Schrödinger operator with constant potential.

The third chapter, titled *The Steklov spectrum of cuboids* [**26**] has been written together with Alexandre Girouard, Iosif Polterovich and Alessandro Savo. We study the Steklov spectrum, *i.e.* the spectrum of the Dirichlet-to-Neumann operator on cuboids of any dimension. Eigenvalue asymptotics for this operator had not been very much studied on domains whose boundaries are not smooth and cuboids provide a first example of such domains. The spectrum is discrete and only accumulates at infinity, and we obtain a two-term Weyl's law for the Steklov spectrum. We also obtain an isoperimetric inequality for the first non-trivial eigenvalue. Finally, we prove that some sequence of eigenfunctions concentrates along edges, which are subsets of measure zero, a phenomenon named scarring.

In the last chapter, titled *Eigenvalue optimisation on flat tori and lattice points in anisotropically expanding domains* [**53**], we turn our attenton to the spectrum of the Laplacian on flat tori. We obtain bounds on $R(\lambda)$ depending on the injectivity radius. We then use those bounds to obtain that any sequence of flat tori $\mathbb{T}_k$ maximising the kth eigenvalue of the Laplacian must degenerate when dimension is inferior or equal to 10. To do so, we have stated the problem at hand in terms of counting points of $\mathbb{Z}^d$ inside anisotropically expanding domains, generalising results of Yuri Kordyukov and Andrei Yakovlev [**49**].

Keywords : Spectral geometry; spectral asymptotics; geometry of numbers; density of states; Steklov problem; Weyl's law; asymptotic optimisation

Table des matières

Table des figures

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Chapitre 1

Introduction et résultats principaux

Introduction et résultats principaux

1.1. L'histoire conjointe de la géométrie des nombres et de la théorie spectrale

La géométrie des nombres étudie les relations d'inclusions entre les points d'un treillis Γ, par exemple $\mathbb{Z}^d$, et les ensembles convexes de $\mathbb{R}^d$. La géométrie des nombres a pris son essor, et son nom, après l'apport de H. Minkowski au début du 20e siècle [**57**]. Toutefois, l'un des problèmes les plus connus du domaine a été étudié par C. F. Gauss au 19e siècle [**24**]. Il s'agit du problème du cercle de Gauss où on dénombre asymptotiquement et en moyenne le nombre de façons dont un nombre peut être représenté comme la somme de deux carrés. Géométriquement, le nombre $r_2(\mu)$ de façons d'écrire un nombre $\mu \in \mathbb{N}$ comme la somme de deux carrés est le même que le nombre de points de $\mathbb{Z}^2$ situés sur un cercle de rayon $\sqrt{\mu}$. Calculer la moyenne de r_2 pour des nombres inférieurs à ρ^2 revient plutôt à compter le nombre de ces points à l'intérieur d'un disque de rayon ρ. On note cette moyenne[1]

$$n_2(\rho) := \sum_{\mu < \rho^2} r_2(\mu).$$

Le problème se généralise rapidement au nombre $r_d(\mu)$ de façons d'écrire μ comme la somme de d carrés, où l'on comptera plutôt le nombre de points de $\mathbb{Z}^d$ sur une sphère de rayon $\sqrt{\mu}$. Dans ce cas, $n_d(\rho)$ est la fonction de compte des points à l'intérieur d'une boule de rayon ρ.

Déjà, Gauss a utilisé une méthode géométrique pour estimer n_2, et cette méthode s'applique aussi à n_d. À chaque élément $\boldsymbol{\mu} := (\mu_1, \ldots, \mu_d)$ dans $\mathbb{Z}^d$, on associe le cube de coin $\boldsymbol{\mu}$ et de volume 1, $(\mu_1, \mu_1 + 1] \times \ldots \times (\mu_d, \mu_d + 1]$. On voit que le volume de l'ensemble des cubes associés aux points à l'intérieur de la boule de rayon ρ est le même que le nombre de ces points, et que ce volume est approximativement égal au volume de la boule de rayon ρ, $\omega_d \rho^d$, où

$$\omega_d = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2} + 1)}$$

est le volume de la boule de rayon 1. On peut aussi borner l'erreur faite par cette approximation en notant qu'elle est confinée au volume d'une bande de largeur $\sqrt{d}$ autour de la boule. On

[1] Formellement, pour obtenir la moyenne on diviserait cette quantité par ρ^2, mais comme on s'intéresse au comportement asymptotique en ρ cela ne change rien. La convention du domaine est préservée ici.

obtient donc que

$$n_d(\rho) = \omega_d \rho^d + O\left(\rho^{d-1}\right). \tag{1.1.1}$$

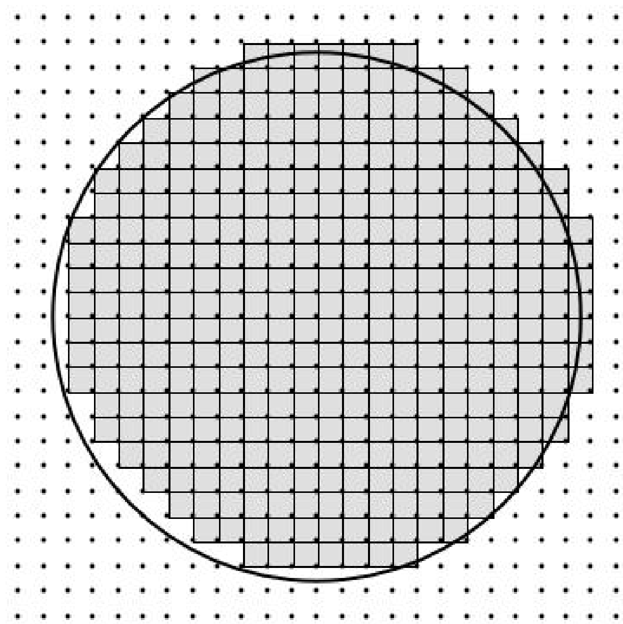

FIGURE 1.1. Problème du cercle, $\rho = 10.5$

Un lecteur averti pourrait noter dans la figure 1.1 qu'on a sous-estimé en haut à droite et surestimé en bas à gauche. On peut donc s'attendre à ce que le terme d'erreur dans l'équation (1.1.1) soit dans les faits plus petit. C'est en effet le cas et l'exposant a rapidement été abaissé à $2/3$ pour n_2 par W. Sierpiński [**68**]. Plusieurs ont ensuite travaillé sur la question, notamment J. G. van der Corput [**20**], G. H. Hardy [**31**], E. Landau [**55**], M. N. Huxley [**37, 38, 39**] puis finalement J. Bourgain et N. Watt [**11**], lesquels ont obtenu la meilleure borne à ce jour : pour tout $\varepsilon > 0$, on a que

$$n_2(\rho) = \pi\rho^2 + O\left(\rho^{\frac{517}{824}+\varepsilon}\right). \tag{1.1.2}$$

D'un autre côté, Hardy et Landau ont indépendemment démontré que l'exposant ne pouvait pas être $1/2$, et ont conjecturé [**32**] que

$$n_2(\rho) = \pi\rho^2 + O\left(\rho^{\frac{1}{2}+\varepsilon}\right) \tag{1.1.3}$$

pour tout $\varepsilon > 0$. Comme $517/824 \approx 0.62743$, la conjecture est loin d'être réalisée, mais elle n'est pas infirmée : la meilleure borne inférieure est due à K. Soundararajan qui démontre [**73**] qu'il existe une suite $\rho_k \to \infty$ et une constante c telle que

$$\left|n_2(\rho_k) - \pi\rho_k^2\right| \geq c\frac{\rho_k^{1/2}(\log\rho_k)^{1/4}(\log_2\rho_k)^{\frac{3}{4}(2^{4/3}-1)})}{(\log_3\rho_k)^{5/8}}. \tag{1.1.4}$$

Pour $d \geq 5$, l'exposant optimal est connu ; F. Götze démontre [**29**] que

$$n_d(\rho) = \omega_d\rho^d + O\left(\rho^{d-2}\right) \tag{1.1.5}$$

et que l'exposant $d-2$ ne peut pas être amélioré.

En 1905, Lord Rayleigh a fait le lien entre la géométrie des nombres et la théorie spectrale des opérateurs différentiels elliptiques [**66**]. Ceux-ci permettent de modéliser plusieurs systèmes physiques, par exemple la vibration d'une membrane ou le ballotement de l'eau dans un contenant. En théorie spectrale, on s'intéresse au spectre d'un opérateur $H : C^\infty(\Omega) \to C^\infty(\Omega)$, *i.e.* à l'ensemble

$$\operatorname{spec}(H) := \left\{\lambda : (H-\lambda)^{-1} \text{ n'existe pas en tant qu'opérateur borné.}\right\}.$$

Le théorème spectral dit, en bref, que comprendre spec(H) nous permet de connaître le comportement de l'opérateur H. Sous certaines hypothèses sur H, son spectre est discret, borné inférieurement, composé des valeurs propres de H et ne s'accumule qu'à l'infini. Dans ce cas, plutôt que d'étudier directement l'ensemble spec(H) on étudie plutôt la fonction de compte

$$N(\lambda; H) := \#\left(\operatorname{spec}(H) \cap (-\infty,\lambda)\right),$$

où on compte « avec multiplicité » : si une valeur propre de H est répétée alors elle est comptée plusieurs fois. Connaître $N(\lambda; H)$ c'est connaître spec(H) et *vice versa*; on s'intéresse au comportement asymptotique de la fonction de compte lorsque $\lambda \to \infty$.

En étudiant les fréquences de vibrations atomiques dans [**66**], Lord Rayleigh en est venu à considérer le problème aux valeurs propres suivant pour le laplacien sur le cube $\Omega = [0,a]^3$:

$$\begin{cases} (\Delta + \lambda)u = 0 & \text{dans } \Omega; \\ u = 0 & \text{sur } \partial\Omega. \end{cases} \tag{1.1.6}$$

En utilisant la méthode de séparation des variables, il a pu décrire les valeurs propres comme étant

$$\lambda_{\ell,m,n} = \frac{\pi^2}{a^2}\left(\ell^2 + m^2 + n^2\right), \qquad (\ell,m,n) \in \mathbb{N}^3. \tag{1.1.7}$$

Compter les valeurs propres du laplacien sur Ω correspond au problème du cercle de Gauss en dimension 3, à la différence qu'on ne compte que les cas où chacun des ℓ, m, n est strictement positif. Cette subtilité a par ailleurs fait qu'une erreur de calcul s'est glissée dans les travaux de Lord Rayleigh, qui a été corrigée par J. H. Jeans [**44**]. Si plutôt qu'un cube on considère le même problème aux valeurs propres sur un carré ou un hypercube de dimension d, on verrait apparaître la version en dimension d du problème du cercle de Gauss.

Par le principe d'inclusion-exclusion, le nombre de points de $\mathbb{N}^d$ dans une boule de rayon $\frac{a}{\pi}\sqrt{\lambda}$ est 2^d fois le nombre de points de $\mathbb{Z}^d$ dans cette même boule, moins les points situés sur les hyperplans de coordonnées. En d'autre termes, on obtient que

$$N(\lambda;\Delta;\Omega) = \frac{\omega_d}{(2\pi)^d}\operatorname{Vol}_d(\Omega)\lambda^{d/2} - \frac{\omega_{d-1}}{4(2\pi)^{d-1}}\operatorname{Vol}_{d-1}(\partial\Omega)\lambda^{\frac{d-1}{2}} + o\left(\lambda^{\frac{d-1}{2}}\right). \tag{1.1.8}$$

On peut observer que dans la dernière équation le volume et l'aire du cube apparaissent dans l'expression asymptotique de la fonction de compte : la géométrie d'un objet et le spectre du laplacien sur celui-ci sont reliés. La loi de Weyl à deux termes, conjecturée par H. Weyl [**81**] et prouvée dans sa forme la plus forte par V. Ivrii [**42**], rend ce dernier énoncé précis : si Ω est un domaine à la frontière lisse dont les trajectoires de billards périodiques ont mesure nulle, alors l'équation asymptotique (1.1.8) est valide pour les valeurs propres de Dirichlet.

Dans cette thèse, trois articles qui cadrent avec la saveur de ces intersections entre la géométrie des nombres et la théorie spectrale sont présentés. Dans le premier article, écrit conjointement avec L. Parnovski, nous avons étudié une généralisation du problème du cercle de Gauss où nous tentions de trouver un exposant optimal similairement à ce qui est obtenu dans l'équation (1.1.1). Dans le second, écrit conjointement avec A. Girouard, I. Polterovich et A. Savo, nous avons étudié le problème de Steklov, étroitement lié au laplacien, sur des cuboïdes de dimension arbitraire, et nous obtenons une asymptotique similaire à celle de l'équation (1.1.8). Finalement, dans le troisième article je reviens à l'équation (1.1.1), qu'on peut généraliser à d'autres réseaux que $\mathbb{Z}^d$ avec des idées similaires. La constante implicite dans (1.1.1) n'est pas uniforme dans le choix du réseau. Nous cherchons donc une dépendence explicite du terme d'erreur en terme des invariants du réseau.

Dans le reste de ce chapitre, j'exposerai brièvement l'une des techniques principales utilisées à travers les trois articles : le passage à la transformée de Fourier pour le calcul de sommes grâce à la formule de sommation de Poisson. Ensuite, je vais exposer brièvement les résultats de chaque article, qui seront ensuite joints à la thèse.

1.2. La formule de sommation de Poisson

La formule de sommation de Poisson a été un outil utilisé crucialement à travers les trois articles qui composent cette thèse. Cette formule a été découverte indépendemment par C. F. Gauss [**25**] et S. D. Poisson [**63**] en dimension 1. Elle relie la somme d'une fonction évaluée sur un réseau Γ, *i.e.* un sous-groupe discret de $\mathbb{R}^d$, à la somme de sa transformée de Fourier

$$\left[\mathscr{F}f\right](\xi) := \int_{\mathbb{R}^d} e^{-2\pi i\xi\cdot x} f(x)\,\mathrm{d}x \tag{1.2.1}$$

évaluée sur le réseau dual

$$\Gamma^* := \left\{\gamma^* \in \mathbb{R}^d : \gamma^* \cdot \Gamma \subset \mathbb{Z}\right\}. \tag{1.2.2}$$

Une version moderne de la formule de sommation de Poisson peut-être trouvée dans [**72**] et se lit comme suit.

Théorème (Formule de sommation de Poisson)**.** *Soit* Γ *un réseau de déterminant* $|\Gamma|$ *et* $f : \mathbb{R}^d \to \mathbb{C}$ *une fonction de Schwartz. Alors, pour tout* $\mathbf{x} \in \mathbb{R}^d$

$$\sum_{\gamma\in\Gamma} f(\mathbf{x}+\gamma) = \frac{1}{|\Gamma|}\sum_{\gamma^*\in\Gamma^*}\left[\mathscr{F}f\right](\gamma^*)e^{2\pi i\mathbf{x}\cdot\gamma^*}. \tag{1.2.3}$$

Bien que ce ne soit pas nécessaire dans le contexte de l'utilisation de ce théorème dans cette thèse, il est possible de demander des contraintes plus faibles que $f \in \mathscr{S}(\mathbb{R}^d)$. En général, elle est valide dès qu'il existe $\varepsilon > 0$ tel que

$$\left|f(x)\right| + \left|\left[\mathscr{F}f\right](x)\right| \ll (1+|x|)^{-d+\varepsilon}, \tag{1.2.4}$$

assurant que les deux sommes dans l'équation (1.2.3) convergent.

L'utilisation de la formule de sommation de Poisson pour calculer des développements asymptotiques est basée sur l'heuristique de la proposition suivante, qui résume la façon dont la formule a été utilisée dans plusieurs contextes *e.g.* [**35, 65, 48**] sans toutefois y avoir été énoncée ainsi.

Proposition 1.2.1. *Soit* $f \in L^1(\mathbb{R}^d)$ *une fonction à support compact dont la transformée de Fourier respecte la borne* $\left[\mathscr{F}f\right](\xi) \ll (1+|\xi|)^{-N}$*, avec* $N > d$*. Notons*

$$\mathbb{E}(f) = \int_{\mathbb{R}^d} f(x)\,\mathrm{d}x. \tag{1.2.5}$$

Alors

$$\sum_{\gamma\in\Gamma} f\left(\frac{\gamma}{\lambda}\right) = \frac{\mathbb{E}(f)}{|\Gamma|}\lambda^d + O\left(\lambda^{d-N}\right). \tag{1.2.6}$$

DÉMONSTRATION. Par la formule de sommation de Poisson ainsi que les propriétés de la transformée de Fourier, on a que

$$\sum_{\gamma \in \Gamma} f\left(\frac{\gamma}{\lambda}\right) = \frac{\lambda^d}{|\Gamma|} \sum_{\gamma^* \in \Gamma^*} \left[\mathscr{F} f\right](\lambda \gamma^*). \tag{1.2.7}$$

Écrivons $\Gamma' = \Gamma^* \setminus \{0\}$. On a alors que

$$\begin{aligned} \sum_{\gamma^* \in \Gamma^*} \left[\mathscr{F} f\right](\lambda \gamma^*) &= \left[\mathscr{F} f\right](0) + \sum_{\gamma^* \in \Gamma'} \left[\mathscr{F} f\right](\lambda \gamma^*) \\ &= \mathbb{E}(f) + O\left(\lambda^{-N} \sum_{\gamma^* \in \Gamma'} \frac{1}{\left|\gamma^*\right|^N}\right). \end{aligned} \tag{1.2.8}$$

Tant qu'on aura choisi $N > d$ la somme converge et on obtient la preuve de la proposition. □

Le problème maintenant se situe quand la transformée de Fourier de f ne respecte pas la borne demandée dans la proposition 1.2.1, empêchant la somme dans l'équation (1.2.8) de converger. C'est le cas dans l'étude du problème du cercle de Gauss ainsi que dans chacun des articles inclus dans cette thèse. La solution consiste alors à lisser f. Pour une fonction ρ, positive, à support dans la boule unité, lisse et de masse 1, on définit la famille de Dirac

$$\rho_\varepsilon(\mathbf{x}) = \frac{1}{\varepsilon^d} \rho\left(\frac{\mathbf{x}}{\varepsilon}\right). \tag{1.2.9}$$

La fonction $f_\varepsilon = \rho_\varepsilon * f$ est lisse et converge ponctuellement presque partout vers f lorsque $\varepsilon \to 0$. De ce fait, on peut s'attendre à approximer la somme de f par celle de f_ε ; on introduit toutefois alors une erreur qui ne nous permet pas d'obtenir des résultats aussi bons ou directs que dans la proposition 1.2.1. Les détails de ces constructions précises sont dictés au cas par cas et changent dans les trois articles présentés ici. On peut par ailleurs noter que la formule de sommation de Poisson peut être utilisée dans d'autres contextes en géométrie des nombres pour obtenir des résultats qui ne sont pas de nature asymptotique, voir par exemple la preuve de Siegel [**71**] du théorème de Minkowski.

1.3. Problèmes spectraux

Nous considérons dans cette thèse trois problèmes spectraux. Dans cette section, nous les décrivons et nous établissons aussi à quel problème de la géométrie des nombres ils correspondent.

1.3.1. Le laplacien sur un tore plat

Soit $\Gamma \subset \mathbb{R}^d$ un réseau et $\mathbb{T}_\Gamma = \mathbb{R}^d/\Gamma$ le tore plat qui est associé. On étudie le problème aux valeurs propres

$$-\Delta\varphi = \lambda\varphi \tag{1.3.1}$$

sur $C^\infty(\mathbb{T}_\Gamma)$. On peut indicer les fonctions propres φ par les éléments γ^* du réseau dual Γ^* :

$$\varphi_{\gamma^*}(\mathbf{x}) = \exp(2\pi i\gamma^* \cdot \mathbf{x}); \qquad \lambda_{\gamma^*} = 4\pi^2 \left|\gamma^*\right|^2. \tag{1.3.2}$$

On obtient donc alors que le spectre du laplacien sur $\mathbb{T}_\Gamma$ est donné par

$$\operatorname{spec}(-\Delta; \mathbb{T}_\Gamma) = \left\{4\pi^2 \left|\gamma^*\right|^2 : \gamma^* \in \Gamma^*\right\}. \tag{1.3.3}$$

Dans [**53**], je me suis intéressé au nombre de valeurs propres de $\mathbb{T}_\Gamma$ inférieures à 1. En terme du réseau dual Γ^*, cela correspond à la quantité

$$N\left(\Gamma; B\left(0,(2\pi)^{-1}\right)\right) = \#\left\{\Gamma^* \cap B\left(0,(2\pi)^{-1}\right)\right\}, \tag{1.3.4}$$

et j'obtiens une formule asymptotique en terme du déterminant du réseau Γ, lequel est égal au volume de $\mathbb{T}_\Gamma$. Pour chaque k, on peut trouver un réseau[2] $\Gamma_k^\star$ qui maximise le déterminant parmi les réseaux possèdant k points à l'intérieur de $B\left(0,(2\pi)^{-1}\right)$. En étudiant le comportement asymptotique dans l'équation (1.3.4), j'ai pu déterminer le comportement de cet optimiseur à mesure que $k \to \infty$ en dimension inférieure à 10.

1.3.2. Opérateurs sur $L^2(\mathbb{R}^d)$

Décomposons $\mathbb{R}^d = \mathbb{R}^k \times \mathbb{R}^l$, $k + l = d$. Dans [**54**], nous nous sommes intéressés à deux opérateurs. Pour $\mathbf{k} \in [-\pi,\pi]^k$, on considère le laplacien $-\Delta_\mathbf{k}$ agissant sur $L^2(\mathbb{R}^d)$ sous la condition de périodicité tordue dans $\mathbb{R}^k$:

$$f(\mathbf{x}+\mathbf{n},\mathbf{y}) = e^{i\mathbf{k}\cdot\mathbf{n}} f(\mathbf{x},\mathbf{y}) \tag{1.3.5}$$

pour $\mathbf{n} \in 2\pi\mathbb{Z}^k$. Le second est l'hamiltonien de Landau

$$H_d = (D_1 + x_2)^2 + D_2^2 + \ldots + D_d^2, \tag{1.3.6}$$

où $D_j = -i\frac{\partial}{\partial x_j}$. Dans les deux cas, le spectre est assez facile à décrire : on a que

$$\operatorname{spec}\left(-\Delta_\mathbf{k}; \mathbb{R}^k \times \mathbb{R}^l\right) = \left[|\mathbf{k}|^2, \infty\right) \tag{1.3.7}$$

[2] Ce réseau n'est possiblement pas unique.

et que

$$\operatorname{spec}(H_d) = [1,\infty). \tag{1.3.8}$$

Dans les deux cas, on ne peut pas définir une fonction de compte des valeurs propres : le spectre n'est pas discret. On définit toutefois une quantité qui remplit le même rôle, la densité d'états intégrée $N_{\text{ids}}(\lambda)$. Celle-ci est définie par le processus limite suivant. Soit $Q_L^m = [-L,L]^m$ un cube de côté $2L$ centré à l'origine. Pour tout L, dénotons $-\Delta_{\mathbf{k},L}$ le laplacien restreint aux fonctions à support dans $\mathbb{R}^k \times Q_L^l$ muni de conditions à la frontière qui le rendent autoadjoint, et $H_{d,L}$ l'hamiltonien de Landau restreint aux fonctions à support dans Q_L^d toujours soumis à des conditions à la frontière qui le rendent autoadjoint. Les spectres de $-\Delta_{\mathbf{k},L}$ et de $H_{d,L}$ sont discrets, réels et bornés inférieurement et on définit

$$N_{\text{ids}}(\lambda;-\Delta_{\mathbf{k}}) := \lim_{L\to\infty} \frac{N(\lambda;-\Delta_{\mathbf{k},L})}{L^k} \tag{1.3.9}$$

et

$$N_{\text{ids}}(\lambda;H_d) := \lim_{L\to\infty} \frac{N(\lambda;H_{d,L})}{L^d}. \tag{1.3.10}$$

Les propriétés de la densité d'états intégrée ainsi que son indépendance des conditions autoadjointes choisies à la frontière de Q_L^m sont exposées par M. Shubin dans [**69**] et [**70**].

Calculer la densité d'états intégrée peut aussi être ramené à des considérations de la géométrie des nombres. Considérons l'ensemble $A_k = \mathbb{Z}^k \times \mathbb{R}^l \subset \mathbb{R}^d$. Cet ensemble consiste en un ensemble d'hyperplans affines parallèles les uns aux autres et disposés selon le réseau $\mathbb{Z}^k$. Définissons

$$S(\rho;\mathbf{k};d,k) := \operatorname{Vol}_l\left(A_k \cap B(\mathbf{k};\rho)\right), \tag{1.3.11}$$

où $B(\mathbf{k};\rho)$ est une boule de rayon ρ centrée en $\mathbf{k}$. On aura alors que

$$N_{\text{ids}}(\lambda;-\Delta_{\mathbf{k}}) = (2\pi)^{-d} S(\sqrt{\lambda};\mathbf{k};d,k). \tag{1.3.12}$$

Pour la densité d'états intégrée de H_d considérons le domaine parabolique donné par

$$\mathscr{P}(\rho) = \left\{(x_0,x') \in \mathbb{R}^d : 0 \le x_0 \le \rho - \left|x'\right|^2\right\}, \tag{1.3.13}$$

et définissons

$$P(\rho;d,k) = \operatorname{Vol}_l\left(\mathscr{P}(\rho)\cap A_k\right). \tag{1.3.14}$$

Nous avons démontré dans [**54**] que

$$N_{\text{ids}}(\lambda;H_d) = 2^{\frac{-d}{2}}\pi^{1-d} P\left(\frac{\lambda-1}{2};d-1,1\right). \tag{1.3.15}$$

Ainsi, pour calculer l'asymptotique de la densité d'états de $-\Delta_{\mathbf{k}}$ ou de H_d nous calculons plutôt l'asymptotique en ρ de S et de P.

1.3.3. Le problème de Steklov sur des cuboïdes

Un cuboïde de paramètres $a_1,\ldots,a_d > 0$ est l'ensemble $\Omega \subset \mathbb{R}^d$ donné par

$$\Omega := [-a_1,a_1] \times \ldots \times [-a_d,a_d]. \tag{1.3.16}$$

Le problème de Steklov sur Ω est le problème aux valeurs propres

$$\begin{cases} \Delta u = 0 & \text{dans } \Omega, \\ \partial_n u = \sigma u & \text{sur } \partial\Omega, \end{cases} \tag{1.3.17}$$

où ∂_n est la dérivée normale vers l'extérieur. Les valeurs σ pour lesquelles il existe $u \in C^\infty(\Omega)$ non-nulle et satisfaisant (1.3.17) sont les valeurs propres de l'opérateur Dirichlet-vers-Neumann $\mathscr{D}$, qui associe

$$u \in C^\infty(\partial\Omega) \mapsto \partial_n(\widehat{u})\big|_{\partial\Omega}, \tag{1.3.18}$$

où $\widehat{u}$ est l'extension harmonique de u à Ω. Une revue de littérature menée par A. Girouard et I. Polterovich sur le problème de Steklov peut être trouvée dans [**27**]. Tant que la frontière d'un domaine dans $\mathbb{R}^d$ est de classe C^1 par morceaux on a que le spectre de $\mathscr{D}$ est discret, et que la suite de valeurs propres σ_k ne s'accumule qu'à l'infini. On a alors une loi de Weyl pour $\mathscr{D}$ [**1**] disant qu'il existe une constante C_Ω telle que

$$N(\sigma;\mathscr{D}) \sim C_\Omega \sigma^{d-1}. \tag{1.3.19}$$

Lorsque la frontière d'un domaine est lisse, $\mathscr{D}$ est un opérateur pseudodifférentiel elliptique d'ordre 1 et on peut faire un estimé plus précis sur le terme d'erreur

$$R(\sigma) := N(\sigma;\mathscr{D}) - C_\Omega \sigma^{d-1}$$

dans l'équation 1.3.19. Toutefois, la frontière du cuboïde n'est pas lisse ; on ne peut donc pas utiliser directement ces techniques car $\mathscr{D}$ n'est plus alors un opérateur pseudodifférentiel. Nous nous servons du cuboïde comme modèle pour comprendre de manière plus générale les domaines dont la frontière n'est pas lisse.

Pour faire la description des fonctions propres de Steklov sur le cuboïde, il faut introduire un peu de notation. Soit (τ_0,τ_1,τ_2) une tripartition de $S_d = \{1,\ldots,d\}$. Nous avons démontré dans

[**26**] qu'il y a une base complète de fonctions propres de Steklov sur Ω de la forme

$$u(x_1,\ldots,x_d) = \prod_{i\in\tau_0} x_i \prod_{j\in\tau_1} \mathrm{Trig}_j(\alpha_j x_j) \prod_{k\in\tau_2} \mathrm{Hyp}_k(\beta_k x_k), \tag{1.3.20}$$

où $\mathrm{Trig}_j \in \{\sin,\cos\}$ et $\mathrm{Hyp}_k \in \{\sinh,\cosh\}$. Le nombre de fonctions propres où $\tau_0 \neq \emptyset$ est fini, et les coefficients α_j et β_k doivent respecter la condition d'harmonicité

$$\sum_{j\in\tau_1} \alpha_j^2 = \sum_{k\in\tau_2} \beta_k^2. \tag{1.3.21}$$

Définissons pour $a > 0$ et $\ell \in \{0,1\}$ les fonctions

$$T_{a,\ell}(x) = x\cot\Big(ax + \ell\frac{\pi}{2}\Big) = \begin{cases} x\cot(ax) & \text{si } \ell = 0, \\ -x\tan(ax) & \text{si } \ell = 1, \end{cases}$$

et

$$H_{a,\ell}(x) = \begin{cases} x\coth(ax) & \text{si } \ell = 0, \\ x\tanh(ax) & \text{si } \ell = 1. \end{cases}$$

Soit $\ell : S_d \to \{0,1\}$ la fonction qui choisit les fonctions trigonométriques et hyperboliques pour chaque facteur de l'équation (1.3.20). Les coefficients α_i et β_i doivent aussi respecter les conditions de compatibilité

$$\sigma = \begin{cases} a_i^{-1} & \text{pour } i \in \tau_0, \\ T_{a_i,\ell(i)}(\alpha_i) & \text{pour } i \in \tau_1, \\ H_{a_i,\ell(i)}(\beta_i) & \text{pour } i \in \tau_2. \end{cases} \tag{1.3.22}$$

La valeur commune σ de ces équations est la valeur propre associée à la fonction propre en question. Asymptotiquement, nous ne considérons pas les fonctions propres avec un facteur linéaire car elles ne peuvent apparaître qu'en nombre fini. En effet, la valeur propre qui est associée à une fonction propre linéaire dans la variable x_j est soit 0, auquel cas la fonction propre est constante dans toutes les variables, ou bien a_j^{-1}. Comme les a_j sont bornés loin de zéro, il y a un nombre fini de valeurs propres plus petites ou égale à a_j^{-1}. Pour compter les valeurs propres, nous avons donc considéré toutes les partitions $\tau = (\tau_1,\tau_2)$ qui déterminent quels facteurs sont trigonométriques et hyperboliques. Notons $p = |\tau_1|$, nous avons réduit le problème de compter les valeurs propres $N(\sigma;\tau)$ inférieures à σ et correspondant à la partition τ au problème de géométrie des nombres

$$N_\tau(\sigma) = C_\tau \cdot \#\left(\mathbb{N}^p \cap E_\sigma\right), \tag{1.3.23}$$

où C_τ est une constante dépendant de τ et explicitement calculable et où E_σ est un ensemble convexe tel que E_σ converge vers l'ellipsoïde de semi-axes $\{a_j : j \in \tau_1\}$ lorsque $\sigma \to \infty$.

Avec la description précise des fonctions propres et des valeurs propres, nous avons aussi pu répondre à des questions concernant la plus petite valeur propre non triviale et la distribution des fonctions propres.

1.4. Asymptotiques spectrales

Nous avons obtenu dans les trois articles qui composent cette thèse plusieurs théorèmes concernant les asymptotiques des fonctions de compte de valeurs propres ou de la densité d'états intégrée. Dans cette section, nous décrivons ces résultats à la fois en terme du spectre ou des problèmes correspondants en géométrie des nombres. Nous soulèverons aussi plusieurs questions et conjectures qui découlent de ces résultats.

1.4.1. Compte de valeurs propres

1.4.1.1. *Formule asymptotique à deux termes pour le problème de Steklov sur un cuboïde*

Le théorème suivant concerne le comportement quand $\sigma \to \infty$ de la fonction de compte $N(\sigma;\mathscr{D})$ du problème décrit à la section 1.3.3.

Théorème 1 (A. Girouard, J. Lagacé, I. Polterovich, A. Savo [**26**])**.** *Soit $\Omega \subset \mathbb{R}^d$ le cuboïde de paramètres $a_1,\dots,a_d > 0$. Pour $d \geq 3$, la fonction de compte des valeurs propres de Steklov de Ω satisfait la formule asymptotique à deux termes*

$$N(\sigma) = C_1 \operatorname{Vol}_{d-1}(\partial\Omega)\sigma^{d-1} + C_2 \operatorname{Vol}_{d-2}(\partial^2\Omega)\sigma^{d-2} + O\left(\sigma^\eta\right), \tag{1.4.1}$$

quand $\sigma \to \infty$, où $\partial^2\Omega$ est l'union de toutes les faces de dimension $(d-2)$ de Ω. L'exposant $\eta = 2/3$ lorsque $d = 3$ et $\eta = d - 2 - \frac{1}{d-1}$ lorsque $d \geq 4$. Les constantes C_1 et C_2 sont données par

$$C_1 = \frac{\omega_{d-1}}{(2\pi)^{d-1}}$$

et

$$C_2 = \frac{2^{\frac{d-2}{2}}\omega_{d-2}}{(2\pi)^{d-2}} - \frac{2G_{d-1,1}}{\pi^{d-1}} - \frac{\omega_{d-2}}{2(2\pi)^{d-2}},$$

où

$$G_{d-1,1} = \underbrace{\int_0^{\pi/2} \dots \int_0^{\pi/2}}_{d-2} \operatorname{arccot}\left(\prod_{j=1}^{d-2} \csc\theta_j\right) \prod_{k=1}^{d-2} \sin^k(\theta_k)\, \mathrm{d}\theta_1 \dots \mathrm{d}\theta_{d-2}.$$

Pour $d = 2$, la fonction de compte admet la formule asymptotique à un terme

$$N(\sigma) = \pi^{-1}\operatorname{Vol}_1(\partial\Omega)\sigma + O(1).$$

Trouver une formule asymptotique pour un problème spectral permet en général de trouver des *invariants spectraux*, c'est-à-dire des quantités géométriques qu'on peut déterminer à partir du spectre. Ce théorème ne fait pas défaut, ici on démontre que dans la classe des cuboïdes, le volume $d-1$ dimensionnel de la frontière est un invariant spectral du problème de Steklov grâce au premier terme. Le second terme quant à lui nous permet de savoir que le volume $d-2$ dimensionnel des facettes de codimension 2 est lui aussi un invariant spectral. Cela soulève évidemment la question suivante.

Question 1. Est-ce que les volumes $d-k$ dimensionnels des facettes de codimension k sont des invariants spectraux du problème de Steklov?

Les méthodes utilisées pour le calcul de l'équation 1.4.1 dans [**26**] nous indiquent que si on arrivait à diminuer l'exposant η à une valeur strictement plus petite que $d-N$, nous obtiendrions en effet que les volumes $d-k$ dimensionnels des facettes de codimension k seraient des invariants spectraux pour $k \leq N$. Toutefois, bien que nous ne fassions aucune affirmation quant à l'optimalité de la valeur de η, il est particulièrement improbable que nous puissions la diminuer en bas de $d-3$, même en changeant nos techniques.[3] Toutefois, les invariants spectraux ne viennent pas nécessairement que de la fonction de compte. La moyenne de Riesz d'ordre $\gamma \geq 0$ des valeurs propres, définie par

$$\operatorname{Tr}(\mathscr{D}_\Omega - \sigma)_-^\gamma := \sum_{k=0}^{\infty} (\sigma - \sigma_k(\Omega))_+^\gamma \tag{1.4.2}$$

[3]Heuristiquement, les bornes optimales obtenues dans [**29**] l'en empêcherait. En effet, pour des ellipsoïdes de dimension $d-1 \geq 5$, l'exposant $d-3$ dans le reste est optimal, on ne s'attend pas à obtenir mieux.

est une version plus lisse de la fonction de compte pour $\gamma > 0$ et correspond à la fonction de compte lorsque $\gamma = 0$. Il est plus probable de trouver un terme d'erreur plus petit dans une formule équivalente à 1.4.1 pour la moyenne de Riesz, qui aurait à ce moment une formule asymptotique à plus de termes, nous permettant ainsi de déterminer plus d'information géométrique à partir du spectre de Steklov. On peut voir [**28**] pour l'utilisation de la moyenne de Riesz dans un autre contexte pour obtenir un terme d'erreur uniforme assez petit lors de l'étude des problèmes de Neumann et de Dirichlet sur des familles de cuboïdes.

Le cuboïde représente aussi un premier modèle pour l'étude du problème de Steklov sur des domaines dont la frontière n'est pas lisse.

Question 2. Est-ce que des résultats similaire au théorème 1 tiennent pour des domaines plus généraux dont la frontière n'est pas lisse ? Tiennent-ils pour des polyèdres ?

Dans une récente prépublication, V. Ivrii [**43**] donne une réponse au moins partielle à ce type de question. Dans le cas où $d = 2$, M. Levitin, L. Parnovski, I. Polterovich et D. Sher [**56**] se sont penchés sur cette question. On y obtient entre autre, du moins partiellement, que les angles auxquels se rencontrent les arêtes d'un polygone doivent apparaître dans l'asymptotique des valeurs propres.

1.4.1.2. *Formule asymptotique pour les valeurs propres sur le tore*

On peut interpréter le théorème principal de [**29**] en termes de fonctions de compte de valeurs propres sur le tore. On y lirait

Théorème (F. Götze [**29**])**.** *Soit* $\mathbb{T}_\Gamma := \mathbb{R}^d/\Gamma$ *un tore plat orthogonal pour* $d \geq 5$. *La fonction de compte des valeurs propres du laplacien sur* $\mathbb{T}_\Gamma$ *a le comportement asymptotique*

$$N\left(\lambda^2; \mathbb{T}_\Gamma\right) = \frac{\omega_d}{(2\pi)^d} |\Gamma| \lambda^d + O\left(\lambda^{d-2}\right). \tag{1.4.3}$$

L'exposant λ^{d-2} dans l'équation (1.4.3) est optimal, mais la constante implicite dépend du tore. En effet, si on ne considère que des tores de volume 1 et qu'on laisse le rayon d'injectivité du tore, *i.e.* la longueur de sa plus petite géodésique, tendre vers 0 cette constante tend vers l'infini.

On peut renverser la formule asymptotique dans l'équation (1.4.3) pour obtenir

$$\lambda_k(\Gamma) = \frac{4\pi}{(|\Gamma| \omega_d)^{2/d}} k^{2/d} + O(1), \tag{1.4.4}$$

mais une fois de plus la constante implicite dépend du tore. Si on cherche à trouver, asymptotiquement, le tore qui maximise λ_k on ne peut donc pas a priori utiliser la formule (1.4.4). Toutefois, au prix d'avoir un moins bon exposant on peut trouver une formule asymptotiquement valide pour une grande classe de tores.

Parce que compter les valeurs propres d'un tore revient à compter les points d'un réseau dans une boule, le prochain résultat est énoncé en ces termes. Définissons les minima successifs d'un réseau par

$$\mu_j(\Gamma) = \inf(\mu : \dim(\operatorname{span}(\Gamma \cap B_\mu)) \geq j). \tag{1.4.5}$$

J'obtiens le résultat suivant.

Théorème 2 (J. Lagacé [**53**])**.** *Il existe* $\delta_0 > 0$ *tel que pour tout* $\delta \in (0, \delta_0)$ *et* $M > 0$*, il existe* C *tel que pour tout réseau* Γ *respectant* $|\Gamma| \leq 1$ *et*

$$\mu_d(\Gamma) \leq M |\Gamma|^\delta \tag{1.4.6}$$

on a que

$$\left| N(B_1; \Gamma) - \frac{\omega_d}{|\Gamma|} \right| \leq C |\Gamma|^{-1 + \frac{\delta(d-1)}{2d^2}}. \tag{1.4.7}$$

Ce résultat peut être traduit pour les tores pour obtenir une loi de Weyl avec un exposant valide même lorsque le rayon d'injectivité tend vers 0, s'il ne le fait pas trop vite. Le résultat se lit ainsi

Théorème 3. *Il existe* δ_0 *tel que pour tout* $\delta \in (0, \delta_0)$ *et* $c > 0$*, il existe un nombre* C *tel que pour tout* $\lambda > 1$ *et pour tout tore plat* $\mathbb{T}_\Gamma$ *de volume* 1 *respectant*

$$\operatorname{inj}(\mathbb{T}_\Gamma) \geq c \lambda^{-1/2 + \delta} \tag{1.4.8}$$

on a que

$$\left| N(\lambda; \mathbb{T}_\Gamma) - \frac{\omega_d}{(2\pi)^d} \lambda^{d/2} \right| \leq C \lambda^{\frac{d}{2} - \delta \frac{d-1}{4d}}. \tag{1.4.9}$$

Les deux derniers résultats sont optimaux dans le sens suivant : je construis dans [**53**] lorsque $\delta = 0$ une suite de tores qui ne respecteront pas les bornes de l'équation (1.4.7). De plus, je démontre aussi que l'exposant sur le reste dans l'équation (1.4.9) doit tendre vers $d/2$ lorsque $\delta \to 0$.

Ces résultats sont ensuite utilisés pour établir le comportement asymptotique de tores plats maximisant la valeur propre λ_k du laplacien. Il est donc naturel de se demander si on peut trouver des bornes sur le terme d'erreur dans la loi de Weyl qui dépendent explicitement de la géométrie d'une variété. P. Buser obtient dans [**14**] une borne supérieure sur les valeurs propres du laplacien dépendent de la courbure de Ricci et du rayon d'injectivité, alors que S. Y. Cheng [**17**] et P. Buser [**15**] obtiennent des bornes supérieures dépendant de la courbure de Ricci et du diamètre. En observant les relations entre les valeurs propres et les quantités géométriques dans ces inégalités, la question suivante est donc naturelle.

Question 3. Quel contrôle doit-on avoir sur le rayon d'injectivité, la courbure de Ricci et le diamètre d'une variété Ω en fonction du paramètre spectral λ, pour qu'on puisse trouver $\eta < d/2$, dépendant de ces quantités géométriques et tel que

$$\left| N(\lambda;\Omega) - \frac{\omega_d}{(2\pi)^d}\lambda^{d/2} \right| \leq C\lambda^{\eta}? \tag{1.4.10}$$

1.4.2. Densité d'états

Nous décrivons maintenant les résultats asymptotiques obtenus dans [**54**] pour les problèmes décrits dans la section 1.3.2. En particulier, nous considérons l'asymptotique en $\rho \to \infty$ de la fonction $S(\rho;\mathbf{k};d,k)$ de l'équation (1.3.11).

Théorème 4 (J. Lagacé et L. Parnovski, [**54**])**.** *On a la formule asymptotique*

$$S(\rho;\mathbf{k};d,k) = \omega_d \rho^d + R(\rho;\mathbf{k};d,k), \tag{1.4.11}$$

où le terme d'erreur $R(\rho;\mathbf{k}_1;d,k)$ vérifie l'estimé asymptotique

$$R(\rho;\mathbf{k}_1;d,k) = \begin{cases} O\left(\rho^{(d-1)/2}\right) & si\ k < (d+1)/2, \\ O\left(\rho^{(d-1)/2}\log\rho\right) & si\ k = (d+1)/2, \\ O\left(\rho^{d-\frac{2k}{1-d+2k}}\right) & si\ k > (d+1)/2 \end{cases} \tag{1.4.12}$$

uniformément en $\mathbf{k}$.

Par l'équivalence mentionnée dans la section 1.3.2, on obtient alors une formule asymptotique pour la densité d'états intégrée du laplacien soumis aux conditions de périodicité partielles. Une fois qu'on connait le comportement de l'opérateur libre, il est naturel de se demander comment se comporte une perturbation de celui-ci.

Question 4. Soit $H = -\Delta + V$ un opérateur de Schrödinger dont le potentiel V est périodique. Peut-on avoir un développement asymptotique pour $N_{\text{ids}}(\lambda;H)$?

Lorsqu'on étudie H sur $L^2(\mathbb{R}^d)$ sans aucune condition supplémentaire, cette question a été répondue par Parnovski et Shterenberg dans [**59**] et [**60**], où ils ont obtenu un développement asymptotique complet pour $N_{\mathrm{ids}}(\lambda; H)$. Il est toutefois irréaliste d'espérer une telle chose dans ce cas : on ne pourra pas obtenir de terme plus précis que le terme d'erreur dans le cas de l'opérateur libre.

Une autre question qu'on peut se poser est de savoir si les exposants donnés dans l'équation (1.4.12) sont optimaux. On sait que dans le cas où $k = d$, la réponse est non et que l'exposant optimal est $d-2$ [**29**]. Lorsque $k \le (d+1)/2$, on peut toutefois répondre à cette question.

Théorème 5 (J. Lagacé et L. Parnovski, [**54**])**.** *Pour $k > 1$ et ρ suffisamment grand, il existe une constante $C_{d,k} > 0$ et $\mathbf{k}_1 \in \mathbb{T}^k$ tels que*

$$R(\rho; \mathbf{k}_1; d,k) \ge \begin{cases} C_{d,k}\rho^{\frac{d-1-\varepsilon}{2}} & \text{si } d \equiv 1 \pmod 4 \\ C_{d,k}\rho^{\frac{d-1}{2}} & \text{sinon,} \end{cases} \tag{1.4.13}$$

où $\varepsilon > 0$ est n'importe quel nombre réel strictement positif. Lorsque $d \not\equiv 1 \pmod 4$, la borne inférieure $R(\rho; \mathbf{k}_1; d,k) \ge C_{d,k}\rho^{\frac{d-1}{2}}$ tient aussi pour $k = 1$.

Cela nous indique donc que, au moins polynomialement, les exposants de l'équation (1.4.12) sont optimaux lorsque $k \le (d+1)/2$. Cela soulève naturellement la question suivante :

Question 5. Quel est l'exposant optimal dans (1.4.12) pour $k > (d+1)/2$.

Finalement, nous avons aussi étudié l'asymptotique de $N_{\mathrm{ids}}(\lambda; H_d)$ et obtenu le résultat suivant pour le problème correspondant consistant à étudier l'asymptotique de $P(\rho; d,k)$.

Théorème 6 (J. Lagacé et L. Parnovski, [**54**])**.** *Quand $\rho \to \infty$, $P(\rho; d,k)$ admet le développement asymptotique*

$$P(\rho; d,1) = \omega_{d-1} E_{\lfloor \frac{d+1}{4} \rfloor}(\rho) + O(1), \tag{1.4.14}$$

$$P(\rho; d,d) = \frac{2\omega_{d-1}}{d+1} + O\left(\rho^{\frac{d^2-d+2}{2d}}\right). \tag{1.4.15}$$

Puis pour $d > k > \frac{d+2}{2}$ nous avons

$$P(\rho; d,k) = E_{\lfloor \frac{k-1}{4k-2} \rfloor}(\rho) + O(\rho^{\frac{1}{2}\left(d-1-\frac{2k-2}{2k-d}\right)}). \tag{1.4.16}$$

Finalement, si $1 < k \le \frac{d+2}{2}$,

$$P(\rho; d,k) = E_{\lfloor \frac{d-4}{8} \rfloor}(\rho) + O(\rho^{\frac{d+4}{4}}(\log \rho)^{\delta}), \tag{1.4.17}$$

où $\delta = 1$ si $k = \frac{d+2}{2}$ et 0 sinon.

1.5. Optimisation spectrale

Connaître les domaines optimaux pour les valeurs propres d'un opérateur différentiel est une question subtile. Par exemple, pour les valeurs propres du laplacien de Dirichlet on sait par l'inégalité de Faber-Krahn [**23, 51**] qu'à volume fixé, le domaine minimisant la première valeur propre est la boule. On sait aussi par l'inégalité de Krahn-Szegö [**52**] que le domaine minimisant la deuxième valeur propre du laplacien de Dirichlet est l'union de deux boules du même volume. Toutefois, pour les valeurs propres d'ordre supérieur du laplacien, nous ne savons pas identifier le domaine maximisant, ni même garantir que ce soit un ouvert. Dans cette section, nous exposons les résultats d'optimisation obtenus dans cette thèse.

1.5.1. Première valeur propre de Steklov sur les cuboïdes

Nous nous sommes intéressés dans [**26**] à la première valeur propre de Steklov non triviale sur un cuboïde, et à trouver l'optimiseur, s'il existe, pour σ_1 dans cette classe. L'optimisation des valeurs propres de Steklov reste un champ d'étude plutôt ouvert ; on sait par exemple par l'inégalité de Weinstock [**79**] que la première valeur propre de Steklov sur des domaines simplement connexe en dimension 2 est maximisée par le disque, mais qu'en enlevant la condition d'être simplement connexe non seulement le disque n'est plus le maximiseur, mais on ne sait même pas si un tel maximiseur existe.

En général, pour prouver des inégalités par rapport aux valeurs propres il faut d'abord savoir quelle est la forme de la fonction propre. Nous avons donc d'abord identifié la première fonction propre de Steklov sur un cuboïde quelconque.

Théorème 7 (A. Girouard, J. Lagacé, I. Polterovich et A. Savo, [**26**])**.** *Soit Ω un cuboïde de paramètres $a_1 \leq \cdots \leq a_d$. Alors, il existe $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_{d-1})$ et $\alpha_d = |\boldsymbol{\beta}| < \frac{\pi}{2a_d}$ tels que*

$$u(x_1, \ldots, x_d) = \sin(\alpha_d x_d) \prod_{k=1}^{d-1} \cosh(\beta_k x_k)$$

est une fonction propre de Steklov sur Ω associée à la première valeur propre non triviale σ_1.

Nous avons utilisé ce résultat pour obtenir l'inégalité isopérimétrique suivante pour la première valeur propre de Steklov sur les cuboïdes.

Théorème 8 (A. Girouard, J. Lagacé, I. Polterovich et A. Savo, [**26**])**.** *Pour tout cuboïde Ω, dénotons par $\Omega^\star$ le cube de même volume et par $\Omega^\sharp$ le cube de même aire. On aura alors que*

- $\sigma_1(\Omega^\star) \geq \sigma_1(\Omega)$ *; l'égalité n'est atteinte que si* $\Omega^\star = \Omega$ *;*

- $\sigma_1(\Omega^\sharp) \geq \sigma_1(\Omega)$ *; l'égalité n'est atteinte que si* $\Omega^\sharp = \Omega$.

Ce dernier théorème nous permet de déterminer en corollaire que le cube est déterminé par son spectre de Steklov parmi tous les cuboïdes.

Corollaire 9 (A. Girouard, J. Lagacé, I. Polterovich et A. Savo, [**26**])**.** *Soit* $\Omega \subset \mathbb{R}^d$ *un cuboïde Steklov-isospectral au cube* $\Omega_a \subset \mathbb{R}^m$ *dont les côtés sont de longueur* $2a > 0$. *Alors on aura que* $d = m$ *et* $\Omega = \Omega_a$.

1.5.2. Optimisation asymptotique des valeurs propres du laplacien sur des tores plats

Dans l'article [**53**], j'ai étudié le comportement asymptotique des optimiseurs de λ_k parmi les tores plats et les bouteilles de Klein plates alors que $k \to \infty$. Des questions similaires pour les valeurs propres de Dirichlet et de Neumann sur des cuboïdes ont été étudiées récemment par entre autres Antunes, Freitas, Ariturk, Laugesen, van den Berg, Gittins, Bucur et Larson [**2, 4, 8, 7, 28**]. Dans ces articles, on démontre que l'optimiseur tend vers le cuboïde le plus symétrique possible, le cube.

Inspiré d'une construction de Kao, Lai et Osting [**45**], j'ai obtenu que le portrait pour les bouteilles de Klein plates ainsi que pour les tores plats en dimension $d \leq 10$ est plutôt différent dans les deux théorèmes suivants.

Théorème 10 (J. Lagacé, [**53**])**.** *Soit* $\mathbb{T}_k$ *une suite de tores plats de dimension* $d \leq 10$ *et de volume* 1 *maximisant la valeur propre* λ_k *du laplacien parmi les tores plats. Alors, la suite* $\mathbb{T}_k$ *n'a pas de point limite. De plus, pour tout* $\delta > 0$

$$k^{-\frac{(1-d)^2}{d}} \ll \operatorname{inj}(\mathbb{T}_k) \ll k^{-\frac{1}{d}+\delta}. \tag{1.5.1}$$

La borne inférieure est valide en toute dimension d.

Théorème 11 (J. Lagacé, [**53**])**.** *Soit* K_k *une suite de bouteilles de Klein plates de volume* 1 *maximisant la valeur propre* λ_k *du laplacien parmi les bouteilles de Klein plates. Alors, la suite* K_k *n'a pas de point limite. De plus, pour tout* $\delta > 0$

$$k^{-\frac{1}{2}} \ll \operatorname{inj}(K_k) \ll k^{\frac{1}{2}+\delta}. \tag{1.5.2}$$

Une question très naturelle provient du premier de ces deux théorèmes.

Question 6. En dimension $d \geq 11$, est-ce le cas qu'une suite de tores plats $\mathbb{T}_k$ maximisant λ_k dégénère ? Est-elle confinée à un sous-ensemble compact des tores plats ? Peut-on observer les deux comportements ?

J'établis au moins dans [**53**] que la construction utilisée ne peut certainement pas être généralisée aux dimensions supérieures à 11.

1.6. Distribution des fonctions propres

On peut associer à une fonction propre u d'un opérateur pseudodifférentiel une mesure $|u|^2\,\mathrm{d}x$ absolument continue par rapport à la mesure de Lebesgue. On s'intéresse généralement au comportement asymptotique de la mesure $|u_k|^2\,\mathrm{d}x$ alors que $k\to\infty$; le théorème ergodique quantique de Šnirel'man [**77**], Colin de Verdière [**19**] et Zelditch [**82**] dit que sous des hypothèses plutôt faibles, la suite $|u_k|^2\,\mathrm{d}x$ possède une sous-suite de densité 1 qui s'équidistribue sur le domaine.

Dans le cas des fonctions propres de Steklov sur des cuboïdes, nous nous sommes intéressés à des sous-suites exceptionnelles se concentrant sur des sous-ensembles de la frontière de mesure nulle. Ceci est illustré dans le théorème suivant.

Théorème 12 (A. Girouard, J. Lagacé, I. Polterovich et A. Savo, [**26**])**.** *Soit $\Omega\subset\mathbb{R}^d$ un cuboïde de paramètres $a_1,\dots,a_d>0$. Soit $p\in\{1,\dots,d-1\}$ et soit $\tau\in\mathcal{T}_p$. Considérons l'ensemble*

$$X_\tau=\{x=(\mathbf{x}_{\tau_1},\mathbf{x}_{\tau_2})\in\partial\Omega : x_j=\pm a_j \textit{ pour } j\in\tau_2\}.$$

Il existe une suite de fonctions propres $\{u_k\}$ normalisées par leur norme $L^2(\partial\Omega)$ se concentrant sur X_τ et s'équidistribuant sur X_τ au sens suivant : pour tout ensemble mesurable $U\subset X_\tau$ et tout $\varepsilon>0$, soit l'ensemble

$$U_\varepsilon=\{\mathbf{x}=(\mathbf{x}_{\tau_1},\mathbf{x}_{\tau_2})\in\partial\Omega:\mathbf{x}_{\tau_1}\in U \textit{ et } \operatorname{dist}(\mathbf{x},U)<\varepsilon\}.$$

Alors, pour tout $\varepsilon>0$,

$$\lim_{k\to\infty}\int_{U_\varepsilon}|u_k(\mathbf{x})|^2\,dx=\frac{\mathrm{Vol}_p(U)}{\mathrm{Vol}_p(X_\tau)}.$$

Considérant que les cuboïdes nous servent de modèle pour des domaines dont la frontière n'est pas lisse, ce théorème mène naturellement à la question suivante.

Question 7. Soit $A\subset\mathbb{R}^d$ un domaine dont la frontière possède une arête de codimension 2. Est-il vrai qu'une sous-suite des fonctions propres de Steklov sur A se concentre près de cette arête ?

Remarques quant au contenu des articles et à la notation qui y est utilisée

L'un des aléas de la rédaction d'une thèse par articles est que ceux-ci ont en général été écrits sans attention à ce qu'ils fassent partie d'un tout. De ce fait, la notation utilisée variera parfois d'un article à l'autre, mais y sera redéfinie chaque fois. De plus, quelques erreurs typographiques ont été modifiées dans les articles déjà publiés ou soumis ; les résultats énoncés, la méthode de preuve et leur structure sont toutefois conformes au document original.

Notation asymptotique

Les notations asymptotiques sont souvent utilisées au long des articles composant cette thèse sans nécessairement être définies. Elles sont :

- $f = O(g)$ ou $f \ll g$: il existe une constante C telle que $|f(x)| \leq C|g(x)|$ pour x suffisamment grand, l'utilisation d'un indice $f = O_\varepsilon(g)$ indique que C peut dépendre de ε. Si le contexte est clair, ça peut aussi être pour x suffisamment près de 0 ;
- $f \asymp g$: on a à la fois $f \ll g$ et $g \ll f$;
- $f \sim g$: $\lim_{x\to\infty} f(x)/g(x) = 1$, si le contexte est clair la limite peut être en x_0 plutôt ;
- $f = o(g)$: $\lim_{x\to\infty} f(x)/g(x) = 0$, si le contexte est clair la limite peut être en x_0 plutôt.

Chapitre 2

A generalised Gauss circle problem and integrated density of states

par

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A generalised Gauss circle problem and integrated density of states

RÉSUMÉ. Compter les points d'un treillis dans une boule de grand rayon dans l'espace euclidien est un problème classique en théorie analytique des nombres, remontant jusqu'à Gauss. Nous proposons une variante de ce problème : étudier l'asymptotique de la mesure de l'intersection d'un treillis de plans affines et d'une boule. Le terme principale est le volume de la boule ; on étudie la taille du reste. Alors que le problème classique correspond à compter les valeurs propres du laplacien sur le tore, notre variante correspond à la densité d'états intégrée du laplacien sur le produit d'un tore et de l'espace euclidien. Les asymptotiques obtenues sont ensuite utilisées pour calculer la densité d'état de l'opérateur de Schrödinger magnétique à champ constant.
Mots clés : Problème du cercle ; Densité d'états intégrée ; Opérateurs de Schrödinger périodiques.

ABSTRACT. Counting lattice points inside a ball of large radius in Euclidean space is a classical problem in analytic number theory, dating back to Gauss. We propose a variation on this problem : studying the asymptotics of the measure of an integer lattice of affine planes inside a ball. The first term is the volume of the ball ; we study the size of the remainder term. While the classical problem is equivalent to counting eigenvalues of the Laplace operator on the torus, our variation corresponds to the integrated density of states of the Laplace operator on the product of a torus with Euclidean space. The asymptotics we obtain are then used to compute the density of states of the magnetic Schrödinger operator.
Keywords: Circle problem ; Integrated density of states ; Periodic Schrödinger operators.

2.1. Introduction and Main results

The first problem we are considering in this paper has several equivalent formulations.

2.1.1. Number theoretic formulation

For $\rho > 0$ and $\mathbf{k} \in \mathbb{R}^d$, let $B(\rho;\mathbf{k})$ be the ball of radius ρ centered at $\mathbf{k}$. Let $S(\rho;\mathbf{k})$ be the number of integer points inside the disk $B(\rho,\mathbf{k}) \subset \mathbb{R}^2$. The classical *Gauss Circle Problem* consists in estimating the remainder term

$$\widetilde{R}(\rho;0) = S(\rho;0) - \pi\rho^2 \tag{2.1.1}$$

Hardy and (Edmund) Landau have found lower bounds for this problem, while the current best upper bound is given by Huxley in [**39**]. This problem has also been studied for balls of dimension higher than two, see e.g. [**29**], and it is well-known that averaging over the radius of the ball improves regularity of the remainder.

In this paper, we consider a variation on this problem: we estimate the measure of the intersection of affine planes sitting on integer coordinates with balls of large radius in $\mathbb{R}^d$. More precisely, put

$$A_k := \mathbb{Z}^k \times \mathbb{R}^{d-k} \subset \mathbb{R}^d \tag{2.1.2}$$

and let $B^d(\rho,\mathbf{k})$ be a ball in $\mathbb{R}^d$ of radius ρ centred at $\mathbf{k} := (\mathbf{k}_1,\mathbf{k}_2) \in \mathbb{R}^k \times \mathbb{R}^l$, where $k + l = d$. Denote by $S(\rho;\mathbf{k}_1;d,k)$ the l-dimensional volume of the set $B^d(\rho,\mathbf{k}_1) \cap A_k$. A simple observation shows that we have

$$S(\rho;\mathbf{k}_1;d,k) = \omega_l \sum_{\substack{\gamma\in\mathbb{Z}^k \\ |\gamma-\mathbf{k}_1|<\rho}} \left(\rho^2 - |\gamma - \mathbf{k}_1|^2\right)^{l/2}, \tag{2.1.3}$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$. One can see that the integral of $\widetilde{R}(\rho,\mathbf{k})$ over $\mathbf{k}_2 \in \mathbb{T}^l = \mathbb{R}^l/\mathbb{Z}^l$, is the same as the remainder term

$$R := S(\rho;\mathbf{k}_1;d,k) - \omega_d \rho^d, \tag{2.1.4}$$

obtained from Equation (2.1.3). Our aim is to compute an estimate of R for large values of ρ. Before discussing the results, we would like to describe different formulations of this problem.

2.1.2. First spectral theoretic formulation

Let

$$H = -\Delta + V \tag{2.1.5}$$

be a Schrödinger operator acting in $\mathbb{R}^d$ with a smooth real-valued periodic potential V; for simplicity we assume that the lattice of periods is $\Gamma := (2\pi\mathbb{Z})^d$, with dual lattice $\Gamma^\dagger = \mathbb{Z}^d$ Denote the integrated density of states (IDS) of H by $N(\lambda) := N(\lambda;H)$. It can be defined by the formula

$$N(\lambda;H) := \lim_{L\to\infty} \frac{\widetilde{N}(\lambda;H_L)}{L^d}, \tag{2.1.6}$$

where H_L is the restriction of H to the cube $[0,L]^d$ with appropriate self-adjoint boundary conditions and $\widetilde{N}(\lambda,H_L)$ is the counting functions of the (discrete) eigenvalues of H_L. Note that this parameter λ is related to the parameter ρ of the previous section by $\rho = \sqrt{\lambda}$. While this formulation of the IDS is important for Theorem 2.1.5, for periodic V we use a useful equivalent definition.

Following [**67**], we express H as a direct integral

$$H = \int_{\mathbb{T}^d}^{\oplus} H(\mathbf{k})\,\mathrm{d}\mathbf{k}, \tag{2.1.7}$$

where $H(\mathbf{k})$ is H restricted to the space of functions such that $f(x+n) = e^{2\pi i \mathbf{k}\cdot n} f(x)$ for all $n \in \mathbb{Z}^d$ and $x \in \mathbb{R}^d$. Then, one can express $N(\lambda; H)$ in terms of the counting functions of the fibre operators $H(\mathbf{k})$:

$$N(\lambda) := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} N(\lambda; H(\mathbf{k}))\,\mathrm{d}\mathbf{k}, \tag{2.1.8}$$

where $N(\lambda, H(\mathbf{k}))$ is the eigenvalue counting function of $H(\mathbf{k})$. Remarkably, despite the fact that the asymptotic behaviour of $N(\lambda, H(\mathbf{k}))$ for fixed $\mathbf{k}$ and $\lambda \to \infty$ is very irregular (so that even the precise size of the remainder

$$R(\lambda; \mathbf{k}) := N(\lambda, H(\mathbf{k})) - C_d \lambda^{d/2} \tag{2.1.9}$$

is unknown), integration over all quasimomenta $\mathbf{k} \in \mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ makes things extremely regular, so that there exists a complete asymptotic expansion of $N(\lambda)$ in powers of λ as $\lambda \to \infty$, [**59, 60**]. Here, we have denoted

$$C_d = \frac{\omega_d}{(2\pi)^d}, \tag{2.1.10}$$

where

$$\omega_d = \frac{\pi^{d/2}}{\Gamma(1 + d/2)} \tag{2.1.11}$$

is the volume of the unit ball in $\mathbb{R}^d$. The question we want to study is what would happen if, instead of integrating against all quasimomenta, we integrate over a subset of them, say over an affine plane. We write $\mathbf{k} = (\mathbf{k}_1, \mathbf{k}_2)$, where $\mathbf{k}_1 \in \mathbb{T}^k$, $\mathbf{k}_2 \in \mathbb{T}^l$ and define the partial density of states (PDS) as

$$N_p(\lambda; \mathbf{k}_1) = N_p(\lambda; \mathbf{k}_1; d, k) := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^l} N(\lambda, H(\mathbf{k}_1, \mathbf{k}_2))\,\mathrm{d}\mathbf{k}_2. \tag{2.1.12}$$

Our aim is to investigate the asymptotic behaviour of the PDS as $\lambda \to \infty$. Obviously, the regularity at infinity will be improving as l increases and so the larger l is, the more asymptotic terms we are likely to obtain. This asymptotic problem can be treated in two steps:

Step 1. Obtain the asymptotic behaviour of the PDS for unperturbed operator $H^0 := -\Delta$. More precisely, we want to obtain as good an estimate on

$$R^0(\lambda; \mathbf{k}_1; d, k) := N_p^0(\lambda; \mathbf{k}_1; d, k) - C_d \lambda^{d/2} \tag{2.1.13}$$

as possible (of course, superscript 0 refers to the fact that we are dealing with the case $V = 0$). A simple calculation shows that if $k = 0$, then $R^0(\lambda;\mathbf{k}_1;d,0) = 0$, so this step is trivial when dealing with the IDS. In the case of $k > 0$ this step becomes quite non-trivial and interesting. Once we have performed this step, we can move to the next step.

Step 2. Compute (or estimate) the difference

$$N_p(\lambda;\mathbf{k}_1;d,k) - N_p^0(\lambda;\mathbf{k}_1;d,k), \tag{2.1.14}$$

and try to obtain as many asymptotic terms of it as possible. It follows from a simple computation that

$$N_p^0(\lambda;\mathbf{k}_1;d,k) = (2\pi)^{-d} S\left(\sqrt{\lambda};\mathbf{k}_1;d,k\right), \tag{2.1.15}$$

hence the main aim of this paper deals with the first step of this programme; we intend to perform the second step in a separate publication.

2.1.3. Second spectral theoretic formulation

Consider the operator $\widetilde{H} = -\Delta + \widetilde{V}$ acting on $\mathbb{T}^l \times \mathbb{R}^k$ with a smooth potential $\widetilde{V}: \mathbb{T}^l \times \mathbb{R}^k \to \mathbb{R}$. We assume that, as a function on $\mathbb{R}^k$, $\widetilde{V}$ is periodic with the lattice of periods $(2\pi\mathbb{Z})^k$. Then, we have, from the definition of both the IDS and the PDS that

$$N\left(\lambda;\widetilde{H}\right) = \frac{1}{(2\pi)^k}\int_{\mathbb{T}^l} N(\lambda;H(\mathbf{k}_2))\,\mathrm{d}\mathbf{k}_2 \quad = (2\pi)^l N_p(\lambda;0;d,k), \tag{2.1.16}$$

that is to say that the integrated density of states equals the partial density of states up to a constant. If we consider a more general (but also less natural) operator $\widetilde{H}_{\mathbf{k}_1}$, the domain of which consists of functions on $\mathbb{T}^l \times \mathbb{R}^k$ which become periodic after multiplication by $e^{i\mathbf{k}_1\mathbf{x}_1}$, then the IDS of $\widetilde{H}_{\mathbf{k}_1}$ equals, again up to the same constant, $N_p(\lambda;\mathbf{k}_1;d,k)$. We would also like to mention that expression (2.1.3) appears in the study of integer points in anisotropically expanding domains. This has applications in the study of the asymptotic behaviour of the eigenvalue of the Laplace operator on the torus in the adiabatic limit, and was developed in [**47**].

2.1.4. Main results

Our first main result is as follows:

Theorem 2.1.1. *The error term $R\left(\rho;\mathbf{k}_1;d,k\right)$ satisfies the asymptotic estimates*

$$R\left(\rho;\mathbf{k}_1;d,k\right)=\begin{cases} O\left(\rho^{(d-1)/2}\right) & \text{if } k<(d+1)/2,\\ O\left(\rho^{(d-1)/2}\log\rho\right) & \text{if } k=(d+1)/2,\\ O\left(\rho^{d-\frac{2k}{1-d+2k}}\right) & \text{if } k>(d+1)/2\end{cases} \tag{2.1.17}$$

uniformly in $\mathbf{k}_1$.

Remark 2.1.2. Recall that $R\left(\rho;\mathbf{k}_1;d,0\right)=0$ for all values of $\rho,\mathbf{k}_1,d$.

We do not pretend that all of these estimates are optimal, but some of them are, as can be seen from the following result.

Theorem 2.1.3. *For $k>1$ and ρ sufficiently large, there exists a positive constant $C_{d,k}$ and $\mathbf{k}_1\in\mathbb{T}^k$ such that*

$$R\left(\rho;\mathbf{k}_1;d,k\right)\geq\begin{cases} C_{d,k}\rho^{\frac{d-1-\varepsilon}{2}} & \text{if } d\equiv 1\,(\mathrm{mod}\,4)\\ C_{d,k}\rho^{\frac{d-1}{2}} & \text{else,}\end{cases} \tag{2.1.18}$$

where $\varepsilon>0$ is arbitrary. When $d\not\equiv 1\,(\mathrm{mod}\,4)$, the lower bound $R\left(\rho;\mathbf{k}_1;d,k\right)\geq C_{d,k}\rho^{\frac{d-1}{2}}$ holds for $k=1$.

In particular, this theorem means that for $1\leq k<\frac{d+1}{2}$ and $d\not\equiv 1\,(\mathrm{mod}\,4)$, we cannot get improvements on the upper bounds found in Theorem 2.1.1. It also means that for $d\equiv 1\,(\mathrm{mod}\,4)$, $k\neq 1$, we cannot get improvements in the exponent.

Remark 2.1.4. It seems interesting that, after we have integrated $N(\lambda;H(\mathbf{k}))$ $(d-1)/2$ times, additional integrations do not improve the remainder estimate, until we perform the last (d-th) integration, which makes the remainder equal zero.

Open problem. *The results in* [**29**] *imply that for $k=d$, our upper bound is not optimal, but as $d\to\infty$, our upper bound converges to the optimal one, in the sense that $d-\left(d-\frac{2k}{1-d+2k}\right)\to 2$. Hence we may ask what is the optimal upper bound for $k\geq\frac{d+1}{2}$.*

2.1.5. Operators with constant magnetic field

Another type of problems we consider in this paper is the asymptotic behaviour of the density of states of the (Lev) Landau Hamiltonian (Schrödinger operator with constant magnetic field).

Let $D_j = -i\frac{\partial}{\partial x_j}$. Then we define the Landau Hamiltonian H_d as the operator acting in $\mathbb{R}^d$ whose action is given by:

$$H_d = (D_1 + x_2)^2 + D_2^2 + \cdots + D_d^2.$$

Of course, only operators H_2 and H_3 make real physical sense, but for the sake of completeness we will deal with all dimensions.

Let $\Omega^d(\rho)$ for $d \geq 2$ be the parabolic domain in $\mathbb{R}^d$ given by

$$\Omega^d(\rho) := \left\{(x_0, x) \in \mathbb{R}^d : 0 \leq x_0 \leq \rho - |x|^2\right\}. \tag{2.1.19}$$

Defining $P(\rho; d, k)$ analogously to $S(\rho; 0; d, k)$, that is,

$$P(\rho; d, k) = \mathrm{Vol}_l\left(\Omega^d(\rho) \cap A_k\right), \tag{2.1.20}$$

one can see that

$$P(\rho; d, k) = \sum_{j=0}^{\lfloor \rho \rfloor} S\left((\rho - j)^{1/2}; 0; d-1, k-1\right). \tag{2.1.21}$$

The IDS $N(\lambda; H_d)$ is related to $P(\rho; d, k)$ by the following proposition.

Proposition 2.1.5. *Let H_d be the d-dimensional Landau Hamiltonian. Then, its integrated density of states is given by*

$$N(\lambda; H_d) = 2^{\frac{-d}{2}} \pi^{1-d} P\left(\frac{\lambda - 1}{2}; d-1, 1\right) \tag{2.1.22}$$

for $\rho \geq 1$, and 0 *otherwise.*

We get an asymptotic expression for $P(\rho; d, k)$, via the next theorem. Defining $E_0(\rho) := E_0(\rho, d) = \frac{2}{d+1}\rho^{(d+1)/2} + \frac{1}{2}\rho^{(d-1)/2}$ and

$$E_n(\rho) := E_n(\rho, d) = E_0 + \sum_{k=1}^{n} \frac{B_{2k}}{(2k!)} \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d+3-4k}{2}\right)} \rho^{\frac{d+1-4k}{2}},$$

we obtain the following theorem.

Theorem 2.1.6. *As $\rho \to \infty$, $P(\rho; d, k)$ admits the asymptotic expansions:*

$$P(\rho; d, 1) = \omega_{d-1} E_{\left\lfloor \frac{d+1}{4} \right\rfloor}(\rho) + O(1), \tag{2.1.23}$$

$$P(\rho; d, d) = \frac{2\omega_{d-1}}{d+1} + O\left(\rho^{\frac{d^2-d+2}{2d}}\right). \tag{2.1.24}$$

If $k > \frac{d+2}{2}$, we have

$$P(\rho; d, k) = E_{\left\lfloor \frac{k-1}{4k-2} \right\rfloor}(\rho) + O\left(\rho^{\frac{1}{2}\left(d-1-\frac{2k-2}{2k-d}\right)}\right). \tag{2.1.25}$$

Finally, if $k \le \frac{d+2}{2}$,

$$P(\rho; d, k) = E_{\left\lfloor \frac{d-4}{8} \right\rfloor}(\rho) + O\left(\rho^{\frac{d+4}{4}} (\log \rho)^{\delta}\right), \tag{2.1.26}$$

where $\delta = 1$ *if* $k = \frac{d+2}{2}$ *and* 0 *otherwise.*

Replacing the result in Proposition 2.1.5 with the asymptotics in Theorem 2.1.6, we immediately deduce the following corollary.

Corollary 2.1.7. *The integrated density of states of the Landau Hamiltonian on* $\mathbb{R}^3$ *admits the asymptotic expansion*

$$N(\lambda; H_3) = \frac{1}{6\pi^2} \lambda^{3/2} + O(1)$$

for large enough λ.

The rest of the paper is organised as follows: in Section 2 we formulate several results which will be used in the proof of the main theorems, but we will postpone their proofs until Section 6. In Section 3 we prove the upper bounds in the Laplace case, and in Section 4 we obtain lower bounds. Finally, in Section 5 we deal with the magnetic case.

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2.2. Auxiliary results

In order to prove Theorem 2.1.1, it will be useful to give an alternate expression for $S(\rho; \mathbf{k}_1; d, k)$. Let us define the function $\chi : \mathbb{R}^k \to \mathbb{R}$ as

$$\chi(x) = \begin{cases} \left(1 - |x|^2\right)^{l/2} & \text{if } |x| < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2.1}$$

We can then observe that

$$S\left(\rho;0;d,k\right)=\omega_l\rho^l\sum_{n\in\mathbb{Z}^k}\chi\left(n/\rho\right). \tag{2.2.2}$$

We would like to use Poisson's summation formula

$$\sum_{n\in\mathbb{Z}^k}f(n)=\sum_{m\in\mathbb{Z}^k}\widehat{f}(m) \tag{2.2.3}$$

with $f=\chi$. This will allow us to get upper bounds for all $\mathbf{k}_1\in\mathbb{T}^k$, from the relation

$$\left[\mathscr{F}f(\cdot-\mathbf{k}_1)\right](\xi)=e^{-2\pi i\mathbf{k}_1\cdot\xi}\left[\mathscr{F}f\right](\xi), \tag{2.2.4}$$

where $\mathscr{F}$ is the Fourier transform operator. For the rest of this section, we therefore consider $\mathbf{k}_1=0$, and it will be seen in the proof of Lemma 2.2.2 that this assumption is made without loss of generality. In order for Equation (2.2.3) to hold we need to smooth out χ. To do so, we will consider its convolution with Friedrichs' mollifier Ψ_ε, defined in Section 2.6.1. Hence, setting $\chi_\varepsilon=\Psi_\varepsilon*\chi$ we get that

$$\widehat{\chi}_\varepsilon(\xi)=\widehat{\Psi}_\varepsilon(\xi)\,\widehat{\chi}(\xi). \tag{2.2.5}$$

Theorem 2.1.1 follows from two lemmas. The first one finds asymptotic upper and lower bounds for S.

Lemma 2.2.1. *Let χ_ε^+ and χ_ε^- be defined on $\mathbb{R}^k$ by*

$$\chi_\varepsilon^\pm(x)=\frac{1}{(1\mp\varepsilon)^l}\chi_\varepsilon((1\mp\varepsilon)x). \tag{2.2.6}$$

Then, we have that

$$\chi_\varepsilon^-(x)\leq\chi(x)\leq\chi_\varepsilon^+(x) \tag{2.2.7}$$

for all $x\in\mathbb{R}^k$. Immediately, if we define

$$S_\varepsilon^\pm\left(\rho\right)=\omega_l\sum_{n\in\mathbb{Z}^k}\chi_\varepsilon^\pm\left(n/\rho\right), \tag{2.2.8}$$

we get that

$$S_\varepsilon^-\left(\rho\right)\leq S\left(\rho\right)\leq S_\varepsilon^+\left(\rho\right). \tag{2.2.9}$$

The functions $\chi_\varepsilon^\pm$ are smooth, hence we can use Poisson's summation formula to compute the asymptotic expansion of $S_\varepsilon^\pm$. As the Fourier transform of a radial function, $\widehat{\chi}$ is radial as well and the second lemma therefore gives the asymptotic expansion of $\widehat{\chi}(|\xi|)$.

Lemma 2.2.2. *The Fourier transform of* χ *satisfies*

$$\widehat{\chi}(\xi) = \frac{C}{|\xi|^{(d+1)/2}} \cos\left(2\pi|\xi| - \frac{(d+1)\pi}{4}\right) + O\left(|\xi|^{-(d+3)/2}\right) \tag{2.2.10}$$

for some $C > 0$ *as* $|\xi| \to \infty$. *Furthermore, its derivative satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}|\xi|}\widehat{\chi}(\xi) = \frac{\widetilde{C}}{|\xi|^{(d+1)/2}} \sin\left(2\pi|\xi| - \frac{d\pi}{4}\right) + O\left(|\xi|^{-(d+3)/2}\right) \tag{2.2.11}$$

In particular, the asymptotic behaviour of both $\widehat{\chi}(\xi)$ *and its derivative does not depend on the co-dimension* k.

We will postpone the proof of these lemmas until Section 2.6.

2.3. Proof of Theorem 2.1.1

In this section, we prove Theorem 2.1.1 using both Lemmas 2.2.2 and 2.2.1. We have that

$$S_\varepsilon^-(\rho) \le S(\rho) \le S_\varepsilon^+(\rho). \tag{2.3.1}$$

Let us therefore find asymptotic expansions on $S_\varepsilon^\pm$. We shall split those computations in two cases : whether $k \ge (d+1)/2$ or $k < (d+1)/2$

2.3.1. Case 1

Here, we assume that $k \ge (d+1)/2$. Let us find asymptotic expansions on $S_\varepsilon^\pm$. Since χ_ε is a smooth compactly supported function of x, we may use Poisson's summation formula (2.2.3) to obtain

$$S_\varepsilon^\pm = \omega_d \rho^l \sum_{n\in\mathbb{Z}^k} \chi_\varepsilon^\pm(n/\rho) = \omega_d \rho^d \sum_{m\in\mathbb{Z}^k} \widehat{\chi}_\varepsilon^\pm(\rho m). \tag{2.3.2}$$

Since we have that

$$\widehat{\chi}_\varepsilon^\pm(m\rho) = \frac{1}{(1\mp\varepsilon)^d} \widehat{\Psi}(\varepsilon m\rho)\, \widehat{\chi}\left(\frac{m\rho}{1\mp\varepsilon}\right), \tag{2.3.3}$$

we get, assuming $\varepsilon \ll 1/\rho$, that

$$S_\varepsilon^\pm = \omega_d \sum_{m\in\mathbb{Z}^k} (1+O(\varepsilon))\, \rho^d \widehat{\Psi}(\varepsilon m\rho)\, \widehat{\chi}(m\rho) + O\left(\sum_{m\in\mathbb{Z}^k} \varepsilon m \rho^{d+1} \Psi(\varepsilon m\rho) \left|\widehat{\chi}'(m\rho)\right|\right), \tag{2.3.4}$$

which directly implies

$$S_\varepsilon^\pm = \omega_d \rho^d + O\left(\varepsilon\rho^d\right) + O\left(\sum_{\substack{m\in\mathbb{Z}^k\\ |m|\neq 0}} \rho^d \widehat{\Psi}\left(\varepsilon m\rho\right)\left|\widehat{\chi}\left(m\rho\right)\right|\right) + O\left(\sum_{m\in\mathbb{Z}^k} \varepsilon m \rho^{d+1} \Psi\left(\varepsilon m\rho\right)\left|\widehat{\chi}'\left(m\rho\right)\right|\right). \tag{2.3.5}$$

Observe that $\widehat{\Psi}(\xi) = O\left(|\xi|^q\right)$ for any $q \in \mathbb{R}$ whenever $|\xi| > 1$ and bounded for $|\xi| \leq 1$. Recall from Lemma 2.2.2 that $\widehat{\chi}(\xi) = O\left(|\xi|^{-(d+1)/2}\right)$. Hence, choosing $q = \frac{d-2k-1}{2}$, the third summand in (2.3.5) can be split into two terms, becoming

$$O\left(\rho^{(d-1)/2}\left[\sum_{\substack{m\in\mathbb{Z}^k\\ 1\leq|m|\leq 1/\varepsilon\rho}} \frac{1}{|m|^{(d+1)/2}} + \sum_{\substack{m\in\mathbb{Z}^k\\ |m|>1/\varepsilon\rho}} \frac{1}{(\varepsilon\rho)^{(2k+1-d)/2}|m|^{k+1}}\right]\right). \tag{2.3.6}$$

The first sum can be estimated by

$$\begin{aligned}\sum_{\substack{m\in\mathbb{Z}^k\\ 1\leq|m|\leq 1/\varepsilon\rho}} \frac{1}{|m|^{(d+1)/2}} &\sim \int_1^{1/\varepsilon\rho} \frac{r^{k-1}}{r^{(d+1)/2}}\,\mathrm{d}r \\ &= \begin{cases} O\left((\varepsilon\rho)^{\frac{d+1-2k}{2}}\right) & \text{if } k > (d+1)/2, \\ O\left(\log\varepsilon\rho\right) & \text{if } k = (d+1)/2. \end{cases}\end{aligned} \tag{2.3.7}$$

The second sum can be estimated by

$$\begin{aligned}&\sum_{\substack{m\in\mathbb{Z}^k\\ |m|\geq 1/\varepsilon\rho}} \frac{1}{(\varepsilon\rho)^{(2k-d+1)/2}|m|^{k+1}} \\ &\sim \int_{1/\varepsilon\rho}^{\infty} \frac{1}{(\varepsilon\rho)^{(2k-d+1)/2}} \frac{r^{k-1}}{r^{k+1}}\,\mathrm{d}r = O\left((\varepsilon\rho)^{\frac{d+1-2k}{2}}\right).\end{aligned} \tag{2.3.8}$$

As for the last summand, it is easy to see with the same computations and using $\widehat{\Psi}(\xi) = O\left(|\xi|^{\frac{d-2k-3}{2}}\right)$ that the extra power of $\varepsilon\rho$ exactly compensates the extra power of m, and we have that the asymptotic behavior in $\varepsilon\rho$ is the same for all for summands whenever $k > (d+1)/2$. Furthermore, when equality holds, the polynomial component is the same. Therefore, we have to choose $\varepsilon = \rho^{-j}$ such that

$$\varepsilon\rho^d = \rho^{(d-1)/2}\left(\varepsilon\rho\right)^{\frac{d+1-2k}{2}}. \tag{2.3.9}$$

This is achieved exactly when

$$j = \frac{2k}{1-d+2k}. \tag{2.3.10}$$

This gives us the announced asymptotic estimates when $k \geq (d+1)/2$, that is

$$S(\rho) = \begin{cases} \omega_d \rho^d + O\left(\rho^{d-\frac{2k}{1-d+2k}}\right) & \text{if } k > (d+1)/2, \\ \omega_d \rho^d + O\left(\rho^{\frac{d-1}{2}} \log \rho\right) & \text{if } k = (d+1)/2. \end{cases} \tag{2.3.11}$$

2.3.2. Case 2

We now assume that $k < (d+1)/2$. In this case, the first series in equation (2.3.5) converges without any decay from $\widehat{\Psi}$. Hence, the asymptotic expansion for $S_\varepsilon^\pm$ simplifies to

$$S_\varepsilon^\pm = \omega_d \rho^d + O\left(\varepsilon \rho^d\right) + O\left(\rho^{(d-1)/2} \sum_{\substack{m \in \mathbb{Z}^k \\ |m| \neq 0}} \frac{1}{|m|^{(d+1)/2}}\right) + O\left(\rho^{(d-1)/2} \sum_{\substack{m \in \mathbb{Z}^k \\ m \neq 0}} \frac{\varepsilon \rho \widehat{\Psi}(\varepsilon m \rho)}{|m|^{(d-1)/2}}\right). \tag{2.3.12}$$

If $k < \frac{d-1}{2}$, the last series in that previous display also converges. In that case, choosing $\varepsilon = \rho^{-(d+1)/2}$ satisfies Theorem 2.1.1, and choosing ε smaller does not improve the estimate. If $k = \frac{d}{2}$ or $k = \frac{d-1}{2}$, using $\widehat{\Psi}(\xi) = O\left(|\xi|^{-1}\right)$ for $m > (\varepsilon\rho)^{-1}$ yields the same result, finishing the proof.

Note that Equation (2.2.4) ensures that these estimates hold for all $\mathbf{k}_1 \in \mathbb{T}^k$.

2.4. Lower bounds

Let us first follow the argument given in [**21**] for $d = k = 2$. The beginning of the argument is the same, which we add for completeness. Since $R(\rho; \mathbf{k}_1)$ is periodic in $\mathbf{k}_1$ with respect to Γ, we can compute its Fourier coefficients, obtaining

$$\begin{aligned} \int_{\mathbb{T}^k} R(\rho; \mathbf{k}_1) e^{-2\pi i \mathbf{k}_1 \cdot \gamma} \, \mathrm{d}\mathbf{k}_1 &= \int_{\mathbb{T}^k} \left(-\omega_d \rho^d + \rho^l \sum_{\gamma \in \Gamma} \chi\left(\frac{\gamma - \mathbf{k}_1}{\rho}\right) e^{-2\pi i \mathbf{k}_1 \cdot \gamma}\right) \mathrm{d}\mathbf{k}_1 \\ &= \int_{\mathbb{R}^k} \rho^l \chi\left(\frac{\mathbf{k}_1}{\rho}\right) e^{-2\pi i \mathbf{k}_1 \cdot \gamma} \, \mathrm{d}\mathbf{k}_1 \\ &= \rho^d \left[\frac{C}{(\rho|\gamma|)^{(d+1)/2}} \cos\left(2\pi\rho|\gamma| - \frac{(d+1)\pi}{4}\right) + O\left(|\rho\gamma|^{-(d+3)/2}\right)\right], \end{aligned} \tag{2.4.1}$$

from Lemma 2.2.2. Additionally, we have that

$$\int_{\mathbb{T}^k} R(\rho; \mathbf{k}_1) \, \mathrm{d}\mathbf{k}_1 = 0. \tag{2.4.2}$$

Hence, for all $\gamma \in \Gamma \setminus \{0\}$, we have that

$$\begin{aligned}
&\int_{\mathbb{T}^k} |R(\rho;\mathbf{k}_1)|\,\mathrm{d}\mathbf{k}_1 \\
&\geq \max\left(\left|\int_{\mathbb{T}^k} R(\rho;\mathbf{k}_1)\, e^{-2\pi i \mathbf{k}_1\cdot\gamma}\,\mathrm{d}\mathbf{k}_1\right|, \left|\int_{\mathbb{T}^k} R(\rho;\mathbf{k}_1)\, e^{-4\pi i \mathbf{k}_1\cdot\gamma}\,\mathrm{d}\mathbf{k}_1\right|\right) \\
&\geq C\frac{\rho^{\frac{d-1}{2}}}{\gamma^{\frac{d+1}{2}}}\max\left(\left|\cos\left(2\pi\rho|\gamma| - \frac{(d+1)\pi}{4}\right)\right|, \frac{1}{2^{\frac{d+1}{2}}}\left|\cos\left(4\pi\rho|\gamma| - \frac{(d+1)\pi}{4}\right)\right|\right) \qquad (2.4.3) \\
&\quad - c\frac{\rho^{\frac{d-1}{2}}}{\gamma^{\frac{d-1}{2}}}
\end{aligned}$$

for C,c positive constants whose value can change throughout. Whenever $d \not\equiv 1 \,(\mathrm{mod}\, 4)$, we have that

$$0 < \inf_{x\in\mathbb{R}} \max\left(\left|\cos\left(x - \frac{(d+1)\pi}{4}\right)\right|, \left|\cos\left(2x - \frac{(d+1)\pi}{4}\right)\right|\right), \qquad (2.4.4)$$

hence in that case, fixing $\gamma \in \Gamma$, we conclude that there exists ρ^* such that for all $\rho \geq \rho^*$

$$\int_{\mathbb{T}^k} |R(\rho;\mathbf{k}_1)|\,\mathrm{d}\mathbf{k}_1 \geq C\rho^{\frac{d-1}{2}}. \qquad (2.4.5)$$

We conclude that whenever $d \not\equiv 1 \,(\mathrm{mod}\, 4)$,

$$\sup_{\mathbf{k}_1\in\mathbb{T}^k} R(\rho;\mathbf{k}_1) \geq C\rho^{\frac{d-1}{2}}. \qquad (2.4.6)$$

The remaining case, that is when $d \equiv 1 \,(\mathrm{mod}\, 4)$ is more subtle. We use results found in [**61**][Theorem 3.1, Lemma 3.3]. Indeed, from Equation (2.4.3), we have

$$\int_{\mathbb{T}^k} |R(\rho;\mathbf{k}_1)|\,\mathrm{d}\mathbf{k}_1 \geq C\frac{\rho^{\frac{d-1}{2}}}{\gamma^{\frac{d+1}{2}}}\left|\cos\left(2\pi\rho|\gamma| - \frac{\pi}{2}\right)\right| - c\frac{\rho^{\frac{d-1}{2}}}{\gamma^{\frac{d-1}{2}}}. \qquad (2.4.7)$$

From Lemma 3.3 in [**61**], we know that, if $k \geq 2$, for all $\varepsilon > 0$, there exists $\rho_0 > 0$ and $\alpha \in (0,1/2)$ such that for all $\rho > \rho_0$ there exists $\gamma \in \Gamma$ such that $|\gamma| < (2\pi\rho)^{\varepsilon}$ and the distance from $2\rho\left|\gamma\right|$ to an integer is greater than α. Choosing such a γ bounds $\cos\left(2\pi\rho|\gamma| - \pi/2\right)$ away from 0, and we get that

$$\int_{\mathbb{T}^k} |R(\rho;\mathbf{k}_1)|\,\mathrm{d}\mathbf{k}_1 \geq C\rho^{\frac{d-2}{2} - \varepsilon\frac{d+1}{2}}. \qquad (2.4.8)$$

Since $\varepsilon > 0$ is arbitrary, we get the desired result.

2.5. An application to the Landau Hamiltonian

2.5.1. The Landau Hamiltonian

Decomposing $H_d = H_2 \oplus D_{d-2}$, we can first study the problem

$$H_2 u = \lambda u.$$

Consider the definition (2.1.6) for $N(\lambda; H_d)$, with periodic boundary conditions for x_1 and Dirichlet boundary conditions for $x = (x_2, \dots, x_d)$.

For H_2, we can write the solutions as $u(x_1, x_2) = e^{\frac{2\pi i n}{L} x_1} f(x_2)$, which reduces the problem to solving the eigenvalue problem

$$\left((\xi_1 + x_2)^2 + D_2^2\right) f(x_2) = \lambda f(x_2).$$

This is a shifted quantum harmonic oscillator. We have that $\sigma(H_2) = \{2j+1 : j \in \mathbb{N}\}$, each with infinite multiplicity. It is a standard computation, see e.g. [**58**], that

$$N(\lambda; H_2) = \frac{1}{2\pi} \left\lfloor \frac{\lambda - 1}{2} \right\rfloor, \tag{2.5.1}$$

for $\lambda \geq 1$, and 0 otherwise. Extending the methods of [**58**] to higher dimensions, it is again a simple computation to show that for $\lambda \geq 1$,

$$N(\lambda; H_d) = \frac{\omega_{d-2}}{(2\pi)^{d-1}} \sum_{n=0}^{\left\lfloor \frac{\lambda-1}{2} \right\rfloor} (\lambda - 2n - 1)^{(d-2)/2}. \tag{2.5.2}$$

Thus, from the definition of $P(\rho; d, k)$, we have indeed that

$$N(\lambda; H_d) = 2^{\frac{-d}{2}} \pi^{1-d} P\left(\frac{\lambda - 1}{2}; d-1, 1\right). \tag{2.5.3}$$

2.5.2. Computations for general paraboloids

In this section we prove Theorem 2.1.6. Consider the expression

$$P(\rho; d, k) = \sum_{j=1}^{\lfloor \rho \rfloor} S\left((\rho - j)^{1/2}; 0; d-1, k-1\right). \tag{2.5.4}$$

By Theorem 2.1.1, we have

$$\sum_{j=0}^{\lfloor \rho \rfloor} S\left((\rho - j)^{1/2}; 0; d-1, k-1\right) = \sum_{j=0}^{\lfloor \rho \rfloor} \left(\omega_{d-1} (\rho - j)^{(d-1)/2} + O(X(\rho))\right),$$

where

$$X\left(\rho\right)=\begin{cases}\rho^{\frac{1}{2}\left(d-1-\frac{2k-2}{2k-d}\right)} & \text{if } k>\left(d+2\right)/2,\\ \rho^{(d-2)/4}\log\rho & \text{if } k=\left(d+2\right)/2,\\ \rho^{(d-2)/4} & \text{if } 1<k<\left(d+2\right)/2,\\ 0 & \text{if } k=1.\end{cases} \tag{2.5.5}$$

Comparing with the integral, we get that for all X as defined above,

$$\sum_{j=0}^{\lfloor\rho\rfloor} X\left(\rho\right)=O\left(\rho X\left(\rho\right)\right). \tag{2.5.6}$$

For any d, we can use the Euler-Maclaurin formula :

$$\begin{aligned}\sum_{n=a}^{b} f\left(n\right) &= \int_a^b f\left(x\right)\mathrm{d}x+\frac{f\left(a\right)+f\left(b\right)}{2}\\ &+\sum_{k=1}^{p}\frac{B_{2k}}{(2k)!}\left(\left.\frac{\mathrm{d}^{2k-1}f}{\mathrm{d}x}\right|_{x=b}-\left.\frac{\mathrm{d}^{2k-1}f}{\mathrm{d}x}\right|_{x=a}\right)+O\left(\int_a^b\left|\left.\frac{\mathrm{d}^{2p}f}{\mathrm{d}x^{2p}}\right|_{x=t}\mathrm{d}t\right|\right),\end{aligned} \tag{2.5.7}$$

for any integer $p\geq 1$, where B_k is the kth Bernoulli number. Note that for any integer a,

$$\sum_{j=0}^{a}\left(a-j\right)^{(d-1)/2}=\sum_{j=0}^{a} j^{(d-1)/2}. \tag{2.5.8}$$

Hence, by the Euler-Maclaurin formula, we get that

$$\begin{aligned}&\sum_{j=0}^{a}\left(a-j\right)^{(d-1)/2}\\ &=\int_0^a t^{(d-1)/2}\,\mathrm{d}t+\frac{a^{(d-1)/2}}{2}+\sum_{k\leq\frac{d+1}{4}}\frac{B_{2k}}{(2k!)}\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d+3-4k}{2}\right)}a^{\frac{d+1-4k}{2}}+O\left(a^{-1/2}\right).\end{aligned} \tag{2.5.9}$$

Obviously, when d is odd, this last sum is actually finite and the error term vanishes.

When ρ is not an integer, we write $\rho = a + \tau$, where τ is the fractional part. In that case, using the Euler-Maclaurin formula again, we get

$$\begin{aligned}
\sum_{j=0}^{a} \left(a+\tau-j\right)^{(d-1)/2} &= \sum_{j=0}^{a} \left(j+\tau\right)^{(d-1)/2} \\
&= \int_0^a (t+\tau)^{(d-1)/2}\, \mathrm{d}t + \frac{1}{2}\left(\tau^{(d-1)/2} + \rho^{(d-1)/2}\right) \\
&\quad + \sum_{k \le \frac{d+1}{4}} \frac{B_{2k}}{(2k!)} \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d+3-4k}{2}\right)} \left(\rho^{\frac{d+1-4k}{2}} - \tau^{\frac{d+1-4k}{2}}\right) + O(\tau) \\
&= \frac{2}{d+1}\left(\rho^{(d+1)/2} - \tau^{(d+1)/2}\right) + \frac{1}{2}\left(\tau^{(d-1)/2} + \rho^{(d-1)/2}\right) \\
&\quad + \sum_{k \le \frac{d+1}{4}} \frac{B_{2k}}{(2k!)} \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d+3-4k}{2}\right)} \left(\rho^{\frac{d+1-4k}{2}} - \tau^{\frac{d+1-4k}{2}}\right) + O(\tau)\,.
\end{aligned}$$

Let us observe that

$$\begin{aligned}
&\lim_{\rho\to\infty} \frac{\frac{-2}{d+1}\rho^{(d+1)/2} + \sum_{j=0}^{a}\left(\rho-j\right)^{(d-1)/2}}{\frac{1}{2}\rho^{(d-1)/2}} \\
&= \lim_{\rho\to\infty} \frac{-\frac{4}{d+1}\tau^{(d+1)/2} + \tau^{(d-1)/2} + \rho^{(d-1)/2} + O\left(\rho^{(d-3)/2}\right)}{\rho^{\frac{d-1}{2}}} \\
&= 1.
\end{aligned}$$

This is because $\tau = O(1)$. Similarly, if we define $E_0 = \frac{2}{d+1}\rho^{(d+1)/2} + \frac{1}{2}\rho^{(d-1)/2}$ and

$$E_n = E_0 + \sum_{k=1}^{n} \frac{B_{2k}}{(2k!)} \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d+3-4k}{2}\right)} \rho^{\frac{d+1-4k}{2}},$$

we get that

$$\lim_{\rho\to\infty} \frac{-E_n + \sum_{j=0}^{a}\left(\rho-j\right)^{(d-1)/2}}{\rho^{(d-1)/2-2n-1}} = \frac{B_{2(n+1)}}{(2\,(n+1))!}\left(\frac{d-1}{2}\right)_{2n+1}$$

whenever $(d-1)/2 - 2n - 1 > 0$, after which point the contribution of the fractional remainder τ gets more important than the denominator. Hence, we obtain the asymptotic expansion

$$\begin{aligned}
\sum_{j=0}^{\lfloor\rho\rfloor} \left(\rho-j\right)^{(d-1)/2} &= \frac{2}{d+1}\rho^{(d+1)/2} + \frac{1}{2}\rho^{(d-1)/2} \\
&\quad + \sum_{k \le \frac{d-3}{4}} \frac{B_{2k}}{(2k!)} \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d+3-4k}{2}\right)} \rho^{\frac{d+1-4k}{2}} + O(\tau)\,.
\end{aligned} \tag{2.5.10}$$

When $k=1$, we already have that $X(\rho)=0$. Therefore, we have that

$$\frac{P(\rho,d,1)}{\omega_{d-1}} = \frac{2}{d+1}\rho^{(d+1)/2} + \frac{1}{2}\rho^{(d-1)/2} + \sum_{1\le k<\frac{d-3}{4}} \frac{B_{2k}}{(2k!)} \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d+3-4k}{2}\right)} \rho^{\frac{d+1-4k}{2}} + O(\tau),$$

from which we recover a (quite sharp) asymptotic integrated density of states for the magnetic Hamiltonian H_{d+1}.

Let us combine equations (2.5.5) and (2.5.10). When $k=d$, we get that the error term from X is greater than $\frac{d-1}{2}$, and as such,

$$P(\rho;d,d) = \frac{2\omega_{d-1}}{d+1} + O\left(\rho^{\frac{d^2-d+2}{2d}}\right).$$

When $k>\frac{d+2}{2}$, we get that

$$\frac{P(\rho;d,k)}{\omega_{d-1}} = \frac{2}{d+1}\rho^{\frac{d+1}{2}} + \frac{1}{2}\rho^{\frac{d-1}{2}} + \sum_{1\le j<\frac{k-1}{4k-2}} \frac{B_{2j}}{(2j)!} \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d+3-4j}{2}\right)} \rho^{\frac{d+1-4j}{2}} + O\left(\rho^{\frac{1}{2}\left(d-1-\frac{2k-2}{2k-d}\right)}\right).$$

Finally, when $k\le\frac{d+2}{2}$, we get that

$$\frac{P(\rho;d,k)}{\omega_{d-1}} = \frac{2\omega_{d-1}}{d+1}\rho^{\frac{d+1}{2}} + \frac{1}{2}\rho^{\frac{d-1}{2}} + \sum_{1\le j<\frac{d+4}{8}} \frac{B_{2j}}{(2j)!} \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d+3-4j}{2}\right)} \rho^{\frac{d+1-4j}{2}} + O\left(\rho^{\frac{d+4}{4}}(\log\rho)^{\delta}\right),$$

where $\delta=1$ if $k=\frac{d+2}{2}$ and 0 otherwise.

2.6. Proofs of auxiliary results

2.6.1. Smoothing of the cut-off function

Let us define a smooth, even bump function ψ in $C_c^\infty(\mathbb{R})$, supported in $[-1,1]$, such that the integral

$$\int_0^\infty \psi(r)\, r^{k-1}\,\mathrm{d}r = \frac{1}{V_{k-1}}, \tag{2.6.1}$$

where V_{k-1} is the area of the unit sphere in $\mathbb{R}^k$.

Using this function, we can define the radial bump function Ψ_ε on $\mathbb{R}^k$, of total mass 1 to be given by

$$\Psi_\varepsilon(x) = \frac{1}{\varepsilon^k}\psi\left(\frac{|x|}{\varepsilon}\right). \tag{2.6.2}$$

Let $\Psi := \Psi_1$ and $\chi_\varepsilon(x) = \Psi_\varepsilon(x) * \chi(x)$. Its Fourier transform is given by

$$\widehat{\chi_\varepsilon}(\xi) = \widehat{\Psi}(\varepsilon\xi)\,\widehat{\chi}(\xi). \tag{2.6.3}$$

Let χ_ε^+ and χ_ε^- be defined on $\mathbb{R}^k$ by

$$\chi_\varepsilon^\pm(x) = \frac{1}{(1\mp\varepsilon)^l}\chi_\varepsilon((1\mp\varepsilon)x). \tag{2.6.4}$$

We can now proceed with the proof of Lemma 2.2.1.

Proof. To show that $\chi_\varepsilon^-(x) \le \chi(x) \le \chi_\varepsilon^+$, the idea is to obtain $\chi_\varepsilon^\pm(x)$ by averaging $\chi(x)$ on a ball of radius $0<\varepsilon<x$ about each x. To do so, first notice that

$$\begin{aligned}\chi_\varepsilon(x) &\le \sup_{|t|\le\varepsilon}\big(\chi(x-t)\big)\int_{\mathbb{R}^k}\Psi_\varepsilon(x)\,\mathrm{d}x\\ &= \begin{cases}1 & \text{if } |x|\le\varepsilon,\\ \big(1-(|x|-\varepsilon)^2\big)^{\frac{l}{2}} & \text{if } \varepsilon\le|x|\le 1+\varepsilon.\end{cases}\end{aligned} \tag{2.6.5}$$

If we show that

$$\chi_\varepsilon(x) \le (1+\varepsilon)^l\,\chi\left(\frac{x}{1+\varepsilon}\right), \tag{2.6.6}$$

we get the desired lower bound. Indeed, taking $y=\frac{x}{1+\varepsilon}$ in the preceding equation yields

$$\chi(y) \ge \frac{1}{(1+\varepsilon)^l}\chi_\varepsilon\big((1+\varepsilon)y\big) = \chi_\varepsilon^-(y). \tag{2.6.7}$$

Therefore, it only remains to show that (2.6.6) holds for all $x\in\mathbb{R}^k$. First note that if $|x|\ge 1+\varepsilon$, both sides are 0. We shall split the remaining cases in $|x|\le\varepsilon$ and $\varepsilon<|x|<1+\varepsilon$.

Restricting ourselves to the first case, if $|x|=\varepsilon$, we get that

$$\begin{aligned}(1+\varepsilon)^l\,\chi\left(\frac{x}{1+\varepsilon}\right) &= (1+\varepsilon)^l\left(1-\frac{\varepsilon^2}{(1+\varepsilon)^2}\right)^{\frac{l}{2}}\\ &= (1+2\varepsilon)^{\frac{l}{2}}\\ &\ge 1\\ &\ge \chi_\varepsilon^-(x)\end{aligned} \tag{2.6.8}$$

Since $\chi\left(\frac{x}{1+\varepsilon}\right)$ is a decreasing function of $|x|$, we conclude that (2.6.6) holds for $0 \le |x| \le \varepsilon$.

In the case where $\varepsilon < |x| \le 1+\varepsilon$, we need to show that

$$\left(1-(|x|-\varepsilon)^2\right)^{\frac{l}{2}} \le (1+\varepsilon)^l \left(1-\frac{|x|^2}{(1+\varepsilon)^2}\right)^{\frac{l}{2}}. \tag{2.6.9}$$

It is equivalent to show that $1-(|x|-\varepsilon)^2 \le (1+\varepsilon)^2 - |x|^2$. This is the case if

$$\begin{aligned} 1-|x|^2+2|x|\varepsilon-\varepsilon^2 &\le 1+2\varepsilon+\varepsilon^2-|x|^2 \\ \Leftrightarrow 2|x|\varepsilon &\le 2\varepsilon(1+\varepsilon) \\ \Leftrightarrow |x| &\le 1+\varepsilon. \end{aligned} \tag{2.6.10}$$

Since the last line is true by hypothesis, we can conclude that the left-hand side inequality of (2.2.7) is true.

In order to get an upper bound on $\chi(x)$, we proceed in a similar fashion, averaging $\chi_\varepsilon(x)$ on a ball of radius ε around x, which yields

$$\begin{aligned} \chi_\varepsilon(x) &\ge \inf_{|t|<\varepsilon} \chi(x-t) \\ &\ge \begin{cases} \left(1-(|x|+\varepsilon)^2\right)^{\frac{l}{2}} & \text{if } |x|<1-\varepsilon, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{2.6.11}$$

As we did before, it suffices to show that

$$\chi_\varepsilon(x) \ge (1-\varepsilon)^l \chi\left(\frac{x}{1-\varepsilon}\right). \tag{2.6.12}$$

Notice that the left hand side of that equation is 0 whenever $|x| \ge 1-\varepsilon$. Like before, we see that

$$\left(1-(|x|+\varepsilon)^2\right)^{\frac{l}{2}} \ge (1-\varepsilon)^l \left[1-\left(\frac{|x|}{1-\varepsilon}\right)^2\right]^{\frac{l}{2}} \tag{2.6.13}$$

is equivalent to $|x| < 1-\varepsilon$. This concludes the proof. □

2.6.2. Fourier transform of χ

PROOF. Let us compute $\widehat{\chi}(\xi)$. We will split the cases $k=1$, $k=2$, and $k>2$. If $k=1$, then

$$\begin{aligned}\widehat{\chi}(\xi) &= \int_{-1}^{1}\left(1-x^2\right)^{(d-1)/2} e^{-i2\pi x\xi}\,\mathrm{d}x \\ &= \frac{C}{|\xi|^{d/2}} J_{d/2}(2\pi|\xi|) \\ &= \frac{C}{|\xi|^{(d+1)/2}} \cos\left(2\pi|\xi| - \frac{(d+1)\pi}{4}\right) + O\left(|\xi|^{(d+3)/2}\right),\end{aligned} \tag{2.6.14}$$

using [**30**][Eq.3.387 and 8.451], which is the desired result.

We also obtain that, following [**30**][Eq. 3.621]

$$\widehat{\chi}(0) = 2^d B\left(\frac{d+1}{2}, \frac{d+1}{2}\right). \tag{2.6.15}$$

Using identities of the Gamma function, we get that

$$\omega_l 2^d B\left(\frac{d+1}{2}, \frac{d+1}{2}\right) = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} = \omega_d, \tag{2.6.16}$$

which is the desired value.

If $k=2$, then the Fourier transform is given by

$$\widehat{\chi}(\xi) = \int_{\mathbb{R}^2} \chi(x) e^{-i2\pi x\cdot\xi}\,\mathrm{d}x. \tag{2.6.17}$$

Working in polar coordinates, we get that

$$\begin{aligned}\widehat{\chi}(\xi) &= \int_0^1 \int_0^{2\pi} r\left(1-r^2\right)^{(d-2)/2} e^{-i2\pi r|\xi|\cos\theta}\,\mathrm{d}\theta\,\mathrm{d}r \\ &= \int_0^1 r\left(1-r^2\right)^{(d-2)/2} J_0(2\pi|\xi|r)\,\mathrm{d}r \\ &= \frac{C}{|\xi|^{d/2}} J_{d/2}(2\pi|\xi|) \\ &= \frac{C}{|\xi|^{(d+1)/2}} \cos\left(2\pi|\xi| - \frac{(d+1)\pi}{4}\right) + O\left(|\xi|^{(d+3)/2}\right),\end{aligned} \tag{2.6.18}$$

which is the desired result. [**30**][Eq. 8.411, 6.567 and 8.451] were used respectively for an integral formula for the Bessel function, its integral, and its asymptotic expansion.

We also obtain that

$$\widehat{\chi}(0) = \frac{2\pi}{d}. \tag{2.6.19}$$

Using identities of the Gamma function, we get that

$$\omega_l \frac{2\pi}{d} = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} = \omega_d, \tag{2.6.20}$$

which is the desired value. Finally, if $k > 2$, then, working in spherical coordinates, we get that the Fourier transform of χ is, for some constant C,

$$\begin{aligned}
\widehat{\chi}(\xi) &= C\int_0^1\int_0^\pi r^{k-1}\left(1-r^2\right)^{l/2}\sin^{k-2}\theta\, e^{-i2\pi r|\xi|\cos\theta}\,\mathrm{d}\theta\,\mathrm{d}r \\
&= \frac{C}{|\xi|^{(k-2)/2}}\int_0^1 r^{k/2}\left(1-r^2\right)^{l/2} J_{(k-2)/2}\left(2\pi|\xi|r\right)\mathrm{d}r \\
&= \frac{C}{|\xi|^{(k-2)/2}}\frac{1}{|\xi|^{(l+2)/2}} J_{d/2}\left(2\pi|\xi|\right) \\
&= \frac{C}{|\xi|^{(d+1)/2}}\cos\left(2\pi|\xi| - \frac{(d+1)\pi}{4}\right) + O\left(|\xi|^{(d+3)/2}\right).
\end{aligned} \tag{2.6.21}$$

using [**30**][Eq. 8.411] in the first line, which is the desired result.

Additionnally, we have that

$$\begin{aligned}
\widehat{\chi}(0) &= \mathrm{Vol}\left(S^{k-1}\right)\int_0^1 r^{k-1}\left(1-r^2\right)^{(d-k)/2}\mathrm{d}r \\
&= \frac{\pi^{k/2}B\left(\frac{k}{2},\frac{d-k+2}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}.
\end{aligned} \tag{2.6.22}$$

Using identities of the Gamma function, we get that

$$\widehat{\chi}(0)\,\omega_{d-k} = \omega_d \tag{2.6.23}$$

which is once again the desired value.

One can note that in each of those cases, we ignored the trigonometric term to get an upper bound, considering it to be 1. Hence, since translation by $\mathbf{k}_1$ is simply multiplication by a complex exponential in Equation (2.2.3), it can be ignored in just the same fashion.

Finally, we get the result for the derivative using the identity $J'_\nu = \frac{1}{2}\left(J_{\nu-1} - J_{\nu+1}\right)$ and basic trigonometric identities. This completes the proof of Lemma 2.2.2. □

useful

Chapitre 3

The Steklov spectrum of cuboids

par

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The Steklov spectrum of cuboids

RÉSUMÉ. Dans cet article, nous étudions le problème aux valeurs propres de Steklov sur des cuboïdes de n'importe quelle dimension. Nous prouvons une formule asymptotique à deux termes pour la fonction de compte des valeurs propres de Steklov d'un cuboïde en dimension $d \geq 3$. En plus du terme de Weyl standard, nous calculons explicitement le second terme dans l'asymptotique, capturant ainsi la contribution des facettes de dimension $d-2$. Notre approche est basée sur des méthodes utilisées pour le compte des points d'un réseau dans un domaine convexe. Bien que cette stratégie soit similaire à celle utilisée pour les valeurs propres du laplacien de Dirichlet, le cas du problème de Steklov comporte des complications additionnelles. En particulier, il n'est pas clair *a priori* que l'ensemble des fonctions propres admettant une séparation des variables forment une base complète. Nous démontrons cette complétude grâce à une famille de problèmes de Robin auxilliaires. De plus, la correspondance entre les valeurs propres de Steklov et les points d'un réseau n'est pas exacte, il nous font donc procéder à une analyse plus délicate pour obtenir les asymptotiques spectrales. Quelques autres résultats sont aussi présentés, comme une inégalité isopérimétrique pour la première valeur propre de Steklov , certaines propriétés de concentration des fonctions propres de Steklov à haute fréquence, ainsi que l'identification d'un cube parmi les cuboïdes grâce à son spectre de Steklov.
Mots clés : Problème de Steklov ; cuboïdes ; asymptotiques spectrales ; compte de points de réseau.

ABSTRACT. The paper is concerned with the Steklov eigenvalue problem on cuboids of arbitrary dimension. We prove a two-term asymptotic formula for the counting function of Steklov eigenvalues on cuboids in dimension $d \geq 3$. Apart from the standard Weyl term, we calculate explicitly the second term in the asymptotics, capturing the contribution of the $(d-2)$–dimensional facets of a cuboid. Our approach is based on lattice counting techniques. While this strategy is similar to the one used for the Dirichlet Laplacian, the Steklov case carries additional complications. In particular, it is not clear how to establish directly the completeness of the system of Steklov eigenfunctions admitting separation of variables. We prove this result using a family of auxiliary Robin boundary value problems. Moreover, the correspondence between the Steklov eigenvalues and lattice points is not exact, hence more delicate analysis is required to obtain spectral asymptotics. Some other related results are presented, such as an isoperimetric inequality for the first Steklov eigenvalue, a concentration property of high frequency Steklov eigenfunctions and applications to spectral determination of cuboids.
Keywords: Steklov problem ; cuboids ; spectral asymptotics ; lattice counting.

3.1. Introduction and main results

3.1.1. Asymptotics of the Steklov spectrum

The Steklov eigenvalues of a bounded Euclidean domain $\Omega \subset \mathbb{R}^d$ are the real numbers $\sigma \in \mathbb{R}$ for which there exists a nonzero harmonic function $u : \Omega \to \mathbb{R}$ such that $\partial_n u = \sigma u$ on the boundary $\partial\Omega$. Here ∂_n denotes the outward normal derivative, which exists almost everywhere provided the boundary $\partial\Omega$ is Lipschitz. Under this assumption, it is known that for $d \geq 2$ the Steklov spectrum is discrete (see [**1**]) and is given by the increasing sequence of eigenvalues $0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \ldots \nearrow \infty$, where each eigenvalue is repeated according to its multiplicity. The counting function $N : \mathbb{R} \to \mathbb{N}$ is then defined by $N(\sigma) := \#\{j \in \mathbb{N} : \sigma_j < \sigma\}$. For domains with smooth boundary, one can show using pseudodifferential techniques that the counting function satisfies Weyl's law

$$N(\sigma) = \frac{\omega_{d-1}}{(2\pi)^{d-1}} \mathrm{Vol}_{d-1}(\partial\Omega)\sigma^{d-1} + O(\sigma^{d-2}) \quad \text{as } \sigma \nearrow +\infty, \tag{3.1.1}$$

where ω_{d-1} is the measure of the unit ball $B_1(0) \subset \mathbb{R}^{d-1}$. The remainder estimate in (3.1.1) is sharp and attained on a round ball. Moreover, a two-term asymptotic formula for the counting function holds under a non-periodicity condition of the geodesic flow on $\partial\Omega$ (see [**64**, formula (5.1.8)]).

Understanding precise asymptotics for Steklov eigenvalues on domains with singularities, such as corners and edges, is significantly more challenging, since pseudodifferential techniques do not work in this case (see [**27**, Section 3] for a discussion). Using variational methods, one can prove a one-term Weyl asymptotic formula that holds for any piecewise C^1 Euclidean domain (see [**1**]):

$$N(\sigma) = \frac{\omega_{d-1}}{(2\pi)^{d-1}} \mathrm{Vol}_{d-1}(\partial\Omega)\sigma^{d-1} + o(\sigma^{d-1}) \quad \text{as } \sigma \nearrow +\infty. \tag{3.1.2}$$

However, in order to get sharper asymptotics, one needs to understand the contribution of singularities to the counting function. In two dimensions, some results in this direction have been recently obtained in [**56**]. In the present paper we aim to explore the most basic higher-dimensional example: the Euclidean cuboids.

3.1.2. Main result

Given $d \in \mathbb{N}$, the *cuboid*[1] with parameters $a_1, \ldots, a_d > 0$ is defined as a product of the intervals

$$\Omega = (-a_1, a_1) \times (-a_2, a_2) \times \ldots \times (-a_d, a_d) \subset \mathbb{R}^d.$$

If $a_1 = a_2 = \cdots = a_d$ we say that Ω is a *cube*. The main result of this paper is the following theorem.

Theorem 3.1.1. *Let $\Omega \subset \mathbb{R}^d$ be the cuboid with parameters $a_1, \ldots, a_d > 0$. For $d \geq 3$, the counting function of Steklov eigenvalues satisfies a two-term asymptotic formula as $\sigma \to \infty$:*

$$N(\sigma) = C_1 \operatorname{Vol}_{d-1}(\partial\Omega)\sigma^{d-1} + C_2 \operatorname{Vol}_{d-2}(\partial^2\Omega)\sigma^{d-2} + O\left(\sigma^{\eta}\right), \tag{3.1.3}$$

where $\partial^2\Omega$ denotes the union of all the $(d-2)$-dimensional facets of Ω. Here $\eta = 2/3$ for $d = 3$ and $\eta = d - 2 - \frac{1}{d-1}$ for $d \geq 4$. The constants C_1 and C_2 are given by

$$C_1 = \frac{\omega_{d-1}}{(2\pi)^{d-1}}$$

and

$$C_2 = \frac{2^{\frac{d-2}{2}}\omega_{d-2}}{(2\pi)^{d-2}} - \frac{2G_{d-1,1}}{\pi^{d-1}} - \frac{\omega_{d-2}}{2(2\pi)^{d-2}}$$

where

$$G_{d-1,1} = \underbrace{\int_0^{\pi/2} \cdots \int_0^{\pi/2}}_{d-2} \operatorname{arccot}\left(\prod_{j=1}^{d-2} \csc\theta_j\right) \prod_{k=1}^{d-2} \sin^k(\theta_k)\, \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_{d-2}.$$

For $d = 2$, the counting function admits a one-term asymptotics

$$N(\sigma) = \pi^{-1} \operatorname{Vol}_1(\partial\Omega)\sigma + O(1).$$

Remark 3.1.2. It can be shown that $C_2 > 0$ for all $d \geq 3$, see Appendix 3.B. The constants $G_{d,1}$ are special cases of constants $G_{p,q}$ which will be introduced in Section 3.3. The constants $G_{2,1}$ and $G_{3,1}$ can be computed explicitly as

$$G_{2,1} = \frac{1}{2}\left(-1 + \sqrt{2}\right)\pi$$
$$G_{3,1} = \frac{1}{8}(-2 + \pi)\pi.$$

Remark 3.1.3. For $d = 2$, the above asymptotics also follows from [**56**, Corollary 1.6.1].

[1] Cuboids are also often referred to as *boxes*, *d-orthotopes* or *hyperrectangles*. The term "cuboids" appears to be more common in recent literature on spectral geometry (see [**28, 8**]).

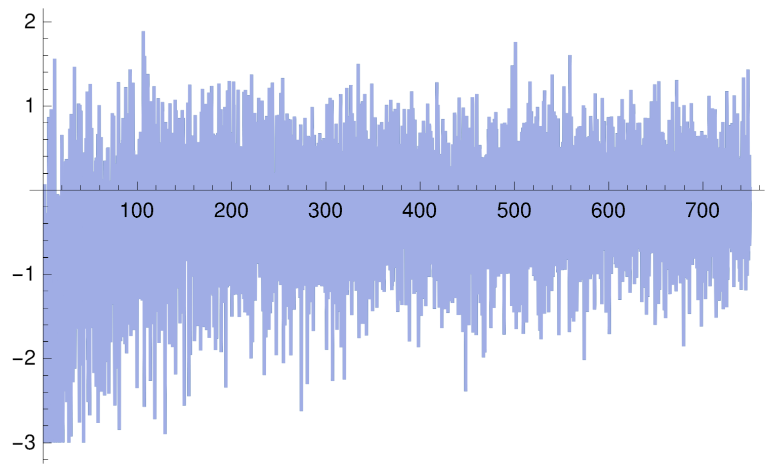

Figure 3.1. $\frac{N(\sigma)-C_1\,\mathrm{Vol}_2(\partial\Omega)\sigma^2-C_2\,\mathrm{Vol}_1(\partial^2\Omega)\sigma}{\sigma^{2/3}}$ for $\sigma < 750$.

Remark 3.1.4. For $d = 3$, Theorem 3.1.1 predicts that

$$\mathscr{R}(\sigma) = \frac{N(\sigma) - C_1\,\mathrm{Vol}_2(\partial\Omega)\sigma^2 - C_2\,\mathrm{Vol}_1(\partial^2\Omega)\sigma}{\sigma^{2/3}}$$

is a bounded function of σ. In order to validate the expression for the constant C_2 obtained in Theorem 3.1.1, we have checked numerically that this claim holds, using the approximate eigenvalues introduced in Section 3.3 on a cube with side lengths 2. Figure 3.1 shows that $|\mathscr{R}(\sigma)| \le 3$ for $\sigma < 750$ which corresponds to approximately a million eigenvalues.

3.1.3. Outline of the proof

The proof of Theorem 3.1.1 is given in Section 3.3. The outline of the argument is as follows. First, we show that the Steklov eigenvalue problem on a cuboid admits separation of variables, see Lemma 3.2.1 below. Separation of variables yields eigenfunctions that are products of trigonometric, hyperbolic and possibly linear factors. One can check that the number of eigenvalues corresponding to eigenfunctions containing linear terms is at most finite, see Theorem 3.2.6. The same theorem also shows that the eigenvalue counting problem can be reduced to a family of approximate lattice counting problems. More specifically, given $1 \le p \le d$, we consider the counting function N_p of eigenvalues corresponding to eigenfunctions with exactly p trigonometric factors. It turns out that for each $p > 1$, the counting function N_p satisfies a two-term asymptotic formula, see Proposition 3.3.1. The functions N_p for $p = d-1$ and $p = d-2$ are the dominant ones. In particular, the main term in (3.1.3) corresponds to the main term in the asymptotics for N_{d-1}. The second term in (3.1.3) is obtained as as a sum of the main term in the asymptotics of N_{d-2} and the second term in N_{d-1}. The latter also splits into two parts: one is the standard contribution of overcounted lattice points (see Lemma 3.3.17), and the other has to do with the geometry of the domain E_σ defined by (3.3.16) arising in the lattice counting problem. While this domain E_σ converges to a ball as $\sigma \to \infty$, the approximation produces an error that contributes to the second term of (3.1.3). This explains why the coefficient C_2 is represented by a sum of three constants. Note that while two of these constants are negative, the coefficient C_2 is always positive, see Appendix 3.B.

3.1.4. Discussion

The second term in Weyl asymptotics (3.1.3) for cuboids could be compared with the corresponding term in the asymptotic expression [**64**, formula (5.1.8)] mentioned earlier, which holds on smooth manifolds with boundary, satisfying a non-periodicity condition. Recall that in the smooth case, the second term is proportional to the integral of the mean curvature of the boundary. A similar interpretation could be given to the second term in (3.1.3), if an analogue of the mean curvature for cuboids is thought of as a δ-function supported on the union of the $(d-2)$-dimensional facets.

It would be very interesting to establish an analogue of Theorem 3.1.1 for arbitrary Euclidean polyhedra and, more generally, for Riemannian manifolds with edges, satisfying certain non-periodicity assumptions. While the present paper was in the final stages of preparation, V. Ivrii [**41**] informed us on his work in progress in this direction. We believe that a two-term Weyl asymptotic formula (3.1.3) holds for any polyhedron in dimension $d \geq 3$, with the coefficients C_1 and C_2 depending on the dimension and the angles between the $(d-1)$-dimensional facets of a polyhedron.

Another promising direction of further research in the subject is to explore the asymptotic expansion for the Steklov heat trace on Euclidean polyhedra, as well as on arbitrary Riemannian manifolds with edges. In particular, one could ask whether the Steklov spectral asymptotics contains information on the lower-dimensional facets of polyhedra. While the Weyl asymptotics does not appear to be accurate enough for that purpose, the Steklov heat trace asymptotics is likely to give a positive answer to this question. We intend to explore it elsewhere.

Remark 3.1.5. The existence of a two-term asymptotic formula for the counting function of Steklov eigenvalues on a cube was claimed earlier in [**62**]. However, the proof of this claim contained a miscalculation invalidating the argument. Indeed, in the beginning of [**62**, Section 3], the authors write down the boundary condition at $x_i = 0$ in case $\beta_i < 0$ and get $c_1\sqrt{|\beta_i|} = \lambda c_2$, while it should be $-c_1\sqrt{|\beta_i|} = \lambda c_2$, since the normal derivative at $x_i = 0$ is $-\partial_i$. Due to this missing minus sign, the authors obtain the equation $\sin(\sqrt{\beta_i}) = 0$ leading to an exact correspondence between Steklov eigenvalues and lattice points. However, in reality this correspondence is only approximate (see subsection 3.2.3), and therefore counting eigenvalues is a significantly more difficult task. Note also that the completeness of eigenfunctions admitting separation of variables was not justified in [**62**].

3.1.5. An isoperimetric inequality for the first Steklov eigenvalue

Given a cuboid $\Omega \subset \mathbb{R}^d$ with parameters $a_1, \ldots, a_d > 0$, let $\Omega^\star$ and $\Omega^\sharp$ be the cubes such that

$$\mathrm{Vol}_{d-1}\,\partial\Omega^\star = \mathrm{Vol}_{d-1}\,\partial\Omega \qquad \text{and} \qquad \mathrm{Vol}_d\,\Omega^\sharp = \mathrm{Vol}_d\,\Omega.$$

Theorem 3.1.6. *For any cuboid* Ω*,*

- $\sigma_1(\Omega^\star) \geq \sigma_1(\Omega)$*, with equality if and only if* $\Omega^\star = \Omega$*;*
- $\sigma_1(\Omega^\sharp) \geq \sigma_1(\Omega)$*, with equality if and only if* $\Omega^\sharp = \Omega$*.*

The proof of the theorem is presented in Section 3.4.3. In a way, it is not surprising that the cube, being the most symmetric of all cuboids, maximizes σ_1 under both volume and surface area restrictions. Theorem 3.1.6 could be compared with the well-known Weinstock's inequality [**79**] stating that the disk is a unique maximizer for σ_1 among planar simply connected domains with a given perimeter (see also a recent generalization of this result for convex domains in higher dimensions obtained in [**13**]), as well as with Brock's result [**12**] which states that balls are unique maximizers among Euclidean domains $\Omega \subset \mathbb{R}^d$ with prescribed d–volume.

It follows from Theorem 3.1.6 that any cube is spectrally determined among all cuboids.

Corollary 3.1.7. *Let* $\Omega \subset \mathbb{R}^d$ *be a cuboid which is isospectral to the cube* $\Omega_a \subset \mathbb{R}^m$ *with side lengths* $2a > 0$*. Then* $d = m$ *and* $\Omega = \Omega_a$*.*

PROOF. It follows from Theorem 3.1.1 that $d = m$ and $\mathrm{Vol}_{d-1}(\partial\Omega) = \mathrm{Vol}_{d-1}(\partial\Omega_a)$. Moreover, since $\sigma_1(\Omega) = \sigma_1(\Omega_a)$, the conclusion follows from the uniqueness of the maximizer in Theorem 3.1.6. □

Note that a similar corollary with an almost identical proof holds for planar simply-connected domains, among which the disk is spectrally determined, using the case of equality in Weinstock's theorem [**79**].

Is still unknown whether there exist nonisometric Steklov isospectral Euclidean domains. Our results imply that if two rectangles are Steklov isospectral, they are isometric.

Corollary 3.1.8. *The Steklov spectrum of a rectangle uniquely determines its side lengths.*

The proof of this corollary is presented in Section 3.4.4. Let us conclude the introduction with the following conjecture:

Conjecture 3.1.9. *Any two Steklov isospectral cuboids are isometric.*

Plan of the paper

In Section 2, we explore the structure of Steklov eigenvalues and eigenfunctions on cuboids. In particular, in subsection 3.2.1 we describe separation of variables and prove that it yields a complete system of Steklov eigenfunctions. In subsection 3.2.2 a classification of eigenfunctions is presented based on the number of linear, trigonometric and hyperbolic terms, which is later used in subsection 3.2.3 to reduce the problem of counting eigenvalues to counting approximate lattice points. Theorem 3.1.1 is proved in Section 3.3. This is the most technicallly involved part of the paper, involving tools from analytic number theory and Fourier analysis. Other results of the paper are proved in Section 3.4. In particular, a somewhat surprising observation that Steklov eigenfunctions may concentrate on lower dimensional facets of cuboids is presented in subsection 3.4.1. Subsections 3.4.3 and 3.4.4 provide the proofs of Theorem 3.1.6 and Corollary 3.1.8. Appendix 3.A contains the proof of an auxiliary Lemma 3.A.1 used in subsection 3.3.4. In Appendix 3.B we justify the positivity of the constant C_2 as stated in Remark 3.1.2.

Remark 3.1.10. Right before submitting our paper on the archive, we learned of the preprint [**76**] which discusses Steklov eigenvalues of rectangles and cuboids of dimension 3. Note that [**76**, Conjecture 3.1] immediately follows from our Proposition 3.4.2.

Acknowledgments

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3.2. Eigenfunctions and separation of variables

3.2.1. Separation of variables

The following lemma shows that the method of separation of variables is applicable to the computation of the Steklov spectrum of a product of compact manifolds with boundary. In particular, we justify completeness of the system of Steklov eigenfunctions admitting separation of variables.

Lemma 3.2.1. *Let M_1 and M_2 be smooth compact Riemannian manifolds with boundary. Let $\sigma \geq 0$ be a Steklov eigenvalue of the product manifold $M = M_1 \times M_2$ with eigenspace $F_\sigma \subset L^2(M)$. There exists a basis $(u^{(1)}, \ldots, u^{(m)})$ of F_σ such that each $u^{(j)} : M_1 \times M_2 \to \mathbb{R}$ is separable:*

$$u^{(j)}(x_1, x_2) = u_1^{(j)}(x_1)\, u_2^{(j)}(x_2), \qquad 1 \leq j \leq m,$$

where $u_1^{(j)} : M_1 \to \mathbb{R}$ and $u_2^{(j)} : M_2 \to \mathbb{R}$.

Proof. Consider the Robin problem with parameter $\sigma \geq 0$ on M

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } M, \\ \partial_n u = \sigma u & \text{on } \partial M. \end{cases}$$

It is well known that the Robin problem on M admits separation of variables. Indeed, it follows from the fact that $L^2(M) = L^2(M_1) \otimes L^2(M_2)$ is a product space, see *e.g.* [**74**, Section 11.5]. The number $\sigma \geq 0$ is a Steklov eigenvalue of M if and only if 0 is an eigenvalue of the Robin problem with parameter σ, and the corresponding eigenspace is the same for both problems. Since one can find a separated eigenbasis for F_σ by virtue of it being a Robin eigenspace on M, it then suffices to use the same basis for F_σ when we consider it as a Steklov eigenspace. □

Remark 3.2.2. It is not easy to show directly that the traces of all separable Steklov eigenfunctions form a basis in $L^2(\partial M)$, since the boundary ∂M of a product manifold is not itself a product manifold.

Remark 3.2.3. Lemma 3.2.1 yields completeness of the system of separable Steklov eigenfunctions on cuboids. Surprisingly, a complete proof of this result has not appeared in the literature even in the case of rectangles. Note that the completeness argument for the square presented in [**27**, Section 3] does not extend to arbitrary rectangles, contrary to the claim made in [**5**, Section 4] and in [**76**]. Indeed, the proof given in [**27**] uses in a crucial way the diagonal symmetries of the square, which allow to use a connection to the vibrating beam problem via mixed Steklov-Neumann-Dirichlet problems on an isosceles right triangle.

Let $d \in \mathbb{N}$ and consider the cuboid Ω with parameters $a_1, \ldots, a_d > 0$. Because Ω is a product of compact intervals, it follows from Lemma 3.2.1 that there exists a complete set $\{u_j\}_{j\in\mathbb{N}_0}$ of separated Steklov eigenfunctions on Ω. Consider a function $u : \Omega \to \mathbb{R}$ given by the product $u(x) = u_1(x_1)\ldots u_d(x_d)$, where $u_j : [-a_j, a_j] \to \mathbb{R}$. Requiring u to be a Steklov eigenfunction with eigenvalue $\sigma \geq 0$ leads to numbers $\lambda_1, \lambda_2, \ldots, \lambda_d \in \mathbb{R}$ such that

$$\begin{cases} u_j'' + \lambda_j u_j = 0 \quad \text{on} \quad (-a_j, a_j), \\ u_j'(a_j) = \sigma u_j(a_j), \\ -u_j'(-a_j) = \sigma u_j(-a_j), \end{cases} \tag{3.2.1}$$

subject to the harmonicity condition

$$\sum_{j=1}^{d} \lambda_j = 0. \tag{3.2.2}$$

The following lemma describes the eigenvalues and eigenfunctions of the auxiliary one-dimensional Steklov spectral problem (3.2.1) with a parameter $\lambda \in \mathbb{R}$.

Lemma 3.2.4. *Let $\lambda \in \mathbb{R}$. The non-zero solutions $\varphi : [-a, a] \to \mathbb{R}$ of the differential equation $\varphi'' + \lambda\varphi = 0$ subject to the boundary conditions*

$$\varphi'(a) = \sigma\varphi(a) \quad \text{and} \quad -\varphi'(-a) = \sigma\varphi(-a)$$

for some constant $\sigma \geq 0$, are constant multiples of one the following functions:

(i) For $\lambda = 0$, $\varphi(t) \equiv 1$ and $\sigma = 0$ or $\varphi(t) = t$ and $\sigma = a^{-1}$.

(ii) For $\lambda = \alpha^2 > 0$, one of

$$\varphi(t) = \sin(\alpha t) \quad \text{with } \sigma = \alpha\cot(\alpha a),$$

$$\varphi(t) = \cos(\alpha t) \quad \text{with } \sigma = -\alpha\tan(\alpha a).$$

In other words, for each $\ell \in \{0, 1\}$, $\sigma = \alpha\cot\left(\alpha a + \ell\frac{\pi}{2}\right)$ is an eigenvalue.

(iii) For $\lambda = -\beta^2 < 0$*, one of*

$$\varphi(t) = \sinh(\beta t) \qquad \textit{with } \sigma = \beta \coth(\beta a)$$

$$\varphi(t) = \cosh(\beta t) \qquad \textit{with } \sigma = \beta \tanh(\beta a).$$

In other words, for each $j \in \{-1,1\}$*,* $\sigma = \beta \tanh(\beta a)^j$ *is an eigenvalue.*

It will be useful to introduce a uniform notation for these eigenvalues. Given $a > 0$ and $\ell \in \{0,1\}$, let

$$T_{a,\ell}(x) = x \cot\left(ax + \ell\frac{\pi}{2}\right) = \begin{cases} x\cot(ax) & \text{for } \ell = 0, \\ -x\tan(ax) & \text{for } \ell = 1, \end{cases}$$

and

$$H_{a,\ell}(x) = \begin{cases} x\coth(ax) & \text{for } \ell = 0, \\ x\tanh(ax) & \text{for } \ell = 1. \end{cases}$$

It follows from Lemma 3.2.4 that separable eigenfunctions are products of linear factors, trigonometric factors (the function sin for $\ell = 0$, and cos for $\ell = 1$) and hyperbolic factors (the function sinh for $\ell = 0$, and cosh for $\ell = 1$). A careful accounting of these will be presented.

3.2.2. Classification of eigenfunctions

It follows from the previous paragraph that there is a complete set of Steklov eigenfunctions which are given by products of linear, trigonometric and hyperbolic factors. They are of the form

$$u(x_1,\ldots,x_d) = \prod_{i\in\tau_0} x_i \prod_{j\in\tau_1} \mathrm{Trig}_j(\alpha_j x_j) \prod_{k\in\tau_2} \mathrm{Hyp}_k(\beta_k x_k) \tag{3.2.3}$$

where τ_0,τ_1,τ_2 are disjoint subsets of $S_d := \{1,2,\ldots,d\}$ such that $\tau_0 \cup \tau_1 \cup \tau_2 = S_d$, and each $\mathrm{Trig}_j \in \{\sin,\cos\}$ and $\mathrm{Hyp}_k \in \{\sinh,\cosh\}$. In order for this function to be a Steklov eigenfunction corresponding to the eigenvalue $\sigma > 0$, the function u must be harmonic. This amounts to the following restatement of condition (3.2.2) in terms of the constants α_j and β_k:

$$\sum_{j\in\tau_1} \alpha_j^2 = \sum_{k\in\tau_2} \beta_k^2. \tag{3.2.4}$$

This equation will be called the *harmonicity condition.* Moreover, the spectral parameter σ has to be the same on each face of the cuboid. By Lemma 3.2.4 this translates into the following equations, called the *compatibility conditions*:

$$\sigma = \begin{cases} a_i^{-1} & \text{for } i \in \tau_0, \\ T_{a_i,\ell(i)}(\alpha_i) & \text{for } i \in \tau_1, \\ H_{a_i,\ell(i)}(\beta_i) & \text{for } i \in \tau_2. \end{cases} \tag{3.2.5}$$

Here the function $\ell : S_d \to \{0,1\}$ is used to specify which trigonometric and hyperbolic functions are used, according to the convention introduced in Lemma 3.2.4. The corresponding eigenfunction (3.2.3) is then given precisely by the product of the factors $u_i : [-a_i, a_i] \to \mathbb{R}$ which are specified by

$$u_i(x_i) = \begin{cases} \mathrm{Trig}_{\ell(i)}(\alpha_i x_i) & \text{for } i \in \tau_1, \\ \mathrm{Hyp}_{\ell(i)}(\beta_i x_i) & \text{for } i \in \tau_2, \\ x_i & \text{otherwise,} \end{cases} \tag{3.2.6}$$

where $\mathrm{Trig}_0 = \sin$, $\mathrm{Trig}_1 = \cos$, $\mathrm{Hyp}_0 = \sinh$ and $\mathrm{Hyp}_1 = \cosh$.

Note that any separated eigenfunction that has a linear factor $u_j(x_j) = x_j$ contributes the eigenvalue $\sigma = a_j^{-1}$ to the spectrum. Since the multiplicity of each eigenvalue is finite, this can occur at most a finite number of times. We summarize the above mentioned facts in the following theorem.

Theorem 3.2.5. *Let $p \in \{1, \ldots, d-1\}$, and let $\mathscr{T}_p$ be the set of all ordered bipartitions $\tau = (\tau_1, \tau_2)$ of $\{1, \ldots, d\}$ in the sets of cardinality p and $q = d - p$. For each $\tau \in \mathscr{T}_p$ and any $\ell : \tau_1 \cup \tau_2 \to \{0,1\}$, let $S_{\tau,\ell}$ be the set of all numbers $\sigma > 0$ for which there exist positive numbers α_i for $i \in \tau_1$ and β_j, for $j \in \tau_2$, which solve*

$$\sigma = T_{a_i,\ell(i)}(\alpha_i) = H_{a_j,\ell(j)}(\beta_j) \qquad \forall i \in \tau_1, j \in \tau_2$$

subject to the constraint

$$\sum_{i \in \tau_1} \alpha_i^2 = \sum_{j \in \tau_2} \beta_j^2.$$

Denote also by S_0 the collection of Steklov eigenvalues corresponding to separated eigenfunctions having a linear factor. Then the Steklov spectrum of a cuboid Ω is given by the union of S_0 which contains at most finitely many elements, and the families $S_{\tau,l}$ for all possible choices of τ and ℓ.

3.2.3. Reduction to approximate lattice counting

We will now give a more precise description of the spectrum by constructing a correspondence between the Steklov eigenvalues of cuboids and the vertices of certain lattices.

Let Ω be a cuboid with parameters $a_1,\ldots,a_d$. Let $p \in \{1,\ldots,d-1\}$ represent the number of trigonometric factors of a separated eigenfunction without linear factors. Each bipartition $\tau = (\tau_1,\tau_2) \in \mathcal{T}_p$ then corresponds to a separated eigenfunction of the form

$$u(x_1,\ldots,x_d) = \prod_{j\in\tau_1} \mathrm{Trig}_j(\alpha_j x_j) \prod_{k\in\tau_2} \mathrm{Hyp}_k(\beta_k x_k). \tag{3.2.7}$$

Let $\mathbb{N}_0 = \{0,1,2,\ldots\}$ be the set of nonnegtive integers. Given $\mathbf{n} \in \mathbb{N}_0^p$, let

$$I_{\mathbf{n}} = I_{\mathbf{n},p,\tau} := \prod_{i\in\tau_1} \left(\frac{n_i\pi}{2a_i}, \frac{(n_i+1)\pi}{2a_i}\right] \subset \mathbb{R}^p.$$

The boxes $I_{\mathbf{n}}$ are fundamental domains of a lattice. The following theorem shows that each box gives rise to a cluster of at most 2^q eigenvalues and, moreover, the boxes $I_{\mathbf{n}}$ with $\mathbf{n} \in \mathbb{N}^p$ and $|\mathbf{n}|$ large enough correspond to precisely 2^q eigenvalues.

Theorem 3.2.6. *Given $p \in \{1,\ldots,d-1\}$, and $q = d-p$, let $\tau \in \mathcal{T}_p$ specify the position of trigonometric and hyperbolic factors of eigenfunctions of the form* (3.2.7). *The following assertions hold:*

(i) *Eigenfunctions of the form* (3.2.7) *form a complete system of Steklov eigenfunctions on a cuboid up to a finite number of eigenfunctions containing linear factors.*

(ii) *For each $\mathbf{n} \in \mathbb{N}^p$, there exist at most 2^q eigenfunctions of the form* (3.2.7) *with $\boldsymbol{\alpha} \in I_{\mathbf{n}}$.*

(iii) *There exists a number $N \in \mathbb{N}$, such that for every $\mathbf{n} \in \mathbb{N}^p$ with $|\mathbf{n}| > N$, there are* exactly *2^q eigenfunctions of the form* (3.2.7) *with $\boldsymbol{\alpha} \in I_{\mathbf{n}}$. The corresponding eigenvalues $\sigma_{\mathbf{n}}^{(k)}$, with $k \in \{1,\ldots,2^q\}$, satisfy*

$$\sigma_{\mathbf{n}}^{(k)} = \frac{|\boldsymbol{\alpha}_{\mathbf{n}}|}{\sqrt{q}} + O\left(|\mathbf{n}|^{-\infty}\right) \tag{3.2.8}$$

for some $\boldsymbol{\alpha}_{\mathbf{n}} \in I_{\mathbf{n}}$, where $f(x) = O(x^{-\infty})$ means that for every N, there is x_0 and C such that for $x > x_0$ $\left|f(x)\right| < Cx^{-N}$.

(iv) *There exist only finitely many eigenfunctions of the form* (3.2.7) *such that $\mathbf{n} \in \mathbb{N}_0^p \setminus \mathbb{N}^p$. For each $\mathbf{n} \in \mathbb{N}_0^p \setminus \mathbb{N}^p$, there are at most 2^q eigenfunctions of the form* (3.2.7) *with $\boldsymbol{\alpha} \in I_{\mathbf{n}}$.*

Assertions (ii) and (iii) essentially say that up to a finite number of boxes, there is always exactly 2^q solutions in the box $I_{\mathbf{n}}$, while assertion (iv) says that while some boxes touching the coordinate hyperplanes $\{x_j = 0\}$ might contain solutions, this will only happen a finite number of times. This means that while all the three cases are needed to fully describe the spectrum, asymptotically we can only count eigenvalues described by (iii), up to a $O(1)$ error.

PROOF OF THEOREM 3.2.6. Assertion (i) is a direct consequence of Lemmas 3.2.1 and 3.2.4. In order to prove assertion (ii), for each $\ell : S_d \to \{0,1\}$ and $\mathbf{n} \in \mathbb{N}^p$ we will show that there exists at most one eigenfunction. Up to a small error, the corresponding eigenvalue will be equal to the norm of a point which is located in the box $I_{2\mathbf{n}+\mathbf{m},p,\tau}$, where $\mathbf{m} \in \{0,1\}^p$ is determined by the restriction of ℓ to τ_1. Together with the choice of ℓ on τ_2, this will account for clusters of at most 2^q eigenvalues corresponding to each of the boxes $I_{\mathbf{n}}$.

Construction of an eigenfunction.

For each $i \in \tau_2$, the function $\beta_i \mapsto H_{a_i,\ell(i)}(\beta_i)$, is increasing and positive for $\beta_i > 0$. It satisfies $H_{a_i,\ell(i)}(\beta_i) = \beta_i + O(\beta_i^{-\infty})$ as $\beta_i \to \infty$ and

$$\lim_{\beta_i \to 0} H_{a_i,\ell(i)}(\beta_i) = \begin{cases} \frac{1}{a_i} & \text{if } \ell(i) = 0, \\ 0 & \text{if } \ell(i) = 1. \end{cases}$$

This implies that the equations

$$H_{a_i,\ell(i)}(\beta_i) = H_{a_j,\ell(j)}(\beta_j) \qquad \forall i,j \in \tau_2 \tag{3.2.9}$$

define a connected curve $C_H = C_{H,p,\tau} \subset \mathbb{R}^q$ (the index H stands for "hyperbolic") which behaves like the diagonal

$$\{\boldsymbol{\beta} \in \mathbb{R}^q : \beta_i = \beta_j \text{ for each } i,j \in \tau_2\}$$

to infinite order as $|\boldsymbol{\beta}| \to \infty$. The common value given by equation (3.2.9) increases monotonically from some $c \geq 0$ to infinity along the curve C_H as it moves away from the origin. In fact, this non-negative constant is

$$c_\ell = \max\{0, a_i^{-1} : i \in \tau_2, \ell(i) = 0\}.$$

On the other hand, for each $i \in \tau_1$ the restricted function

$$T_{a_i,\ell(i)} : \left(\frac{n_i\pi}{a_i} + \frac{\ell(i)\pi}{2a_i}, \frac{n_i\pi}{a_1} + \frac{(\ell(i)+1)\pi}{2a_i} \right] \longrightarrow [0,\infty), \tag{3.2.10}$$

is decreasing and surjective. Hence, for each point $\boldsymbol{\beta} \in C_H \subset \mathbb{R}^q$, there exist unique numbers

$$\alpha_i(\boldsymbol{\beta}) \in \left(\frac{n_i\pi}{a_i} + \frac{\ell(i)\pi}{2a_i}, \frac{n_i\pi}{a_1} + \frac{(\ell(i)+1)\pi}{2a_i} \right] \qquad (\text{ for each } i \in \tau_1)$$

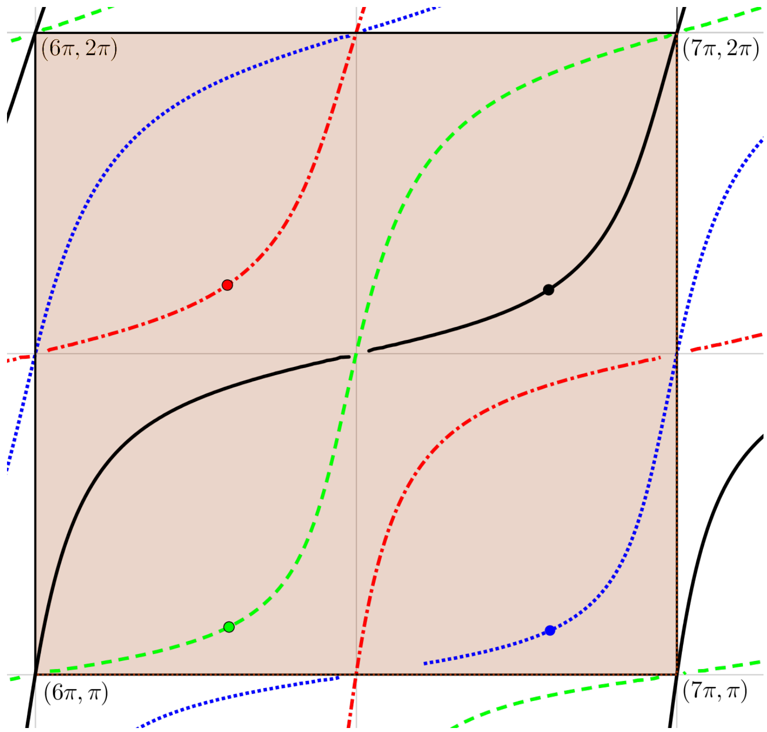

Figure 3.2. Various C_T curves in the situation where $d = 3$, $p = 2$ and $\tau_1 = \{1,2\}$.

such that

$$T_{a_i,\ell(i)}(\alpha_i) = H_{a_j,\ell(j)}(\beta_j) \qquad \forall i \in \tau_1, j \in \tau_2. \tag{3.2.11}$$

This defines an image curve $C_T \subset \mathbb{R}^p$ given by

$$C_T = \{\alpha_i(\boldsymbol{\beta}) : i \in \tau_1, \boldsymbol{\beta} \in C_H\}.$$

In other words, we have defined a continuous map $\boldsymbol{\alpha} : C_H \longrightarrow C_T$ between these two curves. It follows from (3.2.10) that the curve C_T is contained in the box $I_{2\mathbf{n}+\mathbf{m}}$, where $\mathbf{m} \in \{0,1\}^p$ is determined by the restriction of ℓ to τ_1. In particular, as the value of $|\boldsymbol{\beta}|$ increases from its minimal value to $+\infty$ along the curve C_H, the value of $|\boldsymbol{\alpha}(\boldsymbol{\beta})|$ is contained in the compact interval

$$\left[\inf_{\mathbf{x} \in I_{2\mathbf{n}+\mathbf{m}}} |\mathbf{x}|, \sup_{\mathbf{x} \in I_{2\mathbf{n}+\mathbf{m}}} |\mathbf{x}|\right] \subset (0,\infty).$$

Hence, if $\inf_{\mathbf{x} \in I_{2\mathbf{n}+\mathbf{m}}} |\mathbf{x}| > c_\ell$ there will be a point $\boldsymbol{\beta} \in C_H$ such that $\boldsymbol{\alpha} = \alpha(\boldsymbol{\beta})$ satisfy $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$. This amounts to saying that any of the common values given by (3.2.11) is a Steklov eigenvalue of the cuboid. It follows from monotonicity of each factors in Equation (3.2.11) that this solution $(\boldsymbol{\alpha},\boldsymbol{\beta})$ is unique.

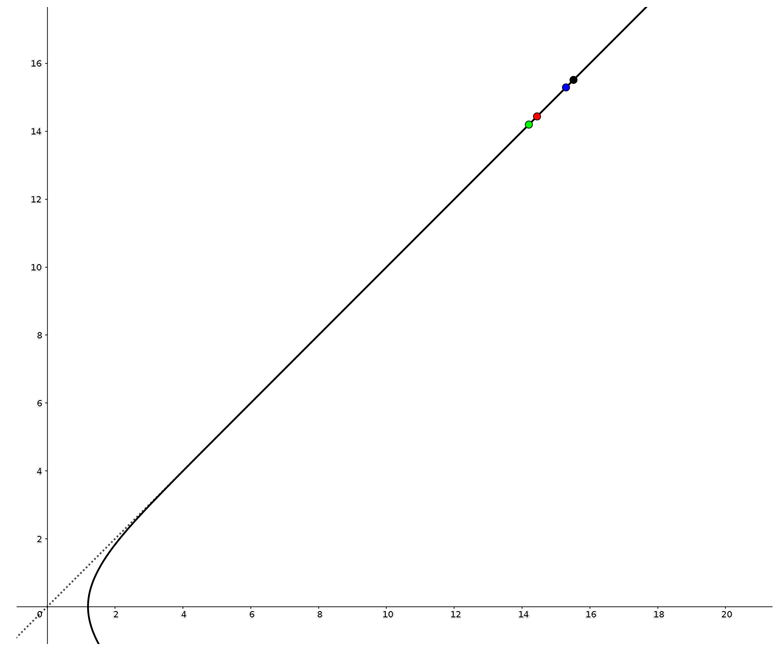

Figure 3.3. The curve C_H corresponding to $\ell(3) = 1$ and $\ell(4) = 0$: $x_3 \tanh(x_3) = x_4 \coth(x_4)$.

Remark 3.2.7. Let $d = 4$, $a_1 = a_2 = a_3 = a_4 = 1$, $p = 2$ and $\tau_1 = (1,2)$. In this case, Figure 3.2 shows the intersections of the four different curves C_T with the boxes $I_{2\mathbf{n}+\mathbf{m}} \subset \mathbb{R}^2$ for $\mathbf{n} = (12,2)$ and $\mathbf{m} \in \{0,1\}^2$. The corresponding curve C_H for the particular choice of the hyperbolic factor given by $\ell(3) = 1$ and $\ell(4) = 0$, is shown on Figure 3.3. On each of these curves, the marked point corresponds to the solution of the compatibility equations. Note that the curves C_T intersect two of the boxes, and the functions $T_{a_i,\ell(i)}$ defined on them are positive in one box and negative in the other. The solutions of the compatibility equations lie on the positive side.

We now turn to assertion (iii). Observe first that there is a uniform bound on c_ℓ hence there is a N such that if $|\mathbf{n}| > N$ then

$$\inf_{\mathbf{x} \in I_{\mathbf{n}}} |\mathbf{x}| > c_\ell.$$

From the previous discussion this ensures that there are exactly 2^q solutions in the box $I_{\mathbf{n}}$. We proceed in two steps for the more quantitative part of the statement. First, we prove that eigenvalues do take the form (3.2.8), and then we show that for all $k \in \{1,2\ldots,2^q\}$ the same $\boldsymbol{\alpha}_{\mathbf{n}}$ works.

Localisation

Fix the restriction $\ell : \tau_2 \to \{0,1\}^q$ for the moment. The various choices of trigonometric factors (represented by the choice of $\ell : \tau_1 \to \{0,1\}$) gives rises to exactly one solution $\boldsymbol{\alpha}_{2\mathbf{n}+\mathbf{m}}$ in each of the of the 2^p boxes $I_{2\mathbf{n}+\mathbf{m}}$, where $\mathbf{m}$ runs over all choices of $\mathbf{m} \in \{0,1\}^p$. For each of these

$\mathbf{m}$, the corresponding eigenvalue is given by any of the functions appearing in Equation (3.2.11) evaluated on any of the coordinates of $(\boldsymbol{\alpha}_{2\mathbf{n}+\mathbf{m}}, \boldsymbol{\beta}_{2\mathbf{n}+\mathbf{m}}) \in \mathbb{R}^p \times \mathbb{R}^q$. It follows that for each $j \in \tau_2$, and $\mathbf{n} \in \mathbb{N}^q$

$$|\boldsymbol{\beta}_{\mathbf{n}}|^2 = \sum_{i \in \tau_2} \beta_{\mathbf{n},i}^2 = q\beta_{\mathbf{n},j}^2 + O\left(|\mathbf{n}|^{-\infty}\right).$$

Hence for each $j \in \tau_2$,

$$\beta_{\mathbf{n},j} = \frac{|\boldsymbol{\beta}_{\mathbf{n}}|}{\sqrt{q}} + O\left(|\mathbf{n}|^{-\infty}\right).$$

The corresponding eigenvalue is therefore given, for any $j \in \tau_2$, by

$$\sigma_{\mathbf{n}} = H_{a_j,\ell(j)}(\beta_{\mathbf{n},j}) = \frac{|\boldsymbol{\beta}_{\mathbf{n}}|}{\sqrt{q}} + O\left(|\mathbf{n}|^{-\infty}\right) = \frac{|\boldsymbol{\alpha}_{\mathbf{n}}|}{\sqrt{q}} + O\left(|\mathbf{n}|^{-\infty}\right),$$

as was announced.

Clustering

If $\ell, \ell' : S_d \to \{0,1\}$ agree on τ_1, it follows from

$$H_{a_j,\ell(j)}(x) - H_{a_j,\ell'(j)}(x) = O\left(x^{-\infty}\right)$$

that the corresponding eigenvalues satisfy

$$\sigma_{\mathbf{n},\ell} - \sigma_{\mathbf{n},\ell'} = O\left(|\mathbf{n}|^{-\infty}\right).$$

The various choices of the restriction $\ell : \tau_2 \to \{0,1\}$ therefore lead to 2^q eigenvalues satisfying

$$\sigma_{\mathbf{n}}^k = \frac{|\boldsymbol{\alpha}_{\mathbf{n}}|}{\sqrt{q}} + O\left(|\mathbf{n}|^{-\infty}\right) \qquad \text{for } k = 1, \ldots, 2^q.$$

Exceptional eigenvalues

For $\mathbf{n} \in \mathbb{N}_0^p \setminus \mathbb{N}^p$ we have that $n_i = 0$ for at least one $i \in \tau_1$. On the interval $\left(0, \frac{\pi}{2a_i}\right]$, the function $T_{a_i,0}$ is positive while $T_{a_i,1}$ is negative, hence an eigenvalue can only correspond to $\ell(i) = 0$. In this case, the range of $T_{a_i,0}$ is $\left[0, a_i^{-1}\right)$. A corresponding eigenvalue is therefore bounded above by a_i^{-1}. There is only a finite number of these, proving assertion (iv).

This concludes the proof of Theorem 3.2.6. □

In the next section we will take up the task of understanding the asymptotic behavior of the counting function $N(\sigma)$.

3.3. Eigenvalue asymptotics

The goal of Section 3.3 is to prove Theorem 3.1.1. The plan is to represent the counting function $N(\sigma)$ as a sum of auxiliary counting functions corresponding to different families of eigenvalues provided by Theorem 3.2.6. Each of those counting functions will be then investigated using lattice counting techniques.

3.3.1. A hierarchy of counting functions

Let $p \in \{1,2,\dots,d-1\}$. Given $\tau = (\tau_1,\tau_2) \in \mathcal{T}_p$ and $\ell : S_d \to \{0,1\}$, define the counting function $N^{\tau,\ell} : \mathbb{R} \to \mathbb{N}$ by

$$N^{\tau,\ell}(\sigma) = \#\{j \in \mathbb{N} : \sigma_j \in S_{\tau,\ell} \text{ and } \sigma_j < \sigma\}.$$

Recall that the bipartition τ defines the location τ_1 of the trigonometric factors, and the location τ_2 of the hyperbolic factors, whereas the function ℓ distinguishes between sin and cos trigonometric factors, and sinh and cosh hyperbolic factors. We also introduce

$$N^{\tau}(\sigma) := \sum_{\ell : S_d \to \{0,1\}} N^{\tau,\ell}(\sigma) \quad \text{and} \quad N_p(\sigma) := \sum_{\tau \in \mathcal{T}_p} N^{\tau}(\sigma). \tag{3.3.1}$$

Since there is only a finite number of eigenfunctions with linear factors, one has

$$N(\sigma) = \sum_{p=1}^{d-1} N_p(\sigma) + O(1).$$

Set $q = d - p$ and let $\partial^q\Omega$ denote the union of p-dimensional facets of a cuboid Ω. Our goal is to prove the following asymptotics for $N_p(\sigma)$.

Proposition 3.3.1. *For each $p = 1,\dots,d-1$, we have:*

$$N_p(\sigma) = \frac{\sqrt{q^p}}{(2\pi)^p}\omega_p \operatorname{Vol}_p(\partial^q(\Omega))\sigma^p + c_p \operatorname{Vol}_{p-1}(\partial^{q+1}\Omega)\sigma^{p-1} + O\left(\sigma^{\eta_p}\right), \tag{3.3.2}$$

where c_p are some explicitly computable constants and

$$\eta_p = \max\left(p-1-\frac{1}{p}, p-2+\frac{2}{p+1}\right) = \begin{cases} 2/3 & \text{if } p = 2, \\ p-1-1/p & \text{otherwise.} \end{cases}$$

We prove Proposition 3.3.1 in subsection 3.3.5.

3.3.2. Quasi-eigenvalues

In this section, we observe that the clustering of eigenvalues in Theorem 3.2.6 allows us to simplify the eigenvalue counting problem. Essentially, we will count every cluster as one eigenvalue with a weight equal to the number of eigenvalues in the cluster.

Definition 3.3.2. Given $p \in S_d$, $q = d - p$, $\tau \in \mathscr{T}_p$, $\ell : S_d \to \{0,1\}$ and $\mathbf{n} \in \mathbb{N}^p$, the number $\frac{|\boldsymbol{\alpha}_\mathbf{n}|}{\sqrt{q}}$ defined in (3.2.8) is called a *quasi-eigenvalue of multiplicity* 2^q.

It is clear from Theorem 3.2.6 that

$$N(\sigma) = \sum_{p=1}^{d-1} 2^q \# \left\{ \mathbf{n} \in \mathbb{N}^p : \frac{|\boldsymbol{\alpha}_\mathbf{n}|}{\sqrt{q}} < \sigma \right\} + O(1). \tag{3.3.3}$$

The factor 2^q accounts for the clustering of eigenvalues around the corresponding quasi-eigenvalue. Note that the $O(1)$ error can be absorbed in the error term in (3.1.3). Therefore, in view of (3.3.3), for our purposes there is no need to distinguish between counting eigenvalues and quasi-eigenvalues.

3.3.3. Eigenfunctions with a single trigonometric factor

Consider first the case $p = 1$.The choice of sin or cos for the trigonometric factor and the choice of the coordinate corresponding to the trigonometric factor yields $2d$ families of eigenfunctions, each having 2^{d-1} possibilities for the choice of the hyperbolic factor. As follows from Theorem 3.2.6, each of the $2d$ families contributes clusters of 2^{d-1} eigenvalues which correspond to the same quasi-eigenvalue. Therefore, as was mentioned earlier, this cluster can be counted for our purposes as a single quasi-eigenvalue of multiplicity 2^{d-1}. The compatibility equations

$$H_{a_i,\ell(i)}(\beta_i) = H_{a_j,\ell(j)}(\beta_j) \quad \forall i,j \in \tau_2 \tag{3.3.4}$$

define a connected curve in $\mathbb{R}^{d-1}$ which goes to infinity along the diagonal while its value increases to $+\infty$. Equating (3.3.4) to $T_{a_k,\ell(k)}$, $k \in \tau_1$ amounts to solving the following equations:

$$\alpha_k \cot(a_k \alpha_k) = \frac{\alpha_k}{\sqrt{d-1}} + O\left(\alpha_k^{-\infty}\right) \qquad \text{if } \ell(k) = 0,$$

and

$$-\alpha_k \tan(a_k \alpha_k) = \frac{\alpha_k}{\sqrt{d-1}} + O\left(\alpha_k^{-\infty}\right) \qquad \text{if } \ell(k) = 1.$$

This yields eigenvalues of the form

$$\sigma_j = \begin{cases} \frac{\pi j}{a_k\sqrt{d-1}} + \frac{1}{a_k\sqrt{d-1}} \operatorname{arccot}((d-1)^{-1/2}) + O\left(j^{-\infty}\right) & \text{if } \ell(k) = 0, \\ \frac{\pi j}{a_k\sqrt{d-1}} + \frac{1}{a_k\sqrt{d-1}} \arctan((d-1)^{-1/2}) + O\left(j^{-\infty}\right) & \text{if } \ell(k) = 1, \end{cases}$$

each with quasi-multiplicity 2^{d-1}. Given that arccot and arctan are bounded functions, and since

$$\operatorname{Vol}_1(\partial^{d-1}\Omega) = 2^d \sum_{j=1}^{d} a_j,$$

we have that

$$N_1(\sigma) = \frac{\omega_1\sqrt{d-1}}{2\pi} \operatorname{Vol}_1(\partial^{d-1}\Omega)\,\sigma + O(1).$$

This concludes the proof of Theorem 3.1.1 for $d = 2$, since $p = 1$ is the only possibility in this case. Observe that for $d = 2$, this is indeed the expected first term of Weyl's law (3.1.2).

3.3.4. Eigenfunctions with many trigonometric factors

In this subsection, we count the number of eigenvalues associated with eigenfunctions with more than one trigonometric factor. The idea is to write the eigenvalues as the norms of points $\boldsymbol{\alpha} \in \mathbb{R}^p$ that are close to some lattice points. The main difficulty is that the compatibility equations are transcendental, making it impossible to explicitly find $\boldsymbol{\alpha}$. We will therefore approximate the eigenvalues in a controlled way, and we will show that this approximation results in a small enough error that could be absorbed in the remainder in the two-term asymptotics for the eigenvalue counting function. Finally, we will use the lattice point counting techniques going back to [**36, 65**], and more recently used in [**54**].

3.3.4.1. *Approximate eigenvalues*

Suppose that $d \geq 3$ and $p \in \{2, \ldots, d-1\}$. Let $\tau \in \mathscr{T}_p$ and $\ell : S_d \to \{0,1\}$ be given.

Given $\mathbf{n} \in \mathbb{N}^p$, it follows from Theorem 3.2.6 and the compatibility equations (3.2.5), that the corresponding solution $\boldsymbol{\alpha} = \boldsymbol{\alpha}_{\mathbf{n}} \in I_{\mathbf{n}}$ satisfies the following for each $i, j \in \tau_1$

$$\alpha_i \cot\left(\alpha_i a_i + \frac{\ell(i)\pi}{2}\right) = \alpha_j \cot\left(\alpha_j a_j + \frac{\ell(j)\pi}{2}\right) = \frac{|\boldsymbol{\alpha}_{\mathbf{n}}|}{\sqrt{q}} + O\left(|\mathbf{n}|^{-\infty}\right).$$

Hence, for each $i \in \tau_1$, we have, choosing the principal branch of arccot, a family of solutions indexed by $\mathbf{n} \in \mathbb{N}^p$

$$\alpha_i a_i = \left(n_i + \frac{\ell(i)}{2}\right)\pi + \operatorname{arccot}\left(\frac{1}{\sqrt{q}}\left[1 + \sum_{j \neq i \in \tau_1} \left(\frac{\alpha_j}{\alpha_i}\right)^2\right]^{1/2}\right) + O\left(|\mathbf{n}|^{-\infty}\right).$$

Since $\alpha_i = \frac{\left(n_i + \frac{\ell(i)}{2}\right)\pi}{a_i} + O(1)$, we can rewrite the previous equation as follows

$$\alpha_i = \frac{\left(n_i + \frac{\ell(i)}{2}\right)\pi}{a_i} + \frac{1}{a_i}\operatorname{arccot}\left(\frac{1}{\sqrt{q}}\left[1 + \sum_{j\neq i}\left(\frac{\frac{\left(n_j + \frac{\ell(j)}{2}\right)\pi}{a_j} + t_{\alpha_j}(\mathbf{n})}{\frac{\left(n_i + \frac{\ell(i)}{2}\right)\pi}{a_i} + t_{\alpha_i}(\mathbf{n})}\right)^2\right]^{1/2}\right) + O\left(|\mathbf{n}|^{-\infty}\right), \tag{3.3.5}$$

where the functions t_{α_j} are bounded. Since $\ell(i)$ ranges over $\{0,1\}$, the solution set to the previous equation is the same as the one to

$$\alpha_i = \frac{n_i\pi}{2a_i} + \frac{1}{a_i}\operatorname{arccot}\left(\frac{1}{\sqrt{q}}\left[1 + \sum_{j\neq i}\left(\frac{\frac{n_j\pi}{2a_j} + t_{\alpha_j}(\mathbf{n})}{\frac{n_i\pi}{2a_i} + t_{\alpha_i}(\mathbf{n})}\right)^2\right]^{1/2}\right) + O\left(|\mathbf{n}|^{-\infty}\right). \tag{3.3.6}$$

Lemma 3.3.3. *Define* $\widetilde{\alpha}_i$ *as*

$$\widetilde{\alpha}_i = \frac{n_i\pi}{2a_i} + \frac{1}{a_i}\operatorname{arccot}\left(\frac{1}{\sqrt{q}}\left[1 + \sum_{j\neq i}\left(\frac{a_i n_j}{a_j n_i}\right)^2\right]^{1/2}\right). \tag{3.3.7}$$

Then,

$$\widetilde{\alpha}_i = \alpha_i + O\left(|\mathbf{n}|^{-1}\right) \tag{3.3.8}$$

PROOF. In Lemma 3.A.1 in the Appendix, take $x_i = \frac{n_i\pi}{a_i}$ and $\psi_i = t_{\alpha_i}$. Then, one readily sees that

$$|\mathbf{x}| \asymp |\mathbf{n}|,$$

where $f \asymp g$ means that $f = O(g)$ and $g = O(f)$. The lemma then follows. □

Note that the right hand side of equation (3.3.7) does not depend on α_i anymore, which makes it easier to analyse.

We now have eigenvalues indexed by $\mathbf{n} \in \mathbb{N}^p$ given by

$$\sigma_{\mathbf{n}} = \sqrt{\frac{1}{q}\sum_{i\in\tau_1}\widetilde{\alpha}_i^2} + O\left(|\mathbf{n}|^{-1}\right). \tag{3.3.9}$$

Definition 3.3.4. The numbers

$$\widetilde{\sigma}_{\mathbf{n}} = \sqrt{\frac{1}{q}\sum_{i\in\tau_1}\widetilde{\alpha}_i^2} \tag{3.3.10}$$

are called the *approximate eigenvalues.*

Remark 3.3.5. Up until now, eigenvalues, quasi-eigenvalues and approximate eigenvalues were indexed by $\mathbf{n} \in \mathbb{N}^p$. In the following two theorems it is convenient to use $n \in \mathbb{N}$ to index them in an ascending order.

The following lemma allows us to estimate the error induced by counting approximate eigenvalues instead of eigenvalues.

Lemma 3.3.6. *Let $(a_n), (b_n)$ be two sequences of positive numbers which tend to infinity. Suppose there exists a number $s > -1$ such that $a_n = b_n + O\left(b_n^{-s}\right)$. Let*

$$N_a(\lambda) = \#\{n : a_n < \lambda\} \qquad \text{and} \qquad N_b(\lambda) = \#\{n : b_n < \lambda\}.$$

Suppose that there exists a number K such that

$$N_a(\lambda) = \sum_{k=0}^{K} c_k \lambda^{p-k} + O\left(\lambda^r\right),$$

with $r < p - K$. Then,

$$N_b(\lambda) = \sum_{k=0}^{K} c_k \lambda^{p-k} + O\left(\lambda^{r'}\right) \tag{3.3.11}$$

where $r' = \max(r, p-1-s)$.

Remark 3.3.7. Note that if $r' \geq p - K$, some of the terms in the sum in (3.3.11) might be absorbed in the error term.

PROOF. Indeed, the assumption on the sequences a_n and b_n implies that there exists $c > 0$ such that

$$N_a\left(\lambda + \frac{c}{\lambda^s}\right) \leq N_b(\lambda) \leq N_a\left(\lambda - \frac{c}{\lambda^s}\right).$$

A direct computation of $N_a(\lambda \pm c\lambda^{-s})$ completes the proof of the lemma. □

Recall now the definition of $N^\tau(\sigma)$ given by (3.3.1). We will write $\widetilde{N}^\tau$ for the counting function of the corresponding approximate eigenvalues.

Lemma 3.3.8. *We have:*

$$\left|\widetilde{N}^\tau(\sigma) - N^\tau(\sigma)\right| = O\left(\sigma^{p-1-1/p}\right).$$

PROOF. Both the eigenvalues and the approximate eigenvalues are, up to a bounded error, the norms of the points of the lattice $\Gamma = \bigoplus_{i=1}^{p} \frac{\pi}{2a_i\sqrt{q}}\mathbb{N}$, repeated 2^q times. Denote by $l_n := \left\{|\boldsymbol{\gamma}| : \boldsymbol{\gamma} \in \Gamma\right\}_n$ the sequence of norms of the points of the lattice Γ arranged in ascending order. It is well known that there is a constant C such that

$$N_l(\sigma) = C\sigma^p + O\left(\sigma^{p-1}\right),$$

where C depends on Γ and N_l denotes the counting function of the sequence l_n as in Lemma 3.3.6. Applying Lemma 3.3.6 with $s = 0$ yields

$$N^{\tau}(\sigma) = 2^q C\sigma^p + O\left(\sigma^{p-1}\right).$$

Reversing this expression tells us that

$$\sigma_n = \left(\frac{n}{2^q C}\right)^{1/p} + o\left(n^{1/p}\right). \tag{3.3.12}$$

From equations (3.3.10) and (3.3.12) we have that

$$\widetilde{\sigma}_n = \sigma_n + O\left(n^{-1/p}\right).$$

Therefore, applying once again Lemma 3.3.6, but this time with $s = 1/p$, yields

$$N^{\tau}(\sigma) = \widetilde{N}^{\tau}(\sigma) + O\left(\sigma^{p-1-1/p}\right). \tag{3.3.13}$$

□

3.3.4.2. *Another representation of the counting function*

For every τ, let us now define a family of sets $E_\sigma \subset \mathbb{R}^p$ with the property that

$$\widetilde{N}^{\tau}(\sigma) = \sum_{\mathbf{n}\in\mathbb{N}^p} 2^q \chi\left(\frac{\mathbf{n}}{\sigma}\right) + O(1), \tag{3.3.14}$$

where $\chi := \chi_\sigma$ is the indicator function of E_σ. Let us define elliptic polar coordinates in $\mathbb{R}^p$ with the convention that $\theta_p = 0$:

$$\begin{aligned} r^2 &= \sum_{i\in\tau_1}\left(\frac{\pi x_i}{2a_i\sqrt{q}}\right)^2, \\ x_j &= r\frac{2a_j\sqrt{q}}{\pi}\cos(\theta_j)\prod_{i<j}\sin(\theta_i). \end{aligned} \tag{3.3.15}$$

We define the family of sets

$$E_\sigma := \left\{(r,\boldsymbol{\theta}) \in \mathbb{R}^p : r^2 + \frac{2r}{\sigma}\sum_{j\in\tau_1}\frac{1}{a_j}g_j(\boldsymbol{\theta}) + \frac{H(\boldsymbol{\theta})}{\sigma^2} < 1\right\}, \tag{3.3.16}$$

with

$$g_j(\boldsymbol{\theta}) := \cos\theta_j \prod_{i<j}\sin\theta_i \operatorname{arccot}\left(\frac{1}{\sqrt{q}}\left[1+\sum_{i\neq j}\left(\frac{x_i}{x_j}\right)^2\right]^{1/2}\right), \tag{3.3.17}$$

and

$$H = H(\boldsymbol{\theta}) = \sum_{j\in\tau_1}\frac{1}{a_j^2}\operatorname{arccot}\left(\frac{1}{\sqrt{q}}\left[1+\sum_{i\neq j}\left(\frac{x_i}{x_j}\right)^2\right]^{1/2}\right)^2.$$

From equation (3.3.10), we can observe that the evaluation of χ at $\sigma^{-1}\mathbf{n}$ in coordinates (3.3.15) is 1 if and only if $\widetilde{\sigma}_{\mathbf{n}} < \sigma$. If $|\mathbf{n}| > N$ as in Theorem 3.2.6, there are 2^q solutions close to any order to $\widetilde{\sigma}_{\mathbf{n}}$. This achieves our stated goal of equation (3.3.14). Let us now prove a few properties of the set E_σ that will be required in the sequel.

Lemma 3.3.9. *There exists σ_0 such that for $\sigma > \sigma_0$ the set E_σ is strictly convex and the principal curvatures of ∂E_σ are positive and uniformly bounded away from* 0*. Furthermore, all the derivatives of the principal curvatures tend to* 0 *as $\sigma \to \infty$.*

PROOF. From equation (3.3.16) ∂E_σ is the level set of a function F satisfying

$$\begin{aligned} F(r,\boldsymbol{\theta}) &= r^2 + O\left(\sigma^{-1}\right), \\ [\nabla F(\mathbf{x})]_i &= \frac{\pi x_i}{a_i\sqrt{q}} + O\left(\sigma^{-1}\right), \\ \operatorname{Hess} F &= \operatorname{diag}\left(\frac{\pi}{a_i\sqrt{q}}\right)_{i\in\tau_1} + O\left(\sigma^{-1}\right), \end{aligned} \tag{3.3.18}$$

with the error estimates uniform in ∂E_σ. This yields that for σ large enough, the second fundamental form of ∂E_σ is positive, with its smallest eigenvalue uniformly bounded away from 0. This implies the claim on the principal curvatures, which in turn implies strict convexity.

As for the derivatives of the principal curvatures, they are the derivatives of the eigenvalues of $\operatorname{Hess} F$. Observe that rg_j and H are smooth away from the origin, hence all their derivatives are bounded on ∂E_σ. This implies that the derivatives of $\operatorname{Hess} F$ are $O\left(\sigma^{-1}\right)$, hence the derivatives of its eigenvalues as well and they go to 0 as $\sigma \to \infty$. ☐

This argument also yields the following corollary.

Corollary 3.3.10. *The product of the principal curvatures of ∂E_σ is uniformly bounded away from zero for σ large enough.*

3.3.4.3. *Poisson Summation Formula*

In this section, we use the general scheme of the proof of [**54**, Theorem 1.1]. Recall that

$$\begin{aligned} N^\tau(\sigma) &= \sum_{\mathbf{n}\in\mathbb{N}^p} 2^q \chi\left(\frac{\mathbf{n}}{\sigma}\right) + O(1) \\ &= 2^{q-p} \sum_{\mathbf{n}\in\mathbb{Z}^p} \chi\left(\frac{\mathbf{n}}{\sigma}\right) + R_\tau(\sigma) + O(1), \end{aligned} \tag{3.3.19}$$

where $R_\tau(\sigma)$ is the error term induced by the overcounting of points on hyperplanes with one vanishing coordinate.

Our goal is now to compute the terms appearing in equation (3.3.19) using the Poisson summation formula which states, under sufficient smoothness assumptions that

$$\sum_{\mathbf{n}\in\mathbb{Z}^p} f(\mathbf{n}) = \sum_{\mathbf{m}\in\mathbb{Z}^p} \widehat{f}(\mathbf{m}) \tag{3.3.20}$$

where the Fourier transform is given by

$$\widehat{f}(\boldsymbol{\xi}) := \int_{\mathbb{R}^p} f(\mathbf{x}) e^{-2\pi i \mathbf{x}\cdot\boldsymbol{\xi}}\,\mathrm{d}\mathbf{x}.$$

However, χ is not regular enough for us to use the Poisson summation formula, hence we need to mollify it. Let us introduce a nonnegative function $\psi \in C_c^\infty(\mathbb{R})$ supported in $[-1,1]$ and such that

$$\int_0^\infty \psi(r) r^{p-1}\,\mathrm{d}r = \frac{1}{V_{p-1}},$$

with V_{p-1} being the volume of the $p-1$ dimensional unit sphere in $\mathbb{R}^p$. We then define a family $\Psi_\epsilon : \mathbb{R}^p \to \mathbb{R}$ of radial bump functions of total mass 1 by

$$\Psi_\epsilon(\mathbf{x}) = \frac{1}{\epsilon^p}\psi\left(\frac{|\mathbf{x}|}{\epsilon}\right).$$

Set $\Psi := \Psi_1$ Consider the smooth function $\chi_\epsilon = \Psi_\epsilon * \chi$. Note that

$$\widehat{\Psi}_\epsilon(\boldsymbol{\xi}) = \widehat{\Psi}(\epsilon\boldsymbol{\xi})$$

We now prove the following lemma.

Lemma 3.3.11. *Let $\chi_\epsilon^+, \chi_\epsilon^- : \mathbb{R}^p \to \mathbb{R}$ be defined by*

$$\chi_\epsilon^+(\mathbf{x}) = \chi_\epsilon\big((1-\eta_+\epsilon)\mathbf{x}\big)$$

$$\chi_\epsilon^-(\mathbf{x}) = \chi_\epsilon\big((1+\eta_-\epsilon)\mathbf{x}\big)$$

for some $\eta_-,\eta_+ > 0$. One can choose η_-,η_+ in such a way that for all σ large enough

$$\chi_\epsilon^-(\mathbf{x}) \le \chi(\mathbf{x}) \le \chi_\epsilon^+(\mathbf{x})$$

for all $\mathbf{x}\in\mathbb{R}^p$ and all $\epsilon > 0$ small enough.

Proof. For the first inequality, observe that

$$\begin{aligned}\chi_\epsilon((1+\eta_-\epsilon)\mathbf{x}) &= \int_{\mathbb{R}^p} \chi(\mathbf{y})\Psi_\epsilon((1+\eta_-\epsilon)\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y} \\ &= \int_{B_{(1+\eta_-\epsilon)\mathbf{x}}(\epsilon)} \chi(\mathbf{y})\Psi_\epsilon((1+\eta_-\epsilon)\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y} \\ &\le \sup_{B_{(1+\eta_-\epsilon)\mathbf{x}}(\epsilon)} \chi(\mathbf{y}).\end{aligned}$$

Hence, to show that $\chi_\epsilon((1+\eta_-\epsilon)\mathbf{x}) \le \chi(\mathbf{x})$ for all $\mathbf{x}$, by convexity of E_σ it is sufficient to show that for all $\mathbf{x} \in \partial E_\sigma$, there exists η_-, independent of σ such that the following holds for each $\epsilon > 0$ small enough

$$B_{(1+\eta_-\epsilon)\mathbf{x}}(\epsilon) \cap E_\sigma = \varnothing.$$

Note that for all $\mathbf{x} \in \partial E_\sigma$, we have that

$$\operatorname{dist}((1+t)\mathbf{x}, \partial E_\sigma) = (\mathbf{x} \cdot \mathcal{N}_{\partial E_\sigma}(\mathbf{x}))\, t + O\left(t^2\right), \tag{3.3.21}$$

where $\mathcal{N}_{\partial E_\sigma}$ is the Gauss map of the boundary. To see this, denote by $T_\mathbf{x}\partial E_\sigma$ the tangent hyperplane of ∂E_σ at $\mathbf{x}$, and by $P_\mathbf{x}$ the orthogonal projection on that hyperplane. We have by the triangle inequality that

$$|\operatorname{dist}((1+t)\mathbf{x}, \partial E_\sigma) - \operatorname{dist}((1+t)\mathbf{x}, T_\mathbf{x}\partial E_\sigma)| \le \operatorname{dist}(P_\mathbf{x}((1+t)\mathbf{x}), \partial E_\sigma).$$

We observe that $\operatorname{dist}((1+t)\mathbf{x}, T_\mathbf{x}\partial E_\sigma) = (\mathbf{x} \cdot \mathcal{N}_{\partial E_\sigma}(\mathbf{x}))\, t$. Let F, as before, be the function in $\mathbb{R}^p$ such that the set $F \equiv 1$ coincides with ∂E_σ. Taking the Taylor expansion of F around $\mathbf{x}$, we have that

$$\operatorname{dist}(P_\mathbf{x}((1+t)\mathbf{x}), \partial E_\sigma) \le \|\operatorname{Hess} F(\mathbf{x})\|_\infty |P_\mathbf{x}((1+t)\mathbf{x})|^2 = O\left(t^2\right),$$

where we used that $\|\operatorname{Hess} F(\mathbf{x})\|_\infty$ is bounded uniformly for $\sigma > \sigma_0$ and $\mathbf{x} \in \partial E_\sigma$. Note that the strict convexity of ∂E_σ and equation (3.3.18) imply that $\mathbf{x} \cdot \mathcal{N}_{\partial E_\sigma}(\mathbf{x})$ is bounded away from zero uniformly for $\sigma > \sigma_0$. This implies that we can choose η_- large enough and independent in σ such that indeed

$$B_{(1+\eta_-\epsilon)\mathbf{x}}(\epsilon) \cap E_\sigma = \varnothing.$$

For the second inequality, we have

$$\begin{aligned} \chi_\epsilon((1-\eta_+\epsilon)\mathbf{x}) &= \int_{\mathbb{R}^p} \chi(\mathbf{y}) \Psi_\epsilon((1-\eta_+\epsilon)\mathbf{x} - \mathbf{y})\, \mathrm{d}\mathbf{y} \\ &= \int_{B_{(1-\eta_+\epsilon)\mathbf{x}}(\epsilon)} \chi(\mathbf{y}) \Psi_\epsilon((1-\eta_+\epsilon)\mathbf{x} - \mathbf{y})\, \mathrm{d}\mathbf{y} \\ &\ge \inf_{B_{(1-\eta_+\epsilon)\mathbf{x}}(\epsilon)} \chi(\mathbf{y}). \end{aligned}$$

Hence, to show that $\chi(\mathbf{x}) \le \chi_\epsilon((1-\eta_+)\mathbf{x})$, it is sufficient to show that for all $\mathbf{x} \in \partial E_\sigma$, there exists η_+ independent of σ such that

$$B_{(1-\eta_+)\mathbf{x}}(\epsilon) \subset E_\sigma.$$

Using once again equation (3.3.21) and arguing exactly as above yields the desired number η_+.

$\square$

The following is an immediate corollary of the previous lemma:

Corollary 3.3.12. *We have that*

$$\sum_{\mathbf{n}\in\mathbb{Z}^p} \chi_\epsilon^- \left(\frac{\mathbf{n}}{\sigma}\right) \le \sum_{\mathbf{n}\in\mathbb{Z}^p} \chi\left(\frac{\mathbf{n}}{\sigma}\right) \le \sum_{\mathbf{n}\in\mathbb{Z}^p} \chi_\epsilon^+ \left(\frac{\mathbf{n}}{\sigma}\right).$$

We will now apply the Poisson summation formula (3.3.20) to $\chi_\epsilon^\pm$, which are smooth functions. This yields, using the basic properties of the Fourier transform,

$$\begin{aligned}\sum_{\mathbf{n}\in\mathbb{Z}^p} \chi_\epsilon^\pm \left(\frac{\mathbf{n}}{\sigma}\right) &= \sigma^p \sum_{\mathbf{m}\in\mathbb{Z}^p} \widehat{\chi}_\epsilon^\pm (\sigma\mathbf{m}) \\ &= \sigma^p \sum_{\mathbf{m}\in\mathbb{Z}^p} (1+O(\epsilon))\, \widehat{\chi}\left(\frac{\sigma\mathbf{m}}{1\mp\eta_\pm\epsilon}\right) \widehat{\Psi}\left(\frac{\epsilon\mathbf{m}\sigma}{1\mp\eta_\pm\epsilon}\right) \\ &= \sigma^p \operatorname{Vol}(E_\sigma) + O\left(\epsilon\sigma^p\right) \\ &\quad + O\left(\sum_{\substack{\mathbf{m}\in\mathbb{Z}^p\\ \mathbf{m}\neq 0}} \sigma^p \widehat{\chi}\left(\frac{\sigma\mathbf{m}}{1\mp\eta_\pm\epsilon}\right) \widehat{\Psi}\left(\frac{\epsilon\mathbf{m}\sigma}{1\mp\eta_\pm\epsilon}\right)\right).\end{aligned} \tag{3.3.22}$$

Note that for this expression to hold, we will need to later choose $\epsilon = o(1)$. Since Ψ is a Schwartz function, its Fourier transform is also Schwartz, hence to find estimates on the asymptotic behaviour of equation (3.3.22), we only need to find bounds on $\widehat{\chi}$. This is done in the following Lemma.

Lemma 3.3.13. *For σ large enough, the Fourier transform of χ satisfies the upper bound*

$$\widehat{\chi}(\boldsymbol{\xi}) = O\left(|\boldsymbol{\xi}|^{-\frac{d+1}{2}}\right). \tag{3.3.23}$$

PROOF. For σ large enough, the set E_σ is strictly convex and has smooth boundary. Therefore, following [**40**, Theorem 2.29] we have that for any function $f \in C^\infty(\mathbb{R}^p)$ such that $f \neq 0$ on ∂E_σ,

$$\int_{E_\sigma} f(\mathbf{x}) e^{-2\pi i \mathbf{x}\cdot\boldsymbol{\xi}}\,\mathrm{d}\mathbf{x} = O\left(|\boldsymbol{\xi}|^{-\frac{d+1}{2}}\right),$$

where the implicit constants depend on the product of the principal curvatures of ∂E_σ and stay bounded as long as the principal curvatures are bounded away from 0. Hence, by equation (3.3.18), these constants will be uniformly bounded for σ large enough. Applying this result with $f(\mathbf{x}) \equiv 1$ yields the desired result. □

Remark 3.3.14. Note that the estimates and the error terms obtained in [**40**, Theorem 2.29] depend on the bounds on the derivatives of the principal curvatures. By Lemma 3.3.9 the derivatives of the principal curvatures of ∂E_σ tend to zero as $\sigma \to \infty$, and therefore they could be bounded uniformly for $\sigma > \sigma_0$.

We now find the dependence on ϵ of the third summand in (3.3.22). We will choose the optimal value of ϵ such that the second and the third terms are both as small as possible. Splitting the third summand into two terms we use equation (3.3.23) and the fact that $\widehat{\Psi}$ is a Schwartz function to obtain

$$O\left(\sum_{\substack{\mathbf{m}\in\mathbb{Z}^p\\ m\neq 0}} \sigma^p \widehat{\chi}\left(\frac{\mathbf{m}\sigma}{1\mp\eta_\pm\epsilon}\right)\widehat{\Psi}\left(\frac{\epsilon\mathbf{m}\sigma}{1\mp\eta_\pm\epsilon}\right)\right) = O\left(\sum_{0<|\mathbf{m}|\le(\epsilon\sigma)^{-1}} \frac{\sigma^{\frac{p-1}{2}}\left(1\mp\eta_\pm\epsilon\right)^{\frac{p+1}{2}}}{|\mathbf{m}|^{\frac{p+1}{2}}}\right.$$
$$\left. + \sum_{|\mathbf{m}|>(\epsilon\sigma)^{-1}} \frac{\sigma^{\frac{p-1}{2}}\left(1\mp\eta_\pm\epsilon\right)^{\frac{p+1}{2}+N}}{|\mathbf{m}|^{\frac{p+1}{2}+N}(\sigma\epsilon)^N}\right),$$

for an arbitrary $N > 0$ which will be fixed below. Assuming that ϵ is small and and taking into account that the summands on the right hand side are decreasing in $|\mathbf{m}|$, we may estimate the first of those sums by

$$\begin{aligned}\sum_{0<|\mathbf{m}|\le(\epsilon\sigma)^{-1}} \frac{\sigma^{\frac{p-1}{2}}\left(1\mp\eta_\pm\epsilon\right)^{\frac{p+1}{2}}}{|\mathbf{m}|^{\frac{p+1}{2}}} &\asymp \sigma^{\frac{p-1}{2}}\int_1^{(\epsilon\sigma)^{-1}} \frac{r^{p-1}}{r^{\frac{p+1}{2}}}\,\mathrm{d}r\\ &= O\left(\epsilon^{\frac{1-p}{2}}\right).\end{aligned}$$

The second of those sums can be estimated, for N large enough that the integral converges, by

$$\begin{aligned}\sum_{|\mathbf{m}|>(\epsilon\sigma)^{-1}} \frac{\sigma^{\frac{p-1}{2}}\left(1\mp\eta_\pm\epsilon\right)^{\frac{p+1}{2}+N}}{|\mathbf{m}|^{\frac{p+1}{2}+N}(\sigma\epsilon)^N} &\asymp \sigma^{\frac{p-1}{2}}(\sigma\epsilon)^{-N}\int_{(\epsilon\sigma)^{-1}}^{\infty} \frac{r^{p-1}}{r^{\frac{p+1}{2}+N}}\,\mathrm{d}r\\ &= O\left(\epsilon^{\frac{1-p}{2}}\right).\end{aligned}$$

The optimal ϵ to make both $\sigma^p\epsilon$ and $\epsilon^{\frac{1-p}{2}}$ as small as possible is

$$\epsilon = \sigma^{\frac{-2p}{1+p}},$$

yielding that

$$\sum_{\mathbf{n}\in\mathbb{Z}^p} \chi_\epsilon^\pm\left(\frac{\mathbf{n}}{\sigma}\right) = \sigma^p \operatorname{Vol}(E_\sigma) + O\left(\sigma^{p-2+\frac{2}{1+p}}\right). \tag{3.3.24}$$

We now compute the volume of E_σ.

Lemma 3.3.15. *Let* $\Sigma = \mathbb{S}^{p-1} \cap \mathbb{R}^p_+$. *We have:*

$$\mathrm{Vol}_p(E_\sigma) = \frac{2^p\sqrt{q}^p}{\pi^p}\omega_p \prod_{j\in\tau_1} a_j - \frac{2^{2p}\sqrt{q}^p G_{p,q}}{\pi^p\sigma} \sum_{j\in\tau_1}\prod_{i\neq j} a_i + O\big(\sigma^{-2}\big), \tag{3.3.25}$$

where

$$G_{p,q} = \int_\Sigma g_j(\boldsymbol{\theta})\,\mathrm{d}\boldsymbol{\theta}, \tag{3.3.26}$$

for any of the functions g_j *defined by equation* (3.3.17).

Remark 3.3.16. Note that $G_{p,q}$ does not depend on j by the symmetry of the construction of g_j.

PROOF. By symmetry, we have that

$$\mathrm{Vol}(E_\sigma) = \frac{2^{2p}\sqrt{q}^p}{\pi^p}\int_\Sigma\int_0^{\rho(\boldsymbol{\theta})} r^{p-1}\prod_{j\in\tau_1} a_j\,\mathrm{d}r\,\mathrm{d}\boldsymbol{\theta}$$

where $\rho(\boldsymbol{\theta})$ is the unique positive root (in r) of the equation

$$r^2 + \frac{2r}{\sigma}\sum_{j\in\tau_1}\frac{g_j(\boldsymbol{\theta})}{a_j} + \frac{H}{\sigma^2} - 1 = 0.$$

One can observe that

$$\rho(\boldsymbol{\theta}) = 1 - \frac{1}{\sigma}\sum_{j\in\tau_1}\frac{g_j(\boldsymbol{\theta})}{a_j} + O\big(\sigma^{-2}\big).$$

Thus, we get that

$$\mathrm{Vol}(E_\sigma) = \frac{2^{2p}}{\pi^p}\prod_{j\in\tau_1} a_j \int_\Sigma \frac{1}{p} - \frac{1}{\sigma}\sum_{j\in\tau_1}\frac{g_j(\boldsymbol{\theta})}{a_j} + O\big(\sigma^{-2}\big)\,\mathrm{d}\boldsymbol{\theta}.$$

Integrating and replacing in the previous equation the definition of $G_{p,q}$ in equation (3.3.26) yields

$$\mathrm{Vol}(E_\sigma) = \frac{2^p}{\pi^p}\omega_p \prod_{j\in\tau_1} a_j - \frac{2^{2p}G_{p,q}}{\pi^p\sigma}\sum_{j\in\tau_1}\prod_{i\neq j} a_i + O\big(\sigma^{-2}\big). \tag{3.3.27}$$

□

Finally, we have to take into account the points that we have overcounted with coefficient $1/2$ on the hyperplanes $\{x_i = 0\}$. This is given in the following lemma.

Lemma 3.3.17. *The number of overcounted points on the hyperplanes* $\{x_i = 0\}$ *is*

$$R_\tau(\sigma) = \frac{\sqrt{q^p}2^p\omega_{p-1}\sigma^{p-1}}{4(2\pi)^{p-1}}\sum_{j\in\tau_1}\prod_{i\neq j} a_i + O\big(\sigma^{p-2}\big). \tag{3.3.28}$$

PROOF. One can observe that R_τ is given by

$$R_\tau(\sigma) = \frac{1}{2}\sum_{i\in\tau_1} \#\left\{\sigma^{-1}\mathbb{N}^{p-1}\cap E_\sigma \cap \{x_i = 0\}\right\}$$

Since E_σ is convex, rough lattice point counting estimates due to Gauss tell us that

$$R_\tau(\sigma) = \frac{1}{2}\sigma^{p-1}\sum_{i\in\tau_1} \mathrm{Vol}_{p-1}\left(E_\sigma \cap \{x_i = 0\}\right) + O\left(\sigma^{p-2}\right).$$

Computing the volumes in the same way as in the proof of the previous lemma yields the desired result. □

3.3.5. Proof of Proposition 3.3.1.

Recall that $\widetilde{N}_p$ is given by

$$\widetilde{N}_p(\sigma) = \sum_{\tau\in\mathscr{T}_p} \widetilde{N}^\tau(\sigma).$$

Observe that

$$\sum_{\tau\in\mathscr{T}_p} 2^{p+q}\prod_{j\in\tau_1} a_j = \mathrm{Vol}_p(\partial^q(\Omega))$$

and that

$$\begin{aligned}\sum_{\tau\in\mathscr{T}_p}\sum_{j\in\tau_1}\prod_{i\neq j} 2^{p+q} a_i &= (q+1)2^{p+q}\sum_{\tau\in\mathscr{T}_{p-1}}\prod_{j\in\tau_1} a_j \\ &= (q+1)\,\mathrm{Vol}_{p-1}(\partial^{q+1}\Omega)).\end{aligned} \tag{3.3.29}$$

Combining these two formulas with equations (3.3.19), (3.3.24) and Lemmas 3.3.15, 3.3.17, yields

$$\widetilde{N}_p(\sigma) = \frac{\sqrt{q^p}}{(2\pi)^p}\omega_p \mathrm{Vol}_p(\partial^q(\Omega))\sigma^p + c_p \mathrm{Vol}_{p-1}(\partial^{q+1}\Omega)\sigma^{p-1} + O\left(\sigma^{p-2+\frac{2}{p+1}}\right).$$

Using equation (3.3.29), we have that $c_p = c_p' + c_p''$, where

$$c_p' = -\frac{(q+1)\sqrt{q^p}G_{p,q}}{\pi^p}$$

comes from the second term in equation (3.3.25) and

$$c_p'' = -\frac{(q+1)\sqrt{q^p}\omega_{p-1}}{4(2\pi)^{p-1}}$$

is obtained from the principal term in equation (3.3.28).

We then have from equation (3.3.13) that

$$N_p(\sigma) = \frac{\sqrt{q^p}}{(2\pi)^p}\omega_p \mathrm{Vol}_p(\partial^q(\Omega))\sigma^p + c_p \mathrm{Vol}_{p-1}(\partial^{q+1}\Omega)\sigma^{p-1} + O\left(\sigma^{\eta_p}\right),$$

where

$$\eta_p = \max\left(p-1-1/p, p-2+\frac{2}{p+1}\right)$$
$$= \begin{cases} 2/3 & \text{if } p=2, \\ p-1-1/p & \text{otherwise.} \end{cases}$$

This completes the proof of Proposition 3.3.1.

3.3.6. Proof of Theorem 3.1.1.

Recall now that

$$N(\sigma) = \sum_{p=1}^{d-1} N_p(\sigma) + O(1).$$

Hence, applying the previous results we get

$$\begin{aligned} N(\sigma) &= N_{d-1}(\sigma) + N_{d-2}(\sigma) + O\left(\sigma^{\eta_{d-1}}\right) \\ &= \frac{1}{(2\pi)^{d-1}}\omega_{d-1}\,\mathrm{Vol}_{d-1}(\partial(\Omega))\sigma^{d-1} + c_{d-1}\,\mathrm{Vol}_{d-2}(\partial^2\Omega)\sigma^{d-2} \\ &\quad + \frac{2^{\frac{d-2}{2}}}{(2\pi)^{d-2}}\omega_{d-2}\,\mathrm{Vol}_{d-2}(\partial^2(\Omega))\sigma^{d-2} + O\left(\sigma^{\eta_{d-1}}\right) \\ &= C_1\,\mathrm{Vol}_{d-1}(\partial\Omega)\sigma^{d-1} + C_2\,\mathrm{Vol}_{d-2}(\partial^2\Omega)\sigma^{d-2} + O\left(\eta_{d-1}\right). \end{aligned}$$

We can write explicitly $C_2 = c'_{d-1} + c''_{d-1} + \frac{2^{\frac{d-2}{2}}\omega_{d-2}}{(2\pi)^{d-2}}$ to get indeed that

$$C_2 = \frac{2^{\frac{d-2}{2}}\omega_{d-2}}{(2\pi)^{d-2}} - \frac{2G_{d-1,1}}{\pi^{d-1}} - \frac{\omega_{d-2}}{2(2\pi)^{d-2}}$$

when $d \geq 3$ and that

$$N(\sigma) = \frac{\omega_1}{2\pi}\mathrm{Vol}_1(\partial\Omega)\sigma + O(1)$$

when $d = 2$.

We can now give explicit expressions for the constants $G_{p,q}$:

$$\begin{aligned} G_{p,q} &= \int_0^{\pi/2}\cdots\int_0^{\pi/2} \operatorname{arccot}\left(\frac{1}{\sqrt{q}}\left[1+\sum_{j=1}^{p-1}\cot^2\theta_j\prod_{i>j}\csc^2\theta_i\right]^{1/2}\right)\prod_{k=1}^{p-1}\sin^k(\theta_k)\,\mathrm{d}\theta_1\ldots\mathrm{d}\theta_{p-1} \\ &= \int_0^{\pi/2}\cdots\int_0^{\pi/2} \operatorname{arccot}\left(\frac{1}{\sqrt{q}}\prod_{j=1}^{p-1}\csc\theta_j\right)\prod_{k=1}^{p-1}\sin^k(\theta_k)\,\mathrm{d}\theta_1\ldots\mathrm{d}\theta_{p-1}. \end{aligned}$$

In particular, calculating the integrals for $q = 1, p = 2$ and $q = 1, p = 3$, we get:

$$G_{2,1} = \frac{1}{2}\left(-1+\sqrt{2}\right)\pi$$
$$G_{3,1} = \frac{1}{8}(-2+\pi)\pi$$

This concludes the proof of Theorem 3.1.1.

3.4. Further results

3.4.1. Concentration of eigenfunctions

In this section, we discuss the behaviour of the eigenfunctions, more precisely how they scar on the lower-dimensional facets of a cuboid. This is made precise in the following theorem, where we will slightly abuse notation and denote by u_k both a Steklov eigenfunction and its boundary trace.

Theorem 3.4.1. *Let* $\Omega \subset \mathbb{R}^d$ *be the cuboid with parameters* $a_1, \dots, a_d > 0$. *Let* $p \in \{1, \dots, d-1\}$ *and let* $\tau \in \mathcal{T}_p$. *Consider the set*

$$X_\tau = \{x = (\mathbf{x}_{\tau_1}, \mathbf{x}_{\tau_2}) \in \partial\Omega : x_j = \pm a_j \text{ for } j \in \tau_2\}.$$

Then, there exists a sequence of $L^2(\partial\Omega)$*-normalised eigenfunctions* $\{u_k\}$ *concentrating on* X_τ *and getting equidistributed around* X_τ *in the following sense: for each measurable* $U \subset X_\tau$ *and every* $\epsilon > 0$, *consider the set*

$$U_\epsilon = \{\mathbf{x} = (\mathbf{x}_{\tau_1}, \mathbf{x}_{\tau_2}) \in \partial\Omega : \mathbf{x}_{\tau_1} \in U \text{ and } \operatorname{dist}(\mathbf{x}, U) < \epsilon\}.$$

Then, for every $\epsilon > 0$,

$$\lim_{k\to\infty} \int_{U_\epsilon} |u_k(\mathbf{x})|^2 \, dx = \frac{\operatorname{Vol}_p(U)}{\operatorname{Vol}_p(X_\tau)}.$$

For example, on a cuboid of dimension 3, the set X_τ is a union of four parallel edges in case $p = 1$, while for $p = 2$ it is a union of two opposite faces.

PROOF. Without loss of generality, we will suppose that U is a subset of one of the connected components of X_τ, say the one where $x_j = a_j$ for all $j \in \tau_2$. For $k \in \mathbb{N}$, let $\mathbf{k} = (k, \dots, k) \in \mathbb{R}^p$ and consider the pair $(\boldsymbol{\alpha}^{(\mathbf{k})}, \boldsymbol{\beta}^{(\mathbf{k})})$ satisfying the compatibility and harmonicity conditions

$$\alpha_i^{(k)} \cot\left(\alpha_i^{(k)} a_i\right) = \beta_j^{(k)} \tanh\left(\beta_j^{(k)} a_j\right) \qquad \forall i \in \tau_1, j \in \tau_2$$
$$\sum_{i\in\tau_1} \left(\alpha_i^{(k)}\right)^2 = \sum_{j\in\tau_2} \left(\beta_j^{(k)}\right)^2$$

with $\boldsymbol{\alpha}^{(\mathbf{k})} \in I_{2\mathbf{k}}$. Note that this corresponds to choosing $\ell(i) = 0$ for all $i \in \tau_1$ and $\ell(j) = 1$ for all $j \in \tau_2$. Since

$$\left(\sum_{i\in\tau_1}\left(\alpha_i^{(k)}\right)^2\right)^{1/2} = k\overbrace{\left(\sum_{i\in\tau_1}\left(\frac{\pi}{2a_i}\right)^2\right)^{1/2}}^{A} + O(1) = Ak + O(1)$$

we have that for all $j \in \tau_2$,

$$\beta_j^{(k)} = \frac{A}{\sqrt{q}}k + O(1).$$

Let $v_k(\mathbf{x})$ be the associated eigenfunction, and observe that

$$\begin{aligned}
v_k(\mathbf{x})^2 &= \prod_{i\in\tau_1}\sin^2\left(\alpha_i^{(k)}x_i\right)\prod_{j\in\tau_2}\cosh^2\left(\beta_j^{(k)}x_j\right)\\
&= \frac{1}{2^p}\prod_{i\in\tau_1}\left(1-\cos\left(2\alpha_i^{(k)}x_i\right)\right)\prod_{j\in\tau_2}\cosh^2\left(\beta_j^{(k)}x_j\right),\\
&= \frac{1}{2^p}\prod_{i\in\tau_1}\left(1-\cos\left(\left(\frac{\pi k}{a_i}+O(1)\right)x_i\right)\right)\prod_{j\in\tau_2}\cosh^2\left(\left(\frac{A}{\sqrt{q}}k+O(1)\right)x_j\right).
\end{aligned}$$

Defining the normalised eigenfunction

$$u_k = \frac{v_k}{\|v_k\|_{L^2(\partial\Omega)}},$$

we estimate both $\|v_k\|^2 := \|v_k\|^2_{L^2(\partial\Omega)}$ and $\int_{U_\epsilon} v_k(x)^2\,\mathrm{d}x$. For $\|v_k\|^2$, we have that

$$\begin{aligned}
\|v_k\|^2 &= \frac{1}{2^p}\prod_{i\in\tau_1}\int_{-a_i}^{a_i}1-\cos\left(\left(\frac{\pi k}{a_i}+O(1)\right)x_i\right)\mathrm{d}x_i\prod_{j\in\tau_2}\int_{-a_j}^{a_j}\cosh^2\left(\beta_jx_j\right)\mathrm{d}x_j\\
&= \frac{1}{2^d}\left(\mathrm{Vol}_p(X_\tau)+o(1)\right)\prod_{j\in\tau_2}\int_{-a_j}^{a_j}\cosh^2\left(\beta_jx_j\right)\mathrm{d}x_j
\end{aligned} \tag{3.4.1}$$

from the Riemann-Lebesgue lemma and the fact that

$$\mathrm{Vol}(X_\tau) = 2^q\prod_{i\in\tau_1}\int_{-a_i}^{a_i}\mathrm{d}x_i.$$

Furthermore, for all $j \in \tau_2$ we have that

$$\begin{aligned}
\int_{-a_j}^{a_j}\cosh^2\left(\beta_jx_j\right)\mathrm{d}x_j &= \frac{1}{4}\int_{-a_j}^{a_j}e^{2\left(\frac{A}{\sqrt{q}}k+O(1)\right)x_j}+e^{-2\left(\frac{A}{\sqrt{q}}k+O(1)\right)x_j}+2\,\mathrm{d}x_j\\
&= \frac{\sqrt{q}}{4Ak}e^{2\frac{A}{\sqrt{q}}ka_j}\left(1+o(1)\right).
\end{aligned} \tag{3.4.2}$$

Setting $C = \frac{\sqrt{q}}{4A}$, equations (3.4.1) and (3.4.2) yield together that

$$\|v_k\|^2 = \frac{C^q}{2^dk^q}\mathrm{Vol}_p(X_\tau)\left(\prod_{j\in\tau_2}e^{2\frac{A}{\sqrt{q}}ka_j}\right)(1+o(1)) \tag{3.4.3}$$

We now also compute the integral of v_k^2 on U_ϵ where we get, in a similar fashion to (3.4.1) that

$$\int_{U_\epsilon} v_k(x)^2 \,\mathrm{d}x = \frac{1}{2^p}\left(\mathrm{Vol}_p(U) + o(1)\right) \prod_{j\in\tau_2} \int_{a_j-\epsilon}^{a_j} \cosh^2\left(\beta_j x_j\right) \mathrm{d}x_j \tag{3.4.4}$$

We also have that

$$\int_{a_j-\epsilon}^{a_j} \cosh^2\left(\beta_j x_j\right) \mathrm{d}x_j = \frac{C}{2} e^{2\frac{A}{\sqrt{q}} k a_j} \left(1 + o(1)\right), \tag{3.4.5}$$

where once again $C = \frac{\sqrt{q}}{4A}$. Together, equations (3.4.4) and (3.4.5) yield

$$\int_{U_\epsilon} v_k(\mathbf{x}) \,\mathrm{d}\mathbf{x} = \frac{C^q}{2^d k^q} \mathrm{Vol}_p(U) \left(\prod_{j\in\tau_2} e^{2\frac{A}{\sqrt{q}} k a_j} \right) \left(1 + o(1)\right). \tag{3.4.6}$$

Finally, putting equations (3.4.3) and (3.4.6) together yields indeed that

$$\lim_{k\to\infty} \int_{U_\epsilon} u_k(x)^2 \,\mathrm{d}x = \lim_{k\to\infty} \int_{U_\epsilon} \frac{v_k(x)^2}{\|v_k\|^2} \,\mathrm{d}x = \frac{\mathrm{Vol}_p(U)}{\mathrm{Vol}_p(X_\tau)},$$

concluding the proof.

□

3.4.2. The first eigenfunction

In this section, we investigate the lowest nonzero eigenvalue σ_1 on the cuboid. Let us first find the form of an eigenfunction u associated with σ_1. By Courant's nodal theorem u has exactly 2 nodal domains. Thus, one of the factors u_j will have 2 nodal domains on the interval $[-a_j, a_j]$ and all the other factors only one nodal domain. In other words there is one odd factor, and all the others are positive even functions. We show the following proposition.

Proposition 3.4.2. *Suppose that* $a_1 \le \ldots \le a_d$. *Then there is* $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_{d-1})$ *and* $\alpha_d = |\boldsymbol{\beta}| < \frac{\pi}{2a_d}$ *such that*

$$u(x_1, \ldots, x_d) = \sin(\alpha_d x_d) \prod_{k=1}^{d-1} \cosh(\beta_k x_k)$$

is an eigenfunction with eigenvalue σ_1.

PROOF. We will fist show that u is a product of one sine factor and $d-1$ hyperbolic cosine factors. Suppose that one of the trigonometric factors was a cosine. Let us study the number of nodal domains of $\cos(\alpha x_j)$ on the interval $[-a_j, a_j]$. By the Steklov boundary condition we have that

$$\cos(\alpha a_j) = -\sigma\alpha \sin(\alpha a_j),$$

There are three possible cases, whether $\sin(\alpha a_j)$ is equal to, greater than or smaller than 0. Since the eigenvalue $\sigma_0 = 0$ is simple, if $\sin(\alpha a_j) = 0$ it would imply that $\cos(\alpha a_j) = 0$, which is impossible.

If $\sin(\alpha_j a_j) > 0$, we have that $\cos(\alpha a_j)$ is negative. This would imply that the function $\cos(\alpha x)$ has changed sign on $[0, a_j]$ and since it is even it will have at least two zeroes on $[-a_j, a_j]$, that is at least three nodal domains, in contradiction with Courant's nodal theorem.

Finally, if $\sin(\alpha a_j) < 0$, this implies that $\alpha a_j > \pi$, meaning that $\cos(\alpha x_j)$ has changed sign at least once on $[0, a_j]$. This implies once again that there are at least three nodal domains, completing the proof that no factor is cosine.

Since there can only be one odd factor, if one is linear all the other factors are a combination of cosine and hyperbolic cosine. We just proved that none of the factors are cosine, and it is impossible for a product of linear functions with only hyperbolic cosines to respect the harmonicity condition (3.2.2). We therefore deduce that the only odd factor of u is a sine, and by the above discussion all of the other factors are hyperbolic cosine. This implies that there exists some $1 \leq j \leq d$, α_j and β_k, $k \neq j$ such that

$$u(x_1, \dots, x_d) = \sin(\alpha_j x_j) \prod_{k \neq j} \cosh(\beta_k x_k),$$

and $\alpha_j a_j < \pi/2$. The compatibility equations (3.2.5) hence become

$$\alpha_j \cot(\alpha_j a_j) = \beta_k \tanh(\beta_k a_k)$$
$$\alpha_j^2 = |\boldsymbol{\beta}|^2 = \sum_{k \neq j} \beta_k^2,$$

and σ_1 is any member of the first equality. We show that σ_1 is smallest when a_j is the largest side, i.e. $a_j = a_d$. Suppose not. Then, there is $1 \leq k \leq d-1$ such that an eigenvalue associated with

$$v(x_1, \dots, x_d) = \sin(|\boldsymbol{\gamma}| x_j) \prod_{k \neq j} \cosh(\gamma_j x_j).$$

is smaller than the one associated with

$$u(x_1, \dots, x_d) = \sin(|\boldsymbol{\beta}| x_d) \prod_{k \neq d} \cosh(\beta_k x_k).$$

The compatibility equations imply that for all $k \neq j$ and $k \neq d$,

$$\gamma_k \tanh(\gamma_k a_k) < \beta_k \tanh(\beta_k a_k).$$

Since $x\tanh(ax)$ is an increasing function, we deduce that $\gamma_k \le \beta_k$ for all such k. However, we also have that

$$|\boldsymbol{\gamma}|\cot(|\boldsymbol{\gamma}|a_k) < |\boldsymbol{\beta}|\cot(|\boldsymbol{\beta}|a_d)$$

and since $x\cot(ax)$ is decreasing on its first period and $a_k \le a_d$, this implies that $|\boldsymbol{\gamma}| > |\boldsymbol{\beta}|$. From this, we therefore have that

$$\beta_j^2 + \sum_{k\neq j,d} \beta_k^2 < \gamma_d^2 + \sum_{k\neq j,d} \gamma_k^2.$$

Since for all $k \neq j,d$ we have that $\gamma_k < \beta_k$, we therefore deduce that $\beta_j < \gamma_d$. However, once again using the compatibility conditions, we have that

$$\gamma_d \tanh(\gamma_d a_d) < \beta_j \tanh(\beta_j a_j).$$

Since $a_d > a_j$, by monotonicity of $x\tanh(ax)$ we deduce that $\gamma_d < \beta_j$, a contradiction. Hence, we have that the first eigenfunction is, taking into account that $\alpha_d = |\boldsymbol{\beta}|$,

$$u(x_1,\ldots,x_d) = \sin(|\boldsymbol{\beta}|x_d)\prod_{j=1}^{d-1}\cosh(\beta_j x_j),$$

□

concluding the proof of the proposition.

3.4.3. Proof of Theorem 3.1.6

The first eigenvalue is given by the following min-max principle :

$$\sigma_1(\Omega) = \inf_{\substack{u\in C^\infty(\Omega)\\ \int_{\partial\Omega} u=0}} R_\Omega[u] = \inf_{\substack{u\in C^\infty(\Omega)\\ \int_{\partial\Omega} u=0}} \frac{\int_\Omega |\nabla u|^2}{\int_{\partial\Omega} u^2}.$$

Denote by Ω_0 the cube $[-1,1]^d$. Then, for any cuboid $\Omega = [-a_1,a_1]\times\ldots\times[-a_d,a_d]$ we have that

$$\int_\Omega f(x)\,\mathrm{d}x = \int_{\Omega_0} f(a_1x_1,\ldots,a_dx_d)\prod_{i=1}^d a_i\,\mathrm{d}x$$

and

$$\begin{aligned}\int_{\partial\Omega} f(x)\,\mathrm{d}x &= \sum_{j=1}^d \int_{\partial\Omega\cap\{x_j=\pm a_j\}} f(x)\,\mathrm{d}x \\ &= \sum_{j=1}^d \int_{\partial\Omega_0\cap\{x_j=\pm1\}} f(a_1x_1,\ldots,a_dx_d)\prod_{i\neq j} a_i\,\mathrm{d}x.\end{aligned} \tag{3.4.7}$$

This allows us to consider integration only on Ω_0 for R_Ω. Observe that the eigenspace of $\sigma_1(\Omega_0)$ has dimension d, and that a basis for it is given by

$$u_j(x_1,\ldots,x_d) = \sin(|\beta| x_d) \prod_{i\neq j} \cosh(\beta_i x_i).$$

The eigenfunctions u_j are orthogonal to constants in the scalar product given by the rescaled integral (3.4.7). Indeed, on all faces where the sin factor is not constant, the integral vanishes since it is an odd function. On the pair of faces where the sin factor is constant, we have that $u_j(x_1,\ldots,a_j,\ldots,x_d) = -u_j(x_1,\ldots,-a_j,\ldots,x_d)$ hence the integrals cancel out on these two faces.

Consider the eigenfunction

$$u = \sum_{j=1}^{d} u_j.$$

It is easy to see that the integral of u^2 on any face of Ω_0 is identical, and we have that $R_{\Omega_0}[u] = \sigma_1(\Omega_0)$. We now compute

$$\begin{aligned}\frac{1}{R_\Omega[u]} &= \frac{\sum_{j=1}^d \prod_{i\neq j} a_i \int_{\partial\Omega_0 \cap \{x_j=\pm 1\}} u^2 \,\mathrm{d}x}{\prod_{j=1}^d a_j \int_{\Omega_0} |\nabla u|^2 \,\mathrm{d}x} \\ &= \frac{1}{R_{\Omega_0}[u]} \frac{\frac{1}{d}\sum_{j=1}^d \prod_{i\neq j} a_i}{\prod_{j=1}^d a_j}.\end{aligned}$$

Fix the volume $\mathrm{Vol}_d(\Omega) = \mathrm{Vol}_d(\Omega_0)$, hence $\prod_j a_j = 1$. Then, from the inequality of arithmetic and geometric means,

$$\frac{R_{\Omega_0}[u]}{R_\Omega[u]} = \frac{1}{d}\sum_{j=1}^{d}\prod_{i\neq j} a_i \geq \left(\prod_{j=1}^{d} a_j^{d-1}\right)^{1/d} = 1,$$

with equality if and only if for all j,k, $\prod_{i\neq j} a_i = \prod_{i\neq k} a_i$, which is true if and only if $a_j = a_k$ for all j,k, which implies in turn that $\sigma_1(\Omega) \leq \sigma_1(\Omega_0)$, with equality if and only if Ω is a cube.

On the other hand, fix the area, $\mathrm{Vol}_{d-1}(\Omega) = \mathrm{Vol}_{d-1}(\Omega_0)$, hence $\sum_j \prod_{i\neq j} a_i = d$. Then,

$$\frac{R_{\Omega_0}[u]}{R_\Omega[u]} = \left(\prod_j a_j\right)^{-1} = \left(\prod_{j=1}^{d}\prod_{i\neq j} a_i\right)^{\frac{d(1-d)}{d}} \geq \left(\frac{1}{d}\sum_{j=1}^{d}\prod_{i\neq j} a_i\right)^{\frac{1-d}{d}} = 1,$$

with equality in the same case as before. Once again, this implies that $\sigma_1(\Omega) \leq \sigma_1(\Omega_0)$, with equality if and only if Ω is a cube.

3.4.4. Proof of Corollary 3.1.8

We want to show that among all rectangles, the Steklov spectrum determines the lengths a_1,a_2 of its sides. From spectral asymptotics, the perimeter of the rectangle is obtained, giving $L = a_1 + a_2$, supposing without loss of generality that $a_1 \le a_2$. On the other hand, we have σ_1, and we know that it is the smallest root of

$$\sigma_1 = \alpha \cot(\alpha a_1) = \alpha \tanh(\alpha a_2).$$

Rewriting these to yield a_2 as a function of α, L and σ_1 gives

$$a_2 = f(\alpha) = \frac{1}{\alpha} \operatorname{arctanh}\left(\frac{\sigma_1}{\alpha}\right) \tag{3.4.8}$$

and

$$a_2 = g(\alpha) = L - \frac{1}{\alpha} \operatorname{arccot}\left(\frac{\sigma_1}{\alpha}\right). \tag{3.4.9}$$

Given σ_1 and L, the intersection of these curves yield possible values a_2 for α. We now show that they intersect at only one point. Equation (3.4.8) is defined for $\alpha > \sigma_1$ and taking the derivative yields

$$f'(\alpha) = -\frac{\operatorname{arctanh}\left(\frac{\sigma_1}{\alpha}\right)}{\alpha^2} - \frac{\sigma_1}{\alpha^3\left(1 - \frac{\sigma_1^2}{\alpha^2}\right)}, \tag{3.4.10}$$

which is always negative for $\alpha > \sigma_1$, hence f is decreasing. We now show that g is increasing on $[\sigma_1, \infty)$. We have that

$$g'(\alpha) = \frac{\operatorname{arccot}\left(\frac{\sigma_1}{\alpha}\right)}{\alpha^2} - \frac{\sigma_1}{\alpha^3\left(1 + \frac{\sigma_1^2}{\alpha^2}\right)}.$$

Thus, g' is positive if

$$\alpha \operatorname{arccot}\left(\frac{\sigma_1}{\alpha}\right)\left(1 + \frac{\sigma_1^2}{\alpha^2}\right) - \sigma_1 \ge 0.$$

However, we have that

$$\alpha \operatorname{arccot}\left(\frac{\sigma_1}{\alpha}\right)\left(1 + \frac{\sigma_1^2}{\alpha^2}\right) - \sigma_1 \ge \frac{\pi}{4}\alpha + \frac{\pi}{4}\frac{\sigma_1^2}{\alpha^2} - \sigma_1$$

hence we need to have that $\alpha^2 - \frac{4\sigma_1\alpha}{\pi} + \sigma_1^2 \ge 0$. This quantity is positive at $\alpha = \sigma_1$ since $2 \ge 4/\pi$ and it is increasing since

$$2\alpha > \frac{4\sigma_1}{\pi}$$

for $\alpha \ge \sigma_1$. We conclude that g is increasing. This implies that f and g have exactly one intersection point, say at α_0. We have that $a_2 = f(\alpha_0) = g(\alpha_0)$ and $a_1 = L - a_2$. Note that since the square maximises σ_1 and since the eigenvalues are continuous functions of the side lengths of a rectangle this means that among all rectangles with given area or perimeter, σ_1 is a decreasing function of a_2.

3.A. Auxiliary result

Lemma 3.A.1. *Let*

$$f_i(\mathbf{x}) = \operatorname{arccot}\left(c\left[1+\sum_{j\neq i}\left(\frac{x_j}{x_i}\right)^2\right]^{1/2}\right).$$

for some $c > 0$ *and where by convention* $\operatorname{arccot}(\infty) = 0$*, and let* $\psi : \mathbb{R}^p \to \mathbb{R}^p$ *be a bounded function. Then,*

$$\left|f_i(\mathbf{x}+\psi(\mathbf{x})) - f_i(\mathbf{x})\right| = O\left(|\mathbf{x}|^{-1}\right). \tag{A.1}$$

Proof. We have that

$$\left|f_i(\mathbf{x}) - f_i(\mathbf{x}_0)\right| = O\left(|\mathbf{x}-\mathbf{x}_0||\nabla f(\mathbf{x}_0)|\right).$$

Consider spherical coordinates $(r,\theta_1,\ldots,\theta_{p-1})$

$$r = |\mathbf{x}|,$$

$$x_j = r\cos(\theta_j)\prod_{i<j}\sin(\theta_i),$$

where by convention $\theta_p = 0$.

Denote $\mathbf{x} = (r,\boldsymbol{\theta})$ and $\mathbf{x}+\psi(\mathbf{x}) = (r_\psi,\boldsymbol{\theta}_\psi)$. It is clear that since ψ is bounded we have that

$$|\boldsymbol{\theta}-\boldsymbol{\theta}_\psi| = O\left(r^{-1}\right).$$

Indeed, from planar geometry we get that

$$\tan(|\boldsymbol{\theta}-\boldsymbol{\theta}_\psi|) \le \frac{\sup_{\mathbf{x}\in\mathbb{R}^p}\psi(\mathbf{x})}{r}.$$

One can observe that the functions in Equation (A.1) depend only on $\boldsymbol{\theta}_\psi$ and $\boldsymbol{\theta}$. Hence, showing that the gradient is bounded in $\boldsymbol{\theta}$ implies that $|f_i(\mathbf{x}+\psi(\mathbf{x})) - f_i(\mathbf{x})| = O\left(r^{-1}\right)$.

By symmetry, we can suppose without loss of generality that $i = p$ in Equation (A.1). Then, using repeatedly the identity $1+\cot^2\theta = \csc^2\theta$ we have that

$$f_p(\mathbf{x}) = \operatorname{arccot}\left(c\prod_{j=1}^{p-1}\csc\theta_j\right).$$

Now, we have that

$$\partial_{\theta_j} f_p(x) = c\frac{\cot\theta_j \prod_{k=1}^{p-1}\csc\theta_k}{1+c^2\prod_{k=1}^{p-1}\csc^2\theta_k}.$$

This is bounded since when $\theta_j \to n\pi$, the singularities are of the same order on the numerator and denominator while when it is any other $\theta_i \to n\pi$, the singularities are of order 1 in the numerator and 2 in the denominator. This concludes the proof. □

3.B. Positivity of the constant C_2

We can rewrite C_2 as

$$C_2 = \frac{(2^{\frac{d+2}{2}}-2)\pi\omega_{d-2} - 2^{d+1}G_{d-1,1}}{2(2\pi)^{d-1}},$$

and we need to show that $C_2 > 0$ for $d \geq 3$. This will be done by showing that

$$\frac{(2^{\frac{d+2}{2}}-2)\pi\omega_{d-2}}{2^{d+1}G_{d-1,1}} > 1. \tag{B.1}$$

Let us first observe that the integrand in $G_{d-1,1}$ is positive and that for any $\boldsymbol{\theta} \in [0,\pi/2]^{d-2}$, we have that

$$\operatorname{arccot}\left(\prod_{j=1}^{d-2}\csc\theta_j\right) \leq \operatorname{arccot}(1) < 1.$$

Hence,

$$\begin{aligned}
G_{d-1,1} &= \int_0^{\pi/2}\dots\int_0^{\pi/2} \operatorname{arccot}\left(\prod_{j=1}^{d-2}\csc\theta_j\right)\prod_{k=1}^{d-2}\sin^k(\theta_k)\,\mathrm{d}\theta_1\dots\mathrm{d}\theta_{d-2} \\
&\leq \prod_{k=1}^{d-2}\int_0^{\pi/2}\sin^k(\theta_k)\,\mathrm{d}\theta_k \\
&= \frac{2^{2-d}\pi^{\frac{2-d}{2}}}{\Gamma\left(\frac{d}{2}\right)}.
\end{aligned}$$

The last equality is true for $d = 3$, and is seen to be true for all $d \geq 3$ by induction using the identity [**30**, 3.621 (1)]

$$\int_0^{\pi/2}\sin^k(\theta)\,\mathrm{d}\theta = 2^{k-1}B\left(\frac{k+1}{2},\frac{k+1}{2}\right).$$

and the Gamma function duplication identity

$$\Gamma(\mu)\Gamma\left(\mu+1/2\right) = 2^{1-2\mu}\sqrt{\pi}\,\Gamma(2\mu).$$

Using the fact that

$$\omega_{d-2} = \frac{\pi^{\frac{d-2}{2}}}{\Gamma\left(\frac{d}{2}\right)}$$

and replacing in Equation (B.1) we have that

$$\frac{(2^{\frac{d+2}{2}} - 2)\pi\omega_{d-2}}{2^{d+1} G_{d-1,1}} \geq \frac{(2^{\frac{d+2}{2}} - 2)\pi}{8} > 1$$

for all $d \geq 3$, concluding the proof that $C_2 > 0$.

Chapitre 4

Eigenvalue optimisation on flat tori and lattice points in anisotropically expanding domains

par

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Eigenvalue optimisation on flat tori and lattice points in anisotropically expanding domains

RÉSUMÉ. Dans cet article, nous nous sommes intéressés à la maximisation de la k-ième valeur propre du laplacien parmi les tores plats de volume 1 en dimension d alors que k tend vers l'infini. Nous montrons qu'en toute dimension, un tore maximal existe pour chaque k, mais que n'importe quelle suite de maximiseurs dégénère lorsque k tend vers l'infini dès que la dimension est inférieure à 10. De plus, nous obtenons des bornes supérieures et inférieures pour le rayon d'injectivité d'une suite de tores maximisants. Nous montrons que le même taux de dégeneres-cence peut être observé pour une suite de bouteilles de Klein plates maximisant la k-ième valeur propre du laplacien. Ces résultats contrastent avec ceux qui ont été récemment obtenus par Gittins et Larson, qui montrent que des suites de cuboïdes optimaux, pour les conditions frontières de Dirichlet ou de Neumann, convergent vers le cube, peu importe la dimension. Nous obtenons nos résultats grâce à des asymptotiques de Weyl explicites qui tiennent tant et aussi longtemps que le rayon d'injectivité tend vers zéro suffisemment lentement en termes du paramètre spectral. Nous réduisons le problème à celui du compte de points d'un réseau dans des domaines étirés de manière anisotropique, wet nous généralisons les méthodes de Yu. Kordyukov et de A. Yakovlev en considérant des domaines qui sont étirés à des vitesses différentes dans différentes directions.
Mots clés : Optimisation asymptotique ; asymptotique spectrale ; compte de points de treillis.

ABSTRACT. This paper is concerned with the maximisation of the kth eigenvalue of the Laplacian amongst flat tori of unit volume in dimension d as k goes to infinity. We show that in any dimension maximisers exist for any given k, but that any sequence of maximisers degenerates as k goes to infinity when the dimension is at most 10. Furthermore, we obtain specific upper and lower bounds for the injectivity radius of any sequence of maximisers. We show that the same rate of degeneracy is also exhibited by sequences of flat Klein bottles maximising the kth eigenvalue of the Laplacian. These results contrast with those obtained recently by Gittins and Larson, stating that sequences of optimal cuboids for either Dirichlet or Neumann boundary conditions converge to the cube no matter the dimension. We obtain these results via explicit Weyl asymptotics that hold as long as the injectivity radius goes to zero slowly enough in terms of to the spectral parameter. We reduce the problem at hand to counting lattice points inside anisotropically expanding domains, where we generalise methods of Yu. Kordyukov and A. Yakovlev by considering domains that expand at different rates in various directions.
Keywords: Asymptotic optimisation ; spectral asyptotics ; lattice counting.

4.1. Introduction and main results

Let (M,g) be a smooth closed Riemannian manifold of dimension d we study the Laplace eigenvalue problem

$$\Delta u + \lambda u = 0.$$

The eigenvalues of the Laplacian form a discrete, nondecreasing sequence, repeating every eigenvalue according to multiplicity,

$$0 = \lambda_0(M,g) \le \lambda_1(M,g) \le \ldots \nearrow \infty$$

accumulating only at infinity.

4.1.1. Asymptotic eigenvalue optimisation

In this paper, we study the maximisation problem

$$\Lambda_k^\star(\mathscr{G}) := \sup_{g\in\mathscr{G}} \Lambda_k(M,g) := \sup_{g\in\mathscr{G}} \mathrm{Vol}_g(M)^{2/d} \lambda_k(M,g), \tag{4.1.1}$$

where $\mathscr{G}$ is some class of metrics on M. This problem has been studied extensively for $k = 1$ in many settings: closed manifolds, manifolds with Neumann boundary conditions, and manifolds with Dirichlet boundary conditions in which case one minimises Λ_k. Note that for closed manifolds it only makes sense to maximise Λ_k. Indeed, for any k one can find a sequence of metrics g_n of unit volume such that $\Lambda_k(M,g_n) \to 0$ as $n \to \infty$ by considering a sequence of metrics that degenerate to a disjoint union of $k+1$ closed manifolds touching at a point.

An interesting feature is that the extremisers for low eigenvalues are in general very symmetric. Indeed, the Faber-Krahn inequality [**23, 51, 52**] and the Szegö-Weinberger inequality [**75, 78**] imply that the ball is the extremiser for Λ_1 with Dirichlet or Neumann, respectively, boundary conditions in any dimension. In the case of closed manifolds, Hersch has shown [**34**] that the round sphere is the maximiser for Λ_1 amongst two-dimensional spheres.

For higher eigenvalues on domains, one does not expect those symmetries to appear. Indeed, A. Berger has shown [**9**] that disks or union of disks can minimise Λ_k on domains in the plane with Dirichlet boundary conditions only finitely many times. Furthermore, numerical experiments of Antunes and Freitas [**3**] suggest that optimal domains in $\mathbb{R}^2$ may not exhibit many symmetries for $k \ge 5$. However, the same authors investigated in [**2**] the behaviour of optimal domains as k goes to infinity. More specifically, they showed that amongst rectangles with Dirichlet boundary condition, the sequence of rectangles minimising Λ_k converges to the square in the Hausdorff metric. This has led to a series of papers [**7, 8, 28**] culminating in a proof by Gittins and Larson, who show that in any dimension and with either Neumann or Dirichlet boundary conditions the sequence of optimal cuboids converges to the cube.

For closed manifolds, without any restriction on the metric one does not even have a maximiser. Indeed, Colbois and Dodziuk have shown in [**18**] that amongst all metrics of fixed volume on a manifold, one can make λ_1 as large as possible. For metrics on closed surfaces, one does not necessarily expect the sequence of maximising metrics to converge to a smooth metric. For instance, Karpukhin, Nadirashvili, Penskoi and Polterovich [**46**] obtained in a recent preprint that the maximising metric on the two-dimensional sphere for the kth Laplace eigenvalue degenerates to a union of k kissing round spheres.

We study the maximisation problem (4.1.1) for metrics on two classes of closed manifold. The first one is the class $\mathscr{M}$ of flat metrics on tori in dimension d. Let $\mathscr{L} = \mathrm{GL}_d(\mathbb{R})/\mathrm{GL}_d(\mathbb{Z})$ be the set of lattices in $\mathbb{R}^d$ equipped with the quotient topology. We identify $\mathscr{M}$ with $\mathscr{L}$ since

$$\mathscr{M} = \left\{\mathbb{T}_\Gamma = \mathbb{R}^d/\Gamma : \Gamma \in \mathscr{L}\right\}.$$

As such, convergence in $\mathscr{M}$ will be identified with convergence in $\mathscr{L}$. We study the properties of maximisers to (4.1.1) in $\mathscr{L}_0$ the subset of all lattices with unit determinant, which corresponds to subset $\mathscr{M}_0$ of flat tori with unit volume.

The second class that we study is the set $\mathscr{E}$ of flat metrics on Klein bottles. Flat Klein bottles are quotients of two-dimensional flat rectangular tori and as such are described by the two-parameters family

$$\mathscr{E} := \left\{K(a,b) := \left(\mathbb{R}^d/(a\mathbb{Z} \oplus b\mathbb{Z})\right)/\sim : (a,b) \in \mathbb{R}^2_+\right\},$$

where $\sim$ is the relation $(x,y) \sim (x + a/2, b - y)$. Once again, we study the properties of maximisers of (4.1.1) in the class $\mathscr{E}_0$ of Klein bottles with unit volume, *i.e.* the family $K(a,b)$ where $ab = 2$.

Before discussing asymptotic properties of maximisers to the problem (4.1.1), we start by proving that such maximisers do exist.

Theorem 4.1.1. *For all $k \in \mathbb{N}$, there exist $\mathbb{T}_k^\star \in \mathscr{M}_0$ and $K_k^\star \in \mathscr{E}_0$ maximising the variational problems*

$$\Lambda_k^\star(\mathscr{M}) = \sup_{\mathbb{T}_\Gamma \in \mathscr{M}} \Lambda_k(\mathbb{T}_\Gamma).$$

and

$$\Lambda_k^\star(\mathscr{E}) = \sup_{K\in\mathscr{E}} \Lambda_k(K).$$

The behaviour of maximisers for tori and Klein bottles contrasts both with the results obtained for cuboids where the optimal cuboid converges to the cube and with the degeneracy results of [**18**] and [**46**]. Indeed, we show that for tori of dimension $2 \le d \le 10$, the sequence of optimisers has no limit points. However, we also show that this degeneracy can happen without changing the curvature as in [**46**], or in [**18**].

Furthermore, we obtain a rate of degeneracy in terms of the injectivity radius. This is similar to the results in [**28**] where the rate of convergence to the cube is given. The range $2 \le d \le 10$ are the dimensions for which the volume of the unit ball ω_d is larger than $\omega_1 = 2$. In higher dimensions, the same type of result may hold, but the degeneracy certainly doesn't happen in the same way.

Theorem 4.1.2. *In dimension* $2 \le d \le 10$*, no flat torus is a limit point of a sequence* $\{\mathbb{T}_k^\star\}$*. For all* $\delta > 0$*, we have that the injectivity radius of* $\mathbb{T}_k^\star$ *respects*

$$k^{-\frac{(1-d)^2}{d}} \ll \operatorname{inj}(\mathbb{T}_k^\star) \ll k^{-\frac{1}{d}+\delta}. \tag{4.1.2}$$

The lower bound is valid for all dimensions $d \in \mathbb{N}$.

Remark 4.1.3. In dimension 2, the lower bound and the upper bound are, at least to polynomial order, the same. The discrepancy in higher dimension between the upper and lower bounds is due to the fact that we find lower bounds on both the first and last successive minima of the associated dual lattice Γ^*, defined in equation (4.1.11). The lower bound on the last successive minima of Γ^* gives directly an upper bound on the first successive minima of Γ via Banaszczyk's transference theorem, and this quantity corresponds to the injectivity radius of $\mathbb{T}_\Gamma$. The lower bound on the first successive minimum of Γ^*, does not give a lower bound on the injectivity radius so directly. Indeed, the lower bound on the first successive minimum of Γ^* yields an upper bound on the last successive minimum of Γ, which in turns provides trivial upper bounds on the last $d-1$ successive minima of Γ, which are necessary to use Minkowski's Theorem. The need for these trivial upper bounds reduces the strength of the estimation.

Our methods also allow us to study sequences of optimisers in the moduli space $\mathscr{E}$ of flat Klein bottles. Indeed, we also have degeneracy in this case, and we can also describe the rate of degeneracy.

Theorem 4.1.4. *No flat Klein bottle is a limit point of a sequence $\{K_k^\star\}$. Furthermore, for all $\delta > 0$, we have that the injectivity radius of $K_k^\star$ respects*

$$k^{-\frac{1}{2}} \ll \mathrm{inj}(K_k^\star) \ll k^{-\frac{1}{2}+\delta}. \tag{4.1.3}$$

4.1.2. Explicit exponent for the remainder in Weyl's law

In the papers [**2, 7, 8, 28**] on optimal cuboids a prominent feature consisted in finding uniform bounds on the eigenvalue counting function

$$N(\lambda; M) = \#\{\lambda_k(M) < \lambda\}.$$

Weyl's law states that for any fixed (M,g) the counting function $N(\lambda; M)$ enjoys the asymptotics

$$N(\lambda; M) = \frac{\omega_d}{(2\pi)^d}\lambda^{\frac{d}{2}} + R(\lambda; M),$$

where $R(\lambda; M) = o\left(\lambda^{\frac{d}{2}}\right)$ and ω_d is the volume of a unit ball in dimension d. Under the hypothesis that periodic geodesics have measure 0 in the cosphere bundle of M, Duistermaat and Guillemin [**22**] have shown that the remainder in equation (4.1.2) satisfies

$$R(\lambda; M) = o\left(\lambda^{\frac{d-1}{2}}\right). \tag{4.1.4}$$

Note that the size of $R(\lambda; M)$ depends on the geometry of M in a non trivial way. Indeed, for any fixed λ one can find a sequence g_n of metrics on M such that $N(\lambda; (M,g_n)) \to \infty$ as $n \to \infty$ for the same reason one can make λ_k arbitrarily small. However, one can still ask under what geometric conditions on M does there exists a function $R(\lambda)$ such that

$$N(\lambda; M) = \frac{\omega_d}{(2\pi)^d}\lambda^{\frac{d}{2}} + R(\lambda) \tag{4.1.5}$$

with $R(\lambda) = O(\lambda^\tau)$ independent of M, with $\tau < d/2$. The search for this type of uniform bounds was a prominent feature in the above mentioned papers [**2, 7, 8, 28**]. The presence of the boundary allowed them to derive a two-term Weyl type bound; closed manifolds do not exhibit this behaviour.

In [**14**, Theorem 6.2], Buser has obtained bounds on the eigenvalue λ_k of a closed manifolds, valid when k was large enough in terms of the injectivity radius, see also [**33**, equation 1.2.5] where this result is reformulated in terms of the counting function. The following theorem states that we can find explicit bounds on the exponent in (4.1.5) depending on the injectivity radius.

Theorem 4.1.5. *There is δ_0 such that for all $\delta \in (0,\delta_0)$ and $c > 0$, there is a constant C such that for all $\lambda > 1$ and all flat tori $\mathbb{T}_\Gamma \in \mathscr{M}_0$ respecting*

$$\operatorname{inj}(\mathbb{T}_\Gamma) \geq c\lambda^{-1/2+\delta}$$

we have that

$$\left| N(\lambda; \mathbb{T}_\Gamma) - \frac{\omega_d}{(2\pi)^d}\lambda^{d/2} \right| \leq C\lambda^{\frac{d}{2}-\delta\frac{d-1}{4d}}. \tag{4.1.6}$$

Moreover, for $d = 2$ and any flat Klein bottle $K(a,b) \in \mathscr{E}_0$ respecting

$$\operatorname{inj}(K(a,b)) \geq c\lambda^{-1/2+\delta}$$

we have that

$$\left| N(\lambda; K(a,b)) - \frac{1}{4\pi}\lambda \right| \leq C\lambda^{1-\frac{\delta}{8}}.$$

We make the following two remarks as to the sharpness of those results.

Remark 4.1.6. The condition on the injectivity radius in the previous theorem is sharp. Indeed, as part of the proof of Theorem 4.1.2 we will construct an explicit sequence of flat tori $\mathbb{T}_k \in \mathscr{M}_0$ such that

$$\operatorname{inj}(\mathbb{T}_k) = \frac{\lambda_{2k}(\mathbb{T}_k)^{-1/2}}{2\pi}$$

whose eigenvalue counting functions satisfy

$$\left| N(\lambda_{2k}(\mathbb{T}_k); \mathbb{T}_k) - \frac{\omega_d}{(2\pi)^d}\lambda_{2k}(\mathbb{T}_k)^{d/2} \right| \gg \lambda^{d/2}.$$

In fact, one will be able to compute explicitly

$$\left| N(\lambda_{2k}(\mathbb{T}_k); \mathbb{T}_k) - \frac{\omega_1}{(2\pi)^d}\lambda_{2k}(\mathbb{T}_k)^{d/2} \right| = 2d - 1,$$

and $\omega_1 \neq \omega_d$.

Remark 4.1.7. It is impossible to find an exponent in the remainder in equation (4.1.6) that does not go to $d/2$ as $\delta \to 0$. In the appendix, explicit sequences will be constructed that will depend on δ such that the remainder has exponent arbitrarily close to $d/2$.

4.1.3. Lattice points inside domains

We translate the problem at hand in the language of lattice point counting. The spectrum of the Laplacian on a flat torus is given by

$$\sigma(\mathbb{T}_\Gamma) = \left\{4\pi^2 \left|\gamma^*\right|^2 : \gamma^* \in \Gamma^*\right\}, \tag{4.1.7}$$

where Γ^* is the lattice dual to Γ defined by

$$\Gamma^* := \left\{\gamma^* \in \mathbb{R}^d : (\gamma^*, \Gamma) \subset \mathbb{Z}\right\}.$$

Similarly, the spectrum of the Laplacian on a flat Klein bottle is giveni in [**10**] to be

$$\sigma(K(a,b)) := \left\{4\pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) : (m,n) \in \mathbb{Z} \times \mathbb{N}_0, (m,n) \neq (2\ell+1,0)\right\}. \tag{4.1.8}$$

A classical problem in the geometry of numbers consists in counting the number of points of an isotropically shrinking lattice $\Gamma_\lambda := \lambda^{-1}\Gamma$ inside a domain Ω containing the origin as $\lambda \to \infty$. This dates back to the Gauss circle problem and has been studied in great details for various type of domains over the years. Denote

$$|\Omega| = \mathrm{Vol}_d(\Omega) \quad \text{and} \quad |\Gamma| = \det(A_\Gamma),$$

where A_Γ is any matrix such that $A_\Gamma \mathbb{Z}^d = \Gamma$. In general, one aims for asymptotics of the form

$$N(\Omega; \Gamma_\lambda) := \#(\Omega \cap \Gamma_\lambda) = \frac{|\Omega|}{|\Gamma_\lambda|} + R(\lambda; \Omega; \Gamma), \tag{4.1.9}$$

where

$$R(\lambda; \Omega; \Gamma) = O\left(|\Gamma_\lambda|^{-\eta}\right) \tag{4.1.10}$$

with $\eta < 1$ The implicit constant in the righthand side of equation (4.1.10) depends on the geometry of Ω, the geometry of its boundary, and on Γ. In general, given non compact families of lattices or domains, the implicit constant is not uniform and therefore the formula (4.1.9) cannot be used directly to find extremisers to $N(\Omega; \Gamma_\lambda)$ for large λ. Note that maximising this counting function does not makes sense, even while keeping the lattice determinant and the volume of the domain fixed. Indeed, for a fixed Ω containing the origin and ε small enough the lattice $\varepsilon^{d-1}\mathbb{Z} \oplus \varepsilon^{-1}\mathbb{Z}^{d-1}$ has arbitrarily many points in Ω and determinant 1.

We formulate the results of the two previous sections in terms of lattices. From the fact that

$$\#\left\{\mathbb{Z}^d \cap A_\Gamma^{-1}(B_1)\right\} = \#\left\{A_\Gamma \mathbb{Z}^d \cap B_1\right\},$$

the following two questions are equivalent.

- What's the largest lattice determinant of a lattice with at least k points in B_1?
- What's the smallest area of an ellipsoid enclosing at least k points of the lattice $\mathbb{Z}^d$?

Symmetry of ellipsoids or lattices with respect to the transformation $x \mapsto -x$ means that no generality is lost by asking these questions for only even (or odd) k. Let us order elements of any lattice as

$$\Gamma = \{\gamma_k : k \in \mathbb{N}_0\}$$

by $\gamma_0 = 0$ and $\gamma < \widetilde{\gamma}$ if $|\gamma| < |\widetilde{\gamma}|$, and if their norms are equal by lexicographic order. The scaling invariance of the problem is made explicit by studying maximisers to the functional

$$\widetilde{\Lambda}_k(\Gamma) = |\Gamma|^{-1/d} \left|\gamma_k\right|.$$

We obtain the following restatement of Theorem 4.1.1 in terms of lattices.

Theorem 4.1.8 (Lattice version of Theorem 4.1.1)**.** *For every $k \in \mathbb{N}$, there exists $\Gamma_k^\star \in \mathscr{L}$ maximising $\widetilde{\Lambda}_k$.*

Remark 4.1.9. The maximiser in the previous theorem is not unique, in particular if Γ is a maximiser, then $\mu\Gamma$ is also one. We will, depending on what is pertinent at the right moment, either normalise them by determinant or by $\left|\gamma_k\right|$. Note that even within $\mathscr{L}_0$ unicity is not guaranteed.

We now study properties of the maximisers $\Gamma_k^\star$. The degeneracy of a sequence $\Gamma_k^\star$ is given in terms of their *successive minima*, the lattice invariants $\mu_j(\Gamma)$ defined for $1 \le j \le d$ by

$$\mu_j(\Gamma) = \inf\{\mu : \dim(\operatorname{span}(\Gamma \cap B_\mu)) \ge j\}. \tag{4.1.11}$$

We prove the following restatement of Theorem 4.1.2.

Theorem 4.1.10 (Lattice version of Theorem 4.1.2)**.** *Let $\{\Gamma_k^\star\} \subset \mathscr{L}_0$ be a sequence of maximisers of $\widetilde{\Lambda}_k$ normalised by $\left|\Gamma_k^\star\right| = 1$, in dimension $d \le 10$. Then, the following holds.*

(1) The sequence $\Gamma_k^\star$ has no limit points.

(2) The successive minima of the sequence $\Gamma_k^\star$ satisfy the asymptotic bounds

$$\mu_1\left(\Gamma_k^\star\right) \gg k^{-1+\frac{1}{d}}$$

and

$$\mu_d\left(\Gamma_k^\star\right) \gg k^{\frac{1}{d}-\delta}.$$

for any $\delta > 0$.

This will be proved thanks to the following restatement of Theorem 4.1.5 in terms of lattices.

Theorem 4.1.11 (Lattice version of Theorem 4.1.5)**.** *There is* $\delta_0 > 0$ *such that for all* $\delta \in (0, \delta_0)$ *and* $M > 0$*, there exists C such that for any lattice* Γ *respecting* $|\Gamma| \leq 1$ *and*

$$\mu_d(\Gamma) \leq M|\Gamma|^\delta \tag{4.1.12}$$

we have that

$$\left|N(B_1;\Gamma) - \frac{\omega_d}{|\Gamma|}\right| \leq C|\Gamma|^{-1+\frac{\delta(d-1)}{2d^2}}. \tag{4.1.13}$$

4.1.4. Plan of the paper and sketch of the proofs

We start in Section 4.2 by exposing general facts about lattices that will be used in the sequel. More specifically, we describe the relevant lattice invariants and state theorems of Minkowski and Banaszczyk that are important later, for ease of reference.

In Section 4.3, we prove Theorems 4.1.8 and 4.1.10. Inspired by a construction of Kao, Lai and Osting [**45**] in dimension 2, we produce in Section 4.3.2 in any dimension a sequence of lattices Θ_{2k} such that

$$|\theta_{2k-1}| = |\theta_{2k}| = \left(\frac{2k}{\omega_1}\right)^{1/d}. \tag{4.1.14}$$

However, Theorem 4.1.11 implies that for any lattice Γ of unit determinant whose successive minima satisfy the bounds (4.1.12) and (4.1.13) then there is a constant C such that

$$\left|\gamma_{2k-1}\right| = \left|\gamma_{2k}\right| \leq \left(\frac{2k}{\omega_d}\right)^{1/d} + Ck^\tau$$

with $\tau < 1/d$ and ω_d the volume of the unit ball. One can see that while the sequence ω_d converges to 0 as $d \to \infty$, it is initially increasing. Indeed, for all $2 \leq d \leq 10$, we have that $\omega_d > \omega_1$.

In Section 4.4, we will show that the spectral theoretic versions of Theorems 4.1.1, 4.1.2 and 4.1.5 are implied by Theorems 4.1.8, 4.1.10 and 4.1.11 using Banaszczyk's transference theorem 4.2.2 and Minkowski's successive minima theorem 4.2.1.

In Section 4.5, we switch gears and describe Theorem 4.1.11 in terms of points of $\mathbb{Z}^d$ sitting inside anisotropically expanding domains. These were studied by Yu. Kordyukov and A. Yakovlev in a series of papers [**47, 48, 49, 50**] and we generalise their results and methods to our setting.

In Section 4.6, we prove two propositions about the number of points of a lattice sitting inside anisotropically expanding domains using the Poisson summation formula method under some conditions on the way the lattice and the domain expansion interact. One of the propositions yields asymptotics uniform in the rates of expansion but not in the directions, the other yields uniform bounds in the directions of expansion but not in the rates, it has, however, an explicit dependence on those rates. In the classical version of this problem, one uses global estimates on the Fourier transform of the indicator of a convex set to obtain bounds on the counting function of lattice points inside an expanding domain. It is, however, not possible to make this kind of computations uniformly when the expansion is anisotropic. The main idea, inspired by [**49**] is to only use Fourier transform estimates along the subspace where the expansion is the fastest and to use trivial L^∞ estimates in the orthogonal complement.

Finally in Appendix 4.A we discuss the sharpness of the exponents obtained for the remainders in Theorems 4.1.11 and 4.1.5.

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4.2. Some facts about lattices in $\mathbb{R}^d$

For most standard results on lattices, one can see [**16**]. The set of all full-rank lattices in $\mathbb{R}^d$ can be realised as $\mathscr{L} = \mathrm{GL}_d(\mathbb{R})/\mathrm{GL}_d(\mathbb{Z})$, equipped with the quotient topology. A lattice $\Gamma \in \mathscr{L}$ is identified with its generator matrix A_Γ, *i.e.* the matrix such that $A_\Gamma \mathbb{Z}^d = \Gamma$. Every lattice determines uniquely a flat torus $\mathbb{T}_\Gamma = \mathbb{R}^d/\Gamma$.

Two relevant lattice invariants that are of interest in this paper are the determinant (or volume) and the successive minima. The determinant is defined as

$$|\Gamma| := \det A_\Gamma = \mathrm{Vol}_d(\mathbb{T}_\Gamma).$$

By convention, we assign to the trivial lattice a volume of 1. The successive minima $\mu_j(\Gamma)$ are defined for $1 \le j \le d$ as

$$\mu_j(\Gamma) := \inf\left\{\mu : \dim(\mathrm{span}(\Gamma \cap B_\mu)) \ge j\right\}.$$

Note that μ_j is always attained, *i.e.* there is always $\gamma \in \Gamma$ such that $\mu_j(\Gamma) = |\gamma|$. Furthermore, the first successive minimum gives the injectivity radius of the associated torus, *i.e.*

$$\mu_1(\Gamma) = \mathrm{inj}(\mathbb{T}_\Gamma).$$

The successive minima of a lattice and the determinant are related through a theorem of Minkowski.

Theorem 4.2.1 (Minkowski's sucessive minima theorem)**.** *Let* $\mu_1,\dots,\mu_d$ *be the successive minima of a lattice* Γ*. Then, there exists constants* $c,C>0$ *such that*

$$c\,|\Gamma|\leq\prod_{j=1}^{d}\mu_j\leq C\,|\Gamma|\,.$$

To any lattice Γ we associate the dual lattice

$$\Gamma^*=\left\{\gamma^*\in\mathbb{R}^d:(\gamma^*,\Gamma)\subset\mathbb{Z}\right\}.$$

The operation $*$ is a continuous involution on $\mathscr{L}$; hence a set $\mathscr{K}\subset\mathscr{L}$ is compact if and only if $\mathscr{K}^*$ is. Let A_Γ be the generating matrix for Γ, then $A_{\Gamma^*}=(A_\Gamma^*)^{-1}$; from this we infer that $|\Gamma^*|=|\Gamma|^{-1}$.

The following theorem from Banaszczyk [**6**] is also useful in the sequel and relates the successive minima of Γ and those of Γ^*.

Theorem 4.2.2 (Banaszczyk's transference theorem)**.** *For any* $1\leq j\leq d$*, the following inequalities hold between the successive minimas of the lattices* Γ *and* Γ^* *:*

$$1\leq\mu_j(\Gamma)\mu_{d-j+1}(\Gamma^*)\leq d.$$

The lattice invariants can be used to characterise compactness in $\mathscr{L}$, by Mahler's selction theorem [**16**][Theorems 5.3, 5.4 and Lemma 8.3]. This theorem states that a set $\mathscr{K}\subset\mathscr{L}$ is compact if and only if the determinant is bounded and the first minimum μ_1 is bounded away from zero on $\mathscr{K}$. Equivalently, it is compact if and only if the determinant is bounded away from zero and μ_d is bounded on $\mathscr{K}$. Compactness in the moduli space of all flat tori is obtained by identifying a torus with its lattice.

Definition 4.2.3. A sequence of lattices $\{\Gamma_k\}$ is said to *degenerate* if either $|\Gamma_k|\to\infty$ or if $\mu_1(\Gamma_k)\to 0$. In other words, it degenerates if it is not contained in some compact set in $\mathscr{L}$.

4.3. Optimal lattices

In this section, we prove Theorems 4.1.8 and 4.1.10 assuming Theorem 4.1.11. Order elements of a lattice Γ with respect to their norms and by lexicographic order whenever the norms are equal. We write $\Gamma=\left\{\gamma_k:k\in\mathbb{N}_0\right\}$. We study sequences of lattices maximising the functionals

$$\widetilde{\Lambda}_k(\Gamma)=|\Gamma|^{-1/d}|\gamma_k|.$$

Note that for any lattice Γ and $m \geq 1$ we have that $\widetilde{\Lambda}_{2m-1}(\Gamma) = \widetilde{\Lambda}_{2m}(\Gamma)$; we will therefore only consider maximisers for even k.

4.3.1. Proof of Theorem 4.1.8

Consider a maximising sequence $\{\Gamma_n\}$ for $\widetilde{\Lambda}_k$. Without loss of generality from the definition of $\widetilde{\Lambda}_k$ we may suppose that $|\Gamma_n| = 1$ for all n. Suppose that $\mu_1(\Gamma_n) \to 0$. Then, for some n we have that $\mu_1(\Gamma_n) < 1/k$. Let $\gamma \in \Gamma_n$ be a lattice point realising $\mu_1(\Gamma_n)$. Then, $1 > |k\gamma| > \left|\gamma_k\right|$. However, the kth element of $\mathbb{Z}^d$ has norm greater than 1, contradicting that $\{\Gamma_n\}$ was a maximising sequence. By Mahler's selection theorem, $\{\Gamma_n\}$ has a convergent subsequence, and by continuity of the norm and the determinant, it converges to a maximiser for $\widetilde{\Lambda}_k$.

□

4.3.2. Lattices with large $\widetilde{\Lambda}_k$

In this section we study a specific sequence of lattices that we will use as a measuring stick for other sequences of lattices. Note that we make no claim of these lattices being the optimisers. Consider the lattices

$$\Theta_{2k} = k^{-1+\frac{1}{d}}\mathbb{Z} \oplus k^{\frac{1}{d}}\mathbb{Z}^{d-1}.$$

Then, we have

$$|\theta_{2k-1}| = |\theta_{2k}| = k^{1/d}$$

and

$$|\Theta_{2k}| = 1.$$

In particular, we have that

$$\widetilde{\Lambda}_{2k}(\Theta_{2k}) = k^{1/d}$$

which will be the quantity to beat. Observe that the sequence Θ_{2k} degenerates and that

$$\mu_d(\Theta_{2k}) = k^{1/d}.$$

4.3.3. Proof of Theorem 4.1.10

Fix $\delta > 0$. Denote by $\{\Gamma_k\} \subset \mathscr{L}_0$ a sequence of lattices such that $\mu_d(\Gamma_k) \ll k^{1/d-\delta}$. We will show that under such conditions, Γ_k cannot be a maximiser for $\widetilde{\Lambda}_k$ infinitely often. We show that for large k and any fixed $t > 0$,

$$\#(B_{k^{1/d}-t} \cap \Gamma_{2k}) > 2k,$$

implying that

$$\widetilde{\Lambda}_{2k}(\Gamma_{2k}) \leq k^{1/d} - t < \widetilde{\Lambda}_{2k}(\Theta_{2k}).$$

We have that

$$\#\left(B_{k^{1/d}-t} \cap \Gamma_{2k}\right) = \#\left(B_1 \cap \frac{1}{k^{1/d}-t}\Gamma_{2k}\right),$$

that

$$\mu_d\left(\frac{1}{k^{1/d}-t}\Gamma_{2k}\right) \ll k^{-\delta},$$

and that

$$\left|\frac{1}{k^{1/d}-t}\Gamma_{2k}\right| = k^{-1}\left(1 - tk^{-1/d}\right)^{-d}.$$

We therefore satisfy the hypotheses of Theorem 4.1.11 and therefore get

$$\#\left(B_{k^{1/d}-t} \cap \Gamma_{2k}\right) = \omega_d(1 - tk^{-1/d})^d k + O\left(k^\tau\right)$$

for some $\tau < 1$. For $2 \leq d \leq 10$, we have that $\omega_d > \omega_1 = 2$. Hence, there is K such that for $k > K$

$$\#(B_{k^{1/d}-t} \cap \Gamma_{2k}) < 2k,$$

proving that there is a finite number of maximisers in the sequence $\{\Gamma_k\}$. This implies that for any $\delta > 0$, any sequence of normalised maximisers respect $\mu_d(\Gamma_k) \gg k^{1/d-\delta}$, also implying that the sequence degenerates.

For the lower bound on $\mu_1(\Gamma_k)$, any sequence Γ_k normalised by determinants such that $\mu_1(\Gamma_{2k}) < k^{-1+1/d}$ has that

$$\widetilde{\Lambda}_{2k}(\Gamma_{2k}) \leq k\mu_1(\Gamma_{2k}) < \widetilde{\Lambda}_{2k}(\Theta_{2k}),$$

hence this is not a sequence of maximisers.

□

4.4. From lattices to tori

In this section we prove the spectral theoretic versions of Theorems 4.1.1, 4.1.2 and 4.1.5, as well as Theorem 4.1.4. For any lattice Γ we denote by γ_k^* the kth ordered element of the dual lattice Γ^*. Since $\lambda_k(\mathbb{T}_\Gamma) = 4\pi^2 \left|\gamma_k^*\right|^2$ and $\mathrm{Vol}(\mathbb{T}_\Gamma) = |\Gamma^*|^{-1}$, we have that

$$\Lambda_k(\mathbb{T}_\Gamma) = \left(2\pi\widetilde{\Lambda}_k(\Gamma^*)\right)^2.$$

Since these quantities are positive the problem of maximising Λ_k on flat tori is the same as the problem of maximising $\widetilde{\Lambda}_k$ on the dual lattices of those tori.

4.4.1. Proof of Theorem 4.1.1

By Theorem 4.1.8 there exists a lattice $\Gamma_k^\star$ maximising $\widetilde{\Lambda}_k$. The torus with lattice $\Gamma = (\Gamma_k^\star)^*$ is therefore a maximiser for Λ_k.

For flat Klein bottles, we have from equation (4.1.8) that the eigenvalues of $K(a,b)$ are continuous in the parameters a and b. Normalising by $ab = 2$, it is easy to see that for any k, $\lambda_k(K(a,b))$ goes to 0 when either a or b goes to zero. Hence for any fixed k we can restrict ourselves to a compact subset of the parameters a,b and the maximiser exists.

□

4.4.2. Proof of Theorem 4.1.2

Denote by $\Gamma_k^\star$ a sequence of optimal lattices with unit determinant for Λ_k and denote by $\mathbb{T}_k^\star$ the corresponding optimal torus $T_k^\star = \mathbb{R}^d/(\Gamma_k^\star)^*$. Since compactness of a set $\mathscr{K} \subset \mathscr{L}_0$ is equivalent to compactness of the set of duals $\mathscr{K}^*$, we have that the sequence of optimal tori degenerates.

We now turn to the geometric constraints. Recall that $\mathrm{inj}(\mathbb{T}_k^\star) = \mu_1((\Gamma_k^\star)^*)$. By Banaszczyk's transference theorem, we have that

$$\mu_1((\Gamma_k^\star)^*) \le \frac{d}{\mu_d(\Gamma_k^\star)}.$$

Hence, from the lower bound for $\mu_d(\Gamma_k^\star)$ in Theorem 4.1.10 we have that for any $\delta > 0$,

$$\mathrm{inj}(\mathbb{T}_k^\star) = \mu_1((\Gamma_k^\star)^*) \ll k^{-1/d+\delta}.$$

On the other hand, by Minkowski's successive minima theorem, there is a constant C such that

$$\mu_d(\Gamma_k^\star) \le C\mu_1 {\Gamma_k^\star}^{1-d}$$
$$\ll k^{\frac{(1-d)^2}{d}}.$$

Once again, Banaszczyk's transference theorem yields

$$\mathrm{inj}(T_k^\star) \gg k^{-\frac{(1-d)^2}{d}},$$

finishing the proof.

□

4.4.3. Proof of Theorem 4.1.4

For flat Klein bottles, observe that the injectivity radius of $K(a,b)$ is given by

$$\mathrm{inj}(K(a,b)) = \min(a, b/2).$$

Let $\Gamma(a,b)$ be the lattice defined by

$$\Gamma(a,b) := \frac{2\pi}{a}\mathbb{Z} \oplus \frac{2\pi}{b}\mathbb{Z}$$

It is not hard to see that $\Gamma(a,b)$ has the property

$$N(\lambda; K(a,b)) = \frac{1}{2}\#\left(\Gamma(a,b) \cap B_{\sqrt{\lambda}}\right) + O(1).$$

Indeed, let $\Xi(a,b)$ be the set

$$\Xi(a,b) := \left(\frac{2\pi}{a}\mathbb{Z} \oplus \frac{2\pi}{b}N_0\right) \setminus \left\{\frac{2\pi}{a}(2\ell+1, 0) : \ell \in \mathbb{Z}\right\}.$$

Then, the spectrum of $K(a,b)$ is the same as the square of the norm elements of $\Xi(a,b)$. However, it is easy to see that if we take the union of $\Xi(a,b)$ and $-\Xi(a,b)$, we recover $\Gamma(a,b)$ except for points of the form $\left(\frac{2(2\ell+1)\pi}{a}, 0\right)$, but we added twice the elements of the form $\left(\frac{4\pi\ell)}{a}, 0\right)$. Hence, we have that

$$\left|\#(\Gamma(a,b) \cap B_{\sqrt{\lambda}}) - \#\left(\Xi(a,b) \cap B_{\sqrt{\lambda}}\right) - \#\left(-\Xi(a,b) \cap B_{\sqrt{\lambda}}\right)\right| \le 3.$$

Now, for rectangular lattices we have that $\mu_1(\Gamma(a,b) = 2\pi\min(a^{-1}, b^{-1})$ and $\mu_2(\Gamma(a,b)) = 2\pi\max(a^{-1}, b^{-1})$. The rest of the analysis is performed exactly in the same way as for flat tori.

□

4.4.4. Proof of Theorem 4.1.5

Fix $\delta > 0$, $c > 0$ and $\lambda \geq 2\pi$. Let $\mathbb{T}_\Gamma$ be any flat torus of unit volume such that

$$\mathrm{inj}(\mathbb{T}_\Gamma) \geq c\lambda^{-1/2+\delta}.$$

We have from equation (4.1.7) that

$$N(\lambda; T_\Gamma) = \#\left(2\pi\lambda^{-1/2}\Gamma^* \cap B_1\right)$$

Denote by Γ_λ^* the rescaled lattice $2\pi\lambda^{-1/2}\Gamma^*$. By Banaszczyk's transference theorem and scaling properties of lattice invariants we have that

$$\mu_d\left(\Gamma_\lambda^*\right) \leq \frac{d\lambda^{-1/2}}{2\pi\,\mathrm{inj}(\mathbb{T}_\Gamma)} \ll \lambda^{-\delta} \ll \left|\Gamma_\lambda^*\right|^{2\delta/d}.$$

The lattice Γ_λ^* therefore satisfies the hypotheses of Theorem 4.1.11, hence we have that

$$N(\lambda; T_\Gamma) = \frac{\omega_d}{\left|\Gamma_\lambda^*\right|} + O\left(\left|\Gamma_\lambda^*\right|^{-1+\frac{\delta(d-1)}{2d^2}}\right)$$

Plugging into the previous equation the determinant

$$\left|\Gamma_\lambda^*\right| = \frac{(2\pi)^d}{\lambda^{d/2}}$$

yields the desired asymptotic in Theorem 4.1.5.

□

4.5. Anisotropically expanding domains

We now ground the statement of Theorem 4.1.11 in terms of the counting of lattice points sitting inside anisotropically expanding domains developped by Yu. Kordyukov and A. Yakovlev in [**47, 48, 49, 50**]. Consider the decomposition of $\mathbb{R}^d$ as

$$\mathbb{R}^d := E := \bigoplus_{j=1}^{m} V_j = V_1 \oplus V',$$

with $\dim(V_j) = d_j$ and $d_1 + \ldots + d_m = d$. We will use E to refer to a specific decomposition for $\mathbb{R}^d$. For $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_m)$ consider the linear transformation $T_{\boldsymbol{\varepsilon}}$ given by

$$T_{\boldsymbol{\varepsilon}} = \sum_{j=1}^{m} \varepsilon_j^{-1}\mathbf{x}_j$$

with $\mathbf{x}_j \in V_j$. Without loss of generality we suppose that $\varepsilon_1 < \ldots < \varepsilon_m$. We say that the transformation $T_{\boldsymbol{\varepsilon}}$ is anisotropic if not all ε_j are equal. We denote the set of all such transformations $\mathscr{T}_E$, and by $\mathscr{T}$ the union of all such transformations over decompositions E.

For Ω a bounded subset of Euclidean space and $\Gamma \in \mathscr{L}$, denote

$$n_{\boldsymbol{\varepsilon}}(\Omega;\Gamma;\mathbf{y}) := \#\left(\Gamma \cap \left(T_{\boldsymbol{\varepsilon}}\Omega + \mathbf{y}\right)\right) = \#\left(T_{\boldsymbol{\varepsilon}}^{-1}(\Gamma - \mathbf{y}) \cap \Omega\right).$$

Kordyukov and Yakovlev have studied asymptotics for $n_{\boldsymbol{\varepsilon}}$ in the specific case where a subspace V of $\mathbb{R}^d$ is fixed, and Ω is stretched along its orthogonal complement. In our notation, this corresponds to $E = V_1 \oplus V_2$ with $\varepsilon_1 \to 0$ and $\varepsilon_2 = 1$.

In our case, the expansion is happening at different rates along different subspaces. We split the remainder of this section in three parts. First, we describe asymptotics for $n_{\boldsymbol{\varepsilon}}$ of two types : uniform in the ε_j but not the decomposition E; then uniform in the decomposition but with an explicit dependence on the ε_j. Then, we show that from the perspective of the counting function, we can describe any lattice using the transformations $T_{\boldsymbol{\varepsilon}}$. Finally, we derive Theorem 4.1.11 from Proposition 4.5.3.

4.5.1. Lattice points inside anisotropically expanding domains

The results we obtain hinge on the two following conditions. The first one is a generic condition on the relation between the dual lattice Γ^* and the decomposition E.

Condition A. The lattice Γ^* is in general position with respect to V', that is

$$\Gamma^* \cap V' = \{0\}.$$

The second condition is a normalisation condition. Indeed, we are interested in asymptotics given in terms of the volume of the expanded domain, and in how the ratios $\varepsilon_i/\varepsilon_j$ influence the asymptotics.

Condition B. We assume that $\boldsymbol{\varepsilon} = (\varepsilon^{\alpha_1}, \ldots, \varepsilon^{\alpha_m})$, with $\alpha_1 \geq \ldots \alpha_m$ and

$$\sum_{j=1}^{m} \alpha_j d_j = d,$$

which implies in particular $\alpha_1 \geq 1$. We say that the condition is satisfied weakly if α_m is allowed to be zero. We say that the condition is satisfied δ-strongly for some $\delta > 0$ if it is required that $\alpha_m \geq \delta$.

Remark 4.5.1. The previous condition implies that the volume of the expanded domain $T_{\boldsymbol{\varepsilon}}\Omega$ is given by

$$|T_{\boldsymbol{\varepsilon}}\Omega| = |\Omega|\,\varepsilon^{-d}.$$

We obtain asymptotics on the counting function $n_{\boldsymbol{\varepsilon}}$, with and without appealing to Condition A. We obtain the two following propositions.

Proposition 4.5.2. *Let Ω be a strictly convex open subset of $\mathbb{R}^d$. Suppose that Γ respects Condition A and that $\boldsymbol{\varepsilon}$ respects Condition B weakly. Then,*

$$n_{\boldsymbol{\varepsilon}}(\Omega;\Gamma;\mathbf{y}) = \frac{|\Omega|}{|\Gamma|}\varepsilon^{-d} + O\left(\varepsilon^{-\eta}\right) \tag{4.5.1}$$

with $\eta \le d - \frac{1}{d}$ and the constants uniform in d_j, α_j and $\mathbf{y}$.

Proposition 4.5.3. *Let Ω be a strictly convex open subset of $\mathbb{R}^d$. Suppose that $\boldsymbol{\varepsilon}$ satisfies Condition B δ-strongly for some $\delta > 0$. Then,*

$$n_{\boldsymbol{\varepsilon}}(\Omega;\Gamma;\mathbf{y}) = \frac{|\Omega|}{|\Gamma|}\varepsilon^{-d} + O\left(\varepsilon^{-d+\delta\frac{d-1}{2d}}\right)$$

with the implicit constants depending only on δ.

Remark 4.5.4. Condition B is a normalisation. In Propositions 4.5.2 and Lemma 4.5.3, if $\sum_{j=1}^m \alpha_j d_j = c$ making the change of variable $\varepsilon \mapsto \varepsilon^{d/c}$ will yield the right normalisation and allow us to deduce the asymptotics

$$n_{\boldsymbol{\varepsilon}}(\Omega;\Gamma;\mathbf{y}) = \frac{|\Omega|}{|\Gamma|}\varepsilon^{-c} + O\left(\varepsilon^{-\eta_c}\right)$$

with $\eta_c < c - \frac{c}{d^2}$ from Proposition 4.5.2 and

$$n_{\boldsymbol{\varepsilon}}(\Omega;\Gamma;\mathbf{y}) = \frac{|\Omega|}{|\Gamma|}\varepsilon^{-c} + O\left(\varepsilon^{-c+\frac{\delta c}{d}\frac{d-1}{2d}}\right)$$

from Proposition 4.5.3.

4.5.2. From $\mathscr{T}$ to lattices

We start by showing that we can restrict ourselves to lattices of the form $T_{\boldsymbol{\varepsilon}}^{-1}\mathbb{Z}^d$ in our investigation of Theorem 4.1.11.

Lemma 4.5.5. *For every $\Gamma \in \mathscr{L}$, there exists a decomposition*

$$\mathbb{R}^d = E = \bigoplus_{j=1}^m V_j$$

and $T_{\boldsymbol{\varepsilon}} \in \mathcal{T}_E$ such that

$$N(\Gamma; B_1) = n_{\boldsymbol{\varepsilon}}\left(B_1; \mathbb{Z}^d; 0\right). \tag{4.5.2}$$

For every $T_{\boldsymbol{\varepsilon}} \in \mathcal{T}$, there exists Γ, unique up to orthogonal transformation, such that equation (4.5.2) *holds.*

PROOF. Let $A_\Gamma \in \mathrm{GL}_d(\mathbb{R})$ be such that $A_\Gamma \mathbb{Z}^d = \Gamma$. Then,

$$\begin{aligned}\sum_{\gamma\in\Gamma} \mathbf{1}_{B_1}(\gamma) &= \sum_{\gamma\in\Gamma} \mathbf{1}_{A_\Gamma(B_1)}\left(A_\Gamma^{-1}\gamma\right) \\ &= \sum_{\mathbf{n}\in\mathbb{Z}^d} \mathbf{1}_{A_\Gamma(B_1)}(\mathbf{n})\end{aligned}$$

Observe now that since $B_1 = \left\{\mathbf{x} \in \mathbb{R}^d : \mathbf{x}^*\mathbf{x} \le 1\right\}$, then

$$A_\Gamma(B_1) = \left\{\mathbf{x} \in \mathbb{R}^d : \mathbf{x}^*(A_\Gamma^*)^{-1} A_\Gamma^{-1}\mathbf{x} \le 1\right\}.$$

Since $(A_\Gamma A_\Gamma^*)^{-1}$ is symmetric definite positive, it can be diagonalised as

$$(A_\Gamma A_\Gamma^*)^{-1} = U^* D^{1/2} D^{1/2} U$$

with U orthogonal. Let $\boldsymbol{\varepsilon} = \mathrm{diag}(D^{1/2})$ and V_j be eigenspaces of $(A_\Gamma A_\Gamma^*)^{-1}$. Since $N(\Gamma; B_1)$ is invariant under orthogonal transformations of Γ, we have that

$$N(\Gamma; 1) = N(U\Gamma; 1) = \#\left\{A_\Gamma^{-1} U\Gamma \cap A_\Gamma^{-1} B_1\right\} = n_{\boldsymbol{\varepsilon}}(B_1).$$

On the other hand, this process can be inverted : given $T_{\boldsymbol{\varepsilon}}$, we take Γ to be the lattice with generating matrix $T_{\boldsymbol{\varepsilon}}^{-1}$. Unicity up to orthogonal transformation is obtained from the fact that a Gram matrix uniquely determines a basis up to orthogonal transformation. □

The previous lemma allows us to consider only the lattices of the form $\Gamma = T_{\boldsymbol{\varepsilon}}^{-1}\mathbb{Z}^d$. The following proposition relates the lattice invariants to the associated transformation $T_{\boldsymbol{\varepsilon}}$.

Lemma 4.5.6. *Let Γ be a lattice in $\mathcal{L}$. Then, for any $T_{\boldsymbol{\varepsilon}} \in \mathcal{T}$ such that $\Gamma = T_{\boldsymbol{\varepsilon}}^{-1} U\mathbb{Z}^d$ for some orthogonal transformation U we have that*

$$|\Gamma| = \det\left(T_{\boldsymbol{\varepsilon}}^{-1}\right) = \prod_{j=1}^{m} \varepsilon_j^{d_j}.$$

and that the following bounds hold for the successive minima $\mu_1(\Gamma)$ and $\mu_d(\Gamma)$:

$$\varepsilon_1 \le \mu_1(\Gamma) \le \mu_d(\Gamma) \le \varepsilon_m.$$

Furthermore, one can choose $T_{\boldsymbol{\varepsilon}}$ *such that* $\Gamma = T_{\boldsymbol{\varepsilon}}^{-1} U \mathbb{Z}^d$ *and*

$$\mu_1(\Gamma) \leq \frac{d^{5/2}}{2}\varepsilon_1 \qquad \textit{and} \qquad \mu_d(\Gamma) \geq \frac{2}{d^{3/2}}\varepsilon_m.$$

PROOF. Without loss of generality, since the determinant and successive minima are invariant under orthogonal transformations we suppose that $U = I$. The assertion on determinants holds by definition. Let $\mathbf{n}$ be any non-zero element of $\mathbb{Z}^d$, and write $\mathbf{n} = \mathbf{n}_1 + \ldots + \mathbf{n}_m$ with $\mathbf{n}_j \in V_j$. Then,

$$\begin{aligned} \left| T_{\boldsymbol{\varepsilon}}^{-1}\mathbf{n} \right|^2 &= \sum_{j=1}^m \varepsilon_j^2 \left| \mathbf{n}_j \right|^2 \\ &\geq \varepsilon_1^2 \sum_{j=1}^m \left| \mathbf{n}_j \right|^2 \\ &= \varepsilon_1^2 |\mathbf{n}|^2 . \end{aligned}$$

Since $\mathbf{n} \neq 0$, we have that $\mu_1(\Gamma) \geq \varepsilon_1$. For the upper bound on μ_d, observe that any $T_{\boldsymbol{\varepsilon}}$ sends bases of $\mathbb{R}^d$ to bases of $\mathbb{R}^d$. As such, from the definition of μ_d we have that

$$\begin{aligned} \mu_d(\Gamma) &\leq \sup_{\mathbf{n} \in \mathbb{Z}^d \setminus \{0\}} \frac{\left| T_{\boldsymbol{\varepsilon}}^{-1}\mathbf{n} \right|}{|\mathbf{n}|}, \\ &\leq \varepsilon_m. \end{aligned}$$

We now obtain the lower bound on μ_d for a specific $T_{\boldsymbol{\varepsilon}}$. There is a basis of Γ whose elements all have norm smaller than $\frac{d\mu_d(\Gamma)}{2}$ [**16**, Lemma V.8]. Let $T_{\boldsymbol{\varepsilon},\Gamma}^{-1}$ be the square root of the diagonalised Gram matrix G_Γ associated to that basis. By Cauchy-Schwartz, the entries of the Gram matrices G_Γ all bounded by $\frac{d^2\mu_d(\Gamma)^2}{4}$. Let $\nu_d(G_\Gamma)$ be the largest eigenvalue of G_Γ. It satisfies the bound

$$\nu_d\left(G_\Gamma\right) \leq \sqrt{\operatorname{tr}(G_\Gamma^* G_\Gamma)} \leq \frac{d^3\mu_d(\Gamma)^2}{4}.$$

Note that the eigenvalues of G_Γ are the same as those of $T_{\boldsymbol{\varepsilon},\Gamma}^{-2}$, hence we have that

$$\varepsilon_m \leq \frac{d^{3/2}}{2}\mu_d(\Gamma),$$

yielding the desired result. For the upper bound on μ_1, observe that a generating matrix for Γ^* is $T_{\boldsymbol{\varepsilon}}$. Hence, by the previous argument we have that

$$\mu_d(\Gamma^*) \geq \frac{2}{d^{3/2}}\varepsilon_1^{-1}.$$

From Banaszczyk's transference theorem, we can then infer that

$$\mu_1(\Gamma) \leq d\mu_d(\Gamma^*)^{-1} \leq \frac{d^{5/2}}{2}\varepsilon_1,$$

finishing the proof.

□

4.5.3. Proof of Theorem 4.1.11

Fix $\delta > 0$ and $M > 0$ and let Γ be any lattice such that $|\Gamma| \leq 1$ and $\mu_d(\Gamma) \leq M|\Gamma|^{\delta}$. From Lemmas 4.5.5, one can find a decomposition E of $\mathbb{R}^d$ and a transformation $T_{\boldsymbol{\varepsilon}} \in \mathscr{T}_E$ such that

$$N(\Gamma; B_1) = n_{\boldsymbol{\varepsilon}}(B_1; \mathbb{Z}^d; 0).$$

Furthermore, normalising $\det(T_{\boldsymbol{\varepsilon}}^{-1}) = \varepsilon^{-d}$, we have by Lemma 4.5.6 that one can choose $T_{\boldsymbol{\varepsilon}}$ in such a way that

$$\varepsilon_m \leq \frac{d^{3/2}}{2}\mu_d(\Gamma) \leq \frac{Md^{3/2}}{2}\varepsilon^{-\frac{\delta}{d}}.$$

We therefore satisfy the hypotheses of Proposition 4.5.3 and we deduce that

$$N(\Gamma; B_1) = \omega_d \varepsilon^{-d} + O\left(\varepsilon^{-d + \frac{\delta(d-1)}{2d}}\right).$$

Plugging in $|\Gamma| = \varepsilon^d$ gives the desired asymptotics.

□

4.6. Asymptotic estimates

In this section, we prove Propositions 4.5.2 and 4.5.3 using the Poisson summation formula. The first two steps of the proof of both theorems are exactly the same, we will differentiate the proofs in the third step. We follow the structure set out by the author and Parnovski in [**54**]. The first thing we have to do is a mollification of $\mathbf{1}_\Omega$ so that it is smooth enough for the Poisson summation formula to be used, and we will get estimates from above and below using the mollified functions.

4.6.1. Mollification

Let $\rho \in C_c^\infty(\mathbb{R}^d)$ be a non-negative bump function supported in the unit ball and such that

$$\int_{\mathbb{R}^d} \rho(\mathbf{x})\,\mathrm{d}\mathbf{x} = 1.$$

We also let $\mathbf{h} = (h_1, \ldots, h_m)$ be a set of parameters to be fixed later, and we set

$$\rho_{\mathbf{h}}(\mathbf{x}) = \frac{1}{h_1^{d_1} \cdots h_m^{d_m}} \rho(T_{\mathbf{h}}(\mathbf{x})). \tag{4.6.1}$$

Note that $\rho_{\mathbf{h}}$ is supported in the ellipsoid

$$E_{\mathbf{h}} = \{\mathbf{x} \in V : \|T_{\mathbf{h}}\mathbf{x}\| < 1\}.$$

For any function $f : \mathbb{R}^d \to \mathbb{R}$ let $f^{(\mathbf{h})}$ be the mollification of f by $\rho_{\mathbf{h}}$, that is

$$f^{(\mathbf{h})}(\mathbf{x}) = \left[f * \rho_{\mathbf{h}}\right](\mathbf{x}) = \int_{\mathbb{R}^d} f(\mathbf{x}-\mathbf{y})\rho_{\mathbf{h}}(\mathbf{y})\, \mathrm{d}\mathbf{y}.$$

Let us now approximate $\mathbf{1}_{\Omega}$ by smooth functions. For any set B define the sets

$$B_{\mathbf{h}} = \bigcup_{\mathbf{x}\in B} (\mathbf{x} + E_{\mathbf{h}}) \qquad \text{and} \qquad B_{-\mathbf{h}} = \mathbb{R}^d \setminus \left(\mathbb{R}^d \setminus B\right)_{\mathbf{h}}.$$

The following lemma will be needed about these sets.

Lemma 4.6.1. *Let* $\boldsymbol{\varepsilon}\mathbf{h} = (\varepsilon_1 h_1, \ldots, \varepsilon_m h_m)$ *and* $B \subset V$*. Then,*

$$T_{\boldsymbol{\varepsilon}}(B)_{\pm\mathbf{h}} = T_{\boldsymbol{\varepsilon}}(B_{\pm\boldsymbol{\varepsilon}\mathbf{h}}).$$

PROOF. It follows simply from linearity of $T_{\boldsymbol{\varepsilon}}$ and the fact that $T_{\boldsymbol{\varepsilon}}E_{\mathbf{h}} = E_{\boldsymbol{\varepsilon}\mathbf{h}}$. □

We now prove that $\mathbf{1}^{(h)}$ provides a good approximation to $\mathbf{1}$.

Lemma 4.6.2. *Let* $\Omega \subset \mathbb{R}^d$ *and* $\mathbf{x} \in \mathbb{R}^d$*. Then,*

$$\mathbf{1}^{(\mathbf{h})}_{T_{\boldsymbol{\varepsilon}}(\Omega)_{-\mathbf{h}}}(\mathbf{x}) \le \mathbf{1}_{T_{\boldsymbol{\varepsilon}}(\Omega)}(\mathbf{x}) \le \mathbf{1}^{(\mathbf{h})}_{T_{\boldsymbol{\varepsilon}}(\Omega)_{\mathbf{h}}}(\mathbf{x}). \tag{4.6.2}$$

PROOF. For any set B we have that

$$0 \le \mathbf{1}^{(\mathbf{h})}_B \le 1.$$

Hence, to show the right most inequality in (4.6.2) it suffices to show that for any $\mathbf{x} \in T_{\boldsymbol{\varepsilon}}(\Omega)$ we have that $\mathbf{1}^{(\mathbf{h})}_{T_{\boldsymbol{\varepsilon}}(\Omega)_{\mathbf{h}}}(\mathbf{x}) = 1$. By definition $\mathbf{x} + E_{\mathbf{h}} \subset T_{\boldsymbol{\varepsilon}}(\Omega)_{\mathbf{h}}$ hence

$$\mathbf{1}^{(\mathbf{h})}_{(T_{\boldsymbol{\varepsilon}}\Omega)_{\mathbf{h}}}(\mathbf{x}) = \int_{E_{\mathbf{h}}} \rho_{\mathbf{h}}(\mathbf{y})\, \mathrm{d}\mathbf{y} = 1.$$

To prove the left-most inequality in (4.6.2), it suffices to show that for any $\mathbf{x} \in E \setminus T_{\boldsymbol{\varepsilon}}(\Omega)$ we have that $\mathbf{1}^{(\mathbf{h})}_{T_{\boldsymbol{\varepsilon}}(\Omega)_{-\mathbf{h}}}(\mathbf{x}) = 0$. We have that

$$\mathbf{x} + E_{\mathbf{h}} \subset (\mathbb{R}^d \setminus T_{\boldsymbol{\varepsilon}}(\Omega))_{\mathbf{h}},$$

and $\mathbf{1}_{(T_{\boldsymbol{\varepsilon}}(\Omega))_{-\mathbf{h}}}$ is supported in the complement of that set. Hence,

$$\begin{aligned}\mathbf{1}^{(\mathbf{h})}_{T_{\boldsymbol{\varepsilon}}(\Omega)_{\mathbf{h}})}(\mathbf{x}) &= \int_{E_{\mathbf{h}}} \mathbf{1}_{T_{\boldsymbol{\varepsilon}}(\Omega)_{-\mathbf{h}}}(\mathbf{x}-\mathbf{y})\rho_{\mathbf{h}}(\mathbf{y})\, \mathrm{d}\mathbf{y} \\ &= 0,\end{aligned}$$

finishing the proof. □

The following corollary follows directly from the previous lemma.

Corollary 4.6.3. *Defining*

$$n_{\boldsymbol{\varepsilon}}^{\pm}(\Omega) = \sum_{\gamma\in\Gamma} \mathbf{1}^{(\mathbf{h})}_{T_{\boldsymbol{\varepsilon}}(\Omega)_{\pm\mathbf{h}}}(\gamma),$$

the inequalities

$$n_{\boldsymbol{\varepsilon}}^{-}(\Omega) \leq n_{\boldsymbol{\varepsilon}}(\Omega) \leq n_{\boldsymbol{\varepsilon}}^{+}(\Omega)$$

hold for all $\boldsymbol{\varepsilon}$.

4.6.2. Fourier transform estimates

Let V be a subspace of $\mathbb{R}^d$ and write $\mathbf{x} = \mathbf{x}_V + \mathbf{x}'$ for any $\mathbf{x} \in \mathbb{R}^d$. We define the V-Fourier transform of a sufficiently rapidly decaying function f as

$$\left[\mathscr{F}_V f\right](\boldsymbol{\xi}_V, \mathbf{x}') = \int_V e^{-2\pi i \mathbf{x}_V \cdot \boldsymbol{\xi}_V} f(\mathbf{x}_V, \mathbf{x}')\, \mathrm{d}\mathbf{x}_V.$$

When $V = \mathbb{R}^d$, we will write $\left[\mathscr{F} f\right] := \left[\mathscr{F}_{\mathbb{R}^d} f\right]$. We obtain estimates for the decay of $\left[\mathscr{F} f\right](\mathbf{x})$ in terms of $\left[\mathscr{F}_V f\right]$. Observe that

$$\begin{aligned}
\left|\left[\mathscr{F} f\right](\boldsymbol{\xi})\right| &= \left|\int_{\mathbb{R}^d} e^{-2\pi i \mathbf{x}\cdot\boldsymbol{\xi}} f(\mathbf{x})\, \mathrm{d}\mathbf{x}\right|, \\
&= \left|\int_{V^\perp} e^{-2\pi i \mathbf{x}'\cdot\boldsymbol{\xi}'} \int_V e^{-2\pi i \mathbf{x}_V\cdot\boldsymbol{\xi}_V} f(\mathbf{x})\, \mathrm{d}\mathbf{x}_V\, \mathrm{d}\mathbf{x}'\right|, \\
&= \left|\int_{V^\perp} e^{-2\pi i \mathbf{x}'\cdot\boldsymbol{\xi}'} \left[\mathscr{F}_V f\right](\boldsymbol{\xi}_V, \mathbf{x}')\, \mathrm{d}\mathbf{x}'\right|, \\
&\leq \int_{V^\perp} \left|\left[\mathscr{F}_V f\right](\boldsymbol{\xi}_V, \mathbf{x}')\right| \mathrm{d}\mathbf{x}'.
\end{aligned} \tag{4.6.3}$$

From this we get the following lemma.

Lemma 4.6.4. *Let* Ω *be a bounded domain, and* V *be a subspace of dimension* d_V *such that the intersection* $\Omega \cap V$ *is strictly convex. Then,*

$$[\mathscr{F}\mathbf{1}_\Omega](\boldsymbol{\xi}) = O\left(\left|\boldsymbol{\xi}_1\right|^{-\frac{d_V+1}{2}}\right).$$

PROOF. Standard results about the Fourier transform of the indicator of a strictly convex set (see e.g. [**40**, Theorem 2.29]) tell us that

$$[\mathscr{F}_V \mathbf{1}_\Omega](\boldsymbol{\xi}_1, \mathbf{x}') = O([)\,\mathbf{x}']\left|\boldsymbol{\xi}_1\right|^{-\frac{d_V+1}{2}}.$$

From equation (4.6.3), we have that

$$|[\mathscr{F}\mathbf{1}]_\Omega(\boldsymbol{\xi})| \leq \int_{V^\perp} \left|[\mathscr{F}_V\mathbf{1}]_\Omega(\boldsymbol{\xi}_V,\mathbf{x}')\right| \mathrm{d}\mathbf{x}'.$$

Since $[\mathscr{F}_V\mathbf{1}_\Omega](\boldsymbol{\xi}_1,\mathbf{x}')$ is compactly supported in $\mathbf{x}'$, we obtain the desired result, finishing the proof. □

4.6.3. Poisson summation formula

Let us apply the Poisson summation formula to the smoothed sums $n^\pm_{\boldsymbol{\varepsilon}}(\Omega;\Gamma;\mathbf{y})$. Denote $\Gamma' = \Gamma^* \setminus \{0\}$ to obtain

$$\begin{aligned} n^\pm_{\boldsymbol{\varepsilon}}(\Omega,\mathbf{y};0) &= \sum_{\gamma\in\Gamma} \mathbf{1}^{(\mathbf{h})}_{T_{\boldsymbol{\varepsilon}}(\Omega)_{\pm\mathbf{h}}+\mathbf{y}}(\gamma) = \frac{1}{|\Gamma|}\sum_{\gamma^*\in\Gamma^*}\left[\mathscr{F}\mathbf{1}^{(\mathbf{h})}_{T_{\boldsymbol{\varepsilon}}(\Omega)_{\pm\mathbf{h}}+\mathbf{y}}\right](\gamma^*); \\ &= \frac{1}{|\Gamma|}\left[\mathscr{F}\mathbf{1}^{(\mathbf{h})}_{T_{\boldsymbol{\varepsilon}}(\Omega)_{\pm\mathbf{h}}+\mathbf{y}}\right](0) + \Sigma(\boldsymbol{\varepsilon},\mathbf{h},\mathbf{y}) \end{aligned} \tag{4.6.4}$$

Observe that

$$\left[\mathscr{F}\mathbf{1}^{(\mathbf{h})}_{T_{\boldsymbol{\varepsilon}}(\Omega)_{\pm\mathbf{h}}+\mathbf{y}}\right](\boldsymbol{\xi}) = e^{i\mathbf{y}\cdot\boldsymbol{\xi}}\left[\mathscr{F}\mathbf{1}^{(\mathbf{h})}_{T_{\boldsymbol{\varepsilon}}(\Omega)_{\pm\mathbf{h}}+\mathbf{y}}\right](\boldsymbol{\xi})\left[\mathscr{F}\rho_{\mathbf{h}}\right](\boldsymbol{\xi}-\mathbf{y}). \tag{4.6.5}$$

Since we will only find bounds using the absolute values of the terms in the previous equation, and since

$$\left[\mathscr{F}\mathbf{1}^{(\mathbf{h})}_{T_{\boldsymbol{\varepsilon}}(\Omega)_{\pm\mathbf{h}}+\mathbf{y}}\right](0) = \left[\mathscr{F}\mathbf{1}^{(\mathbf{h})}_{T_{\boldsymbol{\varepsilon}}(\Omega)_{\pm\mathbf{h}}}\right](0)$$

we suppose without loss of generality that $\mathbf{y} = 0$.

We first turn our attention to $\left[\mathscr{F}\mathbf{1}^{(\mathbf{h})}_{T_{\boldsymbol{\varepsilon}}(\Omega)_{\pm\mathbf{h}}}\right](0)$. Using properties of the Fourier transform, we have from Condition B and Lemma 4.6.1 that

$$\left[\mathscr{F}\mathbf{1}_{T_{\boldsymbol{\varepsilon}}(\Omega)_{\pm\mathbf{h}}}\right](\xi) = \varepsilon^{-d}\left[\mathscr{F}\mathbf{1}_{\Omega_{\boldsymbol{\varepsilon}\mathbf{h}}}\right](T_{\boldsymbol{\varepsilon}}(\boldsymbol{\xi}))$$

and from equation (4.6.1) that

$$\left[\mathscr{F}\rho_{\mathbf{h}}\right](\boldsymbol{\xi}) = \left[\mathscr{F}\rho\right]\left(T^{-1}_{\mathbf{h}}(\boldsymbol{\xi})\right). \tag{4.6.6}$$

Hence, the first term in equation (4.6.4) is given by

$$\begin{aligned} \left[\mathscr{F}\mathbf{1}^{(\mathbf{h})}_{T_{\boldsymbol{\varepsilon}}(\Omega)_{\pm\mathbf{h}}}\right](0) &= \frac{\varepsilon^{-d}}{|\Gamma|}\left[\mathscr{F}\mathbf{1}_{\Omega_{\boldsymbol{\varepsilon}\mathbf{h}}}\right](T_{\boldsymbol{\varepsilon}}(0))\left[\mathscr{F}\rho\right]\left(T^{-1}_{\mathbf{h}}(0)\right) \\ &= \frac{\varepsilon^{-d}}{|\Gamma|}\mathrm{Vol}(\Omega_{\pm\boldsymbol{\varepsilon}\mathbf{h}}) \end{aligned}$$

As long as all the $\varepsilon_j h_j$ remain bounded, we have that there exists a constant C such that

$$\mathrm{Vol}(\Omega_{\boldsymbol{\varepsilon}\mathbf{h}} \setminus \Omega) \leq C\left(\sum_{j=1}^{m} \varepsilon_j h_j\right),$$

and

$$\mathrm{Vol}(\Omega \setminus \Omega_{-\boldsymbol{\varepsilon}\mathbf{h}}) \leq C\left(\sum_{j=1}^{m} \varepsilon_j h_j\right),$$

hence, writing $\Omega_{\boldsymbol{\varepsilon}\mathbf{h}} = \Omega \cup (\Omega_{\boldsymbol{\varepsilon}\mathbf{h}} \setminus \Omega)$ and $\Omega = \Omega_{-\boldsymbol{\varepsilon}\mathbf{h}} \cup (\Omega \setminus \Omega_{-\boldsymbol{\varepsilon}\mathbf{h}})$ we have

$$\left[\mathscr{F}\mathbf{1}^{(\mathbf{h})}_{T_{\boldsymbol{\varepsilon}}(\Omega)_{\pm\mathbf{h}}}\right](0) = \frac{\varepsilon^{-d}}{|\Gamma|}\mathrm{Vol}(\Omega) + O\left(\sum_{k=1}^{m} \varepsilon_k^{1-d_k} h_k \prod_{j\neq k} \varepsilon_j^{-d_j}\right).$$

Let us now study $\Sigma(\boldsymbol{\varepsilon},\mathbf{h},0)$ in equation (4.6.4). Using equations (4.6.5) and (4.6.6) we deduce that

$$\Sigma(\boldsymbol{\varepsilon},\mathbf{h},0) = \varepsilon^{-d} \sum_{\gamma^*\in\Gamma'} \left[\mathscr{F}\mathbf{1}_{\Omega_{\boldsymbol{\varepsilon}\mathbf{h}}}\right]\left(T_{\boldsymbol{\varepsilon}}(\gamma^*)\right)\left[\mathscr{F}\rho\right]\left(T_{\mathbf{h}}^{-1}(\gamma^*)\right).$$

We have that $\left[\mathscr{F}\rho_{\mathbf{h}}\right]$ is a Schwartz function, *i.e.* for any N

$$\left[\mathscr{F}\rho_{\mathbf{h}}\right](\boldsymbol{\xi}) = O\left((1+|\boldsymbol{\xi}|)^{-N}\right),$$

hence we have that

$$\Sigma(\boldsymbol{\varepsilon},\mathbf{h},0) = \varepsilon^{-d} \sum_{\gamma^*\in\Gamma'} \frac{\left[\mathscr{F}\mathbf{1}_{\Omega_{\boldsymbol{\varepsilon}\mathbf{h}}}\right]\left(T_{\boldsymbol{\varepsilon}}(\gamma^*)\right)}{\left(1+(h_1|\mathbf{x}_1|)^N+\ldots+(h_m|\mathbf{x}_m|)^N\right)}.$$

From here, the proof of Propositions 4.5.2 and 4.5.3 splits.

4.6.4. Proof of Proposition 4.5.2

Due to Condition A, for all $\gamma^* \in \Gamma'$ we have that $\gamma_1^* \neq 0$. From Lemma 4.6.4 with $V = V_1$, we obtain the bound

$$\begin{aligned}
\Sigma(\boldsymbol{\varepsilon},\mathbf{h},0) &\ll \varepsilon^{-d} \sum_{\gamma^*\in\Gamma'} \frac{\varepsilon_1^{\frac{d_1+1}{2}}}{|\gamma_1^*|^{\frac{d_1+1}{2}}\left(1+(h_1\left|\gamma_1^*\right|)^N+\ldots+(h_m\left|\gamma_m^*\right|)^N\right)} \\
&\ll \varepsilon_1^{-\frac{d_1-1}{2}} \prod_{j=2}^{m} \varepsilon_j^{-d_j} \int_{\mathbb{R}^d} \frac{|\mathbf{x}_1|^{-\frac{d_1+1}{2}}}{\left(1+(h_1|\mathbf{x}_1|)^N+\ldots+(h_m|\mathbf{x}_m|)^N\right)}\,\mathrm{d}\mathbf{x} \\
&\ll (\varepsilon_1 h_1)^{-\frac{d_1-1}{2}} \prod_{j=2}^{m} (\varepsilon_j h_j)^{-d_j}.
\end{aligned} \tag{4.6.7}$$

Combining with the estimate on $\left[\mathscr{F}\mathbf{1}^{(\mathbf{h})}_{T_{\boldsymbol{\varepsilon}}(\Omega)_{\pm\mathbf{h}}}\right](0)$, we have that

$$n^{\pm}_{\boldsymbol{\varepsilon}}(\Omega) = \frac{\varepsilon^{-d}}{|\Gamma|}\operatorname{Vol}(\Omega) + O\left(\sum_{k=1}^{m} \varepsilon_k^{1-d_k} h_k \prod_{j\neq k} \varepsilon_j^{-d_j}\right) + O\left((\varepsilon_1 h_1)^{-\frac{d_1-1}{2}} \prod_{j=2}^{m} (\varepsilon_j h_j)^{-d_j}\right).$$

We now need to find the values $h_1,\dots,h_m$ that make the remainders in the last equation small. This will be done by making both

$$\varepsilon_k^{1-d_k} h_k \prod_{j\neq k} \varepsilon_j^{-d_j}, \quad 1\le k\le m$$

and

$$(\varepsilon_1 h_1)^{-\frac{d_1-1}{2}} \prod_{j=2}^{m} (\varepsilon_j h_j)^{-d_j}$$

as small as possible. From Condition B we have that $\varepsilon_k = \varepsilon^{\alpha_k}$ and we look for h_k of the form $h_k = \varepsilon^{\beta_k}$ for some $\beta_k \ge 0$. This gives us the following linear optimisation problem: maximise the minimum η of the functions

$$\alpha_k + \beta_k - \sum_{j=1}^{m} \alpha_j d_j \qquad 1\le k\le m \tag{4.6.8}$$

and

$$(\alpha_1+\beta_1)\left(\frac{1-d_1}{2}\right) - \sum_{j=2}^{m} (\alpha_j+\beta_j) d_j. \tag{4.6.9}$$

under the constraints $\beta_k \ge 0$. The unconstrained problem is solved when all the quantities in displays (4.6.8) and (4.6.9) are made equal. This yields $\alpha_k+\beta_k = \alpha_l+\beta_l$ for all k,l, so let us denote this quantity by σ. By hypothesis, we have that

$$\sum_{j=1}^{m} \alpha_j d_j = \sum_{j=1}^{m} d_j = d,$$

hence making (4.6.9) equal to (4.6.8) gives

$$\sigma - d = \sigma\left(\frac{1+d_1}{2} - d\right),$$

that is

$$\sigma = \frac{2d}{1+2d-d_1}.$$

Taking $\beta_k = \sigma - \alpha_k$ will hence solve the constrained problem if $\alpha_1 \le \sigma$. In such a case, we obtain that the value η in equation (4.5.1) will be

$$\eta = d - \sigma \le d - 1,$$

hence can be bounded away from d uniformly in α_k and d_k.

We now deal with the case where $\alpha_1 > \sigma$. We choose β_k in the following way:

$$\beta_k = \begin{cases} \frac{1+d_1}{2d} & \text{when } \alpha_k < \frac{1+d_1}{2d} \\ 0 & \text{otherwise.} \end{cases}$$

Then, the quantities in display (4.6.8) can be bounded for $1 \le k \le m$ as

$$\begin{aligned} \alpha_k + \beta_k - \sum_{j=1}^{m} \alpha_j d_j &\ge \frac{1+d_1}{2d} - d, \\ &\ge -d + \frac{1}{d}. \end{aligned}$$

Finally, the quantity in display (4.6.9) can be bounded as

$$\begin{aligned} (\alpha_1+\beta_1)\left(\frac{1-d_1}{2}\right) - \sum_{j=2}^{m}(\alpha_j+\beta_j)d_j &\ge \frac{\alpha_1(1+d_1)}{2} - d - (d-d_1)\frac{1+d_1}{2d} \\ &\ge -d + \frac{d_1(1+d_1)}{2d} \\ &\ge -d + \frac{1}{d}, \end{aligned}$$

where we used the fact that $\alpha_1 > \sigma \ge 1$. Taking $\eta = d - \frac{1}{d}$, this finishes the proof of Proposition 4.5.2.

□

4.6.5. Proof of Propositon 4.5.3

This time, we use Lemma 4.6.4 with $V = \mathbb{R}^d$ to obtain

$$\Sigma(\boldsymbol{\varepsilon},\mathbf{h},0) \ll \varepsilon^{-d} \sum_{\gamma^* \in \Gamma'} \frac{\left|T_{\boldsymbol{\varepsilon}}\gamma^*\right|^{\frac{1-d}{2}}}{\left(1 + (h_1\left|\gamma_1^*\right|)^N + \ldots + (h_m\left|\gamma_m^*\right|)^N\right)}.$$

Since $\boldsymbol{\varepsilon}$ satisfies Condition B δ-strongly, we have that $\left|T_{\boldsymbol{\varepsilon}}\gamma^*\right|^{\frac{1-d}{2}} \ll \varepsilon^{\delta\frac{d-1}{2}}\left|\gamma^*\right|^{\frac{1-d}{2}}$. This yields the bounds

$$\begin{aligned} \Sigma(\boldsymbol{\varepsilon},\mathbf{h},0) &\ll \varepsilon^{-d+\delta\frac{d-1}{2}} \sum_{\gamma^* \in \Gamma'} \frac{\left|\gamma^*\right|^{\frac{1-d}{2}}}{\left(1 + (h_1\left|\gamma_1^*\right|)^N + \ldots + (h_m\left|\gamma_m^*\right|)^N\right)} \\ &\ll \varepsilon^{-d+\delta\frac{d-1}{2}} \sum_{\gamma^* \in \Gamma'} \int_{|\mathbf{x}| \ge \mu_1(\Gamma)} \frac{|\mathbf{x}|^{\frac{1-d}{2}}}{\left(1 + (h_1\left|\mathbf{x}_1^*\right|)^N + \ldots + (h_m\left|\mathbf{x}_m^*\right|)^N\right)} \\ &\ll \varepsilon^{\delta\frac{d-1}{2}} \prod_{j=1}^{m} (\varepsilon_j h_j)^{-d_j}. \end{aligned}$$

Combining once more with the estimate on $\left[\mathscr{F}\mathbf{1}^{(\mathbf{h})}_{T_{\boldsymbol{\varepsilon}}(\Omega)_{\pm\mathbf{h}}}\right](0)$, we have that

$$\begin{aligned} n^{\pm}_{\boldsymbol{\varepsilon}}(\Omega) = \frac{\varepsilon^{-d}}{|\Gamma|}\operatorname{Vol}(\Omega) + O\left(\sum_{k=1}^{m} \varepsilon_k^{1-d_k} h_k \prod_{j\neq k} \varepsilon_j^{-d_j}\right) \\ + O\left(\varepsilon^{\delta\frac{d-1}{2}} \prod_{j=1}^{m} (\varepsilon_j h_j)^{-d_j}\right). \end{aligned}$$

This time, the linear optimisation problem consists of minimising the quantities

$$\alpha_k + \beta_k - d \qquad 1 \le k \le m \tag{4.6.10}$$

and

$$\delta\frac{d-1}{2} - d - \sum_{j=1}^{m} \beta_k d_k. \tag{4.6.11}$$

Choosing

$$\beta_k = \begin{cases} \delta\frac{d-1}{2d} & \text{if } \alpha_k \le \delta\frac{d-1}{2d}, \\ 0 & \text{else,} \end{cases}$$

we obtain on display (4.6.10) the bound

$$\alpha_k + \beta_k - d \ge -d + \delta\frac{d-1}{2d}.$$

On the other hand, observe that whenever $\delta < 1$, we have that $\alpha_1 > \delta\frac{d-1}{2d}$. This implies that the quantities in display (4.6.11) can be bounded as

$$\begin{aligned} \delta\frac{d-1}{2} - d - \sum_{j=1}^{m} \beta_k d_k &\ge -d + \delta\frac{d-1}{2} - \delta\frac{(d-1)^2}{2d} \\ &= -d + \delta\frac{d-1}{2d}. \end{aligned}$$

This yields the desired exponent in Proposition 4.5.3, finishing the proof. □

4.A. Sharpness of the constraints in Theorem 4.1.11 and 4.1.5

Let $\varepsilon_k \to 0$ be a sequence to be determined later. Consider the sequence of lattices

$$\Gamma_k := \varepsilon_k^{\frac{d-\delta}{d-1}} \mathbb{Z}^{d-1} \oplus \varepsilon_k^{\delta}\mathbb{Z}.$$

For the sequel we need to distinguish the dimension of balls, let us denote by B_ρ^d the ball of radius ρ in dimension d. We can rewrite the counting function $N(\Gamma_k; B_1^d)$ as

$$\begin{aligned} N(\Gamma_k; B_1^d) &= \sum_{-1 \le \varepsilon_k^\delta j \le 1} \# \left\{ \left(\varepsilon_k^\delta j \oplus \varepsilon^{\frac{d-\delta}{d-1}} \mathbb{Z}^{d-1} \right) \cap B_1^d \right\} \\ &= \sum_{-1 \le \varepsilon_k^\delta j \le 1} \# \left\{ \varepsilon^{\frac{d-\delta}{d-1}} \mathbb{Z}^{d-1} \cap B_{\rho_k}^{d-1} \right\} \end{aligned}$$

where $\rho_k = \sqrt{1 - \left(\varepsilon_k^\delta j\right)^2}$. For ε_k small enough, we have that the terms in the sum can be estimated to yield

$$\begin{aligned} N(\Gamma_k; B_1^d) &= \sum_{-1 \le \varepsilon_k^\delta j \le 1} \left(\omega_{d-1} \varepsilon_k^{-d+\delta} \left(1 - \left(\varepsilon_k^\delta j \right)^2 \right)^{\frac{d-1}{2}} + O\left(\varepsilon_k^{-d+\delta+\tau_d} \right) \right) \\ &= \sum_{-1 \le \varepsilon_k^\delta j \le 1} \omega_{d-1} \varepsilon_k^{-d+\delta} \left(1 - \left(\varepsilon_k^\delta j \right)^2 \right)^{\frac{d-1}{2}} + O\left(\varepsilon_k^{-d+\tau_d} \right) \end{aligned}$$

where $\tau_d \ge \frac{2d}{1+d}$. The quantity in equation (4.A) was studied in [**54**]. Observe that for all $d \ge 2$, the quantity being summed is regular enough that sum over it's Fourier transform will converge without mollification. Hence we have directly from the Poisson summation formula that

$$N(\Gamma_k; 1) = \omega_d \varepsilon_k^{-d} + O\left(\varepsilon^{-d+\tau_d} \right) + 2\varepsilon^{-d} \sum_{j=1}^\infty \left(\frac{\varepsilon_k^\delta}{j} \right)^{d/2} J_{d/2}\left(2\pi \varepsilon_k^{-\delta} j \right).$$

For ε_k small enough, the terms in the last sum are bounded above by those of a series in $j^{-\frac{1+d}{2}}$. When $d \ge 3$, the first term is larger than the rest of the sum, hence choosing the sequence ε_k^δ such that $2\pi\varepsilon_k^{-\delta}$ is a zero of $J'_{d/2}$ yields that

$$2\varepsilon^{-d} \sum_{j=1}^\infty \left(\frac{\varepsilon_k^\delta}{j} \right)^{d/2} J_{d/2}\left(2\pi \varepsilon_k^{-\delta} j \right) \gg \varepsilon^{-d+\delta\frac{1+d}{2}}.$$

For $d = 2$, we write

$$J_1(|\xi|) = C|\xi|^{-1/2} \cos\left(|\xi| - \frac{3\pi}{4} \right) + O\left(|\xi|^{-3/2} \right).$$

This means that one needs to choose ε_k such that $\cos\left(2\pi\varepsilon_k^{-\delta} j - \frac{3\pi}{4}\right)$ is greater than $\sqrt{2}/2$ for $j = 1$ and $j = 2$, which can be done infinitely often as $\varepsilon_k \to 0$.

Note that $\mu_d(\Gamma_k) = |\Gamma_k|^{\delta/d}$, and that

$$\left| N(\Gamma_k; B_1) - \frac{\omega_d}{|\Gamma_k|} \right| \gg |\Gamma_k|^{-1+\delta\frac{1+d}{2d}},$$

implying that it is indeed impossible in Theorem 4.1.11 to have an exponent in the remainder that doesn't converge to -1 as $\delta \to 0$. This in turns implies as well that it is impossible in Theorem 4.1.5 to have an exponent in the remainder that doesn't converge to $d/2$ as $\delta \to 0$.

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