Gale's Fixed Tax for Exchanging Houses*

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Abstract

We consider the taxation of exchanges among a set of agents where each agent owns one object. Agents may have different valuations for the objects and they need to pay taxes for exchanges. Using basic properties, we show that if pairwise (or some) exchanges of objects are allowed, then *all* exchanges (in any possible manner) must be feasible. Furthermore, whenever any agent exchanges his object, he pays the same fixed tax (a lump sum payment which is identical for all agents) independently of which object he consumes. Gale's top trading cycles algorithm finds the final allocation using the agents' valuations adjusted with the fixed tax. Our mechanisms are in stark contrast to Clarke-Groves taxation schemes or the max-med schemes proposed by Sprumont (2013).

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1 Introduction

A large literature on house exchange problems has been developed since the pioneering work of Shapley and Scarf (1974). These problems contain a finite set of agents, each of whom is endowed with a single house. Agents are willing to take part in cyclical exchanges if they are better off by such trades. The key assumption in the model is that monetary transactions are not allowed. In spite of its simplicity, house exchange models have been demonstrated to be powerful in many real-life applications. Maybe the best-known examples are the design of kidney exchange programs (Roth et al., 2004) and school choice mechanisms (Abdulkadiroğlu and Sönmez, 2003). However, even if it is natural to abstain from monetary transfers in some settings, it is very unnatural in others. In a real-life house exchange problem, for example, it is not unlikely that local authorities tax house exchanges. Even in the absence of such tax, it is not unlikely that monetary transfers are needed to compensate for differences in

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house values. This paper considers the latter of these situations, namely a house exchange problem with monetary transfers where the transfers are non-positive meaning that agents pay a non-negative tax whenever being involved in a house exchange. Before detailing the model and the main results of the paper, we provide a more general motivation for the type of problem considered here.

In recent years, many online services that facilitate house exchange have been developed. Even if most of these online services arrange temporary trades of vacation homes (e.g., `HomeExchange.com`), there are some alternative websites where persons are helped to perform permanent home swaps. For example, on the UK based site `EasyHouseExchange.com`, homeowners list their properties by, e.g., uploading photographs and detailed descriptions of their houses (including estimated market values), and state what housing they are looking for in return. The main idea is to create trading cycles among house owners. Most of these websites saw the light of the day in the global financial crises in 2008–2009. For example, Sergei Naumov who, in 2009, was the CEO of one of the largest US house exchange platforms `GoSwap.org` stated that:

> "*Since the housing market tanked, homeowners wishing to upgrade to bigger homes, downsize or relocate have become more open to the idea of making a home swap.*"[1]

The main reason for the increased popularity in permanent house exchanges during the financial crises was that some persons lived in houses that they no longer were able to afford. Because they also were unable to find a buyer, it was better to downsize to a smaller house than declaring bankruptcy even if this resulted in a financial loss. Persons with a more advantageous financial situation were not late to take advantage of this situation. Consequently, permanent trading opportunities emerged. However, even if a homeowner is involved in a permanent cyclical trade, this does by no means imply that the homeowner can avoid paying taxes. For example, experts at `EasyHouseExchange.com` informed their customers that:

> "*Any transaction should be dealt with in the same way that a traditional sale or purchase would be.*"[2]

The framework analyzed in this paper can be thought of as a situation where a social planner attempts to design a tax schedule for house exchange. In relation to the `EasyHouseExchange.com` example and related online services, the findings in this paper can be applied to better understand exactly how to design a tax schedule for house owners involved in permanent home swaps. As will be apparent, these options are very limited, at least if the social planner is interested in a tax scheme satisfying a number of natural and desirable properties.

Each agent is endowed with a single indivisible object and has quasi-linear preferences over consumption bundles. Here, a consumption bundle is a pair consisting of one object and a tax attached to that object. The aim for the social planner is to define a mechanism or, equivalently, a tax schedule that, based on the self-reported preferences of the agents, determines the trades and the taxes. However, such a mechanism is not unique. Consequently, the social planner has the option to restrict the set of possible mechanisms by requiring that the outcome of the mechanism should satisfy a number of desirable properties. These properties are informally described below.

[1]See `www.bankrate.com/finance/real-estate/home-swap-tough-market-1.aspx`. Retrieved June 1, 2018.

[2]See the article "Fair Trade?" in Financial Times (May 17, 2009).

Individual rationality says that each agent weakly prefers his consumption bundle to his endowment and paying no tax. Strategy-proofness ensures that agents honestly report their true preferences over consumption bundles (to the social planner). Constrained efficiency says that the rule is efficient on its range of allocations. Consistency says that the rule is robust subject to the departure of a set of agents with their allotments when those coincide with their endowments. Anonymity says that whenever objects are reassigned, then the names of the agents do not matter.

It turns out that consistency and anonymity are too strong in our context as only the no-trade rule (where all agents always keep their endowments and pay zero tax) will satisfy one of these in conjunction with our first three basic requirements. Therefore, we only require consistency or anonymity in "unambiguous" situations in the following sense: weak consistency says that when a subset of agents trade objects and any agent strictly prefers his allotment to any outside of this subset, then the rule should be robust when allocating only the endowments of that subset; and weak anonymity says that when all agents strictly prefer their allotments to any other allotment and no agent keeps his endowment, then the names of the agents do not matter.

The main innovation of this paper is the introduction of a class of taxation rules called Gale's fixed-tax core rules. Two types of economies are considered. In the first type of economy, there are only two agents and, consequently, these agents either exchange their houses with each other or keep their own hoses. The second type of economy contains more than two agents. In the former of these economies, it will be demonstrated that the above properties (without constrained efficiency) characterize the familiar Clarke-Groves mechanisms (Clarke, 1971; Groves, 1973) satisfying our constraints.[3] In those mechanisms, the tax paid by an agent depends only on the valuations of the other agent and the two agents exchange objects only if the valuations for each other's object exceed the taxes (and an agent keeping his endowment pays no tax). Note that, unlike the classical characterizations of the class of Clarke-Groves mechanisms, e.g., Green and Laffont (1979), Moulin (1986), and Roberts (1979), efficiency is not imposed. Consequently, the payment function and the assignment function must simultaneously be deduced from our desiderata.

In the larger economy with more than two agents, the picture changes dramatically. In particular, two ingredients play a key role. The first is the no-trade rule that prescribes that all agents keep their endowments and pay no taxes. The second is a given number $\alpha \geq 0$, henceforth referred to as Gale's fixed-tax (a lump sum payment which is identical for all agents). Given the number α, each agent's valuations induce a weak ordinal ranking over the objects. More precisely, an object is weakly preferred over another object if and only if the valuation of the first object minus Gale's fixed-tax is greater than or equal to the valuation of the second object minus Gale's fixed-tax. When the induced rankings are strict, Gale's fixed-tax core rule finds the assignment by applying Gale's top trading cycles mechanism (first defined in Shapley and Scarf, 1974), and agents who keep their endowments pay zero tax whereas agents exchanging their endowment pay the fixed-tax α. The main result of the paper (Theorem 1) shows that any rule satisfying the above properties (individual rationality, strategy-proofness, constrained efficiency, weak consistency and weak anonymity) must be either the no-trade rule or a Gale's fixed-tax core rule. Existence of such rules is demonstrated (Theorem 2) by using a construction from Saban and Sethuraman (2013), called the Highest

[3] Pápai (2003) studies fair prices in bidding settings.

Priority Object (HPO) algorithm.[4]

Our model and the results derived from it relate to and extend previous work in the literature. A more general observation is that in order to satisfy individual rationality in a model where agents provide resources from their endowment, Clarke-Groves mechanisms must allow for agents to sometimes receive payments. While this is natural in some environments, it is infeasible in others. For example, patients pay for kidney exchanges via insurance, but it is illegal in all countries except Iran for persons to sell organs, thus ruling out Clarke-Groves mechanisms for this application. For quasi-linear preferences, Sun and Yang (2003) found a rule that identifies the unique Pareto efficient and envy-free allocation, for any given economy, by minimizing payments subject to exogenously given lower bounds on payments.[5] When these lower bounds are set to zero, this rule fails individual rationality and, therefore, the "utilitarian criterion" of Clarke-Groves schemes is not the limiting factor here.

Instead of insisting on pointwise efficiency, Sprumont (2013) showed that the max-med mechanisms are constrained optimal among all anonymous, strategy-proof, and envy-free mechanisms. Note that Gale's fixed-tax core rules are fundamentally different from the max-med mechanisms since consistency cannot be applied to max-med mechanism as they are defined only for two agents. Furthermore, neither the rule defined by Sun and Yang (2003) nor Clarke-Groves mechanisms satisfy consistency because they both generalize the "price externality" feature seen in second-price auctions (Vickrey, 1961). That is, the losing bidders determine the price paid by the winner, and by considering sub-populations containing the winner and some different sets of losing bidders, each population might generate a different price. However, Ehlers (2014) demonstrated that no efficient, individually rational, and strategy-proof rule can be consistent. Consequently, we do not aim for full consistency but rather for a conditional version. The above cited rules still fail this weaker condition while this paper uncovers a continuum of rules that satisfy it.

Miyagawa (2001) shows, in a setting where positive transfers to agents are allowed, that any mechanism satisfying individual rationality, strategy-proofness, ontoness, and non-bossiness must be a fixed price core rule. Under such rule, any agent has a personalized price for any object and if involved in an exchange, the transfer is equal to his personalized price of the object he consumes. By adding budget-balance, these personalized prices are represented by a price vector and an agent's transfer is equal to the difference between his personalized price and the price of the object he consumes. The ontoness axiom means that all exchanges are possible. Note that the mechanisms considered in this paper do not have any property pertaining to the thickness of its range. Furthermore, as it turns out, except for the no-trade rule, fixed-tax core rules, by satisfying constrained-efficiency, violate non-bossiness. This is also due to the fact that in showing the existence of fixed-tax core rules satisfying the above properties, we employ recent contributions on house exchange with indifferences and no monetary transfers. In this context Jaramillo and Manjunath (2012) have shown that there exist rules satisfying individual rationality, efficiency and strategy-proofness.[6] The allocations in the range of a fixed-tax core rule correspond to all possible assignments in the house exchange model (because the same fixed-tax is always paid), and constrained efficiency in this model becomes efficiency in that model. Indeed, as demonstrated in this paper, the rules proposed by Jaramillo and Manjunath (2012) satisfy all of the above defined properties.

[4] This class of algorithms generalizes the procedure found by Jaramillo and Manjunath (2012).

[5] These results was later proved on a more general preference domain by Andersson and Svensson (2008).

[6] Independently Alcalde-Unzu and Molis (2011) have proposed another class of rules satisfying these properties.

The remaining part of this paper is organized as follows. Section 2 presents the model and some desirable properties. Section 3 introduces Gale's fixed-tax core rules and states our main result. Section 4 shows the existence of rules satisfying our properties. Section 5 contains some general remarks, e.g., a characterization of the Clarke-Groves mechanisms in our context for the two-agent economy, and a discussion about the implications for the presented results on a more general preference domain than the quasi-linear.

2 Agents, Preferences and Allocations

Let $N = \{1, ..., n\}$ denote the finite universal set of agents. Agent i owns object i and N also denotes the set of indivisible objects. Let $e : N \to N$ denote the endowment vector such that $e_i = i$ for all $i \in N$. For all $N' \subseteq N$, let $e_{N'} = (e_i)_{i \in N'}$. Agent i's utility function $u_i \in \mathbb{R}^N$ assigns utility u_{ij} for receiving object j. We set $u_{ii} = 0$. Let $\mathcal{U}_i$ denote the set of all utility functions for i. For all $N' \subseteq N$, let $\mathcal{U}_{N'} = \times_{i \in N'} \mathcal{U}_i$. A consumption bundle is a tuple (j, t_i) where $j \in N$ and $t_i \in \mathbb{R}_+$, i.e. agent i pays the tax t_i for consuming j and his utility from consuming (j, t_i) is given by $u_{ij} - t_i$.

Given $N' \subseteq N$, a list $u = (u_i)_{i \in N'}$ of individual utility functions (where $u_i \in \mathcal{U}_i$ for all $i \in N'$) is a (utility) profile (for N'). The set of utility profiles having the above properties is denoted by $\mathcal{U} = \cup_{N' \subseteq N} \mathcal{U}_{N'}$.

Given $N' \subseteq N$, a (feasible) assignment $\mathrm{a} : N' \to N'$ assigns every agent $i \in N'$ an object $j \in N'$ such that $\mathrm{a}_i \neq \mathrm{a}_j$ for all $i \neq j$ (where a_i denotes the object assigned to agent i). Note that any feasible assignment (for N') assigns every agent one object and all objects are assigned.

Given $N' \subseteq N$, an *allocation (for N')* consists of an assignment a and a tax vector $\mathrm{t} = (\mathrm{t}_i)_{i \in N'} \in \mathbb{R}_+^{N'}$, denoted by (a, t) for short. Here t_i denotes the tax agent i is paying in allocation (a, t) and $(\mathrm{a}_i, \mathrm{t}_i)$ denotes i's allotment in (a, t). Let $\mathcal{A}_{N'}$ denote the set of all allocations for N' and $\mathcal{A} = \cup_{N' \subseteq N} \mathcal{A}_{N'}$. An allocation rule φ is a pair (a, t) choosing for each $N' \subseteq N$ and each utility profile $u \in \mathcal{U}_{N'}$ an allocation $(a(u), t(u)) \in \mathcal{A}_{N'}$. We say that two rules $\varphi = (a, t)$ and $\bar{\varphi} = (\bar{a}, \bar{t})$ are equivalent if for all $N' \subseteq N$ and all $u \in \mathcal{U}_{N'}$ we have $u_{ia_i(u)} - t_i(u) = u_{i\bar{a}_i(u)} - \bar{t}_i(u)$ for all $i \in N'$, i.e. for any utility profile the two chosen allocations are utility-equivalent for all agents.

Under the no-trade rule NT, each agent keeps his endowment and no taxes are paid, i.e. for all $N' \subseteq N$ and all $u \in \mathcal{U}_{N'}$, $NT(u) = (e_{N'}, 0_{N'})$ (where $0_{N'} = (0, \ldots, 0)$).

2.1 Properties

In the following we introduce some basic properties for an allocation rule $\varphi = (a, t)$. Individual rationality says that nobody should be worse off than keeping his endowment and paying no tax.

Individual Rationality: For all $N' \subseteq N$, all $u \in \mathcal{U}_{N'}$, and all $i \in N'$, $u_{ia_i(u)} - t_i(u) \geq 0$.

Obviously, if $a_i(u) = i$, then we have $u_{ii} - t_i(u) = -t_i(u) \geq 0$, and by $t_i(u) \geq 0$, we obtain $t_i(u) = 0$. Thus, agent i pays *no tax* (or zero tax) if i keeps his endowment.

Strategy-proofness says that truth-telling is a weakly dominant strategy and because agents' preferences are private information, this property ensures that the mechanism's chosen allocations are based on the true preferences.

Strategy-Proofness: For all $N' \subseteq N$, all $u \in \mathcal{U}_{N'}$, all $i \in N'$ and all $u'_i \in \mathcal{U}_i$, $u_{ia_i(u)} - t_i(u) \geq u_{ia_i(u'_i,u_{-i})} - t_i(u'_i, u_{-i})$.

Constrained efficiency says that the rule is efficient on its range. Given $N' \subseteq N$, let $\mathcal{A}^{\varphi}_{N'}$ denote the range of rule φ for N', i.e. $\mathcal{A}^{\varphi}_{N'} = \{(a(u), t(u)) | u \in \mathcal{U}_{N'}\}$. Let $\mathcal{A}^{\varphi} = \cup_{N' \subseteq N} \mathcal{A}^{\varphi}_{N'}$.

Constrained Efficiency: For all $N' \subseteq N$ and all $u \in \mathcal{U}_{N'}$, if $\varphi(u) = (a(u), t(u))$, then there exists no $(\hat{a}, \hat{t}) \in \mathcal{A}^{\varphi}_{N'}$ such that for all $i \in N'$, $u_{i\hat{a}_i} - \hat{t}_i \geq u_{ia_i(u)} - t_i(u)$ with strict inequality holding for some $j \in N'$.

Consistency[7] requires that if the endowments of some set of agents S are allocated among S, then for the problem restricted to S the objects and taxes are chosen in the same way. This is an appealing property, as it insulates groups of agents from each other, when these agents do not trade with each other. Given $S \subseteq N' \subseteq N$ and $u \in \mathcal{U}_{N'}$, let $u|_S = (u_i)_{i \in S}$ and $\varphi_S(u) = (\varphi_i(u))_{i \in S}$.

Consistency: For all $S \subseteq N' \subseteq N$ and all $u \in \mathcal{U}_{N'}$, if $\cup_{i \in S}\{a_i(u)\} = S$, then $\varphi(u|_S) = \varphi_S(u)$.

Unfortunately, as we show later in Corollary 1, consistency is very strong in conjunction with our other properties as basically only the no-trade rule will satisfy them.[8] Thus, we study instead a weaker property. This imposes consistency only for groups of agents that prefer their object to the objects outside the group. This corresponds to weakening the insulation effect, which is undesirable, but the effect is preserved for the groups that value it most.

Weak Consistency: For all $S \subseteq N' \subseteq N$ and all $u \in \mathcal{U}_{N'}$, if $\cup_{i \in S}\{a_i(u)\} = S$ and for all $i \in S$ and all $j \in N' \backslash S$, $u_{ia_i(u)} - t_i(u) > u_{ia_j(u)} - t_j(u)$, then $\varphi(u|_S) = \varphi_S(u)$.

Note that by definition, if $\cup_{i \in S}\{a_i(u)\} = S$, then consistency requires both $\varphi(u|_S) = \varphi_S(u)$ and $\varphi(u|_{N' \backslash S}) = \varphi_{N' \backslash S}(u)$ whereas weak consistency does not necessarily constrain the rule for $N' \setminus S$.

Let $\sigma : N' \to N''$ be a permutation. For any utility profile u for N', let $\sigma(u)$ denote the utility profile for N'' where both the names of the agents and their endowments are relabeled according to σ.[9] Similarly, σ is used for relabeling assignments and tax vectors.

Anonymity: For all $N', N'' \subseteq N$ with $|N'| = |N''|$, all $u \in \mathcal{U}_{N'}$ and all permutations $\sigma : N' \to N''$, if $\varphi(u) = (a(u), t(u))$, then $\varphi(\sigma(u)) = (\sigma(a(u)), \sigma(t(u)))$.

Anonymity simply says that the chosen allocations are symmetric, i.e. they do not depend on the names of the agents. In the context of exchange anonymity *does not imply* that agents with symmetric utility functions are treated equally: for instance, if two agents i and j

[7]See Thomson (1992, 2009) for in-depth surveys of consistency.

[8]This is not surprising. For instance, in the context of allocating indivisible objects (without monetary transfers), Ehlers and Klaus (2007) show that basically only mixed dicator-pairwise-exchange rules satisfy consistency in conjunction with strategy-proofness and efficiency.

[9]Formally, for all $i, j, k \in N'$ we have $u_{ij} - t_i \geq u_{ik} - t'_i$ iff $u_{\sigma(i)\sigma(j)} - t_i \geq u_{\sigma(i)\sigma(k)} - t'_i$.

have symmetric utility functions in the sense that $u_{ij} = u_{ji}$ and $u_{il} = u_{jl}$ for all $l \neq i, j$, anonymity *does not imply* that agents i and j are treated equally (unless i and j form a pairwise exchange). Anonymity only has power in "completely symmetric" situations.

Similar to weak consistency, we just require anonymity when each agent strictly prefers his allotment to any other agent's allotment (and any agent strictly prefers his allotment to keeping his endowment).

Weak Anonymity: For all $N', N'' \subseteq N$ with $|N'| = |N''|$, all $u \in \mathcal{U}_{N'}$ and all permutations $\sigma : N' \to N''$, if $\varphi(u) = (a(u), t(u))$ and for all $i \in N'$ and all $j \in N' \backslash \{i\}$, $u_{ia_i(u)} - t_i(u) > \max\{0, u_{ia_j(u)} - t_j(u)\}$, then $\varphi(\sigma(u)) = (\sigma(a(u)), \sigma(t(u)))$.

Note that (i) consistency implies weak consistency and (ii) anonymity implies weak anonymity (but the reverse implications are not true). Similarly to consistency, as we show later in Corollary 1, anonymity will turn out to be too strong and basically only the no-trade rule will satisfy anonymity and our other properties.

3 Gale's Fixed-Tax Core Rules

In the following, we define (Gale's) fixed-tax core rules. Let $\alpha \geq 0$ be Gale's fixed tax (a lump sum payment which is identical for all agents). Given $i \in N' \subseteq N$ and $u \in \mathcal{U}_{N'}$, we define the relation $R_i(u_i, \alpha)$ over N' as follows: for all $j, k \in N'$,

(i) $jR_i(u_i, \alpha)k \Leftrightarrow u_{ij} - \alpha \geq u_{ik} - \alpha$;

(ii) $jR_i(u_i, \alpha)i \Leftrightarrow u_{ij} - \alpha \geq u_{ii}$; and

(iii) $iR_i(u_i, \alpha)j \Leftrightarrow u_{ii} \geq u_{ij} - \alpha$.

Let $P_i(u_i, \alpha)$ denote the strict ranking associated with $R_i(u_i, \alpha)$. Given $N' \subseteq N$ and $u \in \mathcal{U}_{N'}$, let $R_{N'}(u, \alpha) = (R_i(u_i, \alpha))_{i \in N'}$. Based on the fixed tax α, each utility profile induces "ordinal" rankings over the endowments. We say that $R_{N'}(u, \alpha)$ is strict (over acceptable objects) if for all distinct $i, j, k \in N'$, $jR_i(u_i, \alpha)kR_i(u_i, \alpha)i$ implies $jP_i(u_i, \alpha)kP_i(u_i, \alpha)i$ and $iR_i(u_i, \alpha)j$ implies $iP_i(u_i, \alpha)j$. Now if the induced preferences are strict, then we may apply Gale's top trading cycles algorithm[10] in order to find the unique core assignment. For strict $R_{N'}(u, \alpha)$, let $C(R_{N'}(u, \alpha))$ denote the unique core assignment.

Definition 1. A rule φ is a (Gale's) fixed-tax core rule if there exists $\alpha \geq 0$ such that for all $N' \subseteq N$ and all $u \in \mathcal{U}_{N'}$,

1. for all $i \in N'$, if $a_i(u) \neq i$, then $t_i(u) = \alpha$,
2. for all $i \in N'$, if $a_i(u) = i$, then $t_i(u) = 0$, and
3. if $R_{N'}(u, \alpha)$ is strict, then $a(u) = C(R_{N'}(u, \alpha))$.

In words, a rule is a fixed-tax core rule if there exists a fixed tax α such that for any utility profile, if an agent does not keep his endowment, then he pays the fixed tax α, the agents who keep their endowment pay zero, and for any utility profile that induces strict ordinal rankings, the core assignment of objects is chosen.

[10]The Appendix defines the HPO Algorithm, which reduces to Gale's top trading cycles algorithm when the induced preferences are strict.

Theorem 1. *If rule φ satisfies individual rationality, strategy-proofness, constrained efficiency, weak anonymity and weak consistency, then φ is a fixed-tax core rule or φ is the no-trade rule.*

3.1 Proof of Theorem 1

Obviously, if $\varphi = NT$, then Theorem 1 is true. Let $\varphi \neq NT$. We need additional notation. Note that any assignment consists of cyclic exchanges or cycles. Formally, in assignment a, a cycle c is a sequence of distinct agents, $c = (i_1, i_2, \ldots, i_k)$ such that $\mathrm{a}_{i_l} = i_{l+1}$ for all $l \in \{1, \ldots, k-1\}$, and $\mathrm{a}_{i_k} = i_1$. Then k is the length of cycle c. We use the convention to write c for both the cycle c and the coalition of agents belonging to cycle c. Let $\mathcal{C}_k$ denote the set of all cycles of length k, and $\mathcal{C} = \cup_{k\in\{2,\ldots,|N|\}}\mathcal{C}_k$ the set of all cycles of length at least two. Let $\mathcal{C}_k^{\varphi} = \{c \in \mathcal{C}_k : \text{there exists } u \in \mathcal{U}_c \text{ such that } a(u) = c\}$. Similarly we define $\mathcal{C}^{\varphi} = \cup_{k\in\{2,\ldots,|N|\}}\mathcal{C}_k^{\varphi}$.

Next we show that if a cycle of length k belongs to $\mathcal{C}_k^{\varphi}$, then all cycles of length k belong to $\mathcal{C}_k^{\varphi}$ and all agents must pay the same tax $\alpha(k)$ in any cycle of length k.

Lemma 1. *Let $c = (1, 2, \ldots, k) \in \mathcal{C}_k^{\varphi}$. Then $\mathcal{C}_k^{\varphi} = \mathcal{C}_k$ and there exists $\alpha(k) \geq 0$ such that for all $c' \in \mathcal{C}_k$ and all $u \in \mathcal{U}_{c'}$, if $a(u) = c'$, then for all $i \in c'$, $t_i(u) = \alpha(k)$.*

Proof: Suppose that $(c, \mathrm{t}), (c, \mathrm{t}') \in \mathcal{A}_c^{\varphi}$ with $\mathrm{t} \neq \mathrm{t}'$. By constrained efficiency, for some $i, j \in c$, $\mathrm{t}_i < \mathrm{t}'_i$ and $\mathrm{t}_j > \mathrm{t}'_j$. Let $y = 1 + \max_{l\in\{1,\ldots,k\}}\{\mathrm{t}_l, \mathrm{t}'_l\}$. Let $u \in \mathcal{U}_c$ be such that for all $i \in c$, $u_{ii+1} = y$, $u_{ii} = 0$ and $u_{ij} = -1$ for $j \neq i, i+1$. By $(c, \mathrm{t}) \in \mathcal{A}_c^{\varphi}$ and constrained efficiency, $a(u) = c$. Let $i \in c$. We show that $u_{ii+1} - t_i(u) > 0$: suppose not; then by individual rationality and $u_{ii+1} = y$, $t_i(u) = y$; let $u'_i \in \mathcal{U}_i$ be such that $u'_{ii+1} = y - \frac{1}{2}$ and $u'_{ij} = -1$ for all $j \neq i, i+1$, and $u' = (u'_i, u_{-i})$. By strategy-proofness and individual rationality, $a_i(u') = i$ and $t_i(u') = 0$. Thus, by construction and individual rationality, $a(u') = e_c$ and $t_l(u') = 0$ for all $l \in c$. This is now a contradiction to constrained efficiency as $(c, \mathrm{t}) \in \mathcal{A}_c^{\varphi}$ and $u'_{ll+1} > \mathrm{t}_l$ for all $l \in c$.

Thus, for all $i \in c$, $u_{ii+1} - t_i(u) > 0$, and $a(u) = c$. By construction, $u_{ia_i(u)} - t_i(u) > \max\{0, u_{ia_j(u)} - t_j(u)\}$ for all $j \neq i$. Now by weak anonymity, for all $i, j \in c$, $t_i(u) = t_j(u) \equiv \alpha(k)$. Because $\mathrm{t} \neq \mathrm{t}'$, then either $\mathrm{t} \neq (\alpha(k))_{i\in c}$ or $\mathrm{t}' \neq (\alpha(k))_{i\in c}$, say $\mathrm{t}' \neq (\alpha(k))_{i\in c}$. If for all $i \in c$, $\mathrm{t}'_i \geq \alpha(k)$, then by constrained efficiency the allocation (c, t') can never be chosen for utility profiles of coalition c, which is a contradiction to $(c, \mathrm{t}') \in \mathcal{A}_c^{\varphi}$. Thus, for some $i \in c$, $\alpha(k) = t_i(u) > \mathrm{t}'_i$. But then using strategy-proofness and constrained efficiency yields a contradiction: i may report $u'_{ii+1} = \frac{1}{2}(t_i(u) + \mathrm{t}'_i)$ and $u'_{ij} = u_{ij}$ for $j \neq i+1$, and then by strategy-proofness and individual rationality, $a(u'_i, u_{-i}) = e_c$ and $t(u'_i, u_{-i}) = 0_c$, which is a contradiction to constrained efficiency by $(c, \mathrm{t}') \in \mathcal{A}_c^{\varphi}$.

Hence, for all $(c, \mathrm{t}), (c, \mathrm{t}') \in \mathcal{A}_c^{\varphi}$ we have $\mathrm{t} = \mathrm{t}'$, and $\alpha(k) = \mathrm{t}'_i = \mathrm{t}_i$ for all $i \in c$. Let $c' \in \mathcal{C}_k$ and $\sigma : c \to c'$ be a permutation. By weak anonymity and $a(u) = c$, we obtain $a(\sigma(u)) = c'$ and for all $i \in c'$, $t_i(\sigma(u)) = \alpha(k)$. Hence, $\mathcal{C}_k^{\varphi} = \mathcal{C}_k$. □

In the following we show $\mathcal{C}^{\varphi} = \mathcal{C}$.

Since $\varphi \neq NT$, we must have $\mathcal{C}^{\varphi} \neq \emptyset$. By weak anonymity and strategy-proofness, if $\mathcal{C}_2^{\varphi} \neq \emptyset$, then $\mathcal{C}_2^{\varphi} = \mathcal{C}_2$ and in any pairwise trade agents pay the tax $\alpha(2)$ for two-agent utility profiles.

Let $c = (1, 2, \ldots, k)$. We say that $u \in \mathcal{U}_c$ is *c-cyclic* if for each $i \in c$, $u_{ii+1} > u_{ii-1} > \alpha(2)$ and $u_{ij} < 0$ for all $j \in c\backslash\{i, i+1\}$.

Lemma 2. *Let $C_2^\varphi \neq \emptyset$. Then for all $k \in \{3, \ldots, |N|\}$, $\mathcal{C}_k^\varphi = \mathcal{C}_k$.*

Proof: Suppose that for some $k \in \{3, \ldots, |N|\}$, $\mathcal{C}_k^\varphi = \emptyset$. Let $c = (1, 2, \ldots, k)$ and fix a c-cyclic $u \in \mathcal{U}_c$. By $\mathcal{C}_k^\varphi = \emptyset$, we have $a(u) \neq c, c'$.

Because u is c-cyclic and from *individual rationality*, the only admissible trading arrangement is a mix of pairwise trading and keeping one's endowment.

First, suppose that there exists $i \in \{1, \ldots, k\}$ such that $a_i(u) = i$, say $i = 2$. By individual rationality, $a_2(u) = 2$ and $t_2(u) = 0$. Let $u_2' \in \mathcal{U}_2$ be such that $u_{21}' = u_{21}$ and $u_{2l}' < 0$ for all $l \in c\backslash\{1, 2\}$. Let $u' = (u_2', u_{-2})$. By individual rationality, $a_2(u') \in \{1, 2\}$. If $a_2(u') = 1$, then by strategy-proofness and both $a_2(u) = 2$ and $t_2(u) = 0$, $t_2(u') = u_{21}$. But then choose $u_2'' \in \mathcal{U}_2$ such that $u_{21}' > u_{21}'' > \alpha(2)$ and $u_{2l}'' = u_{2l}'$ for all $l \in c\backslash\{1\}$. Let $u'' = (u_2'', u_{-2})$. Then by strategy-proofness and individual rationality, $a_2(u'') = 2$ and $t_2(u'') = 0$. Thus, without loss of generality, we may suppose for u' that $a_2(u') = 2$ and $t_2(u') = 0$ (by individual rationality). By $\mathcal{C}_k^\varphi = \emptyset$, we have $a_1(u') \in \{1, k\}$.

If $a_1(u') = 1$, then let $u_1'' \in \mathcal{U}_1$ be such that $u_{12}'' = u_{12}$ and $u_{1l}'' < 0$ for all $l \in c\backslash\{1, 2\}$. Let $u'' = (u_1'', u_{-1}')$. By strategy-proofness, individual rationality and $\mathcal{C}_k^\varphi = \emptyset$, $a_1(u'') = 1$ and $t_1(u'') = 0$. Hence, by individual rationality, $a_2(u'') = 2$ and $t_2(u'') = 0$. But now by weak consistency, $a(u_{\{1,2\}}'') = (1, 2)$, which is a contradiction to constrained efficiency as $((2, 1), (\alpha(2), \alpha(2)) \in \mathcal{A}_{(2,1)}^\varphi$.

If $a_1(u') = k$, then using the same argument as above, strategy-proofness and weak consistency (because 1 may deviate as above, obtain 2 and pay $\alpha(2)$), we must have

$$u_{12} - \alpha(2) \leq u_{1k} - t_1(u') < u_{12} - t_1(u')$$

where the first inequality follows from strategy-proofness and the second one from $u_{1k} < u_{12}$. Thus, we have $t_1(u') < \alpha(2)$. Now let $u_1'' \in \mathcal{U}_1$ be such that $u_{1k}'' - t_1(u') > 0 > u_{1k}'' - \alpha(2)$ and $u_{1l}'' < 0$ for all $l \in c\backslash\{1, k\}$. Let $u'' = (u_1'', u_{-1}')$. By strategy-proofness, $a_1(u'') = k$ and $t_1(u'') = t_1(u')$. By $\mathcal{C}_k^\varphi = \emptyset$, we have $a_k(u'') = 1$. But now by construction, $u_{1k}'' - t_1(u'') > u_{1a_j(u'')}'' - t_j(u'')$ for all $j \in c\backslash\{1, k\}$. If $u_{k1} - t_k(u'') > u_{ka_j(u'')} - t_j(u'')$ for all $j \in c\backslash\{1, k\}$, then by weak consistency, $\varphi(u''|_{\{1,k\}}) = \varphi_{\{1,k\}}(u'')$. Thus, $a_1(u''|_{\{1,k\}}) = k$ and $t_1(u''|_{\{1,k\}}) = t_1(u'') = t_1(u') < \alpha(2)$, which is a contradiction to the fact that in any pairwise trade for two-agent utility profiles agents pay the tax $\alpha(2)$. Otherwise ($u_{k1} - t_k(u'') \leq u_{ka_j(u'')} - t_j(u'')$ for some $j \in c\backslash\{1, k\}$), choose $u_k''' \in \mathcal{U}_k$ such that $u_{k1}''' = u_{k1} + 1$ and $u_{kl}''' < 0$ for all $l \in c\backslash\{1, k\}$, and let $u''' = (u_k''', u_{-k}'')$. Now by strategy-proofness, $a_k(u''') = 1$ and $t_k(u''') = t_k(u'')$. By $\mathcal{C}_k^\varphi = \emptyset$ and individual rationality, $a_1(u''') = k$ and $0 \leq t_1(u''') < \alpha(2)$. But now as above we use weak consistency to derive a contradiction to the fact that in any pairwise trade for two-agent utility profiles agents pay the tax $\alpha(2)$.

Thus, for all $i \in \{1, \ldots, k\}$ we have $a_i(u) \neq i$. Hence, k is even and $a(u)$ must consist of $\frac{k}{2}$ pairwise exchanges (and $k \geq 4$). Without loss of generality, let $a_1(u) = k$.

If $t_1(u) < \alpha(2)$, then let $u_1' \in \mathcal{U}_1$ be such that $u_{1k}' - t_1(u) > 0 > u_{1k}' - \alpha(2)$ and $u_{1l}' < 0$ for all $l \in c\backslash\{1, k\}$. Let $u' = (u_1', u_{-1})$. By strategy-proofness, $a_1(u') = k$ and $t_1(u') = t_1(u)$. Thus, by $\mathcal{C}_k^\varphi = \emptyset$, we have $a_k(u') = 1$. Now we derive a contradiction as above using weak consistency and the fact that in any pairwise trade for two-agent utility profiles agents pay $\alpha(2)$.

If $t_1(u) \geq \alpha(2)$, then let $u_1' \in \mathcal{U}_1$ be such that $u_{12}' = u_{12}$ and $u_{1l}' < 0$ for all $l \in c\backslash\{1, 2\}$. Let $u' = (u_1', u_{-1})$. By individual rationality, $a_1(u') \in \{1, 2\}$. If $a_1(u') = 2$, then by strategy-proofness, $u_{12} - t_1(u') \leq u_{1k} - t_1(u) < u_{12} - t_1(u)$ which implies $t_1(u') > t_1(u) \geq \alpha(2)$. But then choose $u''_1 \in \mathcal{U}_1$ such that $u''_{12} - t_1(u') < 0 < u''_{12} - \alpha(2)$ and $u''_{1l} < 0$ for all $l \in c\backslash\{1,2\}$. Let $u'' = (u''_1, u_{-1})$. Now from strategy-proofness and individual rationality, it follows $a_1(u'') = 1$ and $t_1(u'') = 0$.

Thus, without loss of generality, let $t_1(u) \geq \alpha(2)$, $u' = (u'_1, u_{-1})$ be such that $u'_{12} > \alpha(2)$, $u'_{1l} < 0$ for all $l \in c\backslash\{1,2\}$, and both $a_1(u') = 1$ and $t_1(u') = 0$. Now by weak consistency and strategy-proofness, we cannot have $a_2(u') = 2$ (otherwise let $u''_2 \in \mathcal{U}_2$ be such that $u''_{21} = u_{21}$ and $u''_{2l} < 0$ for all $j \neq 1,2$, and then we use strategy-proofness and weak consistency for $(u''_2, u{-1}')$ to derive a contradiction as above). Thus, by $\mathcal{C}^{\varphi}_k = \emptyset$, $a_2(u') = 3$ and $a_3(u') = 2$. Now we can use the same arguments as above for 3 in the role of 1 to deduce $t_3(u') \geq \alpha(2)$ and (without loss of generality) for $u''_3 \in \mathcal{U}_3$ such that $u''_{34} > \alpha(2)$ and $u''_{3l} < 0$ for all $l \in c\backslash\{3,4\}$ and $u'' = (u''_3, u'_{-3})$, we have $a_3(u'') = 3$ and $t_3(u'') = 0$ (and $a_2(u'') \neq 3$). Again by strategy-proofness and weak consistency, we cannot have $a_4(u'') = 4$. If $k = 4$, then $a_4(u'') = 1$ and by $\mathcal{C}^{\varphi}_k = \emptyset$, $a_1(u'') = k$, which is a contradiction to individual rationality (by $u'_{1k} < 0$). If $k > 4$, then by weak consistency and $\mathcal{C}^{\varphi}_k = \emptyset$, $a_4(u'') = 5$ and $a_5(u'') = 4$. Then using the same arguments as above for 5 in the role of 1, we obtain a contradiction since k is finite and even.

Thus, $\mathcal{C}^{\varphi}_k \neq \emptyset$. Now weak consistency implies $\mathcal{C}^{\varphi}_k = \mathcal{C}_k$. □

Note that by Lemma 1, all agents pay in any cycle of length k the same fixed tax $\alpha(k)$. Let $\alpha(2) = \alpha$. We aim to show that whenever a cycle c of length k forms, all agents pay the fixed tax $\alpha(k) = \alpha$.

Lemma 3. *For all $N' \subseteq N$ and all $u \in \mathcal{U}_{N'}$, if $a(u)$ contains the cycle $c = (i_1, \ldots, i_k)$ of length k, then for all $l = 1, \ldots, k$, $t_{i_l}(u) = \alpha(k) = \alpha$.*

Proof: Without loss of generality, let $c = (1, 2, \ldots, k) \in \mathcal{C}^{\varphi}_k$. By Lemma 1, all agents pay the fixed tax $\alpha(k)$ in cycle c for all $u \in \mathcal{U}_c$ such that $a(u) = c$, and by weak anonymity, $\mathcal{C}^{\varphi}_k = \mathcal{C}_k$.

We show $\alpha(k) = \alpha$. Suppose not, i.e. $\alpha(k) < \alpha$ or $\alpha(k) > \alpha$.

First, suppose $\alpha(k) < \alpha$. Consider the c-cyclic utility profile $u_c \in \mathcal{U}_c$ for the cycle $c = (1, \ldots, k)$ such that $\alpha(k) < u_{ii-1} < u_{ii+1} < \alpha$ for all $i \in c$. Let $c' = (k, \ldots, 1)$. Suppose that $a(u) \neq c, c', e_c$. Because in any cycle c all agents pay the fixed tax $\alpha(k)$, constrained efficiency implies $u_{ia_i(u)} - t_i(u) > u_{ii+1} - \alpha(k) > 0$ for some $i \in c$. But then $a_i(u) \neq i$ and

$$u_{ii+1} - t_i(u) \geq u_{ia_i(u)} - t_i(u) > u_{ii+1} - \alpha(k),$$

which implies $\alpha(k) > t_i(u)$. Let $u'_i \in \mathcal{U}_i$ be such that $\alpha(k) > u'_{ia_i(u)} > t_i(u)$ and $0 > u'_{il}$ for all $l \neq i, a_i(u)$. Let $u' = (u'_i, u_{-i})$. By strategy-proofness, $a_i(u') = a_i(u)$ and $t_i(u') = t_i(u)$. But then by indivual rationality and $\alpha(k) > u'_{ia_i(u)}$, i is part of a pairwise exchange under u' with agent $j = a_i(u')$ (and $j \neq i$). Let $u''_j \in \mathcal{U}_j$ be such that $u''_{ji} = u_{ji}$ and $u''_{jl} = -1$ for all $l \neq i, j$. Let $u'' = (u''_j, u_{-j})$. By strategy-proofness, $a_j(u'') = a_j(u') = i$. By individual rationality, $a_i(u'') = j$ and $t_i(u'') < \alpha(k) < \alpha$. This is now a contradiction to weak consistency (since $u'_{ia_i(u')} < \alpha(k) < \alpha$ and in any pairwise trade for two-agent utility profiles agents pay α). Hence, $a(u) \in \{c, c', e_c\}$. By constrained efficiency and the fact that in any cycle of length k all agents pay the same fixed tax $\alpha(k)$, we have $a(u) = c$.

Let $u'_2 \in \mathcal{U}_2$ be such that $\alpha(k) < u'_{23} < u'_{21} < \alpha$ and $u'_{2l} = -1$ for $l \neq 1, 2, 3$. Let $u' = (u'_2, u_{-2})$. Suppose that $a(u') \neq c, c'$. If for some $i \in c\backslash\{2\}$, $u_{ia_i(u')} - t_2(u') > u_{ii+1} - \alpha(k)$, then we use the same arguments as above to derive a contradiction. Otherwise, by constrained efficiency, $u'_{2a_2(u')} - t_2(u') > u'_{23} - \alpha(k) > 0$ and 2 is involved in a pairwise trade under u'. Let

$\hat{u} \in \mathcal{U}_c$ be such that for all $i \in c$, $\hat{u}_{ia_i(u')} = u'_{ia_i(u')}$ and $\hat{u}_{il} < 0$ for $l \neq i, a_i(u')$. If there is any trading under $\hat{u}$, then it must be pairwise as $a(u') \neq c, c'$. If there is any pairwise trade under $\hat{u}$, then using weak consistency gives together with Lemma 1 gives us a contradiction as for all $i, l \in c$ we have $\hat{u}_{il} < \alpha$. Thus, $a_i(\hat{u}) = i$ for all $i \in c$ which is a contradiction to constrained efficiency as $\hat{u}_{ia_i(u')} - t_i(u') \geq 0$ for all $i \in c$ and $\hat{u}_{2a_2(u')} - t_2(u') > 0$ (and $(a(u'), t(u')) \in \mathcal{A}^{\varphi}_c$). Hence, $a(u') \in \{c, c'\}$.

Suppose that $a(u') = c$. Because c is a cycle of length k, $t_2(u') = \alpha(k)$. Let $u''_2 \in \mathcal{U}_2$ be such that $u''_{21} = u'_{21}$ and $u''_{2l} = -1$ for $l \neq 1, 2$. Let $u'' = (u''_2, u_{-2})$. If $a(u'') = c'$, then $t_2(u'') = \alpha(k)$. But now we have $u'_{21} - \alpha(k) > u'_{23} - \alpha(k)$, a contradiction to strategy-proofness. Thus, $a(u'')$ consists of a mix of pairwise trading and keeping one's endowment (and $a(u'')$ contains at least one pairwise trade by constrained efficiency). But then using the same profile $\hat{u}$ as above we derive a contradiction using weak consistency and individual rationality as $u''_{ij} < \alpha$ for all $i, j \in c$.

Hence, $a(u') = c'$ and $a_1(u') = k$. Let $u''_1 \in \mathcal{U}_1$ be such that $u''_{12} = u_{12}$ and $u''_{1l} = -1$ for $l \neq 1, 2$. Let $u'' = (u''_1, u'_{-1})$. If $a(u'') = c$, then $t_1(u'') = \alpha(k)$. But now we have $u'_{12} - \alpha(k) > u'_{1k} - \alpha(k)$, a contradiction to strategy-proofness. Thus, $a(u'')$ consists of a mix of pairwise trading and keeping one's endowment (and $a(u'')$ contains at least one pairwise trade by constrained efficiency). But then using the same profile $\hat{u}$ as above we derive a contradiction using weak consistency and individual rationality as $u''_{ij} < \alpha$ for all $i, j \in c$. Hence, $\alpha(k) < \alpha$ is not possible.

Second, suppose $\alpha(k) > \alpha$. Consider the same type of utility profile u for the cycle $c = (1, \ldots, k)$ as in Lemma 2 such that $\alpha < u_{ii-1} < u_{ii+1} < \alpha(k)$ for all $i \in c$. Because in cycles of length k the fixed tax $\alpha(k)$ is paid, by individual rationality, $a(u)$ consists of a mix of pairwise trading and keeping one's endowment. Then using the same arguments as in the proof of Lemma 2 yields a contradiction.

Finally, let $i \in N' \subseteq N$ and $u \in \mathcal{U}_{N'}$ be such that $a_i(u) \neq i$. We show $t_i(u) = \alpha$. Suppose $t_i(u) < \alpha$. Let $u'_i \in \mathcal{U}_i$ be such that $u'_{ia_i(u)} = \frac{1}{2}(\alpha + t_i(u))$ and $u'_{il} = -1$ for all $l \neq a_i(u), i$. Let $u' = (u'_i, u_{-i})$. By strategy-proofness, $a_i(u') = a_i(u)$ and $t_i(u') = t_i(u)$. Let $j \in N'$ be such that $a_j(u') = i$, and $u''_j \in \mathcal{U}_j$ be such that $u''_{ja_j(u')} = u_{ja_j(u')} + 1$ and $u''_{jl} = -1$ for all $l \neq a_j(u'), j$. Let $u'' = (u''_j, u'_{-j})$. By strategy-proofness, $a_j(u'') = i$ and $t_j(u'') = t_j(u')$. By individual rationality and our construction, $a_i(u'') = a_i(u')$ and $t_i(u'') < \alpha$. If $a_i(u'') = j$, then applying weak consistency yields a contradiction because in all pairwise exchanges the fixed tax α is paid. If $a_i(u'') \neq j$, then let $a_h(u'') = j$ and $u'''_h \in \mathcal{U}_h$ be such that $u'''_{hj} = u_{hj} + 1$ and $u'''_{hl} = -1$ for all $l \neq j, h$. Then we derive the same conclusions as above. At some point we arrive at a profile $\tilde{u}$ such that $a(\tilde{u})$ contains the cycle c, $i \in c$, $t_i(\tilde{u}) < \alpha$, and $\tilde{u}_{ha_h(\tilde{u})} - t_h(\tilde{u}) \geq 0 > \tilde{u}_{hj} - t_j(\tilde{u})$ for all $h \in c$ and all $j \in N' \backslash c$. Then applying weak consistency yields a contradiction because in the exchange c the fixed tax α is paid by all agents belonging to c.

Thus, for all $i \in N'$, if $a_i(u) \neq i$, then $t_i(u) \geq \alpha$. Let $\hat{u} \in \mathcal{U}_{N'}$ be such that (i) $\hat{u}_{ia_i(u)} - \alpha > 0 > \hat{u}_{ij}$ for all $j \in N' \backslash \{i, a_i(u)\}$ and all $i \in N'$ with $a_i(u) \neq i$ and (ii) $0 > \hat{u}_{ij}$ for all $j \in N' \backslash \{i\}$ and all $i \in N'$ with $a_i(u) = i$. Then by constrained efficiency and weak consistency, $a(\hat{u}) = a(u)$ and for all $i \in N$ with $a_i(u) \neq i$, $t_i(\hat{u}) = \alpha$. Thus, the assignment $a(u)$ together with everybody, who does not keep his endowment, paying α belongs to $\mathcal{A}^{\varphi}_{N'}$.

Suppose that $t_j(u) > \alpha$ for some $j \in N'$. Because $t_i(u) \geq \alpha$ for all $i \in N'$ such that $a_i(u) \neq i$, the assignment $a(u)$ together with everybody, who does not keep his endowment, paying α belongs to $\mathcal{A}^{\varphi}_{N'}$ Pareto dominates $(a(u), t(u))$, which is a contradiction to constrained efficiency. □

Thus, we have shown that if agent i does not keep his endowment, then i pays the fixed tax α.

Lemma 4. *For all $N' \subseteq N$ and all $u \in \mathcal{U}_{N'}$, if $R_{N'}(u,\alpha)$ is strict, then $a(u) = C(R_{N'}(u,\alpha))$.*

Proof. Let $c = (i_1, \ldots, i_k)$ be a top cycle in $R_{N'}(u,\alpha)$. Suppose that c is not part of $a(u)$, say $a_{i_k}(u) \neq i_1$. Because $R_{N'}(u,\alpha)$ is strict and in any exchange agents pay the fixed tax α, then $u_{i_k i_1} - \alpha > u_{i_k a_{i_k}(u)} - t_{i_k}(u)$. Let $u'_{i_k} \in \mathcal{U}_{i_k}$ be such that $u'_{i_k i_1} = u_{i_k i_1}$ and $u'_{i_k l} < 0$ for all $l \neq i_1, i_k$. By strategy-proofness and the fact that in cyclical exchanges the fixed tax α is paid, $a_{i_k}(u'_{i_k}, u_{-i_k}) = i_k$ and $t_{i_k}(u'_{i_k}, u_{-i_k}) = 0$. Note that $R_{N'}(u'_{i_k}, u_{-i_k}, \alpha)$ is strict and c remains a top cycle under $R(u'_{i_k}, u_{-i_k}, \alpha)$. Thus, $a_{i_{k-1}}(u'_{i_k}, u_{-i_k}) \neq i_k$. Similar as above $u_{i_{k-1}}$ can be replaced $u'_{i_{k-1}} \in \mathcal{U}_{i_{k-1}}$ such that $u'_{i_{k-1} i_k} = u_{i_{k-1} i_k}$ and $u'_{i_{k-1} l} < 0$ for all $l \neq i_{k-1}, i_k$. Then we arrive at a profile $u' = (u'_{\{i_1,\ldots,i_k\}}, u_{-\{i_1,\ldots,i_k\}})$ where c is still a top cycle under $R_{N'}(u',\alpha)$ but all agents in c receive their endowments, i.e. for $l \in \{i_1, \ldots, i_k\}$, $a_l(u') = l$ and $t_l(u') = 0$. By construction, for all $l \in \{i_1, \ldots, i_k\}$ and all $j \in N' \backslash \{i_1, \ldots, i_k\}$, $0 > u_{l a_j(u')} - t_j(u')$. Thus, by weak consistency, for all $l \in \{i_1, \ldots, i_k\}$, $a_l(u'_{\{i_1,\ldots,i_k\}}) = l$ and $t_l(u'_{\{i_1,\ldots,i_k\}}) = 0$. This is a contradiction to constrained efficiency because c is top cycle under $R_{N'}(u',\alpha)$ and for all $l \in \{1, \ldots, k\}$, $u'_{i_l i_{l+1}} - \alpha = u_{i_l i_{l+1}} - \alpha > 0$.

Thus, c must be part of $a(u)$. Consider a top cycle in $N' \backslash c$, say $c' = (j_1, \ldots, j_m)$. If c' is not part of u, then we can do the same as above: let $u'_{j_m} \in \mathcal{U}_{j_m}$ be such that $u'_{j_m j_1} = u_{j_m j_1}$ and $u'_{j_m l} < 0$ for all $l \neq j_1, j_m$, and $u' = (u'_{j_m}, u_{-j_m})$. Then under u' the cycle c remains a top cycle in the strict $R_{N'}(u',\alpha)$, and thus by the above, $a(u')$ contains c. But then by strategy-proofness and the fact that in cyclical exchanges the fixed tax α is paid, $a_{j_m}(u') = j_m$ and $t_{j_m}(u') = 0$. Note that $R_{N'}(u',\alpha)$ is strict and c remains a top cycle under $R_{N'}(u',\alpha)$ and c' is a top cycle in $N' \backslash c$. Now the same arguments as above yield a contradiction to weak consistency and constrained efficiency. □

We have shown that if $\mathcal{C}_2^{\varphi} \neq \emptyset$, then by Lemma 3 and Lemma 4, φ is a fixed-tax core rule. Our final lemma completes the proof of Theorem 1.

Lemma 5. *If $\mathcal{C}^{\varphi} \neq \emptyset$, then $\mathcal{C}_2^{\varphi} \neq \emptyset$.*

Proof. Suppose that $\mathcal{C}_2^{\varphi} = \emptyset$. By $\mathcal{C}^{\varphi} \neq \emptyset$, let k be minimal such that $\mathcal{C}_k^{\varphi} \neq \emptyset$ and for all $l \in \{2, \ldots, k-1\}$, $\mathcal{C}_l^{\varphi} = \emptyset$. By weak anonymity, $\mathcal{C}_k^{\varphi} = \mathcal{C}_k$. Let $c = (1, \ldots, k)$ and $c' = (k, \ldots, 1)$. By Lemma 1, there exists a unique symmetric fixed tax $\alpha(k)$ for cycles of length k. Let $u \in \mathcal{U}_c$ be such that (i) $u_{21} > u_{23} > \alpha(k)$ and $u_{2l} = -1$ for $l \neq 1, 2, 3$ and (ii) for all $i \in c \backslash \{2\}$, $u_{ii+1} > u_{ii-1} > \alpha(k)$ and $u_{il} = -1$ for $l \neq i-1, i, i+1$. If $a(u) \neq c, c'$, then $a(u)$ is a mix of pairwise trading and keeping one's endowment. Then we use a similar argument as in Lemma 3, via a profile like $\hat{u}$, to deduce a contradiction to the hypothesis that $\mathcal{C}_2^{\varphi} = \emptyset$. By constrained efficiency and our choice of k, $a(u) \in \{c, c'\}$.

First, let $a(u) = c$. Then by Lemma 1, $t_2(u) = \alpha(k)$. Let $u'_2 \in \mathcal{U}_2$ be such that $u'_{21} = u_{21}$ and $u'_{2l} = -1$ for $l \neq 1, 2$. By constrained efficiency and our choice of k, $a(u'_2, u_{-2}) = c'$ and $t_2(u'_2, u_{-2}) = \alpha(k)$. But this is now a contradiction to strategy-proofness as $u_{21} - \alpha(k) > u_{23} - \alpha(k)$.

Second, let $a(u) = c'$. Let $u'_1 \in \mathcal{U}_1$ be such that $u'_{12} = u_{12}$ and $u'_{1l} = -1$ for $l \neq 1, 2$. By constrained efficiency and our choice of k, $a(u'_1, u_{-1}) = c$ and $t_1(u'_1, u_{-1}) = \alpha(k)$. But this is now a contradiction to strategy-proofness as $u_{12} - \alpha(k) > u_{1k} - \alpha(k)$. $\square$

Note that Lemma 3, Lemma 5 and individual rationality imply 1. and 2. of Definition 1, and this together with Lemma 4 implies 3. of Definition 1.

Remark 1. For all $i \in N$, $u_i \in \mathcal{U}_i$ is a vector $u_i \in \mathbb{R}^N$ such that $u_{ii} = 0$. Then (strictly speaking) for $N' \subsetneq N$ and $u \in \mathcal{U}_{N'}$, the allocation $\varphi(u)$ may depend on the utilities over $N \backslash N'$. Our definition of a rule did not exclude this. However, as one may check, the proof of Theorem 1 did not require this.[11]

4 Existence

Theorem 1 showed if a rule φ satisfies individual rationality, weak anonymity, strategy-proofness, constrained efficiency and weak consistency, then φ is a fixed-tax core rule or φ is the no-trade rule. Let φ be a fixed-tax core rule with fixed tax $\alpha \geq 0$. Note that the range of φ for $N' \subseteq N$ is given by $\mathcal{A}^{\varphi}_{N'} = \{(\hat{a}, \hat{t}) \in \mathcal{A}_{N'} | \hat{t}_i = 0 \text{ if } \hat{a}_i = i \text{ and } \hat{t}_i = \alpha \text{ otherwise}\}$. Let $\mathcal{W}_i$ denote the set of all weak ordinal rankings over N. Under φ, agent i's possible consumption bundles are $(i, 0)$ and (j, α) with $j \neq i$. Thus, agent i's utility functions induce *all* weak ordinal rankings over his consumption bundles or over N, i.e. $\{R_i(u_i, \alpha) | u_i \in \mathcal{U}_i\} = \mathcal{W}_i$.

Now in order to establish, for the fixed tax α, the existence of a rule satisfying our properties, we use a construction of Saban and Sethuraman (2013), called the *Highest Priority Object (HPO)* algorithm. This is a class of algorithms that generalizes the procedure found by Jaramillo and Manjunath (2012), which was the first to demonstrate the existence of individually rational, strategy-proof and efficient rules for the model of house exchange with indifferences and no monetary transfers.

Given $N' \subseteq N$, let $\mathcal{O}_{N'}$ denote the set of feasible assignments for N'. Let $f : \cup_{N' \subseteq N} \mathcal{W}_{N'} \to \cup_{N' \subseteq N} \mathcal{O}_{N'}$ be assignment rule. Then

(i) f is assignment-individually-rational iff for all $N' \subseteq N$ and all $R \in \mathcal{W}_{N'}$, we have $f_i(R) R_i i$ for all $i \in N'$,

(ii) f is assignment-strategy-proof iff for all $N' \subseteq N$, all $R \in \mathcal{W}_{N'}$, all $i \in N'$ and all $R'_i \in \mathcal{W}_i$, $f_i(R) R_i f_i(R'_i, R_{-i})$, and

(iii) f is assignment-efficient iff for all $N' \subseteq N$ and all $R \in \mathcal{W}_{N'}$, there exists no feasible assignment $\mathrm{a} \in \mathcal{O}_{N'}$ such that $\mathrm{a}_i R_i f_i(R)$ for all $i \in N'$ with strict preference holding for some $j \in N'$.

Fix an assignment rule f belonging to the class of Highest Priority Object Algorithms (we define this class formally in the Appendix). Then by Saban and Sethuraman (2013), f is assignment-individually-rational, assignment-strategy-proof and assignment-efficient.

Given $\alpha \geq 0$, Gale's fixed α-tax rule $\varphi^f = (a^f, t^f)$ based on f is defined as follows: for all $N' \subseteq N$ and all $u \in \mathcal{U}_{N'}$, we have $a^f(u) = f(R_{N'}(u, \alpha))$ and for all $i \in N'$, $t^f(u) = 0$ if $a^f_i(u) = i$ and $t^f(u) = \alpha$ otherwise.

[11] Indeed, if two agents are indifferent between a pairwise trade and keeping their endowments, our requirements do not pin down the chosen allocation, and in such situations either the agents keep their endowment and trade pairwise depending on the utilities over other houses.

Theorem 2. *Let $\alpha \geq 0$ and f be an assignment rule belonging to the class of HPO algorithms. Then the fixed α-tax rule φ^f based on f satisfies individual rationality, strategy-proofness, constrained efficiency, weak consistency and weak anonymity.*

Proof: It is straightforward to check that φ^f satisfies individual rationality, strategy-proofness and constrained efficiency because f is assignment-individually-rational, assignment-strategy-proof and assignment-efficient.

For weak consistency, let $S \subseteq N' \subseteq N$ and $u \in \mathcal{U}_{N'}$ (setting $R = R_{N'}(u, \alpha)$) be such that $S = \cup_{i \in S}\{a_i^f(u)\} = \cup_{i \in S}\{f_i(R)\}$ and $u_{if_i(R)} - t_i^f(u) > u_{ij} - t_j^f(u)$ for all $i \in S$ and all $j \in N' \backslash S$. This means that for all $i \in S$ and all $j \in N' \backslash S$, iP_ij. But then in the HPO-algorithm, agents in S do not point to agents in $N \backslash S$ (as agents always point to one of their most preferred objects) and any pointing from the agents in $N' \backslash S$ to agents in S is irrelevant. Thus, $a^f(u|_S) = f(R|_S) = f_S(R) = a_S^f(u)$ and (by definition) $t^f(u|_S) = t_S^f(u)$. Hence, φ^f satisfies weak consistency.

For weak anonymity, suppose that for some $N' \subseteq N$ and $u \in \mathcal{U}_{N'}$ we have for all $i \in N'$ and all $j \in N' \backslash \{i\}$, $a_i^f(u) - t_i^f(u) > 0$ and $a_i^f(u) - t_i^f(u) > a_j^f(u) - t_j(u)$. Setting $R_{N'} = R_{N'}(u, \alpha)$, this means for all $i \in N'$, $f_i(R_{N'})P_ii$ and $f_i(R_{N'})P_ij$ for all $j \in N' \backslash \{i, f_i(R_{N'})\}$. But then $f(R_{N'})$ is the unique efficient assignment for $R_{N'}$. Now for any permutation $\sigma : N' \to N''$ (where $N'' \subseteq N$ and $|N''| = |N'|$, $\sigma(f(R_{N'}))$ remains the unique efficient assignment for $\sigma(R_{N'})$, and thus, by efficiency of f we have $a^f(\sigma(u)) = f(\sigma(R_{N'})) = \sigma(f(R_{N'})) = \sigma(a^f(u))$ and (by definition) $t^f(\sigma(u)) = \sigma(t^f(u))$. Hence, φ^f satisfies weak anonymity. □

The agents-optimal mechanism in Theorem 2 is fixed-tax rule with $\alpha = 0$ (call it the zero-tax rule) and the agents-worst mechanism in Theorem 2 is the no-trade rule. Both these rules are worst for the mechanism designer (the government) in terms of monetary transfers from the agents to the mechanism because no taxes collected. Of course, this disregards consumer surplus and other welfare-enhancing considerations.

5 Discussion

5.1 Two Agents

In the following we will introduce regular taxation schemes. Loosely speaking any such scheme is based on a non-increasing function.

Let $I = [\underline{x}, \overline{x}]$. Let $g : I \to \mathbb{R}_+$ be a non-increasing function such that $g(\overline{x}) \geq \overline{x}$. Note that for any such function we have $I \leq g(I)$[12] and that g may contain points of discontinuities, i.e. for $x \in I$ we may have[13] $g(x-) \geq g(x) > g(x+)$ or $g(x-) > g(x) \geq g(x+)$ (where both inequalities may be strict and we set both $g(\underline{x}-) = +\infty$ and $g(\overline{x}+) = \overline{x}$). For our purposes, a function $g^{-1} : [\overline{x}, +\infty) \to I$ is an "inverse" of g if, for each $z \in [\overline{x}, +\infty)$, $g^{-1}(z) \in cl(\{x \in I : g(x-) \geq z \geq g(x+)\})$. Note that g^{-1} is not the inverse of g in the usual sense because g^{-1} is defined over $[\overline{x}, +\infty)$ and not only $g(I)$. Furthermore, for some $x', x'' \in I$ we may have $g(x') = g(x'') = z$ and $x' \neq x''$, i.e. $g^{-1}(z)$ may select x' or x'' (or possibly other elements in $cl(\{x \in I : g(x-) \geq z \geq g(x+)\})$). Let $\mathcal{G} = \{g : I \to \mathbb{R}_+ : g$ is non-increasing, $g(\overline{x}) \geq \overline{x}$, and g^{-1} is defined as above$\}$.

[12] We use the usual convention that for two sets J and J' we write $J \leq J'$ if $z \leq z'$ for all $z \in J$ and all $z' \in J'$.

[13] Here we use the convention $g(x-) = \lim_{\epsilon \to 0} g(x - \epsilon)$ and $g(x+) = \lim_{\epsilon \to 0} g(x + \epsilon)$.

Let $g \in \mathcal{G}$. For each $u \in \mathcal{U}$, we define first a "hypothetical" (regular) tax $h(u)$ in order to check whether exchanging objects makes both agents better off. Below the tax $h_2(u)$ for agent 2 is defined in dependance of u_{12}, agent 1's valuation for 2's object.

(i) If $u_{12} \notin I \cup [\overline{x}, +\infty)$, then $h_2(u) = +\infty$.

(ii) If $u_{12} \in I$, then $h_2(u) = g(u_{12})$.

(iii) If $u_{12} \in [\overline{x}, +\infty)$, then $h_2(u) = g^{-1}(u_{12})$.

Note that for $u_{12} > g(I)$ we have $g^{-1}(u_{12}) = \underline{x}$. In a symmetric way we define $h_1(u)$.

Now a regular tax rule checks first whether the agents' valuations for the other object exceed the hypothetical tax or not. If both valuations exceed the hypothetical taxes, then they exchange their objects and they pay these taxes. If not, then both agents keep their endowments and they pay no taxes. Formally, the regular tax rule $\phi^g = (a^g, t^g)$ is defined as follows: for all $u \in \mathcal{U}$, (i) if both $u_{12} \geq h_1(u)$ and $u_{21} \geq h_2(u)$, then $a^g(u) = (2,1)$ and $t^g(u) = h(u)$, and (ii) otherwise $a^g(u) = e$ and $t^g(u) = (0,0)$. The following is straightforward and left to the reader.

Proposition 1. *Let $N = \{1,2\}$. Any regular tax rule and the no-trade rule satisfy individual rationality, strategy-proofness, weak consistency and weak anonymity.*

Note that regular tax rules do not necessarily satisfy constrained efficiency.

Example 1. Let $I = [0,1]$ and for all $x \in I$, $g(x) = 2 - x$. If agents 1 and 2 report u^2 (with $u^2_{ii} = 0$ and $u^2_{ij} = 2$), then $a^g(u^2) = (2,1)$ and $t^g(u^2) = (0,0)$. If agents 1 and 2 report u^1 (with $u^1_{ii} = 0$ and $u^1_{ij} = 1$), then $a^g(u^1) = (2,1)$ and $t^g(u^1) = (1,1)$. Now for u^1, ϕ^g violates constrained efficiency because both agents strictly prefer $\phi^g(u^2)$ to $\phi^g(u^1)$.

Regular tax rules are just Clarke-Groves schemes satisfying our constraints.[14]

5.2 Independence

Proposition 1 shows that constrained efficiency is independent from the other properties in Theorem 1.

Example 2. Let $N = \{1, \ldots, n\}$. Use the same construction as for Theorem 2 just with the difference that any agent pays 2α when he keeps his endowment. Any such rule satisfies all properties of Theorem 1 except for individual rationality. Note that such a rule is not a fixed-tax rule because the assignment $C(R_{N'}(u,\alpha))$ is not necessarily chosen: agents pay for keeping their endowment and might instead prefer buying another house while for not paying any tax, then they keep their endowment.

Example 3. Let $N = \{1,2,3\}$, and $0 < \alpha(3) < \alpha(2)$. In any cycle c of length 3, agents pay $\alpha(3)$, and in any cycle of length 2, agents pay $\alpha(2)$. For all $u \in \mathcal{U}_N$, if there exists a cycle $c = (i_1, i_2, i_3)$ of length 3 such that $u_{i_l i_{l+1}} - \alpha(3) \geq 0$ for all $l = 1,2,3$, then (choose some cycle of length 3, say c) $a(u) = c$ and for all $i \in N$, $t_i(u) = \alpha(3)$. Otherwise a two-cycle is chosen (and the other agent keeps his endowment). For $N' = \{i_1, i_2\}$, if $u_{i_1 i_2} - \alpha(2) \geq 0$ and $u_{i_2 i_1} - \alpha(2) \geq 0$, then $a(u) = (i_2, i_1)$ and $t(u) = (\alpha(2), \alpha(2))$ (and otherwise $a(u) = e_{\{i_1,i_2\}}$ and

[14] See for instance Nisan (2007, Theorem 9.36) and Sprumont (2013, Lemma 1).

$t(u) = (0,0)$). Then φ satisfies all the properties in Theorem 1 except for strategy-proofness (because agents might disagree on which cycle of length 3 to choose, like in the proof of Lemma 3).

Example 4. Let $N = \{1,\ldots,n\}$ and $\alpha > 0$. For all $N' \subseteq N$ and all $u \in \mathcal{U}_{N'}$, (i) if $|N'|$ is odd, then $\varphi(u) = NT(u)$ and (ii) if $|N'|$ is even, let $\varphi(u)$ be the allocation chosen by an HPO-algorithm having as fixed tax α. Then this rule satisfies all properties of Theorem 1 except for weak consistency.

Example 5. Let $N = \{1,\ldots,n\}$, $c = (1,2,\ldots,n)$ and $\beta \in \mathbb{R}^N_+$ be a vector non-negative payments. For all $N' \subseteq N$ and all $u \in \mathcal{U}_{N'}$, (i) if $N' = N$ and for all $i \in N$, $u_{ii+1} - \beta_i \geq 0$, then $\varphi(u) = (c,\beta)$ and (ii) otherwise $a(u) = e_{N'}$ and $t_i(u) = 0$ for all $i \in N'$. Then this rule satisfies all properties of Theorem 1 except for weak anonymity.

5.3 Anonymity and Consistency

If we strengthen weak consistency to consistency or weak anonymity to anonymity, then no fixed-tax core rules satisfies our properties and we are only left with the no-trade rule (if there are at least 7 agents).

Corollary 1. *Let $|N| \geq 7$.*

(i) A rule φ satisfies individual rationality, strategy-proofness, constrained efficiency, consistency and weak anonymity if and only if φ is the no-trade rule.

(ii) A rule φ satisfies individual rationality, strategy-proofness, constrained efficiency, weak consistency and anonymity if and only if φ is the no-trade rule.

Proof: In showing (i), suppose that $\varphi \neq NT$. Then $\mathcal{C}^\varphi \neq \emptyset$. By the proof of Theorem 1, we then have $\mathcal{C}^\varphi = \mathcal{C}$. Then a must be an assignment rule satisfying individual rationality, strategy-proofness, efficiency and consistency. By Ehlers (2014, Proposition 2 (b)) no such rule exists. Hence, $\varphi = NT$.

In showing (ii), suppose that $\varphi \neq NT$. Then $\mathcal{C}^\varphi \neq \emptyset$. By the proof of Theorem 1, we then have $\mathcal{C}^\varphi = \mathcal{C}$ and in any exchange the fixed tax $\alpha \geq 0$ is paid. Then a must be an assignment rule satisfying individual rationality, strategy-proofness, efficiency and anonymity. Consider $N' = \{1,2,3\}$ and $u \in \mathcal{U}_{N'}$ such that $u_{12} = 2+\alpha = u_{32}$, $u_{13} = 1+\alpha = u_{31}$, and $u_{21} = 2+\alpha = u_{23}$. Then u induces the following ordinal rankings: $R_1(u_1,\alpha) : 231$, $R_2(u_2,\alpha) : [13]2$ and $R_3(u_3,\alpha) : 213$. By (constrained) efficiency, $a(u) = (2,3,1)$ or $a(u) = (3,1,2)$. But then considering the permutation $\sigma : N' \to N'$ such that $\sigma(1) = 3$, $\sigma(2) = 2$ and $\sigma(3) = 1$ gives us a contradiction to anonymity because $\sigma(u) = u$ and $\sigma(R) = R$, but if $a(u) = (2,3,1)$, then $\sigma(2,3,1) = (3,1,2) \neq a(u)$, and if $a(u) = (3,1,2)$, then $\sigma(3,1,2) = (2,3,1) \neq a(u)$. Hence, $\varphi = NT$. □

5.4 General Preferences

One may check that all our results remain true if agents have general preferences over consumption bundles: for all $i \in N$, let $\mathcal{B}_i = \{(i,0)\} \cup \{(j,t_i) : j \in N\backslash\{i\} \text{ and } t_i \geq 0\}$ and $\mathcal{R}_i$ denote the set of all preference relations on $\mathcal{R}_i$ being (i) complete and transitive, (ii) monotonic: for all $j \in N\backslash\{i\}$ and all $0 \leq t_i < t'_i$ we have $(j,t_i)P_i(j,t'_i)$ and (iii) bounded: for all $j \in N\backslash\{i\}$, all $k \in N$ and all $t_i, t'_i \geq 0$, if $(j,t_i)P_i(k,t'_i)$, then there exists $t''_i > t_i$ such that

$(k, t'_i) P_i (j, t''_i)$. As quasi-linear preferences are a subset of $\mathcal{R}_i$, it is easy to check that Theorem 1 and its proof remain true for general preferences. Similarly, Theorem 2 continues to hold on the general preference domain and existence is guaranteed. Note that here, instead of using for $i \in N$, $R_i(u_i, \alpha)$, we use

$$R_i | \{(i,0)\} \cup \{(j,\alpha) : j \in N \backslash \{i\}\},$$

which is the restriction of R_i to the consumption bundles i may receive under Gale's fixed tax rule based on the fixed tax α.

APPENDIX

A The HPO Algorithms

In order to establish, for some flat tax α, the existence of a rule satisfying our axioms, we use a construction of Saban and Sethuraman (2013), called the *Highest Priority Object (HPO)* algorithm. In this section, we elaborate on these algorithms for the sake of completeness.

We shall first describe the algorithm in words. Note that efficient exchange in the presence of indifferences is much more complicated than in their absence. Any allocation will decompose into trading cycles, but these will not be the simple "top cycles" used in the classical algorithm. Rather than find these cycles directly, the literature has employed a familiar, simpler strategy: having agents trade until all gains are exhausted. That is, unlike in the Top Trading Cycles algorithm, agents are required to stay in the market even after they have traded. This is because trading within their thick indifference set may benefit others while not harming them.

There are two phases in each step of the generic HPO algorithm, *removal and update* and *improvement*. During *removal and update*, the algorithm removes the agents who are holding one of their favorite objects, among those remaining, and whose participation in further trading cycles cannot benefit others. These agents are then permanently assigned the object they hold and sent away. The remaining agents update their preferences, given that some objects are no longer available. Any agent holding an object they value at least as much as all remaining objects is called *satisfied*.

During *improvement*, trading cycles are executed. A single agent may trade several times, and hold several different objects, before finally being removed in the *removal and update* phase.

We first make formal the *removal and update* phase. Because agents may hold several different objects before leaving the algorithm, we can no longer conflate agents with objects. Let Ω be the set of (remaining) objects and $\mu : N \to \Omega$ a one-to-one assignment of agents to objects. Given μ and preference profile R, the **ttc graph**, denoted $\mathcal{G}(N, \mu, R)$, has vertices N and directed edges $\{(i,j) : \forall \omega \in \Omega, \mu(j)\ R_i\ \omega\}$. Note that the ttc graph may have loops, as agents may hold their favorite object and remain in the algorithm. As in the body text, we write $(i, i^2, i^3, \ldots, j)$ to refer to the directed path $\{(i, i^2), (i^2, i^3), \ldots, (i^{n-1}, j)\}$. A **sink**, S, of a generic directed graph $\mathcal{G}$ is a (strongly) connected component: for each $i, j \in S$, there is a directed path $(i, i^2, i^3, \ldots, j) \subseteq \mathcal{G}$ with $\{i, i^2, i^3, \ldots, i^{n-1}, j\} \subseteq S$, and for each $i \in S$, $j \notin S$, there is no such path from i to j. A **terminal sink** S^T is a sink with the property that for each $i \in S^T$, $(i,i) \in \mathcal{G}$. Agents in a terminal sink of $\mathcal{G}(N, \mu, R)$ are satisfied. Moreover, they do not belong to any (directed) circuit that includes someone who *is not* satisfied. Thus, they cannot contribute to any Pareto improving trades and so are permanently assigned the objects they hold and are removed. The remaining agents have their preferences updated so that the objects just removed are no longer in their preference ranking.

The *improvement* phase consists mainly of selecting a simple graph, in which each node has out-degree 1, from the starting graph $\mathcal{G}(N, \mu, R)$. Let L be the possibly-empty set of labeled agents. The phase may begin with some agents labeled, depending on the previous step in the algorithm. In the first step, no agents begin labelled. If there are any labeled agents, they select the same agent they pointed to in the last step. Thus, for these agents, ties are broken by history. Next, agents who are not satisfied break ties based on the name of the objects, with everyone using a common order $\prec$. Finally, satisfied agents break ties in a more complicated manor. Label all agents whose ties are already broken, so at this point, all the previously-labelled and unsatisfied agents are labelled. Recursively, perform the following operations: 1) Select an unlabelled agent who is pointing to a labelled agent, breaking ties in this selection by the name of the object each holds; 2) Break the selected agent's ties first by eliminating all objects held by unlabelled agents and then using the name of the objects (and $\prec$); 3) Label the selected agent.

The set of labelled agents will expand to the entire set of agents, at which point all ties have been broken, and we are left with a simple subgraph $\mathcal{G} \subseteq \mathcal{G}(N, \mu, R)$. Now remove all labels, and execute all trading cycles. We must apply labels again for use in the next step. For each unsatisfied agent, j, who *did not* just trade, identify the longest path of satisfied agents $(i_1, i_2, \ldots, j) \subseteq \mathcal{G}(N, \mu, R)$. Label each of these satisfied agents. This completes one step of the algorithm. Proceed now to the next step, beginning with *removal and update*.

We also give a complete, formal description of the algorithm in two figures. Algorithm 1 contains subroutines necessary to run HPO, while the HPO algorithm is Algorithm 2. The algorithms are written in pseudocode, so the meaning of "=" is what it means in programming: "set the name on the left hand side to refer to the value stored on the right." Thus, the potentially confusing sentence $N = N \setminus S$ means, "henceforth, symbol N refers to what was *previously* meant by $N \setminus S$." The order $\prec$ is on the names of the objects.

The core of the algorithm consists of the repeated application of three subroutines, Prune(), Subgraph(), and Trade(). Prune() is run first, and removes all terminal sinks from consideration, making the sub-allocation for those agents final. Subgraph() performs the tie-breaking described above. Finally, Trade() executes trading cycles, and should only be passed graphs that are the output of Subgraph(), as it cannot process overlapping cycles.

Given $R \in \mathcal{W}_N$, $\Omega = N$, and for the strict priority order $\prec$ on Ω, let $f^{\prec}(R)$ denote the output of the HPO Algorithm. For all $N' \subseteq N$ and $R' \in \mathcal{W}_{N'}$, let $\Omega' = N'$ and $\prec|_{\Omega'}$ denote the restriction of $\prec$ to Ω', and let $f^{\prec}(R')$ denote the output of the HPO Algorithm when applied to R' and $\prec|_{\Omega'}$. Now by Saban and Sethuraman (2013), $f^{\prec}$ is assignment-individually-rational, assignment-strategy-proof and assignment-efficient. The argument in the proof of Theorem 2 shows that $f^{\prec}$ satisfies weak anonymity.

In order to see weak consistency of $f^{\prec}$, let $S \subseteq N' \subseteq N$ and $R \in \mathcal{W}_{N'}$ be such that $\cup_{i \in S}\{f_i^{\prec}(R)\} = S$ and for all $i \in S$ and all $j \in N' \setminus S$, $f_i^{\prec}(R) P_i f_j^{\prec}(R)$. By $\cup_{i \in S}\{f_i^{\prec}(R)\} = S$, the last condition implies for all $i \in S$ and all $j \in N' \setminus S$, $f_i^{\prec}(R) P_i j$. Thus, if $i \in S$ belongs to the terminal sink S^T in the HPO Algorithm applied to R and $\prec|_{N'}$, then $S^T \subseteq S$. But then applying the HPO algorithm to R_S and $\prec|_S$ yields the same terminal sinks for

Algorithm 1 Subroutines

procedure PRUNE(N, Ω, μ, R)
  $G = \mathcal{G}(N, \mu, R)$
  while there is a terminal sink S of G **do**
    $(N, \Omega, \mu, R) = \big(N \setminus S,\ \Omega \setminus \mu(S),\ \mu|_{N\setminus S},\ (R_i|_{\mu(N\setminus S)})_{i \in N \setminus S}\big)$ ▷ Remove and update
    $G = \mathcal{G}(N, \mu, R)$ ▷ Repeat with new graph
  end while
  return (N, Ω, μ, R)
end procedure

L is the set of labelled agents, possibly empty.
G^L is a graph storing the edges that were previously selected for the labelled agents. It might also be empty.
The subgraph selection automatically chooses G^L, and then builds upon it.

procedure SUBGRAPH$(N, \Omega, \mu, R, L, G^L)$
First the unsatisfied agents point, with ties broken by $\prec$
  for $i \in N \setminus L, (i, i) \notin \mathcal{G}(N, \mu, R)$ **do**
    $\omega = \min_{\prec} \max_{R_i} \Omega$ ▷ Break i's ties with $\prec$
    $G^L = G^L \cup \{(i, \mu^{-1}(\omega))\}$ ▷ i points to whomever holds ω
    $L = L \cup \{i\}$.
  end for
Now the satisfied agents point
  while there is an unlabelled agent, $i \in N \setminus L$ **do**
    $G = \mathcal{G}(N, \mu, R)$
    $A = \{i \in N \setminus L : \exists j \in L, (i, j) \in G\}$ ▷ Agents pointing to labelled agents. This set is not empty, for otherwise there would be a terminal sink.
    $i = \mu^{-1}\Big[\min_{\prec} \mu(A)\Big]$ ▷ i holds the highest priority object among A
    $\Omega = \mu(L)$ ▷ i will point to the object of a labelled agent
    $\omega = \min_{\prec} \max_{R_i} \Omega$ ▷ Break ties with $\prec$.
    $G^L = G^L \cup \{(i, \mu^{-1}(\omega))\}$ ▷ Increase the simple subgraph G^L
    $L = L \cup \{i\}$ ▷ i is labelled
  end while
  return G^L ▷ The output is the simple subgraph G^L
end procedure

procedure TRADE(G, μ)
  for circuits $(i^1, i^2, \ldots, i^n, i^1) \subseteq G$ **do**
    for $k \in \{1, \ldots, n\}$ **do**
      $\mu(i^k) = \mu(i^{k+1}) \mod n$
    end for
  end for
  return μ
end procedure

Algorithm 2 The HPO Algorithm

$L = \emptyset$
$G^L = \emptyset$
$\mu = e$
while $N \neq \emptyset$ **do**
    $(N, \Omega, \mu, R) = \textsc{Prune}(\mathcal{G}(N, \mu, R))$
    $L = L \setminus N$
    $G^L = G^L|_N$
    $G = \textsc{Subgraph}(N, \Omega, \mu, R, L, G^L)$
    $\alpha = \textsc{Trade}(G, \mu)$
    $L = \{i \in N : (i, j) \in G, (i, i) \in \mathcal{G}(N, \mu, R), \alpha(j) = \mu(j)\}$ ▷ Agents labelled for next step
    $G^L = \{(i, j) : i \in L, (i, j) \in G\}$ ▷ Labeled agents' tie-breakers stored
    $\mu = \alpha$ ▷ Trade updated
end while

the agents belonging to S (because they do not point to any objects in $N' \backslash S$)[15] and hence, $f^{\prec}(R_S) = f_S^{\prec}(R)$, which is the desired conclusion.

[15] The careful reader may check that Saban and Sethuraman (2013, Claim 1) is here useful.

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