Université de Montréal

# Extensions supersymétriques des équations structurelles des supervariétés plongées dans des superespaces

par

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# Extensions supersymétriques des équations structurelles des supervariétés plongées dans des superespaces

présentée par

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# SOMMAIRE

Le but de cette thèse par articles est d'étudier certains aspects géométriques des supervariétés associées aux systèmes supersymétriques intégrables. Ce travail a abouti en quatre articles publiés et un article présentement soumis dans des revues internationales avec des comités de lecture. Dans le premier article, deux extensions supersymétriques des équations de Gauss–Weingarten et de Gauss–Codazzi pour des surfaces plongées dans des superespaces euclidiens ont été construites. Cela a permis de fournir une caractérisation géométrique de telles surfaces avec des vecteurs tangents linéairement indépendants orientés dans la direction des déplacements infinitésimaux des dérivées fermioniques covariantes. De plus, une étude des symétries des versions supersymétriques des équations de Gauss–Codazzi a permis de construire des solutions invariantes au moyen de la méthode de réduction par symétrie impliquant les variables bosoniques et fermioniques, ce qui a mené à des surfaces non triviales, par exemple des surfaces à courbure de Gauss nulle. Dans le second article, l'extension aux cas supersymétriques d'une conjecture énonçant les conditions nécessaires pour qu'un système soit intégrable au sens de la théorie des solitons a été formulée. Cela a été accompli en introduisant un nouvel opérateur de projection et en comparant les symétries du système original avec celles du problème linéaire associé. Cette conjecture a été appliquée à certains exemples et un paramètre « spectral » fermionique a été introduit dans un des systèmes. Dans le troisième article, deux versions supersymétriques de la formule de Fokas–Gel'fand pour l'immersion de surfaces solitoniques dans une superalgèbre de Lie ont été construites. La caractérisation géométrique de la fonction d'immersion, présentée dans cet article, a permis d'investiguer les comportements des surfaces associées. Ces considérations théoriques ont été appliquées à l'équation de sine-Gordon supersymétrique pour laquelle des surfaces à courbure de Gauss constante et de type Weingarten non linéaire ont été obtenues. Le quatrième article est dévoué aux propriétés d'intégrabilité de l'équation de sine-Gordon supersymétrique et à la construction de solutions multisolitoniques explicites. Deux types de problèmes linéaires spectraux, une version supersymétrique d'un ensemble d'équations de Riccati couplées et la transformation d'auto-Bäcklund, tous équivalents à l'équation de sine-Gordon supersymétrique, ont été étudiés. De plus, une analyse détaillée de la énième transformation de Darboux a permis de trouver des solutions multisolitoniques non triviales de l'équation de sine-Gordon supersymétrique. Ces solutions ont été utilisées pour investiguer la version supersymétrique bosonique de la formule d'immersion de Sym–Tafel. Dans le cinquième article, une nouvelle caractérisation géométrique de la formule d'immersion de Fokas–Gel'fand est présentée. Afin d'accomplir cela, trois différents types de problèmes linéaires spectraux sont étudiés, un impliquant les dérivées fermioniques covariantes, un impliquant les dérivées par rapport aux variables bosoniques et un impliquant les dérivées par rapport aux variables fermioniques. Cette caractérisation géométrique implique huit coefficients linéairement indépendants pour les première et deuxième formes fondamentales, contrairement à trois dans le troisième article, ce qui mène à une géométrie plus riche dans le sens où les supervariétés caractérisées de type unidimensionnel (« curve-like ») dans le troisième article sont de type multidimensionnel dans le cinquième article.

Mots clefs : Systèmes intégrables ; Systèmes supersymétriques ; Supervariétés ; Surfaces conformément paramétrisées ; Superalgèbres de Lie ; Équations de Gauss–Weingarten et de Gauss–Codazzi ; Formules d'immersion de Sym–Tafel et de Fokas–Gel'fand ; Équation de sine-Gordon supersymétrique ; Transformations de Bäcklund et de Darboux supersymétriques ; Réduction par symétrie.

# SUMMARY

The goal of this thesis consisting of articles is to study certain geometric aspects of supermanifolds associated with integrable suspersymmetric systems. This work is contained in four published articles and one currently submitted article in international peer-reviewed journals. In the first article, two supersymmetric extensions of the Gauss–Weingarten and Gauss–Codazzi equations for surfaces immersed in Euclidean superspaces were constructed. This allowed us to provide a geometric characterization of such surfaces with linearly independent tangent vectors oriented in the directions of the infinitesimal displacement of the fermionic covariant derivatives. In addition, a study of the symmetries of the supersymmetric versions of the Gauss–Codazzi equations led to the construction of invariant solutions, involving bosonic and fermionic variables, through the symmetry reduction method, which led to nontrivial surfaces, e.g. vanishing Gauss curvature surfaces. In the second article, a conjecture stating the necessary conditions for a system to be integrable in the sense of soliton theory was extended to the supersymmetric cases. This was accomplished by introducing a new projection operator and by comparing the symmetries of the original system to those of the associated linear problem. This conjecture was applied to some examples and a fermionic "spectral" parameter was introduced in one of the systems. In the third article, two supersymmetric versions of the Fokas–Gel'fand formula for the immersion of soliton surfaces in Lie superalgebras were constructed. The geometric characterization of the immersion function presented in this article allowed us to investigate the behavior of the associated surfaces. These theoretical considerations were applied to the supersymmetric sine-Gordon equation, for which constant Gaussian curvature surfaces and nonlinear-type surfaces were obtained. The fourth article was devoted to integrability properties of the supersymmetric sine-Gordon equation and to the construction of explicit multisoliton solutions. Two types of linear spectral problems, a set of coupled super-Riccati equations

and the auto-Bäcklund transformation, all equivalent to the supersymmetric sine-Gordon equation, were studied. In addition, a detailed analysis of the nth Darboux transformations allowed us to find nontrivial multisoliton solutions of the supersymmetric sine-Gordon equation. These solutions were used to investigate the bosonic supersymmetric version of the Sym–Tafel immersion formula. In the fifth article, a new geometric characterization of the Fokas–Gel'fand immersion formula was presented. In order to do this, three different types of linear spectral problems were studied, one involving the covariant fermionic derivatives, one involving the bosonic variable derivatives and one involving the fermionic variable derivatives. This geometric characterization involves eight linearly independent coefficients for both the first and second fundamental forms, in constrast with three such coefficients in the third article, which leads to a richer geometry in the sense that curve-like supermanifolds in the third article are of higher dimensions in the fifth article.

Keywords : Integrable systems ; Supersymmetric systems ; Supermanifolds ; Conformally parametrized surfaces ; Lie superalgebras ; Gauss–Weingarten and Gauss–Codazzi equations ; Sym–Tafel and Fokas–Gel'fand immersion formulas ; Supersymmetric sine-Gordon equation ; Supersymetric Bäcklund and Darboux transformations ; Symmetry reduction.

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# Chapitre 1

## INTRODUCTION

### 1.1. Mise en contexte

En physique, en chimie, en biologie et en mathématique, la résolution des équations différentielles est d'une grande importance afin d'obtenir des modèles quantitatifs. Plusieurs techniques et théories ont été développées au fil des années. Une méthode utile est l'utilisation de transformations laissant un système d'équations différentielles invariant, c'est-à-dire l'utilisation des symétries. En 1888, Sophus Lie a commencé à développer une théorie faisant intervenir des transformations infinitésimales laissant invariant un système d'équations différentielles. Cette théorie a été reprise et continuée par plusieurs chercheurs (voir [**86**, **95**] et leurs références) et a mené à l'étude des algèbres de Lie et des groupes de Lie. Au moyen des symétries qui peuvent être représentées en termes d'algèbres de Lie, il est possible de réduire l'ordre d'un système d'équations différentielles ordinaires ou, dans le cas d'un système d'équations différentielles partielles, il est possible de réduire le nombre de variables indépendantes. La méthode de réduction par symétrie a été d'un grand impact, surtout pour les équations différentielles non linéaires. Une version détaillée de la théorie moderne des symétries peut être trouvée dans le livre d'Olver [**95**]. Il est à noter que pour les équations différentielles partielles, la réduction entraîne une perte de généralité dans les solutions, dans le sens où les solutions réduites sont des solutions particulières ne menant pas à la solution générale contrairement aux équations différentielles ordinaires. En 1915, Emmy Noether a établi le lien entre les symétries et les lois de conservation d'un système d'équations différentielles [**94**]. Ce lien a eu de grands impacts en physique, entre autres du point de vue de l'interprétation des symétries. Par exemple, la symétrie de translation temporelle est liée à la conservation de l'énergie, les translations spatiales sont liées à la conservation de la quantité de mouvement et les rotations sont liées à la conservation du moment cinétique.

Certains types de systèmes possédant des propriétés de symétrie intéressantes apparaissent fréquemment dans des modèles physiques. Par exemple, les systèmes d'équations différentielles ordinaires intégrables (au sens de Liouville) admettent une représentation hamiltonienne, c'est-à-dire

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \qquad i = 1, ..., n, \tag{1.1.1}$$

où, en mécanique classique, $H = H(q_i, p_i)$ représente l'hamiltonien, les variables $q_i$ sont associées à la position et les variables $p_i$ sont associées à l'impulsion dans un espace de dimension $n$ (et un espace de phase de dimension $2n$). Pour que ce type de système soit qualifié de complètement intégrable (au sens de Liouville), il doit exister $n-1$ fonctions $f_j$, $j = 1, ..., n-1$, linéairement indépendantes en involution avec elles-mêmes ainsi que l'hamiltonien (s'il est stationnaire) sous le crochet de Poisson, c'est-à-dire qu'il existe $n$ fonctions $f_i$, $i = 1, ..., n$ $(f_n = H)$ telles que

$$0 = \{f_j, f_k\} = \sum_{i=1}^{n} \left( \frac{\partial f_j}{\partial p_i} \frac{\partial f_k}{\partial q_i} - \frac{\partial f_j}{\partial q_i} \frac{\partial f_k}{\partial p_i} \right), \qquad i, j, k = 1, ..., n. \tag{1.1.2}$$

Les fonctions $f_i$ sont appelées des intégrales premières du système. Récemment (voir [**92**] et ses références), certains systèmes hamiltoniens possédant un nombre $m > n$ d'intégrales premières ont été étudiés. Ces systèmes sont qualifiés de superintégrables.

Pour le cas des systèmes d'équations différentielles partielles, une représentation de la forme

$$D_x \Psi = U \Psi, \qquad D_y \Psi = V \Psi \tag{1.1.3}$$

est le point de départ pour les systèmes intégrables au sens solitonique où $D_x, D_y$ représentent les dérivées totales par rapport à $x, y$, respectivement. Cette théorie, développée depuis 1968, énonce que les matrices potentielles $U$ et $V$ doivent pouvoir être étendues de façon non triviale à une famille à un paramètre satisfaisant

$$D_y U(\lambda) - D_x V(\lambda) + [U(\lambda), V(\lambda)] = 0, \tag{1.1.4}$$

qui est équivalent au système d'équations différentielles partielles original. Le paramètre $\lambda$ est appelé le paramètre spectral associé au problème linéaire spectral (1.1.3). L'équation (1.1.4) est appelée la représentation de courbure nulle ou, dans la théorie des solitons, la représentation de Lax ou de Zakharov–Shabat. Historiquement, Lax [**84**] a introduit le concept de paire de Lax pour l'équation de Korteweg–de Vries afin de construire des solutions exactes au moyen de la méthode de diffusion inverse. En 1974, Zakharov et Shabat [**123**] ont montré que la méthode fonctionnait, entre autres, pour l'équation non linéaire de Schrödinger. Par la suite, plusieurs autres systèmes intégrables ont été découverts, notamment l'équation de sine-Gordon,

$$\partial_x \partial_y \theta = \sin \theta, \tag{1.1.5}$$

qui apparaît dans la géométrie différentielle pour les surfaces à courbure de Gauss négative et constante [**103**]. Plusieurs méthodes d'investigation et de propriétés associées aux systèmes intégrables ont été étudiées. Par exemple, les transformations de Bäcklund [**5**] et de Darboux [**34**] permettent d'obtenir des solutions non triviales des modèles associés. Les formules d'immersion de Sym–Tafel [**112**], de Cieslinski–Doliwa [**26**, **49**] et de Fokas–Gel'fand [**54**, **55**, **72**] pour les surfaces plongées dans une algèbre de Lie permettent d'étudier la géométrie sous-jacente associée à un modèle intégrable.

Indépendamment, l'utilisation des symétries a mené à l'étude de nouveaux champs d'intérêt en physique. Par exemple, comme l'équation de Schrödinger n'est pas invariante sous des boosts relativistes, Dirac [**46**] a cherché une équation d'évolution relativiste pour les systèmes de particules de spin 1/2, ce qui a mené à l'équation de Dirac. Par la suite, Goldfand et Likhtman [**61**] ainsi que Gervais et Sakita [**59**] ont essayé d'étendre le groupe de Poincaré (translations spatiales, rotations spatiales, et boosts relativistes) pour tenir compte d'un lien entre les particules bosoniques et fermioniques. Le formalisme qui est a été produit utilise une algèbre graduée faisant intervenir des variables de Grassmann. Berezin [**11**] a essayé de formuler une description de la mécanique quantique à la fois pour les particules bosoniques (spin entier) et les particules fermioniques (spin demi-entier). La théorie de la supersymétrie commença à voir le jour. De nos jours, plusieurs systèmes d'équations différentielles faisant intervenir des variables bosoniques et fermioniques ont été étudiés, par exemple, la version supersymétrique de l'équation de Korteweg-de Vries [**83**,**88**,**90**] et l'équation de sine-Gordon supersymétrique [**7**, **24**, **51**, **60**, **62**, **87**, **88**, **106**–**108**],

$$D_+ D_- \phi = i \sin \phi, \qquad D_\pm = \partial_{\theta^\pm} - i\theta^\pm \partial_{x_\pm}, \tag{1.1.6}$$

où les variables $\theta^\pm$ sont à caractère fermionique et les variables $x_\pm$ sont à caractère bosonique. L'équation (1.1.6) est invariante sous les transformations supersymétriques

$$J_\pm = \partial_{\theta^\pm} + i\theta^\pm \partial_{x_\pm}, \tag{1.1.7}$$

qui sont générées par la superalgèbre de Lie

$$\tilde{\theta}^{\pm} = \theta^{\pm} + i\xi^{\pm}, \qquad \tilde{x}_{\pm} = x_{\pm} + i\xi^{\pm}\theta^{\pm}. \tag{1.1.8}$$

En somme, très peu est connu sur les systèmes supersymétriques intégrables comparativement aux systèmes « classiques » intégrables. Cette thèse tentera de répondre à certaines questions et d'étudier certaines propriétés des systèmes d'équations supersymétriques, particulièrement pour les systèmes intégrables au sens solitonique.

## 1.2. Méthodologie et objectifs

L'objectif général de cette thèse est d'étudier les supervariétés associées avec les systèmes supersymétriques intégrables. Nous avons concentré notre étude sur les équations structurelles de formules d'immersion de supervariétés dans différents superespaces, leur résolution ainsi que les caractéristiques géométriques associées. Cette étude a abouti en cinq articles. Plus particulièrement, nous avons étudié les différents concepts ci-dessous :

- La construction des équations structurelles de versions supersymétriques des équations de Gauss–Weingarten et de Gauss–Codazzi pour les surfaces conformément paramétrisées.
- La formulation d'une caractérisation géométrique pour les surfaces décrites par les versions supersymétriques des équations de Gauss–Weingarten et de Gauss–Codazzi.
- L'étude des symétries des versions supersymétriques des équations de Gauss–Codazzi et de Gauss–Weingarten.
- La résolution des versions supersymétriques des équations de Gauss–Codazzi par la méthode de réduction par symétrie pour obtenir des solutions invariantes.
- Les conditions nécessaires pour qu'un système supersymétrique soit intégrable au sens solitonique.
- La construction de versions supersymétriques de la formule d'immersion de Fokas–Gel'fand pour les surfaces et les supervariétés solitoniques associées à un modèle intégrable plongées dans une superalgèbre de Lie.
- La caractérisation géométrique associée aux versions supersymétriques de la formule d'immersion de Fokas–Gel'fand.
- L'étude des propriétés d'intégrabilité de l'équation de sine-Gordon supersymétrique afin d'obtenir des solutions.

Une surface (ou variété) est dite intégrable lorsque les équations de Gauss–Codazzi associées sont intégrables. En général, l'étude des liens entre les surfaces et les équations intégrables a mené à de nouvelles méthodes pour étudier leur géométrie, leur construction ainsi que leur résolution, par exemple : les transformations de Bäcklund et de Darboux ainsi que les principes de superposition non linéaires (voir [**103**] et les références à l'intérieur).

Pour ce faire, nous utilisons le formalisme de l'algèbre de Grassmann $\mathbb{G}$ qui sert de base pour les modèles supersymétriques. Cette algèbre est générée par un ensemble d'éléments $\xi_i$, $i = 1, ..., n$, et le corps des réels $\mathbb{R}$ (ou des complexes $\mathbb{C}$). Les éléments fermioniques satisfont les propriétés

$$\xi_i \xi_j + \xi_j \xi_i = 0, \qquad 1 \cdot \xi_i = \xi_i \cdot 1 = \xi_i, \qquad i, j = 1, ..., n, \tag{1.2.1}$$

où 1 est l'unité dans le corps des réels $\mathbb{R}$ (ou des complexes $\mathbb{C}$). Lorsque l'indice $i = j$, nous obtenons la propriété

$$(\xi_i)^2 = 0, \tag{1.2.2}$$

qui reproduit certaines propriétés des particules à spin demi-entier, d'où l'appellation fermionique. L'algèbre de Grassmann peut être décomposée en deux parties (« body » et « soul »), c'est-à-dire qu'un élément $a$ dans l'algèbre de Grassmann peut s'écrire

$$a = a_{\text{body}} + a_{\text{soul}}, \tag{1.2.3}$$

où $a_{\text{body}}$ appartient au corps $\mathbb{R}$ (ou $\mathbb{C}$) et $a_{\text{soul}}$ fait intervenir au moins un générateur $\xi_i$ de l'algèbre de Grassmann dans chaque terme. Si la partie $a_{\text{body}}$ est non nulle, alors il est possible de diviser par $a$. Il est aussi possible de décomposer l'algèbre de Grassmann en partie bosonique (paire) et fermionique (impaire), c'est-à-dire, pour un élément $a$ de l'algèbre de Grassmann, nous pouvons écrire

$$a = a_{\text{bosonique}} + a_{\text{fermionique}}. \tag{1.2.4}$$

La partie bosonique possède la propriété qu'elle commute avec l'ensemble de l'algèbre de Grassmann $\mathbb{G}$. Par exemple, pour quatre générateurs $\xi_i$, $i = 1, 2, 3, 4$, un élément bosonique $b$ prendra la forme

$$\begin{gathered} b = b_0 + b_1\xi_1\xi_2 + b_2\xi_1\xi_3 + b_3\xi_1\xi_4 + b_4\xi_2\xi_3 + b_5\xi_2\xi_4 + b_6\xi_3\xi_4 + b_7\xi_1\xi_2\xi_3\xi_4, \\ b_j \in \mathbb{C}, \qquad j = 0, ..., 7. \end{gathered} \tag{1.2.5}$$

À l'inverse, l'ensemble des éléments fermioniques est l'ensemble complémentaire de la partie bosonique. La partie fermionique commute avec les éléments bosoniques et elle anticommute avec les éléments fermioniques. Lorsqu'un élément possède uniquement une partie bosonique ou une partie fermionique, cet élément est décrit comme étant homogène. Les éléments homogènes possèdent les propriétés de multiplications similaires aux fonctions paires et impaires, c'est-à-dire

$$\text{bosonique} \cdot \text{bosonique} = \text{bosonique},$$
$$\text{bosonique} \cdot \text{fermionique} = \text{fermionique},$$
$$\text{fermionique} \cdot \text{fermionique} = \text{bosonique}.$$

Le degré d'un élément homogène est défini comme étant

$$\deg(a) = \begin{cases} 0 & \text{si } a \text{ est bosonique}, \\ 1 & \text{si } a \text{ est fermionique}. \end{cases} \tag{1.2.6}$$

Une matrice $M$ de dimension $(m+n) \times (m+n)$, donnée par des sous-blocs,

$$M = \begin{pmatrix} A_{m\times m} & B_{m\times n} \\ C_{n\times m} & D_{n\times n} \end{pmatrix}, \tag{1.2.7}$$

est appelée une supermatrice lorsque les éléments de la matrice sont tous homogènes et que les éléments de $A_{m\times m}$ sont de même degré que ceux de $D_{n\times n}$, mais pas de même degré que $B_{m\times n}$ et $C_{n\times m}$. Une supermatrice est dite bosonique lorsque les éléments de la sous-matrice $A_{m\times m}$ sont bosoniques et une supermatrice est dite fermionique lorsque les éléments de $A_{m\times m}$ sont fermioniques. Similairement, le degré d'une supermatrice est défini par

$$\deg(M) = \begin{cases} 0 & \text{si } M \text{ est une supermatrice bosonique}, \\ 1 & \text{si } M \text{ est une supermatrice fermionique}. \end{cases} \tag{1.2.8}$$

Une superalgèbre de Lie $\mathfrak{g}$ est composée d'un superespace vectoriel et d'un superbracket de Lie sous lequel $\mathfrak{g}$ est fermée sous son action, c'est-à-dire

$$M_1M_2 - (-1)^{\deg(M_1)\deg(M_2)}M_2M_1 = M_3 \in \mathfrak{g}, \qquad \forall M_1, M_2 \in \mathfrak{g}. \tag{1.2.9}$$

À titre d'exemple, la superalgèbre de Lie $\mathfrak{gl}(m|n, \mathbb{G})$ est composée de l'ensemble des supermatrices de dimension $(m+n) \times (m+n)$ et la superalgèbre de Lie $\mathfrak{sl}(m|n, \mathbb{G})$ est la sous-algèbre de $\mathfrak{gl}(m|n, \mathbb{G})$ qui satisfait

$$\mathrm{str}(M) = 0, \qquad \forall M \in \mathfrak{sl}(m|n, \mathbb{G}). \tag{1.2.10}$$

La supertrace est définie de la façon suivante :

$$\mathrm{str}(M) = \mathrm{tr}\left(E^{\deg(M)+1}M\right), \tag{1.2.11}$$

où la matrice $E$ prend la forme

$$E = \begin{pmatrix} I_{m\times m} & 0 \\ 0 & -I_{n\times n} \end{pmatrix} \tag{1.2.12}$$

et $I$ est la matrice identité. Le superdéterminant d'une matrice $M$, telle que définie à l'équation (1.2.7), est donné par

$$\mathrm{sdet}(M) = \frac{\det(A - BD^{-1}C)}{\det(D)} = \frac{\det(A)}{\det(D - CA^{-1}B)}, \tag{1.2.13}$$

à condition que les inverses des déterminants de $A$ et $D$ soient bien définis. Le supergroupe de Lie $GL(m|n, \mathbb{G})$, associé à la superalgèbre $\mathfrak{gl}(m|n, \mathbb{G})$, est composé de l'ensemble des supermatrices inversibles (bosoniques) de dimension $(m+n) \times (m+n)$. Une présentation plus détaillée du formalisme de l'algèbre de Grassmann peut être trouvée dans [**11**, **33**, **43**, **56**, **118**].

Dans le premier article [**B1**], nous avons étudié et construit des versions supersymétriques des équations de Gauss–Weingarten et de Gauss–Codazzi. Nous considérons une fonction d'immersion $F$ pour une surface dans un superespace euclidien. Nous considérons deux cas : la version bosonique où $F$ est à valeur bosonique et la version fermionique où $F$ est à valeur fermionique. Dans les deux cas, les vecteurs tangents à la surface sont considérés dans la direction des dérivées covariantes fermioniques $D_\pm$, c'est-à-dire $D_\pm F$, où

$$D_\pm = \partial_{\theta^\pm} - i\theta^\pm \partial_{x_\pm}. \tag{1.2.14}$$

Les variables $x_\pm$ sont à valeur bosonique et les variables $\theta^\pm$ sont à valeur fermionique. (La notation des indices $+$ et $-$ peut être remplacée par 1 et 2, respectivement, dans ce qui suit.) Avec le produit scalaire euclidien $\langle \cdot, \cdot \rangle$, nous pouvons définir les éléments de la métrique $g_{ij}$ par

$$\langle D_i F, D_j F \rangle = g_{ij} f, \tag{1.2.15}$$

où $f$ est une fonction de $x_\pm$ sans partie body dans la version bosonique et une fonction bosonique arbitraire de $x_\pm$ dans la version fermionique. Nous considérons aussi un vecteur normal unitaire $N$ satisfaisant

$$\langle D_i F, N \rangle = 0, \qquad \langle N, N \rangle = 1. \tag{1.2.16}$$

De là, nous choisissons une paramétrisation conforme telle que les éléments de la métrique sont donnés par

$$g_{ii} = 0, \qquad g_{12} = \frac{1}{2} e^{\phi}, \tag{1.2.17}$$

où $\phi$ est un superchamp bosonique. Les coefficients de la deuxième forme fondamentale, définis par

$$\langle D_j D_i F, N \rangle = b_{ij} f, \tag{1.2.18}$$

sont donnés par

$$b_{11} = Q^+, \qquad b_{22} = Q^-, \qquad b_{12} = \frac{1}{2}e^{\phi}H, \tag{1.2.19}$$

où $Q^{\pm}$ sont associés au différentiel de Hopf ($Q\mathrm{d}z^2$ dans le cas classique) et $H$ représente la courbure moyenne. À partir de l'hypothèse

$$D_jD_iF = \Gamma_{ij}{}^{k}D_kF + b_{ij}fN, \qquad D_iN = b_i{}^{k}D_kF + \omega_iN, \tag{1.2.20}$$

et d'une version supersymétrique du repère mobile,

$$\Omega = \begin{pmatrix} D_+F \\ D_-F \\ N \end{pmatrix}, \tag{1.2.21}$$

nous avons construit les versions bosonique et fermionique des équations de Gauss–Weingarten et, par leur condition de compatibilité, de Gauss–Codazzi.

Par la suite, nous avons étudié les symétries des versions supersymétriques des équations de Gauss–Codazzi que nous avons comparées à leur équivalent classique. Nous avons décomposé les superalgèbres de Lie en sommes directes et semi-directes. De plus, nous avons classifié les sous-algèbres unidimensionnelles par classes de conjugaison en utilisant la formule de Baker–Campbell–Hausdorff

$$e^XYe^{-X} = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + ... \tag{1.2.22}$$

et une adaptation de la méthode décrite dans [**120**] faisant intervenir les variables fermioniques. En particulier, pour les algèbres décrites par des sommes directes, nous considérons la méthode du twist de Goursat adaptée pour tenir compte des variables fermioniques et bosoniques. Afin d'obtenir des solutions invariantes des versions supersymétriques (bosonique et fermionique) des équations de Gauss–Codazzi, nous utilisons la méthode de réduction par symétrie. Nous avons choisi des représentants des classes de conjugaison et nous avons trouvé les invariants, les orbites, les équations réduites ainsi qu'une solution invariante des équations réduites.

Dans le deuxième article [**B2**], nous avons étudié les conditions nécessaires pour qu'un système (supersymétrique) soit intégrable au sens de la théorie des solitons. Pour ce faire, nous avons précisé une conjecture déjà existante présentée dans [**85**] à l'aide d'un opérateur de projecteur et nous l'avons généralisée pour tenir compte des variables aux valeurs fermioniques et bosoniques. Nous avons dû trouvé les symétries des systèmes linéarisés en utilisant une adaptation du critère de symétrie pour tenir compte des variables bosoniques et fermioniques. Pour plusieurs systèmes, nous avons comparé les symétries du système non linéaire et les symétries de leur système linéarisé.

Dans le troisième article [**B3**], nous avons étudié deux versions supersymétriques de la formule d'immersion de Fokas–Gel'fand pour des surfaces plongées dans une superalgèbre de Lie. Nous considérons un système supersymétrique intégrable et son problème linéaire spectral associé de la forme

$$D_j\Psi = U_j\Psi, \qquad j = 1, 2, \tag{1.2.23}$$

où $D_j$ sont les dérivées covariantes telles que définies à l'équation (1.2.14), $U_j$ est la supermatrice fermionique potentielle dans la superalgèbre de Lie et $\Psi$ est la fonction d'onde dans le supergroupe associé. Les deux versions sont séparées en termes de déformations bosoniques et de déformations fermioniques. Dans le cas bosonique, nous considérons des déformations de la forme

$$\tilde{U}_j = U_j + \epsilon A_j, \qquad \tilde{\Psi} = \Psi(I + \epsilon F), \tag{1.2.24}$$

où $\epsilon$ est un paramètre bosonique infinitésimal ou nilpotent d'ordre 2. Dans le cas fermionique, nous considérons des déformations de la forme

$$\tilde{U}_j = U_j + \epsilon E A_j, \qquad \tilde{\Psi} = \Psi(I + \epsilon E F), \tag{1.2.25}$$

où $\epsilon$ est un paramètre fermionique et la matrice $E$ est définie à l'équation (1.2.12). Pour les deux versions supersymétriques, nous avons construit la caractérisation géométrique à partir seulement des matrices de déformation des supermatrices potentielles. Nous avons décrit les coefficients de la métrique à partir de la superforme de Killing basée sur la supertrace. Pour obtenir les coefficients de la deuxième forme fondamentale, nous avons donné une expression explicite pour le vecteur normal unitaire qui prend la forme d'une supermatrice bosonique. À partir de ces coefficients, nous pouvons calculer la courbure moyenne pour les deux versions supersymétriques et la courbure de Gauss dans le cas bosonique basées sur la caractérisation fournie dans le premier article. Ces considérations théoriques seront utilisées pour étudier la version supersymétrique de l'équation de sine-Gordon. Il est à noter que les surfaces générées représentent en fait une famille de surfaces à un paramètre grâce au paramètre spectral. La présence du paramètre spectral est cruciale dans les diverses propriétés d'intégrabilité des systèmes intégrables.

Dans le quatrième article [**B4**], nous étudions les diverses propriétés d'intégrabilité de la version supersymétrique de l'équation de sine-Gordon dans le but d'obtenir des solutions multisolitoniques permettant d'investiguer la version supersymétrique de la formule d'immersion de Sym–Tafel. Nous commençons par construire un problème linéaire spectral faisant intervenir les dérivées covariantes fermioniques, puis nous fournissons la transformation permettant d'obtenir un second problème linéaire spectral faisant intervenir les dérivées bosoniques. À partir des problèmes linéaires spectraux, nous investiguons les transformations d'auto-Bäcklund et de Darboux afin d'obtenir des solutions multisolitoniques non triviales à partir d'une solution triviale. Les solutions trouvées sont alors utilisées pour étudier les surfaces solitoniques associées à une déformation spectrale, c'est-à-dire que la formule d'immersion prend la forme

$$F = \Psi^{-1}\beta(\lambda)\partial_\lambda\Psi. \tag{1.2.26}$$

Dans le cinquième article [**B5**], nous étudions une nouvelle caractérisation géométrique des versions supersymétriques de la formule d'immersion de Fokas–Gel'fand pour les supervariétés dans une superalgèbre de Lie. Contrairement au troisième article [**B3**] où nous considérons seulement le problème linéaire spectral faisant intervenir les dérivées covariantes, nous considérons trois types de problèmes linéaires spectraux, c'est-à-dire faisant intervenir les dérivées covariantes fermioniques, les dérivées bosoniques ou les dérivées fermioniques, respectivement

$$D_\pm\Psi = U_\pm\Psi, \qquad D_{x_\pm}\Psi = V_\pm\Psi, \qquad D_{\theta^\pm}\Psi = W_\pm\Psi. \tag{1.2.27}$$

Pour ce faire, nous établissons les liens entre les trois types et nous étudions les conséquences sur les conditions de courbure nulle. De là, nous construisons deux versions supersymétriques de la formule d'immersion de Fokas–Gel'fand, une bosonique et une fermionique. Les déformations pour les deux versions prennent la forme

$$\begin{gathered} \tilde{\Psi} = \Psi(I + \epsilon F), \\ \tilde{U}_\pm = U_\pm + \epsilon A_\pm, \qquad \tilde{V}_\pm = V_\pm + \epsilon B_\pm, \qquad \tilde{W}_\pm = W_\pm + \epsilon C_\pm. \end{gathered} \tag{1.2.28}$$

Alors, nous formulons une nouvelle caractérisation géométrique dans les directions des variables indépendantes bosoniques et fermioniques basée sur la forme de Killing précédemment utilisée. Nous obtenons les coefficients de la première et de la deuxième formes fondamentales et, par conséquent, les courbures de Gauss et moyenne lorsque définies. Nous appliquons ces considérations théoriques à l'équation de sine-Gordon supersymétrique.

## 1.3. Plan de la thèse

Le corps de cette thèse est composé de cinq chapitres, chacun composé d'un article ayant été publié dans un journal international avec un comité de lecture (Journal of Physics A : Mathematical and Theoretical, SIGMA and Journal of

Physics : Conference Series). Chacun de ces articles est rédigé de façon individuelle et peut être compris sans avoir à se référer aux autres articles. Ces cinq articles composant le corps du texte sont présentés dans l'ordre qu'ils ont été rédigés.

Plus précisément, l'article [**B1**] du chapitre 2, intitulé *Supersymmetric versions of the equations of conformally parametrized surfaces*, a été publié dans *Journal of Physics A : Mathematical and Theoretical*. La section 2.1 de cet article comprend une introduction et une mise en contexte du problème étudié. La section 2.2 résume l'obtention des équations de Gauss–Weingarten et de Gauss–Codazzi classiques et étudie les symétries associées. La section 2.3 introduit des notions de base de l'algèbre de Grassmann qui seront utilisées dans cet article. Dans la section 2.4, nous construisons les versions supersymétriques (bosonique et fermionique) des équations de Gauss–Weingarten et de Gauss–Codazzi. Dans la section 2.5, nous discutons de la caractérisation géométrique des surfaces étudiées. La section 2.6 est consacrée à l'étude des symétries ponctuelles de Lie des versions supersymétriques (bosonique et fermionique) des équations de Gauss–Weingarten et de Gauss–Codazzi. La section 2.7 présente une classification des sous-algèbres de Lie unidimensionnelles trouvées dans la section 2.6. Dans la section 2.8, nous trouvons des solutions invariantes à l'aide de la méthode de réduction par symétrie et la section 2.9 présente les conclusions et les perspectives futures de cet article. Les listes des sous-algèbres de Lie unidimensionnelles classifiées par classe de conjugaison pour les versions supersymétriques des équations de Gauss–Codazzi sont présentées en annexe à la fin du chapitre.

L'article [**B2**] du chapitre 3, intitulé *On the integrability of supersymmetric versions of the structural equations for conformally parametrized surfaces*, a été publié dans le journal SIGMA pour l'édition spéciale *On Exact Solvability and Symmetry Avatars in Honour of Luc Vinet*. La section 3.1 de cet article comprend une introduction et une mise en contexte des systèmes supersymétriques intégrables. Dans la section 3.2, nous discutons des symétries des équations de Gauss–Weingarten et de Gauss–Codazzi classiques. Dans la section 3.3, nous introduisons des notions de base utilisées sur les variables à valeur de Grassmann. Dans la section 3.4, nous rappelons les résultats obtenus dans l'article [**B1**] du chapitre 2 sur les versions supersymétriques des équations de Gauss–Weingarten et de Gauss–Codazzi. Dans la section 3.5, nous étendons la conjecture [**85**] aux modèles supersymétriques et nous fournissons trois exemples où nous avons eu à trouver les symétries de leur problème linéaire ; la version bosonique des équations de Gauss-Codazzi, la version supersymétrique de l'équation de sine-Gordon et la version fermionique des équations de Gauss–Codazzi. La section 3.6 présente les conclusions de cet article et certaines perspectives futures.

L'article [**B3**] du chapitre 4, intitulé *Supersymmetric versions of the Fokas–Gel'fand formula for immersion*, a été publié dans *Journal of Physics A : Mathematical and Theoretical*. La section 4.1 de cet article présente une introduction sur les systèmes supersymétriques intégrables et sur la formule d'immersion de Fokas–Gel'fand classique. Dans la section 4.2, nous résumons les résultats portant sur la formule d'immersion de Fokas–Gel'fand classique. La section 4.3 présente les conventions et les notions de base liées aux algèbres de Grassmann. Dans la section 4.4, nous construisons et analysons les versions supersymétriques de la formule d'immersion de Fokas–Gel'fand, pour l'immersion bosonique et l'immersion fermionique dans des superalgèbres de Lie. Dans la section 4.5, nous utilisons ces considérations théoriques pour investiguer la version supersymétrique de l'équation de sine-Gordon. Nous considérons cinq déformations : une déformation bosonique du paramètre spectral, une déformation par une jauge prenant la forme d'une supermatrice bosonique, une déformation bosonique liée à une symétrie ponctuelle de Lie, une déformation par une jauge prenant la forme d'une supermatrice fermionique et une déformation fermionique liée à une symétrie ponctuelle de Lie. Les conclusions et quelques perspectives futures de cet article sont présentées dans la section 4.6.

L'article [**B4**] du chapitre 5, intitulé *On integrability aspects of the supersymmetric sine-Gordon equation*, a été publié dans *Journal of Physics A : Mathematical and Theoretical*. La section 5.1 présente une introduction sur les propriétés des systèmes (supersymétriques) intégrables. Dans la section 5.2, nous étudions des propriétés de la version supersymétrique de l'équation de sine-Gordon : la densité lagrangienne, la construction de deux problèmes linéaires spectraux, une version supersymétrique des équations de Riccati couplées équivalentes à la version supersymétrique de sine-Gordon et les transformations d'auto-Bäcklund et de Darboux. Dans la section 5.3, nous utilisons la transformation de Darboux pour obtenir des solutions multisolitoniques non triviales. Ces solutions permettent d'étudier une version supersymétrique de la formule d'immersion de Sym–Tafel. La section 5.4 est dédiée aux conclusions et aux perspectives futures rattachées à cet article.

L'article [**B5**] du chapitre 6, intitulé *On geometric aspects of the supersymmetric Fokas–Gel'fand immersion formula*, a été publié dans *Journal of Physics A : Mathematical and Theoretical*. La section 6.1 est constituée d'une introduction sur les versions supersymétriques et classique de la formule d'immersion de Fokas–Gel'fand. Dans la section 6.2, nous présentons un résumé de la formule d'immersion de Fokas–Gel'fand classique et de sa caractérisation géométrique. La section 6.3 établit les différentes conventions liées aux algèbres de Grassmann et décrit brièvement les notions utilisées. La section 6.4 est séparée en trois parties. Dans la première partie, nous construisons et étudions les liens entre trois types de problèmes linéaires spectraux et leur condition de courbure nulle. La seconde partie, la section 6.4.1, est consacrée aux déformations bosoniques de la version supersymétrique de la formule d'immersion de Fokas–Gel'fand et la troisième partie, la section 6.4.2, est dédiée aux déformations fermioniques de la version supersymétrique de la formule d'immersion de Fokas–Gel'fand. Dans la section 6.5, nous fournissons une caractérisation géométrique pour les versions bosonique et fermionique de la formule d'immersion de Fokas–Gel'fand. Dans la section 6.6, nous appliquons ces considérations théoriques à l'équation de sine-Gordon supersymétrique pour cinq cas similaires à ceux étudiés dans l'article [**B3**], dans la section 4.5. Les conclusions et les perspectives futures de cet article sont présentées dans la section 6.7.

De plus, le chapitre 7 fait état des conclusions générales de la recherche effectuée dans le cadre de cette thèse et il met en lumière les contributions originales. Plusieurs perspectives futures sont également proposées. La bibliographie est présentée de façon alphabétique, à l'exception des articles apparaissant dans cette thèse qui sont présentés dans l'ordre d'apparition dans cette thèse et où un $B$ est ajouté aux références. Additionnellement, trois articles sont présentés en annexe. Le premier article présenté [**B6**] est un compte-rendu de conférence avec comité de lecture dans le cadre de la *XXIII International Conference on Integrable Systems and Quantum Symmetries* (ISQS-23) tenu à Prague qui a été publié dans *Journal of Physics : Conferences Series*. Les deux articles [**B7**, **B8**] ont été soumis séparément au *Journal of Physics A : Mathematical and Theoretical* dans un court délai et les examinateurs ont demandé de restructurer les deux articles en un seul, ce qui a abouti en l'article [**B1**] du chapitre 2.

# Chapitre 2

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## SUPERSYMMETRIC VERSIONS OF THE EQUATIONS OF CONFORMALLY PARAMETRIZED SURFACES

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**Résumé**

L'objectif de cet article est de formuler deux extensions supersymétriques distinctes des équations de Gauss–Weingarten et de Gauss–Codazzi pour les surfaces conformément paramétrisées plongées dans un superespace de Grassmann ; une en termes d'un superchamp bosonique et l'autre en termes d'un superchamp fermionique. Nous effectuons cette analyse en utilisant un formalisme de superespace/superchamp et une version supersymétrique du repère mobile sur une surface. Contrairement au cas classique où il y a trois équations de Gauss–Codazzi, nous obtenons six équations dans le cas bosonique supersymétrique et quatre équations dans le cas fermionique supersymétrique. Dans le cas fermionique, les équations de Gauss–Codazzi ressemblent à la forme des équations de Gauss–Codazzi classiques. Nous déterminons les algèbres de Lie de symétrie des équations de Gauss–Codazzi classiques qui sont de dimension infinie et nous effectuons une classification en sous-algèbres unidimensionnelles de la sous-algèbre maximale de dimension finie. Par la suite, nous calculons les superalgèbres de Lie ponctuelles de symétrie des équations de Gauss–Codazzi supersymétriques bosoniques et fermioniques, puis nous classifions les sous-algèbres unidimensionnelles par classes de conjugaison. Nous utilisons la méthode de réduction par symétrie pour trouver les invariants, les orbites et les systèmes réduits pour deux sous-algèbres du cas classique, pour deux sous-algèbres du cas bosonique supersymétrique et pour deux sous-algèbres du cas fermionique supersymétrique. Nous trouvons des solutions explicites à ces systèmes réduits, ce qui correspond à différentes surfaces plongées dans un superespace de Grassmann. Dans le cadre de ce travail, pour les versions supersymétriques des équations de Gauss–Codazzi, une interprétation géométrique des résultats est discutée.

**Abstract**

The objective of this paper is to formulate two distinct supersymmetric extensions of the Gauss—Weingarten and Gauss—Codazzi equations for conformally parametrized surfaces immersed in a Grassmann superspace, one in terms of a bosonic superfield and the other in terms of a fermionic superfield. We perform this analysis using a superspace-superfield formalism together with a supersymmetric version of a moving frame on a surface. In constrast with the classical case, where we have three Gauss–Codazzi equations, we obtain six such equations in the bosonic supersymmetric case and four such equations in the fermionic supersymmetric case. In the fermionic case the Gauss–Codazzi equations resemble the form of the classical Gauss–Codazzi equations. We determine the Lie symmetry algebra of the classical Gauss–Codazzi equations to be infinite-dimensional and perform a subalgebra classification of the one-dimensional subalgebras of its largest finite-dimensional subalgebra. We then compute superalgebras of Lie point symmetries of the bosonic and fermionic supersymmetric Gauss–Codazzi equations respectively, and classify the one-dimensional subalgebras of each superalgebra into conjugacy classes. We then use the symmetry reduction method to find invariants, orbits and reduced systems for two one-dimensional subalgebras for the classical case, two one-dimensional subalgebras for the bosonic supersymmetric case and two one-dimensional subalgebras for the fermionic supersymmetric case. We find explicit solutions of these reduced supersymmetric systems, which correspond to different surfaces immersed in a Grassmann superspace. Within this framework for the supersymmetric versions of the Gauss–Codazzi equations, a geometrical interpretation of the results is discussed.

## 2.1. Introduction

In the last three decades, a number of supersymmetric (SUSY) extensions of classical and quantum mechanical models, describing several physical phenomena, have been developed and group-invariant solutions of these SUSY systems have been found (e.g. [**13**, **37**, **76**, **79**]). Recently, this method was further generalized to encompass hydrodynamic-type systems (see e.g. [**36**, **50**, **67**, **74**]).

Their SUSY extensions were established and their group-invariant solutions were constructed. SUSY versions of the Chaplygin gas in (1+1)- and (2+1)-dimensions were formulated by R. Jackiw *et al*, derived from parametrizations of the action for a superstring and a Nambu-Goto membrane respectively (see [**78**] and references therein). It was suggested that a quark-gluon plasma may be described by non-Abelian fluid mechanics [**117**]. In addition, SUSY extensions have been formulated for a number of integrable equations , including among others the Korteweg-de Vries equation [**24**, **83**, **88**, **90**], the Kadomtsev-Petviashvili equation [**89**], the Sawada-Kotera equation [**116**] and the sine-Gordon and sinh-Gordon equations [**3**,**31**,**63**,**64**,**69**,**107**,**108**,**121**]. Various approaches have been used to construct supersoliton solutions, such as the inverse scattering method, Bäcklund and Darboux transformations for odd and even superfields, Lax formalism in a superspace and generalized versions of the symmetry reduction method (SRM). A number of supersoliton and multi-supersoliton solutions were determined by a Crum-type transformation [**64**,**91**,**107**] and it was found that there exist infinitely many local conserved quantities. A connection was established between the super-Darboux transformations and super-Bäcklund transformations which allow for the construction of supersoliton solutions [**3**, **24**, **63**, **69**, **88**, **108**, **116**].

Despite the progress made in the investigation of nonlinear SUSY systems, this area of mathematics does not yet have as solid a theoretical foundation as the classical theory of differential equations. This is related primarily to the fact that, due to the nature of Grassmann variables, the principle of superposition of solutions obtained from the method of characteristics cannot be applied to nonlinear SUSY systems. In most cases, analytic methods for solving quasilinear SUSY systems of equations lead to the construction of classes of solutions that are more restricted than the general solution. One can attempt to construct more restricted classes of solutions which depend on some arbitrary functions and parameters by requiring that the solutions be invariant under certain group transformations of the original system. The main advantages of the group properties appear when group analysis makes it possible to construct regular algorithms for finding certain classes of solutions without referring to any additional considerations but proceeding directly from the given system of partial differential equations (PDEs). A systematic computational method for constructing the group of symmetries of a given system of PDEs has been developed by many authors (see e.g. G. W. Bluman and S. C. Anco [**17**], P. A. Clarkson and P. Winternitz [**30**], P. Olver [**95**], and D. Sattinger and O. Weaver [**105**]) and a broad review of recent developments in the SUSY theory can be found in several books (e.g. J. F. Cornwell [**33**], B. DeWitt [**43**], D. S. Freed [**56**], V. Kac [**80**] and V. S. Varadarajan [**118**]). The methodological approach adopted in this paper is based on the use of the SRM to find solutions of the PDEs which are invariant under subgroups of a Lie supergroup of point transformations. By a symmetry supergroup of a SUSY system of PDEs, we mean a local SUSY Lie supergroup $G$ transforming both the independent and dependent variables of the considered SUSY system of equations in such a way that $G$ transforms given solutions of the system to new solutions of the same system. The Lie superalgebra of such a supergroup is represented by vector fields and their prolongation structures. The standard algorithms for determining the symmetry algebra of a system of equations and classifying its subalgebras have been extended in order to deal with SUSY models (see e.g. [**80**, **100**, **120**]).

Recent studies of the geometric properties of surfaces associated with holomorphic and non-holomorphic solutions of the SUSY bosonic Grassmann sigma models have been performed [**41**, **42**, **104**, **122**]. A gauge-invariant formulation of these SUSY models in terms of orthogonal projectors allows one to obtain explicit solutions and consequently to study the geometry of their associated surfaces. In differential geometry, parametrized surfaces are described in terms of a moving frame satisfying the Gauss–Weingarten (GW) equations, which are linear PDEs. Their compatibility conditions are the Gauss–Codazzi (GC) equations. A representation of nonlinear equations in the form of the GC equations is the starting point in the theory of integrable (soliton) surfaces arising from infinitesimal deformations of integrable differential equations and describing the behaviour of soliton solutions. The construction and analysis of such surfaces associated with integrable systems in several areas of mathematical physics provide new tools for the investigation of nonlinear phenomena described by these systems. In this setting, it is our objective to perform a systematic analysis of SUSY versions of the GW and GC equations. The formulation of a SUSY extension of the GW and GC equations has already been accomplished for the specific case of bosonic Grassmann sigma models [**41**, **42**]. It is of considerable interest to consider such extensions for the general case of the GW and GC equations.

The purpose of this paper is to formulate two distinct SUSY extensions of the GW and GC equations, one using a bosonic superfield and the other using a fermionic superfield, for conformally parametrized surfaces in the superspace $\mathbb{R}^{(n_b|n_f)}$. The SUSY versions of these equations are formulated through the use of a superspace-superfield formalism. The considered surfaces are parametrized by the vector field $\mathcal{F}$ and the normal vector field $\mathcal{N}$, which are replaced in the SUSY version by their corresponding superfields $F$ and $N$ in $\mathbb{R}^{(n_b|n_f)}$. This allows us to formulate the SUSY extensions of the structural equations for the immersion of conformally parametrized surfaces explicitly in matrix form. We establish explicit forms of the SUSY GW equations satisfied by the moving frame on these surfaces. The result is independent of the parametrization. This allows us to examine various geometric properties of the studied immersions, such as the first and second fundamental forms of the surfaces (and therefore the mean and Gaussian curvatures).

The paper is organized as follows. The symmetry algebra of the classical GC equations is determined and a subalgebra classification of its one-dimensional subalgebras is performed in section 2.2. In section 2.3, we introduce the basic properties of Grassmann algebras and Grassmann variables and introduce the notation that will be used in what follows. In section 2.4, we construct the bosonic and fermionic SUSY extensions of the GW and GC equations. In section 2.5, we discuss certain geometric aspects of the conformally parametrized SUSY surfaces. We provide expressions for the first and second fundamental forms and the Gaussian and mean curvatures, which are required for a geometrical interpretation of the invariant solutions. In section 2.6, we determine Lie superalgebras of point symmetries of the SUSY GC equations for both the bosonic and fermionic cases. Section 2.7 involves a classification of the one-dimensional subalgebras of both Lie superalgebras into conjugacy classes. In section 2.8, we provide examples of invariant solutions of the supersymmetric Gauss–Codazzi equations obtained by the SRM. Finally, in section 2.9, we present the conclusions and discuss possible future developments in this field.

## 2.2. Symmetries of conformally parametrized surfaces

The system of PDEs describing the moving frame $\Omega = (\partial\mathcal{F}, \bar{\partial}\mathcal{F}, \mathcal{N})^T$ on a smooth conformally parametrized surface in 3-dimensional Euclidean space satisfies the following GW equations

$$\partial\Omega = V_1\Omega, \qquad \bar{\partial}\Omega = V_2\Omega, \tag{2.2.1}$$

where the matrices $V_1$ and $V_2$ are given by

$$V_1 = \begin{pmatrix} \partial u & 0 & Q \\ 0 & 0 & \frac{1}{2}He^u \\ -H & -2Qe^{-u} & 0 \end{pmatrix}, \qquad V_2 = \begin{pmatrix} 0 & 0 & \frac{1}{2}He^u \\ 0 & \bar{\partial}u & \bar{Q} \\ -2\bar{Q}e^{-u} & -H & 0 \end{pmatrix}. \tag{2.2.2}$$

Here $\partial$ and $\bar{\partial}$ are the partial derivatives with respect to the complex coordinates $z = x + iy$ and $\bar{z} = x - iy$, respectively. The conformal parametrization of a surface is given by a vector-valued function $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) : \mathcal{R} \to \mathbb{R}^3$ (where $\mathcal{R}$ is a Riemann surface) which satisfies the following normalization for the tangent vectors $\partial\mathcal{F}$ and $\bar{\partial}\mathcal{F}$ and the unit normal $\mathcal{N}$

$$\begin{aligned} &\langle\partial\mathcal{F},\partial\mathcal{F}\rangle = \langle\bar{\partial}\mathcal{F},\bar{\partial}\mathcal{F}\rangle = 0, \qquad &&\langle\partial\mathcal{F},\bar{\partial}\mathcal{F}\rangle = \frac{1}{2}e^u, \\ &\langle\partial\mathcal{F},\mathcal{N}\rangle = \langle\bar{\partial}\mathcal{F},\mathcal{N}\rangle = 0, \qquad &&\langle\mathcal{N},\mathcal{N}\rangle = 1. \end{aligned} \tag{2.2.3}$$

We define the quantities

$$Q = \langle\partial^2\mathcal{F},\mathcal{N}\rangle \in \mathbb{C}, \qquad H = 2e^{-u}\langle\partial\bar{\partial}\mathcal{F},\mathcal{N}\rangle \in \mathbb{R}, \tag{2.2.4}$$

where $Qdz^2$ and $\bar{Q}d\bar{z}^2$ are the Hopf differentials and $H$ is the mean curvature function. Here, the bracket $\langle\cdot,\cdot\rangle$ denotes the scalar product in 3-dimensional Euclidean space $\mathbb{R}^3$

$$\langle a,b\rangle = a_1b_1 + a_2b_2 + a_3b_3. \tag{2.2.5}$$

So, the GW equations for a moving frame $\Omega$ on a surface have to obey the following system of equations

$$\begin{gathered} \partial^2\mathcal{F} = \partial u\partial\mathcal{F} + Q\mathcal{N}, \qquad \partial\bar{\partial}\mathcal{F} = \frac{1}{2}He^u\mathcal{N}, \qquad \bar{\partial}^2\mathcal{F} = \bar{\partial}u\bar{\partial}\mathcal{F} + \bar{Q}\mathcal{N}, \\ \partial\mathcal{N} = -H\partial\mathcal{F} - 2e^{-u}Q\bar{\partial}\mathcal{F}, \qquad \bar{\partial}\mathcal{N} = -2e^{-u}\bar{Q}\partial\mathcal{F} - H\bar{\partial}\mathcal{F}. \end{gathered} \tag{2.2.6}$$

The first and second fundamental forms are given by

$$I = \langle d\mathcal{F},d\mathcal{F}\rangle = \left\langle \frac{e^u}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}dz\\ d\bar{z}\end{pmatrix}, \begin{pmatrix}dz\\ d\bar{z}\end{pmatrix}\right\rangle = e^u\left\langle\begin{pmatrix}dx\\ dy\end{pmatrix}, \begin{pmatrix}dx\\ dy\end{pmatrix}\right\rangle, \tag{2.2.7}$$

and

$$I\!I = \langle d^2\mathcal{F},\mathcal{N}\rangle = \left\langle\begin{pmatrix}Q+\bar{Q}+e^uH & i(Q-\bar{Q})\\ i(Q-\bar{Q}) & -(Q+\bar{Q})+e^uH\end{pmatrix}\begin{pmatrix}dx\\ dy\end{pmatrix}, \begin{pmatrix}dx\\ dy\end{pmatrix}\right\rangle, \tag{2.2.8}$$

respectively. The principal curvatures $k_1$ and $k_2$ are the eigenvalues of the matrix

$$B = e^{-u}\begin{pmatrix}Q+\bar{Q}+e^uH & i(Q-\bar{Q})\\ i(Q-\bar{Q}) & -(Q+\bar{Q})+e^uH\end{pmatrix}. \tag{2.2.9}$$

We obtain the following expressions for the mean and Gaussian curvatures

$$H = \frac{1}{2}(k_1+k_2) = \frac{1}{2}\mathrm{tr}(B), \tag{2.2.10}$$

$$\mathcal{K} = k_1k_2 = \det(B) = H^2 - 4Q\bar{Q}e^{-2u}. \tag{2.2.11}$$

Umbilic points on a surface take place when $H^2 - \mathcal{K} = 0$ which implies that $|Q|^2 = 0$. The compatibility conditions of the GW equations (2.2.1) are the GC equations

$$\bar{\partial}V_1 - \partial V_2 + [V_1,V_2] = 0, \tag{2.2.12}$$

(the bracket $[\cdot,\cdot]$ denotes the commutator) which reduce to the following three differential equations for the quantities $Q$, $H$ and $e^u$

$$\begin{aligned} &\partial\bar{\partial}u + \frac{1}{2}H^2e^u - 2Q\bar{Q}e^{-u} = 0, && \text{(the Gauss equation)} \\ &\partial\bar{Q} - \frac{1}{2}e^u\bar{\partial}H = 0, \qquad \bar{\partial}Q - \frac{1}{2}e^u\partial H = 0 && \text{(the Codazzi equations).} \end{aligned} \tag{2.2.13}$$

These equations are the necessary and sufficient conditions for the existence of conformally parametrized surfaces in 3-dimensional Euclidean space $\mathbb{R}^3$ with the fundamental forms given by (2.2.7) and (2.2.8). A review of systematic computational methods for constructing surfaces for a given moving frame can be found in several books (e.g. [**6**, **47**, **82**, **99**, **115**]). Equations (2.2.1), (2.2.2) and (2.2.13) allow us to formulate explicitly the structural equations for the immersion directly in matrix terms. However, it is non-trivial to identify those surfaces which have an invariant geometrical characterization [**19**, **47**, **82**, **99**, **115**]. The task of finding an increasing number of solutions of the GW and GC equations is related to the group properties of these systems of equations. The methodological approach adopted here is based on the SRM for PDEs invariant under a Lie group $G$ of point transformations. Using the Maple program, we find that the symmetry group of the classical GC equations (2.2.13) consists of conformal scaling transformations. The corresponding symmetry algebra $\mathcal{L}_1$ is spanned by the vector fields

$$\begin{aligned} X(\eta) &= \eta(z)\partial_z + \eta'(z)(-2Q\partial_Q - U\partial_U), \\ Y(\zeta) &= \zeta(\bar{z})\partial_{\bar{z}} + \zeta'(\bar{z})(-2\bar{Q}\partial_{\bar{Q}} - U\partial_U), \\ e_0 &= -H\partial_H + Q\partial_Q + \bar{Q}\partial_{\bar{Q}} + 2U\partial_U, \end{aligned} \tag{2.2.14}$$

where $\eta'(\cdot)$ and $\zeta'(\cdot)$ are the derivatives of $\eta(\cdot)$ and $\zeta(\cdot)$ with respect to their arguments and where we have used the notation $e^u = U$. The commutation relations are

$$\begin{aligned} [X(\eta_1), X(\eta_2)] &= (\eta_1\eta_2' - \eta_1'\eta_2)\partial_z + (\eta_1''\eta_2 - \eta_1\eta_2'')(2Q\partial_Q + U\partial_U), \\ [Y(\zeta_1), Y(\zeta_2)] &= (\zeta_1\zeta_2' - \zeta_1'\zeta_2)\partial_{\bar{z}} + (\zeta_1''\zeta_2 - \zeta_1\zeta_2'')(2\bar{Q}\partial_{\bar{Q}} + U\partial_U), \\ [X(\eta), Y(\zeta)] &= 0, \qquad [X(\eta), e_0] = 0, \qquad [Y(\zeta), e_0] = 0. \end{aligned} \tag{2.2.15}$$

Since the vector fields $X(\eta), Y(\zeta)$ and $e_0$ form an Abelian algebra, they determine that the algebra $\mathcal{L}_1$ can be decomposed as a direct sum of two infinite-dimensional Lie algebras together with a one-dimensional algebra generated by $e_0$, i.e.

$$\mathcal{L}_1 = \{X(\eta)\} \oplus \{Y(\zeta)\} \oplus \{e_0\}. \tag{2.2.16}$$

This algebra represents a direct sum of two copies of the Virasoro algebra together with the one-dimensional subalgebra $\{e_0\}$. Assuming that the functions $\eta$ and $\zeta$

are analytic in some open subset $\mathcal{D} \subset \mathbb{C}$, we can develop $\eta$ and $\zeta$ as power series with respect to their arguments and provide a basis for $\mathcal{L}_1$. The largest finite-dimensional subalgebra $L_1$ of the algebra $\mathcal{L}_1$ is spanned by the seven generators

$$\begin{aligned} &e_0 = -H\partial_H + Q\partial_Q + \bar{Q}\partial_{\bar{Q}} + 2U\partial_U, \qquad e_1 = \partial_z, \qquad e_2 = \partial_{\bar{z}}, \\ &e_3 = z\partial_z - 2Q\partial_Q - U\partial_U, \qquad e_5 = z^2\partial_z - 4zQ\partial_Q - 2zU\partial_U, \\ &e_4 = \bar{z}\partial_{\bar{z}} - 2\bar{Q}\partial_{\bar{Q}} - U\partial_U, \qquad e_6 = \bar{z}^2\partial_{\bar{z}} - 4\bar{z}\bar{Q}\partial_{\bar{Q}} - 2\bar{z}U\partial_U, \end{aligned} \tag{2.2.17}$$

with non-zero commutation relations

$$\begin{aligned} &[e_1, e_3] = e_1, \qquad [e_1, e_5] = 2e_3, \qquad [e_3, e_5] = e_5, \\ &[e_2, e_4] = e_2, \qquad [e_2, e_6] = 2e_4, \qquad [e_4, e_6] = e_6. \end{aligned} \tag{2.2.18}$$

This seven-dimensional Lie subalgebra $L_1$ can be decomposed as a direct sum of two simple subalgebras together with a one-dimensional algebra generated by $e_0$,

$$L_1 = \{e_1, e_3, e_5\} \oplus \{e_2, e_4, e_6\} \oplus \{e_0\}. \tag{2.2.19}$$

Therefore, the classification of the subalgebras of $L_1$ consists of two copies of a 3-dimensional Lie algebra together with the center $\{e_0\}$. This 3-dimensional Lie algebra corresponds to the algebra $A_{3,8}$ in the classification of J. Patera and P. Winternitz [**97**] which is isomorphic to $\mathfrak{su}(1,1)$. The resulting classification of the subalgebras of $L_1$ into conjugacy classes, performed according to the methods described in [**120**], is given by the following list of representative subalgebras $L_{1,j}$

$$\begin{gathered} L_{1,0} = \{e_0\}, \qquad L_{1,1} = \{e_1\}, \qquad L_{1,2} = \{e_3\}, \qquad L_{1,3} = \{e_1 + e_5\}, \\ L_{1,4} = \{e_2\}, \qquad L_{1,5} = \{e_4\}, \qquad L_{1,6} = \{e_2 + e_6\}, \\ L_{1,7} = \{e_1 + \epsilon e_2\}, \qquad L_{1,8} = \{e_1 + \epsilon e_4\}, \qquad L_{1,9} = \{e_2 + e_6 + \epsilon e_1\}, \\ L_{1,10} = \{e_3 + \epsilon e_2\}, \qquad L_{1,11} = \{e_3 + a e_4\}, \qquad L_{1,12} = \{e_2 + e_6 + a e_3\}, \\ L_{1,13} = \{e_1 + e_5 + \epsilon e_2\}, \qquad L_{1,14} = \{e_1 + e_5 + a e_4\}, \\ L_{1,15} = \{e_1 + e_5 + a(e_2 + e_6)\}, \end{gathered} \tag{2.2.20}$$

where $\epsilon = \pm 1$ and $a \neq 0$ are parameters. The center of $L_1$, $\{e_0\}$, can be added to any of the subalgebras given above, say $L_{1,j} = \{A\}$, to produce a twisted subalgebra of the form $L'_{1,j} = \{A + be_0\}$, where $b \neq 0$. The symmetry reductions associated with the subalgebras (2.2.20) lead to systems of ordinary differential equations (ODEs). These reduced systems were analyzed systematically as a single generic symmetry reduction in [**32**], where the GC equations (2.2.13) were reduced to the most general Painlevé P6 form (containing two or three arbitrary parameters).

In the following sections, the symmetry properties of the classical GW and GC equations are compared with their bosonic and fermionic SUSY counterparts.

## 2.3. Preliminaries on Grassmann algebras

The mathematical background formalism is based on the theory of supermanifolds as presented in [**11**, **16**, **33**, **43**, **45**, **56**, **80**, **114**, **118**, **119**] and can be summarized as follows. The starting point in our consideration is a complex Grassmann algebra $\Lambda$ involving a finite or infinite number of Grassmann generators $(\xi_1, \xi_2, ...)$. The number of Grassmann generators of $\Lambda$ is not essential provided that there is a sufficient number of them to make any formula encountered meaningful. The Grassmann algebra $\Lambda$ can be decomposed into its even and odd parts

$$\Lambda = \Lambda_{\text{even}} + \Lambda_{\text{odd}}, \tag{2.3.1}$$

where $\Lambda_{\text{even}}$ includes all terms involving a product of an even number of generators $\xi_k$, i.e. $1, \xi_1\xi_2, \xi_1\xi_3, ...$, while $\Lambda_{\text{odd}}$ includes all terms involving a product of an odd number of generators $\xi_k$, i.e. $\xi_1, \xi_2, \xi_3, ..., \xi_1\xi_2\xi_3, ...$ The elements of $\Lambda$ are called supernumbers. The even supernumbers, variables, fields, etc are assumed to be elements of the even part $\Lambda_{\text{even}}$ of the underlying abstract real (complex) Grassmann ring $\Lambda$. The odd supernumbers, variables, fields, etc lie in its odd part $\Lambda_{\text{odd}}$. In the context of supersymmetry, the spaces $\Lambda$ and/or $\Lambda_{\text{even}}$ replace the field of complex numbers. The Grassmann algebra can also be decomposed as

$$\Lambda = \Lambda_{\text{body}} + \Lambda_{\text{soul}}, \tag{2.3.2}$$

where

$$\Lambda_{\text{body}} = \Lambda^0[\xi_1, \xi_2, ...] \simeq \mathbb{C}, \qquad \Lambda_{\text{soul}} = \sum_{k \geqslant 1} \Lambda^k[\xi_1, \xi_2, ...]. \tag{2.3.3}$$

Here $\Lambda^0[\xi_1, \xi_2, ...]$ refers to all terms that do not involve any of the generators $\xi_i$, while $\Lambda^k[\xi_1, \xi_2, ...]$ refers to all terms that involve products of $k$ generators (for instance, if we have 4 generators $\xi_1, \xi_2, \xi_3, \xi_4$, then $\Lambda^2[\xi_1, \xi_2, \xi_3, \xi_4]$ refers to all terms involving $\xi_1\xi_2$, $\xi_1\xi_3$, $\xi_1\xi_4$, $\xi_2\xi_3$, $\xi_2\xi_4$ or $\xi_3\xi_4$). The bodiless elements in $\Lambda_{\text{soul}}$ are non-invertible because of the $\mathbb{Z}_0^+$-grading of the Grassmann algebra. If the number of Grassmann generators $\mathfrak{K}$ is finite, bodiless elements are nilpotent of degree at most $\mathfrak{K}$. In this paper, we assume that $\mathfrak{K}$ is arbitrarily large but finite. Our analysis is based on the global theory of supermanifolds as described in [**12**, **101**, **102**].

Next, in our consideration, we use a $\mathbb{Z}_2$-graded complex vector space $V$ which has even basis elements $u_i$, $i = 1, 2, ..., N$, and odd basis elements $v_\mu$, $\mu = 1, 2, ..., N$, and construct $W = \Lambda \otimes_{\mathbb{C}} V$. We are interested in the even part of $W$

$$W_{\text{even}} = \left\{ \sum_i a_i u_i + \sum_\mu \underline{\alpha}_\mu v_\mu \;\middle|\; a_i \in \Lambda_{\text{even}}, \underline{\alpha}_\mu \in \Lambda_{\text{odd}} \right\}. \tag{2.3.4}$$

Clearly, $W_{\text{even}}$ is a $\Lambda_{\text{even}}$ module which can be identified with $\Lambda_{\text{even}}^{\times N} \times \Lambda_{\text{odd}}^{\times M}$ (consisting of $N$ copies of $\Lambda_{\text{even}}$ and $M$ copies of $\Lambda_{\text{odd}}$). We associate with the original basis, consisting of $u_i$ and $v_\mu$ (although $v_\mu \notin W_{\text{even}}$), the corresponding functionals

$$E_j : W_{\text{even}} \to \Lambda_{\text{even}} : E_j \left( \sum_i a_i u_i + \sum_\mu \underline{\alpha}_\mu v_\mu \right) = a_j, \tag{2.3.5}$$

$$\Upsilon_\nu : W_{\text{even}} \to \Lambda_{\text{odd}} : \Upsilon_\nu \left( \sum_i a_i u_i + \sum_\mu \underline{\alpha}_\mu v_\mu \right) = \underline{\alpha}_\nu, \tag{2.3.6}$$

and view them as the coordinates (even and odd respectively) on $W_{\text{even}}$. Any topological manifold locally diffeomorphic to a suitable $W_{\text{even}}$ is called a supermanifold.

The transitions to even and odd coordinates between different charts on the supermanifold are assumed to be even- and odd-valued superanalytic or at least $G^\infty$ functions on $W_{\text{even}}$. A comprehensive definition of the classes of supersmooth functions $G^\infty$ and superanalytic functions $G^\omega$ can be found in [**102**], definition 2.5. We note that superanalytic functions are those that can be expanded into a convergent power series in even and odd coordinates, whereas the definition of the $G^\infty$ function is a more involved analogue of functions on manifolds. Any $G^\infty$ function can be expanded into products of odd coordinates in a Taylor-like expansion but the coefficients, being functions of even and odd coordinates, may not necessarily be analytic (see e.g. [**102**]).

The super-Minkowski space can be viewed as such a supermanifold globally diffeomorphic to $\Lambda_{\text{even}}^{\times 2} \times \Lambda_{\text{odd}}^{\times 2}$ with even light-cone coordinates $x_+, x_-$ and odd coordinates $\theta^+, \theta^-$. Here $x_+$ and $x_-$ are linear combinations of terms involving an even number of generators : $1, \xi_1\xi_2, \xi_1\xi_3, \xi_1\xi_4, ..., \xi_2\xi_3, \xi_2\xi_4, ..., \xi_1\xi_2\xi_3\xi_4, ...$ On the other hand, $\theta^+$ and $\theta^-$ are linear combinations of terms involving an odd number of generators : $\xi_1, \xi_2, \xi_3, \xi_4, ..., \xi_1\xi_2\xi_3, \xi_1\xi_2\xi_4, \xi_1\xi_3\xi_4, \xi_2\xi_3\xi_4, ...$ The SUSY transformation (2.4.5) and (2.4.6) in the next section can be viewed as a particular change of coordinates on $\mathbb{R}^{(1,1|2)}$ which transforms solutions of the SUSY GW equations (2.4.20) and SUSY GC equations, (2.4.36) respectively, into solutions of the same systems in new coordinates. A smooth superfield is a $G^\infty$ function from $\mathbb{R}^{(n_b|n_f)}$ to $\Lambda$ (where $n_b$ and $n_f$ are the numbers of bosonic and fermionic coordinates, respectively). It can be expanded in powers of the odd coordinates $\theta^+$ and $\theta^-$ giving a decomposition in terms of even superfields

$$\chi_{\text{even}} : \Lambda_{\text{even}}^{\times 2} \to \Lambda_{\text{even}},$$

and odd superfields

$$\chi_{\text{odd}} : \Lambda_{\text{even}}^{\times 2} \to \Lambda_{\text{odd}}.$$

In this paper, we use the convention that partial derivatives involving odd variables satisfy the Leibniz rule

$$\partial_{\theta^\pm}(hg) = (\partial_{\theta^\pm} h)g + (-1)^{\deg(h)} h(\partial_{\theta^\pm} g), \tag{2.3.7}$$

where the degree of a homogeneous supernumber is given by

$$\deg(h) = \begin{cases} 0 \text{ if } h \text{ is even}, \\ 1 \text{ if } h \text{ is odd}, \end{cases} \tag{2.3.8}$$

and we have used the notation

$$f_{\theta^+\theta^-} = \partial_{\theta^-}\left(\partial_{\theta^+} f\right). \tag{2.3.9}$$

The partial derivatives with respect to the odd coordinates satisfy $\partial_{\theta^i}\theta^j = \delta_i^j$ where the indices $i$ and $j$ each stand for $+$ or $-$ and $\delta_i^j$ is the Kronecker delta function. The operators $\partial_{\theta^\pm}$, $J_\pm$ and $D_\pm$, in equations (2.4.3) and (2.4.4) change the parity of a bosonic function to a fermionic function and vice versa. For example, if $\phi$ is an even function, then $\partial_{\theta^+}\phi$ is an odd superfield while $\partial_{\theta^+}\partial_{\theta^-}\phi$ is an even superfield and so on. The chain rule for a Grassmann-valued composite function $f(g(x_+))$ is

$$\frac{\partial f}{\partial x_+} = \frac{\partial g}{\partial x_+}\frac{\partial f}{\partial g}. \tag{2.3.10}$$

The interchange of mixed derivatives (with proper respect to the ordering of odd variables) is assumed throughout. For further details see e.g. the books by Cornwell [**33**], DeWitt [**43**], Freed [**56**], Varadarajan [**118**] and references therein.

## 2.4. SUSY extensions of the GW and GC equations

The purpose of this section is to establish two different SUSY extensions of the GW and GC equations, one using a bosonic superfield representation and the other using a fermionic superfield representation of a surface in a superspace ($\mathbb{R}^{(2,1|2)}$ for the bosonic extension and $\mathbb{R}^{(1,1|3)}$ for the fermionic extension). Let $\mathcal{S}$ be a smooth simply connected surface in a Minkowski superspace with the bosonic light-cone coordinates $x_+ = \frac{1}{2}(t+x)$ and $x_- = \frac{1}{2}(t-x)$ together with the fermionic (anti-commuting) variables $\theta^+$ and $\theta^-$ such that

$$(\theta^+)^2 = (\theta^-)^2 = \theta^+\theta^- + \theta^-\theta^+ = 0. \tag{2.4.1}$$

We assume that the surface $\mathcal{S}$ is conformally parametrized by a vector-valued superfield $F(x_+, x_-, \theta^+, \theta^-)$ (bosonic in the case of the bosonic SUSY extension, fermionic in the case of the fermionic SUSY extension,) which can be decomposed as

$$\begin{aligned} F = F_m(x_+, x_-) + \theta^+ \varphi_m(x_+, x_-) \\ + \theta^- \psi_m(x_+, x_-) + \theta^+\theta^- G_m(x_+, x_-), \quad m = 1, 2, 3 \end{aligned} \tag{2.4.2}$$

For the bosonic SUSY extensions, the functions $F_m$ and $G_m$ are bosonic (even Grassmann)-valued, while the functions $\varphi_m$ and $\psi_m$ are fermionic (odd Grassmann)-valued. Conversely, for the fermionic SUSY extension, the functions $F_m$ and $G_m$ are fermionic-valued, while the functions $\varphi_m$ and $\psi_m$ are bosonic-valued. Here, the fields $F_m$, $\varphi_m$, $\psi_m$ and $G_m$ are the four parts of the truncated power series with respect to $\theta^+$ and $\theta^-$ of the $m^{\text{th}}$ component of the vector superfield $F$. Power series with respect to $\theta^+$ and $\theta^-$ are truncated since the fermionic variables $\theta^+$ and $\theta^-$ satisfy (2.4.1). Also, let $D_+$ and $D_-$ be the covariant superspace derivatives

$$D_\pm = \partial_{\theta^\pm} - i\theta^\pm \partial_{x_\pm}. \tag{2.4.3}$$

The covariant derivatives $D_\pm$ have the property that they anticommute with the differential SUSY operators

$$J_+ = \partial_{\theta^+} + i\theta^+ \partial_{x_+}, \qquad J_- = \partial_{\theta^-} + i\theta^- \partial_{x_-}, \tag{2.4.4}$$

which generate the SUSY transformations

$$x_+ \to x_+ + i\underline{\eta}_+ \theta^+, \qquad \theta^+ \to \theta^+ + i\underline{\eta}_+, \tag{2.4.5}$$

and

$$x_- \to x_- + i\underline{\eta}_- \theta^-, \qquad \theta^- \to \theta^- + i\underline{\eta}_-, \tag{2.4.6}$$

respectively. Here $\underline{\eta}_+$ and $\underline{\eta}_-$ are odd-valued parameters. The four operators, $D_+$, $D_-$, $J_+$ and $J_-$ satisfy the anticommutation relations

$$\begin{gathered} \{J_n, J_m\} = 2i\delta_{mn}\partial_{x_m}, \quad \{D_m, D_n\} = -2i\delta_{mn}\partial_{x_m}, \\ \{J_m, D_n\} = 0, \quad m, n = 1, 2 \end{gathered} \tag{2.4.7}$$

where $\delta_{ij}$ is the Kronecker delta function and $\{\cdot, \cdot\}$ denotes the anticommutator, unless otherwise noted. Here, the values 1 and 2 of the indices $m$ and $n$ stand for $+$ and $-$, respectively. Therefore we have the following relations

$$D_\pm^2 = -i\partial_\pm, \qquad J_\pm^2 = i\partial_\pm. \tag{2.4.8}$$

The conformal parametrization of the surface $\mathcal{S}$ in the superspace $\mathbb{R}^{(n_b|n_f)}$ is assumed to give the following normalization of the superfield $F$

$$\langle D_iF, D_jF\rangle = g_{ij}f, \qquad i,j = 1,2. \tag{2.4.9}$$

Here the values 1 and 2 of the indices $i$ and $j$ stand for $+$ and $-$, respectively. The scalar product $\langle \cdot, \cdot \rangle$ in (2.4.9) is defined in the same way as in equation (2.2.5), taking into account the property (2.4.1) regarding the odd-valued variables $\theta^+$ and $\theta^-$, and taking values in the Grassmann algebra $\Lambda$. Hence the bosonic functions $g_{ij}$ of $x_+$, $x_-$, $\theta^+$ and $\theta^-$ are given by

$$g_{11} = 0, \qquad g_{12} = \frac{1}{2}e^{\phi}, \qquad g_{21} = \frac{\epsilon}{2}e^{\phi}, \qquad g_{22} = 0, \tag{2.4.10}$$

where $\epsilon = -1$ in the bosonic case and $\epsilon = 1$ in the fermionic case. Therefore, in the bosonic case, $f$ is a bodiless bosonic function (i.e. $f \in \Lambda_{\text{soul}}$) of $x_+$ and $x_-$ which is nilpotent of order $k$. In the fermionic case, $f$ is a bosonic function which may or may not be bodiless. In the bosonic case, the bodiless function $f(x_+, x_-)$ has been introduced since the normalization $\langle D_+F, D_-F\rangle$ contains only terms with products of generators $\xi_i$ and the exponential contains a term which involves no generator. One should note that the equations (2.4.9) are identically satisfied for $i = j$ in the bosonic extension, which is not the case in the fermionic extension. Indeed, in the scalar product (2.2.5), we have the sum of the squares of each $m^{th}$ component of the tangent vector superfield $D_iF$. Since the square of a fermionic function vanishes, each of the terms in the scalar product is identically zero, i.e.

$$\langle D_iF, D_iF\rangle = 0. \tag{2.4.11}$$

In the case of the mixed scalar product, the normalization condition is

$$\langle D_+F, D_-F\rangle = \frac{1}{2}e^{\phi}f. \tag{2.4.12}$$

It is interesting to note that, by construction, the metric coefficients $g_{ij}$ of the bosonic extension are antisymmetric for $i \neq j$, i.e.

$$g_{ij} = -g_{ji}. \tag{2.4.13}$$

This is in contrast with the fermionic case where the coefficients of the induced metric $g_{ij}$ are symmetric in the indices $i$ and $j$, i.e.

$$g_{ij} = g_{ji}. \tag{2.4.14}$$

The superfield $\phi$ is assumed to be bosonic and can be decomposed as the following power series in the fermionic variables $\theta^+$ and $\theta^-$

$$\phi = u(x_+, x_-) + \theta^+\gamma(x_+, x_-) + \theta^-\delta(x_+, x_-) + \theta^+\theta^- v(x_+, x_-), \tag{2.4.15}$$

where $u$ and $v$ are bosonic-valued functions, while $\gamma$ and $\delta$ are fermionic-valued functions. Through a power expansion in $\theta^+$ and $\theta^-$ we find the exponential form

$$\begin{aligned} e^{\phi} &= e^{u}(1 + \theta^+\gamma + \theta^-\delta + \theta^+\theta^-(v - \gamma\delta)), \\ e^{-\phi} &= e^{-u}(1 - \theta^+\gamma - \theta^-\delta - \theta^+\theta^-(v + \gamma\delta)). \end{aligned} \tag{2.4.16}$$

The tangent vector superfields $D_\pm F$ together with the normal bosonic superfield $N(x_+, x_-, \theta^+, \theta^-)$, which can be decomposed as

$$\begin{aligned} N = {} & N_m(x_+, x_-) + \theta^+\alpha_m(x_+, x_-) \\ & + \theta^-\beta_m(x_+, x_-) + \theta^+\theta^- H_m(x_+, x_-), \quad m = 1, 2, 3 \end{aligned} \tag{2.4.17}$$

form a moving frame $\Omega$ on the surface $\mathcal{S}$ in the superspace. Here, the bosonic-valued fields $N_m$ and $H_m$ and the fermionic-valued fields $\alpha_m$ and $\beta_m$ are the four parts of the truncated power series with respect to $\theta^+$ and $\theta^-$ of the $m^{th}$ component of the vector superfield $N$. This normal superfield $N$ has to satisfy the conditions

$$\langle D_i F, N\rangle = 0, \qquad \langle N, N\rangle = 1, \qquad i = 1, 2. \tag{2.4.18}$$

We now derive the bosonic and fermionic SUSY versions of the GW and GC equations. We assume that we can decompose the second-order covariant derivatives of $F$ and first-order derivatives of $N$ in terms of the tangent vectors $D_+F$ and $D_-F$ and the unit normal $N$,

$$\begin{aligned} D_j D_i F &= \Gamma_{ij}{}^{k} D_k F + b_{ij} f N, \\ D_i N &= b^k{}_i D_k F + \omega_i N, \end{aligned} \qquad i, j, k = 1, 2 \tag{2.4.19}$$

where the coefficients $\omega_i$ and $\Gamma_{ij}{}^{k}$ are fermionic functions. However, the functions $b_{ij}$ and $b^k{}_i$ are bosonic-valued in the bosonic extension and are fermionic-valued in the fermionic extension. The SUSY GW equations for the moving frame $\Omega$ on a surface are given by

$$D_+\Omega = A_+\Omega, \qquad D_-\Omega = A_-\Omega, \qquad \Omega = \begin{pmatrix} D_+F \\ D_-F \\ N \end{pmatrix}, \tag{2.4.20}$$

where the $3\times 3$ supermatrices $A_+$ and $A_-$ are

$$A_+ = \begin{pmatrix} {\Gamma_{11}}^1 & {\Gamma_{11}}^2 & b_{11}f \\ {\Gamma_{21}}^1 & {\Gamma_{21}}^2 & b_{21}f \\ {b^1}_1 & {b^2}_1 & \omega_1 \end{pmatrix}, \qquad A_- = \begin{pmatrix} {\Gamma_{12}}^1 & {\Gamma_{12}}^2 & b_{12}f \\ {\Gamma_{22}}^1 & {\Gamma_{22}}^2 & b_{22}f \\ {b^1}_2 & {b^2}_2 & \omega_2 \end{pmatrix}. \tag{2.4.21}$$

The conformally parametrized surface $\mathcal{S}$ satisfies the normalization conditions (2.4.9) and (2.4.18) for the superfields $F$ and $N$, and we define the quantities $Q^+, Q^-$ and $H$ to be

$$b_{11} = Q^+, \qquad b_{12} = \frac{1}{2}e^{\phi}H, \qquad b_{21} = -\frac{1}{2}e^{\phi}H, \qquad b_{22} = Q^-, \tag{2.4.22}$$

which gives the relations

$$\langle D_+^2 F, N\rangle = Q^+ f, \qquad \langle D_- D_+ F, N\rangle = \frac{1}{2}e^{\phi}Hf, \qquad \langle D_-^2 F, N\rangle = Q^- f. \tag{2.4.23}$$

The coefficients of the second fundamental form $b_{ij}$ have the property

$$b_{ij} = -b_{ji}, \qquad \text{for} \qquad i \neq j. \tag{2.4.24}$$

To obtain a relation between the functions $b_{ij}$ and ${b^k}_j$, we make use of the relation

$$\langle D_j D_i F, N\rangle = D_j\langle D_i F, N\rangle - \epsilon\langle D_i F, D_j N\rangle = -\epsilon\langle D_i F, D_j N\rangle, \tag{2.4.25}$$

and by substituting $D_j N$ into its decomposition (2.4.19) we get the relation

$$\left(g_{ik}{b^k}_j + \epsilon b_{ij}\right) f = 0. \tag{2.4.26}$$

We can obtain the coefficients $\omega_i$ from the derivative of $\langle N, N\rangle = 1$, i.e.

$$0 = D_i\langle N, N\rangle = \langle D_i N, N\rangle + \langle N, D_i N\rangle = 2\omega_i\langle N, N\rangle = 2\omega_i, \tag{2.4.27}$$

from which we obtain

$$\omega_i = 0. \tag{2.4.28}$$

Also, we can make use of the identities

$$\begin{aligned} D_k(\frac{1}{2}e^{\phi}f) &= D_k\langle D_+F, D_-F\rangle = \langle D_k D_+ F, D_- F\rangle - \langle D_+F, D_k D_- F\rangle \\ &= {\Gamma_{1k}}^1\langle D_+F, D_-F\rangle + {\Gamma_{2k}}^2\langle D_+F, D_-F\rangle, \end{aligned} \tag{2.4.29}$$

which lead to

$$D_k f = ({\Gamma_{1k}}^1 + {\Gamma_{2k}}^2 - D_k\phi)f. \tag{2.4.30}$$

From equation (2.4.30) we can compute the compatibility condition on the function $f$

$$\{D_+, D_-\}f = \left(D_-{\Gamma_{11}}^1 + D_-{\Gamma_{21}}^2 + D_+{\Gamma_{12}}^1 + D_+{\Gamma_{22}}^2\right) f = 0. \tag{2.4.31}$$

Hence we define the Christoffel symbols of the first kind $\Gamma_{ijk}$ to be

$$\Gamma_{ijk} f = \langle D_j D_i F, D_k F \rangle. \tag{2.4.32}$$

By construction, the Christoffel symbols of the first and second kind ($\Gamma_{ijk}$ and ${\Gamma_{ij}}^k$, respectively) are antisymmetric under permutation of the indices $i$ and $j$, i.e.

$$\Gamma_{ijk} = -\Gamma_{jik}, \qquad {\Gamma_{ij}}^k = -{\Gamma_{ji}}^k, \quad \text{for} \quad i \neq j. \tag{2.4.33}$$

The relations between the Christoffel symbols of first and second kind are given by

$$\Gamma_{ijk} f = \langle D_i D_j F, D_k F \rangle = {\Gamma_{ij}}^l \langle D_l F, D_k F \rangle \tag{2.4.34}$$

or

$$(\Gamma_{ijk} - {\Gamma_{ij}}^l g_{lk}) f = 0. \tag{2.4.35}$$

The compatibility conditions of the SUSY GW equations are the SUSY GC equations, given by

$$\begin{aligned}
\{D_+, D_-\}\Omega =& D_+(A_-\Omega) + D_-(A_+\Omega), \\
=& D_+A_-\Omega + \begin{pmatrix} -{\Gamma_{11}}^1 & -{\Gamma_{11}}^2 & -\epsilon b_{11} f \\ {\Gamma_{12}}^1 & {\Gamma_{12}}^2 & -\epsilon b_{21} f \\ -\epsilon {b^1}_1 & -\epsilon {b^2}_1 & 0 \end{pmatrix} D_+\Omega \\
&+ D_-A_+\Omega + \begin{pmatrix} -{\Gamma_{12}}^1 & -{\Gamma_{12}}^2 & -\epsilon b_{12} f \\ -{\Gamma_{22}}^1 & -{\Gamma_{22}}^2 & -\epsilon b_{22} f \\ -\epsilon {b^1}_2 & -\epsilon {b^2}_2 & 0 \end{pmatrix} D_-\Omega \\
=& D_+A_-\Omega - EA_-ED_+\Omega + D_-A_+\Omega - EA_+ED_-\Omega.
\end{aligned}$$

So we have

$$D_+A_- + D_-A_+ - \{EA_+, EA_-\} = 0, \tag{2.4.36}$$

up to the addition of a non-zero matrix $P$ such that $P\Omega = 0$, where $E$ is the diagonal matrix

$$E = \pm \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \epsilon \end{pmatrix}. \tag{2.4.37}$$

### 2.4.1. Bosonic extension

We now construct the SUSY GW and SUSY GC equations for the bosonic extension. From the relations (2.4.26)), the coefficients ${b^i}_j$ are given by

$$ {b^1}_1 = H, \qquad {b^2}_1 = 2e^{-\phi}Q^+, \qquad {b^1}_2 = -2e^{-\phi}Q^-, \qquad {b^2}_2 = H, \tag{2.4.38} $$

up to an additional bosonic bodiless function $\zeta_1 \neq 0$ such that $\zeta_1 f = 0$ and where the ${b^k}_j$ are the mixed coefficients of the second fundamental form. Therefore the SUSY GW equations (2.4.20) take the form

$$ \begin{gathered} D_+\Omega = A_+\Omega, \qquad D_-\Omega = A_-\Omega, \\ A_+ = \begin{pmatrix} {\Gamma_{11}}^1 & {\Gamma_{11}}^2 & Q^+ f \\ -{\Gamma_{12}}^1 & -{\Gamma_{12}}^2 & -\frac{1}{2}e^{\phi}Hf \\ H & 2e^{-\phi}Q^+ & 0 \end{pmatrix}, \\ A_- = \begin{pmatrix} {\Gamma_{12}}^1 & {\Gamma_{12}}^2 & \frac{1}{2}e^{\phi}Hf \\ {\Gamma_{22}}^1 & {\Gamma_{22}}^2 & Q^- f \\ -2e^{-\phi}Q^- & H & 0 \end{pmatrix}, \end{gathered} \tag{2.4.39} $$

where the matrices $A_\pm$ are in the Bianchi form [**15**]. The compatibility condition of the SUSY GW equations is

$$ D_+A_- + D_-A_+ - \{EA_+, EA_-\} = 0, \tag{2.4.40} $$

up to the addition of a non-zero matrix $P$ such that $P\Omega = 0$, where

$$ E = \pm \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{2.4.41} $$

Using the subblock notation, the matrices $A_\pm$ can also be written as

$$ A_+ = \left(\begin{array}{cc|c} {\Gamma_{11}}^1 & {\Gamma_{11}}^2 & Q^+ f \\ -{\Gamma_{12}}^1 & -{\Gamma_{12}}^2 & -\frac{1}{2}e^{\phi}Hf \\ \hline H & 2e^{-\phi}Q^+ & 0 \end{array}\right) = \left(\begin{array}{c|c} A_f^+ & I_{b_1}^+ \\ \hline I_{b_2}^+ & 0 \end{array}\right), \tag{2.4.42} $$

$$ A_- = \left(\begin{array}{cc|c} {\Gamma_{12}}^1 & {\Gamma_{12}}^2 & \frac{1}{2}e^{\phi}Hf \\ {\Gamma_{22}}^1 & {\Gamma_{22}}^2 & Q^- f \\ \hline -2e^{-\phi}Q^- & H & 0 \end{array}\right) = \left(\begin{array}{c|c} A_f^- & I_{b_1}^- \\ \hline I_{b_2}^- & 0 \end{array}\right), \tag{2.4.43} $$

where $A_f^+$ and $A_f^-$ are $2 \times 2$ matrices with fermionic entries, $I_{b_1}^+$ and $I_{b_1}^-$ are two-component column vectors with bosonic entries, and $I_{b_2}^+$ and $I_{b_2}^-$ are two-component row vectors with bosonic entries. Let us consider the moving frame $\Psi = (\psi_f, \psi_b)^T$ where $\psi_f$ is a two-component fermionic vector and $\psi_b$ is a bosonic scalar. From the GW equations for the moving frame $\Psi$, with the matrices given by (2.4.42) and (2.4.43), we obtain

$$D_+\Psi = A_+\Psi, \qquad D_-\Psi = A_-\Psi. \tag{2.4.44}$$

The compatibility conditions for the $\psi_f$ and $\psi_b$ lead us to the four equations

$$\begin{gathered} D_+A_f^- + D_-A_f^+ + I_{b_1}^- I_{b_2}^+ + I_{b_1}^+ I_{b_2}^- - \{A_f^+, A_f^-\} = 0, \\ -A_f^- I_{b_1}^+ + D_+ I_{b_1}^- + I_{b_1}^- \eta_f^+ - A_f^+ I_{b_1}^- + D_- I_{b_1}^+ + I_{b_1}^+ \eta_f^- = 0, \\ D_+ I_{b_2}^- + I_{b_2}^- A_f^+ - \eta_f^- I_{b_2}^+ + D_- I_{b_2}^+ + I_{b_2}^+ A_f^- - \eta_f^+ I_{b_2}^- = 0, \\ I_{b_2}^+ I_{b_1}^- + D_-\eta_f^+ + I_{b_2}^- I_{b_1}^+ + D_+\eta_f^- = 0. \end{gathered} \tag{2.4.45}$$

The zero curvature condition (ZCC) corresponding to the equations (2.4.45) is an equivalent matrix form of (2.4.40).

The ZCC (2.4.40) leads us to the bosonic SUSY GC equations which consist of the following six linearly independent equations for the matrix components

$$\begin{aligned} &(i) && D_-({\Gamma_{11}}^1) + D_+({\Gamma_{22}}^2) + D_+({\Gamma_{12}}^1) - D_-({\Gamma_{12}}^2) = 0, \\ &(ii) && D_-({\Gamma_{11}}^1) - {\Gamma_{11}}^2{\Gamma_{22}}^1 + D_+({\Gamma_{12}}^1) + {\Gamma_{12}}^2{\Gamma_{12}}^1 \\ &&& \quad + \frac{1}{2}H^2 e^{\phi} f - 2Q^+Q^- e^{-\phi} f = 0, \\ &(iii) && Q^+{\Gamma_{22}}^2 - {\Gamma_{11}}^2 Q^- + D_-Q^+ - Q^+D_-\phi + \frac{1}{2}e^{\phi}D_+H = 0, \\ &(iv) && Q^-{\Gamma_{11}}^1 - {\Gamma_{22}}^1 Q^+ + D_+Q^- - Q^-D_+\phi - \frac{1}{2}e^{\phi}D_-H = 0, \\ &(v) && D_-({\Gamma_{11}}^2) - {\Gamma_{12}}^1{\Gamma_{11}}^2 - {\Gamma_{11}}^2{\Gamma_{22}}^2 - {\Gamma_{11}}^1{\Gamma_{12}}^2 \\ &&& \quad + D_+({\Gamma_{12}}^2) + 2Q^+Hf = 0, \\ &(vi) && D_+({\Gamma_{22}}^1) + {\Gamma_{12}}^2{\Gamma_{22}}^1 - {\Gamma_{22}}^1{\Gamma_{11}}^1 + {\Gamma_{22}}^2{\Gamma_{12}}^1 \\ &&& \quad - D_-({\Gamma_{12}}^1) + 2Q^-Hf = 0. \end{aligned} \tag{2.4.46}$$

The Grassmann-valued PDEs (2.4.46) involve eleven dependent functions of the independent variables $x_+$, $x_-$, $\theta^+$ and $\theta^-$ including the four bosonic functions $\phi$, $H$, $Q^\pm$ and the six fermionic functions ${\Gamma_{ij}}^k$ together with one dependent bodiless bosonic function $f$ of $x_+$ and $x_-$. It is interesting to note that the equation (2.4.46.i) is the compatibility condition of the function $f$ given in equation (2.4.31). Under the above assumptions we obtain the following result.

**Proposition 2.4.1** (Structural bosonic SUSY equations).
*For any vector bosonic superfields* $F(x_+, x_-, \theta^+, \theta^-)$ *and* $N(x_+, x_-, \theta^+, \theta^-)$ *satisfying the normalization conditions* (2.4.9), (2.4.10), (2.4.18) *and* (2.4.23), *the moving frame* $\Omega = (D_+F, D_-F, N)^T$ *on a smooth conformally parametrized surface immersed in the superspace* $\mathbb{R}^{(2,1|2)}$ *satisfies the bosonic SUSY GW equations* (2.4.39). *The ZCC* (2.4.40), *which is the compatibility condition of the bosonic SUSY GW equations* (2.4.39) *expressed in terms of the matrices* $A_+$ *and* $A_-$, *is equivalent to the bosonic SUSY GC equations* (2.4.46).

### 2.4.2. Fermionic extension

We now discuss a derivation of the SUSY GW and SUSY GC equations for the fermionic extension. Conditions on the Christoffel symbols of the second kind ${\Gamma_{ij}}^k$ can be obtained by taking derivatives of (2.4.9), i.e.

$$\begin{aligned} 0 &= D_i\langle D_jF, D_jF\rangle = \langle D_iD_jF, D_jF\rangle + \langle D_jF, D_iD_jF\rangle \\ &= 2\langle D_iD_jF, D_jF\rangle = 2{\Gamma_{ji}}^k g_{kj}. \end{aligned} \tag{2.4.47}$$

Therefore we have

$${\Gamma_{ji}}^k = 0, \qquad \text{for} \qquad j \neq i \tag{2.4.48}$$

and since the Christoffel symbols are antisymmetric under a permutation of indices $i$ and $j$ (see equation (2.4.33)), we get

$${\Gamma_{ji}}^k = 0, \qquad \text{if} \qquad i \neq k \quad \text{or} \quad j \neq k. \tag{2.4.49}$$

Using the last result on equations (2.4.30) and (2.4.31) we get

$$D_kf = ({\Gamma_{k(k)}}^{(k)} - D_k\phi)f, \qquad \{D_+, D_-\}f = (D_+{\Gamma_{22}}^2 + D_-{\Gamma_{11}}^1)f = 0. \tag{2.4.50}$$

Also the Christoffel symbols of the first kind are given by

$$\begin{aligned} &\Gamma_{111} = 0, \quad \Gamma_{112} = \frac{1}{2}e^{\phi}{\Gamma_{11}}^1, \quad \Gamma_{121} = 0, \qquad \Gamma_{211} = 0, \\ &\Gamma_{122} = 0, \quad \Gamma_{212} = 0, \qquad \Gamma_{221} = \frac{1}{2}e^{\phi}{\Gamma_{22}}^2, \quad \Gamma_{222} = 0, \end{aligned} \tag{2.4.51}$$

up to the addition of a fermionic function $\zeta_2 \neq 0$ which has the property $\zeta_2 f = 0$. The fermionic quantities ${b^i}_j$ take the form

$${b^1}_1 = H, \qquad {b^2}_1 = -2e^{-\phi}Q^+, \qquad {b^1}_2 = -2e^{-\phi}Q^-, \qquad {b^2}_2 = -H, \tag{2.4.52}$$

up to the addition of a fermionic function $\zeta_3 \neq 0$ which has the property $\zeta_3 f = 0$. Therefore, the fermionic-valued matrices $A_\pm$ in the SUSY GW equations (2.4.20) take the form

$$
\begin{aligned}
D_+\begin{pmatrix} D_+F \\ D_-F \\ N \end{pmatrix} &= \begin{pmatrix} {\Gamma_{11}}^1 & 0 & Q^+f \\ 0 & 0 & -\frac{1}{2}e^{\phi}Hf \\ H & -2e^{-\phi}Q^+ & 0 \end{pmatrix}\begin{pmatrix} D_+F \\ D_-F \\ N \end{pmatrix}, \\
D_-\begin{pmatrix} D_+F \\ D_-F \\ N \end{pmatrix} &= \begin{pmatrix} 0 & 0 & \frac{1}{2}e^{\phi}Hf \\ 0 & {\Gamma_{22}}^2 & Q^-f \\ -2e^{-\phi}Q^- & -H & 0 \end{pmatrix}\begin{pmatrix} D_+F \\ D_-F \\ N \end{pmatrix}.
\end{aligned}
\tag{2.4.53}
$$

The compatibility condition of the SUSY GW equations (2.4.53) are given by

$$
D_+A_- + D_-A_+ - \{A_+, A_-\} = 0, \tag{2.4.54}
$$

up to the addition of a non-zero matrix $P$ such that $P\Omega = 0$. In component form these equations are

$$
\begin{aligned}
&(i) && D_-({\Gamma_{11}}^1) + 2e^{-\phi}Q^+Q^-f = 0, \\
&(ii) && \left[D_-Q^+ + \frac{1}{2}e^{\phi}D_+H + Q^+(D_-\phi - {\Gamma_{22}}^2)\right] f = 0, \\
&(iii) && D_+({\Gamma_{22}}^2) + 2e^{-\phi}Q^-Q^+f = 0, \\
&(iv) && \left[D_+Q^- - \frac{1}{2}e^{\phi}D_-H + Q^-(D_+\phi - {\Gamma_{11}}^1)\right] f = 0, \\
&(v) && D_+Q^- - \frac{1}{2}e^{\phi}D_-H + Q^-(D_+\phi - {\Gamma_{11}}^1) = 0, \\
&(vi) && D_-Q^+ + \frac{1}{2}e^{\phi}D_+H + Q^+(D_-\phi - {\Gamma_{22}}^2) = 0.
\end{aligned}
\tag{2.4.55}
$$

If the equations (2.4.55.vi) and (2.4.55.v) hold, then equations (2.4.55.ii) and (2.4.55.iv) are identically satisfied respectively. By adding (2.4.55.i) and (2.4.55.iii) we obtain

$$
D_-({\Gamma_{11}}^1) + D_+({\Gamma_{22}}^2) = 0, \tag{2.4.56}
$$

which is the compatibility condition for $f$, corresponding to (2.4.50). Therefore the SUSY GC equations are reduced to the four linearly independent equations

$$
\begin{aligned}
&(i) && D_+({\Gamma_{22}}^2) + D_-({\Gamma_{11}}^1) = 0, \\
&(ii) && D_-({\Gamma_{11}}^1) + 2e^{-\phi}Q^+Q^-f = 0, \\
&(iii) && D_+Q^- - \frac{1}{2}e^{\phi}D_-H + Q^-(D_+\phi - {\Gamma_{11}}^1) = 0, \\
&(iv) && D_-Q^+ + \frac{1}{2}e^{\phi}D_+H + Q^+(D_-\phi - {\Gamma_{22}}^2) = 0.
\end{aligned}
\tag{2.4.57}
$$

The Grassmann-valued PDEs (2.4.57) involve six dependent functions of the independent variables $x_+$, $x_-$, $\theta^+$ and $\theta^-$ including one bosonic function $\phi$ and the five fermionic functions ${\Gamma_{11}}^1$, ${\Gamma_{22}}^2$, $H$ and $Q^{\pm}$ together with one bosonic function $f$ of $x_+$ and $x_-$.

**Proposition 2.4.2** (Structural fermionic SUSY equations)**.**
*For any vector fermionic superfield $F(x_+, x_-, \theta^+, \theta^-)$ and bosonic superfield $N(x_+, x_-, \theta^+, \theta^-)$ satisfying the normalization conditions* (2.4.9), (2.4.10), (2.4.18) *and* (2.4.23)*, the bosonic moving frame $\Omega = (D_+F, D_-F, N)^T$ on a smooth conformally parametrized surface immersed in the superspace $\mathbb{R}^{(1,1|3)}$ satisfies the fermionic SUSY GW equations* (2.4.53)*. The ZCC* (2.4.54)*, which is the compatibility condition of the fermionic SUSY GW equations* (2.4.53) *expressed in terms of the matrices $A_+$ and $A_-$, is equivalent to the fermionic SUSY GC equations* (2.4.57)*.*

If we consider the case where $f$ is a bosonic constant in the fermionic extension, then from equations (2.4.50) we have that

$$ {\Gamma_{11}}^1 = D_+\phi, \qquad {\Gamma_{22}}^2 = D_-\phi, \tag{2.4.58} $$

up to the addition of a function $\zeta_4 \neq 0$ with the property $\zeta_4 f = 0$. Also the compatibility condition on $f$ is then identically satisfied. The SUSY GC equations become

$$ \begin{aligned} &(i) \qquad && D_-D_+\phi + 2e^{-\phi}Q^+Q^-f = 0, \\ &(ii) \qquad && D_+Q^- - \frac{1}{2}e^{\phi}D_-H = 0, \\ &(iii) \qquad && D_-Q^+ + \frac{1}{2}e^{\phi}D_+H = 0, \end{aligned} \tag{2.4.59} $$

which resemble the classical GC equations (2.2.13) taking into account that the $H^2$ term vanishes. The equations (2.4.59) contain terms whose signs differ from those of the classical equations. We get an underdetermined system of three PDEs for four dependent variables $H$, $Q^{\pm}$ and $\phi$. One should note that for this special case the SUSY GW equations

$$ \begin{aligned} D_+\begin{pmatrix} D_+F \\ D_-F \\ N \end{pmatrix} &= \begin{pmatrix} D_+\phi & 0 & Q^+f \\ 0 & 0 & -\frac{1}{2}e^{\phi}Hf \\ H & -2e^{-\phi}Q^+ & 0 \end{pmatrix} \begin{pmatrix} D_+F \\ D_-F \\ N \end{pmatrix}, \\ D_-\begin{pmatrix} D_+F \\ D_-F \\ N \end{pmatrix} &= \begin{pmatrix} 0 & 0 & \frac{1}{2}e^{\phi}Hf \\ 0 & D_-\phi & Q^-f \\ -2e^{-\phi}Q^- & -H & 0 \end{pmatrix} \begin{pmatrix} D_+F \\ D_-F \\ N \end{pmatrix}, \end{aligned} \tag{2.4.60} $$

are also similar to the classical GW equations up to some sign differences and the multiplication of some elements by the function $f$.

## 2.5. Geometric aspects of conformally parametrized SUSY surfaces

In this section, we discuss certain aspects of Grassmann variables in conjunction with differential geometry and SUSY analysis. Let us define the differential fermionic operators

$$d_{\pm} = d\theta^{\pm} + idx_{\pm}\partial_{\theta^{\pm}}, \tag{2.5.1}$$

where $d_+$ and $d_-$ are the infinitesimal displacements in the direction of $D_+$ and $D_-$, respectively. These operators are anticommuting, i.e. $\{d_+, d_-\} = 0$. In order to compute the first and second fundamental forms, we have assumed that $(d\theta^j \lrcorner \partial_{\theta^i}) = 0$ for $i, j = 1, 2$. For SUSY conformally parametrized surfaces, the first fundamental form is given by

$$\begin{aligned} I &= \left\langle \begin{pmatrix} d_+ & d_- \end{pmatrix}, \begin{pmatrix} d_+ & d_- \end{pmatrix} \begin{pmatrix} \langle D_+F, D_+F\rangle & \langle D_+F, D_-F\rangle \\ -\langle D_+F, D_-F\rangle & \langle D_-F, D_-F\rangle \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} d_+ & d_- \end{pmatrix}, \begin{pmatrix} d_+ & d_- \end{pmatrix} \begin{pmatrix} g_{11}f & g_{12}f \\ -g_{12}f & g_{22}f \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} d_+ & d_- \end{pmatrix}, \begin{pmatrix} d_+ & d_- \end{pmatrix} Rf \right\rangle \\ &= f\left(d_+^2 g_{11} + 2d_+d_-g_{12} + d_-^2 g_{22}\right) = fd_+d_-e^{\phi}, \end{aligned} \tag{2.5.2}$$

where the $2 \times 2$ bosonic-valued matrix $R$ is given by

$$R = \begin{pmatrix} g_{11} & g_{12} \\ -g_{12} & g_{22} \end{pmatrix} = \frac{1}{2}e^{\phi} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.5.3}$$

The discriminant $g$ is defined to be

$$g = g_{11}g_{22} + (g_{12})^2 = \frac{1}{4}e^{2\phi}. \tag{2.5.4}$$

Hence the covariant metric is given by

$$g_{ij}g^{jk} = \delta_i^k, \qquad \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.5.5}$$

such that

$$g^{11} = g^{22} = 0, \qquad g^{12} = \epsilon g^{21} = 2e^{-\phi}, \tag{2.5.6}$$

where $\epsilon = -1$ in the bosonic case and $\epsilon = 1$ in the fermionic case. The second fundamental form is

$$\begin{aligned}
I\!I &= \left\langle \begin{pmatrix} d_+ & d_- \end{pmatrix}, \begin{pmatrix} d_+ & d_- \end{pmatrix} \begin{pmatrix} \langle D_+^2 F, N\rangle & \langle D_- D_+ F, N\rangle \\ -\langle D_- D_+ F, N\rangle & \langle D_-^2 F, N\rangle \end{pmatrix} \right\rangle \\
&= \left\langle \begin{pmatrix} d_+ & d_- \end{pmatrix}, \begin{pmatrix} d_+ & d_- \end{pmatrix} \begin{pmatrix} b_{11} f & b_{12} f \\ -b_{12} f & b_{22} f \end{pmatrix} \right\rangle \\
&= \left\langle \begin{pmatrix} d_+ & d_- \end{pmatrix}, \begin{pmatrix} d_+ & d_- \end{pmatrix} S f \right\rangle \\
&= f\left(d_+^2 b_{11} + 2 d_+ d_- b_{12} + d_-^2 b_{22}\right) = f\left(d_+^2 Q^+ + d_+ d_-(e^\phi H) + d_-^2 Q^-\right),
\end{aligned} \tag{2.5.7}$$

where the matrix $S$ is given by

$$S = \begin{pmatrix} b_{11} & b_{12} \\ -b_{12} & b_{22} \end{pmatrix} = \begin{pmatrix} Q^+ & \frac{1}{2} e^\phi H \\ -\frac{1}{2} e^\phi H & Q^- \end{pmatrix}. \tag{2.5.8}$$

The discriminant $b$ is defined to be

$$b = b_{11} b_{22} + (b_{12})^2 = Q^+ Q^- + \frac{1}{4} e^{2\phi} H^2. \tag{2.5.9}$$

One should note that the term in $H^2$ vanishes in the fermionic case since $H$ is a fermionic-valued function. Making use of (2.5.4) and (2.5.9), the Gaussian and mean curvatures are defined as

$$\begin{aligned}
\mathcal{K} = \det(S R^{-1}) &= \frac{b_{11} b_{22} + (b_{12})^2}{g_{11} g_{22} + (g_{12})^2} = 4 e^{-2\phi} Q^+ Q^- + H^2, \\
H &= \frac{1}{2} \operatorname{tr}(S R^{-1}).
\end{aligned} \tag{2.5.10}$$

In the fermionic case, $\mathcal{K}$ is a bosonic bodiless function and the term $H^2$ vanishes.

Under the above assumptions on the SUSY versions of the GC equations (2.4.46) or (2.4.57) we can provide a SUSY analogue of the Bonnet theorem.

**Proposition 2.5.1** (SUSY extension of the Bonnet theorem)**.**
*Given a SUSY conformal metric, $M = f d_+ d_- e^\phi$, of a smooth conformally parametrized surface $\mathcal{S}$, the Hopf differentials $d_\pm^2 Q^\pm$ and a mean curvature function $H$ defined on a Riemann surface $\mathcal{R}$ satisfying the GC equations (*(2.4.46) *for the bosonic case,* (2.4.57) *for the fermionic case), there exists a vector-valued immer-*

*sion function,*

$$F^b = (F^b_1, F^b_2, F^b_3) : \tilde{\mathcal{R}} \to \mathbb{R}^{(2,1|2)},$$
$$F^f = (F^f_1, F^f_2, F^f_3) : \tilde{\mathcal{R}} \to \mathbb{R}^{(1,1|3)}, \tag{2.5.11}$$

*(bosonic or fermionic, respectively) with the fundamental forms*

$$I = f d_+ d_- e^{\phi}, \qquad I\!I = f(d_+^2 Q^+ + d_+ d_-(He^{\phi}) + d_-^2 Q^-), \tag{2.5.12}$$

*where* $\tilde{\mathcal{R}}$ *is the universal covering of the Riemann surface* $\mathcal{R}$ *and* $\mathbb{R}^{(n_b|n_f)}$ *is the superspace. The immersion function* $F$ *is unique up to affine transformations in the superspace.*

The proof of this proposition is analogous to that given in [**21**]. Note that it is straightforward to construct surfaces in the superspace related to integrable equations. However, it is non-trivial to identify those surfaces which have an invariant geometrical characterization. A list of such surfaces is known in the classical case [**19**] but, to our knowledge, an identification of such surfaces is an open problem in the case of surfaces immersed in the superspace.

## 2.6. Symmetries of the SUSY GC equations

By a symmetry supergroup $G$ of a SUSY system, we mean a local supergroup of transformations acting on the Cartesian product $\mathcal{X} \times \mathcal{U}$ of supermanifolds, where $\mathcal{X}$ is the space of four independent variables $(x_+, x_-, \theta^+, \theta^-)$ and $\mathcal{U}$ is the space of dependent superfields. For the bosonic case, $\mathcal{U}$ is the space of eleven dependent superfields $\mathcal{U} = (\phi, H, Q^+, Q^-, R^+, R^-, S^+, S^-, T^+, T^-, f)$, where we have used the abbreviated notation for the Christoffel symbols of the second kind

$$R^+ = {\Gamma_{11}}^1, \qquad R^- = {\Gamma_{11}}^2, \qquad S^+ = {\Gamma_{12}}^1, \qquad S^- = {\Gamma_{12}}^2,$$
$$T^+ = {\Gamma_{22}}^1, \qquad T^- = {\Gamma_{22}}^2. \tag{2.6.1}$$

For the fermionic case, $\mathcal{U}$ is the space of seven dependent superfields $\mathcal{U} = (\phi, H, Q^+, Q^-, R^+, T^-, f)$, where we have used the notation (2.6.1) for the non-zero Christoffel symbols of the second kind ${\Gamma_{ij}}^k$. Solutions of the SUSY GC equations, (2.4.46) for the bosonic case or (2.4.57) for the fermionic case, are mapped to solutions of equations (2.4.46) or (2.4.57), respectively by the action of the supergroup $G$ on the functions in $\mathcal{U}$. When we perform the symmetry reductions, we need to take into consideration the fact that the bosonic function $f$ introduced in (2.4.9) depends only on $x_+$ and $x_-$ or is constant. If $G$ is a Lie supergroup as described in [**80**] and [**120**], it can be associated with its Lie superalgebra whose elements are infinitesimal symmetries of the SUSY GC equations. We have made use of the theory described in the book by Olver [**95**] in order to determine superalgebras of infinitesimal symmetries for both the bosonic and fermionic SUSY GC equations.

The bosonic SUSY GC equations (2.4.46) are invariant under the Lie superalgebra $\mathfrak{g}$ generated by the following eight infinitesimal vector fields

$$\begin{aligned} C_0 &= H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} - 2f\partial_f, \\ K_0 &= -H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} + 2\partial_\phi, \\ K_1^b &= -2x_+\partial_{x_+} - \theta^+\partial_{\theta^+} + R^+\partial_{R^+} + 2R^-\partial_{R^-} + S^-\partial_{S^-} \\ &\qquad -T^+\partial_{T^+} + 2Q^+\partial_{Q^+} + \partial_\phi, \\ K_2^b &= -2x_-\partial_{x_-} - \theta^-\partial_{\theta^-} - R^-\partial_{R^-} + S^+\partial_{S^+} + 2T^+\partial_{T^+} \\ &\qquad +T^-\partial_{T^-} + 2Q^-\partial_{Q^-} + \partial_\phi, \\ P_+ &= \partial_{x_+}, \qquad\qquad P_- = \partial_{x_-}, \\ J_+ &= \partial_{\theta^+} + i\theta^+\partial_{x_+}, \qquad J_- = \partial_{\theta^-} + i\theta^-\partial_{x_-}. \end{aligned} \tag{2.6.2}$$

The generators $P_+$ and $P_-$ represent translations in the bosonic variables $x_+$ and $x_-$ while $K_1^b$, $K_2^b$, $K_0$ and $C_0$ generate dilations on both even and odd variables. In addition, we recover the SUSY operators $J_+$ and $J_-$ which were identified previously in equation (2.4.4). The commutation (anticommutation in the case of two fermionic operators) relations of the superalgebra $\mathfrak{g}$ of the SUSY GC equations (2.4.46) are given in table 2.1 for the case $D_\pm f \neq 0$. The Lie superalgebra $\mathfrak{g}$ can

TABLE 2.1. Commutation table for the Lie superalgebra $\mathfrak{g}$ spanned by the vector fields (2.6.2). In the case of two fermionic generators $J_+$ and/or $J_-$ we have anticommutation rather than commutation.

| | $K_1^b$ | $P_+$ | $J_+$ | $K_2^b$ | $P_-$ | $J_-$ | $K_0$ | $C_0$ |
|---|---|---|---|---|---|---|---|---|
| $K_1^b$ | 0 | $2P_+$ | $J_+$ | 0 | 0 | 0 | 0 | 0 |
| $P_+$ | $-2P_+$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $J_+$ | $-J_+$ | 0 | $2iP_+$ | 0 | 0 | 0 | 0 | 0 |
| $K_2^b$ | 0 | 0 | 0 | 0 | $2P_-$ | $J_-$ | 0 | 0 |
| $P_-$ | 0 | 0 | 0 | $-2P_-$ | 0 | 0 | 0 | 0 |
| $J_-$ | 0 | 0 | 0 | $-J_-$ | 0 | $2iP_-$ | 0 | 0 |
| $K_0$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $C_0$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

be decomposed into the following combination of direct and semi-direct sums

$$\mathfrak{g} = \{\{K_1^b\} \ni \{P_+, J_+\}\} \oplus \{\{K_2^b\} \ni \{P_-, J_-\}\} \oplus \{K_0\} \oplus \{C_0\}. \tag{2.6.3}$$

In equation (2.6.3) the brace bracket $\{\cdot,\cdot\}$ denotes the set of all elements within. It should be noted that $K_0$ and $C_0$ constitute the center of the Lie superalgebra $\mathfrak{g}$.

The fermionic SUSY GC equations (2.4.57) are invariant under the following six bosonic symmetry generators

$$\begin{aligned}
P_+ &= \partial_{x_+}, \qquad P_- = \partial_{x_-},\\
C_0 &= H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} - 2f\partial_f,\\
K_0 &= -H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} + 2\partial_\phi,\\
K_1^f &= -2x_+\partial_{x_+} - \theta^+\partial_{\theta^+} + 2Q^+\partial_{Q^+} + R^+\partial_{R^+} + \partial_\phi,\\
K_2^f &= -2x_-\partial_{x_-} - \theta^-\partial_{\theta^-} + 2Q^-\partial_{Q^-} + T^-\partial_{T^-} + \partial_\phi,
\end{aligned} \tag{2.6.4}$$

together with the three fermionic generators

$$J_+ = \partial_{\theta^+} + i\theta^+\partial_{x_+}, \qquad J_- = \partial_{\theta^-} + i\theta^-\partial_{x_-}, \qquad W = \partial_H. \tag{2.6.5}$$

The symmetry generators $W$ and $P_\pm$ represent a fermionic translation of $H$ and bosonic translations in the $x_\pm$ direction respectively, $J_\pm$ represent the SUSY transformations and $C_0$, $K_0$, $K_1^f$ and $K_2^f$ represent dilations. The commutation table (anticommutation for two fermionic symmetries) for the generators of the superalgebra $\mathfrak{h}$ of equations (2.4.57) is given in table 2.2. The decomposition of the

TABLE 2.2. Commutation table for the Lie superalgebra $\mathfrak{h}$ spanned by the vector fields (2.6.4) and (2.6.5). In the case of two fermionic generators $J_+$ and/or $J_-$ and/or $W$ we have anticommutation rather than commutation.

| | $K_1^f$ | $P_+$ | $J_+$ | $K_2^f$ | $P_-$ | $J_-$ | $K_0$ | $C_0$ | $W$ |
|---|---|---|---|---|---|---|---|---|---|
| $K_1^f$ | 0 | $2P_+$ | $J_+$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $P_+$ | $-2P_+$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $J_+$ | $-J_+$ | 0 | $2iP_+$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $K_2^f$ | 0 | 0 | 0 | 0 | $2P_-$ | $J_-$ | 0 | 0 | 0 |
| $P_-$ | 0 | 0 | 0 | $-2P_-$ | 0 | 0 | 0 | 0 | 0 |
| $J_-$ | 0 | 0 | 0 | $-J_-$ | 0 | $2iP_-$ | 0 | 0 | 0 |
| $K_0$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $W$ |
| $C_0$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $-W$ |
| $W$ | 0 | 0 | 0 | 0 | 0 | 0 | $-W$ | $W$ | 0 |

superalgebra composed of (2.6.4) and (2.6.5) is given by

$$\mathfrak{h} = \{\{K_1^f\} \supset\!\!\!\!\!\!\;\, \{P_+, J_+\}\} \oplus \{\{K_2^f\} \supset\!\!\!\!\!\!\;\, \{P_-, J_-\}\} \oplus \{\{K_0, C_0\} \supset\!\!\!\!\!\!\;\, \{W\}\}. \tag{2.6.6}$$

However, if we consider the case where $D_\pm f = 0$, the equations (2.4.59) are invariant under the five bosonic generators

$$\begin{aligned} P_+ &= \partial_{x_+}, \qquad P_- = \partial_{x_-}, \\ K_0 &= -H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} + 2\partial_\phi, \\ \hat{K}_1^f &= -2x_+\partial_{x_+} - \theta^+\partial_{\theta^+} + 2Q^+\partial_{Q^+} + \partial_\phi, \\ \hat{K}_2^f &= -2x_-\partial_{x_-} - \theta^-\partial_{\theta^-} + 2Q^-\partial_{Q^-} + \partial_\phi, \end{aligned} \tag{2.6.7}$$

and the three fermionic generators given by

$$J_+ = \partial_{\theta^+} + i\theta^+\partial_{x_+}, \qquad J_- = \partial_{\theta^-} + i\theta^-\partial_{x_-}, \qquad W = \partial_H. \tag{2.6.8}$$

The commutation table for the generators of the superalgebra of equations (2.4.59) is given in table 2.3. The Lie superalgebra $\hat{\mathfrak{h}}$, generated by (2.6.7) and (2.6.8), can

Table 2.3. Commutation table for the Lie superalgebra $\hat{\mathfrak{h}}$ spanned by the vector fields (2.6.7) and (2.6.8). In the case of two fermionic generators $J_+$ and/or $J_-$ and/or $W$ we have anticommutation rather than commutation.

| | $\hat{K}_1^f$ | $P_+$ | $J_+$ | $\hat{K}_2^f$ | $P_-$ | $J_-$ | $K_0$ | $W$ |
|---|---|---|---|---|---|---|---|---|
| $\hat{K}_1^f$ | $0$ | $2P_+$ | $J_+$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $P_+$ | $-2P_+$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $J_+$ | $-J_+$ | $0$ | $2iP_+$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $\hat{K}_2^f$ | $0$ | $0$ | $0$ | $0$ | $2P_-$ | $J_-$ | $0$ | $0$ |
| $P_-$ | $0$ | $0$ | $0$ | $-2P_-$ | $0$ | $0$ | $0$ | $0$ |
| $J_-$ | $0$ | $0$ | $0$ | $-J_-$ | $0$ | $2iP_-$ | $0$ | $0$ |
| $K_0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $W$ |
| $W$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $-W$ | $0$ |

be decomposed into the following combination of direct and semi-direct sums

$$\hat{\mathfrak{h}} = \{\{\hat{K}_1^f\} \ni \{P_+, J_+\}\} \oplus \{\{\hat{K}_2^f\} \ni \{P_-, J_-\}\} \oplus \{\{K_0\} \ni \{W\}\}. \tag{2.6.9}$$

In both the bosonic and fermionic cases, the one-dimensional subalgebras of the respective superalgebra can be classified into conjugacy classes, as we proceed to do in the next section.

## 2.7. One-dimensional subalgebras of the symmetry superalgebras of the SUSY GC equations

In this section, we perform a classification of the one-dimensional subalgebras of the Lie superalgebras of infinitesimal transformations $\mathfrak{g}$ and $\mathfrak{h}$ into conjugacy classes under the action of their respective supergroups, $\exp(\mathfrak{g})$ generated by

(2.6.2), and $\exp(\mathfrak{h})$ generated by (2.6.4) and (2.6.5). The significance of such a classification resides in the fact that conjugate subgroups necessarily lead to invariant solutions which are equivalent in the sense that they can be transformed from one to the other by a suitable symmetry. Therefore, it is not necessary to compute reductions with respect to algebras which are conjugate to each other.

When constructing a list of representative one-dimensional subalgebras, it would be inconsistent to consider the $\mathbb{R}$ or $\mathbb{C}$ span of the generators (2.6.2) or (2.6.4) and (2.6.5) because we multiply the odd generators $J_+$, $J_-$ and (in the fermionic case) $W$ by the odd parameters $\underline{\mu}$, $\underline{\eta}$ and $\underline{\zeta}$ respectively in the classifications listed in section 2.10. Therefore, one is naturally led to consider a superalgebra which is a supermanifold in the sense presented in section 2.3. This means that $\mathfrak{g}$ and $\mathfrak{h}$ contain any sums of even combinations of the bosonic generators (i.e. multiplied by even parameters including real or complex numbers) and odd combinations of fermionic generators (i.e. multiplied by odd parameters in $\Lambda_{\text{odd}}$). At the same time $\mathfrak{g}$ and $\mathfrak{h}$ are $\Lambda_{\text{even}}$ Lie modules. This fact can lead to the following complication. For a given $X$ in $\mathfrak{g}$ or $\mathfrak{h}$, the subalgebras $\mathfrak{X}$ and $\mathfrak{X}'$ spanned by $X$ and $X' = aX$ with $a \in \Lambda_{\text{even}} \backslash \mathbb{C}$ are not isomorphic in general, i.e. $\mathfrak{X}' \subset \mathfrak{X}$.

It should be noted that subalgebras obtained by multiplying other subalgebras by bodiless elements of $\Lambda_{\text{even}}$ do not provide us with anything new for the purpose of symmetry reduction. It is not particularly useful to consider a subalgebra of the form e.g. $\{P_+ + \underline{\eta}_1\underline{\eta}_2 P_-\}$, since there is no limit to the number of odd parameters $\underline{\eta}_k$ that can be used to construct even coefficients. While such subalgebras may allow for more freedom in the choice of invariants, we then encounter the problem of non-standard invariants [**68**, **69**]. Such non-standard invariants, which do not lead to standard reductions or invariant solutions, are found for several other SUSY hydrodynamic-type systems, e.g. in [**66**, **67**].

In what follows, we will assume throughout the computation of the non-isomorphic one-dimensional subalgebras that the non-zero bosonic parameters are invertible (i.e. behave essentially like ordinary real or complex numbers.) In order to classify the Lie superalgebras (2.6.2) and (2.6.4) and (2.6.5) under the action of their respective supergroups, we make use of the techniques for classifying direct and semi-direct sums of algebras described in [**120**] and generalize them to superalgebras involving both even and odd generators. In the case of direct sums, we use the Goursat twist method generalized to the case of a superalgebra.

In the bosonic case, the superalgebra (2.6.3) contains two isomorphic copies of the 3-dimensional algebra $\mathfrak{g}^{(1)} = \{\{K_1^b\} \niplus \{P_+, J_+\}\}$ (the other copy being $\mathfrak{g}^{(2)} = \{\{K_2^b\} \niplus \{P_-, J_-\}\}$) together with the one-dimensional algebras $\{K_0\}$ and $\{C_0\}$, which constitute the center of the Lie superalgebra $\mathfrak{g}$. This fact allows us to adapt the classification for 3-dimensional algebras as described in [**97**]. So we begin our classification by considering the twisted one-dimensional subalgebras of $\mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(2)}$. Under the action of a one-parameter group generated by the vector field

$$X = \alpha K_1^b + \beta P_+ + \underline{\eta} J_+ + \delta K_2^b + \lambda P_- + \underline{\rho} J_-, \tag{2.7.1}$$

where $\alpha, \beta, \delta, \lambda \in \Lambda_{\text{even}}$ and $\underline{\eta}, \underline{\rho} \in \Lambda_{\text{odd}}$, the one-dimensional subalgebra

$$Y = P_+ + aP_-, \qquad a \in \Lambda_{\text{even}}$$

transforms under the Baker-Campbell-Hausdorff formula

$$Y \to \text{Ad}_{\exp(X)} Y = Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] + \frac{1}{3!}[X, [X, [X, Y]]] + ... \tag{2.7.2}$$

to $e^{-2\alpha} P_+ + e^{-2\delta} aP_-$. Hence we get that $\{P_+ + aP_-\}$ is isomorphic to $\{P_+ + e^{2\alpha - 2\delta} aP_-\}$. By a suitable choice of $\alpha$ and $\delta$, the factor $e^{2\alpha-2\delta} a$ can be rescaled to either 1 or $-1$. Hence, we obtain a twisted subalgebra $\mathfrak{g}_{14} = \{P_+ + \epsilon P_-, \epsilon = \pm 1\}$ given in table 2.4 in section 2.10.

As another example, consider a twisted subalgebra of the form $\{P_+ + aK_2^b,\ a \neq 0\}$, where $a \in \Lambda_{\text{even}}$. Through the Baker-Campbell-Hausdorff formula (2.7.2), the vector field $Y = K_2^b + aP_+$ transforms (through the vector field $X$ given in (2.7.1)) to

$$e^X Y e^{-X} = K_2^b + e^{-2\alpha} aP_+ - \frac{\lambda}{\delta}(e^{-2\delta} - 1)P_- - \frac{1}{\delta}(e^{-\delta} - 1)\underline{\rho} J_-. \tag{2.7.3}$$

Through a suitable choice of $\lambda$ and $\underline{\rho}$, the last two terms of (2.7.3) can be eliminated, so we obtain the twisted subalgebra $\mathfrak{g}_{13} = \{K_2^b + \epsilon P_+, \epsilon = \pm 1\}$. Continuing the classification in an analogous way, we obtain the list of one-dimensional subalgebras given in table 2.4 in section 2.10.

In the fermionic case, the superalgebra (2.6.6) contains two isomorphic copies of the 3-dimensional algebra $\mathfrak{h}^{(1)} = \{\{K_1^f\} \ni \{P_+, J_+\}\}$, the other copy being $\mathfrak{h}^{(2)} = \{\{K_2^f\} \ni \{P_-, J_-\}\}$ together with the three-dimensional algebra $\{\{K_0, C_0\} \ni \{W\}\}$. Therefore, we begin our classification by considering the twisted one-dimensional subalgebras of $\mathfrak{h}^{(1)} \oplus \mathfrak{h}^{(2)}$. Under the action of a one-parameter group generated by the vector field

$$X = \alpha K_1^f + \beta P_+ + \underline{\eta} J_+ + \delta K_2^f + \lambda P_- + \underline{\rho} J_-, \tag{2.7.4}$$

where $\alpha, \beta, \delta, \lambda \in \Lambda_{\text{even}}$ and $\underline{\eta}, \underline{\rho} \in \Lambda_{\text{odd}}$, the one-dimensional subalgebra $Y = P_+ + aP_-, a \in \Lambda_{\text{even}}$ transforms under the Baker-Campbell-Hausdorff formula (2.7.2) to $e^{-2\alpha} P_+ + e^{-2\delta} aP_-$. By an appropriate choice of $\alpha$ and $\delta$, the factor

$e^{2\alpha-2\delta}a$ can be rescaled to either 1 or $-1$. Hence, we get a twisted subalgebra $\mathfrak{h}_{14} = \{P_+ + \epsilon P_-, \epsilon = \pm 1\}$ given in table 2.5 in section 2.10.

As another example, consider a twisted algebra of the form $\{K_1^f + \underline{\zeta} W\}$, where $\underline{\zeta}$ is a fermionic parameter. Through the Baker-Campbell-Hausdorff formula (2.7.2), the vector field $Y = K_1^f + \underline{\zeta} W$ transforms through

$$X = \alpha K_1^f + \beta P_+ + \underline{\eta} J_+ + \gamma K_2^f + \delta P_- + \underline{\lambda} J_- + \rho K_0 + \sigma C_0 + \underline{\tau} W, \tag{2.7.5}$$

(where $\alpha, \beta, \gamma, \delta, \rho, \sigma \in \Lambda_{\text{even}}$ and $\underline{\eta}, \underline{\lambda}, \underline{\tau} \in \Lambda_{\text{odd}}$) to

$$e^X Y e^{-X} = K_1^f + e^{\rho-\sigma} \underline{\zeta} W - \frac{\beta}{\alpha}(e^{2\alpha} - 1) P_+ - \frac{1}{\alpha}(e^{\alpha} - 1) \underline{\eta} J_+. \tag{2.7.6}$$

Through a suitable choice of $\beta$ and $\underline{\eta}$, the last two terms of the expression (2.7.6) can be eliminated, so we obtain the twisted subalgebra $\mathfrak{h}_{32} = \{K_1^f + \underline{\zeta} W\}$ given in table 2.5 in section 2.10. Continuing the classification in a similar way, involving twisted and non-twisted subalgebras according to [**120**], we obtain the list of one-dimensional subalgebras given in table 2.5 in section 2.10. These representative subalgebras allow us to determine invariant solutions of the bosonic and fermionic SUSY GC equations, (2.4.46) and (2.4.57) respectively, using the SRM.

For the specific fermionic case where $f$ is constant (i.e. the SUSY GC equations (2.4.59)), the one-dimensional subalgebras of the resulting Lie symmetry superalgebra (2.6.7) and (2.6.8) can be found by taking the limit where the coefficients of $C_0$ tend to zero in the subalgebras listed in table 2.5 and withdrawing repeated subalgebras, while rescaling appropriately.

## 2.8. Invariant solutions of the SUSY GC equations

We now make use of the SRM in order to obtain invariant solutions of the bosonic and fermionic SUSY GC equations. For each of the two SUSY GC systems (bosonic and fermionic) we select two representative subalgebras from the corresponding list of subalgebras (table 2.4 and table 2.5, respectively) and construct group invariant solutions. For each subalgebra, the invariants and corresponding group orbits are calculated. Next, the unknown superfield functions in $\mathcal{U}$ are expanded in terms of the fermionic symmetry variables $\eta$ and $\sigma$ (involving $\theta^+$ and $\theta^-$, respectively) with coefficients depending on the bosonic symmetry variable $\xi$. Each component $\Upsilon$ of $\mathcal{U}$ is expressed in terms of invariants in the form

$$\Upsilon = u_0(\xi) + \eta u^+(\xi) + \sigma u^-(\xi) + \eta\sigma u_1(\xi). \tag{2.8.1}$$

For the sake of simplicity, in what follows, we only consider the case where $u^+ = u^- = 0$. Substituting these expanded forms of the superfields $\mathcal{U}$ into the SUSY GC equations (2.4.46) and (2.4.57) we reduce these equations to many possible differential subsystems involving even and odd functions. Solving these subsystems, we determine the invariant solutions and provide some geometrical interpretation of the associated surfaces.

For the bosonic SUSY GC equations, we present the following two examples.

**Example 1.** In the case of the subalgebra $\mathfrak{g}_{39} = \{P_+ + \epsilon P_- + aK_0,\ \epsilon = \pm 1,\ a \neq 0\}$, the orbit of the group of the bosonic SUSY GC equations (2.4.46) can be parametrized as follows

$$\begin{aligned}
&H = e^{-ax_+} h(\xi, \theta^+, \theta^-), && \\
&Q^+ = e^{ax_+} q^+(\xi, \theta^+, \theta^-), && S^+ = s^+(\xi, \theta^+, \theta^-), \\
&Q^- = e^{ax_+} q^-(\xi, \theta^+, \theta^-), && S^- = s^-(\xi, \theta^+, \theta^-), \\
&R^+ = r^+(\xi, \theta^+, \theta^-), && T^+ = t^+(\xi, \theta^+, \theta^-), \\
&R^- = r^-(\xi, \theta^+, \theta^-), && T^- = t^-(\xi, \theta^+, \theta^-), \\
&\phi = 2ax_+ + \varphi(\xi, \theta^+, \theta^-), && f = \psi(\xi),
\end{aligned} \tag{2.8.2}$$

where the functions $H, Q^\pm, R^\pm, S^\pm, T^\pm$ and $\phi$ are expressed in terms of the bosonic symmetry variable $\xi = x_- - \epsilon x_+$ and the fermionic symmetry variables $\theta^+$ and $\theta^-$. A corresponding invariant solution is given by

$$\begin{aligned}
&H = e^{-ax_+}\left[h_0 + \theta^+\theta^- 2il_0 e^{\xi}\right], \\
&Q^+ = e^{ax_+}\Big[l_0 e^{2\xi} + l_1 e^{\xi} \\
&\qquad + \theta^+\theta^-\left(\frac{1}{2} i e^{\xi}(a h_0 + \epsilon (h_0)_\xi) + l_0 e^{2\xi}\varphi_1 + l_1 e^{\xi}\varphi_1\right)\Big], \\
&Q^- = e^{ax_+}\Big[\frac{\epsilon l_0}{a\epsilon - 1} + l_2 e^{(1-a\epsilon)\xi} \\
&\qquad + \theta^+\theta^-\left(-\frac{1}{2} i e^{\xi}(h_0)_\xi + \frac{\epsilon l_0}{a\epsilon - 1}\varphi_1 + l_2 e^{(1-a\epsilon)\xi}\varphi_1\right)\Big], \\
&R^- = b_1 \underline{S}_0^+, \quad R^+ = b_2 \underline{S}_0^+, \quad S^+ = \underline{S}_0^+, \quad S^- = \underline{S}_0^+, \quad T^- = b_3 \underline{T}_0^+, \\
&T^+ = b_4 \underline{S}_0^+, \quad \phi = 2ax_+ + \xi + \theta^+\theta^-\varphi_1, \qquad f = \psi, \\
&l_0 = \underline{a}_0 \underline{S}_0^+, \qquad l_1 = \underline{a}_2 \underline{S}_0^+, \qquad l_2 = \underline{a}_2 \underline{S}_0^+, \qquad h_0 = \underline{c}_0 \underline{S}_0^+,
\end{aligned} \tag{2.8.3}$$

where $h_0$, $\varphi_1$ and $\psi$ are functions of the symmetry variable $\xi = x_- - \epsilon x_+$ and where $l_0, l_1, l_2$ and $b_1, b_2, b_3, b_4$ are bosonic constants, while $\underline{S}_0^+, \underline{c}_0$ and $\underline{a}_0, \underline{a}_1, \underline{a}_2$ are fermionic constants.

The first and second fundamental forms of the surface $\mathcal{S}$ associated with (2.8.3) are given by

$$\begin{aligned}
I &= \psi d_+ d_- \left[e^{2ax_+ + \xi}\left(1+\theta^+\theta^-\varphi_1\right)\right],\\
I\!I &= \psi e^{ax_+}\Big\{d_+d_-\left[e^{\xi}\left(h_0+\theta^+\theta^-(2il_0e^{\xi}+h_0\varphi_1)\right)\right]\\
&\quad + d_+^2\left[l_0e^{2\xi}+l_1e^{\xi}+\theta^+\theta^-\left(\frac{1}{2}ie^{\xi}(ah_0+\epsilon(h_0)_\xi)+l_0e^{2\xi}\varphi_1+l_1e^{\xi}\varphi_1\right)\right]\\
&\quad + d_-^2\left[\frac{\epsilon l_0}{a\epsilon-1}+l_2e^{(1-a\epsilon)\xi}\right.\\
&\quad \left.+\theta^+\theta^-\left(-\frac{1}{2}ie^{\xi}(h_0)_\xi+\frac{\epsilon l_0}{a\epsilon-1}\varphi_1+l_2e^{(1-a\epsilon)\xi}\varphi_1\right)\right]\Big\}.
\end{aligned}\tag{2.8.4}$$

The Gaussian curvature takes the form

$$\begin{aligned}
\mathcal{K} &= e^{-2ax_+}\Big[h_0^2+\theta^+\theta^-4ih_0l_0e^{\xi}\\
&\quad +4(l_0e^{2\xi}+l_1e^{\xi})(\frac{\epsilon l_0}{a\epsilon-1}+l_2e^{(1-a\epsilon)\xi})e^{-2\xi}(1-\theta^+\theta^-2\varphi_1)\\
&\quad +4\theta^+\theta^-(l_0e^{2\xi}+l_1e^{\xi})\left(-\frac{1}{2}ie^{\xi}(h_0)_\xi+\frac{\epsilon l_0}{a\epsilon-1}\varphi_1+l_2e^{(1-a\epsilon)\xi}\varphi_1\right)e^{-2\xi}\\
&\quad +4\theta^+\theta^-(\frac{\epsilon l_0}{a\epsilon-1}+l_2e^{(1-a\epsilon)\xi})\\
&\quad \times\left(\frac{1}{2}ie^{\xi}(ah_0+\epsilon(h_0)_\xi)+l_0e^{2\xi}\varphi_1+l_1e^{\xi}\varphi_1\right)e^{-2\xi}\Big].
\end{aligned}\tag{2.8.5}$$

The subalgebra of the classical GC equation (2.2.13) analogous to $\mathfrak{g}_{39}$ is $L'_{1,7}=\{e_1+\epsilon e_2+ae_0, \epsilon=\pm1, a\neq0\}$, whose corresponding invariant solution is given by

$$\begin{aligned}
H(z,\bar z)&=k_0v(\xi)^{-1/2}e^{a/2(\bar z-3z)}, & Q(z,\bar z)&=\frac{1}{2}k_0v(\xi)^{1/2}e^{a/2(z+\bar z)},\\
U(z,\bar z)&=e^{2az}v(\xi), & \bar Q(z,\bar z)&=\frac{1}{2}k_0v(\xi)^{1/2}e^{a/2(z+\bar z)},
\end{aligned}\tag{2.8.6}$$

where the symmetry variable is $\xi=\bar z-\epsilon z$ and the function $v$ of $\xi$ satisfies the ODE

$$v_{\xi\xi}=\frac{(v_\xi)^2}{v}+k_0^2ve^{a\xi}.\tag{2.8.7}$$

For this classical solution, the Gaussian curvature vanishes, in contrast to the SUSY case.

**Example 2.** For the subalgebra $\mathfrak{g}_{76} = \{K_1^b + (a - \frac{1}{2})K_0 + \frac{1}{2}C_0,\ a \neq \frac{1}{2}\}$ we obtain the following parametrization of the orbit of the group

$$\begin{aligned}
&H = (x_+)^{(a-1)/2} h(x_-, \eta, \theta^-), && \\
&Q^+ = (x_+)^{-(a+2)/2} q^+(x_-, \eta, \theta^-), && S^+ = s^+(x_-, \eta, \theta^-), \\
&Q^- = (x_+)^{-a/2} q^-(x_-, \eta, \theta^-), && S^- = (x_+)^{-1/2} s^-(x_-, \eta, \theta^-), \\
&R^+ = (x_+)^{-1/2} r^+(x_-, \eta, \theta^-), && T^+ = (x_+)^{1/2} t^+(x_-, \eta, \theta^-), \\
&R^- = (x_+)^{-1} r^-(x_-, \eta, \theta^-), && T^- = t^-(x_-, \eta, \theta^-), \\
&e^\phi = (x_+)^{-a} \varphi(x_-, \eta, \theta^-), && f = (x_+)^{1/2} \psi(x_-),
\end{aligned} \tag{2.8.8}$$

where the bosonic symmetry variable is $x_-$ and the fermionic symmetry variables are $\eta = (x_+)^{-1/2}\theta^+$ and $\theta^-$. A corresponding invariant solution of the bosonic SUSY GC equations (2.4.46) takes the form

$$\begin{aligned}
&H = 2iB(x_+)^{(a-2)/2} (\rho)_{x_-} \theta^+\theta^-, \\
&Q^+ = BA(x_-)(x_+)^{-(a+2)/2} \left[1 + (x_+)^{-1/2}\theta^+\theta^- G(x_-)\right] \rho(x_-), \\
&Q^- = \frac{2B}{a}(x_+)^{-a/2} \left[1 + (x_+)^{-1/2}\theta^+\theta^- G(x_-)\right], \quad R^+ = (x_+)^{-1/2} l_1 \underline{R}_0^+, \\
&R^- = (x_+)^{-1} l_2 \underline{R}_0^-, \quad S^+ = T^- = \underline{T}_0^-, \quad S^- = T^+ = 0, \\
&e^\phi = A(x_-)(x_+)^{-a}(1 + (x_+)^{-1/2}\theta^+\theta^- G(x_-)), \qquad f = (x_+)^{1/2}\psi(x_-),
\end{aligned} \tag{2.8.9}$$

where $B = \underline{l}_0 \underline{R}_0^+ \underline{R}_0^- \underline{T}_0^-$ and $l_1, l_2, l_3$ are bosonic constants, while $\underline{l}_0$, $\underline{R}_0^\pm$ and $\underline{T}_0^-$ are fermionic constants. Here, $A$, $G$, $\rho$ and $\psi$ are arbitrary bosonic functions of the symmetry variable $x_-$. The function $A$ contains a part in $\Lambda_{\text{body}}$ but $\psi$ is a bodiless function.

The corresponding first and second fundamental forms for the surface $\mathcal{S}$ given by (2.8.9) are

$$I = \psi d_+ d_- \left[A(x_+)^{-(2a+1)/2}\left(1 + (x_+)^{-1/2}\theta^+\theta^- G\right)\right], \tag{2.8.10}$$

and

$$\begin{aligned}
I\!I = (d_+)^2 &\left[AB(x_+)^{-(a+2)/2}\rho\left(1 + (x_+)^{-1/2}\theta^+\theta^- G\right)\right] \\
&+ 2id_+d_- \left[AB(x_+)^{-1}\theta^+\theta^-\rho'\right] \\
&+ (d_-)^2 \left[\frac{2B}{a}(x_+)^{-a/2}\left(1 + (x_+)^{-1/2}\theta^+\theta^- G\right)\right].
\end{aligned} \tag{2.8.11}$$

Consequently, the Gaussian curvature $\mathcal{K}$ and the mean curvature $H$ of the associated surface $\mathcal{S}$ are not constant. The Gaussian curvature is given by

$$\mathcal{K} = \frac{8B}{aA}(x_+)^{a-1}\rho\left(1 + (x_+)^{-1/2}\theta^+\theta^- G\right). \tag{2.8.12}$$

Since $H^2 = 0$, it follows that the surface $\mathcal{S}$ admits umbilic points along the curve defined by $\mathcal{K} = 0$, which lies on the surface $\mathcal{S}$. The subalgebra of the Lie algebra for the classical GC equation (2.2.13) analogous to subalgebra $\mathfrak{g}_{76}$ is $L'_{1,2} = \{e_3 + ae_0\}$. The corresponding invariant solution is given by

$$\begin{aligned} &H(z,\bar{z}) = l_0 e^{-a(z+\bar{z})}, \qquad Q(z,\bar{z}) = k_0 e^{a(z+\bar{z})}, \\ &U(z,\bar{z}) = \frac{-2k_0}{l_0} e^{2a(z+\bar{z})}, \qquad \bar{Q}(z,\bar{z}) = k_0 e^{a(z+\bar{z})}, \qquad k_0, l_0 \in \mathbb{R}. \end{aligned} \tag{2.8.13}$$

In contrast to the bosonic SUSY case (2.8.9), the Gaussian curvature $\mathcal{K}$ vanishes for the classical solution (2.8.13) associated with the subalgebra $L'_{1,2}$. In both cases however, the mean curvature function $H$ is non-zero.

For the fermionic SUSY GC equations (2.4.57), we present the following two examples.

**Example 3.** In the case of the subalgebra $\mathfrak{h}_{124} = \{P_+ + \epsilon P_- + aK_0, \epsilon = \pm 1, a \neq 0\}$, the orbit of the corresponding group of the fermionic SUSY GC equations (2.4.57) can be parametrized as follows

$$\begin{aligned} &H = e^{-ax_+}[h_0(\xi) + \theta^+\theta^- h_1(\xi)], \qquad R^+ = r_0^+(\xi) + \theta^+\theta^- r_1^+(\xi), \\ &Q^+ = e^{ax_+}[q_0^+(\xi) + \theta^+\theta^- q_1^+(\xi)], \qquad T^- = r_0^-(\xi) + \theta^+\theta^- r_1^-(\xi), \\ &Q^- = e^{ax_+}[q_0^-(\xi) + \theta^+\theta^- q_1^-(\xi)], \qquad \phi = \varphi_0(\xi) + \theta^+\theta^-\varphi_1(\xi) + 2ax_+, \\ &f = \psi(\xi), \end{aligned} \tag{2.8.14}$$

where the fermionic functions $H, Q^\pm, R^+$ and $T^-$ are expressed in terms of the bosonic symmetry variable $\xi = x_+ - \epsilon x_-$ and the fermionic symmetry variables $\theta^+$ and $\theta^-$, while the bosonic functions $\varphi_0, \varphi_1$ and $\psi$ are expressed in terms of $\xi$ only. A corresponding invariant solution is given by

$$\begin{aligned} &H = -2\underline{C}_0^+\underline{C}_0^- e^{-ax_+}\left[\epsilon e^{-\varphi_0}\underline{m}_0^+ + i\theta^+\theta^-(e^{-\varphi_0}\underline{m}_0^+)_\xi\right], \\ &Q^+ = -e^{ax_+}\underline{C}_0^+\underline{C}_0^-\left[\underline{m}_0^+ + i\theta^+\theta^-((\underline{m}_0^+)_\xi + \epsilon a\underline{m}_0^+)\right], \\ &Q^- = e^{ax_+}\underline{C}_0^+\underline{C}_0^-\left[\underline{m}_0^- + i\theta^+\theta^-(\epsilon a\underline{m}_0^- + (\underline{m}_0^-)_\xi)\right], \\ &\phi = \varphi_0(\xi) + i\theta^+\theta^-(\varphi_0)_\xi + 2ax_+, \quad R^+ = \underline{C}_0^+, \quad T^- = \underline{C}_0^-, \quad f = \psi(\xi), \end{aligned} \tag{2.8.15}$$

where the fermionic functions $\underline{m}_0^+$, $\underline{m}_0^-$ and the bosonic function $\varphi_0$ of the symmetry variable $\xi$ satisfy the differential constraint

$$[e^{-\varphi_0}(\underline{m}_0^- - \epsilon\underline{m}_0^+)]_\xi + \epsilon a\underline{m}_0^- e^{-\varphi_0} = 0. \tag{2.8.16}$$

Here $\psi$ is an arbitrary bosonic function of $\xi$, while $\underline{C}_0^+$ and $\underline{C}_0^-$ are arbitrary fermionic constants.

The first and second fundamental forms of the surface $\mathcal{S}$ associated with the solution (2.8.15) are given by

$$
\begin{aligned}
I =& \psi e^{\varphi_0+2ax_+}d_+d_-\left[1+\theta^+\theta^-\varphi_1\right],\\
II =& e^{ax_+}\underline{C}_0^+\underline{C}_0^-\psi\Big[d_+^2\left(\underline{m}_0^+ + i\theta^+\theta^-\left[(\underline{m}_0^+)_\xi+\epsilon a\underline{m}_0^+\right]\right)\\
&-2d_+d_-\left(\epsilon\underline{m}_0^+ + i\theta^+\theta^-\left[(\underline{m}_0^+)_\xi-\epsilon i\varphi_1\underline{m}_0^+-(\varphi_0)_\xi\underline{m}_0^+\right]\right)\\
&+d_-^2\left(\underline{m}_0^- + i\theta^+\theta^-\left[(\underline{m}_0^-)_\xi+\epsilon a\underline{m}_0^-\right]\right)\Big].
\end{aligned}
\tag{2.8.17}
$$

The Gaussian curvature (2.5.10) takes the form

$$
\mathcal{K}=0. \tag{2.8.18}
$$

In particular when $a=0$, which corresponds to the subalgebra $\mathfrak{h}_{14}=\{P_+ + \epsilon P_-\}$, the orbits of the group of the fermionic SUSY GC equations (2.4.57) can be parametrized in such a way that $H, Q^\pm, R^+$ and $T^-$ are fermionic functions of the bosonic symmetry variable $\xi = x_- - \epsilon x_+$, and the fermionic coordinates $\theta^+$ and $\theta^-$ while $\phi$ is a bosonic function of $\xi,\theta^+$ and $\theta^-$, and $\psi$ is a bosonic function of $\xi$ only. Under the assumption that the unknown functions take the form

$$
\begin{aligned}
H &= h_0(\xi)+\theta^+\theta^-h_1(\xi), & R^+ &= r_0^+(\xi)+\theta^+\theta^-r_1^+(\xi),\\
Q^\pm &= q_0^\pm(\xi)+\theta^+\theta^-q_1^\pm(\xi), & \phi &= \varphi_0(\xi)+\theta^+\theta^-\varphi_1(\xi),\\
T^- &= t_0^-(\xi)+\theta^+\theta^-t_1^-(\xi), & f &= \psi(\xi),
\end{aligned}
\tag{2.8.19}
$$

the corresponding invariant solution of the fermionic SUSY GC equations (2.4.57) is given by

$$
\begin{aligned}
H =& 2\underline{C}_0^-\underline{C}_0^+\underline{l}\left[\int e^{-\varphi_0}d\xi+i\theta^+\theta^-e^{-\varphi_0}\right]+\underline{C}, \qquad \epsilon=1,\\
Q^+ =& \underline{C}_0^-\underline{C}_0^+\underline{l}e^{\varphi_0}\int e^{-\varphi_0}d\xi+\underline{C}_0^-B_0^+e^{\varphi_0}\\
&+i\theta^+\theta^-\underline{C}_0^-\left[\underline{C}_0^+\underline{l}\left(e^{\varphi_0}(\varphi_0)_\xi\int e^{-\varphi_0}d\xi+1\right)+B_0^+e^{\varphi_0}(\varphi_0)_\xi\right],\\
Q^- =& \underline{C}_0^+\underline{C}_0^-\underline{l}e^{\varphi_0}\int e^{-\varphi_0}d\xi+\underline{C}_0^+B_0^-e^{\varphi_0}\\
&+i\theta^+\theta^-\underline{C}_0^+\left[\underline{C}_0^-\underline{l}\left(e^{\varphi_0}(\varphi_0)_\xi\int e^{-\varphi_0}d\xi+1\right)+B_0^-e^{\varphi_0}(\varphi_0)_\xi\right],\\
R^+ =& \underline{C}_0^+, \qquad T^-=\underline{C}_0^-,\\
\phi =& \varphi_0(\xi)+i\theta^+\theta^-(\varphi_0(\xi))_\xi, \qquad f=\psi(\xi),
\end{aligned}
\tag{2.8.20}
$$

where $\varphi_0$ and $\psi$ are bosonic functions of the symmetry variable $\xi = x_- - x_+$, while $\underline{C}_0^\pm, \underline{C}$ and $\underline{l}$ are arbitrary fermionic constants and $B_0^\pm$ are bosonic constants satisfying the algebraic constraint

$$\underline{C}_0^+ B_0^- + \underline{C}_0^- B_0^+ = 0. \tag{2.8.21}$$

For the solution (2.8.20), the tangent vectors are linearly dependent, so the immersion defines curves instead of surfaces.

**Example 4.** For the subalgebra $\mathfrak{h}_{35} = \{K_1^f + aK_0 + bC_0, a \neq 0, b \neq 0\}$, we obtain the following parametrization of the orbit of the corresponding supergroup of the fermionic SUSY GC equations (2.4.57)

$$\begin{aligned}
&H = (x_+)^{(a-b)/2}[h_0(x_-) + \eta\theta^- h_1(x_-)],\\
&R^+ = (x_+)^{-1/2}[r_0^+(x_-) + \eta\theta^- r_1^+(x_-)],\\
&Q^+ = (x_+)^{-(a+b+2)/2}[q_0^+(x_-) + \eta\theta^- q_1^+(x_-)],\\
&T^- = t_0^-(x_-) + \eta\theta^- t_1^-(x_-),\\
&Q^- = (x_+)^{-(a+b)/2}[q_0^-(x_-) + \eta\theta^- q_1^-(x_-)],\\
&\phi = \varphi_0(x_-) + \eta\theta^-\varphi_1(x_-) - \frac{2a+1}{2}\ln x_+,\\
&f = (x_+)^b\psi(x_-),
\end{aligned}$$

where the bosonic symmetry variable is $x_-$ and the fermionic symmetry variables are $\eta = (x_+)^{-1/2}\theta^+$ and $\theta^-$. A corresponding invariant solution of the fermionic SUSY GC equations (2.4.57) has the form

$$\begin{aligned}
&H = (x_+)^{(a-b)/2} e^{A_0(a-b)x_-/2E_0}\,\Big[\underline{C}\\
&\qquad + i(x_+)^{-1/2}\theta^+\theta^-(a-b+1)A_0\underline{C}_0^+ e^{A_0 x_-/2E_0}\Big],\\
&Q^+ = \underline{C}_0^+(x_+)^{-(a+b+2)/2}\left[E_0 + (x_+)^{-1/2}\theta^+\theta^- E_1\right],\\
&R^+ = \underline{C}_0^+(x_+)^{-1/2}, \quad T^- = \underline{C}_0^+, \qquad f = (x_+)^b\psi(x_-),\\
&Q^- = A_0\underline{C}_0^+(x_+)^{-(a+b)/2}\left[1 + (x_+)^{-1/2}\theta^+\theta^-\frac{E_1}{E_0}\right],\\
&\phi = \frac{A_0}{2E_0}(b-a-1)x_- + (x_+)^{-1/2}\theta^+\theta^-\varphi_1(x_-) - \frac{2a+1}{2}\ln x_+,
\end{aligned} \tag{2.8.22}$$

where the bosonic function $\varphi_1$ of $x_-$ satisfies the constraint

$$\underline{C}_0^+\varphi_1 = \frac{E_1}{E_0}\underline{C}_0^+ + i\frac{(a-b)}{4E_0}\underline{C}e^{-A_0x_-/2E_0}, \tag{2.8.23}$$

and where $\psi$ is an arbitrary bosonic function of $x_-$. Here $\underline{C}_0^+$ and $\underline{C}$ are arbitrary fermionic constants while $E_0, E_1$ and $A_0$ are arbitrary bosonic constants.

The first and second fundamental forms of the surface $\mathcal{S}$ (2.8.22) are given by

$$\begin{aligned}I =&(x_+)^{(2b-2a-1)/2}\exp\left(\frac{A_0}{2E_0}(b-a-1)x_-\right)\psi\\&\times d_+d_-(1+(x_+)^{-1/2}\theta^+\theta^-\varphi_1),\\ I\!I =&(x_+)^{(b-a)/2}\psi\Big[\underline{C}_0^+(x_+)^{-1}d_+^2(E_0+(x_+)^{-1/2}\theta^+\theta^-E_1)\\&+A_0\underline{C}_0^+d_-^2(1+(x_+)^{-1/2}\theta^+\theta^-E_1/E_0)\\&+(x_+)^{-1/2}e^{A_0x_-/2E_0}d_+d_-\,(\underline{C}\\&+\ (x_+)^{-1/2}\theta^+\theta^-\left[i\underline{C}\varphi_1+(a-b+1)A_0\underline{C}_0^+e^{A_0x_-/2E_0}\right])\Big].\end{aligned}$$

The Gaussian curvature (2.5.10) takes the form

$$\mathcal{K}=0. \tag{2.8.24}$$

Note that in our fermionic SUSY adaptation of the classical geometric interpretation of surfaces in $\mathbb{R}^3$, the surfaces obtained in the two examples are composed of parabolic points.

## 2.9. Conclusions

In this paper, we formulate bosonic and fermionic SUSY extensions of the GW and GC equations for smooth conformally parametrized surfaces immersed in a Grassmann superspace ($\mathbb{R}^{(2,1|2)}$ for the bosonic extension and $\mathbb{R}^{(1,1|3)}$ for the fermionic extension). For both SUSY extensions, the GW equations are determined by a moving frame formalism. In the bosonic case, the potential matrices $A_+$ and $A_-$ can be written in subblock form involving both bosonic and fermionic components. In contrast, in the fermionic case, the matrices $A_+$ and $A_-$ are expressed in terms of fermionic quantities only. In the bosonic case, the ZCC for the SUSY GW equations lead to six linearly independent SUSY GC equations in five bosonic functions and six fermionic functions. For the fermionic case, there are four linearly independent SUSY GC equations in two bosonic functions and five fermionic functions. It is interesting to note that, in the latter case, the SUSY GW and SUSY GC equations resemble the form of the classical equations. In the bosonic case, the ZCC involves a diagonal matrix $E$ in addition to the potential matrices $A_+$ and $A_-$. This diagonal matrix becomes the identity matrix in the fermionic case. In both cases, the induced metric involves a bosonic function $f$ of $x_+$ and $x_-$. In the bosonic case, $f$ is bodiless and nilpotent of order $k$, while in the fermionic case the function $f$ may or may not be bodiless.

For both SUSY GC systems we determine Lie superalgebras of infinitesimal symmetries which generate Lie point symmetry transformations. For both SUSY extensions the symmetry superalgebras include translations in the bosonic independent variables $x_+$ and $x_-$, four dilations and the SUSY operators $J_+$ and $J_-$. The fermionic case contains an additional translation in the direction of the mean curvature $H$. It should be noted that, in the classical case, the Lie point symmetry algebra contains two copies of the Virasoro algebra, whereas such Virasoro algebras do not appear for either the bosonic or the fermionic SUSY GC equations. For both cases, a classification of all one-dimensional subalgebras of the symmetry superalgebra into conjugacy classes is performed. It should be observed that the symmetries of both the classical and the bosonic SUSY GC equations contain a center, whereas the symmetries of the fermionic SUSY GC equations do not. Consequently, the classification lists (by conjugacy classes under the action of the associated supergroup) contain 99 one-dimensional subalgebras for the bosonic case and 199 one-dimensional subalgebras for the fermionic case. For each of the two SUSY extensions we construct two invariant solutions of the SUSY GC equations and compare them with solutions of the classical GC equations invariant under similar one-dimensional subalgebras.

The first and second fundamental forms for conformally parametrized surfaces in the superspace are established for both the bosonic and fermionic SUSY extensions of the GC equations. The determinants of the induced metric differs in their signs from the classical case. Also, we establish an analogue of the Bonnet theorem for the SUSY GC equations.

This research could be extended in several directions. It could be beneficial to compute an exhaustive list of all symmetries of the SUSY GC equations and to compare them to the classical case. The computation of such a list would require the development of a computer Lie algebra application capable of handling both even and odd Grassmann variables. Another open problem to be considered is whether all integrable SUSY systems possess non-standard invariants. It could also be worth attempting to establish a SUSY analogue of Noether's theorem in order to study the conservation laws of such SUSY models. Finally, it would be interesting to investigate how integrable characteristics, such as Hamiltonian structure and conserved quantities manifest themselves in surfaces for the SUSY cases. These subjects will be investigated in our future work.

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## 2.10. Annexe. Classification of the one-dimensional subalgebras of the Lie superalgebras (2.6.2) and (2.6.4) and (2.6.5)

Table 2.4. Classification of the one-dimensional subalgebras of the symmetry superalgebra $\mathfrak{g}$ of the bosonic SUSY GW equations (2.4.46) into conjugacy classes. Here $\epsilon = \pm 1$, the parameters $a, b, m$ are non-zero bosonic constants and $\underline{\mu}$ and $\underline{\nu}$ are non-zero fermionic constants.

| No. | Subalgebra | No. | Subalgebra |
|---|---|---|---|
| $\mathfrak{g}_1$ | $\{K_1^b\}$ | $\mathfrak{g}_2$ | $\{P_+\}$ |
| $\mathfrak{g}_3$ | $\{\underline{\mu}J_+\}$ | $\mathfrak{g}_4$ | $\{P_+ + \underline{\mu}J_+\}$ |
| $\mathfrak{g}_5$ | $\{K_2^b\}$ | $\mathfrak{g}_6$ | $\{P_-\}$ |
| $\mathfrak{g}_7$ | $\{\underline{\nu}J_-\}$ | $\mathfrak{g}_8$ | $\{P_- + \underline{\nu}J_-\}$ |
| $\mathfrak{g}_9$ | $\{K_1^b + aK_2^b\}$ | $\mathfrak{g}_{10}$ | $\{K_1^b + \epsilon P_-\}$ |
| $\mathfrak{g}_{11}$ | $\{K_1^b + \underline{\nu}J_-\}$ | $\mathfrak{g}_{12}$ | $\{K_1^b + \epsilon P_- + \underline{\nu}J_-\}$ |
| $\mathfrak{g}_{13}$ | $\{K_2^b + \epsilon P_+\}$ | $\mathfrak{g}_{14}$ | $\{P_+ + \epsilon P_-\}$ |
| $\mathfrak{g}_{15}$ | $\{P_+ + \underline{\nu}J_-\}$ | $\mathfrak{g}_{16}$ | $\{P_+ + \epsilon P_- + \underline{\nu}J_-\}$ |
| $\mathfrak{g}_{17}$ | $\{K_2^b + \underline{\mu}J_+\}$ | $\mathfrak{g}_{18}$ | $\{P_- + \underline{\mu}J_+\}$ |
| $\mathfrak{g}_{19}$ | $\{\underline{\mu}J_+ + \underline{\nu}J_-\}$ | $\mathfrak{g}_{20}$ | $\{P_- + \underline{\mu}J_+ + \underline{\nu}J_-\}$ |
| $\mathfrak{g}_{21}$ | $\{K_2^b + \epsilon P_+ + \underline{\mu}J_+\}$ | $\mathfrak{g}_{22}$ | $\{P_+ + \epsilon P_- + \underline{\mu}J_+\}$ |
| $\mathfrak{g}_{23}$ | $\{P_+ + \underline{\mu}J_+ + \underline{\nu}J_-\}$ | $\mathfrak{g}_{24}$ | $\{P_+ + \epsilon P_- + \underline{\mu}J_+ + \underline{\nu}J_-\}$ |
| $\mathfrak{g}_{25}$ | $\{K_0\}$ | $\mathfrak{g}_{26}$ | $\{K_1^b + aK_0\}$ |
| $\mathfrak{g}_{27}$ | $\{K_0 + \epsilon P_+\}$ | $\mathfrak{g}_{28}$ | $\{K_0 + \underline{\mu}J_+\}$ |
| $\mathfrak{g}_{29}$ | $\{K_0 + \epsilon P_+ + \underline{\mu}J_+\}$ | $\mathfrak{g}_{30}$ | $\{K_2^b + aK_0\}$ |
| $\mathfrak{g}_{31}$ | $\{K_0 + \epsilon P_-\}$ | $\mathfrak{g}_{32}$ | $\{K_0 + \underline{\nu}J_-\}$ |
| $\mathfrak{g}_{33}$ | $\{K_0 + \epsilon P_- + \underline{\nu}J_-\}$ | $\mathfrak{g}_{34}$ | $\{K_1^b + aK_2^b + bK_0\}$ |
| $\mathfrak{g}_{35}$ | $\{K_1^b + aK_0 + \epsilon P_-\}$ | $\mathfrak{g}_{36}$ | $\{K_1^b + aK_0 + \underline{\nu}J_-\}$ |
| $\mathfrak{g}_{37}$ | $\{K_1^b + aK_0 + \epsilon P_- + \underline{\nu}J_-\}$ | $\mathfrak{g}_{38}$ | $\{K_2^b + aK_0 + \epsilon P_+\}$ |
| $\mathfrak{g}_{39}$ | $\{K_0 + \epsilon_1 P_+ + \epsilon_2 P_-\}$ | $\mathfrak{g}_{40}$ | $\{K_0 + \epsilon P_+ + \underline{\nu}J_-\}$ |
| $\mathfrak{g}_{41}$ | $\{K_0 + \epsilon_1 P_+ + \epsilon_2 P_- + \underline{\nu}J_-\}$ | $\mathfrak{g}_{42}$ | $\{K_2^b + aK_0 + \underline{\mu}J_+\}$ |
| $\mathfrak{g}_{43}$ | $\{K_0 + \epsilon P_- + \underline{\mu}J_+\}$ | $\mathfrak{g}_{44}$ | $\{K_0 + \underline{\mu}J_+ + \underline{\nu}J_-\}$ |
| $\mathfrak{g}_{45}$ | $\{K_0 + \epsilon P_- + \underline{\mu}J_+ + \underline{\nu}J_-\}$ | $\mathfrak{g}_{46}$ | $\{K_2^b + aK_0 + \epsilon P_+ + \underline{\mu}J_+\}$ |
| $\mathfrak{g}_{47}$ | $\{K_0 + \epsilon_1 P_+ + \epsilon_2 P_- + \underline{\mu}J_+\}$ | $\mathfrak{g}_{48}$ | $\{K_0 + \epsilon P_+ + \underline{\mu}J_+ + \underline{\nu}J_-\}$ |
| $\mathfrak{g}_{49}$ | $\{K_0 + \epsilon_1 P_+ + \epsilon_2 P_- + \underline{\mu}J_+ + \underline{\nu}J_-\}$ | $\mathfrak{g}_{50}$ | $\{C_0\}$ |
| $\mathfrak{g}_{51}$ | $\{K_1^b + aC_0\}$ | $\mathfrak{g}_{52}$ | $\{C_0 + \epsilon P_+\}$ |
| $\mathfrak{g}_{53}$ | $\{C_0 + \underline{\mu}J_+\}$ | $\mathfrak{g}_{54}$ | $\{C_0 + \epsilon P_+ + \underline{\mu}J_+\}$ |
| $\mathfrak{g}_{55}$ | $\{K_2^b + aC_0\}$ | $\mathfrak{g}_{56}$ | $\{C_0 + \epsilon P_-\}$ |

TABLE 2.4. (Continued)

| No. | Subalgebra |
|---|---|
| $\mathfrak{g}_{57}$ | $\{C_0 + \underline{\nu} J_-\}$ |
| $\mathfrak{g}_{58}$ | $\{C_0 + \epsilon P_- + \underline{\nu} J_-\}$ |
| $\mathfrak{g}_{59}$ | $\{K_1^b + aK_2^b + bC_0\}$ |
| $\mathfrak{g}_{60}$ | $\{K_1^b + aC_0 + \epsilon P_-\}$ |
| $\mathfrak{g}_{61}$ | $\{K_1^b + aC_0 + \underline{\nu} J_-\}$ |
| $\mathfrak{g}_{62}$ | $\{K_1^b + aC_0 + \epsilon P_- + \underline{\nu} J_-\}$ |
| $\mathfrak{g}_{63}$ | $\{K_2^b + aC_0 + \epsilon P_+\}$ |
| $\mathfrak{g}_{64}$ | $\{C_0 + \epsilon_1 P_+ + \epsilon_2 P_-\}$ |
| $\mathfrak{g}_{65}$ | $\{C_0 + \epsilon P_+ + \underline{\nu} J_-\}$ |
| $\mathfrak{g}_{66}$ | $\{C_0 + \epsilon_1 P_+ + \epsilon_2 P_- + \underline{\nu} J_-\}$ |
| $\mathfrak{g}_{67}$ | $\{K_2^b + aC_0 + \underline{\mu} J_+\}$ |
| $\mathfrak{g}_{68}$ | $\{C_0 + \epsilon P_- + \underline{\mu} J_+\}$ |
| $\mathfrak{g}_{69}$ | $\{C_0 + \underline{\mu} J_+ + \underline{\nu} J_-\}$ |
| $\mathfrak{g}_{70}$ | $\{C_0 + \epsilon P_- + \underline{\mu} J_+ + \underline{\nu} J_-\}$ |
| $\mathfrak{g}_{71}$ | $\{K_2^b + aC_0 + \epsilon P_+ + \underline{\mu} J_+\}$ |
| $\mathfrak{g}_{72}$ | $\{C_0 + \epsilon_1 P_+ + \epsilon_2 P_- + \underline{\mu} J_+\}$ |
| $\mathfrak{g}_{73}$ | $\{C_0 + \epsilon P_+ + \underline{\mu} J_+ + \underline{\nu} J_-\}$ |
| $\mathfrak{g}_{74}$ | $\{C_0 + \epsilon_1 P_+ + \epsilon_2 P_- + \underline{\mu} J_+ + \underline{\nu} J_-\}$ |
| $\mathfrak{g}_{75}$ | $\{K_0 + mC_0\}$ |
| $\mathfrak{g}_{76}$ | $\{K_1^b + aK_0 + mC_0\}$ |
| $\mathfrak{g}_{77}$ | $\{K_0 + mC_0 + \epsilon P_+\}$ |
| $\mathfrak{g}_{78}$ | $\{K_0 + mC_0 + \underline{\mu} J_+\}$ |
| $\mathfrak{g}_{79}$ | $\{K_0 + mC_0 + \epsilon P_+ + \underline{\mu} J_+\}$ |
| $\mathfrak{g}_{80}$ | $\{K_2^b + aK_0 + mC_0\}$ |
| $\mathfrak{g}_{81}$ | $\{K_0 + mC_0 + \epsilon P_-\}$ |
| $\mathfrak{g}_{82}$ | $\{K_0 + mC_0 + \underline{\nu} J_-\}$ |
| $\mathfrak{g}_{83}$ | $\{K_0 + mC_0 + \epsilon P_- + \underline{\nu} J_-\}$ |
| $\mathfrak{g}_{84}$ | $\{K_1^b + aK_2^b + bK_0 + mC_0\}$ |
| $\mathfrak{g}_{85}$ | $\{K_1^b + aK_0 + mC_0 + \epsilon P_-\}$ |
| $\mathfrak{g}_{86}$ | $\{K_1^b + aK_0 + mC_0 + \underline{\nu} J_-\}$ |
| $\mathfrak{g}_{87}$ | $\{K_1^b + aK_0 + mC_0 + \epsilon P_- + \underline{\nu} J_-\}$ |
| $\mathfrak{g}_{88}$ | $\{K_2^b + aK_0 + mC_0 + \epsilon P_+\}$ |
| $\mathfrak{g}_{89}$ | $\{K_0 + mC_0 + \epsilon_1 P_+ + \epsilon_2 P_-\}$ |
| $\mathfrak{g}_{90}$ | $\{K_0 + mC_0 + \epsilon P_+ + \underline{\nu} J_-\}$ |
| $\mathfrak{g}_{91}$ | $\{K_0 + mC_0 + \epsilon_1 P_+ + \epsilon_2 P_- + \underline{\nu} J_-\}$ |
| $\mathfrak{g}_{92}$ | $\{K_2^b + aK_0 + mC_0 + \underline{\mu} J_+\}$ |
| $\mathfrak{g}_{93}$ | $\{K_0 + mC_0 + \epsilon P_- + \underline{\mu} J_+\}$ |
| $\mathfrak{g}_{94}$ | $\{K_0 + mC_0 + \underline{\mu} J_+ + \underline{\nu} J_-\}$ |
| $\mathfrak{g}_{95}$ | $\{K_0 + mC_0 + \epsilon P_- + \underline{\mu} J_+ + \underline{\nu} J_-\}$ |
| $\mathfrak{g}_{96}$ | $\{K_2^b + aK_0 + mC_0 + \epsilon P_+ + \underline{\mu} J_+\}$ |
| $\mathfrak{g}_{97}$ | $\{K_0 + mC_0 + \epsilon_1 P_+ + \epsilon_2 P_- + \underline{\mu} J_+\}$ |
| $\mathfrak{g}_{98}$ | $\{K_0 + mC_0 + \epsilon P_+ + \underline{\mu} J_+ + \underline{\nu} J_-\}$ |
| $\mathfrak{g}_{99}$ | $\{K_0 + mC_0 + \epsilon_1 P_+ + \epsilon_2 P_- + \underline{\mu} J_+ + \underline{\nu} J_-\}$ |

TABLE 2.5. Classification of the one-dimensional subalgebras of the symmetry superalgebra $\mathfrak{h}$ of the equations (2.4.57)) into conjugacy classes. Here $\epsilon = \pm 1$, the parameters $a, b$ are non-zero bosonic constants, $\underline{\mu}$,$\underline{\nu}$,$\underline{\zeta}$ are non-zero fermionic constants.

| No. | Subalgebra | No. | Subalgebra |
|---|---|---|---|
| $\mathfrak{h}_1$ | $\{K_1^f\}$ | $\mathfrak{h}_2$ | $\{P_+\}$ |
| $\mathfrak{h}_3$ | $\{\underline{\mu}J_+\}$ | $\mathfrak{h}_4$ | $\{P_+ + \underline{\mu}J_+\}$ |
| $\mathfrak{h}_5$ | $\{K_2^f\}$ | $\mathfrak{h}_6$ | $\{P_-\}$ |
| $\mathfrak{h}_7$ | $\{\underline{\nu}J_-\}$ | $\mathfrak{h}_8$ | $\{P_- + \underline{\nu}J_-\}$ |
| $\mathfrak{h}_9$ | $\{K_1^f + aK_2^f\}$ | $\mathfrak{h}_{10}$ | $\{K_1^f + \epsilon P_-\}$ |
| $\mathfrak{h}_{11}$ | $\{K_1^f + \underline{\nu}J_-\}$ | $\mathfrak{h}_{12}$ | $\{K_1^f + \epsilon P_- + \underline{\nu}J_-\}$ |
| $\mathfrak{h}_{13}$ | $\{K_2^f + \epsilon P_+\}$ | $\mathfrak{h}_{14}$ | $\{P_+ + \epsilon P_-\}$ |
| $\mathfrak{h}_{15}$ | $\{P_+ + \underline{\nu}J_-\}$ | $\mathfrak{h}_{16}$ | $\{P_+ + \epsilon P_- + \underline{\nu}J_-\}$ |
| $\mathfrak{h}_{17}$ | $\{K_2^f + \underline{\mu}J_+\}$ | $\mathfrak{h}_{18}$ | $\{P_- + \underline{\mu}J_+\}$ |
| $\mathfrak{h}_{19}$ | $\{\underline{\mu}J_+ + \underline{\nu}J_-\}$ | $\mathfrak{h}_{20}$ | $\{P_- + \underline{\mu}J_+ + \underline{\nu}J_-\}$ |
| $\mathfrak{h}_{21}$ | $\{K_2^f + \epsilon P_+ + \underline{\mu}J_+\}$ | $\mathfrak{h}_{22}$ | $\{P_+ + \epsilon P_- + \underline{\mu}J_+\}$ |
| $\mathfrak{h}_{23}$ | $\{P_+ + \underline{\mu}J_+ + \underline{\nu}J_-\}$ | $\mathfrak{h}_{24}$ | $\{P_+ + \epsilon P_- + \underline{\mu}J_+ + \underline{\nu}J_-\}$ |
| $\mathfrak{h}_{25}$ | $\{\underline{\zeta}W\}$ | $\mathfrak{h}_{26}$ | $\{K_0\}$ |
| $\mathfrak{h}_{27}$ | $\{C_0\}$ | $\mathfrak{h}_{28}$ | $\{K_0 + aC_0\}$ |
| $\mathfrak{h}_{29}$ | $\{K_0 + \underline{\zeta}W\}$ | $\mathfrak{h}_{30}$ | $\{C_0 + \underline{\zeta}W\}$ |
| $\mathfrak{h}_{31}$ | $\{K_0 + aC_0 + \underline{\zeta}W\}$ | $\mathfrak{h}_{32}$ | $\{K_1^f + \underline{\zeta}W\}$ |
| $\mathfrak{h}_{33}$ | $\{K_1^f + aK_0\}$ | $\mathfrak{h}_{34}$ | $\{K_1^f + aC_0\}$ |
| $\mathfrak{h}_{35}$ | $\{K_1^f + aK_0 + bC_0\}$ | $\mathfrak{h}_{36}$ | $\{K_1^f + aK_0 + \underline{\zeta}W\}$ |
| $\mathfrak{h}_{37}$ | $\{K_1^f + aC_0 + \underline{\zeta}W\}$ | $\mathfrak{h}_{38}$ | $\{K_1^f + aK_0 + bC_0 + \underline{\zeta}W\}$ |
| $\mathfrak{h}_{39}$ | $\{P_+ + \underline{\zeta}W\}$ | $\mathfrak{h}_{40}$ | $\{K_0 + \epsilon P_+\}$ |
| $\mathfrak{h}_{41}$ | $\{C_0 + \epsilon P_+\}$ | $\mathfrak{h}_{42}$ | $\{K_0 + aC_0 + \epsilon P_+\}$ |
| $\mathfrak{h}_{43}$ | $\{K_0 + \epsilon P_+ + \underline{\zeta}W\}$ | $\mathfrak{h}_{44}$ | $\{C_0 + \epsilon P_+ + \underline{\zeta}W\}$ |
| $\mathfrak{h}_{45}$ | $\{K_0 + aC_0 + \epsilon P_+ + \underline{\zeta}W\}$ | $\mathfrak{h}_{46}$ | $\{\underline{\mu}J_+ + \underline{\zeta}W\}$ |
| $\mathfrak{h}_{47}$ | $\{K_0 + \underline{\mu}J_+\}$ | $\mathfrak{h}_{48}$ | $\{C_0 + \underline{\mu}J_+\}$ |
| $\mathfrak{h}_{49}$ | $\{K_0 + aC_0 + \underline{\mu}J_+\}$ | $\mathfrak{h}_{50}$ | $\{K_0 + \underline{\mu}J_+ + \underline{\zeta}W\}$ |
| $\mathfrak{h}_{51}$ | $\{C_0 + \underline{\mu}J_+ + \underline{\zeta}W\}$ | $\mathfrak{h}_{52}$ | $\{K_0 + aC_0 + \underline{\mu}J_+ + \underline{\zeta}W\}$ |
| $\mathfrak{h}_{53}$ | $\{P_+ + \underline{\mu}J_+ + \underline{\zeta}W\}$ | $\mathfrak{h}_{54}$ | $\{K_0 + \epsilon P_+ + \underline{\mu}J_+\}$ |
| $\mathfrak{h}_{55}$ | $\{C_0 + \epsilon P_+ + \underline{\mu}J_+\}$ | $\mathfrak{h}_{56}$ | $\{K_0 + aC_0 + \epsilon P_+ + \underline{\mu}J_+\}$ |
| $\mathfrak{h}_{57}$ | $\{K_0 + \epsilon P_+ + \underline{\mu}J_+ + \underline{\zeta}W\}$ | $\mathfrak{h}_{58}$ | $\{C_0 + \epsilon P_+ + \underline{\mu}J_+ + \underline{\zeta}W\}$ |
| $\mathfrak{h}_{59}$ | $\{K_0 + aC_0 + \epsilon P_+ + \underline{\mu}J_+ + \underline{\zeta}W\}$ | $\mathfrak{h}_{60}$ | $\{K_2^f + \underline{\zeta}W\}$ |
| $\mathfrak{h}_{61}$ | $\{K_2^f + aK_0\}$ | $\mathfrak{h}_{62}$ | $\{K_2^f + aC_0\}$ |
| $\mathfrak{h}_{63}$ | $\{K_2^f + aK_0 + bC_0\}$ | $\mathfrak{h}_{64}$ | $\{K_2^f + aK_0 + \underline{\zeta}W\}$ |
| $\mathfrak{h}_{65}$ | $\{K_2^f + aC_0 + \underline{\zeta}W\}$ | $\mathfrak{h}_{66}$ | $\{K_2^f + aK_0 + bC_0 + \underline{\zeta}W\}$ |
| $\mathfrak{h}_{67}$ | $\{P_- + \underline{\zeta}W\}$ | $\mathfrak{h}_{68}$ | $\{K_0 + \epsilon P_-\}$ |
| $\mathfrak{h}_{69}$ | $\{C_0 + \epsilon P_-\}$ | $\mathfrak{h}_{70}$ | $\{K_0 + aC_0 + \epsilon P_-\}$ |
| $\mathfrak{h}_{71}$ | $\{K_0 + \epsilon P_- + \underline{\zeta}W\}$ | $\mathfrak{h}_{72}$ | $\{C_0 + \epsilon P_- + \underline{\zeta}W\}$ |

TABLE 2.5. (Continued)

| No. | Subalgebra | No. | Subalgebra |
|---|---|---|---|
| $\mathfrak{h}_{73}$ | $\{K_0 + aC_0 + \epsilon P_- + \underline{\zeta}W\}$ | $\mathfrak{h}_{74}$ | $\{\underline{\nu}J_- + \underline{\zeta}W\}$ |
| $\mathfrak{h}_{75}$ | $\{K_0 + \underline{\nu}J_-\}$ | $\mathfrak{h}_{76}$ | $\{C_0 + \underline{\nu}J_-\}$ |
| $\mathfrak{h}_{77}$ | $\{K_0 + aC_0 + \underline{\nu}J_-\}$ | $\mathfrak{h}_{78}$ | $\{K_0 + \underline{\nu}J_- + \underline{\zeta}W\}$ |
| $\mathfrak{h}_{79}$ | $\{C_0 + \underline{\nu}J_- + \underline{\zeta}W\}$ | $\mathfrak{h}_{80}$ | $\{K_0 + aC_0 + \underline{\nu}J_- + \underline{\zeta}W\}$ |
| $\mathfrak{h}_{81}$ | $\{P_- + \underline{\nu}J_- + \underline{\zeta}W\}$ | $\mathfrak{h}_{82}$ | $\{K_0 + \epsilon P_- + \underline{\nu}J_-\}$ |
| $\mathfrak{h}_{83}$ | $\{C_0 + \epsilon P_- + \underline{\nu}J_-\}$ | $\mathfrak{h}_{84}$ | $\{K_0 + aC_0 + \epsilon P_- + \underline{\nu}J_-\}$ |
| $\mathfrak{h}_{85}$ | $\{K_0 + \epsilon P_- + \underline{\nu}J_- + \underline{\zeta}W\}$ | $\mathfrak{h}_{86}$ | $\{C_0 + \epsilon P_- + \underline{\nu}J_- + \underline{\zeta}W\}$ |
| $\mathfrak{h}_{87}$ | $\{K_0 + aC_0 + \epsilon P_- + \underline{\nu}J_- + \underline{\zeta}W\}$ | $\mathfrak{h}_{88}$ | $\{K_1^f + aK_2^f + \underline{\zeta}W\}$ |
| $\mathfrak{h}_{89}$ | $\{K_0 + aK_1^f + bK_2^f\}$ | $\mathfrak{h}_{90}$ | $\{C_0 + aK_1^f + bK_2^f\}$ |
| $\mathfrak{h}_{91}$ | $\{K_0 + aC_0 + bK_1^f + cK_2^f\}$ | $\mathfrak{h}_{92}$ | $\{K_0 + aK_1^f + bK_2^f + \underline{\zeta}W\}$ |
| $\mathfrak{h}_{93}$ | $\{C_0 + aK_1^f + bK_2^f + \underline{\zeta}W\}$ | $\mathfrak{h}_{94}$ | $\{K_0 + aC_0 + bK_1^f + cK_2^f + \underline{\zeta}W\}$ |
| $\mathfrak{h}_{95}$ | $\{K_1^f + \epsilon P_- + \underline{\zeta}W\}$ | $\mathfrak{h}_{96}$ | $\{K_0 + aK_1^f + \epsilon P_-\}$ |
| $\mathfrak{h}_{97}$ | $\{C_0 + aK_1^f + \epsilon P_-\}$ | $\mathfrak{h}_{98}$ | $\{K_0 + aC_0 + bK_1^f + \epsilon P_-\}$ |
| $\mathfrak{h}_{99}$ | $\{K_0 + aK_1^f + \epsilon P_- + \underline{\zeta}W\}$ | $\mathfrak{h}_{100}$ | $\{C_0 + aK_1^f + \epsilon P_- + \underline{\zeta}W\}$ |
| $\mathfrak{h}_{101}$ | $\{K_0 + aC_0 + bK_1^f + \epsilon P_- + \underline{\zeta}W\}$ | $\mathfrak{h}_{102}$ | $\{K_1^f + \underline{\nu}J_- + \underline{\zeta}W\}$ |
| $\mathfrak{h}_{103}$ | $\{K_0 + aK_1^f + \underline{\nu}J_-\}$ | $\mathfrak{h}_{104}$ | $\{C_0 + aK_1^f + \underline{\nu}J_-\}$ |
| $\mathfrak{h}_{105}$ | $\{K_0 + aC_0 + bK_1^f + \underline{\nu}J_-\}$ | $\mathfrak{h}_{106}$ | $\{K_0 + aK_1^f + \underline{\nu}J_- + \underline{\zeta}W\}$ |
| $\mathfrak{h}_{107}$ | $\{C_0 + aK_1^f + \underline{\nu}J_- + \underline{\zeta}W\}$ | $\mathfrak{h}_{108}$ | $\{K_0 + aC_0 + bK_1^f + \underline{\nu}J_- + \underline{\zeta}W\}$ |
| $\mathfrak{h}_{109}$ | $\{K_1^f + \epsilon P_- + \underline{\nu}J_- + \underline{\zeta}W\}$ | $\mathfrak{h}_{110}$ | $\{K_0 + aK_1^f + \epsilon P_- + \underline{\nu}J_-\}$ |
| $\mathfrak{h}_{111}$ | $\{C_0 + aK_1^f + \epsilon P_- + \underline{\nu}J_-\}$ | $\mathfrak{h}_{112}$ | $\{K_0 + aC_0 + bK_1^f + \epsilon P_- + \underline{\nu}J_-\}$ |
| $\mathfrak{h}_{113}$ | $\{K_0 + aK_1^f + \epsilon P_- + \underline{\nu}J_- + \underline{\zeta}W\}$ | $\mathfrak{h}_{114}$ | $\{C_0 + aK_1^f + \epsilon P_- + \underline{\nu}J_- + \underline{\zeta}W\}$ |
| $\mathfrak{h}_{115}$ | $\{K_0 + aC_0 + bK_1^f + \epsilon P_-$ $+\underline{\nu}J_- + \underline{\zeta}W\}$ | $\mathfrak{h}_{116}$ | $\{K_2^f + \epsilon P_+ + \underline{\zeta}W\}$ |
| $\mathfrak{h}_{117}$ | $\{K_0 + aK_2^f + \epsilon P_+\}$ | $\mathfrak{h}_{118}$ | $\{C_0 + aK_2^f + \epsilon P_+\}$ |
| $\mathfrak{h}_{119}$ | $\{K_0 + aC_0 + bK_2^f + \epsilon P_+\}$ | $\mathfrak{h}_{120}$ | $\{K_0 + aK_2^f + \epsilon P_+ + \underline{\zeta}W\}$ |
| $\mathfrak{h}_{121}$ | $\{C_0 + aK_2^f + \epsilon P_+ + \underline{\zeta}W\}$ | $\mathfrak{h}_{122}$ | $\{K_0 + aC_0 + bK_2^f + \epsilon P_+ + \underline{\zeta}W\}$ |
| $\mathfrak{h}_{123}$ | $\{P_+ + \epsilon P_- + \underline{\zeta}W\}$ | $\mathfrak{h}_{124}$ | $\{P_+ + \epsilon P_- + aK_0\}$ |
| $\mathfrak{h}_{125}$ | $\{P_+ + \epsilon P_- + aC_0\}$ | $\mathfrak{h}_{126}$ | $\{P_+ + \epsilon P_- + aK_0 + bC_0\}$ |
| $\mathfrak{h}_{127}$ | $\{P_+ + \epsilon P_- + aK_0 + \underline{\zeta}W\}$ | $\mathfrak{h}_{128}$ | $\{P_+ + \epsilon P_- + aC_0 + \underline{\zeta}W\}$ |
| $\mathfrak{h}_{129}$ | $\{P_+ + \epsilon P_- + aK_0 + bC_0\underline{\zeta}W\}$ | $\mathfrak{h}_{130}$ | $\{P_+ + \underline{\nu}J_- + \underline{\zeta}W\}$ |
| $\mathfrak{h}_{131}$ | $\{K_0 + \epsilon P_+ + \underline{\nu}J_-\}$ | $\mathfrak{h}_{132}$ | $\{C_0 + \epsilon P_+ + \underline{\nu}J_-\}$ |
| $\mathfrak{h}_{133}$ | $\{K_0 + aC_0 + \epsilon P_+ + \underline{\nu}J_-\}$ | $\mathfrak{h}_{134}$ | $\{K_0 + \epsilon P_+ + \underline{\nu}J_- + \underline{\zeta}W\}$ |
| $\mathfrak{h}_{135}$ | $\{C_0 + \epsilon P_+ + \underline{\nu}J_- + \underline{\zeta}W\}$ | $\mathfrak{h}_{136}$ | $\{K_0 + aC_0 + \epsilon P_+ + \underline{\nu}J_- + \underline{\zeta}W\}$ |
| $\mathfrak{h}_{137}$ | $\{P_+ + \epsilon P_- + \underline{\nu}J_- + \underline{\zeta}W\}$ | $\mathfrak{h}_{138}$ | $\{P_+ + \epsilon P_- + aK_0 + \underline{\nu}J_-\}$ |
| $\mathfrak{h}_{139}$ | $\{P_+ + \epsilon P_- + aC_0 + \underline{\nu}J_-\}$ | $\mathfrak{h}_{140}$ | $\{P_+ + \epsilon P_- + aK_0 + bC_0 + \underline{\nu}J_-\}$ |
| $\mathfrak{h}_{141}$ | $\{P_+ + \epsilon P_- + aK_0 + \underline{\nu}J_- + \underline{\zeta}W\}$ | $\mathfrak{h}_{142}$ | $\{P_+ + \epsilon P_- + aC_0 + \underline{\nu}J_- + \underline{\zeta}W\}$ |
| $\mathfrak{h}_{143}$ | $\{P_+ + \epsilon P_- + aK_0 + bC_0$ $+\underline{\nu}J_- + \underline{\zeta}W\}$ | $\mathfrak{h}_{144}$ | $\{K_2^f + \underline{\mu}J_+ + \underline{\zeta}W\}$ |
| $\mathfrak{h}_{145}$ | $\{K_0 + aK_2^f + \underline{\mu}J_+\}$ | $\mathfrak{h}_{146}$ | $\{C_0 + aK_2^f + \underline{\mu}J_+\}$ |

TABLE 2.5. (Continued)

| No. | Subalgebra | No. | Subalgebra |
|---|---|---|---|
| $\mathfrak{h}_{147}$ | $\{K_0 + aC_0 + bK_2^f + \underline{\mu} J_+\}$ | $\mathfrak{h}_{148}$ | $\{K_0 + aK_2^f + \underline{\mu} J_+ + \underline{\zeta} W\}$ |
| $\mathfrak{h}_{149}$ | $\{C_0 + aK_2^f + \underline{\mu} J_+ + \underline{\zeta} W\}$ | $\mathfrak{h}_{150}$ | $\{K_0 + aC_0 + bK_2^f + \underline{\mu} J_+ + \underline{\zeta} W\}$ |
| $\mathfrak{h}_{151}$ | $\{P_- + \underline{\mu} J_+ + \underline{\zeta} W\}$ | $\mathfrak{h}_{152}$ | $\{K_0 + \epsilon P_- + \underline{\mu} J_+\}$ |
| $\mathfrak{h}_{153}$ | $\{C_0 + \epsilon P_- + \underline{\mu} J_+\}$ | $\mathfrak{h}_{154}$ | $\{K_0 + aC_0 + \epsilon P_- + \underline{\mu} J_+\}$ |
| $\mathfrak{h}_{155}$ | $\{K_0 + \epsilon P_- + \underline{\mu} J_+ + \underline{\zeta} W\}$ | $\mathfrak{h}_{156}$ | $\{C_0 + \epsilon P_- + \underline{\mu} J_+ + \underline{\zeta} W\}$ |
| $\mathfrak{h}_{157}$ | $\{K_0 + aC_0 + \epsilon P_- + \underline{\mu} J_+ + \underline{\zeta} W\}$ | $\mathfrak{h}_{158}$ | $\{\underline{\mu} J_+ + \underline{\nu} J_- + \underline{\zeta} W\}$ |
| $\mathfrak{h}_{159}$ | $\{K_0 + \underline{\mu} J_+ + \underline{\nu} J_-\}$ | $\mathfrak{h}_{160}$ | $\{C_0 + \underline{\mu} J_+ + \underline{\nu} J_-\}$ |
| $\mathfrak{h}_{161}$ | $\{K_0 + aC_0 + \underline{\mu} J_+ + \underline{\nu} J_-\}$ | $\mathfrak{h}_{162}$ | $\{K_0 + \underline{\mu} J_+ + \underline{\nu} J_- + \underline{\zeta} W\}$ |
| $\mathfrak{h}_{163}$ | $\{C_0 + \underline{\mu} J_+ + \underline{\nu} J_- + \underline{\zeta} W\}$ | $\mathfrak{h}_{164}$ | $\{K_0 + aC_0 + \underline{\mu} J_+ + \underline{\nu} J_- + \underline{\zeta} W\}$ |
| $\mathfrak{h}_{165}$ | $\{P_- + \underline{\mu} J_+ + \underline{\nu} J_- + \underline{\zeta} W\}$ | $\mathfrak{h}_{166}$ | $\{K_0 + \epsilon P_- + \underline{\mu} J_+ + \underline{\nu} J_-\}$ |
| $\mathfrak{h}_{167}$ | $\{C_0 + \epsilon P_- + \underline{\mu} J_+ + \underline{\nu} J_-\}$ | $\mathfrak{h}_{168}$ | $\{K_0 + aC_0 + \epsilon P_- + \underline{\mu} J_+ + \underline{\nu} J_-\}$ |
| $\mathfrak{h}_{169}$ | $\{K_0 + \epsilon P_- + \underline{\mu} J_+ + \underline{\nu} J_- + \underline{\zeta} W\}$ | $\mathfrak{h}_{170}$ | $\{C_0 + \epsilon P_- + \underline{\mu} J_+ + \underline{\nu} J_- + \underline{\zeta} W\}$ |
| $\mathfrak{h}_{171}$ | $\{K_0 + aC_0 + \epsilon P_- + \underline{\mu} J_+ + \underline{\nu} J_- + \underline{\zeta} W\}$ | $\mathfrak{h}_{172}$ | $\{K_2^f + \epsilon P_+ + \underline{\mu} J_+ + \underline{\zeta} W\}$ |
| $\mathfrak{h}_{173}$ | $\{K_0 + aK_2^f + \epsilon P_+ + \underline{\mu} J_+\}$ | $\mathfrak{h}_{174}$ | $\{C_0 + aK_2^f + \epsilon P_+ + \underline{\mu} J_+\}$ |
| $\mathfrak{h}_{175}$ | $\{K_0 + aC_0 + bK_2^f + \epsilon P_+ + \underline{\mu} J_+\}$ | $\mathfrak{h}_{176}$ | $\{K_0 + aK_2^f + \epsilon P_+ + \underline{\mu} J_+ + \underline{\zeta} W\}$ |
| $\mathfrak{h}_{177}$ | $\{C_0 + aK_2^f + \epsilon P_+ + \underline{\mu} J_+ + \underline{\zeta} W\}$ | $\mathfrak{h}_{178}$ | $\{K_0 + aC_0 + bK_2^f + \epsilon P_+ + \underline{\mu} J_+ + \underline{\zeta} W\}$ |
| $\mathfrak{h}_{179}$ | $\{P_+ + \epsilon P_- + \underline{\mu} J_+ + \underline{\zeta} W\}$ | $\mathfrak{h}_{180}$ | $\{P_+ + \epsilon P_- + aK_0 + \underline{\mu} J_+\}$ |
| $\mathfrak{h}_{181}$ | $\{P_+ + \epsilon P_- + aC_0 + \underline{\mu} J_+\}$ | $\mathfrak{h}_{182}$ | $\{P_+ + \epsilon P_- + aK_0 + bC_0 + \underline{\mu} J_+\}$ |
| $\mathfrak{h}_{183}$ | $\{P_+ + \epsilon P_- + aK_0 + \underline{\mu} J_+ + \underline{\zeta} W\}$ | $\mathfrak{h}_{184}$ | $\{P_+ + \epsilon P_- + aC_0 + \underline{\mu} J_+ + \underline{\zeta} W\}$ |
| $\mathfrak{h}_{185}$ | $\{P_+ + \epsilon P_- + aK_0 + bC_0 + \underline{\mu} J_+ + \underline{\zeta} W\}$ | $\mathfrak{h}_{186}$ | $\{P_+ + \underline{\mu} J_+ + \underline{\nu} J_- + \underline{\zeta} W\}$ |
| $\mathfrak{h}_{187}$ | $\{K_0 + \epsilon P_+ + \underline{\mu} J_+ + \underline{\nu} J_-\}$ | $\mathfrak{h}_{188}$ | $\{C_0 + \epsilon P_+ + \underline{\mu} J_+ + \underline{\nu} J_-\}$ |
| $\mathfrak{h}_{189}$ | $\{K_0 + aC_0 + \epsilon P_+ + \underline{\mu} J_+ + \underline{\nu} J_-\}$ | $\mathfrak{h}_{190}$ | $\{K_0 + \epsilon P_+ + \underline{\mu} J_+ + \underline{\nu} J_- + \underline{\zeta} W\}$ |
| $\mathfrak{h}_{191}$ | $\{C_0 + \epsilon P_+ + \underline{\mu} J_+ + \underline{\nu} J_- + \underline{\zeta} W\}$ | $\mathfrak{h}_{192}$ | $\{K_0 + aC_0 + \epsilon P_+ + \underline{\mu} J_+ + \underline{\nu} J_- + \underline{\zeta} W\}$ |
| $\mathfrak{h}_{193}$ | $\{P_+ + \epsilon P_- + \underline{\mu} J_+ + \underline{\nu} J_- + \underline{\zeta} W\}$ | $\mathfrak{h}_{194}$ | $\{P_+ + \epsilon P_- + aK_0 + \underline{\mu} J_+ + \underline{\nu} J_-\}$ |
| $\mathfrak{h}_{195}$ | $\{P_+ + \epsilon P_- + aC_0 + \underline{\mu} J_+ + \underline{\nu} J_-\}$ | $\mathfrak{h}_{196}$ | $\{P_+ + \epsilon P_- + aK_0 + bC_0 + \underline{\mu} J_+ + \underline{\nu} J_-\}$ |
| $\mathfrak{h}_{197}$ | $\{P_+ + \epsilon P_- + aK_0 + \underline{\mu} J_+ + \underline{\nu} J_- + \underline{\zeta} W\}$ | $\mathfrak{h}_{198}$ | $\{P_+ + \epsilon P_- + aC_0 + \underline{\mu} J_+ + \underline{\nu} J_- + \underline{\zeta} W\}$ |
| $\mathfrak{h}_{199}$ | $\{P_+ + \epsilon P_- + aK_0 + bC_0 + \underline{\mu} J_+ + \underline{\nu} J_- + \underline{\zeta} W\}$ | | |

# Chapitre 3

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## ON THE INTEGRABILITY OF SUPERSYMMETRIC VERSIONS OF THE STRUCTURAL EQUATIONS FOR CONFORMALLY PARAMETRIZED SURFACES

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**Résumé**

Cet article présente les extensions supersymétriques bosonique et fermionique des équations structurelles décrivant les surfaces conformément paramétrisées plongées dans des superespaces de Grassmann, basées sur des résultats trouvés précédemment par les auteurs. Une analyse détaillée des propriétés de symétrie pour les équations de Gauss–Weingarten classiques et de leurs versions supersymétriques est effectuée. Une généralisation supersymétrique de la conjecture établissant les conditions nécessaires pour qu'un système soit intégrable dans le sens de la théorie des solitons est formulée et illustrée au moyen des exemples de la version supersymétrique de l'équation de sine-Gordon et des équations de Gauss–Codazzi.

**Abstract**

The paper presents the bosonic and fermionic supersymmetric extensions of the structural equations describing conformally parametrized surfaces immersed in a Grasmann superspace, based on the authors' earlier results. A detailed analysis of the symmetry properties of both the classical and supersymmetric versions of the

Gauss–Weingarten equations is performed. A supersymmetric generalization of the conjecture establishing the necessary conditions for a system to be integrable in the sense of soliton theory is formulated and illustrated by the examples of supersymmetric versions of the sine-Gordon equation and the Gauss–Codazzi equations.

## 3.1. Introduction

Over the last decades, the concept of supersymmetry has been used extensively in particle physics and string theory [**9**, **16**, **37**, **45**, **76**, **114**, **119**] as well as in hydrodynamic-type models [**13**, **36**, **50**, **67**, **78**, **83**, **89**, **90**]. Systems involving even and odd Grassmann variables are interesting because even Grassmann variables have properties similar to those of bosonic particles and odd Grassmann variables have properties similar to those of fermionic particles. These particles appear in the standard model, bosons as interaction particles and fermions as matter particles. Supersymmetric (SUSY) extensions have been constructed, for example, for the Korteweg–de Vries equation [**83**,**90**], the Chaplygin gas equation in (1+1)- and (2+1)-dimensions (using parametrizations of the action for a superstring and a Nambu–Goto supermembrane, respectively) [**78**], the scalar Born–Infeld equation [**74**], and the sine-Gordon equation [**24**,**31**,**64**,**69**,**107**,**108**,**121**]. Supersoliton solutions were obtained for a number of SUSY theories through a connection between the super-Bäcklund and super-Darboux transformations [**3**, **24**, **62**, **69**, **88**, **108**, **116**]. A Crum-type transformation was used to determine a number of supersoliton and multisupersoliton solutions, and the existence of infinitely many local conserved quantities was determined [**64**, **91**, **107**]. In many cases, the integrability of supersymmetric systems has been demontrated by finding Lax pairs and conservation laws [**83**, **90**].

Superpositions of solutions of nonlinear SUSY systems are not as well understood as superpositions of solutions of nonlinear classical systems. As a result, SUSY differential equations do not have as extensive a theoretical foundation as classical differential equations. However, the method of prolongation of infinitesimal vector fields for Lie symmetries, the methods for the classification of subalgebras and the symmetry reduction method can, to some extent, be adapted to the case of Grassmann-valued systems of differential equations (see, e.g., [**4**, **69**]).

A supersymmetric generalization of the structural equations (the Gauss, Codazzi and Ricci equations), constructed through the use of the exterior geometry formalism, was proposed in [**8**–**10**, **109**]. This generalization was used to study superstrings and different super-$p$-branes. It was shown [**9**], using the superembedding approach, that these structural equations have their doubly supersymmetric counterparts.

The subject of our investigation is conformally parametrized surfaces immersed in a Grassmann superspace. This study is based on the methodology for the construction of SUSY extensions of the Gauss–Weingarten (GW) and Gauss–Codazzi (GC) equations developed in the authors' previous work [**B1**]. It involves the use of a moving frame formalism, leading to an explicit formulation of the structural equations for surfaces immersed in a Grassmann superspace. These equations constitute the SUSY extensions of the GW and GC equations. In [**B1**] we constructed two distinct extensions (one in terms of a bosonic superfield and the other in terms of a fermionic superfield) for each of these systems. For both SUSY extensions of the GC equations, Lie symmetry superalgebras were determined and the one-dimensional subalgebras of these superalgebras were classified into conjugacy classes under the action of their respective supergroups.

The main task undertaken in this paper is an analysis of the conditions for the existence of soliton and multisoliton solutions of the supersymmetric versions of differential equations. For this purpose we adapt the symmetry group approach to the problem of integrability in the sense of soliton theory to the SUSY case. This approach proved to be effective in the classical case when it was first proposed in the form of a conjecture for point symmetries of the GW and GC equations by D. Levi et al. [**85**] and next developed by J. Cieśliński [**28**, **29**]. It establishes a spectral technique which enables us to explicitly construct one-parameter families of surfaces associated with a given integrable system.

To formulate an analogue of the classical conjecture for the SUSY case we had to determine the symmetries of the GW equations for the classical case as well as for the bosonic and fermionic SUSY extensions and to compare them to the symmetries of the associated GC equations. The conjecture states that, if the set of symmetries of the GC equations is larger than the set of symmetries of the GW equations, then we can introduce a spectral parameter into the GW equations and obtain a Lax pair associated with the GC equations, provided that the spectral parameter cannot be eliminated through a gauge transformation. This introduction can be done through the use of vector fields that are symmetries of the original system, but not symmetries of the associated linear system. We provide an algorithmic procedure for this analysis, facilitating the determination of the integrability of a system under consideration. We illustrate these results with the examples of the SUSY versions of the sine-Gordon equation and the GC equations.

The paper is organized as follows. In section 3.2, we discuss the symmetry properties of the classical GW and GC equations, identify the Lie point symmetry algebras. Section 3.3 is devoted to a brief outline of the properties of Grassmann variables and Grassmann algebras. In section 3.4, we analyze bosonic and fermionic SUSY extensions of the GW and GC equations. In section 3.5, we adapt the classical conjecture distinguishing integrable systems to the SUSY extensions of the GW and GC equations. Finally, in section 3.6, we present possibilities for future research.

## 3.2. Symmetries of the structural equations of conformally parametrized surfaces

Consider a moving frame $\Omega$ on a smooth orientable conformally parametrized surface in 3-dimensional Euclidean space $\mathbb{R}^3$ which satisfies the GW equations

$$\begin{aligned}
\partial\begin{pmatrix}\partial F\\ \bar\partial F\\ N\end{pmatrix} &= \begin{pmatrix}\partial u & 0 & Q\\ 0 & 0 & \frac{1}{2}He^u\\ -H & -2e^{-u}Q & 0\end{pmatrix}\begin{pmatrix}\partial F\\ \bar\partial F\\ N\end{pmatrix}, \qquad \partial\Omega = V_1\Omega,\\
\bar\partial\begin{pmatrix}\partial F\\ \bar\partial F\\ N\end{pmatrix} &= \begin{pmatrix}0 & 0 & \frac{1}{2}He^u\\ 0 & \bar\partial u & \bar Q\\ -2e^{-u}\bar Q & -H & 0\end{pmatrix}\begin{pmatrix}\partial F\\ \bar\partial F\\ N\end{pmatrix}, \qquad \bar\partial\Omega = V_2\Omega,
\end{aligned} \tag{3.2.1}$$

where we define the space $\mathcal{X} = (z, \bar z)$ of independent variables, where $z = x + iy$ and $\bar z = x - iy$ are complex variables, and the space $\mathcal{U} = (H, Q, \bar Q, u)$ of unknown functions. Here $\Omega = (\partial F, \bar\partial F, N)^{\mathrm{T}}$ is a moving frame of a conformally parametrized surface with the vector-valued function $F = (F_1, F_2, F_3)\colon \mathcal{R} \to \mathbb{R}^3$ (where $\mathcal{R}$ is a Riemann surface) satisfying the following normalization for the tangent vectors $\partial F$ and $\bar\partial F$ and the unit normal $N$

$$\begin{aligned}
\langle\partial F, \partial F\rangle = \langle\bar\partial F, \bar\partial F\rangle = 0, \qquad \langle\partial F, \bar\partial F\rangle = \tfrac{1}{2}e^u,\\
\langle\partial F, N\rangle = \langle\bar\partial F, N\rangle = 0, \qquad \langle N, N\rangle = 1,
\end{aligned}$$

where the induced metric of the surface satisfies $I = e^u dz d\bar z$ with local $z$ and $\bar z$ coordinates on $\mathcal{R}$. We have used the abbreviated notation

$$\partial \equiv \partial_z = \tfrac{1}{2}(\partial_x - i\partial_y), \qquad \bar\partial \equiv \partial_{\bar z} = \tfrac{1}{2}(\partial_x + i\partial_y),$$

for the partial derivatives with respect to the complex variables $z$ and $\bar z$, respectively. The bracket $\langle\cdot,\cdot\rangle$ denotes the scalar product in 3-dimensional Euclidean space

$$\langle a, b\rangle = \sum_{i=1}^{3} a_i b_i, \qquad a, b \in \mathbb{C}^3. \tag{3.2.2}$$

The quantities $Q$, $\bar{Q}$ and $H$ in equations (3.2.1) involve the second derivatives of the immersion function $F$ and are defined as follows

$$Q = \langle \partial^2 F, N\rangle \in \mathbb{C}, \qquad H = 2e^{-u}\langle \partial\bar{\partial} F, N\rangle \in \mathbb{R},$$

where the differentials $Qdz^2$ and $\bar{Q}d\bar{z}^2$ defined on the Riemann sphere $\mathcal{R}$ are called Hopf differentials while $H$ is the mean curvature function of the surface.

The Gauss–Codazzi equations, which are the zero curvature condition (ZCC) for the potential matrices $V_1$ and $V_2$ taking values in a Lie algebra, are

$$\bar{\partial} V_1 - \partial V_2 + [V_1, V_2] = 0,$$

which reduce to the following three linearly independent equations

$$\begin{gathered} \partial\bar{\partial} u + \tfrac{1}{2}H^2 e^u - 2Q\bar{Q}e^{-u} = 0, \qquad \text{(the Gauss equation)} \\ \partial\bar{Q} = \tfrac{1}{2}e^u \bar{\partial} H, \qquad \bar{\partial} Q = \tfrac{1}{2}e^u \partial H. \qquad \text{(the Codazzi equations)} \end{gathered} \tag{3.2.3}$$

These equations guarantee the existence of conformally parametrized surfaces in $\mathbb{R}^3$. A description of all infinitesimal symmetries of the GC equations was investigated [**B1**] for conformally parametrized surfaces and the results can be summarized as follows.

In the case where the system of the GC equations has maximal rank over $M \subset \mathcal{X} \times \mathcal{U}$, it was found [**B1**] that the set of all infinitesimal Lie point symmetries of the system forms an infinite-dimensional Lie algebra $\mathcal{L}_1$ spanned by the vector fields

$$\begin{aligned} X(\eta) &= \eta(z)\partial_z + \eta'(z)(-2Q\partial_Q - U\partial_U), \\ Y(\zeta) &= \zeta(\bar{z})\partial_{\bar{z}} + \zeta'(\bar{z})(-2\bar{Q}\partial_{\bar{Q}} - U\partial_U), \\ e_0 &= -H\partial_H + Q\partial_Q + \bar{Q}\partial_{\bar{Q}} + 2U\partial_U, \end{aligned}$$

where $\eta$ and $\zeta$ are arbitrary functions of $z$ and $\bar{z}$ respectively, while $\eta'$ and $\zeta'$ are the derivatives of $\eta$ and $\zeta$ with respect to their arguments. Here and subsequently, we use the notation $U = e^u$. The generators $X(\eta)$ and $Y(\zeta)$ are two infinite-dimensional families of conformal transformations, while $e_0$ is a dilation in the dependent variables which constitutes the center of the algebra. The maximal finite-dimensional subalgebra $L_1$ of the algebra $\mathcal{L}_1$ was obtained by expanding $\eta$ and $\zeta$ as power series with respect to their arguments. This algebra $L_1$ is spanned by the seven generators

$$\begin{aligned}
&e_0 = -H\partial_H + Q\partial_Q + \bar{Q}\partial_{\bar{Q}} + 2U\partial_U,\\
&e_1 = \partial_z, \qquad e_3 = z\partial_z - 2Q\partial_Q - U\partial_U, \qquad e_5 = z^2\partial_z - 4zQ\partial_Q - 2zU\partial_U,\\
&e_2 = \partial_{\bar{z}}, \qquad e_4 = \bar{z}\partial_{\bar{z}} - 2\bar{Q}\partial_{\bar{Q}} - U\partial_U, \qquad e_6 = \bar{z}^2\partial_{\bar{z}} - 4\bar{z}\bar{Q}\partial_{\bar{Q}} - 2\bar{z}U\partial_U.
\end{aligned}$$

Let us now perform an analysis of the infinitesimal symmetries of the GW equations (3.2.1). In the case where the system of GW equations has maximal rank over $M \subset \mathcal{X} \times \mathcal{U}$, the set of all infinitesimal symmetries of the system forms an infinite-dimensional Lie algebra $\mathcal{L}_2$ spanned by the vector fields

$$\begin{aligned}
X(\eta) &= \eta(z)\partial_z - \eta'(z)(U\partial_U + 2Q\partial_Q),\\
Y(\zeta) &= \zeta(\bar{z})\partial_{\bar{z}} - \zeta'(\bar{z})(U\partial_U + 2\bar{Q}\partial_{\bar{Q}}),\\
\hat{e}_0 &= -H\partial_H + Q\partial_Q + \bar{Q}\partial_{\bar{Q}} + 2U\partial_U + F_i\partial_{F_i},\\
T_i &= \partial_{F_i}, \qquad \mathcal{D}_i = F_i\partial_{F_{(i)}} + N_i\partial_{N_{(i)}}, \qquad i = 1,2,3,\\
R_{ij} &= (F_i\partial_{F_j} - F_j\partial_{F_i}) + (N_i\partial_{N_j} - N_j\partial_{N_i}), \qquad i < j = 2,3,\\
S_{ij} &= (F_i\partial_{F_j} + F_j\partial_{F_i}) + (N_i\partial_{N_j} + N_j\partial_{N_i}).
\end{aligned} \tag{3.2.4}$$

Here, we have used the notation $\eta'(z) = d\eta/dz$ and $\zeta'(\bar{z}) = d\zeta/d\bar{z}$, where $\eta$ and $\zeta$ are arbitrary functions of $z$ and $\bar{z}$ respectively. The generators in (3.2.4) can be identified as follows : the $T_i$ generate translations in the $F_i$ directions respectively, $R_{ij}$ represent rotations in the direction of the $F_i$ and $N_i$ variables, $S_{ij}$ are local boost transformations and the vector fields $e_0$, $\mathcal{D}_1$ and $\mathcal{D}_2$ correspond to scaling transformations. In addition, we obtain two infinite-dimensional families of infinitesimal transformations generated by $X(\eta)$ and $Y(\zeta)$. The non-zero commutation relations between the generators (3.2.4) are

$$\begin{aligned}
&[X(\eta_1), X(\eta_2)] = (\eta_1\eta_2' - \eta_1'\eta_2)\partial_z + (\eta_1''\eta_2 - \eta_1\eta_2'')(U\partial_U + 2Q\partial_Q),\\
&[Y(\zeta_1), Y(\zeta_2)] = (\zeta_1\zeta_2' - \zeta_1'\zeta_2)\partial_{\bar{z}} + (\zeta_1''\zeta_2 - \zeta_1\zeta_2'')(U\partial_U + 2\bar{Q}\partial_{\bar{Q}}),\\
&[\hat{e}_0, T_i] = -T_i, \qquad [T_i, \mathcal{D}_j] = \delta_{ij}T_i, \qquad [T_i, R_{jk}] = \delta_{ij}T_k - \delta_{ik}T_j,\\
&[T_i, S_{jk}] = \delta_{ij}T_k + \delta_{ik}T_j, \qquad [\mathcal{D}_i, R_{jk}] = \delta_{ij}S_{ik} - \delta_{ik}S_{ij},\\
&[\mathcal{D}_i, S_{jk}] = \delta_{ij}R_{ik} - \delta_{ik}R_{ji}, \qquad [R_{ij}, S_{kl}] = \delta_{jk}S_{il} + \delta_{jl}S_{ik} - \delta_{ik}S_{jl} - \delta_{il}S_{jk},
\end{aligned}$$

where $\delta_{jk}$ is the Kronecker delta function. The Lie algebra $\mathcal{L}_2$ can be decomposed into the direct sum

$$\mathcal{L}_2 = \{X(\eta)\} \oplus \{Y(\zeta)\} \oplus \{\hat{e}_0, T_i, \mathcal{D}_i, R_{ij}, S_{ij}\},$$

which consists of two copies of the Virasoro algebra together with the 13-dimensional algebra generated by $\hat{e}_0$, $T_i$, $\mathcal{D}_i$, $R_{ij}$ and $S_{ij}$. If the functions $\eta$ and $\zeta$ are analytic, they can be expanded as power series with respect to $z$ and $\bar{z}$ respectively. The maximal finite-dimensional subalgebra $L_2$ of $\mathcal{L}_2$ is spanned by the 19 generators

$$\hat{e}_0 = -H\partial_H + Q\partial_Q + \bar{Q}\partial_{\bar{Q}} + 2U\partial_U + F_i\partial_{F_i},$$
$$e_1 = \partial_z, \qquad e_3 = z\partial_z - 2Q\partial_Q - U\partial_U, \qquad e_5 = z^2\partial_z - 4zQ\partial_Q - 2zU\partial_U,$$
$$e_2 = \partial_{\bar{z}}, \qquad e_4 = \bar{z}\partial_{\bar{z}} - 2\bar{Q}\partial_{\bar{Q}} - U\partial_U, \qquad e_6 = \bar{z}^2\partial_{\bar{z}} - 4\bar{z}\bar{Q}\partial_{\bar{Q}} - 2\bar{z}U\partial_U,$$
$$T_i = \partial_{F_i}, \qquad \mathcal{D}_i = F_i\partial_{F_{(i)}} + N_i\partial_{N_{(i)}}, \qquad i = 1,2,3,$$
$$R_{ij} = (F_i\partial_{F_j} - F_j\partial_{F_i}) + (N_i\partial_{N_j} - N_j\partial_{N_i}),$$
$$S_{ij} = (F_i\partial_{F_j} + F_j\partial_{F_i}) + (N_i\partial_{N_j} + N_j\partial_{N_i}), \qquad i < j = 2,3,$$

which have the non-zero commutation relations

$$[e_1, e_3] = e_1, \qquad [e_1, e_5] = 2e_3, \qquad [e_3, e_5] = e_5,$$
$$[e_2, e_4] = e_2, \qquad [e_2, e_6] = 2e_4, \qquad [e_4, e_6] = e_6,$$
$$[\hat{e}_0, T_i] = -T_i, \qquad [T_i, \mathcal{D}_j] = \delta_{ij}T_i, \qquad [T_i, R_{jk}] = \delta_{ij}T_k - \delta_{ik}T_j,$$
$$[T_i, S_{jk}] = \delta_{ij}T_k + \delta_{ik}T_j, \qquad [\mathcal{D}_i, S_{jk}] = \delta_{ij}R_{ik} - \delta_{ik}R_{ji},$$
$$[\mathcal{D}_i, R_{jk}] = \delta_{ij}S_{ik} - \delta_{ik}S_{ij}, \qquad [R_{ij}, S_{kl}] = \delta_{jk}S_{il} + \delta_{jl}S_{ik} - \delta_{ik}S_{jl} - \delta_{il}S_{jk}.$$

The algebra $L_2$ can be decomposed as follows

$$L_2 = \{e_1, e_3, e_5\} \oplus \{e_2, e_4, e_6\} \oplus \{T_i, \mathcal{D}_i, R_{ij}, S_i\} \supset\!\!\!\!+ \{\hat{e}_0\}.$$

In the theory of solitons, there exists a conjecture [**28**, **29**, **85**] to isolate integrable systems which states that this characterization can be performed by comparing the sets of symmetries of the original system and of its associated linear system. In the case where the sets of symmetries of both the original system and the non-parametric linear system (the GW system) are finite-dimensional, we can compare the symmetries of the two systems by defining the differential projection operator $\pi$ as the following operator

$$\pi(L_2) = L_2\omega, \qquad \text{where} \quad \omega = z\partial + \bar{z}\bar{\partial} + H\partial_H + Q\partial_Q + \bar{Q}\partial_{\bar{Q}} + U\partial_U,$$

which involves all independent and dependent variables. Here, $\omega$ is not necessarily an element of $L_1$ or $L_2$. The projection operator $\pi$ has the property that $\pi^n(L_2) = \pi(L_2)$ for any positive integer $n$ and every element of the algebra $L_2$. In fact, we have

$$\pi^2(L_2) = \pi(L_2\omega) = L_2\omega^2 = L_2\omega = \pi(L_2).$$

Under the above assumptions, the conjecture concerning integrable systems proposed in [**28**, **29**, **85**] can be formulated as follows.

**Conjecture 3.2.1.**

(1) *In the case where $L_1 = \pi(L_2)$, the original system is non-integrable in the sense of soliton theory. In the case where there exist reductions of the original system (whose set of symmetries is $L_1'$) and the non-parametric linear system (whose set of symmetries is $L_2'$) such that $L_1' \neq \pi(L_2')$, the reduced subsystem of the original system can be integrable.*

(2) *In the case where $L_1 \subset \pi(L_2)$, the system is a candidate to be integrable (in the sense of soliton theory) if it is possible to introduce a spectral parameter into the linear GW system, which represents a Lax pair, provided that the spectral parameter cannot be eliminated through a gauge transformation.*

It should be noted that, under the above conjecture, the GC equations (3.2.3) do not form an integrable system since $L_1 = \pi(L_2)$.

## 3.3. Certain aspects of Grassmann algebras

We present a brief overview of the concepts related to Grassmann variables and Grassmann algebras. The formalism is based on the theory of supermanifolds as described, e.g., in [**11**, **12**, **16**, **33**, **43**, **56**, **80**, **101**, **102**, **118**]. We consider a complex Grassmann algebra $\Gamma$ involving an arbitrary large (but finite) number $\mathfrak{k}$ of Grassmann generators $(\xi_1, \xi_2, \ldots, \xi_{\mathfrak{k}})$. The exact number of generators is not essential as long as there is a sufficient number of them to make all considered formulas meaningful. The Grassmann algebra $\Lambda$ can be decomposed into its even (bosonic) and odd (fermionic) parts

$$\Lambda = \Lambda_{\text{even}} + \Lambda_{\text{odd}},$$

where $\Lambda_{\text{even}}$ contains all terms involving a product of an even number of generators $\xi_k$, i.e., $1, \xi_1\xi_2, \xi_1\xi_3, \ldots$, while $\Lambda_{\text{odd}}$ contains all terms involving a product of an odd number of generators $\xi_k$, i.e., $\xi_1, \xi_2, \xi_3, \ldots, \xi_1\xi_2\xi_3, \ldots$ The space $\Lambda$ and/or $\Lambda_{\text{even}}$ replaces the field of complex numbers in the context of supersymmetry. The elements of $\Lambda_{\text{even}}$ and $\Lambda_{\text{odd}}$ are called even and odd supernumbers, respectively. An alternative decomposition for the Grassmann algebra $\Lambda$ is

$$\Lambda = \Lambda_{\text{body}} + \Lambda_{\text{soul}},$$

where

$$\Lambda_{\text{body}} = \Lambda^0[\xi_1, \xi_2, \ldots, \xi_{\mathfrak{k}}] \simeq \mathbb{C}, \qquad \Lambda_{\text{soul}} = \sum_{k \geq 1} \Lambda^k[\xi_1, \xi_2, \ldots, \xi_{\mathfrak{k}}].$$

Here $\Lambda^0[\xi_1, \xi_2, \ldots, \xi_{\mathfrak{k}}]$ is used to refer to all elements of $\Lambda$ that do not involve any of the generators $\xi_i$, while $\Lambda^k[\xi_1, \xi_2, \ldots, \xi_{\mathfrak{k}}]$ refers to all elements of $\Lambda$ that contain a product of $k$ generators (for instance, if we have 5 generators $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5$ then $\Lambda^2[\xi_1, \xi_2, \xi_3, \xi_4, \xi_5]$ refers to all terms involving $\xi_1\xi_2$, $\xi_1\xi_3$, $\xi_1\xi_4$, $\xi_1\xi_5$, $\xi_2\xi_3$, $\xi_2\xi_4$, $\xi_2\xi_5$, $\xi_3\xi_4$, $\xi_3\xi_5$ and $\xi_4\xi_5$). Because of the $\mathbb{Z}_0^+$-grading of the Grassmann algebra $\Lambda$, the bodiless elements in $\Lambda_{\text{soul}}$ are non-invertible. Since the number $\mathfrak{k}$ of Grassmann generators is finite, it follows that the bodiless elements are nilpotent of degree at most $\mathfrak{k}$.

In this paper, we use a $\mathbb{Z}_2$-graded complex vector space $V$ with even basis elements $u_i$, $i = 1, 2, \dots, N$ and odd basis elements $v_\mu$, $\mu = 1, 2, \dots, M$ and consider $W = \Lambda \otimes_{\mathbb{C}} V$. The even part of $W$

$$W_{\text{even}} = \left\{ \sum_i a_i u_i + \sum_\mu \underline{\alpha}_\mu v_\mu \,|\, a_i \in \Lambda_{\text{even}},\, \underline{\alpha}_\mu \in \Lambda_{\text{odd}} \right\},$$

is a $\Lambda_{\text{even}}$ module which can be identified with $\Lambda_{\text{even}}^{\times N} \times \Lambda_{\text{odd}}^{\times M}$ (which consists of $N$ copies of $\Lambda_{\text{even}}$ and $M$ copies of $\Lambda_{\text{odd}}$). To the original basis, consisting of the $u_i$ and $v_\mu$ (although $v_\mu \notin W_{\text{even}}$), we associate the corresponding functionals

$$E_j\colon\ W_{\text{even}} \to \Lambda_{\text{even}}\colon E_j \left( \sum_i a_i u_i + \sum_\mu \underline{\alpha}_\mu v_\mu \right) = a_j,$$

$$\Upsilon_\nu\colon\ W_{\text{even}} \to \Lambda_{\text{odd}}\colon \Upsilon_\nu \left( \sum_i a_i u_i + \sum_\mu \underline{\alpha}_\mu v_\mu \right) = \underline{\alpha}_\nu,$$

and view them as the coordinates (even and odd respectively) on $W_{\text{even}}$. Any topological manifold locally diffeomorphic to a suitable $W_{\text{even}}$ is called a supermanifold [**102**]. Super-Minkowski space $\mathbb{R}^{(1,1|2)}$ is an example of such a supermanifold, being globally diffeomorphic to $\Lambda_{\text{even}}^{\times 2} \times \Lambda_{\text{odd}}^{\times 2}$, with bosonic light-cone coordinates $x_+$ and $x_-$, and fermionic coordinates $\theta^+$ and $\theta^-$. Therefore, $x_+$ and $x_-$ are linear combinations of terms containing an even number of generators : 1, $\xi_1\xi_2$, $\xi_1\xi_3$, $\xi_1\xi_4$, ..., $\xi_2\xi_3$, $\xi_2\xi_4$, ..., $\xi_1\xi_2\xi_3\xi_4$, ... In contrast $\theta^+$ and $\theta^-$ are linear combinations of terms containing an odd number of generators : $\xi_1$, $\xi_2$, $\xi_3$, $\xi_4$, ..., $\xi_1\xi_2\xi_3$, $\xi_1\xi_2\xi_4$, $\xi_1\xi_3\xi_4$, $\xi_2\xi_3\xi_4$, ... Any fermionic (odd) variables $\theta^+$ and $\theta^-$ satisfy the relation

$$(\theta^+)^2 = (\theta^-)^2 = \theta^+\theta^- + \theta^-\theta^+ = 0. \tag{3.3.1}$$

The supersymmetry transformations (3.4.3) presented in the next section can be understood as changes in the coordinates of $\mathbb{R}^{(1,1|2)}$ which transform solutions of the SUSY GW equations and the SUSY GC equations, respectively, into solutions of the same equations in new coordinates for both the bosonic and fermionic SUSY extensions. A bosonic or fermionic smooth superfield is a supersmooth $G^\infty$ function from $\mathbb{R}^{(n_{\text{b}}|n_{\text{f}})}$ to $\Lambda$ (the values $n_{\text{b}}$ and $n_{\text{f}}$ of the superspace $\mathbb{R}^{(n_{\text{b}}|n_{\text{f}})}$ stand for the number of bosonic and fermionic Grassmann coordinates respectively).

In this paper we use the convention that partial derivatives involving odd variables obey the following Leibniz rule for the product of two Grassmann-valued functions $h$ and $g$

$$\partial_{\theta^\pm}(hg) = (\partial_{\theta^\pm} h)g + (-1)^{\deg(h)} h(\partial_{\theta^\pm} g),$$

where the degree of a homogeneous supernumber is given by

$$\deg(h) = \begin{cases} 0 & \text{if } h \text{ is even,} \\ 1 & \text{if } h \text{ is odd.} \end{cases}$$

We use the following ordering notation for partial derivatives $f_{\theta^+\theta^-} = \partial_{\theta^-}\partial_{\theta^+} f$. The partial derivatives with respect to the fermionic coordinates satisfy $\partial_{\theta^i}\theta^j = \delta_i^j$, where $\delta_i^j$ is the Kronecker delta function and the indices $i$ and $j$ each stand for $+$ or $-$. The fermionic operators $\partial_{\theta^\pm}$, $J_\pm$ and $D_\pm$ in equations (3.4.1) and (3.4.4) alter the parity of a bosonic function to a fermionic function and vice versa. For instance, if $\phi$ is a bosonic function, then $\partial_{\theta^+}\phi$ is an odd superfield, while $\partial_{\theta^+}\partial_{\theta^-}\phi$ is an even superfield. For a Grassmann-valued composite function $f(g(x_+))$, the chain rule is ordered as follows

$$\frac{\partial f}{\partial x_+} = \frac{\partial g}{\partial x_+}\frac{\partial f}{\partial g}.$$

The interchange of mixed derivatives (with proper respect to the ordering of odd variables) is assumed throughout this paper. Additional details can be found in the books by Cornwell [**33**], DeWitt [**43**], Freed [**56**], Kac [**80**], Varadarajan [**118**] and references therein.

## 3.4. Supersymmetric versions of the Gauss–Weingarten and Gauss–Codazzi equations

In a previous paper [**B1**], we constructed supersymmetric versions of the differential equations which define surfaces in super-Minkowski space. These versions consisted of supersymmetric extensions of the Gauss–Weingarten and Gauss–Codazzi equations using bosonic and fermionic superfields. The purpose of constructing such extensions was to construct surfaces immersed in a superspace ($\mathbb{R}^{(2,1|2)}$ for the bosonic extension and $\mathbb{R}^{(1,1|3)}$ for the fermionic extension). We use the variables $x_\pm = \frac{1}{2}(t \pm x)$ which are the bosonic light-cone coordinates, and $\theta^\pm$ which are fermionic (anticommuting) variables satisfying (3.3.1). Below, we present the outline of our procedure and its main results on which we base our further considerations.

Let $\mathcal{S}$ be a smooth orientable conformally parametrized surface immersed in the superspace given by a vector-valued superfield $F(x_+, x_-, \theta^+, \theta^-)$ which, in view of (3.3.1), can be decomposed as

$$F = F_m(x_+, x_-) + \theta^+\varphi_m(x_+, x_-) + \theta^-\psi_m(x_+, x_-) + \theta^+\theta^-G_m(x_+, x_-),\\ m = 1, 2, 3.$$

In the bosonic case, the functions $F_m$ and $G_m$ are bosonic-valued, while the functions $\varphi_m$ and $\psi_m$ are fermionic-valued. Conversely, in the fermionic case, the functions $F_m$ and $G_m$ are fermionic-valued, while the functions $\varphi_m$ and $\psi_m$ are bosonic-valued. In both cases, we define the covariant superspace derivatives to be

$$D_\pm = \partial_{\theta^\pm} - i\theta^\pm\partial_{x_\pm}. \tag{3.4.1}$$

The conformal parametrization of the surface $\mathcal{S}$ gives the following normalization on the superfield

$$\langle D_iF, D_jF\rangle = fg_{ij}, \qquad \langle D_iF, N\rangle = 0, \qquad \langle N, N\rangle = 1, \qquad i, j = 1, 2, \tag{3.4.2}$$

where $D_\pm F$ are the tangent vector superfields and $N$ is a normal bosonic vector field which can be decomposed in the form

$$N = N_m(x_+, x_-) + \theta^+\alpha_m(x_+, x_-) + \theta^-\beta_m(x_+, x_-) + \theta^+\theta^-H_m(x_+, x_-),\\ m = 1, 2, 3,$$

where $N_m$ and $H_m$ are bosonic functions, while $\alpha_m$ and $\beta_m$ are fermionic functions. In the bosonic case the function $f$ which appears in (3.4.2) is a bodiless bosonic function (i.e., $f \in \Lambda_{\text{soul}}$) of $x_+$ and $x_-$ which is a nilpotent function of some order $k$. In the fermionic case the bosonic function $f$ may be bodiless or not. The values 1 and 2 of the indices $i$ and $j$ stand for $+$ and $-$ respectively. The bracket $\langle\cdot,\cdot\rangle$ denotes the scalar product (3.2.2) for 3-dimensional Euclidean space, where we use the property (3.3.1) for any fermionic variables. This scalar product takes its values in the Grassmann algebra $\Lambda$. The coefficients of the induced bosonic metric function $g_{ij}$ on the surface $\mathcal{S}$ are given by

$$g_{ii} = 0, \qquad g_{12} = \tfrac{1}{2}e^\phi, \qquad g_{21} = \tfrac{1}{2}\epsilon e^\phi, \qquad i = 1, 2,$$

where $\epsilon = 1$ in the fermionic case and $\epsilon = -1$ in the bosonic case. It should be noted that the covariant metric tensor $g_{ij}$ is anti-symmetric in the indices $i$ and $j$ in the bosonic case while it is symmetric in those indices in the fermionic case. Here, the superfield $\phi$ is assumed to be bosonic and can be expanded in terms of the fermionic variables $\theta^+$ and $\theta^-$ :

$$\phi = \phi_0(x_+, x_-) + \theta^+\phi_1(x_+, x_-) + \theta^-\phi_2(x_+, x_-) + \theta^+\theta^-\phi_3(x_+, x_-),$$

where $\phi_0$ and $\phi_3$ are bosonic functions while $\phi_1$ and $\phi_2$ are fermionic functions. The exponential function can be expanded as follows in terms of $\theta^+$ and $\theta^-$ :

$$e^{\pm\phi} = e^{\pm\phi_0}\Big(1 \pm \theta^+\phi_1 \pm \theta^-\phi_2 \pm \theta^+\theta^-\phi_3 - \theta^+\theta^-\phi_1\phi_2\Big).$$

The SUSY extensions of the GW and GC equations are constructed in such a way that they are invariant under the transformations

$$x_\pm \to x_\pm + i\underline{\eta}_\pm\theta^\pm, \qquad \theta^\pm \to \theta^\pm + i\underline{\eta}_\pm, \tag{3.4.3}$$

which are generated by the differential SUSY operators

$$J_\pm = \partial_{\theta^\pm} + i\theta^\pm\partial_{x_\pm}, \tag{3.4.4}$$

respectively. Here $\underline{\eta}_\pm$ are fermionic-valued parameters. The SUSY operators $J_\pm$ satisfy the following anticommutation relations :

$$\{J_n, J_m\} = 2i\delta_{mn}\partial_{x_m}, \qquad \{D_n, D_m\} = -2i\delta_{mn}\partial_{x_m}, \qquad \{J_m, D_n\} = 0,$$
$$D_\pm^2 = -i\partial_\pm, \qquad J_\pm^2 = i\partial_\pm, \qquad m, n = 1, 2,$$

where $\delta_{mn}$ is the Kronecker delta function and the brace brackets denote anticommutation, unless otherwise specified. The values 1 and 2 of the indices $m$ and $n$ stand for $+$ and $-$ respectively. Here and subsequently, summation over repeated indices is understood.

In order to construct the SUSY version of the GW equations we assume that the second-order covariant derivatives of $F$ and the first-order covariant derivatives of the normal unit vector $N$ can be defined in terms of the moving frame $\Omega = (D_+F, D_-F, N)^T$ on a surface $\mathcal{S}$, i.e.,

$$D_jD_iF = \Gamma_{ij}^{\ \ k}D_kF + b_{ij}fN, \qquad D_iN = b_i^{\ k}D_kF + \omega_iN, \qquad i, j, k = 1, 2,$$

where the coefficients $\Gamma_{ij}^{\ \ k}$ and $\omega_i$ are fermionic functions. The functions $b_{ij}$ and $b_i^{\ k}$ are bosonic-valued in the bosonic extension and are fermionic-valued in the fermionic extension. Here, the values 1 and 2 of the indices $i$, $j$ and $k$ stand for $+$ and $-$, respectively. We define the coefficients $b_{ij}$ to be

$$b_{11} = Q^+, \qquad b_{12} = -b_{21} = \tfrac{1}{2}e^\phi H, \qquad b_{22} = Q^-. \tag{3.4.5}$$

In the bosonic extension, the moving frame $\Omega$ contains both bosonic and fermionic components. Under the above assumptions, we obtained the following results [**B1**]

**Proposition 3.4.1.** *For any bosonic vector superfields* $F(x_+, x_-, \theta^+, \theta^-)$ *and* $N(x_+, x_-, \theta^+, \theta^-)$ *satisfying the normalization conditions* (3.4.2) *and* (3.4.5), *the moving frame* $\Omega = (D_+F, D_-F, N)^{\mathrm{T}}$ *on a smooth conformally parametrized surface immersed in the superspace* $\mathbb{R}^{(2,1|2)}$ *satisfies the SUSY GW equations*

$$D_+\Omega = A_+\Omega, \qquad D_-\Omega = A_-\Omega,$$
$$A_+ = \begin{pmatrix} {\Gamma_{11}}^1 & {\Gamma_{11}}^2 & Q^+f \\ -{\Gamma_{12}}^1 & -{\Gamma_{12}}^2 & -\frac{1}{2}e^{\phi}Hf \\ H & 2e^{-\phi}Q^+ & 0 \end{pmatrix},$$
$$A_- = \begin{pmatrix} {\Gamma_{12}}^1 & {\Gamma_{12}}^2 & \frac{1}{2}e^{\phi}Hf \\ {\Gamma_{22}}^1 & {\Gamma_{22}}^2 & Q^-f \\ -2e^{-\phi}Q^- & H & 0 \end{pmatrix}. \tag{3.4.6}$$

*The zero curvature condition*

$$D_+A_- + D_-A_+ - \{EA_+, EA_-\} = 0, \tag{3.4.7}$$

*where*

$$E = \pm \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

*constitutes the GC equations and corresponds to the following six linearly independent equations*

$$\begin{aligned}
&(i)\ \ D_-\left({\Gamma_{11}}^1\right) + D_+\left({\Gamma_{22}}^2\right) + D_+\left({\Gamma_{12}}^1\right) - D_-\left({\Gamma_{12}}^2\right) = 0,\\
&(ii)\ D_-\left({\Gamma_{11}}^1\right) - {\Gamma_{11}}^2{\Gamma_{22}}^1 + D_+\left({\Gamma_{12}}^1\right) + {\Gamma_{12}}^2{\Gamma_{12}}^1 + \frac{1}{2}H^2e^{\phi}f - 2Q^+Q^-e^{-\phi}f = 0,\\
&(iii)\ Q^+{\Gamma_{22}}^2 - {\Gamma_{11}}^2Q^- + D_-Q^+ - Q^+D_-\phi + \frac{1}{2}e^{\phi}D_+H = 0,\\
&(iv)\ \ Q^-{\Gamma_{11}}^1 - {\Gamma_{22}}^1Q^+ + D_+Q^- - Q^-D_+\phi - \frac{1}{2}e^{\phi}D_-H = 0,\\
&(v)\ \ D_-\left({\Gamma_{11}}^2\right) - {\Gamma_{12}}^1{\Gamma_{11}}^2 - {\Gamma_{11}}^2{\Gamma_{22}}^2 - {\Gamma_{11}}^1{\Gamma_{12}}^2 + D_+\left({\Gamma_{12}}^2\right) + 2Q^+Hf = 0,\\
&(vi)\ D_+\left({\Gamma_{22}}^1\right) + {\Gamma_{12}}^2{\Gamma_{22}}^1 - {\Gamma_{22}}^1{\Gamma_{11}}^1 + {\Gamma_{22}}^2{\Gamma_{12}}^1 - D_-\left({\Gamma_{12}}^1\right) + 2Q^-Hf = 0.
\end{aligned} \tag{3.4.8}$$

In the fermionic extension, the moving frame $\Omega$ contains only bosonic components. The fermionic counterpart of proposition 3.4.1 can be summarized as follows.

**Proposition 3.4.2.** *For any fermionic vector superfield* $F(x_+, x_-, \theta^+, \theta^-)$ *and bosonic normal unit vector* $N(x_+, x_-, \theta^+, \theta^-)$ *satisfying the normalization conditions* (3.4.2) *and* (3.4.5), *the bosonic moving frame* $\Omega = (D_+F, D_-F, N)^{\mathrm{T}}$ *on a smooth conformally parametrized surface immersed in the superspace* $\mathbb{R}^{(1,1|3)}$

*satisfies the SUSY GW equations*

$$D_+\begin{pmatrix}D_+F\\ D_-F\\ N\end{pmatrix}=\begin{pmatrix}{\Gamma_{11}}^1 & 0 & Q^+f\\ 0 & 0 & -\frac{1}{2}e^{\phi}Hf\\ H & -2e^{-\phi}Q^+ & 0\end{pmatrix}\begin{pmatrix}D_+F\\ D_-F\\ N\end{pmatrix},$$
$$D_-\begin{pmatrix}D_+F\\ D_-F\\ N\end{pmatrix}=\begin{pmatrix}0 & 0 & \frac{1}{2}e^{\phi}Hf\\ 0 & {\Gamma_{22}}^2 & Q^-f\\ -2e^{-\phi}Q^- & -H & 0\end{pmatrix}\begin{pmatrix}D_+F\\ D_-F\\ N\end{pmatrix}. \tag{3.4.9}$$

*The GC equations, which are equivalent to the ZCC*

$$D_+A_- + D_-A_+ - \{A_+, A_-\} = 0,$$

*reduce to the following four linearly independent equations*

$$\begin{aligned}
&(i)\quad D_+\left({\Gamma_{22}}^2\right) + D_-\left({\Gamma_{11}}^1\right) = 0,\\
&(ii)\quad D_-\left({\Gamma_{11}}^1\right) + 2e^{-\phi}Q^+Q^-f = 0,\\
&(iii)\ D_+Q^- - \frac{1}{2}e^{\phi}D_-H + Q^-\left(D_+\phi - {\Gamma_{11}}^1\right) = 0,\\
&(iv)\ \ D_-Q^+ + \frac{1}{2}e^{\phi}D_+H + Q^+\left(D_-\phi - {\Gamma_{22}}^2\right) = 0.
\end{aligned} \tag{3.4.10}$$

## 3.5. Conjecture on supersymmetric integrable systems

In this section, we formulate a SUSY version of the conjecture 3.2.1 on integrable systems described in section 3.2. A symmetry supergroup $G$ of a SUSY system of equations consists of a local supergroup of transformations acting on a Cartesian product of supermanifolds $\mathcal{X}\times\mathcal{U}$, where $\mathcal{X}$ is the space of four independent variables $(x_+, x_-, \theta^+, \theta^-)$ and $\mathcal{U}$ is the space of dependent superfields.

Let $\mathcal{L}_1$ be a maximal finite-dimensional superalgebra of Lie point symmetries associated with the system of nonlinear partial differential equations (NPDEs) under consideration. Let $\mathcal{L}_2$ be a maximal finite-dimensional superalgebra of Lie point symmetries of the linear system associated with the original system of NPDEs. Let $\pi$ be a projection operator acting on the subalgebra $\mathcal{L}_2$ such that $\pi(\mathcal{L}_2) = \mathcal{L}_2\omega$, where $\omega$ is the differential operator

$$\omega = x_+\partial_{x_+} + x_-\partial_{x_-} + \theta^+\partial_{\theta^+} + \theta^-\partial_{\theta^-} + u^\alpha\partial_{u^\alpha} + \varphi^\beta\partial_{\varphi^\beta}$$

involving all independent bosonic and fermionic variables $(x_+, x_-, \theta^+, \theta^-)$ and all dependent bosonic and fermionic superfields, $u^\alpha$ and $\varphi^\beta$, respectively, appearing in the system of NPDEs. The common symmetries of the NPDEs and the linear spectral problem (LSP), associated with the original system of NPDEs, are the vector fields which span the set

$$\mathcal{L}_3 = \mathcal{L}_1 \cap \pi(\mathcal{L}_2) \neq \varnothing.$$

It should be noted that the set $\mathcal{L}_3$ is not necessarily an algebra. The prolongation of one of these vector fields acting on the LSP has to vanish for all wavefunctions of the LSP. In this case, the integrated form of a two-dimensional surface in a Lie algebra is given by the Fokas–Gel'fand immersion formula [**54**, **55**, **73**], whenever the tangent vectors on the surface are linearly independent. Let us consider the set of vector fields defined by

$$\mathcal{L}_4 = \mathcal{L}_1 \backslash \{\mathcal{L}_1 \cap \pi(\mathcal{L}_2)\}.$$

Here, $\mathcal{L}_4$ consists of all symmetries of $\mathcal{L}_1$ that are not symmetries of $\mathcal{L}_2$. Again, $\mathcal{L}_4$ is not necessarily an algebra. Under the above assumptions, an extension of the conjecture 3.2.1 to SUSY integrable systems can be formulated as follows.

**Conjecture 3.5.1.**

*(1) If $\mathcal{L}_1 = \pi(\mathcal{L}_2)$ then the system of NPDEs is not integrable.*

*(2) If the following conditions are satisfied*

*(a) $\pi(\mathcal{L}_2)$ is a proper subset of $\mathcal{L}_1$, that is*

$$\mathcal{L}_1 \supset \pi(\mathcal{L}_2).$$

*A free parameter can be introduced into the linear system using a symmetry transformation generated by one of the vector fields appearing in $\mathcal{L}_4$.*

*(b) The transformation given in (a) acts in a nontivial way (i.e., cannot be eliminated through an $\mathcal{L}_1$-valued gauge matrix function).*

*Then the system of NPDEs is a candidate to be an integrable system.*

The proposed conjecture is illustrated through the following examples.

**Example 3.5.1.** *The bosonic extension of the GC equations* (3.4.8) *involves eleven unknown functions $\mathcal{U} = (\phi, H, Q^+, Q^-, R^+, R^-, S^+, S^-, T^+, T^-, f)$, where $\phi$, $H$, $Q^+$, $Q^-$, $f$ are bosonic functions while $R^+$, $R^-$, $S^+$, $S^-$, $T^+$, $T^-$ are fermionic functions. In what follows we use the notation*

$$R^+ = {\Gamma_{11}}^1, \qquad R^- = {\Gamma_{11}}^2, \qquad S^+ = {\Gamma_{12}}^1, \qquad S^- = {\Gamma_{12}}^2,$$
$$T^+ = {\Gamma_{22}}^1, \qquad T^- = {\Gamma_{22}}^2.$$

*The action of the supergroup $G$ on the superfields $\mathcal{U}$ of $(x_+, x_-, \theta^+, \theta^-)$ maps solutions of the bosonic version of the SUSY GC equations* (3.4.8) *to solutions of* (3.4.8)*. The bodiless bosonic function $f$ depends only on $x_+$ and $x_-$, in constrast* *with the other listed superfields in $\mathcal{U}$ which can depend on* $(x_+, x_-, \theta^+, \theta^-)$*. Assuming that* $G$ *is a Lie supergroup as described in [**80**, **120**], we found that its associated Lie superalgebra* $\mathfrak{g}_1$*, whose elements are infinitesimal symmetries of the bosonic SUSY GC equations* (3.4.8)*, was generated by the following eight vector fields [**B1**]*

$$\begin{aligned} C_0 &= H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} - 2f\partial_f, \\ K_0 &= -H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} + 2\partial_\phi, \\ K_1^b &= -2x_+\partial_{x_+} - \theta^+\partial_{\theta^+} + R^+\partial_{R^+} + 2R^-\partial_{R^-} \\ &\quad + S^-\partial_{S^-} - T^+\partial_{T^+} + 2Q^+\partial_{Q^+} + \partial_\phi, \\ K_2^b &= -2x_-\partial_{x_-} - \theta^-\partial_{\theta^-} - R^-\partial_{R^-} + S^+\partial_{S^+} \\ &\quad + 2T^+\partial_{T^+} + T^-\partial_{T^-} + 2Q^-\partial_{Q^-} + \partial_\phi, \\ P_+ &= \partial_{x_+}, \qquad P_- = \partial_{x_-}, \\ J_+ &= \partial_{\theta^+} + i\theta^+\partial_{x_+}, \qquad J_- = \partial_{\theta^-} + i\theta^-\partial_{x_-}. \end{aligned} \tag{3.5.1}$$

*The generators* $P_+$ *and* $P_-$ *correspond to translations in the bosonic variables* $x_+$ *and* $x_-$ *respectively. We also have four dilations, of which two,* $C_0$ *and* $K_0$*, involve only the bosonic dependent variables, while the other two,* $K_1^b$ *and* $K_2^b$*, involve both independent and dependent, and both bosonic and fermionic variables. Finally, we also recover the two supersymmetry generators* $J_+$ *and* $J_-$ *which were identified previously in* (3.4.4)*.*

*The algebra* $\mathfrak{g}_1'$ *of infinitesimal symmetries of the SUSY GW equations* (3.4.6) *are spanned by the following vector fields*

$$\begin{aligned} J_\pm &= \partial_{\theta^\pm} + i\theta^\pm\partial_{x_\pm}, \qquad P_\pm = \partial_{x_\pm}, \\ \hat{C}_0 &= H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} - 2f\partial_f + N_i\partial_{N_i}, \\ \hat{K}_0 &= -H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} + 2\partial_\phi - N_i\partial_{N_i}, \\ K_1^b &= -2x_+\partial_{x_+} - \theta^+\partial_{\theta^+} + R^+\partial_{R^+} + 2R^-\partial_{R^-} + S^-\partial_{S^-} \\ &\quad - T^+\partial_{T^+} + 2Q^+\partial_{Q^+} + \partial_\phi, \\ K_2^b &= -2x_-\partial_{x_-} - \theta^-\partial_{\theta^-} - R^-\partial_{R^-} + S^+\partial_{S^+} + 2T^+\partial_{T^+} \\ &\quad + T^-\partial_{T^-} + 2Q^-\partial_{Q^-} + \partial_\phi, \\ G_i &= F_i\partial_{F_i} + N_i\partial_{N_i}, \qquad B_i = \partial_{F_i}, \qquad \textit{for} \quad i = 1,2,3, \\ R_{ij} &= F_i\partial_{F_j} - F_j\partial_{F_i} + N_i\partial_{N_j} - N_j\partial_{N_i}, \qquad i < j = 2,3. \end{aligned} \tag{3.5.2}$$

*Using the projection operator* $\pi(\mathfrak{g}_1') = \mathfrak{g}_1'\omega$ *involving all dependent and independent variables of the SUSY GC equations* (3.4.8)*, where*

$$\omega = x_+\partial_{x_+} + x_-\partial_{x_-} + \theta^+\partial_{\theta^+} + \theta^-\partial_{\theta^-} + \phi\partial_\phi + H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} + f\partial_f + R^+\partial_{R^+} + R^-\partial_{R^-} + S^+\partial_{S^+} + S^-\partial_{S^-} + T^+\partial_{T^+} + T^-\partial_{T^-},$$

*and comparing the resulting vector fields with the generators of* $\mathfrak{g}_1$*, given by* (3.5.1)*, we conclude that* $\mathfrak{g}_1 = \pi(\mathfrak{g}_1')$*, which implies that the SUSY GC equations are non-integrable as in the classical case.*

**Example 3.5.2.** *As another example, we apply the conjecture to the SUSY sine-Gordon equation as formulated in [24]*

$$D_+D_-\Phi = i\sin\Phi, \tag{3.5.3}$$

*where* $\Phi$ *is a bosonic superfield. Its Lie symmetry superalgebra* $\mathfrak{g}_3$ *is spanned by the vector fields*

$$\begin{gathered} P_\pm = \partial_{x_\pm}, \qquad J_\pm = \partial_{\theta^\pm} + i\theta^\pm\partial_{x_\pm}, \\ \mathcal{K} = 2x_+\partial_{x_+} - 2x_-\partial_{x_-} + \theta^+\partial_{\theta^+} - \theta^-\partial_{\theta^-}. \end{gathered} \tag{3.5.4}$$

*The non-parametric linear problem (the GW equations) associated with the SUSY sine-Gordon equation* (3.5.3) *is given by*

$$\begin{gathered} D_\pm\Psi = B_\pm\Psi, \qquad \text{where} \quad \Psi = \begin{pmatrix} \psi_{11} & \psi_{12} & f_{13} \\ \psi_{21} & \psi_{22} & f_{23} \\ f_{31} & f_{32} & \psi_{33} \end{pmatrix}, \\ B_+ = \frac{1}{2}\begin{pmatrix} 0 & 0 & ie^{i\Phi} \\ 0 & 0 & -ie^{-i\Phi} \\ -e^{-i\Phi} & e^{i\Phi} & 0 \end{pmatrix}, \qquad B_- = \begin{pmatrix} iD_-\Phi & 0 & -i \\ 0 & -iD_-\Phi & i \\ -1 & 1 & 0 \end{pmatrix}. \end{gathered} \tag{3.5.5}$$

*Here the* $\psi_{ij}$ *are bosonic superfields and the* $f_{ij}$ *are fermionic superfields,* $i, j = 1, 2, 3$*. The infinitesimal symmetry generators* $\mathfrak{g}_3'$ *of equations* (3.5.5) *are spanned by the vector fields*

$$\begin{gathered} P_\pm = \partial_{x_\pm}, \qquad J_\pm = \partial_{\theta^\pm} + i\theta^\pm\partial_{x_\pm}, \qquad G_1 = \psi_{11}\partial_{\psi_{11}} + \psi_{21}\partial_{\psi_{21}} + f_{31}\partial_{f_{31}}, \\ G_2 = \psi_{12}\partial_{\psi_{12}} + \psi_{22}\partial_{\psi_{22}} + f_{32}\partial_{f_{32}}, \qquad G_3 = f_{13}\partial_{f_{13}} + f_{23}\partial_{f_{23}} + \psi_{33}\partial_{\psi_{33}}. \end{gathered} \tag{3.5.6}$$

*Using the projection operator* $\pi$ *defined as* $\pi(\mathfrak{g}_3') = \mathfrak{g}_3'\omega$*, where*

$$\omega = x_+\partial_{x_+} + x_-\partial_{x_-} + \theta^+\partial_{\theta^+} + \theta^-\partial_{\theta^-} + \Phi\partial_\Phi,$$

*we obtain the relation* $\mathfrak{g}_3 \supset \pi(\mathfrak{g}_3')$*, which implies that the SUSY sine-Gordon equation may be integrable, as in the classical case. The fact that the generator* $\mathcal{K}$ *in* (3.5.4) *does not appear in the symmetries of the linear problem* (3.5.5) *of the*

*SUSY sine-Gordon equations* (3.5.3) *allows us to introduce a bosonic spectral parameter $\lambda$ through $\mathcal{K}$. This is accomplished by introducing a one-parameter group associated with the dilation $\mathcal{K}$ through the transformation $\tilde{x}_+ = \lambda x_+$, $\tilde{x}_- = \lambda^{-1} x_-$, $\tilde{\theta}^+ = \lambda^{1/2}\theta^+$ and $\tilde{\theta}^- = \lambda^{-1/2}\theta^-$, $\lambda = \pm e^{\mu}$, where $\mu \in \Lambda_{\text{even}}$, into the linear system* (3.5.5) *which gives us*

$$D_+\Psi = B_+\Psi, \qquad D_-\Psi = B_-\Psi,$$

$$B_+ = \frac{1}{2\sqrt{\lambda}}\begin{pmatrix} 0 & 0 & ie^{i\Phi} \\ 0 & 0 & -ie^{-i\Phi} \\ -e^{-i\Phi} & e^{i\Phi} & 0 \end{pmatrix}, \qquad B_- = \begin{pmatrix} iD_-\Phi & 0 & -i\sqrt{\lambda} \\ 0 & -iD_-\Phi & i\sqrt{\lambda} \\ -\sqrt{\lambda} & \sqrt{\lambda} & 0 \end{pmatrix},$$

*which coincide with the results found in [**107**]. The ZCC of equation* (3.5.7) *takes the form* (3.4.7)*, where the matrices $A_+$ and $A_-$ are replaced by the matrices $B_+$ and $B_-$, respectively. The connection between the super-Darboux transformations and the super-Bäcklund transformations for the sine-Gordon equation* (3.5.3) *allows the construction of explicit multi-super-soliton solutions [**24**, **64**, **108**].*

**Example 3.5.3.** *The fermionic extension of the GC equations* (3.4.10) *involves seven unknown functions $\mathcal{U} = (\phi, H, Q^+, Q^-, R^+, T^-, f)$ where $\phi$ and $f$ are bosonic functions, while $H$, $Q^+$, $Q^-$, $R^+ = {\Gamma_{11}}^1$, $T^- = {\Gamma_{22}}^2$ are fermionic functions. Proceeding in a similar manner as in the bosonic SUSY case, we obtain a Lie symmetry superalgebra $\mathfrak{g}_2$ consisting of the following six bosonic infinitesimal generators [**B1**]*

$$\begin{aligned} P_+ &= \partial_{x_+}, \qquad P_- = \partial_{x_-}, \\ C_0 &= H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} - 2f\partial_f, \\ K_0 &= -H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} + 2\partial_\phi, \\ K_1^f &= -2x_+\partial_{x_+} - \theta^+\partial_{\theta^+} + 2Q^+\partial_{Q^+} + R^+\partial_{R^+} + \partial_\phi, \\ K_2^f &= -2x_-\partial_{x_-} - \theta^-\partial_{\theta^-} + 2Q^-\partial_{Q^-} + T^-\partial_{T^-} + \partial_\phi, \end{aligned} \tag{3.5.7}$$

*together with the three fermionic generators*

$$J_+ = \partial_{\theta^+} + i\theta^+\partial_{x_+}, \qquad J_- = \partial_{\theta^-} + i\theta^-\partial_{x_-}, \qquad W = \partial_H. \tag{3.5.8}$$

*The generators $W$ and $P_\pm$ correspond to translations in the fermionic variable $H$ and the bosonic variables $x_\pm$ respectively. We obtain two dilations $C_0$ and $K_0$ involving only dependent variables, together with two additional dilations $K_1^f$ and $K_2^f$, which involve both dependent and independent variables. We also recover the two supersymmetry generators $J_+$ and $J_-$.*

*The Lie symmetry algebra* $\mathfrak{g}_2'$ *of the SUSY GW equations* (3.4.9) *is spanned by the vector fields*

$$
\begin{gathered}
P_\pm = \partial_{x_\pm}, \qquad J_\pm = \partial_{\theta^\pm} + i\theta^\pm \partial_{x_\pm}, \\
\hat{C}_0 = H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} - 2f\partial_f + N_i\partial_{N_i}, \\
\hat{K}_0 = -H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} + 2\partial_\phi - N_i\partial_{N_i}, \\
K_1^f = -2x_+\partial_{x_+} - \theta^+\partial_{\theta^+} + 2Q^+\partial_{Q^+} + R^+\partial_{R^+} + \partial_\phi, \\
K_2^f = -2x_-\partial_{x_-} - \theta^-\partial_{\theta^-} + 2Q^-\partial_{Q^-} + T^-\partial_{T^-} + \partial_\phi, \\
G_i = F_i\partial_{F_i} + N_i\partial_{N_i}, \qquad B_i = \partial_{F_i}, \qquad \text{for} \quad i, j = 1, 2, 3, \\
R_{ij} = F_i\partial_{F_j} - F_j\partial_{F_i} + N_i\partial_{N_j} - N_j\partial_{N_i}, \qquad i < j = 2, 3.
\end{gathered}
$$

*By using the projector* $\pi$*, defined as* $\pi(\mathfrak{g}_2') = \mathfrak{g}_2'\omega$*, where*

$$
\begin{aligned}
\omega = x_+\partial_{x_+} + x_-\partial_{x_-} + \theta^+\partial_{\theta^+} + \theta^-\partial_{\theta^-} + H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} \\
+ \phi\partial_\phi + f\partial_f + R^+\partial_{R^+} + T^-\partial_{T^-},
\end{aligned}
$$

*we obtain that the set of symmetries of the SUSY GW equations* (3.4.9) *is a proper subset of the set of Lie symmetries of the SUSY GC equations* (3.4.10)*. More specifically the translation in* $H$ *generated by* $W$ *is not a symmetry of the SUSY GW equations. Therefore we can introduce a fermionic parameter* $\underline{\lambda}$ *in the SUSY GW equations* (3.4.9) *with the potential matrices*

$$
A_+ = \begin{pmatrix} R^+ & 0 & Q^+f \\ 0 & 0 & -\frac{1}{2}e^\phi(H+\underline{\lambda})f \\ H+\underline{\lambda} & -2e^{-\phi}Q^+ & 0 \end{pmatrix},
$$

$$
A_- = \begin{pmatrix} 0 & 0 & \frac{1}{2}e^\phi(H+\underline{\lambda})f \\ 0 & T^- & Q^-f \\ -2e^{-\phi}Q^- & -(H+\underline{\lambda}) & 0 \end{pmatrix}.
$$

*The parameter* $\underline{\lambda}$ *cannot be eliminated through a gauge transformation. This suggests that the fermionic version of the SUSY GC equations* (3.4.10) *may be integrable.*

## 3.6. Concluding remarks and outlook

The objective of this paper was to compare the symmetries of the SUSY GW equations with those of the SUSY GC equations for both the bosonic and fermionic extensions. This comparison allowed us to formulate a generalization of the conjecture establishing the necessary conditions for a system to be integrable in the sense of soliton theory. The symmetry analysis developed in this paper could be extended in several directions. First, it should be noted that the list of symmetries for the bosonic and fermionic SUSY structural equations found in this paper is not necessarily exhaustive since the symmetry criterion has not been proven for equations involving Grassmann variables. A comprehensive list of all symmetries of the bosonic and fermionic SUSY Gauss–Weingarten and SUSY Gauss–Codazzi equations could be compiled. This would require the development of a computer Lie algebra symmetry package capable of handling both bosonic and fermionic symmetries. Another possibility would be to extend the procedure to hypersurfaces in higher dimensions. It could also be worth attempting to establish a SUSY version of Noether's theorem in order to determine conserved quantities, and to derive a SUSY version of the Weierstrass–Enneper formula for the immersion of surfaces in a multidimensional superspace. One could also investigate how characteristics associated with integrable models such as Hamiltonian structures and conserved quantities manifest themselves in the SUSY case. Another worthwhile subject is a variational problem of geometric functionals (e.g., Willmore functionals), which can be interpreted as actions from which we can determine the Euler–Lagrange equations for a given surface immersed in a superspace. Recurrence operators of generalized symmetries of the SUSY GC equations could be used to obtain the recurrence relations for the surfaces. A complete invariant geometrical characterization of these surfaces in the superspace remains to be done. A singularity analysis of the SUSY system under consideration could be performed in connection with Lie groups in order to verify the Painlevé property. This would be motivated by the goal of obtaining explicit analytic solutions. Such analytic solutions can be useful for observing the qualitative behaviour of solutions which would otherwise be difficult to detect numerically. The existence of different types of soliton solutions constitutes such an example. An essential step in the further development of the theory of surfaces associated with SUSY integrable systems would be a generalization of the known formulas for constructing soliton surfaces immersed in Lie algebras, namely the Sym–Tafel [**112**, **113**] and the Fokas–Gel'fand [**55**] formulas. Our procedure for introducing a spectral parameter in the GW equations could make this task feasible.

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# Chapitre 4

## SUPERSYMMETRIC VERSIONS OF THE FOKAS–GEL'FAND FORMULA FOR IMMERSION

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**Résumé**

Dans cet article, nous construisons et investiguons deux versions supersymétriques de la formule de Fokas–Gel'fand pour l'immersion de surfaces bidimensionnelles associées à un système supersymétrique intégrable. La première version implique une déformation infinitésimale de la condition de courbure nulle et du problème linéaire spectral associé à ce système. Cette déformation mène à des surfaces représentées en termes de supermatrices bosoniques plongées dans une superalgèbre de Lie. La seconde version supersymétrique est obtenue en utilisant un paramètre fermionique pour construire des surfaces exprimées en termes de supermatrices fermioniques plongées dans une superalgèbre de Lie. Pour les deux extensions, nous fournissons une caractérisation supersymétrique des surfaces déformées en utilisant la super forme de Killing, en tant que produit scalaire, et un formalisme de super repère mobile. Ces résultats théoriques sont appliqués à l'équation supersymétrique de sine-Gordon dans le but de construire des surfaces supersolitoniques associées à cinq différentes symétries. Nous trouvons les formes intégrées de ces surfaces qui représentent des surfaces à courbure de Gauss constante et des surfaces de type Weingarten non linéaires.

**Abstract**

In this paper, we construct and investigate two supersymmetric versions of the Fokas–Gel'fand formula for the immersion of 2D surfaces associated with a supersymmetric integrable system. The first version involves an infinitesimal deformation of the zero-curvature condition and the linear spectral problem associated with this system. This deformation leads the surfaces to be represented in terms of a bosonic supermatrix immersed in a Lie superalgebra. The second supersymmetric version is obtained by using a fermionic parameter deformation to construct surfaces expressed in terms of a fermionic supermatrix immersed in a Lie superalgebra. For both extensions, we provide a geometrical characterization of deformed surfaces using the super Killing form as an inner product and a super moving frame formalism. The theoretical results are applied to the supersymmetric sine-Gordon equation in order to construct super soliton surfaces associated with five different symmetries. We find integrated forms of these surfaces which represent constant Gaussian curvature surfaces and nonlinear Weingarten-type surfaces.

## 4.1. Introduction

In recent decades, an increasing number of supersymmetric (SUSY) extensions for quantum and classical models have been investigated (see e.g. [**13**,**37**,**76**,**79**]). In particular, super soliton solutions have been determined for SUSY extensions of various integrable systems of partial differential equations (PDEs), such as the SUSY sine-Gordon equation [**3**, **24**, **31**, **63**, **64**, **69**, **121**], the SUSY Korteweg–de Vries equation [**24**, **83**, **88**, **90**], the SUSY Schrödinger equations [**22**, **37**, **76**], the SUSY Sawada–Kotera equations [**116**] and the SUSY Hirota–Satsuma equations [**35**, **98**]. Super soliton solutions were obtained using the connection between the super-Darboux transformations and the super-Bäcklund transformations (see e.g. [**3**, **24**, **63**, **64**, **69**, **88**, **91**, **108**, **116**] and references therein).

Supersymmetric versions of the equations of conformally parametrized surfaces provide rich classes of geometric objects [**38**, **40**, **B1**, **B2**, **B6**]. In fact, until very recently, the formulation of two distinct SUSY extensions of the Gauss–Weingarten and Gauss–Codazzi (GC) equations for conformally parametrized surfaces immersed in a Grassmann superspace, one in terms of a bosonic superfield and the other in terms of a fermionic superfield, were the only known examples [**39**, **77**].

On the other hand, the subalgebras of Lie point symmetries of the bosonic and fermionic SUSY GC equations were recently established in [**B1**, **B2**]. The classification of the 1D subalgebras of each superalgebra into conjugacy classes has been performed. The symmetry reduction method was used to find invariants and reduced systems associated with the SUSY GC extensions [**B1**,**B2**]. This approach allowed us to construct explicit solutions of these reduced SUSY systems, which correspond to different classes of surfaces immersed in a Grassmann superspace. These extensive results make conformally parametrized surfaces a rather special and interesting object of study.

This paper is concerned with the investigation of different geometric aspects of these surfaces obtained in connection with integrable systems. Our main objective is to provide SUSY versions of immersion formulas for constructing large families of surfaces in Lie superalgebras linked with integrable SUSY GC equations. In order to achieve this goal we investigate certain features of point and generalized symmetries of SUSY integrable systems. The construction of the SUSY versions of the Fokas-Gel'fand (FG) formula for the immersion of 2D surfaces in Lie superalgebras is presented in detail. We demonstrate that a SUSY generalization of the classical main result on the immersion of 2D surfaces in a Lie algebra can be constructed. We show for these SUSY extensions that if there exists a common symmetry of the zero-curvature representation (ZCR) of an integrable system and its linear spectral problem (LSP) then the FG immersion formula is applicable in its original form. For this purpose, we write the SUSY version formula for immersion functions of 2D surfaces in Lie superalgebras in terms of vector fields and their prolongations rather than the notion of Fréchet derivatives [**55**, **72**]. In the classical case, the possibility of using a ZCR and its LSP to represent a moving frame on integrable surfaces has yielded many new results concerning the intrinsic geometric properties of such surfaces (see e.g. [**52**, **53**, **55**]). The results obtained for the classical case were so promising that it seemed to be worthwhile to try to extend this method and check its effectiveness for the SUSY case.

One of the purposes of this paper is to formulate a SUSY extension of the FG formula, which is obtained by applying a bosonic infinitesimal deformation to the potential matrices and the wavefunction of the LSP associated with the initial system in such a way that the deformed surface takes the form of a bosonic supermatrix. Next, another SUSY extension is derived using a fermionic parameter deformation of the potential matrices and the wavefunction, which implies that deformed surfaces take the form of a fermionic supermatrix. For both extensions, we provide geometrical characterizations of the deformed surfaces using the super Killing form as an inner product together with a SUSY version of the moving frame on the surface. These theoretical considerations are applied to the SUSY sine-Gordon equation. Surfaces associated with five different symmetries are investigated using the Sym–Tafel immersion formula, two gauge transformations and two Lie point symmetries. For each surface, we provide a geometric characterization via the Gaussian and mean curvatures based on the SUSY versions of the first and second fundamental forms. This is, in short, the aim of the paper.

The paper is organized as follows. In section 4.2, a brief exposition of the classical FG formula for the immersion of 2D surfaces in Lie algebras is presented for integrable systems. Section 4.3 contains an overview of the Grassmann algebra formalism and introduces the notation used in this paper. Section 4.4 is devoted to the construction of two SUSY versions of the FG formula for the immersion of 2D surfaces in Lie superalgebras. More specifically, in section 4.4.1, we formulate the bosonic immersion of super soliton surfaces, while section 4.4.2 describes the fermionic immersion of super soliton surfaces. In section 4.5, surfaces associated with five different symmetries of the SUSY sine-Gordon equation are constructed, namely the Sym–Tafel immersion formula, a bosonic gauge transformation, a bosonic symmetry deformation, a fermionic gauge transformation and a fermionic symmetry deformation, respectively. The conclusions and some possible future developments are presented in section 4.6.

## 4.2. Immersion formula for soliton surfaces

Consider an integrable system of PDEs

$$\Delta[u] = 0, \tag{4.2.1}$$

in two independent variables $x_1, x_2$ and the dependent variables $u^k(x_1, x_2)$, which can be linearized by a matrix LSP given by

$$D_{x_\alpha}\Phi([u],\lambda) = U_\alpha([u],\lambda)\Phi([u],\lambda), \qquad \alpha = 1, 2. \tag{4.2.2}$$

We use the abbreviated notation $[u] = (x_1, x_2, u^k, u^k_J)$ of an element of the jet space, where

$$u^k_J = \frac{\partial^n u^k}{\partial x_{j_1} \dots \partial x_{j_n}}, \qquad J = (j_1, \dots, j_n), \qquad |J| = n, \qquad j_i = 1, 2$$

with the total derivatives

$$D_\alpha = \partial_{x_\alpha} + \sum_J u^k_{J,\alpha} \frac{\partial}{\partial u^k_J}, \qquad \alpha = 1, 2.$$

The compatibility conditions of (4.2.2) are in the form of a zero-curvature condition (ZCC) which is assumed to be valid for all values of the spectral parameter $\lambda \in \mathbb{C}$. This requirement implies that

$$D_2 U_1 - D_1 U_2 + [U_1, U_2] = 0, \tag{4.2.3}$$

which is equivalent to the original PDEs (4.2.1). It was shown [**110**, **112**, **113**] that if a solution $\Phi([u],\lambda)$ of the LSP (4.2.2) is an element of a Lie group $G$ and $U_\alpha([u],\lambda)$ are functions in the associated Lie algebra $\mathfrak{g}$, then the function

$$F([u],\lambda) = \Phi^{-1}([u],\lambda)(\partial_\lambda \Phi([u],\lambda)), \tag{4.2.4}$$

where $\partial_\lambda$ is the partial derivative with respect to the spectral parameter $\lambda$, takes values in the Lie algebra $\mathfrak{g}$. The function $F$ can be interpreted for a fixed value of $\lambda$ as a surface in a Lie algebra $\mathfrak{g}$ provided that the tangent vectors

$$D_\alpha F = \Phi^{-1}(\partial_\lambda U_\alpha)\Phi, \qquad \alpha = 1,2$$

are linearly independent. Such a formula, which was first proposed by Sym [**111**] and Tafel [**113**], and subsequently used by many authors (see e.g. [**19**, **103**]), allowed the link between classical geometry and integrable systems to be established, leading to the requirement that all 2D soliton solutions be represented by a one-parameter family of surfaces parametrized by the spectral parameter [**19**]. Since then, the applicability of the Sym–Tafel formula for immersion to geometric problems of 2D surfaces related to integrable equations has been extended. In particular, new terms have been added to its original form (4.2.4). As proven in [**54**], for any $\mathfrak{g}$-valued matrix functions $A_\alpha([u],\lambda)$, $\alpha = 1,2$, which satisfy

$$D_2 A_1 - D_1 A_2 + [A_1, U_2] + [U_1, A_2] = 0, \tag{4.2.5}$$

there exists a $\mathfrak{g}$-valued immersion function $F$ with tangent vectors given by

$$D_\alpha F = \Phi^{-1} A_\alpha \Phi, \qquad \alpha = 1,2.$$

Whenever the matrix functions $A_1$ and $A_2$ are linearly independent, $F$ is an immersion function for a 2D surface in the Lie algebra $\mathfrak{g}$. As proven in [**55**, **72**], three linearly independent terms which satisfy (4.2.5) are given by

$$A_\alpha = \beta(\lambda)\partial_\lambda U_\alpha + (D_\alpha S + [S, U_\alpha]) + \mathrm{pr}\omega_R U_\alpha \in \mathfrak{g}, \qquad \alpha = 1,2$$

where $\beta(\lambda)$ is an arbitrary scalar function of $\lambda$, $S([u],\lambda)$ is an arbitrary $\mathfrak{g}$-valued function of $[u]$ and $\lambda$, and

$$\omega_R = R^k[u]\partial_{u^k}$$

is the vector field, written in evolutionary form, of the generalized symmetry of the integrable PDEs (4.2.1), while

$$\mathrm{pr}\omega_R = \omega_R + D_J R^k \partial_{u^k_J}$$

is the prolongation of vector field $\omega_R$. Furthermore, it has been proven in [**55**, **72**] that the $\mathfrak{g}$-valued function $F$ can be explicitly integrated as

$$F = \beta(\lambda)\Phi^{-1}(\partial_\lambda \Phi) + \Phi^{-1}S\Phi + \Phi^{-1}(\mathrm{pr}\omega_R \Phi) \tag{4.2.6}$$

as long as $\omega_R$ is a generalized symmetry of the integrable PDEs (4.2.1) and its LSP (4.2.2). The three terms in (4.2.6) correspond to conformal transformations of the spectral parameter $\lambda$ (the Sym–Tafel formula for immersion [**110**, **112**, **113**]), a gauge symmetry of the LSP (due to Cieslinski and Doliwa [**26**, **49**]) and a generalized common symmetry of the ZCC (4.2.3) and the LSP (4.2.2) (proposed by Fokas and Gel'fand [**54**] and further developed in [**55**, **72**]).

The second term in (4.2.6), associated with the gauge symmetry of the LSP (4.2.2)), can be integrated explicitly as

$$F^S = \Phi^{-1}S([u], \lambda)\Phi \in \mathfrak{g},$$

which is consistent with the tangent vectors

$$D_\alpha F^S = \Phi^{-1}(D_\alpha S + [S, U_\alpha])\Phi. \qquad \alpha = 1, 2$$

For $F$ to be an immersion function of a 2D surface, we require that the tangent vectors be linearly independent. Note that any surface $P \in \mathfrak{g}$ can be expressed as $P = \Phi^{-1}S\Phi = F^S$ and hence $F^S$ represents a completely arbitrary surface immersed in a Lie algebra $\mathfrak{g}$. So we can interpret the surface $F^S$ as an arbitrary surface immersed in the Lie algebra $\mathfrak{g}$ written in the frame defined by conjugation by the wavefunction $\Phi$, an element of the Lie group $G$.

The third term in (4.2.6) corresponds to the FG formula for immersion, which is applicable in its original form under the condition that the vector field $\Omega_R$ is a common symmetry of both the original system (4.2.1) and its LSP (4.2.2) [**65**]. In this case the matrices

$$A_\alpha = \mathrm{pr}\omega_R U_\alpha \tag{4.2.7}$$

identically satisfy the determining equation (4.2.5). In the derivation of (4.2.7) we have used the fact that the total derivatives $D_\alpha$ commute with the prolongation of a vector field $\omega_R$ written in evolutionary form [**95**], that is

$$[D_\alpha, \mathrm{pr}\omega_R] = 0, \qquad \alpha = 1, 2.$$

Thus, there exists a $\mathfrak{g}$-valued immersion function $F$ with tangent vectors

$$D_\alpha F = \Phi^{-1}(\mathrm{pr}\omega_R U_\alpha)\Phi.$$

Further, the immersion function $F$ can be integrated as

$$F = \Phi^{-1}(\mathrm{pr}\omega_R \Phi) \in \mathfrak{g},$$

if and only if the vector field $\omega_R$ is also a generalized symmetry of the LSP (4.2.2).

In section 4.4.1 and section 4.4.2, the three terms in the immersion formula (4.2.6) will be used to construct two SUSY versions of the FG formula for the immersion of 2D surfaces in Lie superalgebras.

## 4.3. Preliminaries on Grassmann algebras

In this section, we present an overview of the definitions and formalism used throughout this paper. A more detailed description of Grassmann algebra can be found in [**11**, **16**, **33**, **43**, **45**, **56**, **80**, **114**, **118**, **119**] and the references therein.

The complex Grassmann algebra $\mathbb{G}$ (denoted $\mathbb{C}B_L$ in [**33**]) is a commutative associative algebra generated by a set of odd elements $\xi_j$ together with the unit 1, where

$$\xi_j \xi_k + \xi_k \xi_j = 0 \qquad \text{and} \qquad 1\xi_j = \xi_j.$$

Therefore any odd generator (or any odd element) of $\mathbb{G}$ satisfies the property

$$\xi_j \xi_j = 0 \qquad \text{(no summation)}.$$

An odd element of $\mathbb{G}$ is composed of a linear combinaison of odd products of generators (e.g. $\xi_1 + \xi_1\xi_2\xi_3$), while an even element of $\mathbb{G}$ is composed of a linear combinaison of even products of generators (e.g. $1 + \xi_1\xi_2$). The number of generators for each case is not specified, but we consider that there is a sufficient number of them to make all considered formulas meaningful. The degree of a homogeneous element $a \in \mathbb{G}$ is defined to be

$$\deg(a) = \begin{cases} 0 & \text{for an even element,} \\ 1 & \text{for an odd element.} \end{cases}$$

One can also define the concept of a $(p+q) \times (r+s)$ supermatrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where the submatrices $A$, $B$, $C$ and $D$ are of dimensions $p \times r$, $q \times r$, $p \times s$ and $q \times s$, respectively. The matrix $M$ is said to be an even supermatrix (or an even element of the Lie superalgebra $\mathfrak{gl}(p|q, \mathbb{G})$ if $r = p$ and $q = s$) if the submatrices $A$ and $D$ take their values in the even elements of $\mathbb{G}$ and if $B$ and $C$ take their values in the odd elements of $\mathbb{G}$. Conversely, the matrix $M$ is said to be an odd supermatrix (or an odd element of the Lie superalgebra $\mathfrak{gl}(p|q,\mathbb{G})$ if $r=p$ and $q=s$) if the the submatrices $A$ and $D$ take their values in the odd elements of $\mathbb{G}$ and if $B$ and $C$ take their values in the even elements of $\mathbb{G}$. The degree of a supermatrix is defined similarly to the degree of an element of $\mathbb{G}$, which is

$$\deg(M)=\begin{cases}0 & \text{if } M \text{ is an even supermatrix,}\\ 1 & \text{if } M \text{ is an odd supermatrix.}\end{cases}$$

The set of square $(p+q)\times(p+q)$ supermatrices with complex entries forms the Lie superalgebra $\mathfrak{gl}(p|q,\mathbb{G})$ and any Lie superalgebra $\mathfrak{g}$ has to satisfy the Lie super bracket

$$M_1M_2-(-1)^{\deg(M_1)\deg(M_2)}M_2M_1=M_3\in\mathfrak{g}$$

for any $M_1,M_2\in\mathfrak{g}$. The Lie super bracket will be denoted by the commutator and anticommutator,

$$[M_1,M_2]=M_1M_2-M_2M_1 \qquad \text{and} \qquad \{M_1,M_2\}=M_1M_2+M_2M_1,$$

respectively, depending on the degree of $M_1$ and $M_2$. The associated Lie supergroup $GL(p|q,\mathbb{G})$ is composed of all (even) supermatrices of dimension $(p+q)\times(p+q)$ that are invertible.

In this paper, we use the convention that partial derivatives involving odd variables satisfy the Leibniz rule

$$\partial_{\theta^j}(hg)=(\partial_{\theta^j}h)g+(-1)^{\deg(h)}h\partial_{\theta^j}g,$$

and

$$f_{\theta^2\theta^1}=\partial_{\theta^1}(\partial_{\theta^2}f)=-\partial_{\theta^2}(\partial_{\theta^1}f)=-f_{\theta^1\theta^2}.$$

The partial derivatives with respect to odd coordinates change the parity of an even function to an odd function and vice versa.

In this paper, we do not follow the implicit notation for the odd derivative of a supermatrix (or multiplication by an odd scalar) used in [**33**], e.g.

$$\partial_{\theta^j}M=\begin{pmatrix}\partial_{\theta^j}A & \partial_{\theta^j}B\\ -\partial_{\theta^j}C & -\partial_{\theta^j}D\end{pmatrix}.$$

Therefore, we introduce the matrix $E$ such that

$$E\partial_{\theta^j}M=\begin{pmatrix}\partial_{\theta^j}A & \partial_{\theta^j}B\\ -\partial_{\theta^j}C & -\partial_{\theta^j}D\end{pmatrix}, \qquad E=\begin{pmatrix}I_p & 0\\ 0 & -I_q\end{pmatrix},$$

where $I_p$ is the $p\times p$ identity matrix. One should note that in this paper the terms even and odd are equivalent to bosonic and fermionic, respectively.

By considering the super Killing form $B : \mathfrak{gl}(p|q, \mathbb{G}) \times \mathfrak{gl}(p|q, \mathbb{G}) \to \mathbb{G}$ defined using the supertrace [**33**],

$$\langle M, N\rangle = \alpha \,\mathrm{str}(MN) = \alpha \mathrm{tr}(E^{\deg(MN)+1} MN), \qquad E = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}, \tag{4.3.1}$$

where $\alpha$ is a nonzero real constant (e.g. in the examples of section 4.5, $\alpha = 1/2$) and $M, N$ are supermatrices in $\mathfrak{gl}(p|q, \mathbb{G})$, we can introduce an inner product $\langle \cdot, \cdot \rangle$, which has the following properties :

(1) Left linearity,

$$\langle M + N, P\rangle = \langle M, P\rangle + \langle N, P\rangle.$$

(2) Right linearity,

$$\langle M, N + P\rangle = \langle M, N\rangle + \langle M, P\rangle.$$

(3) Inner permutation,

$$\langle MN, P\rangle = \langle M, NP\rangle.$$

(4) Outer permutation,

$$\langle M, N\rangle = (-1)^{\deg(M)\deg(N)} \langle N, M\rangle.$$

(5) Invariance under group conjugation,

$$\langle S^{-1}MS, S^{-1}NS\rangle = \langle M, N\rangle,$$

for $S \in GL(p|q, \mathbb{G})$.

(6) Supercommutator,

$$\langle M, [N, P]\rangle = \langle [M, N], P\rangle$$

or

$$\langle M, \{N, P\}\rangle = \langle \{M, N\}, P\rangle,$$

depending on the degree of $M$, $N$ and $P$ for $\deg(M) = \deg(N) = \deg(P)$.

One should note from property (6) that the commutator/anticommutator acts as the vector product for the purpose of obtaining an orthogonal vector, e.g.

$$\langle M, [M, N]\rangle = \langle [M, M], N\rangle = 0.$$

## 4.4. SUSY versions of the Fokas–Gel'fand formula

Let $\Delta[u] = 0$ be an integrable system of PDEs in terms of the bosonic independent variables $x_1$ and $x_2$, the fermionic independent variables $\theta^1$ and $\theta^2$, and the SUSY dependent variables $u^k$ and their derivatives. Also, let us assume that there exists an associated LSP of the form

$$\Omega(\lambda,[u]) = D_j\Psi(\lambda,[u]) - U_j(\lambda,[u])\Psi(\lambda,[u]) = 0, \qquad j=1,2 \tag{4.4.1}$$

where $\Psi(\lambda,[u]) \in GL(p|q,\mathbb{G})$, $U_j(\lambda,[u])$ are fermionic supermatrices in $\mathfrak{gl}(p|q,\mathbb{G})$, $\lambda$ is a spectral parameter and the covariant derivatives

$$D_j = \partial_{\theta^j} - i\theta^j\partial_{x_j}, \qquad j=1,2$$

satisfy the properties

$$\{D_1,D_2\}=0, \qquad D_j^2 = -i\partial_{x_j}.$$

The compatibility conditions of the LSP (4.4.1) (i.e. the ZCC) are given by

$$D_1U_2 + D_2U_1 - \{EU_1,EU_2\} = 0, \qquad \text{where} \qquad E = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}, \tag{4.4.2}$$

which, for any value of $\lambda$, is equivalent to the original system of PDEs $\Delta[u]=0$. One should note that the fermionic derivatives

$$J_k = \partial_{\theta^k} + i\theta^k\partial_{x_k}, \qquad k=1,2 \tag{4.4.3}$$

anticommute with the differential generators $D_j$ and generate the SUSY transformations

$$x_k \to x_k + i\gamma\theta^k, \qquad \theta^k \to \theta^k + i\gamma, \qquad k=1,2$$

where $\gamma$ is a odd-valued parameter.

### 4.4.1. SUSY version of the Fokas–Gel'fand formula for bosonic immersion

We now consider an infinitesimal transformation on the potential supermatrices $U_j$,

$$\tilde{U}_1 = U_1 + \epsilon A_1, \qquad \tilde{U}_2 = U_2 + \epsilon A_2, \tag{4.4.4}$$

where the matrices $A_j(\lambda,[u])$ are fermionic supermatrices in $\mathfrak{gl}(p|q,\mathbb{G})$ and $\epsilon$ is an infinitesimal bosonic parameter such that $\epsilon^2$ is negligeable. We also consider the infinitesimal transformation on the wavefunction $\Psi$ given by

$$\tilde{\Psi} = \Psi(I+\epsilon F). \tag{4.4.5}$$

Assuming that these infinitesimal transformations preserve the LSP,

$$D_j\tilde{\Psi} = \tilde{U}_j\tilde{\Psi}, \qquad j=1,2 \tag{4.4.6}$$

we obtain a deformed surface $F(\lambda,[u])$ expressed in terms of bosonic supermatrices of the Lie superalgebra $\mathfrak{gl}(p|q,\mathbb{G})$ under the condition that the tangent vectors are linearly independent. One can determine that the tangent vectors $ED_j(F)$ are

$$ED_j(F) = \Psi^{-1}EA_j\Psi. \tag{4.4.7}$$

Moreover, the compatibility conditions of the tangent vectors (4.4.7) are equivalent to the infinitesimal transformation of the ZCC (4.4.2), which is

$$D_1A_2 + D_2A_1 - \{EA_1, EU_2\} - \{EA_2, EU_1\} = 0. \tag{4.4.8}$$

Let us consider the deformed surface (an analogue of the classical case, see equation (4.2.6))

$$F = \Psi^{-1}\beta(\lambda)(\partial_\lambda\Psi) + \Psi^{-1}ES\Psi + \Psi^{-1}(\mathrm{pr}\omega\Psi). \tag{4.4.9}$$

We use the spectral symmetry generator $\beta(\lambda)\partial_\lambda$, where $\beta(\lambda)$ is an arbitrary function such that $\deg(\beta) = \deg(\lambda)$ (i.e. if $\lambda$ is bosonic, then $\beta(\lambda)$ is bosonic and if $\lambda$ is fermionic, then $\beta(\lambda)$ is fermionic). The gauge $S(\lambda,[u])$ is an even supermatrix in $\mathfrak{gl}(p|q,\mathbb{G})$ and the bosonic generator $\omega$ spans a symmetry transformation for both the ZCC $\Delta[u] = 0$ and the LSP $\Omega(\lambda,[u]) = 0$. Then, the supermatrices $A_j$ take the form

$$A_j = \beta(\lambda)\partial_\lambda U_j + E\left(D_jS + [ES, EU_j]\right) + \left(\mathrm{pr}\omega U_j + \left([D_j, \mathrm{pr}\omega]\Psi\right)\Psi^{-1}\right), \tag{4.4.10}$$

which satisfy equation (4.4.8).

**Proposition 4.4.1.** *Let us assume that there exists an LSP of the form* (4.4.1) *associated with a SUSY integrable system of PDEs* $\Delta[u] = 0$ *such that the ZCC* (4.4.2) *is equivalent to* $\Delta[u] = 0$. *If we consider the bosonic infinitesimal deformations* (4.4.4) *and* (4.4.5) *that preserve both the LSP and the ZCC, i.e.* $A_1, A_2$ *and* $F$ *must satisfy equations* (4.4.6)-(4.4.8)*, then there exists an immersion bosonic supermatrix* $F$ *which defines a 2D surface provided that its tangent vectors* (4.4.7) *are linearly independent.*

**Corollary 4.4.1.** *If one considers a deformed surface* $F$*, as defined in proposition 4.4.1, of the form* (4.4.9)*, then the linearly independent supermatrices* $A_1$ *and* $A_2$ *appearing in the tangent vectors* (4.4.7) *take the form* (4.4.10)*.*

A geometric characterization of the deformed surface $F$ can lead us to a better understanding of the PDEs under investigation. However, an explicit solution for the wavefunction $\Psi$ can, in some cases, be a task harder to accomplish than to solve the initial PDEs. Therefore, by choosing an inner product which is invariant under the automorphism $\mathfrak{g} \rightarrow \Psi^{-1}\mathfrak{g}\Psi$, we can eliminate the wavefunction $\Psi$

and obtain a pseudo-Riemannian immersion formula. Throughout this paper, we consider the super Killing form defined in equation (4.3.1).

Using the super Killing form, we can define the coefficients of the first fundamental form to be

$$g_{ij} = \langle ED_i(F), ED_j(F)\rangle = \langle EA_i, EA_j\rangle,$$

which are bosonic quantities. However, this inner product requires that the coefficients $g_{ii}$ be zero. In order to lift the degeneracy of this inner product, it is convenient to use the alternative definition

$$g_{ii} = \langle EA_i, A_j\rangle, \qquad g_{12} = -g_{21} = \langle EA_1, EA_2\rangle, \tag{4.4.11}$$

which we use throughout the rest of the paper for the bosonic immersion. For both definitions, the first fundamental form is given by

$$I = (d_1)^2 g_{11} + 2d_1 d_2 g_{12} + (d_2)^2 g_{22},$$

where the $d_j$ are fermionic differential forms which are the infinitesimal displacement in the direction of $D_j$ and satisfy the relation

$$\{d_1, d_2\} = 0.$$

These operators are defined as [**B1**]

$$d_j = d\theta^j + i dx_j \partial_{\theta^j}. \qquad j = 1, 2 \tag{4.4.12}$$

In order to construct the second fundamental form, we introduce a unit normal vector $N$ in terms of a bosonic supermatrix which has the properties

$$\langle N, N\rangle = 1, \qquad \langle ED_jF, N\rangle = 0, \qquad j = 1, 2.$$

A unit normal vector can be given by

$$N = \frac{\{ED_1(F), ED_2(F)\}}{\langle\{ED_1(F), ED_2(F)\}, \{ED_1(F), ED_2(F)\}\rangle^{1/2}},$$

or equivalently

$$N = \frac{\Psi^{-1}\{EA_1, EA_2\}\Psi}{\langle\{EA_1, EA_2\}, \{EA_1, EA_2\}\rangle^{1/2}}. \tag{4.4.13}$$

Therefore, the coefficients of the second fundamental form are given by the bosonic quantities

$$b_{ij} = \langle D_j D_i F, N\rangle = \langle D_j A_i - \{EA_i, EU_j\}, \Psi N \Psi^{-1}\rangle, \tag{4.4.14}$$

which have the property $b_{12}=-b_{21}$. The second fundamental form is

$$I\!I=(d_1)^2b_{11}+2d_1d_2b_{12}+(d_2)^2b_{22}.$$

The Gaussian and mean curvatures are given, respectively, by

$$K=\frac{b_{11}b_{22}-b_{12}b_{21}}{g_{11}g_{22}-g_{12}g_{21}}=\frac{b_{11}b_{22}+(b_{12})^2}{g_{11}g_{22}+(g_{12})^2}, \tag{4.4.15}$$

$$H=\frac{b_{11}g_{22}+b_{22}g_{11}-b_{12}g_{21}-b_{21}g_{12}}{2(g_{11}g_{22}-g_{12}g_{21})}=\frac{b_{11}g_{22}+b_{22}g_{11}+2b_{12}g_{12}}{2(g_{11}g_{22}+(g_{12})^2)}, \tag{4.4.16}$$

where both curvatures are bosonic quantities. One should note that the coefficients $g_{ij}$ and $b_{ij}$ are explicitly given using the supermatrices $U_j$, $A_j$ and $N$.

### 4.4.2. SUSY version of the Fokas–Gel'fand formula for fermionic immersion

We now consider a transformation on the potential supermatrices $U_j$,

$$\tilde{U}_1=U_1+\epsilon EA_1,\qquad \tilde{U}_2=U_2+\epsilon EA_2, \tag{4.4.17}$$

where $\epsilon$ is a fermionic parameter, the matrices $A_j(\lambda,[u])$ are bosonic supermatrices in $\mathfrak{gl}(p|q,\mathbb{G})$, together with the transformation on the wavefunction $\Psi$,

$$\tilde{\Psi}=\Psi(I+\epsilon EF), \tag{4.4.18}$$

such that the LSP remains invariant under these transformations, i.e.

$$D_j\tilde{\Psi}=\tilde{U}_j\tilde{\Psi},\qquad j=1,2. \tag{4.4.19}$$

We obtain a deformed surface $F(\lambda,[u])$ expressed in terms of fermionic supermatrices in the Lie superalgebra $\mathfrak{gl}(p|q,\mathbb{G})$ assuming that the tangent vectors are linearly independent. The tangent vectors $ED_j(F)$ are given by

$$ED_j(F)=-\Psi^{-1}EA_j\Psi, \tag{4.4.20}$$

up to the addition of a bosonic supermatrix $R$ such that $\epsilon R=0$. The compatibility conditions of the tangent vectors (4.4.20) are equivalent to the deformation of the ZCC (4.4.2), which is given by

$$D_1A_2+D_2A_1+[EA_1,EU_2]+[EA_2,EU_1]=0. \tag{4.4.21}$$

If one considers the deformed surface, one obtains

$$F=\Psi^{-1}E\beta(\lambda)(\partial_\lambda\Psi)+\Psi^{-1}ES\Psi+\Psi^{-1}E(\mathrm{pr}\omega\Psi). \tag{4.4.22}$$

We use the spectral symmetry generator $\beta(\lambda)\partial_\lambda$, where $\beta(\lambda)$ is an arbitrary function such that $\deg(\beta)=\deg(\lambda)+1 \mod 2$ (i.e. if $\lambda$ is bosonic, then $\beta(\lambda)$ is fermionic and if $\lambda$ is fermionic, then $\beta(\lambda)$ is bosonic). The gauge $S(\lambda,[u])$ is a fermionic supermatrix in $\mathfrak{gl}(p|q,\mathbb{G})$ and the fermionic generator $\omega$ spans a symmetry transformation for both the ZCC $\Delta[u]=0$ and the LSP $\Omega(\lambda,[u])=0$. Then, the supermatrices $A_j$ take the form

$$\begin{aligned} A_j = E\beta(\lambda)\partial_\lambda U_j - E\left(D_jS - \{ES, EU_j\}\right) \\ +E\left(\text{pr}\omega U_j - \left(\{D_j, \text{pr}\omega\}\Psi\right)\Psi^{-1}\right), \end{aligned} \tag{4.4.23}$$

which satisfy equation (4.4.21).

**Proposition 4.4.2.** *Let us assume that there exists an LSP of the form* (4.4.1) *associated with a SUSY integrable system of PDEs* $\Delta[u]=0$ *such that the ZCC* (4.4.2) *is equivalent to* $\Delta[u]=0$*. If we consider the fermionic parameter deformations* (4.4.17) *and* (4.4.18) *that preserve both the LSP and the ZCC, i.e.* $A_1, A_2$ *and* $F$ *must satisfy equations* (4.4.19)-(4.4.21)*, then there exists an immersion fermionic supermatrix* $F$ *which defines a 2D surface provided that its tangent vectors* (4.4.20) *are linearly independent.*

**Corollary 4.4.2.** *If one considers a deformed surface* $F$*, as defined in proposition 4.4.2, of the form* (4.4.22)*, then the linearly independent supermatrices* $A_1$ *and* $A_2$ *appearing in the tangent vectors* (4.4.20) *take the form* (4.4.23)*.*

Using the inner product (4.3.1), we define the coefficients of the first fundamental form to be the bosonic quantities

$$g_{ij} = \langle ED_i(F), ED_j(F)\rangle = \langle EA_i, EA_j\rangle, \tag{4.4.24}$$

with the property $g_{12}=g_{21}$. The first fundamental form is given by

$$I = (d_1)^2 g_{11} + 2d_1d_2g_{12} + (d_2)^2 g_{22},$$

where the $d_j$ are fermionic differential forms which represent the infinitesimal displacement in the direction of $D_j$ and satisfy the relation

$$\{d_1, d_2\} = 0.$$

These operators are defined as in (4.4.12). In order to construct the second fundamental form, we must find a unit normal vector $N$ in terms of a bosonic supermatrix which has the properties

$$\langle N, N\rangle = 1, \qquad \langle ED_jF, N\rangle = 0, \qquad j=1,2.$$

A unit normal vector is

$$N = \frac{[ED_1(F), ED_2(F)]}{\langle [ED_1(F), ED_2(F)], [ED_1(F), ED_2(F)]\rangle^{1/2}},$$

or equivalently

$$N = \frac{\Psi^{-1}[EA_1, EA_2]\Psi}{\langle [EA_1, EA_2], [EA_1, EA_2]\rangle^{1/2}}, \tag{4.4.25}$$

whenever the division is possible. Therefore, the coefficients of the second fundamental form are given by

$$b_{ij} = \langle D_j D_i F, N\rangle = \left\langle -(D_j A_i + [EA_i, EU_j]), \Psi N \Psi^{-1} \right\rangle, \tag{4.4.26}$$

which have the property $b_{12} = -b_{21}$ and are fermionic quantities. The second fundamental form is

$$I\!I = (d_1)^2 b_{11} + 2 d_1 d_2 b_{12} + (d_2)^2 b_{22}.$$

The Gaussian and mean curvatures are given, respectively, by

$$K = \frac{b_{11}b_{22} - b_{12}b_{21}}{g_{11}g_{22} - g_{12}g_{21}} = \frac{b_{11}b_{22} + (b_{12})^2}{g_{11}g_{22} - (g_{12})^2}, \tag{4.4.27}$$

$$H = \frac{b_{11}g_{22} + b_{22}g_{11} - b_{12}g_{21} - b_{21}g_{12}}{2(g_{11}g_{22} - g_{12}g_{21})} = \frac{b_{11}g_{22} + b_{22}g_{11}}{2(g_{11}g_{22} - (g_{12})^2)}, \tag{4.4.28}$$

where $K$ is a bosonic quantity and $H$ is a fermionic one and such that they can be computed only using the supermatrices $U_j$, $A_j$ and $N$.

## 4.5. Example : The SUSY sine-Gordon equation

In this section, we apply the theory described in the previous sections to the SUSY sine-Gordon equation. The SUSY sine-Gordon equation takes the form [**24**, **63**, **64**]

$$D_2 D_1 \phi = i \sin\phi, \tag{4.5.1}$$

where $\phi$ is a bosonic superfunction of $x_1$, $x_2$, $\theta^1$ and $\theta^2$, which can be decomposed as the truncated series

$$\phi = \phi_0(x_1,x_2) + \phi_1(x_1,x_2)\theta^1 + \phi_2(x_1,x_2)\theta^2 + \phi_{12}(x_1,x_2)\theta^1\theta^2,$$

where $\phi_0$ and $\phi_{12}$ are bosonic functions of $x_1, x_2$ and $\phi_1$ and $\phi_2$ are fermionic functions of $x_1, x_2$. The associated LSP [**108**, **B2**]

$$D_j\Psi = U_j\Psi, \qquad j = 1,2$$

$$U_1 = \frac{1}{2\sqrt{\lambda}}\begin{pmatrix} 0 & 0 & ie^{i\phi} \\ 0 & 0 & -ie^{-i\phi} \\ -e^{-i\phi} & e^{i\phi} & 0 \end{pmatrix}, \qquad U_2 = \begin{pmatrix} iD_2\phi & 0 & -i\sqrt{\lambda} \\ 0 & -iD_2\phi & i\sqrt{\lambda} \\ -\sqrt{\lambda} & \sqrt{\lambda} & 0 \end{pmatrix}, \tag{4.5.2}$$

with the bosonic spectral parameter $\lambda$ for which the ZCC satisfies

$$D_1U_2 + D_2U_1 - \{EU_1, EU_2\} = 0, \qquad E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

is equivalent to the SUSY sine-Gordon equation (4.5.1) for any value of $\lambda$.

One way to indroduce the spectral parameter in the potential supermatrices $U_1$ and $U_2$ is to consider the linear problem

$$D_j\hat{\Psi} = \hat{U}_j\hat{\Psi}, \qquad j = 1, 2 \tag{4.5.3}$$

where the matrices $\hat{U}_j$ take the form

$$\hat{U}_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & ie^{i\phi} \\ 0 & 0 & -ie^{-i\phi} \\ -e^{-i\phi} & e^{i\phi} & 0 \end{pmatrix}, \qquad \hat{U}_2 = \begin{pmatrix} iD_2\phi & 0 & -i \\ 0 & -iD_2\phi & i \\ -1 & 1 & 0 \end{pmatrix}.$$

From there, one should note that the vector field

$$\omega = 2x_1\partial_{x_1} - 2x_2\partial_{x_2} + \theta^1\partial_{\theta^1} - \theta^2\partial_{\theta^2}, \tag{4.5.4}$$

is a symmetry generator of the SUSY sine-Gordon equation, but not of the linear problem (4.5.3). This vector field (4.5.4) generates the transformations

$$\tilde{x}_+ = \lambda x_+, \qquad \tilde{x}_- = \lambda^{-1}x_-, \qquad \tilde{\theta}^+ = \lambda^{1/2}\theta^+, \qquad \tilde{\theta}^- = \lambda^{-1/2}\theta^-, \qquad \lambda = \pm e^{\mu},$$

where $\mu$ is a bosonic-valued parameter. By imposing these transformations on the supermatrices $\hat{U}_1$ and $\hat{U}_2$ we obtain, after some computation, the potential supermatrices defined in (4.5.2)). Therefore, the parameter $\lambda$ can play the role of the spectral parameter [**B2**].

In the following three examples we apply the theory from section 4.4.1 to the SUSY sine-Gordon equation and we derive the bosonic immersion of surfaces. In the remaining two examples, we consider the fermionic immersion described in section 4.4.2.

### 4.5.1. Sym–Tafel formula for a bosonic immersion

In this case, we consider the deformed surface that takes the form of the bosonic supermatrix

$$F = \Psi^{-1}\beta(\lambda)\partial_\lambda\Psi \in \mathfrak{sl}(2|1, \mathbb{G}), \tag{4.5.5}$$

with tangent vectors given by

$$ED_jF = \Psi^{-1}E\beta(\lambda)\partial_\lambda U_j\Psi = \Psi^{-1}EA_j\Psi$$

for a bosonic arbitrary function $\beta(\lambda)$. Explicitly, the matrices $A_j$ are linearly independent and take the form

$$A_1 = \frac{-\beta}{4\sqrt{\lambda^3}} \begin{pmatrix} 0 & 0 & ie^{i\phi} \\ 0 & 0 & -ie^{-i\phi} \\ -e^{-i\phi} & e^{i\phi} & 0 \end{pmatrix} = \frac{-\beta}{2\lambda} U_1, \qquad A_2 = \frac{\beta}{2\sqrt{\lambda}} \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & i \\ -1 & 1 & 0 \end{pmatrix}$$

and from equations (4.4.11) we determine the coefficients of the first fundamental form

$$\begin{aligned} g_{11} &= \langle EA_1, A_1 \rangle = \frac{-i\beta^2}{8\lambda^3}, \\ g_{12} = -g_{21} &= \langle EA_1, EA_2 \rangle = -\frac{i\beta^2}{4\lambda^2} \cos\phi, \\ g_{22} &= \langle EA_2, A_2 \rangle = \frac{i\beta^2}{2\lambda}, \end{aligned}$$

such that

$$g = g_{11}g_{22} - g_{12}g_{21} = \frac{\beta^2}{16\lambda} \sin^2\phi.$$

In order to obtain the coefficients of the second fundamental form and the Gaussian and mean curvatures, we first need to compute a unit normal vector $N$ in matrix form, as given by equation (4.4.13),

$$N = \Psi^{-1} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Psi.$$

Therefore, we obtain that from equation (4.4.14) the coefficients of the second fundamental form are given by

$$b_{11} = 0, \qquad b_{12} = \frac{\beta}{2\lambda} \sin\phi, \qquad b_{22} = 0,$$

and consequently the Gaussian and mean curvatures are given by

$$K = \frac{4\lambda^2}{\beta^2}, \qquad \text{and} \qquad H = \frac{-2i\lambda}{\beta} \cot\phi,$$

which we obtain from equations (4.4.15) and (4.4.16), respectively. It should be noted that the Gaussian curvature is constant as for the classical case [**23**], but the sign of the Gaussian curvature differs. By analogy with the classical geometry, if we look for umbilic points using the formula

$$0 = H^2 - K = \frac{-4\lambda^2}{\beta^2} \csc^2\phi$$

we observe that umbilic points do not exist on this surface. Moreover, if we consider a SUSY version of the Euler-Poincaré character

$$\chi = \frac{1}{2\pi}\int\int_{\Omega} d_2 d_1 g^{1/2} K,$$

and since soliton solutions of the SUSY sine-Gordon equation rapidly decay to zero, the Euler-Poincaré character vanishes,

$$\chi = \frac{1}{2\pi}\int_{-\infty}^{\infty} d_2 \int_{-\infty}^{\infty} d_1 \sin\phi = \frac{-i}{2\pi}\int_{-\infty}^{\infty} d_2 \int_{-\infty}^{\infty} d_1 (D_2 D_1 \phi) = 0.$$

In analogy with the lemma 3.5 in [**23**] for soliton solutions of the classical sine-Gordon equation, this demonstrates that the SUSY version of this lemma is still valid. If the soliton solutions of the SUSY sine-Gordon equation (4.5.1) satisfy the conditions that the function $\phi$ and its derivatives tend to zero as the independent variables go to infinity, then we have

$$\int_{-\infty}^{\infty} d_i \sqrt{g} K = 0. \qquad i = 1, 2$$

In comparison with the classical geometry, if the deformed surface (4.5.5) is compact and connected, then it is homeomorphic to a torus since the Euler-Poincaré character vanishes [**48**].

### 4.5.2. Bosonic gauge transformation

Using the bosonic gauge supermatrix

$$S = U_2 D_2 \phi = \sqrt{\lambda} D_2 \phi \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & i \\ -1 & 1 & 0 \end{pmatrix} \in \mathfrak{sl}(2|1, \mathbb{G})$$

for the deformed surface

$$F = \Phi^{-1} E S \Psi \in \mathfrak{sl}(2|1, \mathbb{G}) \qquad (4.5.6)$$

consistent with the linearly independent tangent vectors

$$E D_j F = \Psi^{-1} E A_j \Psi,$$

where

$$A_1 = \begin{pmatrix} -i\cos\phi D_2\phi & ie^{i\phi} D_2\phi & -\sqrt{\lambda}\sin\phi \\ ie^{-i\phi} D_2\phi & -i\cos\phi D_2\phi & \sqrt{\lambda}\sin\phi \\ -i\sqrt{\lambda}\sin\phi & i\sqrt{\lambda}\sin\phi & -2i\cos\phi D_2\phi \end{pmatrix},$$

$$A_2 = -i\sqrt{\lambda}\partial_{x_2}\phi \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & i \\ 1 & -1 & 0 \end{pmatrix},$$

we get the metric coefficients

$$g_{11} = 2i\lambda \sin^2 \phi, \qquad g_{12} = 0, \qquad g_{22} = 2i\lambda(\partial_{x_2}\phi)^2.$$

A unit normal vector $N$ is given by

$$N = \Psi^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Psi.$$

The coefficients of the second fundamental form reduce, after some straightfoward computation, to

$$b_{11} = 2\sin\phi(iD_1\phi D_2\phi + \cos\phi), \qquad b_{12} = -2\cos\phi\partial_{x_2}\phi, \qquad b_{22} = 0.$$

Therefore, we get the non-trivial Gaussian and mean curvatures

$$K = \frac{-\cot^2\phi}{\lambda^2}, \qquad \text{and} \qquad H = \frac{iD_2\phi D_1\phi - \cos\phi}{2\lambda\sin\phi}.$$

This surface represents a nonlinear Weingarten surface since there exists a second-order polynomial relation in $H$ with coefficients depending on $K$,

$$f(K,H) = H^2 - \frac{i}{2}\sqrt{K}H + 2K = 0.$$

#### 4.5.3. Fokas–Gel'fand formula for bosonic immersion

If we consider the bosonic supermatrix

$$F = \Psi^{-1}(\text{pr}\omega\Psi) \in \mathfrak{sl}(2|1,\mathbb{G}), \tag{4.5.7}$$

with a bosonic symmetry generator of both the SUSY sine-Gordon equation and its LSP given by $\partial_{x_1}$ (which generates a translation in the $x_1$ direction) [**B2**], then we obtain the tangent vectors given by

$$ED_jF = \Psi^{-1}E\partial_{x_1}U_j\Psi = \Psi^{-1}EA_j\Psi,$$

where

$$A_1 = \frac{\partial_{x_1}\phi}{2\sqrt{\lambda}} \begin{pmatrix} 0 & 0 & -e^{i\phi} \\ 0 & 0 & -e^{-i\phi} \\ ie^{-i\phi} & ie^{i\phi} & 0 \end{pmatrix}, \qquad A_2 = i\partial_{x_1}D_2\phi \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is interesting to note that the metric coefficients degenerate to a curve-like metric, i.e.

$$g_{11} = \frac{-i}{2\lambda}(\partial_{x_1}\phi)^2, \qquad g_{12} = 0, \qquad g_{22} = -(\partial_{x_1}D_2\phi)^2 = 0.$$

A unit normal vector $N$ is given by

$$N = \Psi^{-1} E \Psi,$$

which leads to the following coefficients of the second fundamental form

$$b_{11} = b_{12} = b_{21} = 0, \qquad b_{22} = 2(\text{pr}\omega_R D_2\phi) D_2\phi.$$

It is also interesting to consider the isotropic normal vector $N$ of the form

$$N = \Psi^{-1} \frac{-i(\partial_{x_1}\phi)^2}{2\lambda} \begin{pmatrix} 0 & 0 & -e^{i\phi} \\ 0 & 0 & e^{-i\phi} \\ -ie^{-i\phi} & ie^{i\phi} & 0 \end{pmatrix} \Psi,$$

so that we get a non-trivial second fundamental form

$$II = (d_1)^2 \frac{-\partial_{x_1}\phi D_1\phi}{\sqrt{2i\lambda}} + d_1 d_2 \frac{\sqrt{2}\partial_{x_1} D_2\phi}{\sqrt{i\lambda}}.$$

Consequently, we find the curvatures

$$K = \frac{-1}{(\partial_{x_1}\phi)^2}, \qquad \text{and} \qquad H = -\left(\frac{i\lambda}{2}\right)^{\frac{1}{2}} \frac{D_1\phi}{\partial_{x_1}\phi}.$$

Such an example of an isotropic normal vector has been obtained and investigated for the classical FG immersion formula [**71**]. Note that for both cases the tangent vectors are linearly independent. So the immersion (4.5.7) defines a surface instead of a curve.

### 4.5.4. Fermionic gauge transformation

We consider the fermionic gauge supermatrix

$$S = \begin{pmatrix} -iD_2\phi & 0 & i\sqrt{\lambda} \\ 0 & iD_2\phi & -i\sqrt{\lambda} \\ -\sqrt{\lambda} & \sqrt{\lambda} & 0 \end{pmatrix} \in \mathfrak{sl}(2|1, \mathbb{G})$$

for the deformed surface

$$F = \Psi^{-1} E S \Psi \in \mathfrak{sl}(2|1, \mathbb{G}), \tag{4.5.8}$$

which is consistent with the tangent vectors

$$ED_j F = -\Psi^{-1} E A_j \Psi = \Psi^{-1}(D_j S - \{ES, EU_j\})\Psi,$$

where the linearly independent matrices $A_j$ take the form

$$A_1 = \begin{pmatrix} -ie^{-i\phi} & -ie^{i\phi} & e^{i\phi}D_2\phi/2\sqrt{\lambda} \\ -ie^{-i\phi} & -ie^{i\phi} & e^{-i\phi}D_2\phi/2\sqrt{\lambda} \\ ie^{-i\phi}D_2\phi/2\sqrt{\lambda} & ie^{i\phi}D_2\phi/2\sqrt{\lambda} & 2i\cos\phi \end{pmatrix},$$

$$A_2 = \begin{pmatrix} \partial_{x_2}\phi & 0 & -2\sqrt{\lambda}D_2\phi \\ 0 & -\partial_{x_2}\phi & -2\sqrt{\lambda}D_2\phi \\ 0 & 0 & 0 \end{pmatrix},$$

such that the tangent vector $ED_1F$ is isotropic. The first fundamental form coefficients are computed from equation (4.4.24)) which give

$$g_{11} = \langle EA_1, EA_1\rangle = 0, \qquad g_{22} = \langle EA_2, EA_2\rangle = (\partial_{x_2}\phi)^2,$$
$$g_{12} = \langle EA_1, EA_2\rangle = -i\sin\phi\partial_{x_2}\phi$$

and the unit normal vector $N$ from equation (4.4.25) is given by

$$N = \Psi^{-1}\begin{pmatrix} 0 & ie^{i\phi} & 2D_2\phi(-i\sqrt{\lambda}\frac{e^{-i\phi}}{\partial_{x_2}\phi} - \frac{e^{i\phi}}{8\sqrt{\lambda}}) \\ -ie^{-i\phi} & 0 & 2D_2\phi(-i\sqrt{\lambda}\frac{e^{i\phi}}{\partial_{x_2}\phi} + \frac{e^{-i\phi}}{8\sqrt{\lambda}}) \\ -ie^{-i\phi}D_2\phi/4\sqrt{\lambda} & ie^{i\phi}D_2\phi/4\sqrt{\lambda} & 0 \end{pmatrix}\Psi.$$

Therefore, the coefficients of the second fundamental form are given by (4.4.26)

$$b_{11} = -iD_1\phi - \frac{i}{2\lambda}D_2\phi\cos\phi + \frac{iD_2\phi}{\partial_{x_2}\phi}\sin\phi$$
$$+\frac{2iD_2\phi}{\partial_{x_2}\phi}\sin\phi\cos 2\phi + 3i\frac{D_2\phi}{\partial_{x_2}\phi}\sin 2\phi\cos\phi,$$
$$b_{12} = -iD_2\phi(1+\cos 2\phi + \frac{1}{2}\sin^2\phi), \qquad b_{22} = 0,$$

which leads us to the Gaussian and mean curvatures of the fermionic immersion,

$$K = 0, \qquad H = \frac{b_{11}}{\sin^2\phi}.$$

This surface admits parabolic points and is a nonlinear Weingarten surface since $H$ is fermionic and such a relation holds

$$f(K,H) = H^2 + \alpha K = 0,$$

where $\alpha$ is an arbitrary bosonic constant.

### 4.5.5. Fokas–Gel'fand formula for fermionic immersion

We investigate the deformed surface $F$ generated by the fermionic differential operator

$$\omega_k = J_k = \partial_{\theta^k} + i\theta^k\partial_{x_k}, \qquad k = 1, 2$$

where $J_k$ is defined in equation (4.4.3) and is a symmetry of both the SUSY sine-Gordon equation and its LSP [**B2**]. The deformed surface

$$F = \Psi^{-1} E(\mathrm{pr}\omega_k \Psi) \in \mathfrak{sl}(2|1, \mathbb{G}) \tag{4.5.9}$$

has tangent vectors of the form

$$ED_j F = -\Psi^{-1} E A_j \Psi,$$

$$A_1 = \frac{\mathrm{pr}\omega_k \phi}{2\sqrt{\lambda}} \begin{pmatrix} 0 & 0 & -e^{i\phi} \\ 0 & 0 & -e^{-i\phi} \\ -ie^{-i\phi} & -ie^{i\phi} & 0 \end{pmatrix}, \qquad A_2 = i\mathrm{pr}\omega_k D_2\phi \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which are linearly independent. One should note that the matrix $A_1$ is an isotropic vector. The metric coefficients are curve-like, i.e.

$$g_{11} = 0, \qquad g_{12} = 0, \qquad g_{22} = -(\mathrm{pr}\omega_k(D_2\phi))^2.$$

A unit normal vector $N$ is given by

$$N = \Psi^{-1} \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Psi$$

and the coefficients of the second fundamental form are

$$b_{11} = \frac{\mathrm{pr}\omega_R \phi}{2\lambda} \cos 2\phi, \qquad b_{12} = b_{21} = b_{22} = 0.$$

We can also use the isotropic normal vector $N$ given by

$$N = \Psi^{-1} \frac{1}{\sqrt{2i}} \begin{pmatrix} 0 & 0 & e^{i\phi} \\ 0 & 0 & -e^{-i\phi} \\ ie^{-i\phi} & -ie^{i\phi} & 0 \end{pmatrix} \Psi,$$

and the coefficients of the second fundamental form do not degenerate to a curve-like form with the coefficients

$$b_{11} = \frac{-D_1\phi(\mathrm{pr}\omega_k\phi)}{\sqrt{2i\lambda}}, \qquad b_{12} = \frac{-i\mathrm{pr}\omega_k(D_2\phi)}{\sqrt{2i\lambda}}, \qquad b_{22} = 0,$$

such that

$$b = b_{11}b_{22} - b_{12}b_{21} = -\frac{(\mathrm{pr}\omega_k(D_2\phi))^2}{2i\lambda}.$$

For the surface (4.5.9), the tangent vectors are linearly independent, so that the immersion defines a surface in $\mathfrak{sl}(2|1, \mathbb{G})$.

## 4.6. Conclusions

In this paper, we have constructed two SUSY versions of the FG formula for the immersion of 2D surfaces in Lie superalgebras. The first (bosonic) SUSY extension is considered for the immersion of a bosonic supermatrix $F$ in the Lie superalgebra $\mathfrak{gl}(p|q, \mathbb{G})$ using a bosonic infinitesimal deformation of the LSP and, therefore, of the ZCC. We have provided the form of the tangent vector based on three different deformations, i.e. transformations of the spectral parameter, invariances under gauge transformations of the wavefunction $\Psi$ and symmetries of both the LSP and the ZCC. From the tangent vectors, we have been able to describe the metric using the super Killing form and to find a unit normal vector $N$ which allows us to describe the coefficients of the second fundamental form in terms only of the potential matrices $U_j$ and their deformations $A_j$. The Gaussian and mean curvatures were determined. The second (fermionic) SUSY version of the FG formula for immersion uses an odd-valued parameter instead of the bosonic infinitesimal parameter. This leads to a fermionic supermatrix $F$ immersed in the Lie superalgebra $\mathfrak{gl}(p|q, \mathbb{G})$. Using similar deformations as those described in section 4.4 for the bosonic formula for immersion, we have determined the form of the tangent vectors together with a unit normal vector. The two fundamental forms and the two curvatures have also been given explicitly in terms of the potential matrices $U_j$ and their deformations $A_j$ via the super Killing form.

The integrable SUSY sine-Gordon equation and its Lax pair have been employed in order to apply the two SUSY versions of the FG formula for immersion. Among these examples, we have considered a bosonic deformation of the spectral parameter for which the Gaussian and mean curvatures resemble the classical case. Moreover, the Gaussian curvature is constant and positive. We have also considered separately the bosonic and fermionic gauge transformations and provided their associated geometric characterizations. Both surfaces are nonlinear Weingarten-type surfaces. The bosonic variable translations and the SUSY transformation symmetries have also been investigated.

This research could be extended in several directions. It would be interesting to investigate other examples of integrable SUSY systems like the SUSY Schrödinger equations or the SUSY Korteweg–de Vries equation and the associated soliton surfaces. Also, the use of different norms and inner products could be applied to get a different approach depending on the physical interpretation of the considered models. Moreover, it would be interesting to explicitly solve the wavefunction $\Psi$ so that we can provide a visual image of the surface. As an additional future perspective, we could investigate how the conserved quantities, such as the Hamiltonian structure, manifest themselves on the surface.

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# Chapitre 5

## ON INTEGRABILITY ASPECTS OF THE SUPERSYMMETRIC SINE-GORDON EQUATION

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**Résumé**

Dans cet article, nous étudions certaines propriétés d'intégrabilité de l'équation de sine-Gordon supersymétrique. Nous construisons des paires de Lax avec des représentations de courbure nulle qui sont équivalentes à l'équation de sine-Gordon supersymétrique. À partir du problème linéaire spectral fermionique, nous dérivons un système couplé d'équations de super Riccati et la transformation d'auto-Bäcklund de l'équation de sine-Gordon supersymétrique. De plus, une description détaillée des transformations de Darboux associées est présentée et des solutions super-multisolitoniques non triviales sont construites. Ces propriétés d'intégrabilité nous permettent de produire une nouvelle caractérisation géométrique explicite de la version bosonique supersymétrique de la formule de Sym–Tafel pour l'immersion de surfaces dans une superalgèbre de Lie. Cette caractérisation est exprimée seulement en termes des variables indépendantes bosoniques et fermioniques.

**Abstract**

In this paper we study certain integrability properties of the supersymmetric sine-Gordon equation. We construct Lax pairs with their zero-curvature representations which are equivalent to the supersymmetric sine-Gordon equation. From the

fermionic linear spectral problem, we derive coupled sets of super Riccati equations and the auto-Bäcklund transformation of the supersymmetric sine-Gordon equation. In addition, a detailed description of the associated Darboux transformation is presented and non-trivial super multisoliton solutions are constructed. These integrability properties allow us to provide new explicit geometric characterizations of the bosonic supersymmetric version of the Sym–Tafel formula for the immersion of surfaces in a Lie superalgebra. These characterizations are expressed only in terms of the independent bosonic and fermionic variables.

## 5.1. Introduction

Over the past four decades, supersymmetric (SUSY) integrable models have generated a great deal of interest in the literature of mathematical physics (see e.g. [**1**,**7**,**13**,**14**,**22**,**37**,**76**,**79**,**83**,**90**,**98**,**116**] and references therein). Their special properties, such as the existence of a linear spectral problem (LSP), the Bäcklund and Darboux transformations and infinite sets of conserved currents, have allowed the construction of analytical supersoliton solutions (e.g. [**7**,**44**,**64**,**69**,**87**,**88**, **108**]). Some of these integrability properties have been investigated for the case of the SUSY sine-Gordon equation, see e.g. [**7**, **24**, **51**, **60**, **63**, **87**, **88**, **106**–**108**]. The approach proposed in this paper goes deeper into certain integrability properties of the SUSY sine-Gordon equation than [**107**, **108**], especially the Darboux transformation of the SUSY sine-Gordon equation, which will later enable us to obtain new results for the bosonic SUSY version of the Sym–Tafel formula for the immersion of surfaces in Lie superalgebras. One should note that $n$-order Darboux transformations were investigated in [**87**, **88**] using Pfaffian solutions. The associated linear system uses $2 \times 2$ matrices which are more compact than our $3 \times 3$ potential supermatrices. However, the linear system used in these articles is not convenient for the bosonic SUSY version of the Sym–Tafel immersion formula since this system uses differential matrix operators instead of supermatrices, as used in our approach. In this paper, we discuss the links between different integrability properties associated with the SUSY sine-Gordon equation. Furthermore, we provide explicit solutions for the SUSY sine-Gordon equation and its LSP which allow us to investigate examples of the bosonic SUSY version of the Sym–Tafel immersion formula through their geometric characterization.

This paper is organized as follows. In section 5.2, we present some of the integrability properties of the SUSY sine-Gordon equation. We derive a fermionic LSP and we link it to different integrability properties, such as a bosonic version of the LSP, equivalent coupled sets of super Riccati equations and the auto-Bäcklund transformations of the SUSY sine-Gordon equation. Moreover, we provide a detailed description of the Darboux transformations associated with the SUSY sine-Gordon equation. In section 5.3, we investigate two examples of the bosonic SUSY version of the Sym–Tafel formula for immersion associated with the SUSY sine-Gordon equation. These examples are obtained using the first iteration of the Darboux transformation and are exclusively written in terms of the bosonic and fermionic independent variables. A characterization of the geometry of each surface is provided.

## 5.2. Integrability aspects of the supersymmetric sine-Gordon equation

Throughout this paper, we follow the notation introduced in section 3 of the paper [**B3**]. A more detailed presentation of the theory of Grassmann algebras can be found in the books [**33**,**43**,**56**,**118**] and references therein. In what follows we do not use the implicit notation for the fermionic derivatives of a $(m|n)$-supermatrix $M$ [**33**], e.g.

$$\partial_\theta M = \begin{pmatrix} \partial_\theta A & \partial_\theta B \\ -\partial_\theta C & -\partial_\theta D \end{pmatrix}.$$

Instead, we introduce a matrix $E$ such that

$$E\partial_\theta M = \begin{pmatrix} \partial_\theta A & \partial_\theta B \\ -\partial_\theta C & -\partial_\theta D \end{pmatrix}, \qquad E = \begin{pmatrix} I_m & 0 \\ 0 & -I_n \end{pmatrix},$$

where the square submatrices $I_m$ and $I_n$ represent the identity matrices of dimension $m$ and $n$, respectively. The chain rule is assumed to be

$$\frac{d}{dv} f(u) = \frac{df}{du}\frac{du}{dv}.$$

According to [**31**], the SUSY sine-Gordon equation (SSGE),

$$D_+D_-s = i\sin s, \tag{5.2.1}$$

is considered for a bosonic superfield $s = s(\theta^+, \theta^-, x_+, x_-)$ with the covariant derivatives

$$D_\pm = \frac{\partial}{\partial\theta^\pm} - i\theta^\pm \frac{\partial}{\partial x_\pm}, \tag{5.2.2}$$

where $\theta^\pm$ are fermonic independent variables and $x_\pm$ are bosonic light-cone coordinates. The fermionic derivatives $D_\pm$ have the properties

$$D_\pm^2 = -i\partial_{x_\pm}, \qquad \{D_+, D_-\} = 0, \tag{5.2.3}$$

where $\{\cdot,\cdot\}$ stands for the anticommutator. The SSGE (5.2.1) can be obtained through the super Euler–Lagrange equation,

$$\frac{\partial}{\partial s}\mathcal{L} + D_+\left(\frac{\partial}{\partial(D_+s)}\mathcal{L}\right) + D_-\left(\frac{\partial}{\partial(D_-s)}\mathcal{L}\right) = 0, \tag{5.2.4}$$

with the Lagrangian density

$$\mathcal{L} = \cos s - \frac{i}{2}D_+sD_-s. \tag{5.2.5}$$

The SSGE (5.2.1) is known to be integrable in the sense of soliton theory [**7**, **24**,**51**,**60**,**107**,**108**]. One can provide an infinite set of locally conserved quantities and a LSP under the form of a differential linear matrix representation. One way to obtain a linearization of the SSGE (5.2.1) is to consider the following problem for the wavefunction $\Phi$ :

$$D_+\Phi = (\mathcal{J}e^{is} + \mathcal{K}e^{-is})\Phi, \qquad D_-\Phi = (\mathcal{M}D_-s + \mathcal{N})\Phi, \tag{5.2.6}$$

where $\mathcal{J},\mathcal{K},\mathcal{M},\mathcal{N}$ are complex-valued matrices and then take the compatibility conditions of $\Phi$ to be equivalent to the SSGE (5.2.1). The resulting algebraic constraints are

$$i\mathcal{J} = [\mathcal{M},\mathcal{J}], \qquad i\mathcal{K} = [\mathcal{K},\mathcal{M}], \qquad \{\mathcal{J},\mathcal{N}\} = -\{\mathcal{K},\mathcal{N}\}, \qquad \frac{1}{2}\mathcal{M} = \{\mathcal{K},\mathcal{N}\}. \tag{5.2.7}$$

One solution to these constraints, which takes its values in the $\mathfrak{sl}(3,\mathbb{C})$ Lie algebra, is

$$\begin{aligned} \mathcal{J} &= \frac{1}{2}\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \mathcal{K} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ -1 & 0 & 0 \end{pmatrix}, \\ \mathcal{M} &= \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{N} = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & i \\ -1 & 1 & 0 \end{pmatrix}. \end{aligned} \tag{5.2.8}$$

To introduce the spectral parameter $\lambda$ in equations (5.2.6), as proposed in [**B2**], we apply the Lie point symmetry transformation of the SSGE (5.2.1),

$$\tilde{x}_+ = \lambda x_+, \quad \tilde{x}_- = \lambda^{-1}x_-, \quad \tilde{\theta}^+ = \lambda^{1/2}\theta^+, \quad \tilde{\theta}^- = \lambda^{-1/2}\theta^-, \quad \lambda = \pm e^{\mu}, \tag{5.2.9}$$

to the linear system, where $\mu$ is any bosonic constant in the Grassmann algebra. One should note that this scaling transformation does not leave the linear system (5.2.6) invariant. Hence, the LSP of the SSGE (5.2.1) takes the form

$$D_\pm\Phi(\lambda,s) = U_\pm(\lambda,s)\Phi(\lambda,s) \tag{5.2.10}$$

where $\lambda$ is the bosonic spectral parameter and $U_\pm$ are fermionic supermatrices taking values in the $\mathfrak{sl}(2|1,\mathbb{G})$ superalgebra, given by [**108**]

$$U_+ = \frac{1}{2\sqrt{\lambda}}\begin{pmatrix} 0 & 0 & ie^{is} \\ 0 & 0 & -ie^{-is} \\ -e^{-is} & e^{is} & 0 \end{pmatrix}, \qquad U_- = \begin{pmatrix} iD_-s & 0 & -i\sqrt{\lambda} \\ 0 & -iD_-s & i\sqrt{\lambda} \\ -\sqrt{\lambda} & \sqrt{\lambda} & 0 \end{pmatrix}. \tag{5.2.11}$$

The wavefunction $\Phi$ is a $(2|1)$-supervector

$$\Phi = \begin{pmatrix} \psi \\ \phi \\ \chi \end{pmatrix}, \qquad (-1)^{\deg(\psi)} = (-1)^{\deg(\phi)} = (-1)^{\deg(\chi)+1}, \tag{5.2.12}$$

such that either $\psi, \phi$ are bosonic superfields and $\chi$ is a fermionic superfield, or $\psi, \phi$ are fermionic superfields and $\chi$ is a bosonic superfield.

The LSP can also be defined using an invertible wavefunction $\Psi$ in the $GL(2|1,\mathbb{G})$ supergroup,

$$D_\pm \Psi(\lambda, s) = U_\pm(\lambda, s)\Psi(\lambda, s). \tag{5.2.13}$$

The compatibility condition of both LSPs (5.2.10) and (5.2.13), i.e.

$$D_+U_- + D_-U_+ - \{EU_+, EU_-\} = 0, \qquad E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{5.2.14}$$

are equivalent to the SSGE (5.2.1).

Moreover, a LSP can be described using the light-cone coordinate derivatives $\partial_{x_\pm}$. From the property (5.2.3), we write the bosonic version of the LSP exclusively in terms of the wavefunction $\Psi$ and the potential matrices $U_\pm$ from the fermionic version of the LSP, i.e.

$$\partial_{x_\pm}\Psi = i(D_\pm U_\pm - (EU_\pm)^2)\Psi = V_\pm \Psi, \tag{5.2.15}$$

where the bosonic supermatrices $V_\pm$ take the forms

$$\begin{aligned} V_+ &= \frac{1}{2}\begin{pmatrix} \frac{1}{2\lambda} & \frac{-1}{2\lambda}e^{2is} & \frac{-i}{\sqrt{\lambda}}e^{is}D_+s \\ \frac{-1}{2\lambda}e^{-2is} & \frac{1}{2\lambda} & \frac{-i}{\sqrt{\lambda}}e^{-is}D_+s \\ \frac{-1}{\sqrt{\lambda}}e^{-is}D_+s & \frac{-1}{\sqrt{\lambda}}e^{is}D_+s & \frac{1}{\lambda} \end{pmatrix} \in \mathfrak{sl}(2|1,\mathbb{G}), \\ V_- &= \begin{pmatrix} i\partial_{x_-}s - \lambda & \lambda & -i\sqrt{\lambda}D_-s \\ \lambda & -i\partial_{x_-}s - \lambda & -i\sqrt{\lambda}D_-s \\ \sqrt{\lambda}D_-s & \sqrt{\lambda}D_-s & -2\lambda \end{pmatrix} \in \mathfrak{sl}(2|1,\mathbb{G}). \end{aligned} \tag{5.2.16}$$

The compatibility conditions of the LSP (5.2.15) correspond to the "classical" version of the zero-curvature conditions, i.e.

$$\partial_{x_+} V_- - \partial_{x_-} V_+ + [V_-, V_+] = 0, \tag{5.2.17}$$

which is satisfied whenever the SSGE (5.2.1) is satisfied.

To obtain coupled sets of super Riccati equations [**51**, **107**], we consider the quantities

$$p = \frac{\phi}{\psi}, \qquad q = \frac{\chi}{\psi}, \tag{5.2.18}$$

where we take $\psi, \phi$ to be bosonic superfields and $\chi$ to be a fermionic superfield. By differentiating them, we get

$$\begin{aligned} D_+ p &= \frac{-i}{2\sqrt{\lambda}} e^{-is} q - \frac{i}{2\sqrt{\lambda}} e^{is} pq, \\ D_- p &= -2i(D_- s)p + i\sqrt{\lambda}(1+p)q, \end{aligned} \tag{5.2.19}$$

together with

$$\begin{aligned} D_+ q &= \frac{-1}{2\sqrt{\lambda}} e^{-is} + \frac{1}{2\sqrt{\lambda}} e^{is} p, \\ D_- q &= \sqrt{\lambda}(p-1) - i(D_- s)q, \end{aligned} \tag{5.2.20}$$

from which one can obtain an infinite set of locally conserved currents [**51**]. The compatibility conditions of both sets of equations are satisfied whenever $s$ is a solution of the SSGE (5.2.1). Moreover, by setting

$$p \to e^{-i(s+\tilde{s})}, \qquad q \to f e^{-\frac{i}{2}(s+\tilde{s})}, \tag{5.2.21}$$

where $f$ is a fermionic superfield, we obtain the auto-Bäcklund transformations of the SSGE [**63**, **107**]

$$\begin{aligned} D_+(s+\tilde{s}) &= \frac{f}{\sqrt{\lambda}} \cos\left(\frac{s-\tilde{s}}{2}\right), \\ D_-(s-\tilde{s}) &= 2\sqrt{\lambda} f \cos\left(\frac{s+\tilde{s}}{2}\right), \\ D_+ f &= \frac{i}{\sqrt{\lambda}} \sin\left(\frac{s-\tilde{s}}{2}\right), \\ D_- f &= -2i\sqrt{\lambda} \sin\left(\frac{s+\tilde{s}}{2}\right). \end{aligned} \tag{5.2.22}$$

The compatibility conditions of these equations are satisfied whenever $s$ and $\tilde{s}$ are solutions of the SSGE (5.2.1). One should note that the auto-Bäcklund transformation (5.2.22) of the SSGE requires an additional fermionic function $f$ due to the oddness of the derivatives $D_\pm$.

It is possible to construct $n$ soliton solutions of the SSGE (5.2.1) from the Darboux transformation using one (trivial) solution of the SSGE together with $n$ solutions of the associated LSP (5.2.10) for $n$ fixed spectral parameters $\lambda_j$, $j = 0, 1, ..., n-1$. One should note that Darboux transformations do not ensure that the newly constructed solutions are linearly independent of the previously constructed solutions. The first iteration of the Darboux transformation for the SSGE (5.2.1) [**87**, **88**, **108**] (similarly to the classical case [**2**, **91**]) is given by

$$s[1] = s - i\ln\left(\frac{\psi_0}{\phi_0}\right), \tag{5.2.23}$$

$$\Phi_j[1] = \begin{pmatrix}\psi_j[1]\\ \phi_j[1]\\ \chi_j[1]\end{pmatrix} = \begin{pmatrix} -\lambda_0\frac{\phi_0}{\psi_0} & \lambda_j & -i\sqrt{\lambda_0\lambda_j}\frac{\chi_0}{\psi_0}\\ \lambda_j & -\lambda_0\frac{\psi_0}{\phi_0} & -i\sqrt{\lambda_0\lambda_j}\frac{\chi_0}{\phi_0}\\ \sqrt{\lambda_0\lambda_j}\frac{\chi_0}{\psi_0} & \sqrt{\lambda_0\lambda_j}\frac{\chi_0}{\phi_0} & -(\lambda_0+\lambda_j)\end{pmatrix}\begin{pmatrix}\psi_j\\ \phi_j\\ \chi_j\end{pmatrix}, \tag{5.2.24}$$

where $s$ is a solution of the SSGE (5.2.1), $\psi_j, \phi_j$ are bosonic solutions and $\chi_j$ is a fermionic solution of the LSP (5.2.10) for the solution $s$ and the fixed spectral parameter $\lambda = \lambda_j$. The new solution $\Phi_j[1]$ of the LSP is given for the system

$$D_\pm\Phi_j[1] = A_\pm(\lambda_j, s[1])\Phi_j[1]. \qquad j = 1, 2, 3, ... \tag{5.2.25}$$

One should note that the solution of the LSP (5.2.25) with the fixed parameter $\lambda_0$ has been used in order to construct the new solution. The index $j = 0$ for the first (or higher) iteration transformation correspond to the trivial solution $\Phi = 0$. Therefore, the solution for the LSP with $\lambda = \lambda_0$ cannot be used to obtain other new solutions.

In order to construct a higher iteration solution of the SSGE (5.2.1), we must "drop" other solutions of the LSP associated with $s$ and $\lambda_j$ for the lowest indices $j$. As an example, let us say that we know a solution $s$ of the SSGE and three solutions $\Phi_0$, $\Phi_1$ and $\Phi_2$ of the LSP associated with the fixed-valued spectral parameters $\lambda_0$, $\lambda_1$ and $\lambda_2$, respectively. Hence, from $\Phi_1$ and $\Phi_2$, and dropping $\Phi_0$, we get respectively $\Phi_1[1]$ and $\Phi_2[1]$. To iterate once more, we can drop $\Phi_1[1]$ to obtain $\Phi_2[2]$. Moreover, from $\Phi_0$, $\Phi_1[1]$ and $\Phi_2[2]$, we can construct three new solutions $s[1]$, $s[2]$ and $s[3]$, respectively. The procedure can be applied $n$ times using $n$ fixed solutions of the LSP and the associated solution of the SSGE (5.2.1) as described in fig. 5.1.

The second Darboux Transformation of the a solution $s$ of the SSGE (5.2.1) is given by

$$s[2] = s - i\ln\left[\frac{\lambda_0\phi_0\psi_1 - \lambda_1\phi_1\psi_0 + i\sqrt{\lambda_0\lambda_1}\chi_0\chi_1}{\lambda_0\phi_1\psi_0 - \lambda_1\phi_0\psi_1 + i\sqrt{\lambda_0\lambda_1}\chi_0\chi_1}\right]. \tag{5.2.26}$$

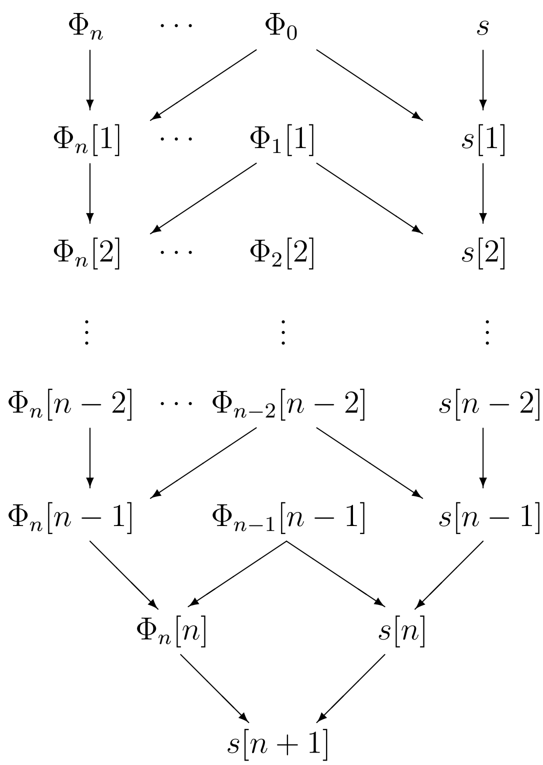


FIGURE 5.1. Diagram describing how to obtain the $(n{+}1)$-iterated solution generated by Darboux transformations using one solution $s$ of the SSGE (5.2.1) and $n+1$ solutions $\Phi_j$ of the LSP (5.2.10) associated with the solution $s$ for the fixed spectral parameters $\lambda_j$, $j = 0, 1, ..., n$.

By repeating the Darboux transformation $n$ times ($n > 2$), we obtain by induction the solution $s[n]$ which takes the form

$$s[n] = s - i \ln \left[ \frac{\sum_{m=0}^{(n+1)/2} (i)^m P(\lambda_{k_j}; \lambda_{k_\nu}) \Delta^1_{k_j} X_{k_\nu}}{\sum_{m=0}^{(n+1)/2} (i)^m P(\lambda_{k_j}; \lambda_{k_\nu}) \Delta^2_{k_j} X_{k_\nu}} \right], \quad \text{when } n \text{ is odd}, \qquad (5.2.27)$$

$$s[n] = s - i \ln \left[ \frac{\sum_{m=0}^{n/2} (i)^m P(\lambda_{k_j}; \lambda_{k_\nu}) \Delta^1_{k_j} X_{k_\nu} + \sum_{m=1}^{n/2-1} P(\lambda_{k_j}; \lambda_{k_\nu}) \frac{(-1)^m}{m!} X_{k_j} X_{k_\nu}}{\sum_{m=0}^{n/2} (i)^m P(\lambda_{k_j}; \lambda_{k_\nu}) \Delta^2_{k_j} X_{k_\nu} + \sum_{m=1}^{n/2-1} P(\lambda_{k_j}; \lambda_{k_\nu}) \frac{(-1)^m}{m!} X_{k_j} X_{k_\nu}} \right],$$
$$\text{when } n \text{ is even},$$

where $k_j$ represents the set of $n - 2m$ ordered indices and $k_\nu$ represents the set of $2m$ ordered indices not in $k_j$. The quantities $\Delta^i_{abc...}$, $X_{abc...}$ and $P(\lambda_a, \lambda_b, ...; \lambda_c, \lambda_d ...)$

are given by

$$\Delta^1_{abc...} = \det \begin{pmatrix} \vdots & \vdots & \vdots & \\ \lambda_a^2\psi_a & \lambda_b^2\psi_b & \lambda_c^2\psi_c & \cdots \\ \lambda_a\phi_a & \lambda_b\phi_b & \lambda_c\phi_c & \cdots \\ \psi_a & \psi_b & \psi_c & \cdots \end{pmatrix},$$

$$\Delta^2_{abc...} = \det \begin{pmatrix} \vdots & \vdots & \vdots & \\ \lambda_a^2\phi_a & \lambda_b^2\phi_b & \lambda_c^2\phi_c & \cdots \\ \lambda_a\psi_a & \lambda_b\psi_b & \lambda_c\psi_c & \cdots \\ \phi_a & \phi_b & \phi_c & \cdots \end{pmatrix}, \tag{5.2.28}$$

$$X_{abcd...} = \sqrt{\lambda_a\lambda_b\lambda_c\lambda_d...}\chi_a\chi_b\chi_c\chi_d...,$$

$$P(\lambda_a, \lambda_b, ...; \lambda_c, \lambda_d...) = (-1)^{n+\alpha}(\lambda_a + \lambda_c)(\lambda_a + \lambda_d)...(\lambda_b + \lambda_c)(\lambda_b + \lambda_d)...,$$

where $P(...\,;...)$ has the following definition when it has no argument $\lambda_k$ on one side or the other of the semicolon

$$P(\lambda_{k_j}; \emptyset) = 1 = P(\emptyset; \lambda_{k_\nu}), \tag{5.2.29}$$

and $\alpha$ is the binary function

$$\alpha = \begin{cases} 0 & \text{if } ab...cd... \text{ are equivalent to an even number} \\ & \text{of cyclic permutations of the ordered indices,} \\ & \\ 1 & \text{if } ab...cd... \text{ are equivalent to an odd number} \\ & \text{of cyclic permutations of the ordered indices.} \end{cases} \tag{5.2.30}$$

In addition, we define $\Delta_\emptyset = 0$.

As examples, we provide the first four Darboux transformations of a known solution $s$ of the SSGE (5.2.1) :

$$s[1] = s - i \ln\left(\frac{\Delta^1_0}{\Delta^2_0}\right),$$

$$s[2] = s - i \ln\left(\frac{\Delta^1_{01} + iX_{01}}{\Delta^2_{01} + iX_{01}}\right),$$

$$s[3] = s - i \ln\Big[\Big(\Delta^1_{012} + iP(\lambda_0; \lambda_1, \lambda_2)\Delta^1_0 X_{12}$$
$$+ iP(\lambda_2; \lambda_0, \lambda_1)\Delta^1_2 X_{01} + iP(\lambda_1; \lambda_0, \lambda_2)\Delta^1_1 X_{02}\Big)$$
$$/ \left(\Delta^2_{012} + iP(\lambda_0; \lambda_1, \lambda_2)\Delta^2_0 X_{12} + iP(\lambda_2; \lambda_0, \lambda_1)\Delta^2_2 X_{01} + iP(\lambda_1; \lambda_0, \lambda_2)\Delta^2_1 X_{02}\right)\Big],$$

$$
\begin{aligned}
s[4] = s - i\ln\Big[\Big(&\Delta^1_{0123} + iP(\lambda_0,\lambda_3;\lambda_1,\lambda_2)\Delta^1_{03}X_{12} + iP(\lambda_0,\lambda_2;\lambda_1,\lambda_3)\Delta^1_{02}X_{13}\\
&+iP(\lambda_0,\lambda_1;\lambda_2,\lambda_3)\Delta^1_{01}X_{23} + iP(\lambda_1,\lambda_3;\lambda_0,\lambda_2)\Delta^1_{13}X_{02}\\
&+iP(\lambda_1,\lambda_2;\lambda_0,\lambda_3)\Delta^1_{12}X_{03} + iP(\lambda_2,\lambda_3;\lambda_0,\lambda_1)\Delta^1_{23}X_{01}\\
&-(P(\lambda_0,\lambda_1;\lambda_2,\lambda_3) + P(\lambda_0,\lambda_2;\lambda_1,\lambda_3) + P(\lambda_0,\lambda_3;\lambda_1,\lambda_2))X_{0123}\Big)\\
&/\Big(\Delta^2_{0123} + iP(\lambda_0,\lambda_3;\lambda_1,\lambda_2)\Delta^2_{03}X_{12} + iP(\lambda_0,\lambda_2;\lambda_1,\lambda_3)\Delta^2_{02}X_{13}\\
&+iP(\lambda_0,\lambda_1;\lambda_2,\lambda_3)\Delta^2_{01}X_{23} + iP(\lambda_1,\lambda_3;\lambda_0,\lambda_2)\Delta^2_{13}X_{02}\\
&+iP(\lambda_1,\lambda_2;\lambda_0,\lambda_3)\Delta^2_{12}X_{03} + iP(\lambda_2,\lambda_3;\lambda_0,\lambda_1)\Delta^2_{23}X_{01}\\
&-(P(\lambda_0,\lambda_1;\lambda_2,\lambda_3) + P(\lambda_0,\lambda_2;\lambda_1,\lambda_3) + P(\lambda_0,\lambda_3;\lambda_1,\lambda_2))X_{0123})\Big].
\end{aligned}
$$

## 5.3. Explicit solutions used for the bosonic supersymmetric Sym–Tafel formula for immersion

By considering the trivial solution of the SSGE (5.2.1),

$$s = 2k\pi, \qquad k \in \mathbb{Z} \tag{5.3.1}$$

the solution $\Phi$ of the LSP for any invertible value of $\lambda_j \in \mathbb{G}$ is given by

$$
\begin{pmatrix}\psi_j\\ \phi_j\\ \chi_j\end{pmatrix} = \begin{pmatrix}
c_j + \left(\frac{-b_j}{2\sqrt{\lambda_j}} - \frac{i\underline{a}_j}{2\sqrt{\lambda_j}}\theta^+ + i\sqrt{\lambda_j}\underline{a}_j\theta^- + \frac{ib_j}{2\sqrt{\lambda_j}}\theta^+\theta^-\right)e^{\eta_j}\\
c_j - \left(\frac{-b_j}{2\sqrt{\lambda_j}} - \frac{i\underline{a}_j}{2\sqrt{\lambda_j}}\theta^+ + i\sqrt{\lambda_j}\underline{a}_j\theta^- + \frac{ib_j}{2\sqrt{\lambda_j}}\theta^+\theta^-\right)e^{\eta_j}\\
\left(\underline{a}_j + \frac{b_j}{2\lambda_j}\theta^+ - b_j\theta^- + i\underline{a}_j\theta^+\theta^-\right)e^{\eta_j}
\end{pmatrix}, \tag{5.3.2}
$$

for j=0,1,2,... , where

$$\eta_j = \frac{x_+}{2\lambda_j} - 2\lambda_j x_- \tag{5.3.3}$$

is a bosonic linear function of $x_+$ and $x_-$, $\underline{a}_j$ is an arbitrary fermionic constant and $b_j, c_j$ are arbitrary bosonic constants. Since the solution (5.3.2) satisfies the LSP (5.2.10) for any value of $\lambda_j$, it is possible to compute a high number of solutions using the Darboux transformations from equation (5.2.27).

In the further examples, we will only consider two non-trivial solutions using the first iteration of the Darboux transformations for the geometric characterization of the bosonic SUSY version of the Sym–Tafel formula for immersion.

According to [**B3**], we now present the bosonic SUSY Sym–Tafel formula for the immersion of solitonic surfaces in Lie superalgebras.

**Proposition 5.3.1.** *Let us assume that there exists a LSP of the form* (5.2.13) *associated with a SUSY integrable systems of partial differential equations* $\Omega = 0$, *where the fermionic potential matrices* $U_\pm$ *take values in the* $\mathfrak{gl}(m|n)$ *Lie superalgebra and the wavefunction* $\Psi$ *takes value in the* $GL(m|n)$ *Lie supergroup. Consider the bosonic infinitesimal deformations*

$$\tilde{U}_\pm = U_\pm + \epsilon\beta(\lambda)\partial_\lambda U_\pm \in \mathfrak{gl}(m|n), \qquad \tilde{\Psi} = \Psi(I + \epsilon F) \in GL(m|n), \tag{5.3.4}$$

*that preserve both the LSP* (5.2.13) *and the zero-curvature condition* (5.2.14) *for an arbitrary bosonic function* $\beta(\lambda)$ *of* $\lambda$, *where* $\partial_\lambda$ *is the derivative with respect to* $\lambda$ *and* $\epsilon$ *is a bosonic infinitesimal parameter whose quadratic terms are neglected. Then, there exists an immersion bosonic supermatrix* $F$ *given by*

$$F = \beta(\lambda)\Psi^{-1}\partial_\lambda\Psi \in \mathfrak{gl}(m|n) \tag{5.3.5}$$

*which defines a two-dimensional surface in a Lie superalgebra whenever its tangent vectors*

$$ED_\pm F = \beta(\lambda)\Psi^{-1}E\partial_\lambda U_\pm\Psi, \qquad E = \begin{pmatrix} I_m & 0 \\ 0 & -I_n \end{pmatrix}, \tag{5.3.6}$$

*are linearly independent, where* $I_m$ *and* $I_n$ *are the identity matrices of dimension* $m$ *and* $n$, *respectively.*

Using the super Killing form defined by the supertrace,

$$\langle A, B\rangle = \frac{1}{2}\mathrm{str}(AB) = \frac{1}{2}\mathrm{tr}(E^{\deg(AB)+1}AB), \tag{5.3.7}$$

we obtain that the metric coefficients are given by the relations [**B3**]

$$\begin{aligned} g_{ii} &= \langle \beta(\lambda)E\partial_\lambda U_\pm, \beta(\lambda)\partial_\lambda U_\pm\rangle, \\ g_{12} = -g_{21} &= \langle \beta(\lambda)E\partial_\lambda U_+, \beta(\lambda)E\partial_\lambda U_-\rangle, \end{aligned} \tag{5.3.8}$$

where $i = 1, 2$ and $1, 2$ stand for $+, -$ respectively, such that the first fundamental form is given by

$$I = (d_+)^2 g_{11} + 2d_+d_-g_{12} + (d_-)^2 g_{22}. \tag{5.3.9}$$

According to [**B1**], the fermionic differential forms $d_\pm$ anticommute with each other,

$$\{d_+, d_-\} = 0, \tag{5.3.10}$$

and represent the infinitesimal displacement in the direction of $D_\pm$. The discriminant $g$ of the metric is given by

$$g = g_{11}g_{22} - g_{12}g_{21} = g_{11}g_{22} + (g_{12})^2. \tag{5.3.11}$$

The unit normal vector $N$ satisfies the relations

$$\langle N, N\rangle = 1, \qquad \langle ED_\pm F, N\rangle = 0. \tag{5.3.12}$$

The vector $N$ can be written only in terms of the tangent vectors,

$$N = \frac{\{ED_+F, ED_-F\}}{\langle\{ED_+F, ED_-F\}, \{ED_+F, ED_-F\}\rangle^{1/2}}, \tag{5.3.13}$$

assuming that the norm,

$$\|\{ED_+F, ED_-F\}\| = \langle\{ED_+F, ED_-F\}, \{ED_+F, ED_-F\}\rangle^{1/2}, \tag{5.3.14}$$

is invertible. The coefficients of the second fundamental form are given by

$$b_{ij} = \langle D_jD_iF, N\rangle = \langle\beta(\lambda)D_j\partial_\lambda U_i - \{\beta(\lambda)E\partial_\lambda U_i, EU_j\}, \Psi N\Psi^{-1}\rangle, \tag{5.3.15}$$

where $i, j = 1, 2$. The indices $1, 2$ stand for $+, -$ respectively. The second fundamental form is

$$II = (d_+)^2 b_{11} + 2d_+d_-b_{12} + (d_-)^2 b_{22}. \tag{5.3.16}$$

The Gaussian and mean curvatures are written respectively as

$$\begin{aligned} K &= \frac{b_{11}b_{22} + (b_{12})^2}{g_{11}g_{22} + (g_{12})^2}, \\ H &= \frac{b_{11}g_{22} + b_{22}g_{11} + 2b_{12}g_{12}}{2(g_{11}g_{22} + (g_{12})^2)}. \end{aligned} \tag{5.3.17}$$

A first solution $s_1[1]$ of the SSGE (5.2.1) is constructed using the first Darboux transformation (5.2.23) for the solutions (5.3.1) and (5.3.2), where the constant $b_0$ is sent to zero. The solution $s_1[1]$ takes the form

$$s_1[1] = 2k\pi - i\ln\left(1 - \frac{i\underline{a}_0 e^{\eta_0}}{c_0\sqrt{\lambda_0}}\theta^+ + \frac{2i\sqrt{\lambda_0}}{c_0}\underline{a}_0 e^{\eta_0}\theta^-\right). \tag{5.3.18}$$

The associated solution $\Phi_1$ of the LSP for $\lambda = \lambda_1$ is obtained from the 1-Darboux transformations (5.2.24),

$$\begin{aligned}
\psi_1[1] &= \left[(\lambda_1-\lambda_0)c_1 - \frac{i}{c_0}\sqrt{\lambda_0\lambda_1}\underline{a}_0\underline{a}_1 e^{\eta_0+\eta_1}\right] \\
&\quad + \left[\frac{i}{2\sqrt{\lambda_1}}(\lambda_1+\lambda_0)\underline{a}_1 e^{\eta_1} - i\sqrt{\lambda_0}\frac{c_1}{c_0}\underline{a}_0 e^{\eta_0}\right]\theta^+ \\
&\quad + \left[-i\sqrt{\lambda_1}(\lambda_1+\lambda_0)\underline{a}_1 e^{\eta_1} + 2i\lambda_0^{3/2}\frac{c_1}{c_0}\underline{a}_0 e^{\eta_0}\right]\theta^- \\
&\quad + \left[\sqrt{\frac{\lambda_0}{\lambda_1}}(\lambda_1+\lambda_0)\frac{\underline{a}_0\underline{a}_1}{c_0}e^{\eta_0+\eta_1}\right]\theta^+\theta^-, \\
\phi_1[1] &= \left[(\lambda_1-\lambda_0)c_1 - \frac{i}{c_0}\sqrt{\lambda_0\lambda_1}\underline{a}_0\underline{a}_1 e^{\eta_0+\eta_1}\right] \qquad (5.3.19) \\
&\quad + \left[\frac{-i}{2\sqrt{\lambda_1}}(\lambda_1+\lambda_0)\underline{a}_1 e^{\eta_1} + i\sqrt{\lambda_0}\frac{c_1}{c_0}\underline{a}_0 e^{\eta_0}\right]\theta^+ \\
&\quad + \left[i\sqrt{\lambda_1}(\lambda_1+\lambda_0)\underline{a}_1 e^{\eta_1} - 2i\lambda_0^{3/2}\frac{c_1}{c_0}\underline{a}_0 e^{\eta_0}\right]\theta^- \\
&\quad + \left[\sqrt{\frac{\lambda_0}{\lambda_1}}(\lambda_1+\lambda_0)\frac{\underline{a}_0\underline{a}_1}{c_0}e^{\eta_0+\eta_1}\right]\theta^+\theta^-, \\
\chi_1[1] &= \left[-(\lambda_0+\lambda_1)\underline{a}_1 e^{\eta_1} + 2\sqrt{\lambda_0\lambda_1}\frac{c_1}{c_0}\underline{a}_0 e^{\eta_0}\right](1+i\theta^+\theta^-).
\end{aligned}$$

The associated pseudo-Riemannian geometry taken from the bosonic SUSY version of the Sym–Tafel immersion formula gives the following coefficients for the metric

$$g_{11} = \frac{-i}{2\lambda}, \qquad g_{12} = -g_{21} = -i, \qquad g_{22} = 2i\lambda, \tag{5.3.20}$$

where the arbitrary function $\beta(\lambda)$ is taken to be $\beta = 2\lambda$. The coefficients $b_{ij}$ of the second fundamental form are given by

$$b_{11} = b_{22} = 0, \qquad b_{12} = -b_{21} = \frac{\underline{a}_0}{c_0}e^{\eta_0}\left(\frac{-1}{\sqrt{\lambda_0}}\theta^+ + 2\sqrt{\lambda_0}\theta^-\right). \tag{5.3.21}$$

The Gaussian curvature $K = 1$ implies that the surface can be classified as a constant positive Gaussian curvature one, which would implies that it would be a sphere in comparison with the classical geometry. However, the mean curvature cannot be computed since the discriminant $g = g_{11}g_{22} - g_{12}g_{21}$ vanishes.

A second 1-Darboux transformation solution $s_2[1]$ is considered using the constraint $\underline{a}_0 = 0$ on solution (5.3.1) and (5.3.2), which leads to the particular solution

$$s_2[1] = 2k\pi - i\ln\Bigg[\left(c_0+\frac{b_0}{2\sqrt{\lambda_0}}e^{\eta_0}\right)^{-1}\left(c_0-\frac{b_0}{2\sqrt{\lambda_0}}e^{\eta_0}\right)$$
$$+2c_0\left(c_0+\frac{b_0}{2\sqrt{\lambda_0}}e^{\eta_0}\right)^{-2}\frac{ib_0}{2\sqrt{\lambda_0}}e^{\eta_0}\theta^+\theta^-\Bigg]. \tag{5.3.22}$$

The associated solution $\Phi_2$ of the LSP for $\lambda = \lambda_1$ is given by

$$\psi_1[1] = \Bigg[\lambda_1 c_1 + \frac{\sqrt{\lambda_1}b_1}{2}e^{\eta_1}$$
$$-\lambda_0\left(c_0+\frac{b_0}{2\sqrt{\lambda_0}}e^{\eta_0}\right)\left(c_1-\frac{b_1}{2\sqrt{\lambda_1}}e^{\eta_1}\right)\left(c_0-\frac{b_0}{2\sqrt{\lambda_0}}e^{\eta_0}\right)^{-1}\Bigg]$$
$$+\Bigg[\left(c_1-\frac{b_1}{2\sqrt{\lambda_1}}e^{\eta_1}\right)\left(c_0-\frac{b_0}{2\sqrt{\lambda_0}}e^{\eta_0}\right)^{-1}\frac{i\sqrt{\lambda_0}b_0}{2}e^{\eta_0}$$
$$-\lambda_0\left(c_0+\frac{b_0}{2\sqrt{\lambda_0}}e^{\eta_0}\right)\left(c_0-\frac{b_0}{2\sqrt{\lambda_0}}e^{\eta_0}\right)^{-1}\frac{ib_1}{2\sqrt{\lambda_1}}+\frac{-i\sqrt{\lambda_1}b_1}{2}e^{\eta_1}$$
$$+\left(c_0+\frac{b_0}{2\sqrt{\lambda_0}}e^{\eta_0}\right)\left(c_1-\frac{b_1}{2\sqrt{\lambda_1}}e^{\eta_1}\right)\left(c_0-\frac{b_0}{2\sqrt{\lambda_0}}e^{\eta_0}\right)^{-2}\frac{ib_0\sqrt{\lambda_0}}{2}e^{\eta_0}\Bigg]\theta^+\theta^-,$$

$$\phi_1[1] = \Bigg[\lambda_1 c_1 - \frac{\sqrt{\lambda_1}b_1}{2}e^{\eta_1}$$
$$-\lambda_0\left(c_0-\frac{b_0}{2\sqrt{\lambda_0}}e^{\eta_0}\right)\left(c_1+\frac{b_1}{2\sqrt{\lambda_1}}e^{\eta_1}\right)\left(c_0+\frac{b_0}{2\sqrt{\lambda_0}}e^{\eta_0}\right)^{-1}\Bigg]$$
$$+\Bigg[\lambda_0\left(c_0-\frac{b_0}{2\sqrt{\lambda_0}}e^{\eta_0}\right)\left(c_0-\frac{b_0}{2\sqrt{\lambda_0}}e^{\eta_0}\right)^{-1}\frac{ib_1}{2\sqrt{\lambda_1}}e^{\eta_1}$$
$$-\left(c_1+\frac{b_1}{2\sqrt{\lambda_1}}e^{\eta_1}\right)\left(c_0+\frac{b_0}{2\sqrt{\lambda_0}}e^{\eta_0}\right)^{-1}\frac{i\sqrt{\lambda_0}b_0}{2}e^{\eta_0}+\frac{i\sqrt{\lambda_1}b_1}{2}e^{\eta_1}$$
$$-\left(c_0-\frac{b_0}{2\sqrt{\lambda_0}}e^{\eta_0}\right)\left(c_1+\frac{b_1}{2\sqrt{\lambda_1}}e^{\eta_1}\right)\left(c_0+\frac{b_0}{2\sqrt{\lambda_0}}e^{\eta_0}\right)^{-2}\frac{i\sqrt{\lambda_0}b_0}{2}e^{\eta_0}\Bigg]\theta^+\theta^-,$$

$$\chi_1[1] = \left[-(\lambda_0+\lambda_1)\frac{b_1e^{\eta_1}}{2\lambda_1}+\sqrt{\frac{\lambda_1}{\lambda_0}}b_0e^{\eta_0}\left(c_0^2-\frac{b_0^2e^{2\eta_0}}{4\lambda_0}\right)^{-1}\left(c_0c_1-\frac{b_0b_1e^{\eta_0+\eta_1}}{4\sqrt{\lambda_0\lambda_1}}\right)\right]\theta^+$$
$$+\left[(\lambda_0+\lambda_1)b_1e^{\eta_1}-2\sqrt{\lambda_0\lambda_1}b_0e^{\eta_0}\left(c_0^2-\frac{b_0^2e^{2\eta_0}}{4\lambda_0}\right)^{-1}\left(c_0c_1-\frac{b_0b_1e^{\eta_0+\eta_1}}{4\sqrt{\lambda_0\lambda_1}}\right)\right]\theta^-.$$

The first fundamental form's coefficients $g_{ij}$ for the associated pseudo-Riemannian geometry of the bosonic SUSY version of the Sym–Tafel immersion formula with

$\beta(\lambda) = 2\lambda$ are given by

$$\begin{aligned} g_{11} =& \frac{-i}{2\lambda}, \qquad g_{22} = 2i\lambda, \\ g_{12} =& - g_{21} = -i\cos(s_2[1]) \\ =& - i\left(c_0^2 - \frac{b_0^2}{4\lambda_0}e^{2\eta_0}\right)^{-1}\left(c_0^2 + \frac{b_0^2}{4\lambda_0}e^{2\eta_0}\right) \\ & - 2\left(c_0^2 - \frac{b_0^2}{4\lambda_0}^{2\eta_0}\right)^{-2}\frac{c_0^2b_0^2}{\lambda_0}e^{\eta_0}\theta^+\theta^- \end{aligned} \tag{5.3.23}$$

and the coefficients $b_{ij}$ of the second fundamental form are

$$\begin{aligned} b_{11} =& b_{22} = 0, \\ b_{12} =& - b_{21} = \sin(s_2[1]) \\ =& i\left(c_0^2 - \frac{b_0^2}{4\lambda_0}e^{2\eta_0}\right)^{-1}\frac{c_0b_0}{\sqrt{\lambda_0}}e^{\eta_0} \\ & + \left(c_0^2 - \frac{b_0^2}{4\lambda_0}e^{2\eta_0}\right)^{-2}\left(c_0^2 + \frac{b_0^2}{4\lambda_0}e^{2\eta_0}\right)\frac{c_0b_0}{\sqrt{\lambda_0}}e^{\eta_0}. \end{aligned} \tag{5.3.24}$$

Both discriminants $g = g_{11}g_{22} - g_{12}g_{21}$ and $b = b_{11}b_{22} - b_{12}b_{21}$ are equal to

$$\begin{aligned} \sin^2(s_2[1]) =& - \left(c_0^2 - \frac{b_0^2}{4\lambda_0}e^{2\eta_0}\right)^{-2}\frac{c_0^2b_0^2}{\lambda_0}e^{2\eta_0} \\ & + 2i\left(c_0^2 - \frac{b_0^2}{4\lambda_0}e^{2\eta_0}\right)^{-3}\left(c_0^2 + \frac{b_0^2}{4\lambda_0}e^{2\eta_0}\right)\frac{c_0^2b_0^2}{\lambda_0}e^{2\eta_0}\theta^+\theta^-. \end{aligned} \tag{5.3.25}$$

The Gaussian curvature $K$ is equal to 1 and the mean curvature takes the non-trivial form

$$\begin{aligned} H =& - i\cot(s_2[1]), \\ =& - \frac{\sqrt{\lambda_0}}{c_0b_0}e^{-\eta_0}\left(c_0^2 - \frac{b_0^2}{4\lambda_0}e^{2\eta_0}\right) - 2\left(c_0^2 - \frac{b_0^2}{4\lambda_0}e^{2\eta_0}\right)^{-1}\frac{c_0b_0}{\sqrt{\lambda_0}}e^{\eta_0}\theta^+\theta^- \\ & + \left(c_0^2 - \frac{b_0^2}{4\lambda_0}e^{2\eta_0}\right)^{-1}\left(c_0^2 + \frac{b_0^2}{4\lambda_0}e^{2\eta_0}\right)^2\frac{\sqrt{\lambda_0}}{c_0b_0}e^{-\eta_0}\theta^+\theta^-, \end{aligned} \tag{5.3.26}$$

in terms of the bosonic quantities $\lambda_0, x_+, x_-$ and $\theta^+\theta^-$. By considering the case where $b_0 = 2\sqrt{\lambda_0}c_0$, we obtain that the body part of the mean curvature is simply given by

$$H_b = \sinh\eta_0. \tag{5.3.27}$$

## 5.4. Conclusions

In this paper, we study the links between some of the integrability properties associated with the SSGE. First, we derive fermionic potential supermatrices in $\mathfrak{sl}(2|1,\mathbb{G})$ which provide a LSP whose zero-curvature condition corresponds to the SSGE. Using this LSP, we construct an equivalent LSP in terms of bosonic derivatives instead of fermionic derivatives, which require that the potential matrices be bosonic supermatrices in $\mathfrak{sl}(2|1,\mathbb{G})$. Moreover, we provide links between the fermionic LSP, coupled sets of super Riccati equations whose compatibility condition is equivalent to the SSGE, and the associated auto-Bäcklund transformation. Furthermore, we provide a comprehensive description of the Darboux transformation associated with the SSGE. This Darboux transformation allows us to provide non-trivial multisoliton solutions of the SSGE.

The bosonic SUSY version of the Sym–Tafel formula for immersion is investigated through examples for the SSGE. Using 1-Darboux transformation solutions, we are able to compute two new examples of geometric characterizations of the associated surfaces immersed in the Lie superalgebra $\mathfrak{gl}(2|1,\mathbb{G})$ exclusively in terms of the fermionic and bosonic independent variables. These two surfaces are linked with spheres in analogy with the classical differential geometry since they have a positive constant Gaussian curvature, $K = 1$.

The subjects addressed in this paper can be extended in many directions. Among them, we can study other SUSY integrable systems based on their integrability properties and evaluate some examples of immersed surfaces in Lie superalgebras. Moreover, it would be interesting to find an invertible wavefunction $\Psi$ so that we could explicitly compute the deformed surfaces $F$, which are written in terms of the wavefunction $\Psi$. From these surfaces, it should be possible to graphically show the shape of the surfaces and see how their characteristics, such as the metric and curvatures, manifest themselves.

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# Chapitre 6

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## ON GEOMETRIC ASPECTS OF THE SUPERSYMMETRIC FOKAS–GEL'FAND IMMERSION FORMULA

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**Résumé**

Dans cet article, nous développons une nouvelle caractérisation géométrique pour les versions supersymétriques de la formule d'immersion de Fokas–Gel'fand pour plonger des supervariétés solitoniques à deux variables indépendantes bosoniques et deux variables indépendantes fermioniques dans des superalgèbres de Lie. L'approche utilisée considère trois types de problèmes linéaires spectraux : un premier utilisant les dérivées covariantes, un deuxième utilisant les dérivés par rapport aux variables bosoniques et un troisième utilisant les dérivées par rapport aux variables fermioniques. Nous démontrons que les deuxième et troisième types de problèmes linéaires spectraux peuvent être obtenus à partir du premier type grâce à des relations spécifiques. Cela nous permet d'investiguer, au moyen des première et deuxième formes fondamentales, la géométrie des supervariétés plongées dans des superalgèbres de Lie. Lorsque cela est possible, les courbures moyenne et de Gauss sont calculées. Ces considérations théoriques sont appliquées à l'équation supersymétrique de sine-Gordon, qui est un système supersymétrique intégrable.

**Abstract**

In this paper, we develop a new geometric characterization for the supersymmetric versions of the Fokas–Gel'fand formula for the immersion of soliton supermanifolds with two bosonic and two fermionic independent variables into Lie superalgebras. The approach used considers three types of linear spectral problems : the first using covariant derivatives, the second using bosonic variable derivatives and the third using fermionic variable derivatives. We demonstrate that the second and third types of linear spectral problems can be obtained from the first type via a specific relation. This allows us to investigate, through the first and second fundamental forms, the geometry of the supermanifolds immersed in Lie superalgebras. Whenever possible, the mean and Gaussian curvatures of the supermanifolds are calculated. These theoretical considerations are applied to the supersymmetric sine-Gordon equation, which is a supersymmetric integrable system.

## 6.1. Introduction

The Fokas–Gel'fand formula for the immersion of surfaces into Lie algebra has been at the intersection of several branches of mathematics, such as differential geometry, Lie groups and Lie algebras, complex variables, global analysis, mathematical physics, integrable systems and spectral theory. The Fokas–Gel'fand immersion formula (FGIF) provides rich classes of geometric objects (see e.g. [**26**, **49**, **54**, **55**, **70**, **72**, **112**]). The continuous deformations of manifolds which are much better understood in the classical case than in their supersymmetric (SUSY) versions for which only few examples are known. Recently, generalizations of the FGIF were constructed to consider bosonic and fermionic deformations associated with SUSY integrable systems [**B3**]. Since soliton supermanifolds associated with SUSY integrable models behave rather differently than the classical smooth manifolds, the rich character of the SUSY versions of the FGIF makes a rather special and interesting object of study. This paper is devoted to the differential geometric apsects of the SUSY FGIF. Our main goal is to provide a self-contained, comprehensive approach to this subject.

The present paper is a follow-up of the investigation performed in [**B3**] concerning the SUSY extensions of the FGIF. In particular, we develop a new geometric characterization of deformation supermanifolds associated with SUSY integrable systems in two bosonic and two fermionic independent variables admitting three types of linear spectral problems (LSPs). This characterization is achieved through the construction of structural equations associated with these supermanifolds. We determine their first and second fundamental forms. We find the explicit expressions, whenever possible, for the mean and Gaussian curvatures, which are expressed solely in terms of the potential matrices of the LSP and a normal unit vector. To perform this analysis, the inner product used is the nondegenerate super Killing form on Lie superalgebras based on the supertrace. Furthermore, we provide links between these three types of LSPs and consequently we find specific relations between their associated zero-curvature conditions (ZCCs). These theoretical considerations are illustrated by the SUSY sine-Gordon equation (SSGE) for which we obtain richer classes of geometric objects for different deformations than the ones obtained in [**B3**].

The paper is organized as follows. In section 6.2, we provide an overview of the classical Fokas–Gel'fand formula for the immersion of surfaces in Lie algebras and its corresponding pseudo-Riemannian geometry. Section 6.3 consists of a summary of the Grassmann algebras and the conventions used in this paper. Section 6.4 is separated into three parts. In the first part, we state the assumptions made on the SUSY integrable system under consideration and we provide a link between the three different types of LSPs used in this paper. In section 6.4.1, we construct the bosonic version of the FGIF through deformation supermatrices and a bosonic deformation parameter. In section 6.4.2, we construct the fermionic version of the FGIF using a fermionic deformation parameter and some deformation supermatrices. Section 6.5 is devoted to the geometric characterization of both the bosonic and fermionic SUSY versions of the FGIF. In section 6.6, we apply these theoretical considerations to a SUSY integrable system, namely the SSGE, in order to obtain a geometric characterization of the deformation surfaces associated with different symmetries of the LSPs. Some conclusions and future perspectives are discussed in section 6.7.

## 6.2. The classical Fokas–Gel'fand immersion formula

Let us consider an integrable system of partial differential equations (PDEs)

$$\Delta[u] = 0, \tag{6.2.1}$$

where we use the jet space notation $[u] = (x_1, x_2, u^1, ..., u^n, \partial_{x_1} u^1, ...)$, which depends on two independent real (or complex) variables $x_1$ and $x_2$, $n$ dependent real (or complex) variables $u^1, ..., u^n$ and some derivatives of the dependent variables up to order $k$. Let us assume that there exists a LSP

$$D_{x_i}\Psi = V_i\Psi, \qquad i = 1, 2, \tag{6.2.2}$$

where $D_{x_i}$ is the total derivative with respect to $x_i$, the potential matrices $V_i = V_i([u], \lambda)$ take their values in a Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(m, \mathbb{C})$, the wavefunction

$\Psi = \Psi([u], \lambda)$ takes its values in the corresponding Lie subgroup $G$ in $GL(m, \mathbb{C})$ and $\lambda$ is a spectral parameter taking values in a subset of $\mathbb{C}$. The compatibility condition of the LSP (6.2.2) forms a ZCC,

$$D_{x_1} V_2 - D_{x_2} V_1 + [V_2, V_1] = 0, \tag{6.2.3}$$

which is assumed to be equivalent to the original integrable system (6.2.1) for any value of the spectral parameter $\lambda$.

To construct the FGIF, we consider an infinitesimal deformation of the potential matrices $V_i$,

$$\tilde{V}_i = V_i + \epsilon B_i \in \mathfrak{g}, \qquad i = 1, 2, \tag{6.2.4}$$

where $B_i = B_i([u], \lambda)$ are $\mathfrak{g}$-valued matrices and $\epsilon$ is a real infinitesimal parameter (so that we neglect any terms involving $\epsilon^2$ or higher order in $\epsilon$), together with the infinitesimal deformation of the wavefunction $\Psi$,

$$\tilde{\Psi} = \Psi(I + \epsilon F) \in G, \tag{6.2.5}$$

where $F = F([u], \lambda)$ is the deformation surface taking its values in the associated Lie algebra $\mathfrak{g}$. We assume that these infinitesimal deformations preserve the LSP (6.2.2), i.e.

$$D_{x_i} \tilde{\Psi} = \tilde{V}_i \tilde{\Psi}, \qquad i = 1, 2, \tag{6.2.6}$$

and that the tangent vectors

$$D_{x_i} F = \Psi^{-1} B_i \Psi, \qquad i = 1, 2 \tag{6.2.7}$$

are linearly independent. If the tangent vectors are not linearly independent, then the surface degenerates to a curve. In order to leave the ZCC (6.2.3) invariant, the deformation matrices $B_i$ must satisfy the relation

$$D_{x_2} B_1 - D_{x_1} B_2 + [B_1, V_2] + [V_1, B_2] = 0, \tag{6.2.8}$$

which is obtained by the compatibility condition of equations (6.2.7) whenever the ZCC (6.2.3) is satisfied. The relation (6.2.8) also coincides with the infinitesimal deformation of the ZCC (6.2.3).

Solutions of equations (6.2.7) and (6.2.8) for the matrices $B_i$ and $F$ of the FGIF can be decomposed into three types of deformations : the spectral deformation [**112**], the common (generalized) symmetry deformation [**54**, **55**, **72**] and the gauge symmetry deformation [**26**, **49**]. The spectral deformation is associated with the Sym–Tafel immersion formula, i.e. the deformation surface $F$ is given by

$$F = \Psi^{-1}\beta(\lambda)\partial_\lambda\Psi, \tag{6.2.9}$$

where $\beta(\lambda)$ is an arbitrary function of the spectral parameter $\lambda$ and $\partial_\lambda$ is the derivative with respect to $\lambda$. The tangent vectors of the surface $F$ immersed in the Lie algebra $\mathfrak{g}$ take the form

$$D_{x_i}F = \Psi^{-1}\beta(\lambda)\partial_\lambda V_i\Psi, \qquad i = 1, 2. \tag{6.2.10}$$

The deformations associated with generalized symmetries of the original system (6.2.1) and its LSP (6.2.2) were introduced by Fokas and Gel'fand [**54**] and then investigated by many authors (see e.g. [**55**, **70**, **72**, **B3**]). The associated deformation surface $F$ takes the form

$$F = \Psi^{-1}\mathrm{pr}(\omega_R)\Psi, \tag{6.2.11}$$

where $\omega_R$ is a vector field which generates a (generalized) Lie symmetry transformation that is common to both the original system (6.2.1) and its LSP (6.2.2) and $\mathrm{pr}(\omega_R)$ stands for the prolongation of the vector field $\omega_R$ in the extended jet space as described in [**95**]. In the classical case, a vector field $\omega_R$ can always be written in the evolutionary form,

$$\omega_R = R[u]\partial_u, \tag{6.2.12}$$

which possesses the convenient property that it commutes with the total derivatives with respect to the independent variables, i.e.

$$[\mathrm{pr}(\omega_R), D_{x_i}] = 0 \tag{6.2.13}$$

(see Proposition 5.12 in [**95**]). Using this evolutionary form of $\omega_R$, the tangent vectors of $F$ are given by

$$D_{x_i}F = \Psi^{-1}\mathrm{pr}(\omega_R)V_i\Psi, \qquad i = 1, 2. \tag{6.2.14}$$

The gauge deformations were introduced by Cieslinski and Doliwa [**26**, **49**] in such a way that

$$F = \Psi^{-1}S\Psi, \tag{6.2.15}$$

where $S = S([u], \lambda)$ is a gauge taking its values in the associated Lie algebra $\mathfrak{g}$. The gauge deformation can be seen as the generator of a left-transformation gauge $e^{\epsilon S}$ of the wavefunction $\Psi$, i.e.

$$\begin{aligned}\lim_{\epsilon^2\to 0}\left(e^{\epsilon S}\Psi\right) &= (I + \epsilon S)\Psi = \Psi\Psi^{-1}(I + \epsilon S)\Psi = \Psi(I + \epsilon\Psi^{-1}S\Psi)\\ &= \Psi(I + \epsilon F).\end{aligned} \tag{6.2.16}$$

The tangent vectors of the surface $F$ in (6.2.15) take the form

$$D_{x_i}F = \Psi^{-1}(D_{x_i}S + [S, V_i])\Psi, \qquad i = 1, 2. \tag{6.2.17}$$

The FGIF is therefore given by the superposition of all three types of deformations in equations (6.2.9), (6.2.11) and (6.2.15), that is

$$F = \Psi^{-1}\beta(\lambda)\partial_\lambda\Psi + \Psi^{-1}\text{pr}(\omega_R)\Psi + \Psi^{-1}S\Psi, \tag{6.2.18}$$

with the tangent vectors given by

$$D_{x_i}F = \Psi^{-1}\left(\beta(\lambda)\partial_\lambda V_i + \text{pr}(\omega_R)V_i + D_{x_i}S + [S, V_i]\right)\Psi, \qquad i = 1, 2. \tag{6.2.19}$$

The spectral deformation and the (generalized) Lie symmetry deformation are not equivalent since it is possible to introduce the spectral parameter $\lambda$ in the LSP (6.2.2) via a symmetry which leaves the original system (6.2.1) invariant but does not leave its LSP (6.2.2) invariant [**B2**]. Hence, the condition for the surface $F$ in (6.2.11) that $\omega_R$ has to be a symmetry of both the original system (6.2.1) and its LSP (6.2.2) is not satisfied. However, it is possible to choose a gauge $S$ which provides a deformation equivalent to either the spectral deformation or the Lie symmetry deformation,

$$S = \beta(\lambda)\partial_\lambda\Psi\Psi^{-1}, \qquad \text{or} \qquad S = \text{pr}(\omega_R)\Psi\Psi^{-1}, \tag{6.2.20}$$

respectively [**70**]. In addition, the gauge deformation contains some additional symmetries of the wavefunction $\Psi$ which leaves the ZCC (6.2.3) invariant.

Even if we do not have an explicit solution for the wavefunction $\Psi$, it is still possible to extract some informations about the surface $F$. Using the Killing form

$$\langle M, N\rangle = \alpha\text{tr}(MN), \qquad M, N \in \mathfrak{g} \tag{6.2.21}$$

(up to a normalization factor $\alpha$), it is possible to geometrically characterize the pseudo-Riemannian surface $F$ since the Killing form is invariant under group conjugation of its argument, i.e.

$$\langle\Psi^{-1}M\Psi, \Psi^{-1}N\Psi\rangle = \langle M, N\rangle, \qquad M, N \in \mathfrak{g}, \qquad \Psi \in G. \tag{6.2.22}$$

The metric coefficients are given by

$$g_{ij} = \langle D_{x_i}F, D_{x_j}F\rangle = \langle B_i, B_j\rangle, \qquad i, j = 1, 2 \tag{6.2.23}$$

and the first fundamental form is given by

$$I = g_{11}dx_1^2 + 2g_{12}dx_1dx_2 + g_{22}dx_2^2. \tag{6.2.24}$$

We introduce a normal unit vector $N$ defined by

$$N = \frac{\Psi^{-1}[B_1, B_2]\Psi}{\|[B_1, B_2]\|} \in \mathfrak{g}, \tag{6.2.25}$$

which has the properties

$$\langle N, N\rangle = 1, \qquad \langle D_{x_i}F, N\rangle = 0, \qquad i = 1, 2. \tag{6.2.26}$$

From this normal unit vector $N$, we can compute the coefficients of the second fundamental form,

$$b_{ij} = \langle D_{x_j} D_{x_j} F, N\rangle, \qquad i, j = 1, 2, \tag{6.2.27}$$

hence

$$II = b_{11}dx_1^2 + 2b_{12}dx_1dx_2 + b_{22}dx_2^2. \tag{6.2.28}$$

If we consider the matrices

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \qquad b = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \tag{6.2.29}$$

then the mean and Gaussian curvatures respectively take the forms

$$\begin{aligned} H &= \frac{1}{2}\mathrm{tr}(bg^{-1}) = \frac{b_{11}g_{22} + b_{22}g_{11} - 2b_{12}g_{12}}{2(g_{11}g_{22} - g_{12}^2)}, \\ K &= \frac{\det(b)}{\det(g)} = \frac{b_{11}b_{22} - b_{12}^2}{g_{11}g_{22} - g_{12}^2}. \end{aligned} \tag{6.2.30}$$

## 6.3. Preliminaries on Grassmann algebras

In this section, we provide some basic notions concerning the Grassmann algebra formalism which are used throughout this paper. A more comprehensive presentation on Grassmann algebras can be found in [**33**, **43**, **56**, **118**] and references therein.

The (complex) Grassmann algebra $\mathbb{G}$ is an associative graded algebra generated by a set of odd elements $\xi_j$, $j = 1, 2, ..., \ell$, together with a unit 1, which have the properties

$$\xi_j\xi_k + \xi_k\xi_j = 0, \qquad 1\xi_j = \xi_j, \qquad j, k = 1, 2, ..., \ell \tag{6.3.1}$$

From the first property in equation (6.3.1), for $j = k$, we obtain that any odd element squared vanishes,

$$(\xi_j)^2 = 0. \tag{6.3.2}$$

We do not specify the number $\ell$ of generators $\xi_j$ of the Grassmann algebra $\mathbb{G}$, but we assume that there is a sufficient number of them to make all formulas meaningful.

All elements in the Grassmann algebra $\mathbb{G}$ can be decomposed into the sum of two homogeneous parts, one even which commutes with all elements of $\mathbb{G}$ and the other one odd which is the complement set of the even set in the Grassmann algebra $\mathbb{G}$. Homogeneous (even or odd) elements satisfy the multiplication properties

$$\begin{aligned} \text{even} \cdot \text{even} &= \text{even}, \\ \text{even} \cdot \text{odd} &= \text{odd}, \\ \text{odd} \cdot \text{odd} &= \text{even}. \end{aligned} \tag{6.3.3}$$

The degree of a homogeneous element $a$ in $\mathbb{G}$ is defined as

$$\deg(a) = \begin{cases} 0 & \text{if } a \text{ is an even element of } \mathbb{G}, \\ 1 & \text{if } a \text{ is an odd element of } \mathbb{G}. \end{cases} \tag{6.3.4}$$

Throughout this paper, one can substitute the words bosonic and fermionic for even and odd, respectively, without distinctions.

A square $(m+n) \times (m+n)$ supermatrix $M$ is defined by block submatrices of homogeneous elements in $\mathbb{G}$ as

$$M = \begin{pmatrix} A_{(m\times m)} & B_{(m\times n)} \\ C_{(n\times m)} & D_{(n\times n)} \end{pmatrix}, \tag{6.3.5}$$

where the degree of the elements in $A$ is the same as those in $D$, but is not the same as those in $B$ and $C$. When the degree of the elements in $A$ is even, the supermatrix $M$ is said to be even and when the degree of the elements in $A$ is odd, the supermatrix $M$ is said to be odd. Supermatrices also satisfy the properties in equations (6.3.3). The degree of a supermatrix $M$ is defined similarly to that of homogeneous elements as given in equation (6.3.4), i.e.

$$\deg(M) = \begin{cases} 0 & \text{if } M \text{ is an even supermatrix}, \\ 1 & \text{if } M \text{ is an od supermatrix}. \end{cases} \tag{6.3.6}$$

A Lie superalgebra $\mathfrak{g}$ is composed of a super vector space together with a Lie super bracket for which $\mathfrak{g}$ is closed under its action, i.e.

$$M_1 M_2 - (-1)^{\deg(M_1)\deg(M_2)} M_2 M_1 = M_3 \in \mathfrak{g}, \qquad \forall M_1, M_2 \in \mathfrak{g}. \tag{6.3.7}$$

In this paper, the Lie super bracket is denoted explicitly by the commutator and anticommutator,

$$[M_1, M_2] = M_1M_2 - M_2M_1, \qquad \{M_1, M_2\} = M_1M_2 + M_2M_1, \tag{6.3.8}$$

respectively, depending on the degrees of $M_1$ and $M_2$. The $\mathfrak{gl}(m|n, \mathbb{G})$ Lie superalgebra is the set of all $(m+n)\times(m+n)$ supermatrices and the $\mathfrak{sl}(m|n, \mathbb{G})$ Lie superalgebra involves the subset of all $(m+n)\times(m+n)$ supermatrices $M$ with the property that their supertrace vanishes, i.e.

$$\operatorname{str}(M) = \operatorname{tr}(E^{\deg(M)+1}M) = 0, \tag{6.3.9}$$

where

$$E = \begin{pmatrix} I_{(m\times m)} & 0 \\ 0 & -I_{(n\times n)} \end{pmatrix}, \tag{6.3.10}$$

and $I_{(m\times m)}$ (and $I_{(n\times n)}$) is the identity matrix of dimension $m$ (and $n$).

The superdeterminant is defined for an even supermatrix $M$ as

$$\operatorname{sdet}(M) = \frac{\det(A - BD^{-1}C)}{\det(D)} = \frac{\det(A)}{\det(D - CA^{-1}B)}, \tag{6.3.11}$$

whenever $\det(A)^{-1}$ and $\det(D)^{-1}$ are well-defined. The $GL(m|n, \mathbb{G})$ Lie supergroup is composed of all invertible (even) $(m+n)\times(m+n)$ supermatrices.

In this paper, we use the following convenient notation for derivatives with respect to an odd variable $\theta$ and multiplication by an odd scalar $\underline{\alpha}$ with a supermatrix $M$ (of the form given in (6.3.5)) :

$$\partial_\theta M = \begin{pmatrix} \partial_\theta A & \partial_\theta B \\ -\partial_\theta C & -\partial_\theta D \end{pmatrix}, \qquad \underline{\alpha}M = \begin{pmatrix} \underline{\alpha}A & \underline{\alpha}B \\ -\underline{\alpha}C & -\underline{\alpha}D \end{pmatrix}, \tag{6.3.12}$$

respectively. Moreover, the associated Leibniz rule for derivatives with respect to an odd variable $\theta$ takes the form

$$\partial_\theta(hg) = \partial_\theta h g + (-1)^{\deg(h)} h \partial_\theta g, \tag{6.3.13}$$

if $h$ and $g$ are scalar functions or supermatrices of $\theta$. One should note that the derivatives with respect to an odd variable change the degree of a (scalar or supermatrix) function. Commutation between a supermatrix $M$ and an odd scalar $\underline{\alpha}$ is given by

$$\underline{\alpha}M = (-1)^{\deg(M)} M\underline{\alpha}. \tag{6.3.14}$$

## 6.4. The supersymmetric Fokas–Gel'fand immersion formula

Let us consider a SUSY integrable system of PDEs

$$\Delta[u]=0, \qquad [u]=(x_+,x_-,\theta^+,\theta^-,u^1,...,u^k,\partial_{x_\pm}u^1,\partial_{\theta^\pm}u^1,...) \tag{6.4.1}$$

written in terms of the bosonic light-cone coordinates $x_+, x_-$, the fermionic independent variables $\theta^+, \theta^-$, the $\mathbb{G}$-valued dependent variables $u^\alpha$ ($\alpha=1,...,k$), and some derivatives of $u^\alpha$.

Let us assume that there exists a fermionic linear spectral problem (FLSP)

$$D_\pm\Psi = U_\pm\Psi, \tag{6.4.2}$$

where the wavefunction $\Psi=\Psi([u],\lambda)$ takes value in a subgroup $G$ of the $GL(m|n,\mathbb{G})$ Lie supergroup and the potential fermionic supermatrices $U_\pm=U_\pm([u],\lambda)$ take values in the associated Lie superalgebra $\mathfrak{g}$, which is a subalgebra of the $\mathfrak{gl}(m|n,\mathbb{G})$ Lie superalgebra. The constant $\lambda$ is the spectral parameter taking its values in a subset of $\mathbb{G}$. The covariant derivatives $D_+, D_-$ are given by

$$D_\pm = D_{\theta^\pm} - i\theta^\pm D_{x_\pm}, \tag{6.4.3}$$

where $D_{\theta^\pm}$ and $D_{x_\pm}$ stand for the total derivatives with respect to $\theta^\pm$ and $x_\pm$, respectively. These fermionic derivatives satisfy the properties

$$D_\pm^2 = -iD_{x_\pm}, \qquad \{D_+,D_-\}=0, \tag{6.4.4}$$

where $\{\cdot,\cdot\}$ stands for the anticommutator. The compatibility condition of equations (6.4.2) imposed by $\{D_+,D_-\}\Psi=0$ is given by the relation

$$\Omega \equiv D_+U_- + D_-U_+ - \{U_+,U_-\}=0, \tag{6.4.5}$$

called the ZCC, which is assumed to be equivalent to the original system (6.4.1) for any possible value of the spectral parameter $\lambda$.

From the FLSP (6.4.2) and the property (6.4.4a), we can construct another LSP which involves the bosonic light-cone coordinate derivatives $D_{x_\pm}$ instead of the covariant derivatives $D_\pm$. This LSP (which, in what follows, is called the $x_\pm$-LSP) takes the form

$$D_{x_\pm}\Psi = V_\pm\Psi, \tag{6.4.6}$$

where the bosonic potential supermatrices $V_\pm=V_\pm([u],\lambda)$ take values in the Lie superalgebra $\mathfrak{g}$ and can be written in terms of the potential supermatrices $U_\pm$ as

$$V_\pm = i\left(D_\pm U_\pm - U_\pm^2\right). \tag{6.4.7}$$

The bosonic potential supermatrices $V_\pm$ satisfy the relation (similar to the classical ZCC (6.2.3))

$$D_{x_+} V_- - D_{x_-} V_+ + [V_-, V_+] = 0, \tag{6.4.8}$$

which is the compatibility condition of the $x_\pm$-LSP (6.4.6). The relation (6.4.8) is satisfied whenever the set of variables $u^\alpha$ is a solution of the original system (6.4.1) and of the ZCC (6.4.5). However, one should note that equation (6.4.8) is not necessarily equivalent to the original system (6.4.1), but it is always equivalent at least to certain differential consequences of the original system (6.4.1), i.e.

$$D_\pm(\Delta[u]) = 0, \qquad \text{or} \qquad D_+ D_-(\Delta[u]) = 0. \tag{6.4.9}$$

Explicitly, equation (6.4.8) is given in terms of the ZCC $\Omega$, from equation (6.4.5), by

$$\begin{aligned} D_+ D_- \Omega + &\{D_+\Omega, U_-\} - \{D_-\Omega, U_+\} + [D_- U_+, \Omega] \\ &+ U_-\Omega U_+ - U_+\Omega U_- + \Omega U_+ U_- - U_- U_+ \Omega = 0. \end{aligned} \tag{6.4.10}$$

We can also construct a third LSP, which involves the fermionic derivatives $D_{\theta^\pm}$, from the FLSP (6.4.2) and the $x_\pm$-LSP (6.4.6). That LSP takes the form

$$D_{\theta^\pm} \Psi = W_\pm \Psi, \tag{6.4.11}$$

which, in what follows, is denoted the $\theta^\pm$-LSP. The fermionic supermatrices $W_\pm = W_\pm([u], \lambda)$ take values in the Lie superalgebra $\mathfrak{g}$ and can be written in terms of the potential supermatrices $U_\pm$ and $V_\pm$ as

$$W_\pm = U_\pm + i\theta^\pm V_\pm. \tag{6.4.12}$$

The ZCC associated with equation (6.4.11) is given by

$$D_{\theta^+} W_- + D_{\theta^-} W_+ - \{W_+, W_-\} = 0 \tag{6.4.13}$$

and can be written in terms of the supermatrices $U_\pm$ and $V_\pm$ as

$$\begin{aligned} &(D_+ U_- + D_- U_+ - \{U_+, U_-\}) + i\theta^+ \left(D_{x_+} U_- - D_- V_+ + [U_-, V_+]\right) \\ &+ i\theta^- \left(D_{x_-} U_+ - D_+ V_- + [U_+, V_-]\right) \\ &+ \theta^+\theta^- \left(D_{x_-} V_+ - D_{x_+} V_- + [V_+, V_-]\right) = 0. \end{aligned} \tag{6.4.14}$$

The second and third sets of terms in equation (6.4.14) represent the mixed compatibility conditions of the FLSP (6.4.2) and the $x_\pm$-LSP (6.4.6), i.e.

$$0 = [D_{x_\pm}, D_\mp]\Psi = \left(D_{x_\pm} U_\mp - D_\mp V_\pm + [U_\mp, V_\pm]\right) \Psi. \tag{6.4.15}$$

These compatibility conditions are equivalent to the relation

$$0 = D_\pm\Omega + [\Omega, U_\pm], \tag{6.4.16}$$

which is in the form of a Lax equation [**75**].

### 6.4.1. The bosonic supersymmetric Fokas–Gel'fand immersion formula

Let us consider a deformation of the three LSPs (6.4.2), (6.4.6) and (6.4.11) which preserves their forms, i.e.

$$D_\pm\tilde{\Psi} = \tilde{U}_\pm\tilde{\Psi}, \qquad D_{x_\pm}\tilde{\Psi} = \tilde{V}_\pm\tilde{\Psi}, \qquad D_{\theta^\pm}\tilde{\Psi} = \tilde{W}_\pm\tilde{\Psi}, \tag{6.4.17}$$

for which their compatibility conditions are equivalent to the original system (6.4.1) for any value of the spectral parameter $\lambda$ and such that the deformed potential supermatrices are given by

$$\tilde{U}_\pm = U_\pm + \epsilon A_\pm, \quad \tilde{V}_\pm = V_\pm + \epsilon V_\pm, \quad \tilde{W}_\pm = W_\pm + \epsilon C_\pm \in \mathfrak{g} \tag{6.4.18}$$

together with the deformed wavefunction $\tilde{\Psi}$ written in terms of a deformed supermanifold $F = F([u], \lambda)$ as

$$\tilde{\Psi} = \Psi(I + \epsilon F) \in G. \tag{6.4.19}$$

The deformation fermionic supermatrices $A_\pm = A_\pm([u], \lambda), C_\pm = C_\pm([u], \lambda)$ and bosonic supermatrices $B_\pm = B_\pm([u], \lambda), F$ take their values in the Lie superalgebra $\mathfrak{g}$ and the bosonic deformation parameter $\epsilon$ is assumed to vanish for quadratic terms, i.e. $\epsilon^2 = 0$. Hence, the bosonic parameter $\epsilon$ is either very small or nilpotent of order 2. The compatibility conditions of the deformed LSPs (6.4.17) impose conditions on the supermatrices $A_\pm$, $B_\pm$ and $C_\pm$ given by the relations

$$\begin{aligned} &D_+A_- + D_-A_+ - \{A_+, U_-\} - \{A_-, U_+\} = 0, \\ &D_{x_+}B_- - D_{x_-}B_+ + [B_-, V_+] + [V_-, B_+] = 0, \\ &D_{\theta^+}C_- + D_{\theta^-}C_+ - \{C_+, W_-\} - \{C_-, W_+\} = 0, \end{aligned} \tag{6.4.20}$$

which are equivalent to the deformation of the three ZCCs (6.4.5), (6.4.8) and (6.4.13), respectively, up to the addition of supermatrices at least of first order in $\epsilon$.

From equations (6.4.2), (6.4.17a), (6.4.18a) and (6.4.19), we can compute the covariant derivatives of the deformed surface $F$, which are given by

$$D_\pm F = \Psi^{-1}A_\pm\Psi \tag{6.4.21}$$

up to the addition of fermionic supermatrices which are at least of first order in $\epsilon$. Similarly, we can compute the tangent vectors of the deformed surfaces $F$ in the $x_{\pm}$- and $\theta^{\pm}$-directions, which take the forms

$$D_{x_{\pm}}F = \Psi^{-1}B_{\pm}\Psi, \qquad D_{\theta^{\pm}}F = \Psi^{-1}C_{\pm}\Psi \tag{6.4.22}$$

up to the addition of supermatrices which are at least of first order in $\epsilon$.

Furthermore, from equations (6.4.21), it is possible to determine the supermatrices $B_{\pm}$ and $C_{\pm}$ explicitly in terms of $A_{\pm}$ and the potential supermatrices $U_{\pm}$, i.e.

$$B_{\pm} = i\left(D_{\pm}A_{\pm} - \{U_{\pm}, A_{\pm}\}\right), \qquad C_{\pm} = A_{\pm} + i\theta^{\pm}B_{\pm}, \tag{6.4.23}$$

which are equivalent to the deformations of equations (6.4.7) and (6.4.12), respectively.

A SUSY analogue of the classical Fokas–Gel'fand theorem containing the main results on the immersion of surfaces in Lie algebras can be formulated. The supermanifold $F$ consists of the superposition of three terms, i.e.

$$F = \beta(\lambda)\Psi^{-1}\partial_{\lambda}\Psi + \Psi^{-1}\mathrm{pr}(\omega_R)\Psi + \Psi^{-1}S\Psi. \tag{6.4.24}$$

The first term refers to the Sym–Tafel immersion formula, which represents a deformation in the spectral parameter $\lambda$ generated by the vector field $\beta(\lambda)\partial_{\lambda}$, where $\beta(\lambda)$ is an arbitrary function of $\lambda$ with the constraint $\deg(\beta) = \deg(\lambda)$. The second term represents a Lie symmetry deformation generated by the bosonic vector field $\omega_R$, which is common to both the original system (6.4.1) (and the ZCC (6.4.5)) and the FSLP (6.4.2). The last term represents a left-transformation of the wavefunction $\Psi$ by a Lie supergroup gauge generated by the conjugated supermatrix $S$ taking its value in the bosonic part of the Lie superalgebra $\mathfrak{g}$. The associated supermatrices $A_{\pm}$, $B_{\pm}$ and $C_{\pm}$ take the forms

$$\begin{aligned}
A_{\pm} &= \beta(\lambda)\partial_{\lambda}U_{\pm} + \Big(\mathrm{pr}(\omega_R)U_{\pm} + [D_{\pm}, \mathrm{pr}(\omega_R)]\Psi\Psi^{-1}\Big) \\
&\qquad\qquad + \left(D_{\pm}S + [S, U_{\pm}]\right), \\
B_{\pm} &= \beta(\lambda)\partial_{\lambda}V_{\pm} + \Big(\mathrm{pr}(\omega_R)V_{\pm} + [D_{x_{\pm}}, \mathrm{pr}(\omega_R)]\Psi\Psi^{-1}\Big) \\
&\qquad\qquad + \Big(D_{x_{\pm}}S + [S, V_{\pm}]\Big), \\
C_{\pm} &= \beta(\lambda)\partial_{\lambda}W_{\pm} + \Big(\mathrm{pr}(\omega_R)W_{\pm} + [D_{\theta^{\pm}}, \mathrm{pr}(\omega_R)]\Psi\Psi^{-1}\Big) \\
&\qquad\qquad + \left(D_{\theta^{\pm}}S + [S, W_{\pm}]\right).
\end{aligned} \tag{6.4.25}$$

Under these assumptions, we have the following statements :

**Proposition 6.4.1.** *If we consider the bosonic deformations* (6.4.18) *and* (6.4.19) *which preserve the LSPs* (6.4.2), (6.4.6) *and* (6.4.11) *and their ZCCs (i.e. satisfy* *equations* (6.4.17) *and* (6.4.20)*) where* $F$*,* $B_\pm$ *are bosonic supermatrices and* $A_\pm$*,* $C_\pm$ *are fermionic supermatrices in the Lie superalgebra* $\mathfrak{g}$ *and* $\epsilon$ *is a bosonic parameter such that* $\epsilon^2$ *vanishes, then there exists an immersion superfield* $F$ *which defines* $(2|1+1)$*-dimensional supermanifolds immersed in the Lie superalgebra* $\mathfrak{g}$ *provided that its tangent vectors* (6.4.22) *are linearly independent.*

**Corollary 6.4.1.** *If we consider the deformed supermanifold* $F$*, as defined in Proposition 6.4.1, in the form* (6.4.24) *where* $\beta(\lambda)$ *is an arbitrary function of the spectral parameter* $\lambda$ *with the constraint* $\deg(\beta) = \deg(\lambda)$*,* $pr(\omega_R)$ *is the prolongation of a bosonic supervector field* $\omega_R$ *which is associated with a common (generalized) symmetry of the original system* (6.4.1) *and the LSPs* (6.4.2)*,* (6.4.6) *and* (6.4.11)*, then* $F$ *is a solution of the equations* (6.4.20) *and* (6.4.21) *for which the supermatrices* $A_\pm$*,* $B_\pm$ *and* $C_\pm$ *take the form* (6.4.25)*.*

### 6.4.2. The fermionic supersymmetric Fokas–Gel'fand immersion formula

Let us consider a deformation of the three LSPs (6.4.2), (6.4.6) and (6.4.11) which leaves these LSPs invariant, i.e.

$$D_\pm \tilde{\Psi} = \tilde{U}_\pm \tilde{\Psi}, \qquad D_{x_\pm} \tilde{\Psi} = \tilde{V}_\pm \tilde{\Psi}, \qquad D_{\theta^\pm} \tilde{\Psi} = \tilde{W}_\pm \tilde{\Psi}, \tag{6.4.26}$$

and whose compatibility conditions are equivalent to the original system (6.4.1) for any value of the spectral parameter $\lambda$. The deformed potential supermatrices are given by

$$\tilde{U}_\pm = U_\pm + \epsilon A_\pm, \qquad \tilde{V}_\pm = V_\pm + \epsilon B_\pm, \qquad \tilde{W}_\pm = W_\pm + \epsilon C_\pm, \quad \in \mathfrak{g} \tag{6.4.27}$$

and the deformed wavefunction $\tilde{\Psi}$ is written in terms of a deformed supermanifold $F = F([u], \lambda)$,

$$\tilde{\Psi} = \Psi(I + \epsilon F) \in G. \tag{6.4.28}$$

The bosonic deformation supermatrices $A_\pm = A_\pm([u], \lambda)$, $C_\pm = C_\pm([u], \lambda)$ and the fermionic deformation supermatrices $B_\pm = B_\pm([u], \lambda)$, $F$ take their values in the Lie superalgebra $\mathfrak{g}$ and $\epsilon$ is a fermionic deformation parameter. The compatibility conditions of the deformed LSPs (6.4.26) impose conditions on the supermatrices $A_\pm$, $B_\pm$ and $C_\pm$ given by the relations

$$\begin{aligned} &D_+ A_- + D_- A_+ + [A_+, U_-] + [A_-, U_+] = 0, \\ &D_{x_+} B_- - D_{x_-} B_+ + [V_-, B_+] + [B_-, V_+] = 0, \\ &D_{\theta^+} C_- + D_{\theta^-} C_+ + [C_-, W_+] + [C_+, W_-] = 0, \end{aligned} \tag{6.4.29}$$

up to the addition of supermatrices which are at least of first order in $\epsilon$. Equations (6.4.29) are equivalent to the deformation of the three ZCCs (6.4.5), (6.4.8) and (6.4.13).

From equations (6.4.2), (6.4.26a), (6.4.27a) and (6.4.28), one can compute the covariant derivatives of the deformed surface $F$, which is given by

$$D_{\pm}F = -\Psi^{-1}A_{\pm}\Psi. \tag{6.4.30}$$

Similarly, we can compute the tangent vectors of the deformed surface $F$ in the $x_{\pm}$- and $\theta^{\pm}$-directions, which take the forms

$$D_{x_{\pm}}F = \Psi^{-1}B_{\pm}\Psi, \qquad D_{\theta^{\pm}}F = -\Psi^{-1}C_{\pm}\Psi \tag{6.4.31}$$

up to the addition of supermatrices which are at least of first order in $\epsilon$.

Furthermore, from equation (6.4.30), it is possible to determine the supermatrices $B_{\pm}$ and $C_{\pm}$ explicitly in terms of $A_{\pm}$ and the potential supermatrices $U_{\pm}$,

$$B_{\pm} = -i\left(D_{\pm}A_{\pm} + [A_{\pm}, U_{\pm}]\right), \qquad C_{\pm} = A_{\pm} - i\theta^{\pm}B_{\pm}, \tag{6.4.32}$$

which are equivalent to the deformations of equations (6.4.7) and (6.4.12), respectively.

Similarly to the bosonic and classical case, the supermanifold $F$ consists of the superposition of three terms, i.e

$$F = \beta(\lambda)\Psi^{-1}\partial_{\lambda}\Psi + \Psi^{-1}\text{pr}(\omega_R)\Psi + \Psi^{-1}S\Psi. \tag{6.4.33}$$

The first term refers to the Sym–Tafel immersion formula, which represents a deformation in the spectral parameter $\lambda$ generated by the vector field $\beta(\lambda)\partial_{\lambda}$, where $\beta(\lambda)$ is an arbitrary function of $\lambda$ with the constraint $\deg(\beta) \neq \deg(\lambda)$. The second term represents a Lie symmetry deformation generated by the fermionic vector field $\omega_R$, which is common to both the original system (6.4.1) (and the ZCC (6.4.5)) and the FLSP (6.4.2). The last term represents a left-transformation of the wavefunction $\Psi$ by a Lie supergroup gauge generated by the conjugated supermatrix $S$ taking its values in the fermionic part of the $\mathfrak{gl}(m|n, \mathbb{G})$ Lie superalgebra.

The associated supermatrices $A_\pm$, $B_\pm$ and $C_\pm$ take the forms

$$\begin{aligned}
A_\pm &= \beta(\lambda)\partial_\lambda U_\pm + \Big(\mathrm{pr}(\omega_R)U_\pm - \{D_\pm, \mathrm{pr}(\omega_R)\}\Psi\Psi^{-1}\Big) \\
&\qquad\qquad\qquad + \left(-D_\pm S + \{S, U_\pm\}\right), \\
B_\pm &= \beta(\lambda)\partial_\lambda V_\pm + \Big(\mathrm{pr}(\omega_R)V_\pm - [D_{x_\pm}, \mathrm{pr}(\omega_R)]\Psi\Psi^{-1}\Big) \\
&\qquad\qquad\qquad + \Big(D_{x_\pm} S + [S, V_\pm]\Big), \\
C_\pm &= \beta(\lambda)\partial_\lambda W_\pm + \Big(\mathrm{pr}(\omega_R)W_\pm - \{D_{\theta^\pm}, \mathrm{pr}(\omega_R)\}\Psi\Psi^{-1}\Big) \\
&\qquad\qquad\qquad + \left(-D_{\theta^\pm} S + \{S, W_\pm\}\right).
\end{aligned} \tag{6.4.34}$$

Under these assumptions, we have the following statements :

**Proposition 6.4.2.** *If we consider the fermionic deformations* (6.4.27) *and* (6.4.28) *which preserve the LSPs* (6.4.2), (6.4.6) *and* (6.4.11) *and their ZCCs, i.e. satisfy equations* (6.4.26) *and* (6.4.29)*, where $F$, $B_\pm$ are fermionic supermatrices and $A_\pm$, $C_\pm$ are bosonic supermatrices taking values in the Lie superalgebra $\mathfrak{g}$ and $\epsilon$ is a fermionic parameter, then there exists an immersion superfield $F$ which defines* $(2|1+1)$*-dimensional supermanifolds immersed in the Lie superalgebra $\mathfrak{g}$ provided that its tangent vectors* (6.4.31) *are linearly independent.*

**Corollary 6.4.2.** *If we consider the deformed supermanifold $F$, as defined in Proposition 6.4.2, in the form* (6.4.33)*, where $\beta(\lambda)$ is an arbitrary function of the spectral parameter $\lambda$ with the constraint* $\deg(\beta) \neq \deg(\lambda)$*, $pr(\omega_R)$ is the prolongation of a fermionic vector superfield $\omega_R$ which is associated with a common (generalized) symmetry of the original system* (6.4.1) *and the LSPs* (6.4.2), (6.4.6) *and* (6.4.11)*, then $F$ is a solution of equations* (6.4.29) *and* (6.4.30) *such that the supermatrices $A_\pm$, $B_\pm$ and $C_\pm$ take the form* (6.4.34)*.*

## 6.5. The geometric characterization of supermanifolds immersed in Lie superalgebras

Different geometrical approaches and forms can be used depending on the interpretation of the physical model or system under investigation [**33**]. In this paper, we focus our study on a SUSY version of the Killing form on Lie superalgebras. This super Killing form is defined using the supertrace,

$$\langle M, N\rangle = \frac{1}{2}\mathrm{str}(MN) = \frac{1}{2}\mathrm{tr}\left(E^{\deg(MN)+1}MN\right), \qquad M, N \in \mathfrak{g} \subset \mathfrak{gl}(m|n, \mathbb{G}), \tag{6.5.1}$$

which provides a pseudo-Riemannian description of the associated geometry whenever it is nondegenerated. The factor $\frac{1}{2}$ has been selected for convenience from the $\mathfrak{sl}(2|1,\mathbb{G})$ Lie superalgebra in the example of the next section. This form possesses the following properties for any $L, M, N \in \mathfrak{g}$ :

(1) Left linearity,

$$\langle L+M, N\rangle = \langle L, N\rangle + \langle M, N\rangle. \tag{6.5.2}$$

(2) Scalar multiplication,

$$\langle \alpha L, M\rangle = \alpha\langle L, M\rangle, \tag{6.5.3}$$

where $\alpha$ is a bosonic or fermionic scalar in $\mathbb{G}$.

(3) Inner permutation,

$$\langle LM, N\rangle = \langle L, MN\rangle. \tag{6.5.4}$$

(4) Outer permutation,

$$\langle L, M\rangle = (-1)^{\deg(L)\deg(M)}\langle M, L\rangle. \tag{6.5.5}$$

(5) Invariance under (super)group conjugation,

$$\langle h^{-1}Lh, h^{-1}Mh\rangle = \langle L, M\rangle, \tag{6.5.6}$$

for $h \in GL(m|n,\mathbb{G})$.

(6) Supercommutator,

$$\langle L, [M, N]\rangle = \langle [L, M], N\rangle, \tag{6.5.7}$$

or

$$\langle L, \{M, N\}\rangle = \langle \{L, M\}, N\rangle, \tag{6.5.8}$$

depending on the degree of $L$, $M$ and $N$ for $\deg(L) = \deg(M) = \deg(N)$.

The super Killing form possesses the property of being invariant under a Lie supergroup conjugation of its arguments, which is of great advantage in our case since the tangent vectors (6.4.22) or (6.4.31) of the deformed surfaces $F$ are conjugated by the wavefunction $\Psi$, an a priori unknown Lie supergroup-valued function. In addition, from the property (6), the commutator/anticommutator can be seen as a vector product, which provides an orthogonal vector.

Throughout this section, we use the abbreviated notation for the indices associated with the independent bosonic and fermionic variables $x_\pm$ and $\theta^\pm$ given by

$$\begin{aligned} &x_+ \to 1, \quad D_{x_+} = D_1, \quad \theta^+ \to 3, \quad D_{\theta^+} = D_3,\\ &x_- \to 2, \quad D_{x_-} = D_2, \quad \theta^- \to 4, \quad D_{\theta^-} = D_4. \end{aligned} \tag{6.5.9}$$

### 6.5.1. The geometric characterization associated with the bosonic Fokas–Gel'fand immersion formula

The metric coefficients associated with the bosonic SUSY FGIF are given by

$$g_{ij} = \langle D_i F, D_j F\rangle, \qquad i, j, = 1, 2, 3, 4. \tag{6.5.10}$$

The 16 coefficients can be written in terms of the supermatrices $B_\pm$ and $C_\pm$ as

$$\begin{aligned}
&g_{11} = \langle B_+, B_+\rangle, && g_{12} = g_{21} = \langle B_+, B_-\rangle, && g_{22} = \langle B_-, B_-\rangle,\\
&g_{13} = g_{31} = \langle B_+, C_+\rangle, && g_{14} = g_{41} = \langle B_+, C_-\rangle, && g_{33} = g_{44} = 0,\\
&g_{23} = g_{32} = \langle B_-, C_+\rangle, && g_{24} = g_{42} = \langle B_-, C_-\rangle, &&\\
&g_{34} = -g_{43} = \langle C_+, C_-\rangle. && &&
\end{aligned} \tag{6.5.11}$$

The metric coefficients can be represented by a bosonic $\mathfrak{gl}(2|2, \mathbb{G})$-valued supermatrix,

$$g \equiv [g_{ij}] = \left(\begin{array}{cc|cc} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{12} & g_{22} & g_{23} & g_{24} \\ \hline g_{13} & g_{23} & 0 & g_{34} \\ g_{14} & g_{24} & -g_{34} & 0 \end{array}\right), \tag{6.5.12}$$

which depends on a maximum of 8 linearly independent coefficients. The first fundamental form is defined to be

$$\begin{aligned}
I = \sum_{i,j=1}^{4} g_{ij} d_i d_j &= g_{11}(dx_+)^2 + 2g_{12}dx_+dx_- + g_{22}(dx_-)^2 + 2g_{34}d\theta^+ d\theta^- \\
&+ 2\left(g_{13}dx^+d\theta^+ + g_{14}dx^+d\theta^- + g_{23}dx_-d\theta^+ + g_{24}dx_-d\theta^-\right).
\end{aligned} \tag{6.5.13}$$

Using equations (6.4.23), we can eliminate the supermatrices $C_\pm$ in the metric coefficients (6.5.11) and reduce some coefficients in terms of the supermatrices $A_\pm$, $B_\pm$, which are given by

$$\begin{aligned}
&g_{13} = i\langle D_+A_+, A_+\rangle + i\theta^+ g_{11}, \qquad g_{14} = \langle B_+, A_-\rangle + i\theta^- g_{12},\\
&g_{23} = \langle B_-, A_+\rangle + i\theta^+ g_{12}, \qquad g_{24} = i\langle D_-A_-, A_-\rangle + i\theta^- g_{22},\\
&g_{34} = \langle A_+, A_-\rangle + i\theta^+ g_{14} - i\theta^- g_{23} + \theta^+\theta^- g_{12}.
\end{aligned} \tag{6.5.14}$$

In order to construct the second fundamental form, we consider a normal unit vector $N$ taking the form of a bosonic $\mathfrak{g}$-valued supermatrix which has the properties

$$\langle N, N\rangle = 1, \qquad \langle D_i F, N\rangle = 0, \qquad i = 1, 2, 3, 4. \tag{6.5.15}$$

The second fundamental form's coefficients are defined by

$$b_{ij} = \langle D_j D_i F, N\rangle, \qquad i,j = 1,2,3,4 \tag{6.5.16}$$

and they can be represented in the same form as equation (6.5.12) with a maximum of 8 linearly independent coefficients,

$$b \equiv [b_{ij}] = \left(\begin{array}{cc|cc} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{12} & b_{22} & b_{23} & b_{24} \\ \hline b_{13} & b_{23} & 0 & b_{34} \\ b_{14} & b_{24} & -b_{34} & 0 \end{array}\right). \tag{6.5.17}$$

The second fundamental form then takes the form

$$\begin{aligned} II =& b_{11}(dx_+)^2 + 2b_{12}dx_+dx_- + b_{22}(dx_-)^2 + 2b_{34}d\theta^+d\theta^- \\ &+ 2\left(b_{13}dx^+d\theta^+ + b_{14}dx^+d\theta^- + b_{23}dx_-d\theta^+ + b_{24}dx_-d\theta^-\right). \end{aligned} \tag{6.5.18}$$

Eliminating the supermatrices $W_\pm$ and $C_\pm$ in (6.5.16), we obtain

$$\begin{aligned} & b_{13} = i\langle D_+^3 F, N\rangle + i\theta^+ b_{11}, && b_{14} = i\langle D_+^2 D_- F, N\rangle + i\theta^- b_{12}, \\ & b_{23} = i\langle D_- D_+^2 F, N\rangle + i\theta^+ b_{12}, && b_{24} = i\langle D_-^3 F, N\rangle + i\theta^- b_{22}, \\ & b_{34} = \langle D_- D_+ F, N\rangle - i\theta^+ b_{14} + i\theta^- b_{23} - \theta^+\theta^- b_{12}. \end{aligned} \tag{6.5.19}$$

Whenever the bosonic supermatrix $g$ in equation (6.5.12) is invertible, we can compute the mean curvature given by

$$H = \frac{1}{4}\mathrm{str}(bg^{-1}) \tag{6.5.20}$$

and if $b$ is also invertible, then the Gaussian curvature can be computed as

$$K = \mathrm{sdet}(bg^{-1}). \tag{6.5.21}$$

### 6.5.2. The geometric characterization associated with the fermionic Fokas–Gel'fand immersion formula

The metric coefficients associated with the fermionic SUSY FGIF are given by

$$g_{ij} = \langle D_i F, D_j F\rangle, \qquad i,j = 1,2,3,4. \tag{6.5.22}$$

The 16 coefficients can be written in terms of the supermatrices $B_\pm$ and $C_\pm$ as

$$\begin{aligned} &g_{11}=g_{22}=0, && g_{12}=-g_{21}=\langle B_+,B_-\rangle,\\ &g_{13}=g_{31}=-\langle B_+,C_+\rangle, && g_{14}=g_{41}=-\langle B_+,C_-\rangle,\\ &g_{23}=g_{32}=-\langle B_-,C_+\rangle, && g_{24}=g_{42}=-\langle B_-,C_-\rangle,\\ &g_{33}=\langle C_+,C_+\rangle, && g_{34}=g_{43}=\langle C_+,C_-\rangle,\\ &g_{44}=\langle C_-,C_-\rangle. \end{aligned} \tag{6.5.23}$$

The metric coefficients can be represented by a bosonic $\mathfrak{gl}(2|2,\mathbb{G})$-valued supermatrix,

$$g\equiv[g_{ij}]=\left(\begin{array}{cc|cc} 0 & g_{12} & g_{13} & g_{14}\\ -g_{12} & 0 & g_{23} & g_{24}\\ \hline g_{13} & g_{23} & g_{33} & g_{34}\\ g_{14} & g_{24} & g_{34} & g_{44} \end{array}\right), \tag{6.5.24}$$

which depends on a maximum of 8 linearly independent coefficients. The first fundamental form is defined to be

$$\begin{aligned} I&=\sum_{i,j=1}^{4} g_{ij}d_id_j\\ &=2\left(g_{13}dx_+d\theta^+ + g_{14}dx_+d\theta^- + g_{23}dx_-d\theta^+ + g_{24}dx_-d\theta^-\right). \end{aligned} \tag{6.5.25}$$

Using equation (6.4.32), we can eliminate the supermatrices $C_\pm$ in some of the coefficients of the metric and reduce them in terms of the supermatrices $A_\pm$ and $B_\pm$, which are given by

$$\begin{aligned} &g_{13}=-\langle B_+,A_+\rangle, && g_{14}=-\langle B_+,A_-\rangle-i\theta^-g_{12},\\ &g_{23}=-\langle B_-,A_+\rangle+i\theta^+g_{12}, && g_{24}=-\langle B_-,A_-\rangle,\\ &g_{33}=\langle A_+,A_+\rangle+2i\theta^+g_{13}, && g_{44}=\langle A_-,A_-\rangle+2i\theta^-g_{24},\\ &g_{34}=\langle A_+,A_-\rangle+i\theta^+g_{14}+i\theta^-g_{23}-\theta^+\theta^-g_{12}. \end{aligned} \tag{6.5.26}$$

In order to construct the second fundamental form, we consider a normal unit vector $N$ which takes the form of a bosonic $\mathfrak{g}$-valued supermatrix and has the following properties :

$$\langle N,N\rangle=1,\qquad \langle D_iF,N\rangle=0,\qquad i=1,2,3,4. \tag{6.5.27}$$

The second fundamental form's coefficients are defined by

$$b_{ij}=\langle D_jD_iF,N\rangle,\qquad i,j=1,2,3,4 \tag{6.5.28}$$

and they can be represented in the same form as equation (6.5.17) but as a fermionic supermatrix with a maximum of 8 linearly independent coefficients,

$$b \equiv [b_{ij}] = \left(\begin{array}{cc|cc} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{12} & b_{22} & b_{23} & b_{24} \\ \hline b_{13} & b_{23} & 0 & b_{34} \\ b_{14} & b_{24} & -b_{34} & 0 \end{array}\right). \tag{6.5.29}$$

By eliminating the supermatrices $W_\pm$ and $C_\pm$ in (6.5.28), we obtain, similarly to the bosonic case,

$$\begin{aligned} b_{13} &= i\langle D_+^3 F, N\rangle + i\theta^+ b_{11}, \qquad & b_{14} &= i\langle D_+^2 D_- F, N\rangle + i\theta^- b_{12}, \\ b_{23} &= i\langle D_- D_+^2 F, N\rangle + i\theta^+ b_{12}, \qquad & b_{24} &= i\langle D_-^3 F, N\rangle + i\theta^- b_{22}, \\ b_{34} &= \langle D_- D_+ F, N\rangle - i\theta^+ b_{14} + i\theta^- b_{23} - \theta^+\theta^- b_{12}. \end{aligned} \tag{6.5.30}$$

Whenever the bosonic supermatrix $g$ is invertible, we can compute the mean curvature,

$$H = \frac{1}{4}\mathrm{str}(bg^{-1}). \tag{6.5.31}$$

However, the Gaussian curvature $K$ is not defined since $b$ is a fermionic supermatrix.

## 6.6. Example : The supersymmetric sine-Gordon equation

Let us consider the SSGE

$$D_+D_-\phi = i\sin\phi, \tag{6.6.1}$$

(where $\phi = \phi(x_+, x_-, \theta^+, \theta^-)$ is a bosonic $\mathbb{G}$-valued function,) which can be obtained through the super Euler–Lagrange equation

$$\frac{\partial}{\partial\phi}L + D_+\left(\frac{\partial}{\partial(D_+\phi)}L\right) + D_-\left(\frac{\partial}{\partial(D_-\phi)}L\right) = 0 \tag{6.6.2}$$

with the Lagrangian density

$$L = \cos\phi - \frac{i}{2}D_+\phi D_-\phi. \tag{6.6.3}$$

The associated FLSP can be constructed from the linear problem [**24**, **B4**]

$$D_+\Psi = \left(Je^{i\phi} + Ke^{-i\phi}\right)\Psi, \qquad D_-\Psi = \left(D_-\phi M + N\right)\Psi, \tag{6.6.4}$$

where $J$, $K$, $M$ and $N$ are complex-valued matrices. By considering the compatibility condition of equation (6.6.4) and using the SSGE (6.6.1), we obtain the algebraic constraints

$$iJ=[M,J],\qquad iK=[K,M],$$
$$\{J,N\}=-\{K,N\},\qquad \frac{1}{2}M=\{K,N\} \tag{6.6.5}$$

for which the compatibility condition of equation (6.6.4) is equivalent to the SSGE (6.6.1). One solution to the constraints (6.6.5) is given by

$$J=\frac{1}{2}\begin{pmatrix}0&0&i\\0&0&0\\0&1&0\end{pmatrix},\qquad K=\frac{1}{2}\begin{pmatrix}0&0&0\\0&0&-i\\1&0&0\end{pmatrix},$$
$$M=\begin{pmatrix}i&0&0\\0&-i&0\\0&0&0\end{pmatrix},\qquad N=\begin{pmatrix}0&0&-i\\0&0&i\\-1&1&0\end{pmatrix} \tag{6.6.6}$$

which takes its values in the $\mathfrak{sl}(3,\mathbb{C})$ Lie algebra. We can introduce a spectral parameter in the linear problem (6.6.4) using the method proposed in [**B2**]. Therefore, by applying the Lie point transformation, which is a symmetry of the SSGE (6.6.1) but not of the linear problem (6.6.4),

$$\tilde{x}_{\pm}=\lambda^{\pm1}x_{\pm},\qquad \tilde{\theta}^{\pm}=\lambda^{\pm1/2}\theta^{\pm},\qquad \lambda=\pm e^{\mu}, \tag{6.6.7}$$

where $\mu$ is an arbitrary bosonic parameter in the Grassmann algebra $\mathbb{G}$, to the linear problem (6.6.4), we obtain that the potential matrices $U_{\pm}$ in the FLSP (6.4.2) take the form [**108**, **B4**]

$$U_+=\frac{1}{2\sqrt{\lambda}}\begin{pmatrix}0&0&ie^{i\phi}\\0&0&-ie^{-i\phi}\\-e^{-i\phi}&e^{i\phi}&0\end{pmatrix},\quad U_-=\begin{pmatrix}iD_-\phi&0&-i\sqrt{\lambda}\\0&-iD_-\phi&i\sqrt{\lambda}\\-\sqrt{\lambda}&\sqrt{\lambda}&0\end{pmatrix}, \tag{6.6.8}$$

which take their values in the $\mathfrak{sl}(2|1,\mathbb{G})$ Lie superalgebra. The associated $x_{\pm}$-LSP (6.4.6) is composed of the $\mathfrak{sl}(2|1,\mathbb{G})$-valued bosonic supermatrices

$$V_+=\frac{1}{2}\begin{pmatrix}-1/2\lambda&e^{2i\phi}/2\lambda&-iD_+\phi e^{i\phi}/\sqrt{\lambda}\\e^{-2i\phi}/2\lambda&-1/2\lambda&-iD_+\phi e^{-i\phi}/\sqrt{\lambda}\\D_+\phi e^{-i\phi}/\sqrt{\lambda}&D_+\phi e^{i\phi}/\sqrt{\lambda}&-1/\lambda\end{pmatrix},$$
$$V_-=\begin{pmatrix}i\partial_{x_-}\phi+\lambda&-\lambda&-i\sqrt{\lambda}D_-\phi\\-\lambda&-i\partial_{x_-}\phi+\lambda&-i\sqrt{\lambda}D_-\phi\\-\sqrt{\lambda}D_-\phi&-\sqrt{\lambda}D_-\phi&2\lambda\end{pmatrix}, \tag{6.6.9}$$

and the $\mathfrak{sl}(2|1,\mathbb{G})$-valued fermionic supermatrices $W_\pm$ in the $\theta^\pm$-LSP (6.4.11) take the forms

$$W_+ = \frac{1}{2\lambda}\begin{pmatrix} -i\theta^+/2 & i\theta^+e^{2i\phi}/2 & i\sqrt{\lambda}e^{i\phi}(1-i\theta^+\partial_{\theta^+}\phi) \\ i\theta^+e^{-2i\phi}/2 & -i\theta^+/2 & -i\sqrt{\lambda}e^{-i\phi}(1+i\theta^+\partial_{\theta^+}\phi) \\ -\sqrt{\lambda}e^{-i\phi}(1+i\theta^+\partial_{\theta^+}\phi) & \sqrt{\lambda}e^{i\phi}(1-i\theta^+\partial_{\theta^+}\phi) & i\theta^+ \end{pmatrix},$$

$$W_- = \begin{pmatrix} i\lambda\theta^- + i\partial_{\theta^-}\phi & -i\lambda\theta^- & -i\sqrt{\lambda}(1+i\theta^-\partial_{\theta^-}\phi) \\ -i\lambda\theta^- & i\lambda\theta^- - i\partial_{\theta^-}\phi & i\sqrt{\lambda}(1-i\theta^-\partial_{\theta^-}\phi) \\ -\sqrt{\lambda}(1-i\theta^-\partial_{\theta^-}\phi) & \sqrt{\lambda}(1+i\theta^-\partial_{\theta^-}\phi) & -i\theta^- \end{pmatrix}.$$

### 6.6.1. The bosonic Sym–Tafel immersion formula

Let us consider the deformation generated by a translation of the spectral parameter $\lambda$, i.e.

$$\beta(\lambda) = 1, \qquad F = \Psi^{-1}\partial_\lambda\Psi. \tag{6.6.10}$$

The deformation supermatrices $A_\pm$ and $B_\pm$ take the forms

$$\begin{aligned} A_+ &= \frac{-1}{4\lambda^{3/2}}\begin{pmatrix} 0 & 0 & ie^{i\phi} \\ 0 & 0 & -ie^{-i\phi} \\ -e^{-i\phi} & e^{i\phi} & 0 \end{pmatrix} = \frac{1}{2\lambda}U_+, \\ A_- &= \frac{1}{2\sqrt{\lambda}}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & i \\ -1 & 1 & 0 \end{pmatrix}, \\ B_+ &= \frac{1}{4\lambda^2}\begin{pmatrix} 1 & -e^{2i\phi} & i\sqrt{\lambda}D_+\phi e^{i\phi} \\ -e^{-2i\phi} & 1 & i\sqrt{\lambda}D_+\phi e^{-i\phi} \\ -\sqrt{\lambda}D_+\phi e^{i\phi} & -\sqrt{\lambda}D_+\phi e^{i\phi} & 2 \end{pmatrix}, \\ B_- &= \begin{pmatrix} 1 & -1 & -iD_-\phi/2\sqrt{\lambda} \\ -1 & 1 & -iD_-\phi/2\sqrt{\lambda} \\ -D_-\phi/2\sqrt{\lambda} & -D_-\phi/2\sqrt{\lambda} & 2 \end{pmatrix}. \end{aligned} \tag{6.6.11}$$

The metric coefficients $g_{ij}$ are given by

$$
\begin{aligned}
&g_{11} = g_{22} = g_{13} = g_{31} = g_{24} = g_{42} = g_{33} = g_{44} = 0,\\
&g_{12} = g_{21} = \frac{1}{4\lambda^2}\left(\cos 2\phi - iD_+\phi D_-\phi\cos\phi\right),\\
&g_{14} = g_{41} = \frac{1}{4\lambda^2}\Big(D_+\phi\sin\phi + i\theta^-\left(\cos 2\phi - iD_+\phi\partial_{\theta^-}\phi\cos\phi\right)\Big),\\
&g_{23} = g_{32} = \frac{1}{4\lambda^2}\Big(-D_-\phi\sin\phi + i\theta^+\left(\cos 2\phi - i\partial_{\theta^+}\phi D_-\phi\cos\phi\right)\Big),\\
&g_{34} = -g_{43} = \frac{i}{4\lambda^2}(\cos\phi + \sin\phi\left(\theta^+\partial_{\theta^+}\phi + \theta^-\partial_{\theta^-}\phi\right)\\
&\qquad + i\theta^+\theta^-\left(\cos 2\phi - i\partial_{\theta^+}\phi\partial_{\theta^-}\phi\cos\phi\right))
\end{aligned}
\tag{6.6.12}
$$

and the metric

$$
g = \begin{pmatrix} 0 & g_{12} & 0 & g_{14}\\ g_{12} & 0 & g_{23} & 0\\ 0 & g_{23} & 0 & g_{34}\\ g_{14} & 0 & -g_{34} & 0 \end{pmatrix}
\tag{6.6.13}
$$

takes its values in the $GL(2|2,\mathbb{G})$ Lie supergroup. Hence, $g$ is invertible.

A normal unit vector is given by

$$
N = \Psi^{-1}\begin{pmatrix} 1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0 \end{pmatrix}\Psi
\tag{6.6.14}
$$

and the coefficients of the second fundamental form are given by

$$
\begin{aligned}
&b_{11} = b_{12} = b_{21} = b_{22} = b_{13} = b_{31} = b_{24} = b_{42} = b_{33} = b_{44} = 0,\\
&b_{14} = b_{41} = -iD_+\phi\frac{1}{2\lambda}\cos\phi, \qquad b_{23} = b_{32} = iD_-\phi\frac{1}{2\lambda}\cos\phi,\\
&b_{34} = -b_{43} = \frac{1}{2\lambda}\Big(\sin\phi - \cos\phi\left(\theta^+\partial_{\theta^+}\phi + \theta^-\partial_{\theta^-}\phi\right)\Big)
\end{aligned}
\tag{6.6.15}
$$

such that

$$
b = \begin{pmatrix} 0 & 0 & 0 & b_{14}\\ 0 & 0 & b_{23} & 0\\ 0 & b_{23} & 0b_{34} & \\ b_{14} & 0 & -b_{34} & 0 \end{pmatrix}.
\tag{6.6.16}
$$

The mean curvature depends linearly on the spectral parameter $\lambda$ and is given by

$$H = 4\lambda \sin\phi \Big(-D_+\phi D_-\phi \tan^2\phi \sec 2\phi - i\left(\theta^+\partial_{\theta^+}\phi + \theta^-\partial_{\theta^-}\phi\right)\tan\phi\sec\phi \\ +\theta^+\theta^- \cos 2\phi \sec^2\phi + i\theta^+\partial_{\theta^+}\phi\theta^-\partial_{\theta^-}\phi \sec\phi(2\tan^2\phi + 1)\Big)$$

and the Gaussian curvature $K$ is not defined since $b$ is not invertible.

### 6.6.2. The translation in $x_\pm$ symmetry deformation

Let us consider the bosonic vector field

$$\omega_R = \partial_{x_\pm}, \qquad F = \Psi^{-1} D_{x_\pm}\Psi \tag{6.6.17}$$

which generates the translation in the direction of $x_\pm$. This deformation is equivalent to a bosonic gauge deformation given by $S = V_\pm$. The deformation supermatrices $A_\pm$ and $B_\pm$ are given by

$$A_+ = \frac{\partial_{x_\pm}\phi}{2\sqrt{\lambda}}\begin{pmatrix} 0 & 0 & -e^{i\phi} \\ 0 & 0 & -e^{-i\phi} \\ ie^{-i\phi} & ie^{i\phi} & 0 \end{pmatrix},$$

$$A_- = i\partial_{x_\pm} D_-\phi \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$B_+ = \frac{1}{2\lambda}\begin{pmatrix} 0 & i\partial_{x_\pm}\phi e^{2i\phi} & \sqrt{\lambda}e^{i\phi}(\partial_{x_\pm}\phi D_+\phi - i\partial_{x_\pm}D_+\phi) \\ -i\partial_{x_\pm}\phi e^{-2i\phi} & 0 & -\sqrt{\lambda}e^{-i\phi}(\partial_{x_\pm}\phi D_+\phi + i\partial_{x_\pm}D_+\phi) \\ \sqrt{\lambda}e^{-i\phi}(-i\partial_{x_\pm}\phi D_+\phi + \partial_{x_\pm}D_+\phi) & \sqrt{\lambda}e^{i\phi}(i\partial_{x_\pm}\phi D_+\phi + \partial_{x_\pm}D_+\phi) & 0 \end{pmatrix},$$

$$B_- = \begin{pmatrix} i\partial_{x_\pm}\partial_{x_-}\phi & 0 & -i\sqrt{\lambda}\partial_{x_\pm}D_-\phi \\ 0 & -i\partial_{x_\pm}\partial_{x_-}\phi & -i\sqrt{\lambda}\partial_{x_\pm}D_-\phi \\ -\sqrt{\lambda}\partial_{x_\pm}D_-\phi & -\sqrt{\lambda}\partial_{x_\pm}D_-\phi & 0 \end{pmatrix}.$$

The metric coefficients $g_{ij}$ are given by

$$
\begin{aligned}
&g_{33} = g_{44} = 0, \qquad g_{11} = \frac{1}{4\lambda^2}(\partial_{x_\pm}\phi)^2, \qquad g_{22} = -(\partial_{x_\pm}\partial_{x_-}\phi)^2, \\
&g_{12} = g_{21} = \left(-i\sin\phi\partial_{x_\pm}\phi D_+\phi + i\cos\phi\partial_{x_\pm}D_+\phi\right)\partial_{x_\pm}D_-\phi, \\
&g_{13} = g_{31} = \frac{1}{2\lambda}\partial_{x_\pm}\phi D_+\partial_{x_\pm}\phi + \frac{1}{4\lambda^2}i\theta^+(\partial_{x_\pm}\phi)^2, \\
&g_{14} = g_{41} = i\theta^- g_{12} = \theta^-\left(\sin\phi\partial_{x_\pm}\phi D_+\phi - \cos\phi\partial_{x_\pm}D_+\phi\right)\partial_{x_\pm}\partial_{\theta^-}\phi, \\
&g_{23} = g_{32} = \partial_{x_\pm}\phi\partial_{x_\pm}D_-\phi\cos\phi \\
&\qquad + \theta^+\left(\sin\phi\partial_{\theta^+}\phi - \cos\phi\partial_{x_\pm}\partial_{\theta^+}\phi\right)\partial_{x_\pm}D_-\phi, \\
&g_{24} = g_{42} = -\partial_{x_\pm}D_-\phi\partial_{x_\pm}\partial_{x_-}\phi - i\theta^-(\partial_{x_\pm}\partial_{x_-}\phi)^2 \\
&g_{34} = -g_{43} = -i\theta^- g_{23} = -i\theta^-\partial_{x_\pm}\phi\partial_{x_\pm}\partial_{\theta^-}\phi\cos\phi \\
&\qquad\qquad + i\theta^+\theta^-\left(\sin\phi\partial_{\theta^+}\phi - \cos\phi\partial_{x_\pm}\partial_{\theta^+}\phi\right)\partial_{x_\pm}\partial_{\theta^-}\phi.
\end{aligned}
\tag{6.6.18}
$$

An $\mathfrak{sl}(2|1,\mathbb{G})$-valued bosonic normal unit vector is given by

$$
N = \Psi^{-1}\begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 2i \end{pmatrix}\Psi. \tag{6.6.19}
$$

Therefore, the second fundamental form's coefficients are given by

$$
\begin{aligned}
&b_{12} = b_{21} = b_{14} = b_{41} = b_{23} = b_{32} = b_{33} = b_{34} = b_{43} = b_{44} = 0, \\
&b_{11} = \frac{-1}{2\lambda}\partial_{x_\pm}D_+\phi D_+\phi, \qquad b_{22} = 2\lambda\partial_{x_\pm}D_-\phi D_-\phi, \\
&b_{13} = b_{31} = \frac{i}{2\lambda}\left(\partial_{x_\pm}\phi D_+\phi - \theta^+\partial_{x_\pm}\partial_{\theta^+}\phi\partial_{\theta^+}\phi\right), \\
&b_{24} = b_{42} = i\theta^- b_{22} = 2i\lambda\theta^-\partial_{x_\pm}\partial_{\theta^-}\phi\partial_{\theta^-}\phi,
\end{aligned}
\tag{6.6.20}
$$

such that

$$
b = \begin{pmatrix} b_{11} & 0 & b_{13} & 0 \\ 0 & b_{22} & 0 & i\theta b_{22} \\ b_{13} & 0 & 0 & 0 \\ 0 & i\theta^- b_{22} & 0 & 0 \end{pmatrix}. \tag{6.6.21}
$$

The mean and Gaussian curvatures are not defined since $g$ is not invertible.

#### 6.6.3. A bosonic gauge deformation

Let us consider the bosonic $\mathfrak{sl}(2|1,\mathbb{G})$-valued gauge

$$S = D_-\phi U_- = \sqrt{\lambda} D_-\phi \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & i \\ -1 & 1 & 0 \end{pmatrix} \tag{6.6.22}$$

with the associated deformation supermatrices

$$A_+ = \sin\phi \begin{pmatrix} D_-\phi & 0 & \sqrt{\lambda} \\ 0 & -D_-\phi & -\sqrt{\lambda} \\ -i\sqrt{\lambda} & i\sqrt{\lambda} & 0 \end{pmatrix}, \qquad A_- = \sqrt{\lambda}\partial_{x_-}\phi \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ i & -i & 0 \end{pmatrix},$$

$$B_+ = \begin{pmatrix} iD_+\phi D_-\phi\cos\phi & 0 & \frac{iD_-\phi}{4\sqrt{\lambda}}(1-e^{2i\phi}) + i\sqrt{\lambda}\cos\phi D_+\phi \\ 0 & -iD_+\phi D_-\phi\cos\phi & \frac{-iD_-\phi}{4\sqrt{\lambda}}(1-e^{-2i\phi}) - i\sqrt{\lambda}\cos\phi D_+\phi \\ \frac{-D_-\phi}{4\sqrt{\lambda}}(e^{-2i\phi}-1) - \sqrt{\lambda}\cos\phi D_+\phi & \frac{D_-\phi}{4\sqrt{\lambda}}(e^{2i\phi}-1) + \sqrt{\lambda}\cos\phi D_+\phi & 0 \end{pmatrix},$$

$$B_- = \sqrt{\lambda} \begin{pmatrix} 0 & 0 & -i\partial_{x_-}D_-\phi - \partial_{x_-}\phi D_-\phi \\ 0 & 0 & i\partial_{x_-}D_-\phi - \partial_{x_-}\phi D_-\phi \\ \partial_{x_-}D_-\phi + i\partial_{x_-}\phi D_-\phi & -\partial_{x_-}D_-\phi + i\partial_{x_-}\phi D_-\phi & 0 \end{pmatrix}.$$

The metric coefficients $g_{ij}$ take the form

$$\begin{aligned} & g_{22} = g_{24} = g_{42} = g_{33} = g_{44} = 0, \qquad g_{11} = 2iD_+\phi D_-\phi\cos\phi\sin^2\phi, \\ & g_{12} = g_{21} = 4iD_-\phi\partial_{x_-}D_-\phi\sin^2\phi, \\ & g_{13} = g_{31} = -\sin^3\phi D_-\phi - 2\theta^+\partial_{\theta^+}\phi D_-\phi\cos\phi\sin^2\phi, \\ & g_{14} = g_{41} = -\sin^2\phi(\partial_{x_-}\phi D_-\phi + 4\theta^-\partial_{\theta^-}\phi\partial_{x_-}\partial_{\theta^-}\phi), \\ & g_{23} = g_{32} = i\theta^+ g_{12} = -4\theta^+ D_-\phi\partial_{x_-}D_-\phi\sin^2\phi, \\ & g_{34} = -g_{43} = i\theta^+ g_{14} \\ & \quad = -i\theta^+\sin^2\phi\partial_{x_-}\phi D_-\phi - 4i\theta^+\theta^-\partial_{\theta^-}\phi\partial_{x_-}\partial_{\theta^-}\phi\sin^2\phi. \end{aligned} \tag{6.6.23}$$

A normal unit vector $N$ is given by the bosonic $\mathfrak{sl}(2|1,\mathbb{G})$-valued supermatrix

$$N = \Psi^{-1} \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 2i \end{pmatrix} \Psi \tag{6.6.24}$$

and the second fundamental form's coefficients are

$$\begin{aligned} &b_{12} = b_{21} = b_{22} = b_{23} = b_{32} = b_{24} = b_{42} = b_{33} = b_{44} = 0,\\ &b_{11} = \frac{i}{2\lambda} D_+\phi D_-\phi \sin\phi,\\ &b_{14} = b_{41} = -2\lambda \cos\phi D_+\phi,\\ &b_{13} = b_{31} = i\theta^+ b_{11} = \frac{-1}{2\lambda}\theta^+\partial_{\theta^+}\phi D_-\phi \sin\phi,\\ &b_{34} = -b_{43} = -2i\lambda(\sin\phi - \theta^+\partial_{\theta^+}\phi\cos\phi). \end{aligned} \tag{6.6.25}$$

The curvatures are not defined since $g$ is not an element of the $GL(2|2,\mathbb{G})$ Lie supergroup.

### 6.6.4. The supersymmetric transformation deformation

Let us consider the SUSY transformation generators

$$J_\pm = \partial_{\theta^{+\pm}} + i\theta^\pm \partial_{x_\pm}, \tag{6.6.26}$$

which generate the associated transformations

$$x_\pm \to x_\pm - i\theta^\pm \underline{\xi}^\pm, \qquad \theta^\pm \to \theta^\pm + i\underline{\xi}^\pm, \tag{6.6.27}$$

where $\underline{\xi}^\pm$ are arbitrary fermonic parameters. The differential operators $J_\pm$ satisfy the properties

$$J_\pm^2 = i\partial_{x_\pm}, \qquad \{J_+, J_-\} = 0, \qquad \{J_\pm, D_\pm\} = 0, \qquad \{J_\pm, D_\mp\} = 0. \tag{6.6.28}$$

We can construct a deformed surface $F$ using this common symmetry as

$$F = \Psi^{-1}\mathrm{pr}(J_\pm)\Psi. \tag{6.6.29}$$

The associated deformation supermatrices are given by

$$A_+ = \frac{\mathrm{pr}(J_\pm)\phi}{2\sqrt{\lambda}}\begin{pmatrix} 0 & 0 & -e^{i\phi}\\ 0 & 0 & -e^{-i\phi}\\ ie^{-i\phi} & ie^{i\phi} & 0 \end{pmatrix}, \qquad A_- = i\mathrm{pr}(J_\pm)D_-\phi\begin{pmatrix} 1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0 \end{pmatrix},$$

$$B_+ = \frac{-1}{2\sqrt{\lambda}} \begin{pmatrix} 0 & \frac{-i}{\sqrt{\lambda}}\mathrm{pr}(J_\pm)\phi e^{2i\phi} & (i\mathrm{pr}(J_\pm)D_+\phi + D_+\phi\mathrm{pr}(J_\pm)\phi)e^{i\phi} \\ \frac{i}{\sqrt{\lambda}}\mathrm{pr}(J_\pm)\phi e^{-2i\phi} & 0 & (i\mathrm{pr}(J_\pm)D_+\phi - D_+\phi\mathrm{pr}(J_\pm)\phi)e^{-i\phi} \\ (\mathrm{pr}(J_\pm)D_+\phi + iD_+\phi\mathrm{pr}(J_\pm)\phi)e^{-i\phi} & (\mathrm{pr}(J_\pm)D_+\phi - iD_+\phi\mathrm{pr}(J_\pm)\phi)e^{i\phi} & 0 \end{pmatrix},$$

$$B_- = \begin{pmatrix} i\mathrm{pr}(J_\pm)\partial_{x_-}\phi & 0 & -i\sqrt{\lambda}\mathrm{pr}(J_\pm)D_-\phi \\ 0 & -i\mathrm{pr}(J_\pm)\partial_{x_-}\phi & -i\sqrt{\lambda}\mathrm{pr}(J_\pm)D_-\phi \\ \sqrt{\lambda}\mathrm{pr}(J_\pm)D_-\phi & \sqrt{\lambda}\mathrm{pr}(J_\pm)D_-\phi & 0 \end{pmatrix}.$$

The metric coefficients are

$$\begin{aligned}
&g_{11} = g_{22} = g_{13} = g_{31} = g_{33} = 0, \\
&g_{12} = -g_{21} = -i\mathrm{pr}(J_\pm)D_+\phi(\cos\phi\mathrm{pr}(J_\pm)D_-\phi + \sin\phi D_+\phi\mathrm{pr}(J_\pm)\phi), \\
&g_{14} = g_{41} = -i\theta^- g_{12} = -\theta^-\mathrm{pr}(J_\pm)D_+\phi(\cos\phi\mathrm{pr}(J_\pm)D_-\phi + \sin\phi D_+\phi\mathrm{pr}(J_\pm)\phi), \\
&g_{23} = g_{32} = \mathrm{pr}(J_\pm)\phi\mathrm{pr}(J_\pm)D_-\phi\cos\phi + \theta^+\mathrm{pr}(J_\pm)D_+\phi(\cos\phi\mathrm{pr}(J_\pm)D_-\phi \\
&\qquad\qquad + \sin\phi D_+\phi\mathrm{pr}(J_\pm)\phi), \\
&g_{24} = g_{42} = \mathrm{pr}(J_\pm)D_-\phi\mathrm{pr}(J_\pm)\partial_{x_-}\phi, \\
&g_{44} = -(\mathrm{pr}(J_\pm)D_-\phi)^2 + 2i\theta^-\mathrm{pr}(J_\pm)D_-\phi\mathrm{pr}(J_\pm)\partial_{x_-}\phi, \\
&g_{34} = g_{43} = i\theta^-\mathrm{pr}(J_\pm)\phi\mathrm{pr}(J_\pm)D_-\phi\cos\phi \\
&\qquad - i\theta^+\theta^-\mathrm{pr}(J_\pm)D_+\phi(\cos\phi\mathrm{pr}(J_\pm)D_-\phi + \sin\phi D_+\phi\mathrm{pr}(J_\pm)\phi)
\end{aligned}$$

and $g$ does not take its value in the $GL(2|2,\mathbb{G})$ Lie supergroup.

A normal unit vector is given by

$$N = \Psi^{-1}\begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 2i \end{pmatrix}\Psi \tag{6.6.30}$$

and the second fundamental form coefficients are

$$\begin{aligned}
&b_{12} = b_{21} = b_{14} = b_{41} = b_{23} = b_{32} = b_{33} = b_{34} = b_{43} = b_{44} = 0, \\
&b_{11} = \frac{-1}{2\lambda}\mathrm{pr}(J_\pm)D_+\phi D_+\phi, \qquad b_{22} = 2\lambda\mathrm{pr}(J_\pm)D_-\phi D_-\phi, \\
&b_{13} = b_{31} = \frac{3i}{2\lambda}\mathrm{pr}(J_\pm)\phi D_+\phi - \frac{i\theta^+}{2\lambda}\mathrm{pr}(J_\pm)D_+\phi D_+\phi, \\
&b_{24} = b_{42} = i\theta^- b_{22} = 2i\lambda\theta^-\mathrm{pr}(J_\pm)D_-\phi D_-\phi,
\end{aligned} \tag{6.6.31}$$

such that

$$b = \begin{pmatrix} b_{11} & 0 & b_{13} & 0 \\ 0 & b_{22} & 0 & i\theta^- b_{22} \\ b_{13} & 0 & 0 & 0 \\ 0 & i\theta^- b_{22} & 0 & 0 \end{pmatrix}. \tag{6.6.32}$$

The mean curvature is not defined since $g$ is not invertible.

### 6.6.5. A fermionic gauge deformation

Let us consider the fermionic $\mathfrak{sl}(2|1,\mathbb{G})$-valued gauge

$$S = D_-\Phi \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \tag{6.6.33}$$

which is associated with the deformation supermatrices

$$\begin{aligned}
A_+ &= \begin{pmatrix} -i\sin\phi & 0 & \frac{-iD_-\phi}{2\sqrt{\lambda}}e^{i\phi} \\ 0 & -i\sin\phi & \frac{iD_-\phi}{2\sqrt{\lambda}}e^{-i\phi} \\ \frac{D_-\phi}{2\sqrt{\lambda}}e^{-i\phi} & \frac{-D_-\phi}{2\sqrt{\lambda}}e^{i\phi} & -2i\sin\phi \end{pmatrix}, \\
A_- &= \begin{pmatrix} i\partial_{x_-}\phi & 0 & i\sqrt{\lambda}D_-\phi \\ 0 & i\partial_{x_-}\phi & -i\sqrt{\lambda}D_-\phi \\ \sqrt{\lambda}D_-\phi & -\sqrt{\lambda}D_-\phi & 2i\partial_{x_-}\phi \end{pmatrix}, \\
B_+ &= \begin{pmatrix} -\sin\phi D_+\phi & 0 & \frac{-i}{2\sqrt{\lambda}}D_+\phi D_-\phi e^{i\phi} \\ 0 & -\sin\phi D_+\phi & \frac{-i}{2\sqrt{\lambda}}D_+\phi D_-\phi e^{-i\phi} \\ \frac{D_+\phi D_-\phi}{2\sqrt{\lambda}}e^{-i\phi} & \frac{D_+\phi D_-\phi}{2\sqrt{\lambda}}e^{i\phi} & 2\sin\phi D_+\phi \end{pmatrix}, \\
B_- &= \partial_{x_-}D_-\phi \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.
\end{aligned} \tag{6.6.34}$$

The metric coefficients are given by

$$\begin{aligned}
&g_{11} = g_{22} = 0, \qquad g_{12} = -g_{21} = \sin\phi D_+\phi\partial_{x_-}D_-\phi, \\
&g_{13} = g_{31} = i\sin^2\phi D_+\phi, \\
&g_{14} = g_{41} = -i\sin\phi D_+\phi(\partial_{x_-}\phi + \theta^-\partial_{x_-}\partial_{\theta^-}\phi), \\
&g_{23} = g_{32} = -i\sin\phi\partial_{x_-}D_-\phi(1 - \theta^+\partial_{\theta^+}\phi), \\
&g_{24} = g_{42} = i\partial_{x_-}D_-\phi\partial_{x_-}\phi, \\
&g_{33} = \sin^2\phi(1 - 2\theta^+\partial_{\theta^+}\phi), \qquad g_{44} = \partial_{x_-}\phi(\partial_{x_-}\phi - 2\theta^-\partial_{x_-}\partial_{\theta^-}\phi), \\
&g_{34} = g_{43} \\
&\quad = \sin\phi(-\partial_{x_-}\phi + \theta^+\partial_{x_-}\phi\partial_{\theta^+}\phi + \theta^-\partial_{x_-}\partial_{\theta^-}\phi + \theta^+\theta^-\partial_{x_-}\partial_{\theta^-}\phi\partial_{\theta^+}\phi).
\end{aligned} \tag{6.6.35}$$

The induced metric on the surface is not invertible since $g_{11} = g_{22} = 0$ and $g_{12}$ is nilpotent.

A normal unit vector is given by

$$N = \Psi^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Psi \tag{6.6.36}$$

and the second fundamental form coefficients are given by

$$\begin{aligned} &b_{11} = b_{12} = b_{21} = b_{22} = b_{14} = b_{41} = 0, \\ &b_{23} = b_{32} = b_{24} = b_{42} = b_{33} = b_{44} = 0, \\ &b_{13} = b_{31} = \frac{-i}{2\lambda} D_+\phi D_-\phi, \qquad b_{34} = -b_{43} = \frac{1}{2} D_-\phi \sin\phi. \end{aligned} \tag{6.6.37}$$

The mean curvature is not defined since $g$ is not invertible.

## 6.7. Conclusions

In this paper, we have constructed bosonic and fermionic versions of the Fokas–Gel'fand formula for the immersion of supermanifolds in Lie superalgebras. We use three different types of LSPs : the FLSP, the $x_\pm$-LSP and the $\theta^\pm$-LSP, instead of only the FLSP as presented in [**B3**]. We provide a link between these three types of LSP so that the $x_\pm$-LSP and $\theta^\pm$-LSP can be computed using only the FLSP. The use of these three types of LSPs provides a more complete geometric characterization of the supermanifolds under investigation. Using the super Killing form on the four $\mathfrak{g}$-valued tangent vectors $D_jF$, $j = 1, 2, 3, 4$, we compute the 16 metric coefficients $g_{ij}$ (instead of the 4 coefficients given in [**B3**]). In addition, for a given normal unit vector $N$, we compute the 16 coefficients $b_{ij}$ of the second fundamental form and if the supermatrix $g = [g_{ij}]$ (in equations (6.5.12) or (6.5.24)) is invertible, then we can compute the mean curvature $H$ and also the Gaussian curvature $K$ in the bosonic immersion formula. As an example, we apply these theoretical considerations to the SSGE. As deformations, we consider the translation in the spectral parameter, the translation in the $x_\pm$-direction, the SUSY transformations generated by $J_\pm$ and two gauge deformations, one bosonic and one fermonic. In each case, we obtain a non-trivial geometry. In the paper [**B3**], the authors obtain curve-like metrics in certain cases even if the tangent vectors are linearly independent. In comparable cases, we obtain more complex structures for the metric and the second fundamental form.

This research can be extended in many directions. One of these directions could be to investigate other SUSY integrable systems, such as the SUSY Korteweg–de Vries (KdV) equation. This particular example would be of great interest since the SUSY KdV equation reduces to the classical KdV equation and the SSGE does not reduce to its classical counterpart unless we impose an additional condition on the bosonic superfield $\phi$. It would also be interesting to investigate if the (SUSY) FGIF is complete in the sense that there are no additional deformations which can be added to the spectral deformations, the common generalized symmetry deformations and the gauge deformations. Finally, it would be interesting to provide a complete list of geometric classes associated with SUSY integrable systems.

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# Chapitre 7

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## CONCLUSIONS ET PERSPECTIVES FUTURES

### 7.1. Conclusions

Dans cette thèse, les supervariétés plongées dans des superespaces sont étudiées afin d'obtenir leurs équations structurelles, puis leur résolution. Ces équations permettent de décrire et d'obtenir une caractérisation géométrique des supervariétés étudiées. Cette recherche examine les supervariétés associées aux extensions supersymétriques de systèmes intégrables et ces résultats sont appliqués entre autres au moyen de l'équation de sine-Gordon supersymétrique. En résumé, nous investiguons deux extensions supersymétriques des équations de Gauss–Weingarten et de Gauss–Codazzi pour les surfaces plongées dans un superespace euclidien $\mathbb{R}^{(1,1|2)} \times \mathbb{G}$. De plus, nous proposons l'extension d'une conjecture [**85**] permettant de savoir si un système (supersymétrique) est intégrable au sens solitonique au moyen des symétries non communes du système et de son problème linéaire. Additionnellement, nous construisons des versions supersymétriques de la formule d'immersion de Fokas–Gel'fand ainsi que leur caractérisation géométrique associée pour les supervariétés solitoniques plongées dans une superalgèbre de Lie. De surcroît, nous concentrons notre étude sur les propriétés de l'équation de sine-Gordon supersymétrique afin d'obtenir des solutions explicites. Dans ce qui suit, nous précisons les éléments neufs qui ont été apportés dans le cadre de ces études doctorales.

Le chapitre 2 est dévoué aux extensions supersymétriques des équations de Gauss–Weingarten et de Gauss–Codazzi pour une fonction d'immersion $F$ de surfaces conformément paramétrisées plongées dans un superespace euclidien $\mathbb{R}^{(1,1|2)} \times$

$\mathbb{G}$. À cette fin, pour les cas bosonique et fermionique, nous avons défini les éléments de la métrique $g_{ij}$ à partir des relations

$$\langle D_1 F, D_2 F\rangle = e^{\phi} f, \qquad \langle D_i F, D_i F\rangle = 0, \qquad i = 1, 2, \tag{7.1.1}$$

où

$$D_1 = D_\theta^+ - i\theta^+ D_{x_+}, \qquad D_2 = D_\theta^- - i\theta^- D_{x_-} \tag{7.1.2}$$

sont les dérivées covariantes fermioniques, $D_i F$ représentent les vecteurs tangents à la surface et $f$ est une fonction nilpotente (c'est-à-dire une fonction sans partie body) dans le cas bosonique qui a dû être introduite pour tenir compte du produit de deux éléments fermioniques, alors que la fonction bosonique $e^{\phi}$ est toujours inversible. De plus, en considérant un vecteur bosonique, normal et unitaire $N$, et l'hypothèse

$$D_j D_i F = \Gamma_{ij}{}^k D_k F + b_{ij} f N, \qquad D_i N = b_i^k D_k F, \qquad i, j, k = 1, 2, \tag{7.1.3}$$

nous obtenons certaines contraintes et certains liens entre les différentes quantités ainsi que les équations de Gauss–Weingarten. Explicitement, pour l'extension bosonique (où $F$ est à valeur bosonique), les équations de Gauss–Weingarten sont données par

$$\begin{aligned} D_1 \begin{pmatrix} D_1 F \\ D_2 F \\ N \end{pmatrix} &= \begin{pmatrix} \Gamma_{11}{}^1 & \Gamma_{11}{}^2 & Q^+ f \\ -\Gamma_{12}{}^1 & -\Gamma_{12}{}^2 & -\frac{1}{2} e^{\phi} H f \\ H & 2e^{-\phi} Q^+ & 0 \end{pmatrix} \begin{pmatrix} D_1 F \\ D_2 F \\ N \end{pmatrix}, \\ D_2 \begin{pmatrix} D_1 F \\ D_2 F \\ N \end{pmatrix} &= \begin{pmatrix} \Gamma_{12}{}^1 & \Gamma_{12}{}^2 & \frac{1}{2} e^{\phi} H f \\ \Gamma_{22}{}^1 & \Gamma_{22}{}^2 & Q^- f \\ -2e^{-\phi} Q^- & H & 0 \end{pmatrix} \begin{pmatrix} D_1 F \\ D_2 F \\ N \end{pmatrix}, \end{aligned} \tag{7.1.4}$$

où $H$ est la courbure moyenne, $Q^{\pm}$ sont associés aux différentiels de Hopf $((d_{\pm})^2 Q^{\pm})$ et $\Gamma_{ij}{}^k$ représentent les symboles de Christoffel du second type. En prenant la condition de compatibilité des équations de Gauss–Weingarten, nous obtenons les équations de Gauss–Codazzi,

$$\begin{aligned}
&(i) && D_2({\Gamma_{11}}^1) + D_1({\Gamma_{22}}^2) + D_1({\Gamma_{12}}^1) - D_2({\Gamma_{12}}^2) = 0,\\
&(ii) && D_2({\Gamma_{11}}^1) - {\Gamma_{11}}^2{\Gamma_{22}}^1 + D_1({\Gamma_{12}}^1) + {\Gamma_{12}}^2{\Gamma_{12}}^1\\
& && \qquad + \frac{1}{2}H^2 e^{\phi} f - 2Q^+Q^- e^{-\phi} f = 0,\\
&(iii) && Q^+{\Gamma_{22}}^2 - {\Gamma_{11}}^2 Q^- + D_2Q^+ - Q^+D_2\phi + \frac{1}{2}e^{\phi}D_1H = 0,\\
&(iv) && Q^-{\Gamma_{11}}^1 - {\Gamma_{22}}^1 Q^+ + D_1Q^- - Q^-D_1\phi - \frac{1}{2}e^{\phi}D_2H = 0,\\
&(v) && D_2({\Gamma_{11}}^2) - {\Gamma_{12}}^1{\Gamma_{11}}^2 - {\Gamma_{11}}^2{\Gamma_{22}}^2 - {\Gamma_{11}}^1{\Gamma_{12}}^2\\
& && \qquad + D_1({\Gamma_{12}}^2) + 2Q^+Hf = 0,\\
&(vi) && D_1({\Gamma_{22}}^1) + {\Gamma_{12}}^2{\Gamma_{22}}^1 - {\Gamma_{22}}^1{\Gamma_{11}}^1 + {\Gamma_{22}}^2{\Gamma_{12}}^1\\
& && \qquad - D_2({\Gamma_{12}}^1) + 2Q^-Hf = 0.
\end{aligned} \tag{7.1.5}$$

Similairement pour l'extension fermionique (où $F$ est à valeur fermionique), nous obtenons les équations de Gauss–Weingarten,

$$\begin{aligned}
D_1\begin{pmatrix} D_1F\\ D_2F\\ N\end{pmatrix} &= \begin{pmatrix} {\Gamma_{11}}^1 & 0 & Q^+f\\ 0 & 0 & -\frac{1}{2}e^{\phi}Hf\\ H & -2e^{-\phi}Q^+ & 0\end{pmatrix}\begin{pmatrix} D_1F\\ D_2F\\ N\end{pmatrix},\\
D_2\begin{pmatrix} D_1F\\ D_2F\\ N\end{pmatrix} &= \begin{pmatrix} 0 & 0 & \frac{1}{2}e^{\phi}Hf\\ 0 & {\Gamma_{22}}^2 & Q^-f\\ -2e^{-\phi}Q^- & -H & 0\end{pmatrix}\begin{pmatrix} D_1F\\ D_2F\\ N\end{pmatrix},
\end{aligned} \tag{7.1.6}$$

et de Gauss–Codazzi

$$\begin{aligned}
&(i) && D_1({\Gamma_{22}}^2) + D_2({\Gamma_{11}}^1) = 0,\\
&(ii) && D_2({\Gamma_{11}}^1) + 2e^{-\phi}Q^+Q^- f = 0,\\
&(iii) && D_1Q^- - \frac{1}{2}e^{\phi}D_2H + Q^-(D_1\phi - {\Gamma_{11}}^1) = 0,\\
&(iv) && D_2Q^+ + \frac{1}{2}e^{\phi}D_1H + Q^+(D_2\phi - {\Gamma_{22}}^2) = 0.
\end{aligned} \tag{7.1.7}$$

Il est à noter que, dans l'extension fermionique pour le cas où $f$ est une constante bosonique, nous obtenons un cas similaire au cas classique à quelques signes près, c'est-à-dire que les équations de Gauss–Weingarten sont données par

$$
\begin{aligned}
D_1 \begin{pmatrix} D_1F \\ D_2F \\ N \end{pmatrix} &= \begin{pmatrix} D_1\phi & 0 & Q^+f \\ 0 & 0 & -\frac{1}{2}e^{\phi}Hf \\ H & -2e^{-\phi}Q^+ & 0 \end{pmatrix} \begin{pmatrix} D_1F \\ D_2F \\ N \end{pmatrix}, \\
D_2 \begin{pmatrix} D_1F \\ D_2F \\ N \end{pmatrix} &= \begin{pmatrix} 0 & 0 & \frac{1}{2}e^{\phi}Hf \\ 0 & D_2\phi & Q^-f \\ -2e^{-\phi}Q^- & -H & 0 \end{pmatrix} \begin{pmatrix} D_1F \\ D_2F \\ N \end{pmatrix},
\end{aligned}
\tag{7.1.8}
$$

et les équations de Gauss–Codazzi sont données par

$$
\begin{aligned}
&(i) && D_2D_1\phi + 2e^{-\phi}Q^+Q^-f = 0, \\
&(ii) && D_1Q^- - \frac{1}{2}e^{\phi}D_2H = 0, \\
&(iii) && D_2Q^+ + \frac{1}{2}e^{\phi}D_1H = 0.
\end{aligned}
\tag{7.1.9}
$$

Comparativement, les équations de Gauss–Weingarten et de Gauss–Codazzi classiques sont données respectivement par

$$
\begin{aligned}
\partial \begin{pmatrix} \partial F \\ \bar{\partial}F \\ N \end{pmatrix} &= \begin{pmatrix} \partial u & 0 & Q \\ 0 & 0 & \frac{1}{2}He^{u} \\ -H & -2Qe^{-u} & 0 \end{pmatrix} \begin{pmatrix} \partial F \\ \bar{\partial}F \\ N \end{pmatrix}, \\
\bar{\partial} \begin{pmatrix} \partial F \\ \bar{\partial}F \\ N \end{pmatrix} &= \begin{pmatrix} 0 & 0 & \frac{1}{2}He^{u} \\ 0 & \bar{\partial}u & \bar{Q} \\ -2\bar{Q}e^{-u} & -H & 0 \end{pmatrix} \begin{pmatrix} \partial F \\ \bar{\partial}F \\ N \end{pmatrix},
\end{aligned}
\tag{7.1.10}
$$

$$
\begin{gathered}
\partial\bar{\partial}u + \frac{1}{2}H^2e^{u} - 2Q\bar{Q}e^{-u} = 0, \\
\partial\bar{Q} - \frac{1}{2}e^{u}\bar{\partial}H = 0, \qquad \bar{\partial}Q - \frac{1}{2}e^{u}\partial H = 0.
\end{gathered}
\tag{7.1.11}
$$

Pour les deux extensions supersymétriques, nous avons spécifié la caractérisation géométrique de telles surfaces au moyen des première et deuxième formes fondamentales et, conséquemment, les courbures moyenne et de Gauss. Afin d'obtenir des solutions aux extensions supersymétriques des équations de Gauss–Weingarten et de Gauss–Codazzi, nous avons trouvé, à l'aide d'une adaptation du critère de symétrie classique aux cas supersymétriques, des générateurs de symétrie (voir les équations (2.6.2) et (2.6.4)-(2.6.5)) que nous avons classifiés en classes de conjugaison d'une façon analogue à la méthode classique où nous avons tenu compte des variables bosoniques et fermioniques. Nous avons obtenu 99 sous-algèbres unidimensionnelles dans le cas bosonique et 199 sous-algèbres unidimensionnelles dans le cas fermionique. Les listes des sous-algèbres se retrouvent dans les tables 2.4 et 2.5, respectivement. De plus, nous avons trouvé des symétries non standards n'admettant pas de réduction parmi ces sous-algèbres comme dans les articles [**68**, **69**]. Pour les équations de Gauss–Codazzi classiques, nous avons trouvé deux sous-algèbres de Lie de dimension infinie de type Virasoro qui ne se retrouvent pas dans les versions supersymétriques. Nous avons classifié ces algèbres par classes de conjugaison pour la sous-algèbre maximale de dimension finie où nous avons obtenu 16 sous-algèbres unidimensionnelles qui sont énumérées à l'équation (2.2.20). En utilisant la méthode de réduction par symétrie pour les superalgèbres de Lie $\mathfrak{g}_{39}$, $\mathfrak{g}_{76}$, $\mathfrak{h}_{124}$ et $\mathfrak{h}_{35}$, nous avons trouvé des solutions explicites qui permettent d'obtenir une caractérisation géométrique non triviale.

Dans le chapitre 3, nous avons étudié le lien entre les symétries d'un système d'équations différentielles non linéaires et celles de son problème linéaire associé, ce qui nous a permis de préciser une conjecture déjà existante (voir la référence [**85**]) concernant l'intégrabilité au sens solitonique et de l'étendre aux modèles supersymétriques. Afin d'étendre cette conjecture, entre autres, nous avons trouvé des algèbres de Lie de symétries ponctuelles des équations de Gauss–Weingarten classiques. Nous obtenons, comme pour les équations de Gauss–Codazzi classiques, une algèbre avec deux sous-algèbres de dimension infinie de type Virasoro ainsi qu'une sous-algèbre de dimension finie faisant aussi intervenir les composantes matricielles de la fonction d'onde (voir l'équation (3.2.4)). Pour l'extension bosonique des équations de Gauss–Weingarten, nous trouvons un ensemble de générateurs de symétries faisant aussi intervenir les composantes matricielles de la fonction d'onde (voir l'équation (3.5.2)) où, pour chaque symétrie de la version bosonique des équations de Gauss–Codazzi, il y a un analogue dans l'ensemble associé à la version bosonique des équations de Gauss–Weingarten. Pour l'extension fermionique, nous avons trouvé les générateurs de symétries de la version fermionique des équations de Gauss–Weingarten tels que donnés aux équations (3.5.7) et (3.5.8) et en les comparant avec ceux de la version fermionique des équations de Gauss–Codazzi, nous observons que la translation de la courbure moyenne $H$ n'apparaît pas comme symétrie du problème linéaire. Nous trouvons aussi des symétries pour l'équation de sine-Gordon supersymétrique et pour son problème linéaire associé (voir les équations (3.5.4) et (3.5.6) respectivement). Une symétrie de changement d'échelle (l'algèbre $\mathcal{K}$), présente pour l'équation de sine-Gordon supersymétrique, n'apparaît pas pour son problème linéaire. Afin de comparer les symétries d'un système et de son problème linéaire, nous introduisons un projecteur s'appliquant par la droite sur les générateurs de symétries du problème linéaire. Ce projecteur prend la forme d'un générateur de changement d'échelle impliquant toutes les variables du système d'équations différentielles non linéaires. Par exemple, pour une équation où $\phi$ est la variable dépendante et $x, y$ sont les variables indépendantes, le projecteur prend la forme

$$\omega = x\partial_x + y\partial_y + \phi\partial_\phi. \tag{7.1.12}$$

Une fois le projecteur appliqué sur l'ensemble des générateurs de symétries du problème linéaire et en ne tenant pas compte des dérivées secondes et des symétries nulles, nous pouvons comparer les symétries du système original et de son problème linéaire afin d'évaluer l'intégrabilité du système au sens solitonique. La conjecture énonçant les conditions nécessaires est la suivante :

**Conjecture 7.1.1.**

*(1) Si l'ensemble de symétries du système original est équivalent à l'ensemble de symétries du problème linéaire projeté, alors le système non linéaire n'est pas intégrable au sens solitonique.*

*(2) Si les conditions suivantes sont satisfaites :*

*(a) Si l'ensemble de symétries du problème linéaire projeté est un sous-ensemble des symétries du système original, alors il est possible d'introduire un paramètre dans le problème linéaire au moyen de la symétrie non commune.*

*(b) La transformation en (a) doit agir de manière non triviale, c'est-à-dire que le paramètre ne peut pas être éliminé par une jauge.*

*alors le système non linéaire est un candidat pour être un système intégrable.*

En appliquant cette conjecture aux exemples ci-dessus, nous obtenons que le cas classique et l'extension bosonique des équations de Gauss–Codazzi ne sont pas intégrables. Par contre, l'extension fermionique des équations de Gauss–Codazzi et l'équation de sine-Gordon supersymétrique sont des candidats à l'intégrabilité. Il est à noter que, dans l'extension fermionique, un paramètre fermionique est généré, ce qui n'avait pas été observé dans la littérature.

Le chapitre 4 constitue une première étude des versions supersymétriques de la formule d'immersion de Fokas–Gel'fand pour les surfaces plongées dans une superalgèbre de Lie associées aux systèmes intégrables supersymétriques. Ces deux versions prennent en considération les déformations bosonique et fermionique, individuellement. Dans les deux cas, les déformations peuvent représenter une déformation spectrale, une déformation générée par les symétries communes au système original et son problème linéaire spectral ou une déformation par transformation de jauge. Plus précisément, dans le cas d'une déformation bosonique, nous considérons une déformation du problème linéaire spectral,

$$D_j\Psi = U_j\Psi, \qquad j=1,2, \tag{7.1.13}$$

de la forme

$$\tilde{\Psi} = \Psi(I+\epsilon F) \in G, \qquad \tilde{U}_j = U_j + \epsilon A_j \in \mathfrak{g}, \qquad j=1,2, \tag{7.1.14}$$

où $\epsilon$ est un paramètre bosonique infinitésimal ou nilpotent d'ordre 2 ($\epsilon^2=0$). De là, nous obtenons les vecteurs tangents à la surface $F$,

$$ED_jF = \Psi^{-1}EA_j\Psi, \qquad j=1,2 \tag{7.1.15}$$

avec la condition de compatibilité

$$D_1A_2 + D_2A_1 - \{EA_1, EU_2\} - \{EA_2, EU_1\} = 0. \tag{7.1.16}$$

Une solution de l'équation (7.1.16) pour la déformation $A_j$ est donnée par

$$\begin{aligned} A_j =& \beta(\lambda)\partial_\lambda U_j + E\left(D_jS + [ES, EU_j]\right) \\ &+ \left(\text{pr}\omega U_j + \left([D_j, \text{pr}\omega]\Psi\right)\Psi^{-1}\right) \in \mathfrak{g} \end{aligned} \tag{7.1.17}$$

et il est possible d'intégrer (7.1.15) pour obtenir la formule d'immersion de la surface

$$F = \Psi^{-1}\beta(\lambda)(\partial_\lambda\Psi) + \Psi^{-1}ES\Psi + \Psi^{-1}(\text{pr}\omega\Psi) \in \mathfrak{g} \tag{7.1.18}$$

plongée dans la superalgèbre de Lie $\mathfrak{g}$. Afin de fournir la caractérisation géométrique de cette surface, nous définissons les éléments de la métrique induite à l'équation (4.4.11), ce qui nous permet d'établir la première forme fondamentale. De plus, nous déterminons une formule pour obtenir le vecteur normal unitaire $N$ (voir l'équation (4.4.13)) afin de calculer les coefficients de la deuxième forme fondamentale (voir l'équation (4.4.14)). À partir des première et deuxième formes fondamentales, nous pouvons calculer les courbures de Gauss et moyenne (voir les équations (4.4.15) et (4.4.16) respectivement). De façon similaire pour les déformations fermioniques, nous considérons une déformation du problème linéaire spectral (7.1.13) sous la forme

$$\tilde{\Psi} = \Psi(I+\epsilon EF) \in G, \qquad \tilde{U}_j = U_j + \epsilon EA_j \in \mathfrak{g}, \qquad j=1,2, \tag{7.1.19}$$

où $\epsilon$ est un paramètre fermionique. De là, nous obtenons les vecteurs tangents à la surface $F$,

$$ED_jF = -\Psi^{-1}EA_j\Psi, \qquad j=1,2 \tag{7.1.20}$$

avec la condition de compatibilité

$$D_1A_2 + D_2A_1 + [EA_1, EU_2] + [EA_2, EU_1] = 0. \tag{7.1.21}$$

Une solution de l'équation (7.1.21) pour la déformation $A_j$ est donnée par

$$\begin{aligned} A_j =& E\beta(\lambda)\partial_\lambda U_j - E\left(D_jS - \{ES, EU_j\}\right) \\ &+ E\left(\mathrm{pr}\omega U_j - \left(\{D_j, \mathrm{pr}\omega\}\Psi\right)\Psi^{-1}\right) \in \mathfrak{g} \end{aligned} \tag{7.1.22}$$

et il est possible d'intégrer (7.1.20) pour obtenir la formule d'immersion de la surface

$$F = \Psi^{-1}E\beta(\lambda)(\partial_\lambda\Psi) + \Psi^{-1}ES\Psi + \Psi^{-1}E(\mathrm{pr}\omega\Psi) \in \mathfrak{g} \tag{7.1.23}$$

plongée dans la superalgèbre de Lie $\mathfrak{g}$. La caractérisation géométrique associée est formulée de façon analogue. Les éléments de la première forme fondamentale sont décrits par l'équation (4.4.24). Les éléments de la deuxième forme fondamentale sont décrits par l'équation (4.4.26) avec un vecteur normal unitaire exprimé en termes des vecteurs tangents à l'équation (4.4.25). Ceci nous permet de calculer les courbures de Gauss et moyenne, voir les équations (4.4.27) et (4.4.28) respectivement. Ces considérations théoriques ont été appliquées au moyen de l'équation de sine-Gordon supersymétrique, un système intégrable dont le problème linéaire spectral est connu [**107**, **108**]. Parmi ces applications, nous avons obtenu des surfaces à courbures de Gauss constantes et des surfaces de type Weingarten non linéaire.

Le chapitre 5 est dédié à l'équation de sine-Gordon supersymétrique. Dans la première partie, nous concentrons notre étude sur les propriétés de type intégrable de cette équation afin de construire des solutions solitoniques qui nous serviront à obtenir une caractérisation géométrique pour la version supersymétrique de la formule d'immersion de Sym–Tafel, c'est-à-dire

$$F = \Phi^{-1}\beta(\lambda)\partial_\lambda\Phi. \tag{7.1.24}$$

Nous commençons par construire un problème linéaire spectral équivalent à celui trouvé dans [**107**, **108**] (voir les équations (5.2.10) et (5.2.11)) en termes des dérivées fermioniques covariantes

$$D_\pm = \partial_{\theta^\pm} - i\theta^\pm\partial_{x_\pm}. \tag{7.1.25}$$

De plus, nous établissons le lien entre cette paire de Lax et celle utilisant les dérivées bosoniques (voir les équations (5.2.15) et (5.2.16)). À partir du premier type de problème linéaire spectral, nous construisons une version supersymétrique d'un ensemble d'équations de Riccati couplées (voir les équations (5.2.19) et (5.2.20)) et les transformations d'auto-Bäcklund (voir l'équation (5.2.22)) associées. Également, nous fournissons une description détaillée de la énième transformation de Darboux de l'équation de sine-Gordon supersymétrique, voir les équations (5.2.23) à (5.2.30), en termes de déterminants de solutions. Une version classique a déjà été fournie par Matveev et Salle [**91**], une première version supersymétrique a été fournie par Liu et Mañas [**87**, **88**] avec des matrices potentielles utilisant des opérateurs différentiels qui ne pouvaient être utilisés pour la formule d'immersion de Sym–Tafel et nous avons approfondi la version de Siddiq et coll [**108**]. À partir de la transformation de Darboux et de la solution triviale $s = 2k\pi$, $k \in \mathbb{Z}$, nous trouvons des solutions multisolitoniques non triviales seulement en termes des variables indépendantes bosoniques et fermioniques, ce qui nous permet d'étudier deux cas de la formule d'immersion supersymétrique de Fokas–Gel'fand pour une transformation bosonique de changement d'échelle du paramètre spectral. Dans ces deux cas, nous obtenons des surfaces à courbure de Gauss constante ($K = 1$), ce qui laisse sous-entendre qu'il s'agit de sphère en analogie avec le cas classique. Pour une solution, nous trouvons une courbure moyenne non triviale (voir l'équation (5.3.26)).

Finalement, dans le chapitre 6, nous continuons l'étude des versions supersymétriques de la forme d'immersion de Fokas–Gel'fand pour les supervariétés solitoniques plongées dans une superalgèbre de Lie. Cette deuxième étude considère les supervariétés avec des vecteurs tangents linéairement indépendants dans les directions des quatre variables indépendantes (deux bosoniques $x_\pm$ et deux fermioniques $\theta^\pm$) plutôt que dans les directions des dérivées covariantes fermioniques $D_\pm = D_{\theta^\pm} - i\theta^\pm D_{x_\pm}$, telle qu'effectuée dans le chapitre 4. Afin d'entamer l'étude de ce sujet, à partir de l'existence d'un problème linéaire spectral utilisant les dérivées covariantes fermioniques $D_\pm$, nous établissons les relations entre les trois types de problèmes linéaires spectraux,

$$D_\pm \Psi = U_\pm \Psi, \qquad D_{x_\pm} \Psi = V_\pm \Psi, \qquad D_{\theta^\pm} \Psi = W_\pm \Psi, \tag{7.1.26}$$

qui sont données par

$$V_\pm = i\left(D_\pm U_\pm - U_\pm^2\right) \in \mathfrak{g}, \qquad W_\pm = U_\pm + i\theta^\pm V_\pm \in \mathfrak{g}. \tag{7.1.27}$$

Par la suite, nous étudions les conséquences différentielles sur les conditions de courbure nulle de chaque problème linéaire spectral (voir les équations (6.4.10) et (6.4.14)). Ensuite, nous abordons la formule d'immersion où nous considérons les déformations de la forme

$$\begin{aligned} \tilde{U}_\pm &= U_\pm + \epsilon A_\pm \in \mathfrak{g}, & \tilde{V}_\pm &= V_\pm + \epsilon B_\pm \in \mathfrak{g}, \\ \tilde{W}_\pm &= W_\pm + \epsilon C_\pm \in \mathfrak{g}, & \tilde{\Psi} &= \Psi(I + \epsilon F) \in G, \end{aligned} \tag{7.1.28}$$

où $\epsilon$ est un paramètre bosonique infinitésimal ou nilpotent d'ordre 2 dans le cas bosonique ou un paramètre fermionique dans le cas fermionique. Pour le cas bosonique, les supermatrices $A_\pm$, $B_\pm$ et $C_\pm$ sont sujettes aux contraintes (6.4.20) et elles sont reliées de la façon suivante :

$$B_\pm = i\left(D_\pm A_\pm - \{U_\pm, A_\pm\}\right), \qquad C_\pm = A_\pm + i\theta^\pm B_\pm. \tag{7.1.29}$$

Il est possible d'intégrer les vecteurs tangents,

$$D_{x_\pm} F = \Psi^{-1} B_\pm \Psi, \qquad D_{\theta^\pm} F = \Psi^{-1} C_\pm \Psi, \tag{7.1.30}$$

pour obtenir la fonction d'immersion de la supervariété

$$F = \beta(\lambda)\Psi^{-1}\partial_\lambda \Psi + \Psi^{-1}\mathrm{pr}(\omega_R)\Psi + \Psi^{-1} S \Psi \in \mathfrak{g} \tag{7.1.31}$$

et les supermatrices $A_\pm$, $B_\pm$ et $C_\pm$ données par l'équation (6.4.25). Similairement, pour le cas fermionique, les supermatrices $A_\pm$, $B_\pm$ et $C_\pm$ sont sujettes aux contraintes (6.4.29) et elles sont reliées de la façon suivante :

$$B_\pm = -i\left(D_\pm A_\pm - [U_\pm, A_\pm]\right), \qquad C_\pm = A_\pm - i\theta^\pm B_\pm. \tag{7.1.32}$$

Il est possible d'intégrer les vecteurs tangents,

$$D_{x_\pm} F = \Psi^{-1} B_\pm \Psi, \qquad D_{\theta^\pm} F = -\Psi^{-1} C_\pm \Psi, \tag{7.1.33}$$

pour obtenir la fonction d'immersion de la supervariété

$$F = \beta(\lambda)\Psi^{-1}\partial_\lambda \Psi + \Psi^{-1}\mathrm{pr}(\omega_R)\Psi + \Psi^{-1} S \Psi \in \mathfrak{g} \tag{7.1.34}$$

et les supermatrices $A_\pm$, $B_\pm$ et $C_\pm$ données par l'équation (6.4.34). Dans les deux cas, les déformations sont basées sur trois types distincts de symétries : les transformations spectrales, les transformations associées aux symétries communes des problèmes linéaires spectraux et de leur condition de courbure nulle ainsi que les transformations de jauges de la fonction d'onde. De plus, une méthode de caractérisation géométrique a été formulée à l'aide de la superforme de Killing (en termes de la supertrace) en tant que produit scalaire. Dans cette caractérisation, nous avons défini les coefficients des première et deuxième formes fondamentales, respectivement,

$$g_{ij} = \langle D_i F, D_j F \rangle, \qquad b_{ij} = \langle D_j D_i F, N \rangle, \qquad i, j = 1, 2, 3, 4. \tag{7.1.35}$$

Il est à noter que les éléments faisant intervenir les indices 3 et 4 dépendent partiellement d'autres éléments (voir les équations (6.5.14) et (6.5.19) pour le cas bosonique ainsi que les équations (6.5.26) et (6.5.30) pour le cas fermionique). À partir de ces coefficients, il est possible de calculer la courbure moyenne à l'aide de la supertrace

$$H = \frac{1}{4}\mathrm{str}(bg^{-1}), \qquad g_{ij} = [g]_{ij}, \quad b_{ij} = [b]_{ij}, \tag{7.1.36}$$

à condition que la métrique $g$ soit une supermatrice invertible. Également, dans le cas bosonique seulement, nous pouvons définir la courbure de Gauss à l'aide du superdéterminant,

$$K = \mathrm{sdet}(bg^{-1}), \tag{7.1.37}$$

si les supermatrices $g$ et $b$ sont invertibles. À titre d'exemple, l'équation de sine-Gordon supersymétrique a été utilisée pour des déformations similaires à celles utilisées dans le chapitre 4. Pour les différentes supervariétés construites, nous obtenons une géométrie plus riche que dans le chapitre 4 dans le sens où nous considérons huit coefficients pour chaque forme fondamentale *a priori* linéairement indépendants dans chaque forme fondamentale contrairement à trois coefficients. Il est à noter que la version bosonique de la formule d'immersion de Fokas–Gel'fand (ainsi que sa caractérisation géométrique une fois adaptée) peut être réduite à sa version classique à condition que le système supersymétrique intégrable et à condition que son problème linéaire spectral utilisant les dérivées bosoniques puissent être réduits aux versions classiques associées lorsque nous considérons la limite où les éléments fermioniques tendent vers zéro.

## 7.2. Perspectives futures

L'étude effectuée dans cette thèse peut être étendue dans plusieurs directions, que ce soit par rapport à la caractérisation géométrique, aux symétries ou dans l'application à différents systèmes intégrables.

Dans la géométrie différentielle classique, une classification des surfaces associées aux systèmes intégrables a été proposée par Bobenko [**19**] en termes des courbures de Gauss $K$ et moyenne $H$. Cette classification est composée de huit cas :

(1) Les surfaces minimales ($H = 0$),

(2) Les surfaces à courbure moyenne constante ($H =$ constante),

(3) Les surfaces à courbure de Gauss positive et constante ($K =$ constante $> 0$),

(4) Les surfaces à courbure de Gauss négative et constante ($K =$ constante $< 0$),

(5) Les surfaces de Bonnet, c'est-à-dire les surfaces possédant des familles non triviales d'isométrie préservant les courbures principales,

(6) Les surfaces à courbure moyenne de type harmonique inverse ($\partial_z\partial_{\bar{z}}(1/H) = 0$, où $z$ est une variable conforme de la première forme fondamentale),

(7) Les surfaces de Bianchi ($\partial_x\partial_y(\sqrt{-K}) = 0$, où $x$ et $y$ sont les coordonnées asymptotiques),

(8) Les surfaces de Bianchi à courbure positive ($\partial_z\partial_{\bar{z}}(\sqrt{K}) = 0$, où $z$ est une coordonnée conforme de la deuxième forme fondamentale).

Il serait intéressant d'adapter ou d'étendre cette classification aux cas supersymétriques. On peut se demander s'il existe de nouvelles classes de surfaces n'apparaissant pas dans le cas classique. Additionnellement, l'étude des courbures de Gauss $K$ et moyenne $H$ a permis de connaître le comportement et certaines propriétés des surfaces étudiées. Par exemple, les surfaces à courbure moyenne nulle ($H = 0$) et les surfaces à courbure moyenne constante ($H =$ constante) ont été étudiées dans plusieurs articles, voir par exemple [**27**, **38**, **93**]. Également, il est possible de classifier des (régions de) surfaces au moyen de la courbure de Gauss $K$, c'est-à-dire qu'il est possible de connaître le comportement d'une surface en connaissant sa courbure de Gauss (c'est-à-dire des points hyperboliques $K < 0$, paraboliques $K = 0$ ou elliptiques $K > 0$). Compte tenu de cela, il serait intéressant d'étudier des surfaces possèdant ces propriétés dans le cas supersymétrique. On peut se demander s'il existe de telles surfaces. Est-ce que ces propriétés ont les mêmes implications ?

Dans le chapitre 2, nous avons considéré un superespace euclidien $\mathbb{R}^{(1+1|2)} \times \mathbb{G}$. Il serait intéressant d'investiguer les versions supersymétriques des équations de Gauss–Weingarten et de Gauss–Codazzi pour d'autres types de superespaces, par exemple un superespace hyperbolique ou sphérique. Dans le cas classique [**20**], ces deux types d'immersion sont représentés par le produit scalaire

$$\langle a, b\rangle = \epsilon a_0 b_0 + \sum_{j=1}^{3} a_i b_i, \tag{7.2.1}$$

où $\epsilon = -1$ dans l'espace hyperbolique $H^3 \subset \mathbb{R}^{3,1}$ et $\epsilon = 1$ dans l'espace sphérique $S^3 \subset \mathbb{R}^4$, tel que les vecteurs satisfont

$$\langle a, a\rangle = \epsilon, \qquad \langle b, b\rangle = \epsilon. \tag{7.2.2}$$

En considérant une paramétrisation conforme d'une fonction d'immersion $F$, c'est-à-dire

$$\langle \partial F, \partial F\rangle = 0, \qquad \langle \partial F, \bar{\partial} F\rangle = \frac{1}{2}e^u, \tag{7.2.3}$$

et le vecteur normal unitaire $N$ satisfaisant

$$\langle \partial F, N\rangle = \langle \bar{\partial} F, N\rangle = \langle F, N\rangle = 0, \qquad \langle N, N\rangle = 1, \tag{7.2.4}$$

nous obtenons les équations de Gauss–Weingarten

$$\partial \begin{pmatrix} \partial F \\ \bar{\partial} F \\ N \\ F \end{pmatrix} = \begin{pmatrix} \partial u & 0 & Q & 0 \\ 0 & 0 & \frac{1}{2}e^u H & -\frac{\epsilon}{2}e^u \\ -H & -2e^{-u}Q & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial F \\ \bar{\partial} F \\ N \\ F \end{pmatrix}, \tag{7.2.5}$$

$$\bar{\partial} \begin{pmatrix} \partial F \\ \bar{\partial} F \\ N \\ F \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{2}e^u H & -\frac{\epsilon}{2}e^u \\ 0 & \bar{\partial} u & \bar{Q} & 0 \\ -2e^{-u}\bar{Q} & -H & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial F \\ \bar{\partial} F \\ N \\ F \end{pmatrix}, \tag{7.2.6}$$

(7.2.7)

où les différentielles de Hopf $Qdz^2$ et la courbure moyenne $H$ sont définis par les coefficients de la deuxième forme fondamentale,

$$Q = \langle \partial^2 F, N\rangle, \qquad H = 2e^{-u}\langle \partial\bar{\partial} F, N\rangle. \tag{7.2.8}$$

La condition de compatibilité de Gauss–Weingarten est donnée par

$$\partial\bar{\partial} u + \frac{1}{2}(H^2 + \epsilon)e^u - 2|Q|^2 e^{-u} = 0, \tag{7.2.9}$$

$$\bar{\partial} Q = \frac{1}{2}e^u \partial H, \tag{7.2.10}$$

qui sont les équations de Gauss–Codazzi. Afin de faire l'extension supersymétrique dans de tels superespaces, il faudra considérer des vecteurs tangents faisant intervenir les variables bosoniques et fermioniques ainsi qu'adapter le produit scalaire afin que les surfaces (ou supervariétés) soient plongées dans une version supersymétrique de $S^3$ ou $H^3$.

Au début du XX$^{\text{e}}$ siècle, une classification des équations différentielles ordinaires du premier ordre (Fuchs [**57**]) et du deuxième ordre (Painlevé [**96**] et Gambier [**58**]) ont été formulées. Dans le cas supersymétrique, une telle classification, incluant les variables fermioniques, n'a pas été effectuée et une version des équations de Painlevé faisant intervenir les variables fermioniques n'a pas été trouvée jusqu'à présent. Il serait intéressant d'étendre cette classification afin de trouver de possibles nouvelles classes d'équations et de voir si des équations de Painlevé supersymétriques existent. De plus, dans le cas classique [**32**], il existe une réduction par symétrie des équations de Gauss–Codazzi qui mène à l'équation de Painlevé P6. Dans le chapitre 2, nous avons produit une liste de sous-algèbres de Lie unidimensionnelles classifiées par classes de conjugaison pour les versions supersymétriques des équations de Gauss–Codazzi. Parmi nos exemples de réduction par symétrie, nous n'avons pas trouvé des équations similaires aux équations de Painlevé faisant intervenir les variables fermioniques. Il vaudrait la peine de trouver une réduction par symétrie analogue au cas classique pour essayer d'obtenir une version supersymétrique de l'équation de Painlevé P6. Également, il est connu [**25**] que l'équation de sine-Gordon (classique) peut être réduite à l'équation P3. On peut donc se demander s'il est possible d'en faire autant pour le cas de l'équation de sine-Gordon supersymétrique. Est-ce que cette forme de l'équation de Painlevé P3 (si elle existe) fera intervenir les variables fermioniques ? Est-ce qu'une extension supersymétrique des transcendants de Painlevé restera une solution pour ces types d'équations ?

Le critère de symétrie n'a pas été étendu ni démontré pour les systèmes supersymétriques, seulement adapté. On peut se demander si le critère de symétrie classique, adapté pour tenir compte des variables fermioniques, représente les conditions nécessaires et suffisantes. De plus, il existe des superalgèbres de Lie non standards telles que trouvées dans [**68**,**69**,**B1**,**B2**]. Il pourrait valoir la peine d'étudier leurs implications. On peut d'autant plus se demander si chaque système supersymétrique (intégrable) en possède. Avec la présence des symétries non standards, est-ce que le (premier) théorème de Noether, reliant les symétries aux lois de conservation, peut être étendu aux modèles supersymétriques ?

Afin d'étudier l'intégrabilité au sens de la théorie des solitons dans le chapitre 3 et par la suite dans les chapitres 4, 5 et 6, nous avons étudié et observé les symétries non communes entre un système supersymétrique intégrable et ses problèmes linéaires spectraux. Dans tous les cas, nous avons observé la présence de seulement une symétrie non commune pour chaque système intégrable. On peut se demander s'il existe plus d'une symétrie non commune par système intégrable. Est-ce que le paramètre fermionique généré par une symétrie non commune dans la version fermionique des équations de Gauss–Codazzi peut être utilisé comme un paramètre spectral ? Si oui, est-ce qu'il existe d'autres systèmes avec un paramètre spectral fermionique ? Peut-on construire des solutions multisolitoniques à partir d'un paramètre spectral fermionique ? Dans la formule d'immersion de Fokas–Gel'fand, est-il possible d'établir un lien entre la déformation spectrale et les déformations liées aux symétries généralisées ? De plus, comme le critère de symétrie n'a pas été démontré pour les systèmes supersymétriques, on peut se demander si les versions supersymétriques de la formule d'immersion de Fokas–Gel'fand sont complètes dans le sens où nous avons les conditions nécessaires et suffisantes. Également, la formule d'immersion de Fokas–Gel'fand permet de plonger des surfaces solitoniques dans une (super)algèbre de Lie. Par contre, il existe une formule d'immersion [**54**] permettant de plonger des surfaces solitoniques dans un groupe de Lie. Il pourrait être intéressant d'adapter cette formule au cas supersymétrique afin d'obtenir les équations structurelles et d'étudier la géométrie sous-jacente et la comparer à la formule d'immersion dans la superalgèbre associée. Il est à noter que les surfaces solitoniques associées au problème linéaire spectral constituent une famille de surfaces paramétrisées par le paramètre spectral [**19**, **112**]. Dans le cas supersymétrique, un tel phénomène est aussi présent.

Dans le cas du chapitre 5, il serait intéressant d'étudier les différentes caractéristiques et d'utiliser des méthodes d'investigation (comme la formule d'immersion de Fokas–Gel'fand) pour d'autres systèmes supersymétriques intégrables. Par exemple, nous pouvons étudier les transformations de Darboux et Bäcklund, les symétries et les lois de conservation, la densité lagrangienne, les problèmes linéaires spectraux, etc. Des exemples de différents systèmes incluent les équations non linéaires de Schrödinger, l'équation de Korteweg–de Vries, les équations de Sawada–Kotera, etc.

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# Annexe A

## SUPERSYMMETRIC VERSIONS AND INTEGRABILITY OF CONFORMALLY PARAMETRIZED SURFACES

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**Résumé**

Le but principal de cet article comprend deux parties. L'une d'elles est la construction de versions supersymétriques (bosonique et fermionique) des équations structurelles pour les surfaces conformément paramétrisées. L'autre est d'investiguer l'intégrabilité au sens de la théorie des solitons au moyen d'une comparaison des symétries des systèmes d'équations différentielles et celles de leur problème linéaire associé. Cet article consiste d'un survol des résultats obtenus dans deux études antécédentes [**B1**, **B2**].

**Abstract**

The main aim of this paper is twofold. One is the construction and analysis of supersymmetric (bosonic and fermionic) versions of the structural equations for conformally parametrized surfaces. The other is the investigation of integrability in the sense of soliton theory via a comparison of the symmetries of the system of differential equations and those of the associated linear problem. This paper consists of an overview of the results obtained in two previous works [**B1**, **B2**].

## A.1. Introduction

Over the last decades, applications of supersymmetry have expanded from particle physic to a large number of domains (see e.g. [**118**] and references therein). A number of supersymmetric (SUSY) extensions of classical and quantum physical models have been investigated, such as the Chaplygin gas model [**78**] (and references therein), the Born–Infled model [**74**], the Korteweg–de Vries equation (e.g. for waves in shallow water in (1+1)-dimensions) [**83**, **88**, **90**], the Kadomtsev–Petviashvili equation (e.g. for waves in shallow water in (2+1)-dimensions) [**89**] and the sine/sinh-Gordon equation (e.g. for crystal dislocation) [**31**, **64**, **69**, **107**, **108**, **121**].

In differential geometry, parametrized surfaces are described in terms of a moving frame satisfying the Gauss–Weingarten (GW) equations and their compatibility conditions are the Gauss–Codazzi (GC) equations. The construction and analysis of such surfaces associated with integrable systems in several areas of mathematical physics provided new tools for the investigation of nonlinear phenomena described by these systems. The analysis of such SUSY versions of the structural equations for conformally parametrized surfaces are one of the main goals of this paper and have been studied in [**B1**].

Methods to solve SUSY differential equations are not as well established as in the classical case. In order to establish new methods and obtain solutions, one can investigate special types of differential equations which are easier to solve in general, like integrable systems (in the sense of the inverse scattering method). In order to solve such equations, we can use the group theory approach, e.g. given in [**95**], and adapt it for Grassmann-valued coordinates and functions. This lead to the second objective of this paper, which is to formulate a conjecture that states that by comparing the symmetries of the (SUSY) differential equations with those of its associated linear problem, it is possible to know if the system is a candidate for integrability. This conjecture, first proposed for classical differential equations [**28**, **29**, **85**], can be extended to SUSY differential equations as stated in [**B2**].

The paper is organized as follow. At the end of this section, an outline of the classical Gauss–Codazzi (CGC) and classical Gauss–Weingarten (CGW) equations for conformally parametrized surfaces is given. In section A.2, a conjecture states that the integrability can be determinated by comparing the symmetries of a system of equations and those of the associated linear problem. This conjecture can be extented to SUSY equations and is applied to the CGC equations and to the SUSY sinh-Gordon equation in this section as examples. In section A.3, the bosonic conformal parametrization of a surface is studied and its integrability is investigated via the conjecture of section A.2. In section A.4, it is the fermionic conformal parametrization of a surface which is studied and its integrability is also investigated. More details on this work can be found in the papers [**B1**, **B2**].

### A.1.1. Classical conformally parametrized surfaces

Let us consider a smooth orientable surface $\mathcal{S}$ immersed in the 3-dimensional Euclidean space $\mathbb{R}^3$ which is conformally parametrized by the function $F = (F_1, F_2, F_3) : \mathcal{R} \to \mathbb{R}^3$. The normalization is given by

$$\begin{aligned} &\langle \partial F, \partial F\rangle = \langle \bar\partial F, \bar\partial F\rangle = 0, \qquad &&\langle \partial F, \bar\partial F\rangle = \frac{1}{2}e^u, \\ &\langle \partial F, N\rangle = \langle \bar\partial F, N\rangle = 0, &&\langle N, N\rangle = 1, \end{aligned} \tag{A.1.1}$$

where $N$ is the normal unit vector and $\partial, \bar\partial$ are the partial derivatives with respect to the complex variables $z, \bar z$ ($z = x + iy$). Here and subsequently, the form $\langle \cdot|\cdot\rangle$ is the traditionnal inner product in $\mathbb{R}^3$. Introducing the notion of the moving frame $\Omega = (\partial F, \bar\partial F, N)^T$ and assuming that the derivatives of each component of the moving frame can be written as linear combinations of $\partial F$, $\bar\partial F$ and $N$, we get the CGW equations

$$\begin{aligned} &\partial \begin{pmatrix} \partial F \\ \bar\partial F \\ N \end{pmatrix} = \begin{pmatrix} \partial u & 0 & Q \\ 0 & 0 & \frac{1}{2}He^u \\ -H & -2e^{-u}Q & 0 \end{pmatrix} \begin{pmatrix} \partial F \\ \bar\partial F \\ N \end{pmatrix}, \qquad \partial\Omega = V_1\Omega, \\ &\bar\partial \begin{pmatrix} \partial F \\ \bar\partial F \\ N \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{2}He^u \\ 0 & \bar\partial u & \bar Q \\ -2e^{-u}\bar Q & -H & 0 \end{pmatrix} \begin{pmatrix} \partial F \\ \bar\partial F \\ N \end{pmatrix}, \qquad \bar\partial\Omega = V_2\Omega, \end{aligned} \tag{A.1.2}$$

where $Q$ (which is associated with the Hopf differential $Qdz^2$) and the mean curvature $H$ are defined by

$$Q = \langle \partial^2 F, N\rangle, \qquad H = 2e^{-u}\langle \partial\bar\partial F, N\rangle. \tag{A.1.3}$$

From the compatibility condition of the CGW equations, we obtain the CGC equations,

$$\begin{aligned} &\partial\bar\partial u + \frac{1}{2}H^2e^u - 2Q\bar Q e^{-u} = 0 \qquad &&\text{(, the Gauss equation)}, \\ &\partial\bar Q = \frac{1}{2}e^u\bar\partial H, \quad \bar\partial Q = \frac{1}{2}e^u\partial H &&\text{(, the Codazzi equations)}. \end{aligned} \tag{A.1.4}$$

## A.2. Investigating integrability via a comparison of symmetries

The conjecture stated further in this section was first proposed by Levy et al. [**85**] and then by Cieslinski [**28, 29**] for classical differential equations. The formulation proposed in this paper uses a new projection operator and is extended to SUSY differential equations. In order to formulate this conjecture, let us define some notation.

Let $L_1$ be the set of all vector fields associated with the Lie point symmetries of the original PDEs $\Delta = 0$. Let $L_2$ be the set of all vector fields associated with the Lie point symmetries of the linear problem (LP) $\Lambda = 0$. Let $\pi_\rho$ be a projection operator acting on a set $L$ of vector fields $\omega$ such that

$$\pi_\rho(L) = \{\omega' | \omega' = \omega\rho\}, \tag{A.2.1}$$

where $\rho$ is in the form of a dilation generator (e.g. $\rho = x_1\partial_{x_1} + x_2\partial_{x_2} + y\partial_y$).

Here and in what follows, in order to compare the sets $L_1$ and $L_2$, we will consider a $\rho$ involving all independent and dependent variables of the PDEs $\Delta = 0$. This choice of $\rho$ implies that $\pi_\rho$ acts as an identity on the set $L_1$, i.e. $\pi_\rho(L_1) = L_1$. The common symmetries of the PDEs $\Delta = 0$ and LP $\Lambda = 0$ are the vector fields which span the set

$$L_3 = L_1 \cap \pi_\rho(L_2) \neq \emptyset. \tag{A.2.2}$$

One should note that the set $L_3$ is not necessarily an algebra. Also, let $L_4$ be the set of all vector fields that generate a symmetry of the PDEs $\Delta = 0$ but do not generate a symmetry of the LP $\Lambda = 0$. The set $L_4$ is define as

$$L_4 = L_1 \backslash L_3. \tag{A.2.3}$$

This set does not form necessarily an algebra.

**Conjecture A.2.1.**

*(1) If $L_1 = \pi(L_2)$ then the PDEs $\Delta = 0$ are not integrable.*

*(2) If the following conditions are satisfied*

*(a) $\pi(L_2)$ is a proper subset of $L_1$, that is*

$$L_1 \supset \pi(L_2). \tag{A.2.4}$$

*A free parameter can be introduced into the linear system using a symmetry transformation generated by one of the vector fields appearing in* $L_4$.

*(b) The transformation given in (a) acts in a nontivial way (i.e. cannot be eliminated through an* $L_1$*-valued gauge matrix function).*

*Then the system of PDEs* $\Delta = 0$ *is a candidate to be an integrable one.*

In order to apply this conjecture to the CGC equations, we must compare the symmetry Lie algebra of these PDEs,

$$\begin{aligned} X(\eta) &= \eta(z)\partial_z + \eta'(z)(-2Q\partial_Q - U\partial_U), \\ Y(\zeta) &= \zeta(\bar{z})\partial_{\bar{z}} + \zeta'(\bar{z})(-2\bar{Q}\partial_{\bar{Q}} - U\partial_U), \\ e_0 &= -H\partial_H + Q\partial_Q + \bar{Q}\partial_{\bar{Q}} + 2U\partial_U, \end{aligned} \tag{A.2.5}$$

with the symmetry Lie algebra of the CGW equations,

$$\begin{aligned} X(\eta) &= \eta(z)\partial_z - \eta'(z)(U\partial_U + 2Q\partial_Q), \\ Y(\zeta) &= \zeta(\bar{z})\partial_{\bar{z}} - \zeta'(\bar{z})(U\partial_U + 2\bar{Q}\partial_{\bar{Q}}), \\ \hat{e}_0 &= -H\partial_H + Q\partial_Q + \bar{Q}\partial_{\bar{Q}} + 2U\partial_U + F_i\partial_{F_i}, \\ T_i &= \partial_{F_i}, \qquad \mathcal{D}_i = F_i\partial_{F_{(i)}} + N_i\partial_{N_{(i)}}, \qquad i = 1,2,3 \\ R_{ij} &= (F_i\partial_{F_j} - F_j\partial_{F_i}) + (N_i\partial_{N_j} - N_j\partial_{N_i}), \qquad i < j = 2,3 \\ S_{ij} &= (F_i\partial_{F_j} + F_j\partial_{F_i}) + (N_i\partial_{N_j} + N_j\partial_{N_i}). \end{aligned} \tag{A.2.6}$$

By comparing these algebras with $\rho$ of the form

$$\rho = z\partial_z + \bar{z}\bar{\partial} + Q\partial_Q + \bar{Q}\partial_{\bar{Q}} + H\partial_H + u\partial_u, \tag{A.2.7}$$

we get that

$$L_1 = \pi_\rho(L_2). \tag{A.2.8}$$

Hence the classical CGC equations are not integrable.

As a second example, we consider the SUSY sinh-Gordon,

$$D_+D_-\Phi = i\sin\Phi, \tag{A.2.9}$$

where the derivatives $D_\pm$ are given by

$$D_\pm = \partial_{\theta^\pm} - i\theta^\pm\partial_{x_\pm}, \tag{A.2.10}$$

$\theta^{\pm}$ being the fermionic independent variables and $x_{\pm}$ being the bosonic independent variables. The associate LP is given by

$$D_{\pm}\Psi = B_{\pm}\Psi, \qquad \text{where} \quad \Psi = \begin{pmatrix} \psi_{11} & \psi_{12} & f_{13} \\ \psi_{21} & \psi_{22} & f_{23} \\ f_{31} & f_{32} & \psi_{33} \end{pmatrix},$$

$$B_{+} = \frac{1}{2}\begin{pmatrix} 0 & 0 & ie^{i\Phi} \\ 0 & 0 & -ie^{-i\Phi} \\ -e^{-i\Phi} & e^{i\Phi} & 0 \end{pmatrix}, \qquad B_{-} = \begin{pmatrix} iD_{-}\Phi & 0 & -i \\ 0 & -iD_{-}\Phi & i \\ -1 & 1 & 0 \end{pmatrix}.$$

One should note that $\Psi$ is an even-supermatrix and $B_{\pm}$ are odd-supermatrices.

The symmetries of the SUSY sinh-Gordon equation and its associated LP are given respectively by

$$\begin{gathered} P_{\pm} = \partial_{x_{\pm}}, \qquad J_{\pm} = \partial_{\theta^{\pm}} + i\theta^{\pm}\partial_{x_{\pm}}, \\ \mathcal{K} = 2x_{+}\partial_{x_{+}} - 2x_{-}\partial_{x_{-}} + \theta^{+}\partial_{\theta^{+}} - \theta^{-}\partial_{\theta^{-}}, \end{gathered} \tag{A.2.11}$$

and by

$$\begin{gathered} P_{\pm} = \partial_{x_{\pm}}, \quad J_{\pm} = \partial_{\theta^{\pm}} + i\theta^{\pm}\partial_{x_{\pm}}, \quad G_1 = \psi_{11}\partial_{\psi_{11}} + \psi_{21}\partial_{\psi_{21}} + f_{31}\partial_{f_{31}}, \\ G_2 = \psi_{12}\partial_{\psi_{12}} + \psi_{22}\partial_{\psi_{22}} + f_{32}\partial_{f_{32}}, \quad G_3 = f_{13}\partial_{f_{13}} + f_{23}\partial_{f_{23}} + \psi_{33}\partial_{\psi_{33}}. \end{gathered} \tag{A.2.12}$$

Using conjecture A.2.1 with $\rho$ of the form

$$\rho = x_{+}\partial_{x_{+}} + x_{-}\partial_{x_{-}} + \theta^{+}\partial_{\theta^{+}} + \theta^{-}\partial_{\theta^{-}} + \Phi\partial_{\Phi}, \tag{A.2.13}$$

we get that $L_1 \supset \pi_{\rho}(L_2)$. So using the symmetry generator $\mathcal{K} \in L_4$, we can introduce a free parameter $\lambda$ in such a way :

$$D_{+}\Psi = B_{+}\Psi, \qquad D_{-}\Psi = B_{-}\Psi,$$

$$B_{+} = \frac{1}{2\sqrt{\lambda}}\begin{pmatrix} 0 & 0 & ie^{i\Phi} \\ 0 & 0 & -ie^{-i\Phi} \\ -e^{-i\Phi} & e^{i\Phi} & 0 \end{pmatrix}, \qquad B_{-} = \begin{pmatrix} iD_{-}\Phi & 0 & -i\sqrt{\lambda} \\ 0 & -iD_{-}\Phi & i\sqrt{\lambda} \\ -\sqrt{\lambda} & \sqrt{\lambda} & 0 \end{pmatrix},$$

where $\lambda$ does not act trivially on the new linear spectral problem. Hence, the SUSY sinh-Gordon equation is a candidate to be integrable.

## A.3. Bosonic conformal parametrization of a surface

Let the surface $\mathcal{S}$ be smooth, orientable and immersed in the superspace $\mathbb{R}^{(2,1|2)}$. We assume that the surface $\mathcal{S}$ is conformally parametrized by a bosonic superfield $F(x_+, x_-, \theta^+, \theta^-)$ with the normalization conditions

$$\langle D_\pm F, D_\pm F\rangle = 0, \qquad \langle D_+F, D_-F\rangle = \frac{1}{2}e^\phi f, \qquad \langle D_-F, D_+F\rangle = -\frac{1}{2}e^\phi f \tag{A.3.1}$$

where $D_\pm$ are defined as in equation (A.2.10), $\phi$ is a bosonic-valued function of $x_+, x_-, \theta^+, \theta^-$, and $f$ is a bodiless bosonic-valued function of $x_+, x_-$. We can introduce a bosonic superfield $N$ which acts as the classical normal unit vector

$$\langle D_\pm F, N\rangle = 0, \qquad \langle N, N\rangle = 1. \tag{A.3.2}$$

The tangent vectors $D_\pm F$ together with the normal superfield $N$ form the SUSY moving frame $\Omega = (D_+F, D_-F, N)^T$. Assuming that the first derivatives of $\Omega$ can be written as linear combinations of the elements of $\Omega$ and with some calculation we get the bosonic case of the supersymmetric Gauss–Weingarten (bSUSYGW) equations

$$\begin{gathered} D_+\Omega = A_+\Omega, \qquad D_-\Omega = A_-\Omega, \\ A_+ = \begin{pmatrix} {\Gamma_{11}}^1 & {\Gamma_{11}}^2 & Q^+ f \\ -{\Gamma_{12}}^1 & -{\Gamma_{12}}^2 & -\frac{1}{2}e^\phi H f \\ H & 2e^{-\phi}Q^+ & 0 \end{pmatrix}, \\ A_- = \begin{pmatrix} {\Gamma_{12}}^1 & {\Gamma_{12}}^2 & \frac{1}{2}e^\phi H f \\ {\Gamma_{22}}^1 & {\Gamma_{22}}^2 & Q^- f \\ -2e^{-\phi}Q^- & H & 0 \end{pmatrix}. \end{gathered} \tag{A.3.3}$$

where the fermionic functions ${\Gamma_{ij}}^k$ are the Christoffel symbols of second kind and where the $Q^\pm$ and the mean curvature $H$ are defined by

$$\begin{gathered} \langle D_+D_+F, N\rangle = Q^+ f, \qquad \langle D_-D_-F, N\rangle = Q^- f, \\ \langle D_-D_+F, N\rangle = -\langle D_+D_-F, N\rangle = \frac{1}{2}e^\phi H f. \end{gathered} \tag{A.3.4}$$

The zero curvature condition of the bSUSYGW equations

$$D_+A_- + D_-A_+ - \{EA_+, EA_-\} = 0, \tag{A.3.5}$$

where

$$E = \pm \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{A.3.6}$$

constitutes the bosonic case of the supersymmetric Gauss–Codazzi (bSUSYGC) equations and corresponds to the following six linearly independent equations

$$\begin{aligned}
&(i) && D_-({\Gamma_{11}}^1) + D_+({\Gamma_{22}}^2) + D_+({\Gamma_{12}}^1) - D_-({\Gamma_{12}}^2) = 0,\\
&(ii) && D_-({\Gamma_{11}}^1) - {\Gamma_{11}}^2{\Gamma_{22}}^1 + D_+({\Gamma_{12}}^1) + {\Gamma_{12}}^2{\Gamma_{12}}^1\\
&&& \quad + \frac{1}{2}H^2 e^{\phi} f - 2Q^+Q^- e^{-\phi} f = 0,\\
&(iii) && Q^+{\Gamma_{22}}^2 - {\Gamma_{11}}^2 Q^- + D_-Q^+ - Q^+D_-\phi + \frac{1}{2}e^{\phi}D_+H = 0,\\
&(iv) && Q^-{\Gamma_{11}}^1 - {\Gamma_{22}}^1 Q^+ + D_+Q^- - Q^-D_+\phi - \frac{1}{2}e^{\phi}D_-H = 0,\\
&(v) && D_-({\Gamma_{11}}^2) - {\Gamma_{12}}^1{\Gamma_{11}}^2 - {\Gamma_{11}}^2{\Gamma_{22}}^2 - {\Gamma_{11}}^1{\Gamma_{12}}^2\\
&&& \quad + D_+({\Gamma_{12}}^2) + 2Q^+Hf = 0,\\
&(vi) && D_+({\Gamma_{22}}^1) + {\Gamma_{12}}^2{\Gamma_{22}}^1 - {\Gamma_{22}}^1{\Gamma_{11}}^1 + {\Gamma_{22}}^2{\Gamma_{12}}^1\\
&&& \quad - D_-({\Gamma_{12}}^1) + 2Q^-Hf = 0.
\end{aligned} \tag{A.3.7}$$

We will now determine whether or not the bSUSYGC equations form an integrable system. In order to lighthen the Christoffel symbols's notation, we write them as

$$\begin{aligned}
R^+ &= {\Gamma_{11}}^1, \quad R^- = {\Gamma_{11}}^2, \quad S^+ = {\Gamma_{12}}^1,\\
S^- &= {\Gamma_{12}}^2, \quad T^+ = {\Gamma_{22}}^1, \quad T^- = {\Gamma_{22}}^2.
\end{aligned} \tag{A.3.8}$$

The vector fields of the Lie point symmetries of the bSUSYGC equations are given by

$$\begin{aligned}
C_0 &= H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} - 2f\partial_f,\\
K_0 &= -H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} + 2\partial_\phi,\\
K_1^b &= -2x_+\partial_{x_+} - \theta^+\partial_{\theta^+} + R^+\partial_{R^+} + 2R^-\partial_{R^-} + S^-\partial_{S^-}\\
&\quad -T^+\partial_{T^+} + 2Q^+\partial_{Q^+} + \partial_\phi,\\
K_2^b &= -2x_-\partial_{x_-} - \theta^-\partial_{\theta^-} - R^-\partial_{R^-} + S^+\partial_{S^+} + 2T^+\partial_{T^+}\\
&\quad +T^-\partial_{T^-} + 2Q^-\partial_{Q^-} + \partial_\phi,\\
P_+ &= \partial_{x_+}, \qquad\qquad P_- = \partial_{x_-},\\
J_+ &= \partial_{\theta^+} + i\theta^+\partial_{x_+}, \qquad J_- = \partial_{\theta^-} + i\theta^-\partial_{x_-}.
\end{aligned} \tag{A.3.9}$$

and the symmetries of the bSUSYGW equations are spanned by

$$\begin{aligned}
P_\pm &= \partial_{x_\pm},\\
J_\pm &= \partial_{\theta^\pm} + i\theta^\pm\partial_{x_\pm},\\
\hat{C}_0 &= H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} - 2f\partial_f + N_i\partial_{N_i},\\
\hat{K}_0 &= -H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} + 2\partial_\phi - N_i\partial_{N_i},\\
K_1^b &= -2x_+\partial_{x_+} - \theta^+\partial_{\theta^+} + R^+\partial_{R^+} + 2R^-\partial_{R^-} + S^-\partial_{S^-}\\
&\qquad -T^+\partial_{T^+} + 2Q^+\partial_{Q^+} + \partial_\phi,\\
K_2^b &= -2x_-\partial_{x_-} - \theta^-\partial_{\theta^-} - R^-\partial_{R^-} + S^+\partial_{S^+} + 2T^+\partial_{T^+}\\
&\qquad +T^-\partial_{T^-} + 2Q^-\partial_{Q^-} + \partial_\phi,\\
G_i &= F_i\partial_{F_i} + N_i\partial_{N_i}, \qquad B_i = \partial_{F_i}, \qquad \text{for } i = 1,2,3\\
R_{ij} &= F_i\partial_{F_j} - F_j\partial_{F_i} + N_i\partial_{N_j} - N_j\partial_{N_i}, \qquad i < j = 2,3.
\end{aligned} \tag{A.3.10}$$

Using conjecture A.2.1 with $\rho$ involving all independent and dependent variables of the bSUSYGC equations, we see that $L_1 = \pi_\rho(L_2)$. Hence the bSUSYGC equations are not integrable.

## A.4. Fermionic conformal parametrization of a surface

Let the surface $\mathcal{S}$ be smooth, orientable and immersed in the superspace $\mathbb{R}^{(1,1|3)}$. We assume that the surface $\mathcal{S}$ is conformally parametrized by a fermionic superfield $F(x_+, x_-, \theta^+, \theta^-)$ with the normalization conditions

$$\langle D_\pm F, D_\pm F\rangle = 0, \qquad \langle D_\pm F, D_\mp F\rangle = \frac{1}{2}e^\phi f, \tag{A.4.1}$$

where $\phi$ is a bosonic-valued function of $x_+, x_-, \theta^+, \theta^-$ and the bosonic-valued function $f(x_+, x_-)$ may be bodiless depending on the structure of $F$. The normalization on the bosonic superfield $N$ is given by

$$\langle D_\pm F, N\rangle = 0, \qquad \langle N, N\rangle = 1. \tag{A.4.2}$$

The tangent vectors $D_\pm F$ together with the normal superfield $N$ form the SUSY moving frame $\Omega = (D_+F, D_-F, N)^T$. Assuming that the first derivatives of the moving frame $\Omega$ can be written as linear combinations of the elements of $\Omega$ and with some calculation we get the fermionic case of the supersymemtric Gauss–Weingarten (fSUSYGW) equations

$$D_+\Omega = A_+\Omega, \qquad D_-\Omega = A_-\Omega,$$

$$D_+\begin{pmatrix} D_+F \\ D_-F \\ N \end{pmatrix} = \begin{pmatrix} {\Gamma_{11}}^1 & 0 & Q^+f \\ 0 & 0 & -\frac{1}{2}e^{\phi}Hf \\ H & -2e^{-\phi}Q^+ & 0 \end{pmatrix}\begin{pmatrix} D_+F \\ D_-F \\ N \end{pmatrix},$$
$$D_-\begin{pmatrix} D_+F \\ D_-F \\ N \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{2}e^{\phi}Hf \\ 0 & {\Gamma_{22}}^2 & Q^-f \\ -2e^{-\phi}Q^- & -H & 0 \end{pmatrix}\begin{pmatrix} D_+F \\ D_-F \\ N \end{pmatrix}. \tag{A.4.3}$$

The fermionic case of the supersymmetric Gauss–Codazzi (fSUSYGC) equations, which are equivalent to the ZCC

$$D_+A_- + D_-A_+ - \{A_+, A_-\} = 0, \tag{A.4.4}$$

reduce to the following four linearly independent equations

$$\begin{aligned} &(i) && D_+({\Gamma_{22}}^2) + D_-({\Gamma_{11}}^1) = 0, \\ &(ii) && D_-({\Gamma_{11}}^1) + 2e^{-\phi}Q^+Q^-f = 0, \\ &(iii) && D_+Q^- - \frac{1}{2}e^{\phi}D_-H + Q^-(D_+\phi - {\Gamma_{11}}^1) = 0, \\ &(iv) && D_-Q^+ + \frac{1}{2}e^{\phi}D_+H + Q^+(D_-\phi - {\Gamma_{22}}^2) = 0. \end{aligned} \tag{A.4.5}$$

Using the abbreviated notation (A.3.8) and conjecture A.2.1, we seek to know if the fSUSYGC equations are integrable. The symmetries of the fSUSYGC equations are spanned by

$$\begin{aligned} &J_+ = \partial_{\theta^+} + i\theta^+\partial_{x_+}, \qquad J_- = \partial_{\theta^-} + i\theta^-\partial_{x_-}, \qquad W = \partial_H \\ &P_+ = \partial_{x_+}, \qquad P_- = \partial_{x_-}, \\ &C_0 = H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} - 2f\partial_f, \\ &K_0 = -H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} + 2\partial_\phi, \\ &K_1^f = -2x_+\partial_{x_+} - \theta^+\partial_{\theta^+} + 2Q^+\partial_{Q^+} + R^+\partial_{R^+} + \partial_\phi, \\ &K_2^f = -2x_-\partial_{x_-} - \theta^-\partial_{\theta^-} + 2Q^-\partial_{Q^-} + T^-\partial_{T^-} + \partial_\phi, \end{aligned} \tag{A.4.6}$$

and the set of vector fields which generates symmetries of the fSUSYGW equations is given by

$$\begin{aligned}
P_\pm &= \partial_{x_\pm}, \qquad J_\pm = \partial_{\theta^\pm} + i\theta^\pm \partial_{x_\pm}, \\
\hat{C}_0 &= H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} - 2f\partial_f + N_i\partial_{N_i}, \\
\hat{K}_0 &= -H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} + 2\partial_\phi - N_i\partial_{N_i}, \\
K_1^f &= -2x_+\partial_{x_+} - \theta^+\partial_{\theta^+} + 2Q^+\partial_{Q^+} + R^+\partial_{R^+} + \partial_\phi, \\
K_2^f &= -2x_-\partial_{x_-} - \theta^-\partial_{\theta^-} + 2Q^-\partial_{Q^-} + T^-\partial_{T^-} + \partial_\phi, \\
G_i &= F_i\partial_{F_i} + N_i\partial_{N_i}, \qquad B_i = \partial_{F_i}, \qquad \text{for } i,j = 1,2,3 \\
R_{ij} &= F_i\partial_{F_j} - F_j\partial_{F_i} + N_i\partial_{N_j} - N_j\partial_{N_i}, \qquad i<j=2,3.
\end{aligned} \tag{A.4.7}$$

Comparing these sets of generators, we see that the symmetry spanned by the vector field $W$ is not common to both the fSUSYGC equations and the fSUSYGW equations. We can use this symmetry to incorporate a free fermionic parameter $\lambda$ which cannot be eliminated through a gauge transformation. The potential matrices $A_\pm$ take the form

$$\begin{aligned}
A_+ &= \begin{pmatrix} {\Gamma_{11}}^1 & 0 & Q^+f \\ 0 & 0 & -\frac{1}{2}e^\phi(H+\lambda)f \\ H+\lambda & -2e^{-\phi}Q^+ & 0 \end{pmatrix}, \\
A_- &= \begin{pmatrix} 0 & 0 & \frac{1}{2}e^\phi(H+\lambda)f \\ 0 & {\Gamma_{22}}^2 & Q^-f \\ -2e^{-\phi}Q^- & -H-\lambda & 0 \end{pmatrix}.
\end{aligned} \tag{A.4.8}$$

Therefore, the system of the fSUSYGC equation is a candidate to be integrable.

## A.5. Conclusion

In this paper we have presented supersymmetric conformal parametrizations of surfaces, one for the bosonic case and another for the ferminonic case. Both the Gauss–Codazzi and Gauss–Weingarten equations have been given for the bosonic and fermionic cases. Moreover, a conjecture on the integrability of systems of differential equations in the sense of soliton theory has been presented. This conjecture allows us to know if a system of differential equations can be integrable by comparing the symmetries of the system with the symmetries of its linear problem. In this paper, examples of this conjecture have been applied to the classical Gauss–Codazzi equations, the supersymmetric sinh-Gordon equation, the bosonic case of the supersymmetric Gauss–Codazzi equations and the fermionic case of the supersymmetric Gauss–Codazzi equations.

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# Annexe B

## SUPERSYMMETRIC VERSION OF THE EQUATIONS OF CONFORMALLY PARAMETRIZED SURFACES

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Cet article a été combiné à l'article [**B8**], apparaissant dans l'annexe C, afin d'être publié dans J. Phys. A : Math. Theor. (voir le chapitre 2 ou [**B1**])

**Résumé**

Dans cet article, nous formulons une extension supersymétrique des équations de Gauss–Weingarten et de Gauss–Codazzi pour les surfaces conformément paramétrisées plongées dans un superespace de Grassmann. Nous effectuons cette analyse en utilisant un formalisme de superespace/superchamp et une version supersymétrique du repère mobile sur une surface. Contrairement au cas classique où nous avons trois équations de Gauss–Codazzi, nous obtenons six équations dans le cas supersymétrique. Nous déterminons l'algèbre de Lie des symétries des équations classiques de Gauss–Codazzi qui est de dimension infinie et nous effectuons une classification des sous-algèbres unidimensionnelles de la sous-algèbre maximale de dimension finie. Par la suite, nous calculons la superalgèbre de Lie des symétries ponctuelle des équations de Gauss–Codazzi supersymétriques et nous classifions les sous-algèbres en classes de conjugaison de cette superalgèbre. Puis, nous utilisons la méthode de réduction par symétrie pour trouver les invariants, les orbites et les systèmes réduits pour deux sous-algèbres unidimensionnelles dans le cas classique et pour trois sous-algèbres unidimensionnelles dans le cas supersymétrique. Par les solutions de ces systèmes réduits, nous obtenons des solutions explicites et des surfaces des équations de Gauss–Codazzi classique et supersymétrique. Nous fournissons une interprétation géométrique des résultats.

**Abstract**

In this paper, we formulate a supersymmetric extension of the Gauss–Weingarten and Gauss–Codazzi equations for conformally parametrized surfaces immersed in a Grassmann superspace. We perform this analysis using a superspace-superfield formalism together with a supersymmetric version of a moving frame on a surface. In constrast to the classical case, where we have three Gauss–Codazzi equations, we obtain six such equations in the supersymmetric case. We determine the Lie symmetry algebra of the classical Gauss–Codazzi equations to be infinite-dimensional and perform a subalgebra classification of the one-dimensional subalgebras of its largest finite-dimensional subalgebra. We then compute a superalgebra of Lie point symmetries of the supersymmetric Gauss–Codazzi equations and classify the one-dimensional subalgebras of this superalgebra into conjugacy classes. We then use the symmetry reduction method to find invariants, orbits and reduced systems for two one-dimensional subalgebras in the classical case and three one-dimensional subalgebras in the supersymmetric case. Through the solutions of these reduced systems, we obtain explicit solutions and surfaces of the classical and supersymmetric Gauss–Codazzi equations. We provide a geometrical interpretation of the results.

## B.1. Introduction

The theory of supersymmetry has been studied extensively over the past decades and has generated a great deal of interest and activity in several areas of mathematics and physics. Applications have been numerous in the field of particle physics (see e.g. [**16**, **37**, **45**, **76**, **114**, **119**]) but it has also been applied to classical fluid dynamics ( [**36**, **50**, **67**, **78**, **83**, **89**, **90**]). Various approaches have been used to construct supersoliton solutions, such as the inverse scattering method, Bäcklund and Darboux transformations for odd and even superfields, Lax formalism in a superspace and generalized versions of the symmetry reduction method. A number of soliton and super multi-soliton solutions were determined by a Crum-type transformation [**64**, **91**, **107**] and it was found that there exist infinitely many local conserved quantities. A connection was established between the super-Darboux transformations and super-Bäcklund transformations which allow for the construction of supersoliton solutions [**3**, **24**, **63**, **69**, **88**, **108**, **116**].

In differential geometry, parametrized surfaces are described in terms of moving frames satisfying the Gauss–Weingarten (GW) equations, which are linear differential equations. Their compatibility conditions are the Gauss–Codazzi (GC) equations. A representation of nonlinear equations in the form of the GC equations is the starting point in the theory of integrable (soliton) surfaces arising from infinitesimal deformations of integrable differential equations and describing the behaviour of soliton solutions. The construction and analysis of such surfaces associated with integrable systems in several areas of mathematical physics provides new tools for the investigation of nonlinear phenomena described by these systems. In this setting, it is our objective to perform a systematic analysis of a supersymmetric (SUSY) version of the GC equations. The formulation of a SUSY extension of the GW and GC equations has already been accomplished for the specific case of bosonic Grassmann sigma models [**41**, **42**, **104**, **122**]. It would be of considerable interest to consider such an extension for general case of the GW and GC equations.

The purpose of this paper is to formulate a SUSY extension of the GW and GC equations for the case of conformally parametrized surfaces in $\mathbb{R}^3$. The SUSY version of these equations is formulated through the use of a superspace-superfield formalism. The considered surfaces are parametrized by the vector field $\mathcal{F}$ and the normal vector field $\mathcal{N}$, which are replaced in the SUSY version by their corresponding bosonic vector superfields $F$ and $N$ in a superspace $\mathbb{R}^{(2,1|2)}$. This allows us to formulate the SUSY extension of the structural equations for the immersion of conformally parametrized surfaces explicitly in matrix form. We establish an explicit form of the SUSY GW equations satisfied by the moving frame on these surfaces. The result is independent of the parametrization. This allows us to examine various geometric properties of the studied immersions, such as the first and second fundamental forms of the surfaces (and therefore the mean and Gaussian curvatures).

Once we have established the SUSY extension of the GW and GC equations, we compute a Lie symmetry superalgebra and classify its one-dimensional subalgebras into conjugacy classes. We then use a generalized version of the symmetry reduction method to determine invariant solutions of our SUSY model. Some geometrical aspects of the obtained results are explored. We demonstrate that the SUSY GW equations for the superframe on the surface resemble the linear system which appears in the soliton theory of the immersion of surfaces in Lie algebras.

The paper is organized as follows. The symmetry algebra of the classical GC equations is determined and a subalgebra classification of its one-dimensional subalgebras is performed in section B.2. Section B.3, we recall the basic properties of Grassmann algebras and Grassmann variables and introduce the notation that will be used in what follows. In section B.4, we construct the supersymmetric extensions of the Gauss–Weingarten and Gauss–Codazzi equations. In section B.5, we discuss certain geometric aspects of the conformally parametrized supersymmetric surface. We provide expressions for the first and second fundamental forms and the Gaussian and mean curvatures, which are required for a geometrical interpretation of the invariant solutions. In section B.6, we determine a Lie superalgebra of symmetries of the supersymmetric Gauss–Codazzi equations. Section B.7 involves a classification of the one-dimensional subalgebras of the Lie superalgebra into conjugacy classes. In section B.8, we provide examples of invariant solutions of the supersymmetric Gauss–Codazzi equations obtained by the symmetry reduction method. Finally, in section B.9, we present the conclusions and discuss possible future developments in this field.

## B.2. Symmetries of conformally parametrized surfaces

The system of partial differential equations (PDEs) describing the moving frame $\Omega = (\partial\mathcal{F}, \bar{\partial}\mathcal{F}, \mathcal{N})^T$ on a conformally parametrized surface in 3-dimensional Euclidean space satisfies the following GW equations

$$\partial\Omega = V_1\Omega, \qquad \bar{\partial}\Omega = V_2\Omega, \tag{B.2.1}$$

where the matrices $V_1$ and $V_2$ are given by

$$V_1 = \begin{pmatrix} \partial u & 0 & Q \\ 0 & 0 & \frac{1}{2}He^u \\ -H & -2Qe^{-u} & 0 \end{pmatrix}, \qquad V_2 = \begin{pmatrix} 0 & 0 & \frac{1}{2}He^u \\ 0 & \bar{\partial}u & \bar{Q} \\ -2\bar{Q}e^{-u} & -H & 0 \end{pmatrix}. \tag{B.2.2}$$

Here $\partial$ and $\bar{\partial}$ are the partial derivatives with respect to the complex coordinates $z = x+iy$ and $\bar{z} = x-iy$, respectively. The conformal parametrization of a surface is given by a vector-valued function $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)^T : \mathcal{R} \to \mathbb{R}^3$ (where $\mathcal{R}$ is a Riemann surface) which satisfies the following normalization for the tangent vectors $\partial\mathcal{F}$ and $\bar{\partial}\mathcal{F}$ and the unit normal $N$

$$\begin{aligned} &\langle \partial\mathcal{F}, \partial\mathcal{F} \rangle = \langle \bar{\partial}\mathcal{F}, \bar{\partial}\mathcal{F} \rangle = 0, \qquad \langle \partial\mathcal{F}, \bar{\partial}\mathcal{F} \rangle = \frac{1}{2}e^u, \\ &\langle \partial\mathcal{F}, \mathcal{N} \rangle = \langle \bar{\partial}\mathcal{F}, \mathcal{N} \rangle = 0, \qquad \langle \mathcal{N}, \mathcal{N} \rangle = 1, \end{aligned} \tag{B.2.3}$$

and we define the quantities

$$Q = \langle \partial^2\mathcal{F}, \mathcal{N} \rangle \in \mathbb{C}, \qquad H = 2e^{-u}\langle \partial\bar{\partial}\mathcal{F}, \mathcal{N} \rangle \in \mathbb{R}. \tag{B.2.4}$$

Here the bracket $\langle\cdot,\cdot\rangle$ denotes the scalar product in 3-dimensional Euclidean space $\mathbb{R}^3$

$$\langle a,b\rangle = a_1b_1 + a_2b_2 + a_3b_3. \tag{B.2.5}$$

So, the GW equations for a moving frame $\Omega$ on a surface have to obey the following system of equations

$$\begin{gathered}\partial^2\mathcal{F} = \partial u\partial\mathcal{F} + Q\mathcal{N}, \qquad \partial\bar{\partial}\mathcal{F} = \frac{1}{2}He^u\mathcal{N}, \qquad \bar{\partial}^2\mathcal{F} = \bar{\partial}u\bar{\partial}\mathcal{F} + \bar{Q}\mathcal{N}, \\ \partial\mathcal{N} = -H\partial\mathcal{F} - 2e^{-u}Q\bar{\partial}\mathcal{F}, \qquad \bar{\partial}\mathcal{N} = -2e^{-u}\bar{Q}\partial\mathcal{F} - H\bar{\partial}\mathcal{F}.\end{gathered} \tag{B.2.6}$$

The first and second fundamental forms are given by

$$I = \langle d\mathcal{F}, d\mathcal{F}\rangle = \left\langle \frac{e^u}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}dz\\ d\bar{z}\end{pmatrix}, \begin{pmatrix}dz\\ d\bar{z}\end{pmatrix}\right\rangle = e^u\left\langle \begin{pmatrix}dx\\ dy\end{pmatrix}, \begin{pmatrix}dx\\ dy\end{pmatrix}\right\rangle, \tag{B.2.7}$$

and

$$I\!I = \langle d^2\mathcal{F}, \mathcal{N}\rangle = \left\langle \begin{pmatrix}Q+\bar{Q}+e^uH & i(Q-\bar{Q})\\ i(Q-\bar{Q}) & -(Q+\bar{Q})+e^uH\end{pmatrix}\begin{pmatrix}dx\\ dy\end{pmatrix}, \begin{pmatrix}dx\\ dy\end{pmatrix}\right\rangle, \tag{B.2.8}$$

respectively. The principal curvatures $k_1$ and $k_2$ are the eigenvalues of the matrix

$$B = e^{-u}\begin{pmatrix}Q+\bar{Q}+e^uH & i(Q-\bar{Q})\\ i(Q-\bar{Q}) & -(Q+\bar{Q})+e^uH\end{pmatrix}. \tag{B.2.9}$$

We obtain the following expressions for the mean and Gaussian curvatures

$$H = \frac{1}{2}(k_1+k_2) = \frac{1}{2}\mathrm{tr}(B), \tag{B.2.10}$$

$$\mathcal{K} = k_1k_2 = \det(B) = H^2 - 4Q\bar{Q}e^{-2u}. \tag{B.2.11}$$

Umbilic points on a surface take place when $H^2 - \mathcal{K} = 0$ which implies that $|Q|^2 = 0$. The compatibility conditions of the GW equations (B.2.1) are the GC equations

$$\bar{\partial}V_1 - \partial V_2 + [V_1, V_2] = 0, \tag{B.2.12}$$

(the bracket $[\cdot,\cdot]$ denotes the commutator) which reduce to the following three differential equations for the quantities $Q$, $H$ and $e^u$

$$\begin{gathered}\partial\bar{\partial}u + \frac{1}{2}H^2e^u - 2Q\bar{Q}e^{-u} = 0, \qquad \text{(the Gauss equation)} \\ \partial\bar{Q} - \frac{1}{2}e^u\bar{\partial}H = 0, \qquad \bar{\partial}Q - \frac{1}{2}e^u\partial H = 0. \qquad \text{(the Codazzi equations)}\end{gathered} \tag{B.2.13}$$

These equations are the necessary and sufficient conditions for the existence of conformally parametrized surfaces in 3-dimensional Euclidean space $\mathbb{R}^3$ with the fundamental forms given by (B.2.7) and (B.2.8). A review of systematic computational methods for constructing surfaces for a given moving frame can be found in several books (e.g. [**6**, **47**, **82**, **99**, **115**]). Equations (B.2.1), (B.2.2) and (B.2.13) allow us to formulate explicitly the structural equations for the immersion directly in matrix terms. However, it is nontrivial to identify those surfaces which have an invariant geometrical characterization [**19**, **47**]. The task of finding an increasing number of solutions of the GW and GC equations is related to the group properties of these systems of equations. Their main advantages appear when group analysis makes it possible to construct regular algorithms for finding certain classes of solutions (describing diverse types of surfaces) without referring to any additional considerations, but proceeding only from the given system of equations. A broad review of recent developments in this subject can be found in several books (see e.g. P. Olver [**95**], D. Sattinger and O. Weaver [**105**], and G. Bluman and S. Kumai [**18**]). The methodological approach adopted here is based on the symmetry reduction method for PDEs invariant under a Lie group $G$ of point transformations. Using the Maple program, we find that the symmetry group of the classical GC equations (B.2.13) consists of conformal scaling transformations. The corresponding symmetry algebra $\mathcal{L}$ is spanned by the vector fields

$$\begin{aligned} X(\eta) &= \eta(z)\partial_z + \eta'(z)(-2Q\partial_Q - U\partial_U), \\ Y(\zeta) &= \zeta(\bar{z})\partial_{\bar{z}} + \zeta'(\bar{z})(-2\bar{Q}\partial_{\bar{Q}} - U\partial_U), \\ e_0 &= -H\partial_H + Q\partial_Q + \bar{Q}\partial_{\bar{Q}} + 2U\partial_U, \end{aligned} \tag{B.2.14}$$

where $\eta'(\cdot)$ and $\zeta'(\cdot)$ are the derivatives of $\eta(\cdot)$ and $\zeta(\cdot)$ with respect to their arguments respectively and where we have used the notation $e^u = U$. The commutation relations are

$$\begin{aligned} &= (\eta_1'\eta_2 - \eta_1\eta_2')\partial_z + (\eta_1''\eta_2 - \eta_1\eta_2'')(2Q\partial_Q + U\partial_U), \\ [Y(\zeta_1), Y(\zeta_2)] &= (\zeta_1'\zeta_2 - \zeta_1\zeta_2')\partial_{\bar{z}} + (\zeta_1''\zeta_2 - \zeta_1\zeta_2'')(2\bar{Q}\partial_{\bar{Q}} + U\partial_U), \\ [X(\eta), Y(\zeta)] = 0, \qquad & [X(\eta), e_0] = 0, \qquad [Y(\zeta), e_0] = 0. \end{aligned} \tag{B.2.15}$$

Since the vector fields $X(\eta), Y(\zeta)$ and $e_0$ form an Abelian algebra, they determine that the algebra $\mathcal{L}$ can be decomposed as a direct sum of two infinite-dimensional Lie algebras together with a one-dimensional algebra generated by $e_0$, i.e.

$$\mathcal{L} = \{X(\eta)\} \oplus \{Y(\zeta)\} \oplus \{e_0\}. \tag{B.2.16}$$

This algebra represents a direct sum of two copies of the Virasoro algebra together with the one-dimensional subalgebra $\{e_0\}$. Assuming that the functions $\eta$ and $\zeta$ are analytic in some open subset $\mathcal{D} \subset \mathbb{C}$, we can develop $\eta$ and $\zeta$ as power series with respect to their arguments and provide a basis for $\mathcal{L}$. The largest finite-dimensional subalgebra $L$ of the algebra $\mathcal{L}$ is spanned by seven generators

$$\begin{gathered} e_0 = -H\partial_H + Q\partial_Q + \bar{Q}\partial_{\bar{Q}} + 2U\partial_U, \\ e_1 = \partial_z, \qquad e_3 = z\partial_z - 2Q\partial_Q - U\partial_U, \\ e_2 = \partial_{\bar{z}}, \qquad e_4 = \bar{z}\partial_{\bar{z}} - 2\bar{Q}\partial_{\bar{Q}} - U\partial_U, \\ e_5 = z^2\partial_z - 4zQ\partial_Q - 2zU\partial_U, \qquad e_6 = \bar{z}^2\partial_{\bar{z}} - 4\bar{z}\bar{Q}\partial_{\bar{Q}} - 2\bar{z}U\partial_U, \end{gathered} \tag{B.2.17}$$

with nonzero commutation relations

$$\begin{gathered} [e_1, e_3] = e_1, \qquad [e_1, e_5] = -2e_3, \qquad [e_3, e_5] = e_5, \\ [e_2, e_4] = e_2, \qquad [e_2, e_6] = -2e_4, \qquad [e_4, e_6] = e_6. \end{gathered} \tag{B.2.18}$$

This 7-dimensional Lie subalgebra $L$ can be decomposed as a direct sum of two simple subalgebras together with a one-dimensional algebra generated by $e_0$,

$$L = \{e_1, e_3, e_5\} \oplus \{e_2, e_4, e_6\} \oplus \{e_0\}. \tag{B.2.19}$$

Therefore, the classification of the subalgebras of $L$ consists of two copies of a 3-dimensional Lie algebra together with the center $\{e_0\}$. This 3-dimensional Lie algebra corresponds to the algebra $A_{3,8}$ in the classification of J. Patera and P. Winternitz [**97**] which is isomorphic to $\mathfrak{su}(1,1)$. The resulting classification of the subalgebras of $L$ into conjugacy classes, performed according to the methods described in [**120**], is given by the following list of representative subalgebras $L_{1,j}$

$$\begin{gathered} L_{1,0} = \{e_0\}, \qquad L_{1,1} = \{e_1\}, \qquad L_{1,2} = \{e_3\}, \qquad L_{1,3} = \{e_1 + e_5\}, \\ L_{1,4} = \{e_2\}, \qquad L_{1,5} = \{e_4\}, \qquad L_{1,6} = \{e_2 + e_6\}, \\ L_{1,7} = \{e_1 + \epsilon e_2\}, \qquad L_{1,8} = \{e_1 + \epsilon e_4\}, \\ L_{1,9} = \{e_2 + e_6 + \epsilon e_1\}, \qquad L_{1,10} = \{e_3 + \epsilon e_2\}, \\ L_{1,11} = \{e_3 + a e_4\}, \qquad L_{1,12} = \{e_2 + e_6 + a e_3\}, \\ L_{1,13} = \{e_1 + e_5 + \epsilon e_2\}, \qquad L_{1,14} = \{e_1 + e_5 + a e_4\}, \\ L_{1,15} = \{e_1 + e_5 + a(e_2 + e_6)\}, \end{gathered} \tag{B.2.20}$$

where $\epsilon = \pm 1$ and $a \neq 0$ are parameters. The center of $L$, $\{e_0\}$, can be added to any of the subalgebras given above, say $L_{1,j} = \{A\}$, to produce a twisted subalgebra of the form $L'_{1,j} = \{A + be_0\}$, where $b \neq 0$. The symmetry reductions associated with the subalgebras (B.2.20) lead to systems of ordinary differential equations (ODEs). These reduced systems were analyzed systematically as a single generic symmetry reduction in [**32**], where the GC equations (B.2.13) were reduced to the most general Painlevé P6 form (containing two or three arbitrary parameters).

## B.3. Preliminaries on Grassmann algebras

The mathematical background formalism is based on the theory of supermanifolds as presented in [**11**, **33**, **43**, **56**, **118**] and can be summarized as follows. The starting point in our consideration is a complex Grassmann algebra $\Lambda$ involving a finite or infinite number of Grassmann generators $(\xi_1, \xi_2, ...)$. The number of Grassmann generators of $\Lambda$ is not essential provided that there is a sufficient number of them to make any formula encountered meaningful. The Grassmann algebra $\Lambda$ can be decomposed into its even and odd parts

$$\Lambda = \Lambda_{\text{even}} + \Lambda_{\text{odd}}. \tag{B.3.1}$$

In the context of supersymmetry, the spaces $\Lambda$ and/or $\Lambda_{\text{even}}$ replace the field of complex numbers. The elements of $\Lambda$ are called supernumbers while elements of its even or odd parts are called even or odd supernumbers respectively. The Grassmann algebra can also be decomposed as

$$\Lambda = \Lambda_{\text{body}} + \Lambda_{\text{soul}}, \tag{B.3.2}$$

where

$$\Lambda_{\text{body}} = \Lambda^0[\xi_1, \xi_2, ...] \simeq \mathbb{C}, \qquad \Lambda_{\text{soul}} = \sum_{k \geqslant 1} \Lambda^k[\xi_1, \xi_2, ...]. \tag{B.3.3}$$

Here $\Lambda^0[\xi_1, \xi_2, ...]$ refers to all terms that do not involve any of the generators $\xi_i$, while $\Lambda^k[\xi_1, \xi_2, ...]$ refers to all terms that involve products of $k$ generators (for instance, if we have 4 generators $\xi_1, \xi_2, \xi_3, \xi_4$, then $\Lambda^2[\xi_1, \xi_2, \xi_3, \xi_4]$ refers to all terms involving $\xi_1\xi_2$, $\xi_1\xi_3$, $\xi_1\xi_4$, $\xi_2\xi_3$, $\xi_2\xi_4$ or $\xi_3\xi_4$). The bodiless elements in $\Lambda_{\text{soul}}$ are non-invertible because of the $\mathbb{Z}_0^+$-grading of the Grassmann algebra. If the number of Grassmann generators $\mathfrak{K}$ is finite, bodiless elements are nilpotent of degree at most $\mathfrak{K}$. In this paper, we assume that $\mathfrak{K}$ is arbitrarily large but finite. Our analysis is based on the global theory of supermanifolds as described in [**12**, **101**, **102**].

Next, in our consideration, we use a $\mathbb{Z}_2$-graded complex vector space $V$ which has even basis elements $u_i$, $i = 1, 2, ..., N$, and odd basis elements $v_\mu$, $\mu = 1, 2, ..., N$, and construct $W = \Lambda \otimes_{\mathbb{C}} V$. We are interested in the even part of $W$

$$W_{\text{even}} = \left\{ \sum_i a_i u_i + \sum_\mu \underline{\alpha}_\mu v_\mu | a_i \in \Lambda_{\text{even}}, \underline{\alpha}_\mu \in \Lambda_{\text{odd}} \right\}. \tag{B.3.4}$$

Clearly, $W_{\text{even}}$ is a $\Lambda_{\text{even}}$ module which can be identified with $\Lambda_{\text{even}}^{\times N} \times \Lambda_{\text{odd}}^{\times M}$ (consisting of $N$ copies of $\Lambda_{\text{even}}$ and $M$ copies of $\Lambda_{\text{odd}}$). We associate with the original basis, consisting of $u_i$ and $v_\mu$ (although $v_\mu \notin W_{\text{even}}$), the corresponding functionals

$$E_j : W_{\text{even}} \to \Lambda_{\text{even}} : E_j \left( \sum_i a_i u_i + \sum_\mu \underline{\alpha_\mu} v_\mu \right) = a_j, \tag{B.3.5}$$

$$\Upsilon_\nu : W_{\text{even}} \to \Lambda_{\text{odd}} : \Upsilon_\nu \left( \sum_i a_i u_i + \sum_\mu \underline{\alpha_\mu} v_\mu \right) = \underline{\alpha}_\nu, \tag{B.3.6}$$

and view them as the coordinates (even and odd respectively) on $W_{\text{even}}$. Any topological manifold locally diffeomorphic to a suitable $W_{\text{even}}$ is called a supermanifold.

The transitions to even and odd coordinates between different charts on the supermanifold are assumed to be even- and odd-valued superanalytic or at least $G^\infty$ functions on $W_{\text{even}}$. A comprehensive definition of the classes of supersmooth functions $G^\infty$ and superanalytic functions $G^\omega$ can be found in [**102**], definition 2.5. We note that superanalytic functions are those that can be expanded into a convergent power series in even and odd coordinates, whereas the definition of the $G^\infty$ function is a more involved analogue of functions on manifolds. Any $G^\infty$ function can be expanded into products of odd coordinates in a Taylor-like expansion but the coefficients, being functions of even and odd coordinates, may not necessarily be analytic (see e.g. [**102**]).

The super-Minkowski space $\mathbb{R}^{(1,1|2)}$ can be viewed as such a supermanifold globally diffeomorphic to $\Lambda_{\text{even}}^{\times 2} \times \Lambda_{\text{odd}}^{\times 2}$ with even light-cone coordinates $x_+, x_-$ and odd coordinates $\theta^+, \theta^-$. Here $x_+$ and $x_-$ are linear combinations of terms involving an even number of generators : $1, \xi_1\xi_2, \xi_1\xi_3, \xi_1\xi_4, ..., \xi_2\xi_3, \xi_2\xi_4, ..., \xi_1\xi_2\xi_3\xi_4, ...$ On the other hand, $\theta^+$ and $\theta^-$ are linear combinations of terms involving an odd number of generators : $\xi_1, \xi_2, \xi_3, \xi_4, ..., \xi_1\xi_2\xi_3, \xi_1\xi_2\xi_4, \xi_1\xi_3\xi_4, \xi_2\xi_3\xi_4, ...$ The supersymmetry transformation (B.4.12) and (B.4.13) in the next section can be viewed as a particular change of coordinates on $\mathbb{R}^{(2,1|2)}$ which transforms solutions of the SUSY GW equations (B.4.16) and GC equations, (B.4.41) respectively, into solutions of the same equation in new coordinates. A bosonic smooth superfield is a $G^\infty$ function from $\mathbb{R}^{(1,1|2)}$ to $\Lambda_{\text{even}}$. It can be expanded in powers of the odd coordinates $\theta^+$ and $\theta^-$ giving a decomposition in terms of even superfields

$$\chi_{\text{even}} : \Lambda_{\text{even}}^{\times 2} \to \Lambda_{\text{even}},$$

and odd superfields

$$\chi_{\text{odd}} : \Lambda_{\text{even}}^{\times 2} \to \Lambda_{\text{odd}}.$$

In this paper, we use the convention that partial derivatives involving odd variables satisfy the Leibniz rule

$$\partial_{\theta^\pm}(hg) = (\partial_{\theta^\pm} h)g + (-1)^{\deg(h)} h(\partial_{\theta^\pm} g), \tag{B.3.7}$$

where

$$\deg(h) = \begin{cases} 0 \text{ if } h \text{ is even,} \\ 1 \text{ if } h \text{ is odd,} \end{cases} \tag{B.3.8}$$

and the notation

$$f_{\theta^+\theta^-} = \partial_{\theta^-}\left(\partial_{\theta^+} f\right). \tag{B.3.9}$$

The partial derivatives with respect to the odd coordinates satisfy $\partial_{\theta^i}\theta^j = \delta_i^j$ where the indices $i$ and $j$ each stand for $+$ or $-$. The operators $\partial_{\theta^\pm}$, $J_\pm$ and $D_\pm$, in equations (B.4.3) and (B.4.11) change the parity of a bosonic function to a fermionic function and vice versa. For example, if $\phi$ is a bosonic function, then $\partial_{\theta^+}\phi$ is an odd superfield while $\partial_{\theta^+}\partial_{\theta^-}\phi$ is an even superfield and so on. For further details see e.g. the books by Cornwell [**33**], DeWitt [**43**], Freed [**56**], Varadarajan [**118**] and references therein. The chain rule for a Grassmann-valued composite function $f(g(x_+))$ is

$$\frac{\partial f}{\partial x_+} = \frac{\partial g}{\partial x_+}\frac{\partial f}{\partial g}. \tag{B.3.10}$$

The interchange of mixed derivatives (with proper respect to the ordering of odd variables) is assumed throughout. The even supernumbers, variables, fields, etc. are assumed to be elements of the even part $\Lambda_{\text{even}}$ of the underlying abstract real (complex) Grassmann ring $\Lambda$. The odd supernumbers, variables, fields, etc. lie in its odd part $\Lambda_{\text{odd}}$.

## B.4. Supersymmetric extension of the Gauss-Weingarten and Gauss-Codazzi equations

In this section, we derive the main elements allowing us to construct surfaces in the superspace $\mathbb{R}^{(2,1|2)}$. Let us consider a SUSY version of the differential equations which define surfaces in two-dimensional Minkowski space with the bosonic light-cone coordinates $x_+ = \frac{1}{2}(t+x)$ and $x_- = \frac{1}{2}(t-x)$ and the fermionic (anti-commuting) variables $\theta^+$ and $\theta^-$ such that

$$(\theta^+)^2 = (\theta^-)^2 = \theta^+\theta^- + \theta^-\theta^+ = 0. \tag{B.4.1}$$

Let $\mathcal{S}$ be a smooth simply connected surface in the superspace $\mathbb{R}^{(2,1|2)}$ which we assume is conformally parametrized in the sense that the surface $\mathcal{S}$ is given by a vector-valued bosonic superfield $F(x_+, x_-, \theta^+, \theta^-)$ satisfying conditions (B.4.5) specified below. Such a superfield can be decomposed in the form

$$\begin{aligned} F = F_m(x_+, x_-) + \theta^+\varphi_m(x_+, x_-) + \theta^-\psi_m(x_+, x_-) \\ +\theta^+\theta^-G_m(x_+, x_-), \quad m = 1, 2, 3 \end{aligned} \tag{B.4.2}$$

Here, the odd-valued fields $\varphi_m$ and $\psi_m$ and the even-valued fields $F_m$ and $G_m$ are the four parts of the power series with respect to $\theta^+$ and $\theta^-$ of the $m^{th}$ component of the vector superfield $F$. Let $D_+$ and $D_-$ be the covariant superspace derivatives

$$D_\pm = \partial_{\theta^\pm} - i\theta^\pm\partial_{x_\pm}. \tag{B.4.3}$$

Then the conformal parametrization of the surface $\mathcal{S}$ in the superspace $\mathbb{R}^{(2,1|2)}$ is assumed to give the following normalization of the superfield $F$

$$\langle D_iF, D_jF\rangle = g_{ij}f, \qquad i, j = 1, 2 \tag{B.4.4}$$

where $f$ is a bodiless bosonic function (i.e. $f \in \Lambda_{soul}$) of $x_+$ and $x_-$ which is nilpotent of order $k$. Here the values 1 and 2 of the indices $i$ and $j$ stand for $+$ and $-$, respectively. The scalar product $\langle\cdot,\cdot\rangle$ in (B.4.4) is defined in the same way as in equation (B.2.5), taking into account the property (B.4.1) regarding the odd-valued variables $\theta^+$ and $\theta^-$, and taking values in the Grassmann algebra $\Lambda$. Hence the bosonic functions $g_{ij}$ of $x_+$, $x_-$, $\theta^+$ and $\theta^-$ are given by

$$g_{11} = 0, \qquad g_{12} = \frac{1}{2}e^\phi, \qquad g_{21} = -\frac{1}{2}e^\phi, \qquad g_{22} = 0. \tag{B.4.5}$$

For the superfield $F$, given by (B.4.2), the equations (B.4.4) are identically satisfied for $i = j$. Indeed, in the scalar product (B.2.5) we have the sum of the squares of each $m^{th}$ component of the vector superfield $D_iF$. Since the square of a fermionic function vanishes, each of the terms in the scalar product is identically zero, i.e. $\langle D_iF, D_iF\rangle = 0$. In the case of the mixed scalar product, the normalization imposes the condition $\langle D_+F, D_-F\rangle = \frac{1}{2}e^\phi f$. It should be noted that in equation (B.4.4), the product $\langle D_iF, D_jF\rangle$ necessarily contains only terms involving some of the generators $\xi_1, \xi_2, \xi_3, ..., \xi_i, ...$ For this reason we include the bodiless function $f(x_+, x_-)$ in the normalization (B.4.4). It is interesting to note that, by construction, the metric coefficients $g_{ij}$ satisfy the property

$$g_{ij} = -g_{ji}. \tag{B.4.6}$$

The superfield $\phi$ is assumed to be bosonic and can be decomposed as the following power series in the odd variables $\theta^+$ and $\theta^-$

$$\phi = u(x_+, x_-) + \theta^+\gamma(x_+, x_-) + \theta^-\delta(x_+, x_-) + \theta^+\theta^-v(x_+, x_-). \tag{B.4.7}$$

Through a power expansion in $\theta^+$ and $\theta^-$ we find the exponential form

$$\begin{aligned} e^{\phi} &= e^{u}(1+\theta^+\gamma+\theta^-\delta+\theta^+\theta^-(v-\gamma\delta)), \\ e^{-\phi} &= e^{-u}(1-\theta^+\gamma-\theta^-\delta-\theta^+\theta^-(v+\gamma\delta)). \end{aligned} \tag{B.4.8}$$

The tangent vector superfields $D_+F$ and $D_-F$ together with the normal bosonic superfield $N(x_+, x_-, \theta^+, \theta^-)$, which can be decomposed as

$$\begin{aligned} N = N_m(x_+, x_-) + \theta^+\alpha_m(x_+, x_-) + \theta^-\beta_m(x_+, x_-) \\ +\theta^+\theta^- H_m(x_+, x_-), \quad m = 1,2,3 \end{aligned} \tag{B.4.9}$$

form a moving frame $\Omega$ on the surface $\mathcal{S}$ in the superspace $\mathbb{R}^{(2,1|2)}$. Here, the even-valued fields $N_m$ and $H_m$ and the odd-valued fields $\alpha_m$ and $\beta_m$ are the four parts of the power series with respect to $\theta^+$ and $\theta^-$ of the $m^{th}$ component of the vector superfield $N$. This normal superfield $N$ has to satisfy the conditions

$$\langle D_iF, N\rangle = 0, \qquad \langle N, N\rangle = 1, \qquad i = 1,2. \tag{B.4.10}$$

For any non-bodiless bosonic function $C$, one can find its inverse

$$C = C_0 + \theta^+C_1 + \theta^-C_2 + \theta^+\theta^-C_3,$$
$$C^{-1} = \frac{1}{C_0} - \theta^+\frac{C_1}{(C_0)^2} - \theta^-\frac{C_2}{(C_0)^2} - \theta^+\theta^-\left(\frac{2C_1C_2}{(C_0)^3} + \frac{C_3}{(C_0)^2}\right),$$

where $C_0$ is non-bodiless. So if each $N_m$ is a non-bodiless bosonic function, we can always normalize $N$ to 1 due to the fact that

$$\begin{aligned} \hat{N} =& \frac{1}{\sqrt{3}}\left(N_m + \theta^+\alpha_m + \theta^-\beta_m + \theta^+\theta^-H_m\right) \\ &\times\left(\frac{1}{N_m} - \theta^+\frac{\alpha_m}{N_m^2} - \theta^-\frac{\beta_m}{N_m^2} - \theta^+\theta^-\left(\frac{2\alpha_m\beta_m}{N_m^3} + \frac{H_m}{N_m^2}\right)\right), \end{aligned}$$

for $m = 1,2,3$, where $\langle\hat{N}, \hat{N}\rangle = 1$ holds. The covariant derivatives $D_+$ and $D_-$ have the property that they anticommute with the differential supersymmetry operators

$$J_+ = \partial_{\theta^+} + i\theta^+\partial_{x_+}, \qquad J_- = \partial_{\theta^-} + i\theta^-\partial_{x_-}, \tag{B.4.11}$$

which generate the SUSY transformations

$$x \to x_+ + i\underline{\eta}_1\theta^+, \qquad \theta^+ \to \theta^+ + i\underline{\eta}_1, \tag{B.4.12}$$

and

$$t \to x_- + i\underline{\eta}_2\theta^-, \qquad \theta^- \to \theta^- + i\underline{\eta}_2, \tag{B.4.13}$$

respectively. Here $\underline{\eta}_1$ and $\underline{\eta}_2$ are odd-valued parameters. The four operators, $D_+$, $D_-$, $J_+$ and $J_-$ satisfy the anticommutation relations

$$\begin{gathered}\{J_n, J_m\} = 2i\delta_{mn}\partial_{x_m}, \qquad \{D_m, D_n\} = -2i\delta_{mn}\partial_{x_m},\\ \{J_m, D_n\} = 0, \qquad m,n = 1,2\end{gathered} \tag{B.4.14}$$

where $\delta_{ij}$ is the Kronecker delta function and $\{\cdot,\cdot\}$ denotes the anticommutator, unless otherwise noted. Here, the values 1 and 2 of the indices $m$ and $n$ stand for $+$ and $-$, respectively. Therefore we have the following relations

$$D_\pm^2 = -i\partial_\pm, \qquad J_\pm^2 = i\partial_\pm. \tag{B.4.15}$$

We now derive the SUSY version of the GW equations. Due to the normalization (B.4.4) and (B.4.10), the moving frame on a surface in the superspace satisfies the following GW equations

$$D_+\Omega = A_+\Omega, \qquad D_-\Omega = A_-\Omega, \tag{B.4.16}$$

where the moving frame $\Omega$ is denoted by

$$\Omega = \begin{pmatrix} D_+F \\ D_-F \\ N \end{pmatrix}. \tag{B.4.17}$$

Here, the first two components of $\Omega$, $D_+F$ and $D_-F$, are fermionic functions while the third component, $N$, is a bosonic function. In order to derive the SUSY GW equations, we assume that we can decompose the second-order covariant derivatives of $F$ and first-order derivatives of $N$ in terms of the tangent vectors $D_+F$ and $D_-F$ and the unit normal $N$,

$$\begin{gathered} D_jD_iF = \Gamma_{ij}{}^{k} D_kF + b_{ij}fN,\\ D_iN = b^k{}_i D_kF + \omega_i N, \end{gathered} \qquad i,j,k = 1,2 \tag{B.4.18}$$

where the coefficients $\omega_i$ and $\Gamma_{ij}{}^{k}$ are fermionic functions, while $b_{ij}$ and $b^k{}_i$ are bosonic functions. Here, the values 1 and 2 of the indices $i$, $j$ and $k$ stand for $+$ and $-$, respectively. We make use of the identities

$$\begin{aligned} 0 &= D_i\langle N,N\rangle = \langle D_iN,N\rangle + \langle N,D_iN\rangle = 2\omega_i\langle N,N\rangle = 2\omega_i,\\ D_k(\frac{1}{2}e^\phi f) &= D_k\langle D_+F, D_-F\rangle = \langle D_kD_+F, D_-F\rangle - \langle D_+F, D_kD_-F\rangle\\ &= \Gamma_{1k}{}^{1}\langle D_+F, D_-F\rangle + \Gamma_{2k}{}^{2}\langle D_+F, D_-F\rangle, \end{aligned} \tag{B.4.19}$$

from which we obtain

$$\omega_i = 0, \qquad D_kf = (\Gamma_{1k}{}^{1} + \Gamma_{2k}{}^{2} - D_k\phi)f, \tag{B.4.20}$$

and the compatibility condition for the bodiless bosonic function $f$ in equation (B.4.20) is

$$\{D_+, D_-\}f = \left(D_-{\Gamma_{11}}^1 + D_-{\Gamma_{21}}^2 + D_+{\Gamma_{12}}^1 + D_+{\Gamma_{22}}^2\right) f = 0. \tag{B.4.21}$$

The conformally parametrized surface $\mathcal{S}$ satisfies the normalization conditions (B.4.4) and (B.4.10) for the superfields $F$ and $N$, and we define the bosonic quantities $Q^+, Q^-$ and $H$ to be

$$b_{11} = Q^+, \qquad b_{12} = \frac{1}{2}e^{\phi}H, \qquad b_{21} = -\frac{1}{2}e^{\phi}H, \qquad b_{22} = Q^-, \tag{B.4.22}$$

which gives the relations

$$\langle D_+^2 F, N\rangle = Q^+ f, \qquad \langle D_- D_+ F, N\rangle = \frac{1}{2}e^{\phi}Hf, \qquad \langle D_-^2 F, N\rangle = Q^- f. \tag{B.4.23}$$

Here the $b_{ij}$ have the property

$$b_{ij} = -b_{ji}, \qquad \text{for} \qquad i \neq j, \tag{B.4.24}$$

and are the coefficients of the second fundamental form. To obtain the bosonic functions ${b^k}_j$, we make use of the relation

$$\langle D_j D_i F, N\rangle = D_j\langle D_i F, N\rangle + \langle D_i F, D_j N\rangle = \langle D_i F, D_j N\rangle, \tag{B.4.25}$$

and by substituting $D_j N$ into its decomposition (B.4.18) we get the relation

$$\left(g_{ik}{b^k}_j - b_{ij}\right) f = 0. \tag{B.4.26}$$

Hence the functions ${b^k}_j$ take the form

$${b^1}_1 = H, \qquad {b^2}_1 = 2e^{-\phi}Q^+, \qquad {b^1}_2 = -2e^{-\phi}Q^-, \qquad {b^2}_2 = H, \tag{B.4.27}$$

up to an additional bosonic bodiless function $l \neq 0$ such that $lf = 0$ and where the ${b^k}_j$ are the mixed coefficients of the second fundamental form. By construction, the Christoffel symbols of the second kind ${\Gamma_{ij}}^k$ have the property

$${\Gamma_{ij}}^k = -{\Gamma_{ji}}^k, \qquad \text{for } i \neq j. \tag{B.4.28}$$

Hence we define the Christoffel symbols of the first kind $\Gamma_{ijk}$ to be

$$\Gamma_{ijk} f = \langle D_j D_i F, D_k\rangle, \tag{B.4.29}$$

so that the relations between the Christoffel symbols of the first and second kinds are

$$(\Gamma_{ijk} - {\Gamma_{ij}}^l g_{lk}) f = 0, \tag{B.4.30}$$

or

$$\Gamma_{ijk} = \Gamma_{ij}{}^{l} g_{lk}, \tag{B.4.31}$$

up to an additional fermionic function $\zeta \neq 0$ such that $\zeta f = 0$. Therefore, the Christoffel symbols of the first kind satisfy the property

$$\Gamma_{ijk} = -\Gamma_{jik}, \qquad \text{for } i \neq j. \tag{B.4.32}$$

The matrices $A_+$ and $A_-$ of the SUSY GW equation (B.4.16) are in the Bianchi form [**15**]

$$A_i = \begin{pmatrix} \Gamma_{1i}{}^{1} & \Gamma_{1i}{}^{2} & b_{1i} f \\ \Gamma_{2i}{}^{1} & \Gamma_{2i}{}^{2} & b_{2i} f \\ b^1{}_i & b^2{}_i & 0 \end{pmatrix}, \qquad i = 1, 2, \tag{B.4.33}$$

and therefore the GW equations (B.4.16) take the form

$$\begin{aligned} & D_+\Omega = A_+\Omega, \qquad D_-\Omega = A_-\Omega, \\ & A_+ = \begin{pmatrix} R^+ & R^- & Q^+ f \\ -S^+ & -S^- & -\frac{1}{2}e^{\phi} H f \\ H & 2e^{-\phi} Q^+ & 0 \end{pmatrix}, \\ & A_- = \begin{pmatrix} S^+ & S^- & \frac{1}{2}e^{\phi} H f \\ T^+ & T^- & Q^- f \\ -2e^{-\phi} Q^- & H & 0 \end{pmatrix}, \end{aligned} \tag{B.4.34}$$

where we define the fermionic functions $R^+ = \Gamma_{11}{}^{1}$, $R^- = \Gamma_{11}{}^{2}$, $S^+ = \Gamma_{12}{}^{1}$, $S^- = \Gamma_{12}{}^{2}$, $T^+ = \Gamma_{22}{}^{1}$ and $T^- = \Gamma_{22}{}^{2}$. The compatibility condition of the GW equations (B.4.34) is

$$\begin{aligned} \{D_+, D_-\}\Omega = & D_+(A_-\Omega) + D_-(A_+\Omega), \\ = & D_+A_-\Omega + \begin{pmatrix} -R^+ & -R^- & Q^+ f \\ S^+ & S^- & -\frac{1}{2}e^{\phi} H f \\ H & 2e^{-\phi} Q^+ & 0 \end{pmatrix} D_+\Omega \\ & + D_-A_+\Omega + \begin{pmatrix} -S^+ & -S^- & \frac{1}{2}e^{\phi} H f \\ -T^+ & -T^- & Q^- f \\ -2e^{-\phi} Q^- & H & 0 \end{pmatrix} D_-\Omega \\ = & D_+A_-\Omega - EA_-ED_+\Omega + D_-A_+\Omega - EA_+ED_-\Omega. \end{aligned}$$

So we have

$$D_+A_- + D_-A_+ - \{EA_+, EA_-\} = 0, \tag{B.4.35}$$

where

$$E = \pm \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{B.4.36}$$

The matrices $A_+$ and $A_-$ of the GW equations can also be written in the Bianchi form using matrix subblock notation

$$A_+ = \left(\begin{array}{cc|c} R^+ & R^- & Q^+ f \\ -S^+ & -S^- & -\frac{1}{2}e^{\phi}Hf \\ \hline H & 2e^{-\phi}Q^+ & 0 \end{array}\right) = \left(\begin{array}{c|c} A_f^+ & I_{b_1}^+ \\ \hline I_{b_2}^+ & 0 \end{array}\right), \tag{B.4.37}$$

$$A_- = \left(\begin{array}{cc|c} S^+ & S^- & \frac{1}{2}e^{\phi}Hf \\ T^+ & T^- & Q^- f \\ \hline -2e^{-\phi}Q^- & H & 0 \end{array}\right) = \left(\begin{array}{c|c} A_f^- & I_{b_1}^- \\ \hline I_{b_2}^- & 0 \end{array}\right), \tag{B.4.38}$$

where $A_f^+$ and $A_f^-$ are $2 \times 2$ matrices with fermionic entries, $I_{b_1}^+$ and $I_{b_1}^-$ are two-component column vectors with bosonic entries, and $I_{b_2}^+$ and $I_{b_2}^-$ are two-component row vectors with bosonic entries.

Indeed, let us consider a moving frame $\Psi = (\psi_f, \psi_b)$ where $\psi_f$ is a two-component fermionic vector and $\psi_b$ is a bosonic scalar. From the GW equation for the moving frame $\Omega$, with the matrices given by (B.4.37) and (B.4.38), we obtain

$$D_+\Psi = A_+\Psi, \qquad D_-\Psi = A_-\Psi. \tag{B.4.39}$$

The compatibility conditions for $\psi_f$ and $\psi_b$ lead us to the four equations

$$\begin{gathered} D_+A_f^- + D_-A_f^+ + I_{b_1}^- I_{b_2}^+ + I_{b_1}^+ I_{b_2}^- - \{A_f^+, A_f^-\} = 0, \\ -A_f^- I_{b_1}^+ + D_+ I_{b_1}^- + I_{b_1}^- \eta_f^+ - A_f^+ I_{b_1}^- + D_- I_{b_1}^+ + I_{b_1}^+ \eta_f^- = 0, \\ D_+ I_{b_2}^- + I_{b_2}^- A_f^+ - \eta_f^- I_{b_2}^+ + D_- I_{b_2}^+ + I_{b_2}^+ A_f^- - \eta_f^+ I_{b_2}^- = 0, \\ I_{b_2}^+ I_{b_1}^- + D_-\eta_f^+ + I_{b_2}^- I_{b_1}^+ + D_+\eta_f^- = 0. \end{gathered} \tag{B.4.40}$$

The ZCC corresponding to the equations (B.4.40) is an equivalent matrix form of (B.4.35).

The zero curvature condition (B.4.35) leads us to the SUSY GC equations which consist of the following six linearly independent equations for the matrix components

$$
\begin{aligned}
&(i) && D_-R^+ + D_+T^- + D_+S^+ - D_-S^- = 0,\\
&(ii) && D_-R^+ - R^-T^+ + D_+S^+ + S^-S^+ + \frac{1}{2}H^2e^{\phi}f\\
& && \quad - 2Q^+Q^-e^{-\phi}f = 0,\\
&(iii) && Q^+T^- - R^-Q^- + D_-Q^+ - Q^+D_-\phi + \frac{1}{2}e^{\phi}D_+H = 0,\\
&(iv) && Q^-R^+ - T^+Q^+ + D_+Q^- - Q^-D_+\phi - \frac{1}{2}e^{\phi}D_-H = 0,\\
&(v) && D_-R^- - S^+R^- - R^-T^- - R^+S^- + D_+S^- + 2Q^+Hf = 0,\\
&(vi) && D_+T^+ + S^-T^+ - T^+R^+ + T^-S^+ - D_-S^+ + 2Q^-Hf = 0.
\end{aligned}
\tag{B.4.41}
$$

The Grassmann-valued partial differential equations (B.4.41) involve eleven dependent functions of the independent variables $x_+$, $x_-$, $\theta^+$ and $\theta^-$ including the four bosonic functions $\phi$, $H$, $Q^\pm$ and the six fermionic functions $R^\pm$, $S^\pm$ , $T^\pm$ together with one dependent bodiless bosonic function $f$ of $x_+$ and $x_-$. It is interesting to note that the equation (B.4.41.i) is the compatibility condition of the function $f$ given in equation (B.4.21). Under the above assumptions we obtain the following result.

**Proposition B.4.1** (Structural SUSY equations for a moving frame on a surface)**.** *For any bosonic superfields $F(x_+, x_-, \theta^+, \theta^-)$ and $N(x_+, x_-, \theta^+, \theta^-)$ satisfying the normalization conditions* (B.4.4)*,* (B.4.5)*,* (B.4.10) *and* (B.4.23)*, the moving frame $\Omega = (D_+F, D_-F, N)^T$ on a surface immersed in the superspace $\mathbb{R}^{(2,1|2)}$ satisfies the SUSY GW equations* (B.4.34)*. The ZCC* (B.4.35)*, which is the compatibility condition of the SUSY GW equations* (B.4.34) *expressed in terms of the matrices $A_+$ and $A_-$, is equivalent to the SUSY GC equations* (B.4.41)*.*

## B.5. Geometric aspects of conformally parametrized supersymmetric surfaces

In this section, we discuss certain aspects of Grassmann variables in conjunction with differential geometry and supersymmetry analysis. We begin by defining the following differential superspace fermionic operators

$$
d_\pm = \frac{1}{2}\left[d\theta^\pm + i dx_\pm \partial_{\theta^\pm}\right], \tag{B.5.1}
$$

where $d_+$ and $d_-$ are the infinitesimal displacements in the direction of $D_+$ and $D_-$, respectively. These operators are anticommuting, i.e. $\{d_+, d_-\} = 0$. For SUSY conformally parametrized surfaces, the first fundamental form is given by

$$I = \left\langle \begin{pmatrix} d_+ & d_- \end{pmatrix}, \begin{pmatrix} d_+ & d_- \end{pmatrix} \begin{pmatrix} \langle D_+F, D_+F\rangle & \langle D_-F, D_+F\rangle \\ \langle D_+F, D_-F\rangle & \langle D_-F, D_-F\rangle \end{pmatrix} \right\rangle. \tag{B.5.2}$$

Making use of the normalization of the tangent vectors $D_\pm F$ given by (B.4.4), we have

$$\begin{aligned} I =& \left\langle \begin{pmatrix} d_+ & d_- \end{pmatrix}, \begin{pmatrix} d_+ & d_- \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{2} f e^{\phi} \\ \frac{1}{2} f e^{\phi} & 0 \end{pmatrix} \right\rangle \\ =& -\frac{1}{2} f d_- d_+ e^{\phi} + \frac{1}{2} f d_+ d_- e^{\phi} \\ =& f d_+ d_- e^{\phi} = 2 f d_+ d_- g_{12}. \end{aligned} \tag{B.5.3}$$

In this SUSY case, the discriminant is defined to be

$$g = g_{11} g_{22} - g_{12} g_{21} = \frac{1}{4} e^{2\phi}. \tag{B.5.4}$$

The elements of the inverse metric form defined as

$$g_{ij} g^{jk} = \delta_i^k, \qquad \begin{pmatrix} g_{11} & g_{21} \\ g_{12} & g_{22} \end{pmatrix} \begin{pmatrix} g^{11} & g^{21} \\ g^{12} & g^{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{B.5.5}$$

are

$$g^{11} = g^{22} = 0, \qquad g^{12} = -g^{21} = -2e^{-\phi}. \tag{B.5.6}$$

The SUSY version of the second fundamental form is

$$I\!I = \left\langle \begin{pmatrix} d_+ & d_- \end{pmatrix}, \begin{pmatrix} d_+ & d_- \end{pmatrix} \begin{pmatrix} \langle D_+^2 F, N\rangle & \langle D_+ D_- F, N\rangle \\ \langle D_- D_+ F, N\rangle & \langle D_-^2 F, N\rangle \end{pmatrix} \right\rangle. \tag{B.5.7}$$

By virtue of the normalization of the tangent vectors $D_\pm F$ and the unit normal $N$, given by (B.4.4) and (B.4.10) respectively, we have

$$\begin{aligned} I\!I =& \left\langle \begin{pmatrix} d_+ & d_- \end{pmatrix}, \begin{pmatrix} d_+ & d_- \end{pmatrix} \begin{pmatrix} Q^+ f & -\frac{1}{2} H e^{\phi} f \\ \frac{1}{2} H e^{\phi} f & Q^- f \end{pmatrix} \right\rangle \\ =& f(d_+^2 Q^+ - \frac{1}{2} d_- d_+ (He^{\phi}) + \frac{1}{2} d_+ d_- (He^{\phi}) + d_-^2 Q^-) \\ =& f(d_+^2 Q^+ + d_+ d_- (He^{\phi}) + d_-^2 Q^-) = f\left(d_+^2 b_{11} + 2 d_+ d_- b_{12} + d_-^2 b_{22}\right). \end{aligned} \tag{B.5.8}$$

In order to compute the first and second fundamental forms, we have assumed that $(d\theta^j \lrcorner\, \partial_{\theta^i}) = 0$, for $i, j = 1, 2$. In the SUSY case, the discriminant is defined to be

$$b = b_{11} b_{22} - b_{12} b_{21} = Q^+ Q^- + \frac{1}{4} H^2 e^{2\phi}. \tag{B.5.9}$$

From the first and second fundamental forms, we can determine the SUSY version of the Gaussian curvature $\mathcal{K}$ and the mean curvature $H$ as follows

$$\mathcal{K} = \det(SR^{-1}) = \frac{b}{g} = 4Q^+Q^-e^{-2\phi} + H^2, \qquad H = \frac{1}{2}tr(SR^{-1}), \qquad \text{(B.5.10)}$$

where the matrices $R$ and $S$ are similar to the matrices in equations (B.5.3) and (B.5.8) respectively (without the function $f$) and can be written as

$$R = \begin{pmatrix} g_{11} & g_{21} \\ g_{12} & g_{22} \end{pmatrix} = \frac{1}{2}e^{\phi}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{pmatrix} = \begin{pmatrix} Q^+ & -\frac{1}{2}He^{\phi} \\ \frac{1}{2}e^{\phi}H & Q^- \end{pmatrix}.$$

The determinant is well-defined for $R$ and $S$ since both $R$ and $S$ are bosonic-valued matrices. Based on the SUSY version of the GC equations (B.4.41) we can provide a SUSY analogue of the Bonnet Theorem.

**Proposition B.5.1** (Supersymmetric extension of the Bonnet theorem)**.** *Given a SUSY conformal metric*

$$M = fd_+d_-e^{\phi} \qquad \text{(B.5.11)}$$

*of a conformally parametrized surface $\mathcal{S}$, the Hopf differentials $d_\pm^2 Q^\pm$ and a mean curvature function $H$ defined on a Riemann surface $\mathcal{R}$ satisfying the GC equations* (B.4.41)*, there exists a vector-valued bosonic immersion function*

$$F = (F_1, F_2, F_3) : \tilde{\mathcal{R}} \to \mathbb{R}^{(2,1|2)}, \qquad \text{(B.5.12)}$$

*with the fundamental forms*

$$I = fd_+d_-e^{\phi}, \qquad I\!I = f(d_+^2Q^+ + d_+d_-(He^{\phi}) + d_-^2Q^-), \qquad \text{(B.5.13)}$$

*where $\tilde{\mathcal{R}}$ is the universal covering of the Riemann surface $\mathcal{R}$ and $\mathbb{R}^{(2,1|2)}$ is the superspace. The immersion function $F$ is unique up to affine transformations in the superspace $\mathbb{R}^{(2,1|2)}$.*

The proof of this proposition is analogous to that given in [**21**]. Note that it is straightforward to construct surfaces on the superspace $\mathbb{R}^{(2,1|2)}$ related to integrable equations. However, it is nontrivial to identify those surfaces which have an invariant geometrical characterization. A list of such surfaces is known in the classical case [**19**] but, to our knowledge, an identification of such surfaces is an open problem in the case of surfaces immersed in the superspace.

## B.6. Symmetries of the supersymmetric Gauss-Codazzi equations

A symmetry supergroup $G$ of a SUSY system is a (local) supergroup of transformations acting on a Cartesian product of supermanifolds $X \times \mathcal{U}$, where $X$ is the space of four independent variables $(x_+, x_-, \theta^+, \theta^-)$ and $\mathcal{U}$ is the space of eleven dependent superfields $(\phi, H, Q^+, Q^-, R^+, R^-,^+, S^-, T^+, T^-, f)$. The action of the group $G$ on the functions $\phi$, $H$, $Q^+$, $Q^-$, $R^+$, $R^-$, $S^+$, $S^-$, $T^+$, $T^-$ and $f$ of $(x_+, x_-, \theta^+, \theta^-)$ maps solutions of (B.4.41) into solution of (B.4.41). Strictly speaking, the bodiless bosonic function $f$ introduced in (B.4.4) depends only on $x_+$ and $x_-$, which has to be taken into consideration when we perform the symmetry reductions. If we assume that $G$ is a Lie supergroup as described in [**30**, **80**, **120**], one can associate it with its Lie superalgebra $\mathfrak{g}$ whose elements are infinitesimal symmetries of (B.4.41). The SUSY GC equations (B.4.41) are invariant under the Lie superalgebra $\mathfrak{g}$ generated by the following eight infinitesimal vector fields

$$\begin{aligned}
C_0 &= H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} - 2f\partial_f, \\
K_0 &= -H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} + 2\partial_\phi, \\
K_1 &= -2x_+\partial_{x_+} - \theta^+\partial_{\theta^+} + R^+\partial_{R^+} + 2R^-\partial_{R^-} + S^-\partial_{S^-} \\
&\quad -T^+\partial_{T^+} + 2Q^+\partial_{Q^+} + \partial_\phi, \\
K_2 &= -2x_-\partial_{x_-} - \theta^-\partial_{\theta^-} - R^-\partial_{R^-} + S^+\partial_{S^+} + 2T^+\partial_{T^+} \\
&\quad +T^-\partial_{T^-} + 2Q^-\partial_{Q^-} + \partial_\phi, \\
P_+ &= \partial_{x_+}, \qquad P_- = \partial_{x_-}, \\
J_+ &= \partial_{\theta^+} + i\theta^+\partial_{x_+}, \qquad J_- = \partial_{\theta^-} + i\theta^-\partial_{x_-}.
\end{aligned} \tag{B.6.1}$$

The generators $P_+$ and $P_-$ represent translations in the bosonic variables $x_+$ and $x_-$ while $K_1$, $K_2$, $K_0$ and $C_0$ generate dilations on both even and odd variables. In addition, we recover the supersymmetry transformations $J_+$ and $J_-$ which were identified previously in (B.4.11). In order to determine this superalgebra of infinitesimal symmetries, we have made use of the theory described in the book by P. Olver [**95**].

The commutation (anticommutation in the case of two fermionic operators) relations of the superalgebra $\mathfrak{g}$ of the SUSY GC equations (B.4.41) are given in table B.1 for the case $D_\pm f \neq 0$.

The Lie superalgebra $\mathfrak{g}$ can be decomposed into the following combination of direct and semi-direct sums

$$\mathfrak{g} = \{\{K_1\} \ni \{P_+, J_+\}\} \oplus \{\{K_2\} \ni \{P_-, J_-\}\} \oplus \{K_0\} \oplus \{C_0\}. \tag{B.6.2}$$

In equation (B.6.2) the braces $\{\cdot, ..., \cdot\}$ denote the set of generators listed in (B.6.1). It should be noted that $K_0$ and $C_0$ constitute the center of the Lie superalgebra $\mathfrak{g}$.

TABLE B.1. Commutation table for the Lie superalgebra $\mathfrak{g}$ spanned by the vector fields (B.6.1). In the case of two fermionic generator $J_+$ and/or $J_-$ we have anticommutation rather than commutation.

| | $K_1$ | $P_+$ | $J_+$ | $K_2$ | $P_-$ | $J_-$ | $K_0$ | $C_0$ |
|---|---|---|---|---|---|---|---|---|
| $K_1$ | 0 | $2P_+$ | $J_+$ | 0 | 0 | 0 | 0 | 0 |
| $P_+$ | $-2P_+$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $J_+$ | $-J_+$ | 0 | $2iP_+$ | 0 | 0 | 0 | 0 | 0 |
| $K_2$ | 0 | 0 | 0 | 0 | $2P_-$ | $J_-$ | 0 | 0 |
| $P_-$ | 0 | 0 | 0 | $-2P_-$ | 0 | 0 | 0 | 0 |
| $J_-$ | 0 | 0 | 0 | $-J_-$ | 0 | $2iP_-$ | 0 | 0 |
| $K_0$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $C_0$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

## B.7. One-dimensional subalgebras of the symmetry superalgebra of the supersymmetric Gauss-Codazzi equations

In this section, we perform a classification of the one-dimensional subalgebras of the Lie superalgebra of infinitesimal transformations $\mathfrak{g}$ into conjugacy classes under the action of the Lie supergroup $\exp(\mathfrak{g})$ generated by (B.6.1). The significance of such a classification resides in the fact that conjugate subgroups necessarily lead to invariant solutions which are equivalent in the sense that they can be transformed from one to the other by a suitable symmetry. Therefore, it is not necessary to compute reductions with respect to algebras which are conjugate to each other.

The significance of the algebra $\mathfrak{g}$ resides in the following facts. It would be inconsistent to consider the $\mathbb{R}$ or $\mathbb{C}$ span of the generators (B.6.1) because we multiply the odd generators $J_+$ and $J_-$ by the odd parameters $\underline{\mu}$ and $\underline{\eta}$ respectively in equation (B.7.4). Therefore, one is naturally led to consider $\mathfrak{g}$ which is a supermanifold in the sense presented in section B.2. This means that $\mathfrak{g}$ contains sums of any even combinations of $P_+$, $P_-$, $K_1$, $K_2$, $K_0$ and $C_0$ (i.e. multiplied by even parameters in $\mathfrak{g}$ including real or complex numbers) and odd combinations of $J_+$ and $J_-$ (i.e. multiplied by odd parameters in $\Lambda_{odd}$). At the same time $\mathfrak{g}$ is a $\Lambda_{even}$ Lie module. This fact can lead to the following complication. For a given $X \in \mathfrak{g}$, the subalgebras $\mathfrak{X}$ and $\mathfrak{X}'$ spanned by $X$ and $X' = aX$ with $a \in \Lambda_{even} \backslash \mathbb{C}$ are not isomorphic in general, i.e. $\mathfrak{X}' \subset \mathfrak{X}$.

Note that the subalgebras obtained from other ones through multiplication by bodiless elements of $\Lambda_{even}$ do not provide us with anything new for the purpose of symmetry reduction. These subalgebras may allow for more freedom in the choice of invariants, but we then encounter the problem of non-standard invariants [**67**–**69**] which are discussed at the end of this section. Note also that it does not appear to be particularly useful to consider a subalgebra of the form e.g. $\{P_+ + \underline{\eta}_1\underline{\eta}_2 P_-\}$.

In what follows, we will assume throughout the computation of the non-isomorphic one-dimensional subalgebras that the nonzero bosonic parameters are invertible (i.e. behave essentially like ordinary real or complex numbers.) In order to classify the Lie superalgebra (B.6.2) under the action of the supergroup generated by $\mathfrak{g}$, we make use of the techniques for classifying direct and semi-direct sums of algebras described in [**120**] and generalize them to superalgebras involving both even and odd generators. In the case of direct sums, we use the Goursat twist method generalized to the case of a superalgebra. Here the superalgebra (B.6.2) contains two isomorphic copies of the 3-dimensional algebra $\mathfrak{g}_1 = \{\{K_1\} \supset\!\!\!\!\!\!+ \{P_+, J_+\}\}$ (the other copy being $\mathfrak{g}_2 = \{\{K_2\} \supset\!\!\!\!\!\!+ \{P_-, J_-\}\}$) together with the one-dimensional algebras $\{K_0\}$ and $\{C_0\}$ which constitute the center of the Lie superalgebra $\mathfrak{g}$. This fact allows us to adapt the classification for 3-dimensional algebras as described in [**97**]. So we begin our classification by considering the twisted one-dimensional subalgebras of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$. Under the action of a one-parameter group generated by the vector field

$$X = \alpha K_1 + \beta P_+ + \underline{\eta} J_+ + \delta K_2 + \lambda P_- + \underline{\rho} J_-, \tag{B.7.1}$$

where $\alpha, \beta, \delta, \lambda \in \Lambda_{\text{even}}$ and $\underline{\eta}, \underline{\rho} \in \Lambda_{\text{odd}}$, the one-dimensional subalgebra

$$Y = P_+ + aP_-, \qquad a \in \Lambda_{\text{even}}$$

transforms under the Baker-Campbell-Hausdorff formula

$$Y \to \text{Ad}_{\exp(X)} Y = Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] + \frac{1}{3!}[X, [X, [X, Y]]] + \ldots \tag{B.7.2}$$

to $e^{-2\alpha} P_+ + e^{-2\delta} aP_-$. Hence we get that $\{P_+ + aP_-\}$ is isomorphic to $\{P_+ + e^{2\alpha - 2\delta} aP_-\}$. By a suitable choice of $\alpha$ and $\delta$, the factor $e^{2\alpha-2\delta} a$ can be re-scaled to either 1 or $-1$. Hence, we obtain a twisted subalgebra $L_{14} = \{P_+ + \epsilon P_-, \epsilon = \pm 1\}$.

As another example, consider a twisted subalgebra of the form $\{P_+ + aK_2,\ a \neq 0\}$, where $a \in \Lambda_{even}$. Through the Baker-Campbell-Hausdorff formula (B.7.2), the vector field $Y = K_2 + aP_+$ transforms (through the vector field $X$ given in (B.7.1)) to

$$e^X Y e^{-X} = K_2 + e^{-2\alpha} aP_+ - \frac{\lambda}{\delta}(e^{-2\delta} - 1)P_- - \frac{1}{\delta}(e^{-\delta} - 1)\underline{\rho} J_-. \tag{B.7.3}$$

Through a suitable choice of $\underline{\lambda}$ and $\underline{\rho}$, the last two terms of (B.7.3) can be eliminated, so we obtain the twisted subalgebra $L_{13} = \{K_2 + \epsilon P_+, \epsilon = \pm 1\}$.

Continuing the classification in an analogous way, we obtain the following list of one-dimensional subalgebras

$$
\begin{aligned}
&L_1 = \{K_1\}, && L_{26} = \{K_1 + aK_0\},\\
&L_2 = \{P_+\}, && L_{27} = \{K_0 + \epsilon P_+\},\\
&L_3 = \{\underline{\mu} J_+\}, && L_{28} = \{K_0 + \underline{\mu} J_+\},\\
&L_4 = \{P_+ + \underline{\mu} J_+\}, && L_{29} = \{K_0 + \epsilon P_+ + \underline{\mu} J_+\},\\
&L_5 = \{K_2\}, && L_{30} = \{K_2 + aK_0\},\\
&L_6 = \{P_-\}, && L_{31} = \{K_0 + \epsilon P_-\},\\
&L_7 = \{\underline{\nu} J_-\}, && L_{32} = \{K_0 + \underline{\nu} J_-\},\\
&L_8 = \{P_- + \underline{\nu} J_-\}, && L_{33} = \{K_0 + \epsilon P_- + \underline{\nu} J_-\},\\
&L_9 = \{K_1 + aK_2\}, && L_{34} = \{K_1 + aK_2 + bK_0\},\\
&L_{10} = \{K_1 + \epsilon P_-\}, && L_{35} = \{K_1 + aK_0 + \epsilon P_-\},\\
&L_{11} = \{K_1 + \underline{\nu} J_-\}, && L_{36} = \{K_1 + aK_0 + \underline{\nu} J_-\},\\
&L_{12} = \{K_1 + \epsilon P_- + \underline{\nu} J_-\}, && L_{37} = \{K_1 + aK_0 + \epsilon P_- + \underline{\nu} J_-\},\\
&L_{13} = \{K_2 + \epsilon P_+\}, && L_{38} = \{K_2 + aK_0 + \epsilon P_+\},\\
&L_{14} = \{P_+ + \epsilon P_-\}, && L_{39} = \{K_0 + \epsilon_1 P_+ + \epsilon_2 P_-\},\\
&L_{15} = \{P_+ + \underline{\nu} J_-\}, && L_{40} = \{K_0 + \epsilon P_+ + \underline{\nu} J_-\},\\
&L_{16} = \{P_+ + \epsilon P_- + \underline{\nu} J_-\}, && L_{41} = \{K_0 + \epsilon_1 P_+ + \epsilon_2 P_- + \underline{\nu} J_-\},\\
&L_{17} = \{K_2 + \underline{\mu} J_+\}, && L_{42} = \{K_2 + aK_0 + \underline{\mu} J_+\},\\
&L_{18} = \{P_- + \underline{\mu} J_+\}, && L_{43} = \{K_0 + \epsilon P_- + \underline{\mu} J_+\},\\
&L_{19} = \{\underline{\mu} J_+ + \underline{\nu} J_-\}, && L_{44} = \{K_0 + \underline{\mu} J_+ + \underline{\nu} J_-\},\\
&L_{20} = \{P_- + \underline{\mu} J_+ + \underline{\nu} J_-\}, && L_{45} = \{K_0 + \epsilon P_- + \underline{\mu} J_+ + \underline{\nu} J_-\},\\
&L_{21} = \{K_2 + \epsilon P_+ + \underline{\mu} J_+\}, && L_{46} = \{K_2 + aK_0 + \epsilon P_+ + \underline{\mu} J_+\},\\
&L_{22} = \{P_+ + \epsilon P_- + \underline{\mu} J_+\}, && L_{47} = \{K_0 + \epsilon_1 P_+ + \epsilon_2 P_- + \underline{\mu} J_+\},\\
&L_{23} = \{P_+ + \underline{\mu} J_+ + \underline{\nu} J_-\}, && L_{48} = \{K_0 + \epsilon P_+ + \underline{\mu} J_+ + \underline{\nu} J_-\},\\
&L_{24} = \{P_+ + \epsilon P_- + \underline{\mu} J_+ + \underline{\nu} J_-\}, &&\\
&L_{25} = \{K_0\}, && L_{49} = \{K_0 + \epsilon_1 P_+ + \epsilon_2 P_- + \underline{\mu} J_+ + \underline{\nu} J_-\},
\end{aligned}
\tag{B.7.4}
$$

where $\epsilon$ is either $\pm 1$, the parameters $a$ and $b$ are non-zero bosonic constants and $\underline{\mu}$ and $\underline{\nu}$ are fermionic constants. In addition to these 49 subalgebras, we also have the subalgebras obtained when, for each of the subalgebras from $L_{25}$ to $L_{49}$, the vector field $K_0$ is replaced by the vector field $C_0$ and/or by the vector $K_0 + mC_0$

for an arbitrary non-zero bosonic constant $m$. For each subalgebra $L_k$ (where $25 \leq k \leq 49$) involving $K_0$, the equivalent subalgebra where $K_0$ is replaced by $C_0$ is labelled $L'_k$, and the equivalent subalgebra where $K_0$ is replaced by $K_0 + mC_0$ is labelled $L''_k$. Since this would lengthen our list considerably we do not list them here. These representative subalgebras allow us to determine invariant solutions of the SUSY GC equations (B.4.41) using the symmetry reduction method.

In addition, we note that for certain one-dimensional subalgebras (e.g. $L_{25}$, $L'_{25}$ and $L''_{25}$), the invariants have a non-standard form in the sense that they do not lead to standard reductions or invariant solutions. Such non-standard invariants were found by the authors for several other SUSY hydrodynamic-type systems, including the SUSY polytropic gas dynamics [**67**], the SUSY sine-Gordon equation [**69**] and SUSY Klein-Gordon polynomial equations [**68**].

## B.8. Invariant solutions of the supersymmetric GC equations

We now make use of the symmetry reduction method (SRM) in order to obtain invariant solutions of the GC equations (B.4.41). For each subalgebra, the superfields $\mathcal{U} = (\phi, H, Q^+, Q^-, R^+, R^-,^+, S^-, T^+, T^-, f)$ are expanded in terms of the various invariants. The dependence of the components of $\mathcal{U}$ on each odd variable $\theta^+$ or $\theta^-$ must be at most linear as the odd variables satisfy (B.4.1). Substituting this decomposition into the GC equations (B.4.41), we obtain reduced partial differential equations for the superfields $\mathcal{U}$ which in turn lead to systems of differential constraints between their component even and odd functions. For instance, if the invariants are given by the bosonic symmetry variable $\xi$ and the fermionic symmetry variables $\eta$ and $\sigma$ (which are expressed in terms of $\theta^+$ and $\theta^-$, respectively), then $\mathcal{U}$ can be decomposed into the form

$$\begin{aligned}
&Q^{\pm} = q_0^{\pm}(\xi) + \eta q_1^{\pm}(\xi) + \sigma q_2^{\pm}(\xi) + \eta\sigma q_3^{\pm}(\xi),\\
&R^{\pm} = r_0^{\pm}(\xi) + \eta r_1^{\pm}(\xi) + \sigma r_2^{\pm}(\xi) + \eta\sigma r_3^{\pm}(\xi),\\
&S^{\pm} = s_0^{\pm}(\xi) + \eta s_1^{\pm}(\xi) + \sigma s_2^{\pm}(\xi) + \eta\sigma s_3^{\pm}(\xi),\\
&T^{\pm} = t_0^{\pm}(\xi) + \eta t_1^{\pm}(\xi) + \sigma t_2^{\pm}(\xi) + \eta\sigma t_3^{\pm}(\xi),\\
&H = h_0(\xi) + \eta h_1(\xi) + \sigma h_2(\xi) + \eta\sigma h_3(\xi),\\
&\phi = \phi_0(\xi) + \eta\phi_1(\xi) + \sigma\phi_2(\xi) + \eta\sigma\phi_3(\xi),\\
&f = \psi(\xi),
\end{aligned} \tag{B.8.1}$$

where $q_0^{\pm}, q_3^{\pm}, r_1^{\pm}, r_2^{\pm}, s_1^{\pm}, s_2^{\pm}, t_1^{\pm}, t_2^{\pm}, h_0, h_3, \phi_0, \phi_3$ and $\psi$ are even-valued functions of $\xi$ while $q_1^{\pm}, q_2^{\pm}, r_0^{\pm}, r_3^{\pm}, s_0^{\pm}, s_3^{\pm}, t_0^{\pm}, t_3^{\pm}, h_1, h_2, \phi_1$ and $\phi_2$ are odd-valued functions of $\xi$.

We now present the following three examples in order to illustrate the geometrical considerations.

**1.** For the subalgebra $L_{39} = \{P_+ + \epsilon P_- + aK_0,\ \epsilon = \pm 1,\ a \neq 0\}$, the orbit of the group of the SUSY GC equations (B.4.41) can be parametrized as follows

$$\begin{aligned}
&H = e^{-ax_+}h(\xi,\theta^+,\theta^-), && \\
&Q^+ = e^{ax_+}q^+(\xi,\theta^+,\theta^-), && S^+ = s^+(\xi,\theta^+,\theta^-),\\
&Q^- = e^{ax_+}q^-(\xi,\theta^+,\theta^-), && S^- = s^-(\xi,\theta^+,\theta^-),\\
&R^+ = r^+(\xi,\theta^+,\theta^-), && T^+ = t^+(\xi,\theta^+,\theta^-),\\
&R^- = r^-(\xi,\theta^+,\theta^-), && T^- = t^-(\xi,\theta^+,\theta^-),\\
&\phi = 2ax_+ + \varphi(\xi,\theta^+,\theta^-), && f = \psi(\xi),
\end{aligned} \tag{B.8.2}$$

where the functions $H, Q^\pm, R^\pm, S^\pm, T^\pm$ and $\phi$ are expressed in terms of the bosonic symmetry variable $\xi = x_- - \epsilon x_+$ and the fermionic symmetry variables $\theta^+$ and $\theta^-$. A corresponding invariant solution is given by

$$\begin{aligned}
H =& e^{-ax_+}\left[h_0 + \theta^+\theta^- 2il_0e^{\xi}\right],\\
Q^+ =& e^{ax_+}\Big[l_0e^{2\xi} + l_1e^{\xi}\\
&+\theta^+\theta^-\left(\frac{1}{2}ie^{\xi}(ah_0 + \epsilon(h_0)_\xi) + l_0e^{2\xi}\varphi_1 + l_1e^{\xi}\varphi_1\right)\Big],\\
Q^- =& e^{ax_+}\Big[\frac{\epsilon l_0}{a\epsilon - 1} + l_2e^{(1-a\epsilon)\xi}\\
&+\theta^+\theta^-\left(-\frac{1}{2}ie^{\xi}(h_0)_\xi + \frac{\epsilon l_0}{a\epsilon - 1}\varphi_1 + l_2e^{(1-a\epsilon)\xi}\varphi_1\right)\Big],\\
R^- =& b_1\underline{S}_0^+, \quad R^+ = b_2\underline{S}_0^+, \quad S^+ = \underline{S}_0^+, \quad S^- = \underline{S}_0^+, \quad T^- = b_3\underline{T}_0^+,\\
T^+ =& b_4\underline{S}_0^+, \quad \phi = 2ax_+ + \xi + \theta^+\theta^-\varphi_1, \qquad f = \psi,\\
l_0 =& \underline{a}_0\underline{S}_0^+, \qquad l_1 = \underline{a}_2\underline{S}_0^+, \qquad l_2 = \underline{a}_2\underline{S}_0^+, \qquad h_0 = \underline{c}_0\underline{S}_0^+,
\end{aligned} \tag{B.8.3}$$

where $h_0$, $\varphi_1$ and $\psi$ are functions of the symmetry variable $\xi = x_- - \epsilon x_+$ and where $l_0, l_1, l_2$ and $b_1, b_2, b_3, b_4$ are bosonic constants, while $\underline{S}_0^+, \underline{c}_0$ and $\underline{a}_0, \underline{a}_1, \underline{a}_2$ are fermionic constants.

The first and second fundamental forms of the surface $\mathcal{S}$ associated with (B.8.3) are given by

$$\begin{aligned}
I =&\psi d_+ d_- \left[e^{2ax_+ + \xi}\left(1+\theta^+\theta^-\varphi_1\right)\right],\\
I\!I =&\psi e^{ax_+}\Big\{d_+d_-\left[e^{\xi}\left(h_0+\theta^+\theta^-(2il_0e^{\xi}+h_0\varphi_1)\right)\right]\\
&+d_+^2\left[l_0e^{2\xi}+l_1e^{\xi}+\theta^+\theta^-\left(\frac{1}{2}ie^{\xi}(ah_0+\epsilon(h_0)_\xi)+l_0e^{2\xi}\varphi_1+l_1e^{\xi}\varphi_1\right)\right]\\
&+d_-^2\Bigg[\frac{\epsilon l_0}{a\epsilon-1}+l_2e^{(1-a\epsilon)\xi}\\
&+\theta^+\theta^-\left(-\frac{1}{2}ie^{\xi}(h_0)_\xi+\frac{\epsilon l_0}{a\epsilon-1}\varphi_1+l_2e^{(1-a\epsilon)\xi}\varphi_1\right)\Bigg]\Big\}.
\end{aligned} \tag{B.8.4}$$

The Gaussian curvature takes the form

$$\begin{aligned}
\mathcal{K} =&e^{-2ax_+}\Big[h_0^2+\theta^+\theta^-4ih_0l_0e^{\xi}\\
&+4(l_0e^{2\xi}+l_1e^{\xi})(\frac{\epsilon l_0}{a\epsilon-1}+l_2e^{(1-a\epsilon)\xi})e^{-2\xi}(1-\theta^+\theta^-2\varphi_1)\\
&+4\theta^+\theta^-(l_0e^{2\xi}+l_1e^{\xi})\left(-\frac{1}{2}ie^{\xi}(h_0)_\xi+\frac{\epsilon l_0}{a\epsilon-1}\varphi_1+l_2e^{(1-a\epsilon)\xi}\varphi_1\right)e^{-2\xi}\\
&+4\theta^+\theta^-(\frac{\epsilon l_0}{a\epsilon-1}+l_2e^{(1-a\epsilon)\xi})\\
&\times\left(\frac{1}{2}ie^{\xi}(ah_0+\epsilon(h_0)_\xi)+l_0e^{2\xi}\varphi_1+l_1e^{\xi}\varphi_1\right)e^{-2\xi}\Big].
\end{aligned} \tag{B.8.5}$$

The subalgebra of the classical GC equation (B.2.13) analogous to $L_{39}$ is $L'_{1,7} = \{e_1+\epsilon e_2+ae_0, \epsilon=\pm 1, a\neq 0\}$, whose corresponding invariant solution is given by

$$\begin{aligned}
&H(z,\bar z)=k_0v(\xi)^{-1/2}e^{a/2(\bar z-3z)}, &&Q(z,\bar z)=\frac{1}{2}k_0v(\xi)^{1/2}e^{a/2(z+\bar z)},\\
&U(z,\bar z)=e^{2az}v(\xi), &&\bar Q(z,\bar z)=\frac{1}{2}k_0v(\xi)^{1/2}e^{a/2(z+\bar z)},
\end{aligned} \tag{B.8.6}$$

where the symmetry variable is $\xi=\bar z-\epsilon z$ and the function $v$ of $\xi$ satisfies the ODE

$$v_{\xi\xi}=\frac{(v_\xi)^2}{v}+k_0^2ve^{a\xi}. \tag{B.8.7}$$

For this classical solution, the Gaussian curvature vanishes, in contrast to the SUSY case.

**2.** For the subalgebra $L'_{27} = \{C_0 + \epsilon P_+,\ \epsilon = \pm 1\}$ we obtain the following orbits of the group

$$\begin{aligned}
&H = e^{\epsilon x_+} h(x_-, \theta^+, \theta^-), \\
&Q^+ = e^{\epsilon x_+} q^+(x_-, \theta^+, \theta^-), S^+ = s^+(x_-, \theta^+, \theta^-), \\
&Q^- = e^{\epsilon x_+} q^-(x_-, \theta^+, \theta^-), S^- = s^-(x_-, \theta^+, \theta^-), \\
&R^+ = r^+(x_-, \theta^+, \theta^-), \qquad T^+ = t^+(x_-, \theta^+, \theta^-), \\
&R^- = r^-(x_-, \theta^+, \theta^-), \qquad T^- = t^-(x_-, \theta^+, \theta^-), \\
&\phi = \varphi(x_-, \theta^+, \theta^-), \qquad f = e^{-2\epsilon x_+} \psi(x_-),
\end{aligned} \tag{B.8.8}$$

where the symmetry variables are $x_-, \theta^+$ and $\theta^-$. An invariant solution of the SUSY GC equations (B.4.41) is given by

$$\begin{aligned}
&H = e^{\epsilon x_+} \left[ h_0(x_-) + h_1(x_-) \theta^+ \theta^- \right], \\
&Q^+ = e^{\epsilon x_+} \left[ B_0^+ + B_1^+ \theta^+ \theta^- \right] \psi(x_-), \\
&Q^- = e^{\epsilon x_+} \left[ B_0^- + B_1^- \theta^+ \theta^- \right] \psi(x_-), \\
&R^+ = 0, \quad R^- = 0, \quad S^+ = \underline{S}_0^+, \quad S^- = a \underline{S}_0^+, \quad T^+ = 0, \\
&T^- = 0, \quad \phi = \varphi_0(x_-) + \varphi_1(x_-) \theta^+ \theta^-, \qquad f = e^{-2\epsilon x_+} \psi(x_-),
\end{aligned} \tag{B.8.9}$$

where $\underline{S}_0^+$ is a fermionic constant while $a$, $B_0^\pm$ and $B_1^\pm$ are bosonic constants. The bosonic functions $h_0$ and $h_1$ obey the relations

$$h_0 = 2i\epsilon (B_1^+ - B_0^+ \varphi_1) e^{-\varphi_0} \psi, \qquad h_1 = 2i\epsilon B_0^- e^{-\varphi_0} \psi. \tag{B.8.10}$$

The bosonic function $\varphi_1$ is given by

$$\varphi_1 = \frac{\epsilon}{(B_0^+)^2} (B_0^- B_1^+ - B_0^+ B_1^-) x_- + k_0, \tag{B.8.11}$$

where $k_0$ is a bosonic constant and the function $\varphi_0$ obeys the ODE

$$\psi_{x_-} = \left( \frac{\epsilon B_0^-}{B_0^+} + \varphi_{0,x_-} \right) \psi, \tag{B.8.12}$$

where $\psi$ is a bodiless bosonic function of order 3 of $x_-$.

The first and second fundamental forms for this surface $\mathcal{S}$ are given by

$$\begin{aligned}
I =& d_+ d_- \left[ e^{\varphi_0 - 2\epsilon x_+} \left( 1 + \varphi_1 \theta^+ \theta^- \right) \psi \right], \\
I\!I =& d_+^2 \left[ \psi^2 e^{-\epsilon x_+} \left( B_0^+ + B_1^+ \theta^+ \theta^- \right) \right] \\
&+ d_+ d_- \left[ \psi e^{\varphi_0 - \epsilon x_+} \left( h_0 + (h_0 \varphi_1 + h_1) \theta^+ \theta^- \right) \right] \\
&+ d_-^2 \left[ \psi^2 e^{-\epsilon x_+} \left( B_0^- + B_1^- \theta^+ \theta^- \right) \right].
\end{aligned} \tag{B.8.13}$$

Consequently, the Gaussian curvature and mean curvature are not constant. The Gaussian curvature is given by

$$\mathcal{K} = e^{2\epsilon x_+}\left[\frac{\psi^2\left(B_0^+B_0^- + \left(B_0^+B_1^- + B_0^-B_1^+\right)\theta^+\theta^-\right)}{e^{2\varphi_0}\left(1+2\varphi_1\theta^+\theta^-\right)} + \left(h_0^2 + 2h_0h_1\theta^+\theta^-\right)\right]. \tag{B.8.14}$$

The umbilic points of the surface $\mathcal{S}$ occur when $\psi^2 B_0^+B_0^- = 0$ and

$$\psi^2(B_0^+B_1^- + B_0^-B_1^+) = 0. \tag{B.8.15}$$

**3.** For the subalgebra $L''_{26} = \{K_1 + (a-\frac{1}{2})K_0 + \frac{1}{2}C_0,\ a \neq \frac{1}{2}\}$ we obtain the following parametrization of the orbit of the group

$$\begin{aligned}
&H = (x_+)^{(a-1)/2}h(x_-,\eta,\theta^-),\\
&Q^+ = (x_+)^{-(a+2)/2}q^+(x_-,\eta,\theta^-), S^+ = s^+(x_-,\eta,\theta^-),\\
&Q^- = (x_+)^{-a/2}q^-(x_-,\eta,\theta^-), \qquad S^- = (x_+)^{-1/2}s^-(x_-,\eta,\theta^-),\\
&R^+ = (x_+)^{-1/2}r^+(x_-,\eta,\theta^-), \qquad T^+ = (x_+)^{1/2}t^+(x_-,\eta,\theta^-),\\
&R^- = (x_+)^{-1}r^-(x_-,\eta,\theta^-), \qquad T^- = t^-(x_-,\eta,\theta^-),\\
&e^\phi = (x_+)^{-a}\varphi(x_-,\eta,\theta^-), \qquad f = (x_+)^{1/2}\psi(x_-),
\end{aligned} \tag{B.8.16}$$

where the bosonic symmetry variable is $x_-$ and the fermionic symmetry variables are $\eta = (x_+)^{-1/2}\theta^+$ and $\theta^-$. A corresponding invariant solution of the SUSY GC equations (B.4.41) takes the form

$$\begin{aligned}
&H = 2iB(x_+)^{(a-2)/2}(\rho)_{x_-}\theta^+\theta^-,\\
&Q^+ = BA(x_-)(x_+)^{-(a+2)/2}\left[1 + (x_+)^{-1/2}\theta^+\theta^-G(x_-)\right]\rho(x_-),\\
&Q^- = \frac{2B}{a}(x_+)^{-a/2}\left[1 + (x_+)^{-1/2}\theta^+\theta^-G(x_-)\right],\\
&R^+ = (x_+)^{-1/2}l_1\underline{R}_0^+, \quad R^- = (x_+)^{-1}l_2\underline{R}_0^-, \quad S^+ = T^- = \underline{T}_0^-,\\
&S^- = T^+ = 0, \qquad f = (x_+)^{1/2}\psi(x_-),\\
&e^\phi = A(x_-)(x_+)^{-a}(1 + (x_+)^{-1/2}\theta^+\theta^-G(x_-)),
\end{aligned} \tag{B.8.17}$$

where $B = \underline{l}_0\underline{R}_0^+\underline{R}_0^-\underline{T}_0^-$ and $l_1, l_2, l_3$ are bosonic constants, while $\underline{l}_0$, $\underline{R}_0^\pm$ and $\underline{T}_0^-$ are fermionic constants. Here, $A$, $G$, $\rho$ and $\psi$ are arbitrary bosonic functions of the symmetry variable $x_-$. However, the function $A$ contains a part in $\Lambda_{\text{body}}$ but $\psi$ is a bodiless function.

The corresponding first and second fundamental forms for the surface $\mathcal{S}$ given by (B.8.17) are

$$I = \psi d_+ d_- \left[A(x_+)^{-(2a+1)/2}\left(1+(x_+)^{-1/2}\theta^+\theta^- G\right)\right], \tag{B.8.18}$$

and

$$\begin{aligned} I\!I = &(d_+)^2\left[AB(x_+)^{-(a+2)/2}\rho\left(1+(x_+)^{-1/2}\theta^+\theta^- G\right)\right] \\ &+ 2id_+d_-\left[AB(x_+)^{-1}\theta^+\theta^-\rho'\right] \\ &+ (d_-)^2\left[\frac{2B}{a}(x_+)^{-a/2}\left(1+(x_+)^{-1/2}\theta^+\theta^- G\right)\right]. \end{aligned} \tag{B.8.19}$$

Consequently, the Gaussian curvature $\mathcal{K}$ and the mean curvature $H$ of the associated surface $\mathcal{S}$ are not constant. The Gaussian curvature is given by

$$\mathcal{K} = \frac{8B}{aA}(x_+)^{a-1}\rho\left(1+(x_+)^{-1/2}\theta^+\theta^- G\right). \tag{B.8.20}$$

Since $H^2 = 0$, it follows that the surface $\mathcal{S}$ admits umbilic points along the curve defined by $\mathcal{K} = 0$, which lies on the surface $\mathcal{S}$. The subalgebra of the Lie algebra for the classical GC equation (B.2.13) analogous to subalgebra $L''_{26}$ is $L'_{1,2} = \{e_3 + ae_0\}$. The corresponding invariant solution is given by

$$\begin{aligned} &H(z,\bar z) = l_0 e^{-a(z+\bar z)}, \qquad Q(z,\bar z) = k_0 e^{a(z+\bar z)}, \\ &U(z,\bar z) = \frac{-2k_0}{l_0}e^{2a(z+\bar z)}, \qquad \bar Q(z,\bar z) = k_0 e^{a(z+\bar z)}, \qquad k_0, l_0 \in \mathbb{R}. \end{aligned} \tag{B.8.21}$$

In contrast to the SUSY case (B.8.17), the Gaussian curvature $\mathcal{K}$ vanishes for the classical solution (B.8.21) associated with the subalgebra $L'_{1,2}$. In both cases however, the mean curvature $H$ is non-zero.

## B.9. Conclusions

The objective of this paper was to construct a supersymmetric extension of the Gauss–Weingarten equations (B.4.34) and the Gauss–Codazzi equations (B.4.41) through a superspace and superfield formalism. The analysis included conformally parametrized surfaces immersed in a Grassmann superspace $\mathbb{R}^{(2,1|2)}$. This analysis allowed us to determine a Lie superalgebra of infinitesimal symmetries which generate Lie point symmetries of the SUSY GC equations (B.4.41). In addition, we also computed the Lie symmetry algebra of the classical GC equations (B.2.13) and classified the one-dimensional subalgebras of its largest finite-dimensional subalgebra into conjugacy classes. Comparing the symmetries of the SUSY GC equations (B.4.41) with those of the classical GC equations (B.2.13), we observe an additional dilation in the SUSY case. More specifically, $K_1$ and $K_2$ in (B.6.1) are supersymmetrized versions of the fields $e_3$ and $e_4$ in (B.2.17) respectively. The generators $K_0$ and $C_0$ in (B.6.1) play the role of center in the SUSY case in the same way that $e_0$ does in the classical case and we did not find any Virasoro algebras in the SUSY case. Next, we performed a group-theoretical analysis in order to classify all of the one-dimensional subalgebras of the obtained superalgebra (B.6.2) into conjugacy classes. Through the use of a generalized version of the symmetry reduction method we demonstrated for three subalgebras in (B.7.4) how to find exact invariant solutions of the SUSY model. A systematic use of the structure of the invariance supergroup of the SUSY GC equations (B.4.41) allowed us to generate (bosonic and/or fermionic) symmetry variables. For certain subalgebras, the invariants had a non-standard structure and therefore did not lead to invariant solutions. This phenomenon of non-standard invariants has also been observed in the analysis of symmetries of SUSY hydrodynamic-type equations. The SRM enabled us to reduce, after some transformations, the basic system of PDEs to many possible reduced PDE systems. We also explored certain geometrical properties of invariant solutions of the SUSY GC equations (B.4.41).

This research could be extended in several other directions. One possibility would be to compute an exhaustive list of all symmetries of the SUSY GC equations and compare them to the classical case and also to apply the above SUSY extension methods to the SUSY GC system in higher dimensions. Due to the complexity of the computations involved, this would require the development of a computer algebra Lie symmetry package capable of handling odd and even Grassmann variables. To the best of our knowledge such a package does not presently exist. Conservation laws are well-established for the construction of the classical parametrized surfaces based on the generalized Weierstrass-Enneper formula for immersion [**81**], but it has been observed that, for the GC equations, such conservation laws are broken in their corresponding SUSY extensions. The problem of determining which quantities $Q^{\pm}$, $R^{\pm}$, $S^{\pm}$, $T^{\pm}$, $H$, $\mathcal{K}$, $\phi$, $f$ are conserved by the SUSY model still remains an open question for the GC equations. Another open problem is to determine whether all integrable SUSY systems possess non-standard invariants. These topics will be investigated in our future work.

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# Annexe C

## FERMIONIC SUPERSYMMETRIC EXTENSION OF THE GAUSS–WEINGARTEN AND GAUSS–CODAZZI EQUATIONS

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Cet article a été combiné à l'article [**B7**], apparaissant dans l'appendix B, afin d'être publié dans J. Phys. A : Math. Theor. (voir le chapitre 2 ou [**B1**])

**Résumé**

Une extension supersymétrique fermionique est établie pour les équations de Gauss–Weingarten et de Gauss–Codazzi décrivant les surfaces conformément paramétrisées plongées dans un superespace de Grassmann. Une analyse de cette extension est effectuée en utilisant un formalisme de superespace/superchamp et une version supersymétrique du repère mobile sur une surface. Contrairement à l'extension supersymétrique bosonique, les équations de Gauss–Codazzi du modèle fermionique supersymétrique ressemblent à la forme classique des équations de Gauss–Codazzi. Par la suite, une superalgèbre de Lie des symétries ponctuelles de ces équations est déterminée et une classification des sous-algèbres unidimensionnelles en classes de conjugaison de cette superalgèbre est présentée. La méthode de réduction par symétrie est utilisée pour obtenir les invariants de groupe, les orbites et les systèmes réduits pour trois sous-algèbres unidimensionnelles choisies. Les solutions explicites de ces systèmes réduits correspondent à différentes surfaces plongées dans un superespace de Grassmann. Dans le cadre de cette étude, pour la version supersymétrique des équations de Gauss–Codazzi, une interprétation géométrique des résultats est discutée.

**Abstract**

A fermionic supersymmetric extension is established for the Gauss–Weingarten and Gauss–Codazzi equations describing conformally parametrized surfaces immersed in a Grassmann superspace. An analysis of this extension is performed using a superspace-superfield formalism together with a supersymmetric version of a moving frame on a surface. In contrast with the bosonic supersymmetric extension, the equations of the fermionic supersymmetric Gauss–Codazzi model resemble the form of the classical equations. Next, a superalgebra of Lie point symmetries of these equations is determined and a classification of the one-dimensional subalgebras of this superalgebra into conjugacy classes is presented. The symmetry reduction method is used to obtain group-invariants, orbits and reduced systems for three chosen one-dimensional subalgebras. The explicit solutions of these reduced systems correspond to different surfaces immersed in a Grassmann superspace. Within this framework for the supersymmetric version of the Gauss–Codazzi equations a geometrical interpretation of the results is discussed.

## C.1. Introduction

In the last three decades, a number of supersymmetric (SUSY) extensions of classical and quantum mechanical models, describing several physical phenomena, have been developed and group-invariant solutions of these SUSY systems have been found (e.g. [**13**, **37**, **76**, **79**]). Recently, this method was further generalized to encompass hydrodynamic-type systems (see e.g. [**36**, **50**, **67**]). Their SUSY extensions were established and their group-invariant solutions were constructed. Supersymmetric versions of the Chaplygin gas in (1+1)- and (2+1)-dimensions were formulated by R. Jackiw et al., derived from parametrizations of the action for a superstring and a Nambu-Goto membrane, respectively (see [**78**] and references therein). It was suggested that a quark-gluon plasma may be described by non-Abelian fluid mechanics [**117**]. In addition, SUSY extensions have been formulated for a number of soliton equations [**24**], including among others the Korteweg–de Vries equation [**83**, **88**, **90**], the Kadomtsev–Petviashvili equation [**89**], the Sawada–Kotera equation [**116**] and the sine-Gordon and sinh-Gordon equations [**3**, **63**, **64**, **107**, **108**].

Despite the progress made in the investigation of nonlinear SUSY systems, this area of mathematics does not yet have as solid a theoretical foundation as the classical theory of differential equations. This is related primarily to the fact that, due to the nature of Grassmann variables, the principle of superposition of solutions obtained from the method of characteristics cannot be applied to nonlinear SUSY systems. In most cases, analytic methods for solving quasilinear SUSY systems of equations lead to the construction of classes of solutions that are more restricted than the general solution. One can attempt to construct more restricted classes of solutions which depend on some arbitrary functions and parameters by requiring that the solutions be invariant under certain group properties of the original system. The main advantages of the group properties appear when group analysis makes it possible to construct regular algorithms for finding certain classes of solutions without referring to any additional considerations but proceeding directly from the given system of partial differential equations (PDEs). A systematic computational method for constructing the group of symmetries of a given system of PDEs has been developed by many authors (see e.g. [**30**, **95**]) and a broad review of recent developments in the SUSY case can be found in several books (e.g. J. F. Cornwell [**33**], D. S. Freed [**56**], V. Kac [**80**], V. S. Varadarajan [**118**] and B. De Witt [**43**]). The methodological approach adopted in this paper is based on the symmetry reduction method (SRM) of PDEs invariant under a SUSY Lie group of point transformations. By a symmetry group of a SUSY system of PDEs, we mean a local SUSY Lie group $G$ transforming both the independent and dependent variables of the considered SUSY system of equations in such a way that a Lie supergroup transforms given solutions of the system to new solutions. The Lie superalgebra of such a group is represented by vector fields and their prolongation structures. The standard algorithms for determining the symmetry algebra of a system of equations and classifying its subalgebras have been extended in order to deal with our SUSY models (see e.g. [**80**, **100**, **120**]).

Recent studies of the geometric properties of surfaces associated with holomorphic and nonholomorphic solutions of the SUSY bosonic Grassmann sigma models have been performed [**41**, **42**, **104**, **122**]. A gauge invariant formulation of these SUSY models in terms of orthogonal projectors allows one to obtain explicit solutions and consequently to study the geometry of their associated surfaces. To pursue this research further, it is convenient to formulate a fermionic SUSY extension of the Gauss–Weingarten (GW) and Gauss–Codazzi (GC) equations for conformally parametrized surfaces immersed in a Grassmann superspace. A similar analysis of surfaces was performed [**B7**] using a formalism of a superspace and bosonic vector superfields together with a supersymmetric version of a moving frame on a surface. The bosonic SUSY extension of the GC equations was given by six equations and this formulation allowed us to discuss in detail a geometric characterization of surfaces, including their fundamental forms and their Gaussian and mean curvatures, and consequently to establish a SUSY version of the Bonnet theorem for surfaces immersed in the superspace $\mathbb{R}^{(2,1|2)}$. The results obtained for the SUSY systems were so promising that it seemed worthwhile to try to apply the above geometric approach, already used for a SUSY extension of the GW and GC equations involving bosonic superfields [**B7**], to a similar extension of the SUSY GC equations built in terms of fermionic superfields. This is, in short, the aim of this paper.

The paper is organized as follows. In section C.2, we present certain basic notions and properties of Grassmann algebras and Grassmann variables and introduce the notation that will be used throughout this paper. In section C.3, we review relevant points from Lie's theory of symmetry groups for the GW and GC equations and apply an algorithm to isolate integrable systems and their associated surfaces. In section C.4, we construct and investigate the SUSY extension of the GW and GC equations. We show that, in contrast with the previously considered bosonic SUSY extension of the GC equations [**B7**], the femionic SUSY extension closely resembles the form of the classical GW and GC equations for moving frames on surfaces appearing in differential geometry. In section C.5, we examine some geometric aspects of conformally parametrized SUSY surfaces. In section C.6, a Lie superalgebra of symmetries of the SUSY GC equations is determined. In section C.7, a systematic classification of the one-dimensional subalgebras of the Lie superalgebra into conjugacy classes is performed. Section C.8 contains three examples of invariant solutions of the SUSY GC equations obtained by the SRM. Finally, in section C.9, we present some conclusions and possibilities for future research.

## C.2. Preliminaries on certain aspects of Grassmann algebras

The basic notions and definitions used in this section, such as the properties of Grassmann algebras and Grassmann variables, together with the introduced notation, will be used subsequently in what follows. A more complete description can be found in severals references (see e.g. [**11, 12, 16, 33, 43, 45, 56, 80, 101, 102, 114, 118, 119**]). For a brief review, we refer the reader to [**B7**], section 3. In this paper we make use of a complex Grassmann algebra $\Lambda$ involving a number of Grassmann generators $(\xi_1, \xi_2, \xi_3, ...)$. The actual number of Grassmann generators of $\Lambda$ does not matter as long as there are enough of them to make all formulas encountered meaningful. Any fermionic (odd) variables $\theta^+$ and $\theta^-$ satisfy the relation

$$(\theta^+)^2 = (\theta^-)^2 = \theta^+\theta^- + \theta^-\theta^+ = 0. \tag{C.2.1}$$

The Grassmann algebra $\Lambda$ can be decomposed into the form

$$\Lambda = \Lambda_{\text{even}} + \Lambda_{\text{odd}}, \tag{C.2.2}$$

where $\Lambda_{\text{even}}$ includes all terms involving a product of an even number of generators $\xi_k$, i.e. $1, \xi_1\xi_2, \xi_1\xi_3, ...,$ while $\Lambda_{\text{odd}}$ includes all terms involving a product of an odd number of generators $\xi_k$, i.e. $\xi_1, \xi_2, \xi_3, ..., \xi_1\xi_2\xi_3, ...$ Also, the Grassmann algebra $\Lambda$ can be decomposed as

$$\Lambda = \Lambda_{\text{body}} + \Lambda_{\text{soul}}, \tag{C.2.3}$$

where $\Lambda_{\text{body}}$ includes all terms that do not include any of the generators $\xi_k$, i.e $\Lambda_{\text{body}}$ is isomorphic to $\mathbb{C}$, while $\Lambda_{\text{soul}}$ includes all terms that include at least one generator $\xi_k$. If $h$ and $g$ are Grassmann quantities, then the partial derivatives involving odd variables, $\theta^+$ and $\theta^-$, satisfy the following Leibniz rule

$$\partial_{\theta^\pm}(hg) = (\partial_{\theta^\pm}h)g + (-1)^{\deg(h)}h(\partial_{\theta^\pm}g), \tag{C.2.4}$$

where

$$\deg(h) = \begin{cases} 0 \text{ if } h \text{ is even,} \\ 1 \text{ if } h \text{ is odd.} \end{cases} \tag{C.2.5}$$

If $f$ is a function involving the variables $\theta^+$ and $\theta^-$ then we use the notation

$$f_{\theta^+\theta^-} = \partial_{\theta^-}(\partial_{\theta^+}f). \tag{C.2.6}$$

The partial derivatives with respect to the odd Grassmann variables $\theta^+$ and $\theta^-$ satisfy

$$\partial_{\theta^j}\theta^k = \delta_j^k, \tag{C.2.7}$$

where $\delta_j^k$ is the Kronecker delta function, and the values 1 and 2 of the indices $j$ and $k$ stand for $+$ and $-$, respectively. If $\theta$ is a fermionic-valued Grassmann variable, then the derivative $\partial_\theta$ will have the following effect on Grassmann-valued functions. If $f$ is a bosonic function, then $\partial_\theta f$ is fermionic. Similarly, if $f$ is a fermionic function, then $\partial_\theta f$ is bosonic. The interchangeability of mixed derivatives (with proper respect to the ordering of odd variables) is assumed throughout.

## C.3. Symmetries of the Gauss–Weingarten equations compared with symmetries of the Gauss–Codazzi equations

Consider the GC equations for conformally parametrized surfaces immersed in 3-dimensional Euclidean space $\mathbb{R}^3$ for the unknown functions $H$, $Q$ and $e^u$

$$\begin{aligned} &\partial\bar{\partial}u + \frac{1}{2}H^2e^u - 2|Q|e^{-u} = 0, && \text{(the Gauss equation)} \\ &\partial\bar{Q} = \frac{1}{2}e^u\bar{\partial}H, \quad \bar{\partial}Q = \frac{1}{2}e^u\partial H, && \text{(the Codazzi equations)} \end{aligned} \tag{C.3.1}$$

whose zero curvature condition (ZCC), with potential matrices $V_1$ and $V_2$ taking values in a Lie algebra, takes the form

$$\bar{\partial}V_1 - \partial V_2 + [V_1, V_2] = 0, \tag{C.3.2}$$

where we have used

$$\partial \equiv \partial_z = \frac{1}{2}(\partial_x - i\partial_y), \quad \bar{\partial} \equiv \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y), \tag{C.3.3}$$

which are the partial derivatives with respect to the complex variables $z = x + iy$ and $\bar{z} = x - iy$, respectively. The bracket $[\cdot, \cdot]$ denotes the commutator. Here, summation over repeated indices ($i = 1, 2, 3$) is understood and is used in what follows unless otherwise specified (by the use of a parenthesis). The conformally parametrized surface with the vector-valued function $F = (F_1, F_2, F_3)^T : \mathcal{R} \to \mathbb{R}^3$ (where $\mathcal{R}$ is a Riemann surface) satisfies the following normalization for the tangent vectors $\partial F$ and $\bar{\partial}F$, and the unit normal $N$

$$\begin{aligned} \langle \partial F, \partial F\rangle = \langle \bar{\partial}F, \bar{\partial}F\rangle = 0, &\qquad \langle \partial F, \bar{\partial}F\rangle = \frac{1}{2}e^u, \\ \langle \partial F, N\rangle = \langle \bar{\partial}F, N\rangle = 0, &\qquad \langle N, N\rangle = 1. \end{aligned} \tag{C.3.4}$$

The coefficients $Q$, $\bar{Q}$ and $H$ of the GC equations (C.3.1) involving the second derivatives of $F$ are defined as

$$Q = \langle \partial^2 F, N\rangle \in \mathbb{C}, \quad \bar{Q} = \langle \bar{\partial}^2 F, N\rangle \in \mathbb{C}, \quad H = 2e^{-u}\langle \partial\bar{\partial}F, N\rangle \in \mathbb{R}. \tag{C.3.5}$$

The bracket $\langle \cdot, \cdot\rangle$ denotes the scalar product in Euclidean space $\mathbb{R}^3$

$$\langle a, b\rangle = a^i b_i. \tag{C.3.6}$$

The matrix representation (C.3.2) is the compatibility condition of a matrix differential linear system for the wavefunction $\Phi$ taking values in the corresponding

Lie group $GL(3,\mathbb{C})$

$$\partial\Phi = V_1\Phi, \qquad \bar{\partial}\Phi = V_2\Phi,$$
$$V_1 = \begin{pmatrix} \partial u & 0 & \frac{1}{\lambda}Q \\ 0 & 0 & \frac{\lambda}{2}He^u \\ -\lambda H & -\frac{2}{\lambda}e^{-u}Q & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 0 & \frac{1}{2\lambda}He^u \\ 0 & \bar{\partial}u & \lambda\bar{Q} \\ -2\lambda e^{-u}\bar{Q} & -\frac{1}{\lambda}H & 0 \end{pmatrix}, \tag{C.3.7}$$

where $\lambda \in \mathbb{C}$ is a unitary constant, $|\lambda| = 1$. Note that we have inserted a free parameter $\lambda$ into this linear system. The linear equations (C.3.7) describing the kinematics of the moving frame $\Phi = (\partial F, \bar{\partial}F, N)^T$ associated with a surface when $\lambda = 1$ are known as the GW equations and are identified as the non-parametric linear problem in differential geometry

$$\begin{aligned} \partial \begin{pmatrix} \partial F \\ \bar{\partial}F \\ N \end{pmatrix} &= \begin{pmatrix} \partial u & 0 & Q \\ 0 & 0 & \frac{1}{2}He^u \\ -H & -2e^{-u}Q & 0 \end{pmatrix} \begin{pmatrix} \partial F \\ \bar{\partial}F \\ N \end{pmatrix}, \\ \bar{\partial} \begin{pmatrix} \partial F \\ \bar{\partial}F \\ N \end{pmatrix} &= \begin{pmatrix} 0 & 0 & \frac{1}{2}He^u \\ 0 & \bar{\partial}u & \bar{Q} \\ -2e^{-u}\bar{Q} & -H & 0 \end{pmatrix} \begin{pmatrix} \partial F \\ \bar{\partial}F \\ N \end{pmatrix}. \end{aligned} \tag{C.3.8}$$

The GC equations (C.3.1) are the necessary and sufficient conditions for the existence of conformally parametrized surfaces in $\mathbb{R}^3$ and, according to the Bonnet theorem [**21**], the immersion function $F$ is unique up to Euclidean motions in $\mathbb{R}^3$. The symmetry algebra $\mathcal{L}$ of the GW equations (C.3.8) is spanned by the following vector fields

$$\begin{aligned} X(\eta) &= \eta(z)\partial_z - \eta'(z)(U\partial_U + 2Q\partial_Q), \\ Y(\zeta) &= \zeta(\bar{z})\partial_{\bar{z}} - \zeta'(\bar{z})(U\partial_U + 2\bar{Q}\partial_{\bar{Q}}), \\ e_0 &= -H\partial_H + Q\partial_Q + \bar{Q}\partial_{\bar{Q}} + 2U\partial_U + F_i\partial_{F_i}, \\ T_i &= \partial_{F_i}, \qquad i = 1,2,3 \\ D_i &= F_i\partial_{F_{(i)}} + N_i\partial_{N_{(i)}}, \\ R_{ij} &= (F_i\partial_{F_j} - F_j\partial_{F_i}) + (N_i\partial_{N_j} - N_j\partial_{N_i}), \qquad i < j = 2,3 \\ S_{ij} &= (F_i\partial_{F_j} + F_j\partial_{F_i}) + (N_i\partial_{N_j} + N_j\partial_{N_i}), \end{aligned} \tag{C.3.9}$$

where we have used the notation $U = e^u$, $\eta'(z) = d\eta/dz$ and $\zeta'(\bar{z}) = d\zeta/d\bar{z}$. Here $\eta$ and $\zeta$ are arbitrary functions of $z$ and $\bar{z}$, respectively.

The symmetry algebra $\mathcal{L}$ spanned by the infinitesimal generators given in (C.3.9) include two infinite-dimensional subalgebras generated by $X(\eta)$ and $Y(\zeta)$. The vector fields $e_0, D_1$ and $D_2$ correspond to three types of dilations, the $T_i$ generate translations in the $F_i$ directions respectively, $R_{ij}$ is the rotation in the

$F_i$ and $N_i$ variables and $S_{ij}$ is a local boost transformation. The symmetry algebra $\mathcal{L}'$ of the linear system (C.3.7) is spanned by the generators $X(\eta)$ and $Y(\zeta)$ of equation (C.3.9) together with a modified version of the generator $e_0$, given by

$$e_0 = -H\partial_H + Q\partial_Q + \bar{Q}\partial_{\bar{Q}} + 2U\partial_U. \tag{C.3.10}$$

It should be noted that the parameter $\lambda$ does not appear in the latter symmetry algebra $\mathcal{L}'$. The commutation relations of the algebra $\mathcal{L}$ are

$$\begin{gathered}
[X(\eta_1), X(\eta_2)] = (\eta_1'\eta_2 - \eta_1\eta_2')\partial_z + (\eta_1''\eta_2 - \eta_1\eta_2'')(U\partial_U + 2Q\partial_Q),\\
[Y(\zeta_1), Y(\zeta_2)] = (\zeta_1'\zeta_2 - \zeta_1\zeta_2')\partial_{\bar{z}} + (\zeta_1''\zeta_2 - \zeta_1\zeta_2'')(U\partial_U + 2\bar{Q}\partial_{\bar{Q}}),\\
\begin{array}{lll}
[X(\eta), Y(\zeta)] = 0, & [X(\eta), e_0] = 0, & [Y(\zeta), e_0] = 0,\\
[X(\eta), T_i] = 0, & [Y(\zeta), T_i] = 0, & [e_0, T_i] = -T_i,\\
[X(\eta), D_i] = 0, & [Y(\zeta), D_i] = 0, & [e_0, D_i] = 0,\\
[X(\eta), R_{ij}] = 0, & [Y(\zeta), R_{ij}] = 0, & [e_0, R_{ij}] = 0,\\
[X(\eta), S_{ij}] = 0, & [Y(\zeta), S_{ij}] = 0, & [e_0, S_{ij}] = 0,
\end{array}\\
[T_i, R_{jk}] = \delta_{ij}T_k - \delta_{ik}T_j, \qquad [T_i, S_{jk}] = \delta_{ij}T_k + \delta_{ik}T_j,\\
[D_i, R_{jk}] = \delta_{ij}S_{ik} - \delta_{ik}S_{ij}, \qquad [D_i, S_{jk}] = \delta_{ij}R_{ik} - \delta_{ik}R_{ji},\\
[T_i, D_j] = \delta_{ij}T_i, \qquad [R_{ij}, S_{kl}] = \delta_{jk}S_{il} + \delta_{jl}S_{ik} - \delta_{ik}S_{jl} - \delta_{il}S_{jk}.
\end{gathered} \tag{C.3.11}$$

Since the vector fields $X(\eta)$ and $Y(\zeta)$ form an Abelian algebra, the vector fields (C.3.9) determine that the algebra $\mathcal{L}$ can be decomposed as a direct sum of two infinite-dimensional Lie algebras together with a 13-dimensional subalgebra generated by $e_0, T_i, D_i, R_{ij}$ and $S_{ij}$, i.e.

$$\mathcal{L} = \{X(\eta)\} \oplus \{Y(\zeta)\} \oplus \{e_0, T_i, D_i, R_{ij}, S_{ij}\}. \tag{C.3.12}$$

This algebra represents a direct sum of two copies of the Virasoro algebra together with the 13-dimensional algebra generated by $e_0, T_i, D_i, R_{ij}$ and $S_{ij}$. Assuming that the functions $\eta$ and $\zeta$ are analytic in some open subset $\mathcal{D} \subseteq \mathbb{C}$, we can develop $\eta$ and $\zeta$ as power series with respect to $z$ and $\bar{z}$, and the results provide a basis for $\mathcal{L}$. The largest finite-dimensional subalgebra $L$ of the algebra $\mathcal{L}$ is spanned by the following 19 generators

$$\begin{gathered}
\hat{e}_0 = -H\partial_H + Q\partial_Q + \bar{Q}\partial_{\bar{Q}} + 2U\partial_U + F_i\partial_{F_i}, \\
e_1 = \partial_z, \quad e_3 = z\partial_z - 2Q\partial_Q - U\partial_U, \quad e_5 = z^2\partial_z - 4zQ\partial_Q - 2zU\partial_U, \\
e_2 = \partial_{\bar{z}}, \quad e_4 = \bar{z}\partial_{\bar{z}} - 2\bar{Q}\partial_{\bar{Q}} - U\partial_U, \quad e_6 = \bar{z}^2\partial_{\bar{z}} - 4\bar{z}\bar{Q}\partial_{\bar{Q}} - 2\bar{z}U\partial_U, \\
T_i = \partial_{F_i}, \qquad i = 1,2,3 \\
D_i = F_i\partial_{F_{(i)}} + N_i\partial_{N_{(i)}}, \\
R_{ij} = (F_i\partial_{F_j} - F_j\partial_{F_i}) + (N_i\partial_{N_j} - N_j\partial_{N_i}), \qquad i < j = 2,3 \\
S_{ij} = (F_i\partial_{F_j} + F_j\partial_{F_i}) + (N_i\partial_{N_j} + N_j\partial_{N_i}),
\end{gathered} \tag{C.3.13}$$

with nonzero commutation relations

$$\begin{gathered}
[e_1, e_3] = e_1, \quad [e_1, e_5] = -2e_3, \quad [e_3, e_5] = e_5, \\
[e_2, e_4] = e_2, \quad [e_2, e_6] = -2e_4, \quad [e_4, e_6] = e_6, \\
[e_0, T_i] = -T_i, \quad [T_i, D_j] = \delta_{ij}T_i, \quad [T_i, R_{jk}] = \delta_{ij}T_k - \delta_{ik}T_j, \\
[T_i, S_{jk}] = \delta_{ij}T_k + \delta_{ik}T_j, \\
[D_i, R_{jk}] = \delta_{ij}S_{ik} - \delta_{ik}S_{ij}, \qquad [D_i, S_{jk}] = \delta_{ij}R_{ik} - \delta_{ik}R_{ji}, \\
[R_{ij}, S_{kl}] = \delta_{jk}S_{il} + \delta_{jl}S_{ik} - \delta_{ik}S_{jl} - \delta_{il}S_{jk}.
\end{gathered} \tag{C.3.14}$$

On the other hand the maximal finite-dimensional Lie algebra $A$ of the GC equations (C.3.1) is spanned by the seven generators [**B7**]

$$\begin{gathered}
e_0 = -H\partial_H + Q\partial_Q + \bar{Q}\partial_{\bar{Q}} + 2U\partial_U, \\
e_1 = \partial_z, \quad e_3 = z\partial_z - 2Q\partial_Q - U\partial_U, \quad e_5 = z^2\partial_z - 4zQ\partial_Q - 2zU\partial_U, \\
e_2 = \partial_{\bar{z}}, \quad e_4 = \bar{z}\partial_{\bar{z}} - 2\bar{Q}\partial_{\bar{Q}} - U\partial_U, \quad e_6 = \bar{z}^2\partial_{\bar{z}} - 4\bar{z}\bar{Q}\partial_{\bar{Q}} - 2\bar{z}U\partial_U,
\end{gathered} \tag{C.3.15}$$

with nonzero commutation relations

$$\begin{gathered}
[e_1, e_3] = e_1, \qquad [e_1, e_5] = -2e_3, \qquad [e_3, e_5] = e_5, \\
[e_2, e_4] = e_2, \qquad [e_2, e_6] = -2e_4, \qquad [e_4, e_6] = e_6.
\end{gathered} \tag{C.3.16}$$

This 7-dimensional Lie subalgebra $A$ can be decomposed as a direct sum of two simple subalgebras together with a one-dimensional algebra generated by $e_0$ :

$$A = \{e_1, e_3, e_5\} \oplus \{e_2, e_4, e_6\} \oplus \{e_0\}. \tag{C.3.17}$$

Therefore the classification of the subalgebras of $A$ into conjugacy classes involves two copies of a 3-dimensional Lie algebra together with the center $\{e_0\}$. This classification was performed in [**B7**].

The conjecture on an integrable class of differential equations in the sense of the soliton theory proposed in [**29**,**85**], states that the set of Lie point symmetries of the non-parametric linear problem is a subset of the set of Lie point symmetries of the original system. If the original system is non-integrable (in the sense of soliton theory), then the sets of symmetries are equal. If the original system is integrable, then there exists at least one symmetry of the original system which is not a symmetry of the non-parametric linear problem. These additional symmetries can be used to introduce a spectral parameter into the linear problem which cannot be eliminated by a gauge transformation. Note that the insertion condition is necessary for one to introduce a spectral parameter in the linear system. In order to compare the Lie point symmetries of the original system and those of the non-parametric linear problem, in the case where both sets of symmetries are finite-dimensional, we define the differential projector operator $\pi$ in the form of a dilation operator involving all independent and dependent variables

$$\pi(L) = L(z\partial + \bar{z}\bar{\partial} + H\partial_H + Q\partial_Q + \bar{Q}\partial_{\bar{Q}} + U\partial_U), \tag{C.3.18}$$

where all the elements of the algebra $L$ are applied on the operator $\omega = z\partial + \bar{z}\bar{\partial} + H\partial_H + Q\partial_Q + \bar{Q}\partial_{\bar{Q}} + U\partial_U$. Note that $\omega$ is not necessarily an element of the algebra $L$ under consideration. This projector $\pi$ has the property that $\pi^n(L) = \pi(L)$ for any finite integer $n \in \mathbb{Z}^+$. In view of the above conjecture, we can characterize the integrability of the original system as follows. We denote by $L_1$ and $L_2$ the sets of Lie point symmetries of the original system and its non-parametric linear problem, respectively. If $L_1 = \pi(L_2)$, then the original system is non-integrable (in the sense of the soliton theory). If there exist reductions of the original system (whose set of symmetries is $L_1'$) and of the non-parametric linear problem (whose set of symmetries is $L_2'$) such that $L_1' \neq \pi(L_2')$, then the reduced subsystem of the original system can be integrable. In view of the above statements, we observe that the ZCC for the GC equations (C.3.1) is not an integrable system, since the algebra $\pi(L)$ and $A$ are equal

$$A = \pi(L) = \{e_1, e_3, e_5\} \oplus \{e_2, e_4, e_6\} \oplus \{e_0\}. \tag{C.3.19}$$

It seems worthwhile to try to extend the conjecture proposed for the classical system to the case of a SUSY version of the GC equations and their reduced SUSY systems. This conjecture can be illustrated with the example of the SUSY sine-Gordon equation [**107**]

$$D_+ D_- \Phi = i \sin\Phi, \qquad D_\pm = \partial_{\theta^\pm} - i\theta^\pm \partial x_\pm, \tag{C.3.20}$$

where $\Phi$ is a real bosonic superfield and $D_\pm$ are covariant derivatives. Here, $x_+$ and $x_-$ are even Grassmann variables which constitute the orthogonal light-cone coordinates $x_\pm = \frac{1}{2}(t \pm x)$, while $\theta^+$ and $\theta^-$ are odd Grassmann variables. The symmetries which leave the SUSY equation (C.3.20) invariant are generated by the infinitesimal vector fields

$$\begin{aligned} P_+ &= \partial_{x_+}, \qquad P_- = \partial_{x_-}, \\ J_+ &= \partial_{\theta^+} + i\theta^+\partial_{x_+}, \qquad J_- = \partial_{\theta^-} + i\theta^-\partial_{x_-}, \\ R &= 2x_+\partial_{x_+} + \theta^+\partial_{\theta^+} - 2x_-\partial_{x_-} - \theta^-\partial_{\theta^-}. \end{aligned} \tag{C.3.21}$$

The non-parametric linear system corresponding to the SUSY sine-Gordon equation (C.3.20) is given by

$$D_+\Psi = A_+\Psi, \qquad D_-\Psi = A_-\Psi, \tag{C.3.22}$$

where

$$A_+ = \frac{1}{2}\begin{pmatrix} 0 & 0 & ie^{i\Phi} \\ 0 & 0 & -ie^{-i\Phi} \\ -e^{-i\Phi} & e^{i\Phi} & 0 \end{pmatrix}, \qquad A_- = \begin{pmatrix} iD_-\Phi & 0 & -i \\ 0 & -iD_-\Phi & i \\ -1 & 1 & 0 \end{pmatrix}. \tag{C.3.23}$$

The mixed derivatives of $\Psi$ are

$$\begin{aligned} D_-D_+\Psi &= D_-(A_+\Psi) = (D_-A_+)\Psi - EA_+E(D_-\Psi) \\ &= (D_-A_+)\Psi - EA_+EA_-\Psi, \\ D_+D_-\Psi &= D_+(A_-\Psi) = (D_+A_-)\Psi - EA_-E(D_+\Psi) \\ &= (D_+A_-)\Psi - EA_-EA_+\Psi, \end{aligned} \tag{C.3.24}$$

where

$$E = \pm\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{C.3.25}$$

Hence, the ZCC of equation (C.3.22) is given by

$$D_-A_+ + D_+A_- - \{EA_+, EA_-\} = 0, \tag{C.3.26}$$

whenever equation (C.3.20) holds. The brackets $\{\cdot,\cdot\}$ denote the anticommutator, unless otherwise indicated. The ZCC (C.3.26) is equivalent to the SUSY sine-Gordon equation (C.3.20). The non-parametric linear system (C.3.22) is invariant under the transformation associated with the vector fields $P_\pm$ and $J_\pm$ but is not invariant with respect to the dilation $R$. This fact allows us to introduce a one parameter group associated with the dilation $R$ through the transformation $\tilde{x}_+ = e^{2\mu}x_+$, $\tilde{x}_- = e^{-2\mu}x_-$, $\tilde{\theta}^+ = e^{\mu}\theta^+$ and $\tilde{\theta}^- = e^{-\mu}\theta^-$, $\mu \in \mathbb{R}$, into the linear system (C.3.22) which gives us

$$D_+\Psi = A_+\Psi, \qquad D_-\Psi = A_-\Psi, \tag{C.3.27}$$

where

$$A_+ = \frac{1}{2\sqrt{\lambda}} \begin{pmatrix} 0 & 0 & ie^{i\Phi} \\ 0 & 0 & -ie^{-i\Phi} \\ -e^{-i\Phi} & e^{i\Phi} & 0 \end{pmatrix},$$
$$A_- = \sqrt{\lambda} \begin{pmatrix} \frac{i}{\sqrt{\lambda}} D_-\Phi & 0 & -i \\ 0 & -\frac{i}{\sqrt{\lambda}} D_-\Phi & i \\ -1 & 1 & 0 \end{pmatrix}. \tag{C.3.28}$$

This result coincides with the one established for the SUSY Lax pair found in [**107**]. Here, $\lambda = e^{2\mu}$ plays the role of a spectral parameter. The connection between the super-Darboux transformations and the super-Bäcklund transformations allows the construction of $n$ explicit supersoliton solutions [**108**]. The results obtained for the SUSY sine-Gordon equations (C.3.20) were so promising, that it seemed to be wothwhile to try to extend this approach and check its effectiveness for the case of a fermionic SUSY extension of the GC equations.

## C.4. Fermionic supersymmetric extension of the Gauss–Weingarten and Gauss–Codazzi equations

The purpose of this section is to establish a SUSY version of the GW and GC equations using a fermionic superfield representation of a surface in the superspace $\mathbb{R}^{(1,1|3)}$. Consider a SUSY version of the differential equations which define surfaces in two-dimensional Minkowski space with the bosonic coordinates $x_+$ and $x_-$, and the fermionic (anti-commuting) variables $\theta^+$ and $\theta^-$. Let $\mathcal{S}$ be a smooth simply connected surface in the superspace $\mathbb{R}^{(1,1|3)}$ which we assume is conformally parametrized in the sense that the surface $\mathcal{S}$ is given by a vector-valued fermionic superfield $F(x_+, x_-, \theta^+, \theta^-)$ satisfying the normalization conditions of function $F$ (C.4.9) specified below. Such a fermionic superfield can be decomposed as

$$\begin{aligned} F(x_+, x_-, \theta^+, \theta^-) = F_l(x_+, x_-) + \theta^+ \varphi_l(x_+, x_-) + \theta^- \psi_l(x_+, x_-) \\ + \theta^+\theta^- G_l(x_+, x_-), \end{aligned} \tag{C.4.1}$$

where $F_l$ and $G_l$ are odd-valued fields, while $\varphi_l$ and $\psi_l$ are even-valued fields (for $l = 1, 2, 3$). In what follows we use the same notation for the fermionic immersion function $F$ as in the classical case. Also let $D_\pm$ be two covariant derivatives given by

$$D_\pm = \partial_{\theta^\pm} - i\theta^\pm \partial_{x_\pm}. \tag{C.4.2}$$

The covariant derivatives $D_+$ and $D_-$ have the property that they anticommute with the differential supersymmetry operators

$$J_+ = \partial_{\theta^+} + i\theta^+\partial_{x_+}, \qquad J_- = \partial_{\theta^-} + i\theta^-\partial_{x_-}, \tag{C.4.3}$$

which generate the SUSY transformations

$$x \to x + i\underline{\eta}_1\theta^+, \qquad \theta^+ \to \theta^+ + i\underline{\eta}_1, \tag{C.4.4}$$

and

$$t \to t + i\underline{\eta_2}\theta^-, \qquad \theta^- \to \theta^- + i\underline{\eta}_2, \tag{C.4.5}$$

respectively. Here $\underline{\eta}_1$ and $\underline{\eta}_2$ are odd-valued parameters. The four operators, $D_+$, $D_-$, $J_+$ and $J_-$ satisfy the following anticommutation relations

$$\begin{gathered}\{J_m, J_n\} = 2i\delta_{mn}\partial_{x_m}, \quad \{D_m, D_n\} = -2i\delta_{mn}\partial_{x_m}, \\ \{J_m, D_n\} = 0, \quad m, n = 1, 2\end{gathered} \tag{C.4.6}$$

where $\delta_{mn}$ is the Kronecker delta function. Therefore we have the following relations

$$D_\pm^2 = -i\partial_\pm, \qquad J_\pm^2 = i\partial_\pm. \tag{C.4.7}$$

The conformal parametrization of the surface in the superspace $\mathbb{R}^{(1,1|3)}$ is assumed to give the following normalization of $F$

$$\langle D_i F, D_j F\rangle = g_{ij} f, \qquad i, j = 1, 2 \tag{C.4.8}$$

where the indices $1, 2$ stand for $+, -$ respectively and $f$ is a bosonic function of $x_+$ and $x_-$. The bosonic metric functions $g_{ij}$ are given by

$$g_{11} = g_{22} = 0, \qquad g_{12} = g_{21} = \frac{1}{2}e^\phi, \tag{C.4.9}$$

which represent the metric coefficients and are symmetric, in contrast to the case in [**B7**]. We have introduced the function $f$ which allows for the possibility that the inner product (C.4.8) may be bodiless. This in turn allows us to consider larger types of surfaces.

Let the bosonic function $\phi$ be decomposed through an expansion in the fermionic variables $\theta^+$ and $\theta^-$ as

$$\phi = \phi_0 + \theta^+\phi_1 + \theta^-\phi_2 + \theta^+\theta^-\phi_3, \tag{C.4.10}$$

where $\phi_0$ and $\phi_3$ are bosonic functions and $\phi_1$ and $\phi_2$ are fermionic functions. The exponential of $\phi$ is given by

$$\begin{aligned} e^{\phi} &= e^{\phi_0}(1+\theta^+\phi_1+\theta^-\phi_2+\theta^+\theta^-(\phi_3-\phi_1\phi_2)), \\ e^{-\phi} &= e^{-\phi_0}(1-\theta^+\phi_1-\theta^-\phi_2-\theta^+\theta^-(\phi_3+\phi_1\phi_2)), \end{aligned} \tag{C.4.11}$$

The tangent vectors $D_+F$ and $D_-F$, together with the normal bosonic superfield $N$, form a bosonic moving frame $\Omega$ on the surface in the superspace $\mathbb{R}^{(1,1|3)}$. The normal vector $N$ can be decomposed as

$$\begin{aligned} N = N_m(x_+,x_-) &+ \theta^+\alpha_m(x_+,x_-) + \theta^-\beta_m(x_+,x_-) \\ &+\theta^+\theta^- H_m(x_+,x_-), \quad m=1,2,3 \end{aligned} \tag{C.4.12}$$

where $N_m$ and $H_m$ are even-valued fields and $\alpha_m$ and $\beta_m$ are odd-valued fields and have to satisfy the normalization

$$\langle D_iF, N\rangle = 0, \qquad \langle N,N\rangle = 1, \qquad i=1,2. \tag{C.4.13}$$

We assume that the second-order covariant derivatives of $F$ and the first-order covariant derivatives of $N$ are spanned by $D_\pm F$ and $N$ in such a way that

$$D_jD_iF = {\Gamma_{ij}}^k D_kF + b_{ij}fN, \qquad D_iN = {b^k}_i D_kF + \omega_i N, \tag{C.4.14}$$

where ${\Gamma_{ij}}^k$, $b_{ij}$, ${b^k}_i$ and $\omega_i$ are all fermionic functions.

The GW equations for the moving frame $\Omega$ on a surface are given by

$$D_+\Omega = A_+\Omega, \qquad D_-\Omega = A_-\Omega, \qquad \Omega = \begin{pmatrix} D_+F \\ D_-F \\ N \end{pmatrix}, \tag{C.4.15}$$

where the $3\times 3$ fermionic-valued matrices $A_+$ and $A_-$ are

$$A_+ = \begin{pmatrix} {\Gamma_{11}}^1 & {\Gamma_{11}}^2 & b_{11}f \\ {\Gamma_{21}}^1 & {\Gamma_{21}}^2 & b_{21}f \\ {b^1}_1 & {b^2}_1 & \omega_1 \end{pmatrix}, \qquad A_- = \begin{pmatrix} {\Gamma_{12}}^1 & {\Gamma_{12}}^2 & b_{12}f \\ {\Gamma_{22}}^1 & {\Gamma_{22}}^2 & b_{22}f \\ {b^1}_2 & {b^2}_2 & \omega_2 \end{pmatrix}. \tag{C.4.16}$$

We can obtain the conditions on $b_{ij}$, ${b^i}_j$ and $\omega_i$ from the derivatives of (C.4.13)

$$\begin{aligned} 0 &= D_i\langle N,N\rangle = \langle D_iN,N\rangle + \langle N, D_iN\rangle = 2\langle D_iN,N\rangle \\ &= 2\omega_i\langle N,N\rangle, \\ 0 &= D_i\langle D_jF,N\rangle = \langle D_iD_jF,N\rangle + \langle D_jF, D_iN\rangle \\ &= b_{ji}f\langle N,N\rangle + {b^k}_i\langle D_jF, D_kF\rangle, \end{aligned} \tag{C.4.17}$$

which imply

$$\omega_i = 0, \qquad f(b_{ji} + g_{jk}{b^k}_i) = 0, \qquad i,j=1,2. \tag{C.4.18}$$

The conditions on ${\Gamma_{ij}}^k$ can be obtained by taking the derivatives of equations (C.4.8). We get

$$\begin{aligned} 0 = D_i\langle D_jF, D_jF\rangle &= \langle D_iD_jF, D_jF\rangle + \langle D_jF, D_iD_jF\rangle \\ &= 2\langle D_iD_jF, D_jF\rangle = 2{\Gamma_{ji}}^k g_{kj}. \end{aligned} \tag{C.4.19}$$

Therefore we have

$${\Gamma_{ji}}^k = 0, \qquad \text{for} \qquad j \neq k \tag{C.4.20}$$

and by construction we have that ${\Gamma_{ji}}^k$ is antisymmetric under a permutation of $i$ and $j$, i.e.

$${\Gamma_{ji}}^k = -{\Gamma_{ji}}^k, \qquad \text{for} \qquad i \neq j. \tag{C.4.21}$$

This implies that

$${\Gamma_{ji}}^k = 0, \qquad \text{if} \qquad i \neq k \text{ or } j \neq k. \tag{C.4.22}$$

Differentiating (C.4.8) by the covariant derivatives $D_i$ we get

$$\begin{aligned} \frac{1}{2}D_i(e^\phi f) = D_i\langle D_+F, D_-F\rangle &= \langle D_iD_+F, D_-F\rangle + \langle D_+F, D_iD_-F\rangle \\ &= {\Gamma_{1i}}^1\langle D_+F, D_-F\rangle + {\Gamma_{2i}}^2\langle D_+F, D_-F\rangle, \\ \frac{1}{2}e^\phi f D_i\phi + \frac{1}{2}e^\phi D_i f &= \frac{1}{2}e^\phi {\Gamma_{1i}}^1 f + \frac{1}{2}e^\phi {\Gamma_{2i}}^2 f, \end{aligned} \tag{C.4.23}$$

which implies

$$D_i f = ({\Gamma_{1i}}^1 + {\Gamma_{2i}}^2 - D_i\phi) f. \tag{C.4.24}$$

However, using equation (C.4.22), we get

$$D_i f = ({\Gamma_{i(i)}}^{(i)} - D_i\phi) f, \qquad i = 1, 2 \tag{C.4.25}$$

where we do not sum over the index $i$. It is interesting to note that if $D_i f = 0$, then we have

$${\Gamma_{11}}^1 = D_+\phi, \qquad {\Gamma_{22}}^2 = D_-\phi, \tag{C.4.26}$$

which is similar to the classical case. Also, one can compute the compatibility condition on $f$ which is given by

$$\{D_+, D_-\} f = (D_+{\Gamma_{22}}^2 + D_-{\Gamma_{11}}^1) f = 0. \tag{C.4.27}$$

The Christoffel symbols of the first kind are defined as

$$\Gamma_{ijk} f = \langle D_jD_iF, D_kF\rangle, \tag{C.4.28}$$

and using equation (C.4.14) we get the following relation between the Christoffel symbols of the first and second kinds

$$\Gamma_{ijk}f = \langle D_jD_iF, D_kF\rangle = \Gamma_{ij}{}^{l}\langle D_lF, D_kF\rangle = \Gamma_{ij}{}^{l}g_{lk}f, \tag{C.4.29}$$

that is

$$f(\Gamma_{ijk} - \Gamma_{ij}{}^{l}g_{lk}) = 0. \tag{C.4.30}$$

So we obtain

$$\begin{aligned} &\Gamma_{111} = 0, \quad \Gamma_{112} = \frac{1}{2}e^{\phi}\Gamma_{11}{}^{1}, \quad \Gamma_{121} = 0, \qquad \Gamma_{211} = 0, \\ &\Gamma_{122} = 0, \quad \Gamma_{212} = 0, \qquad \Gamma_{221} = \frac{1}{2}e^{\phi}\Gamma_{22}{}^{2}, \quad \Gamma_{222} = 0, \end{aligned} \tag{C.4.31}$$

up to the addition of a fermionic function $\zeta \neq 0$ which has the property $\zeta f = 0$.

We define the fermionic quantities $b_{ij}$ to be

$$b_{11} = Q^+, \qquad b_{12} = -b_{21} = \frac{1}{2}e^{\phi}H, \qquad b_{22} = Q^-, \tag{C.4.32}$$

which give the relations

$$\langle D_+^2F, N\rangle = Q^+f, \qquad \langle D_-D_+F, N\rangle = \frac{1}{2}e^{\phi}Hf, \qquad \langle D_-^2F, N\rangle = Q^-f. \tag{C.4.33}$$

Hence, this implies that the fermionic quantities $b^i{}_j$ are

$$b^1{}_1 = H, \qquad b^2{}_1 = -2e^{-\phi}Q^+, \qquad b^1{}_2 = -2e^{-\phi}Q^-, \qquad b^2{}_2 = -H, \tag{C.4.34}$$

up to the addition of a fermionic function $\ell \neq 0$ such that $\ell f = 0$. The Gauss-Weingarten equations take the Bianchi form [**15**]

$$\begin{aligned} D_+\begin{pmatrix} D_+F \\ D_-F \\ N \end{pmatrix} &= \begin{pmatrix} R^+ & 0 & Q^+f \\ 0 & 0 & -\frac{1}{2}e^{\phi}Hf \\ H & -2e^{-\phi}Q^+ & 0 \end{pmatrix}\begin{pmatrix} D_+F \\ D_-F \\ N \end{pmatrix}, \\ D_-\begin{pmatrix} D_+F \\ D_-F \\ N \end{pmatrix} &= \begin{pmatrix} 0 & 0 & \frac{1}{2}e^{\phi}Hf \\ 0 & R^- & Q^-f \\ -2e^{-\phi}Q^- & -H & 0 \end{pmatrix}\begin{pmatrix} D_+F \\ D_-F \\ N \end{pmatrix}, \end{aligned} \tag{C.4.35}$$

where we define $R^+ = \Gamma_{11}{}^1$ and $R^- = \Gamma_{22}{}^2$. The GC equations are the compatibility conditions of the GW equations. The ZCC for equations (C.4.35) in matrix form is

$$D_+A_- + D_-A_+ - \{A_+, A_-\} = 0, \tag{C.4.36}$$

where

$$A_+ = \begin{pmatrix} \Gamma_{11}{}^1 & \Gamma_{11}{}^2 & b_{11}f \\ \Gamma_{21}{}^1 & \Gamma_{21}{}^2 & b_{21}f \\ b^1{}_1 & b^2{}_1 & \omega_1 \end{pmatrix} = \begin{pmatrix} R^+ & 0 & Q^+f \\ 0 & 0 & -\frac{1}{2}e^{\phi}Hf \\ H & -2e^{-\phi}Q^+ & 0 \end{pmatrix},$$
$$A_- = \begin{pmatrix} \Gamma_{12}{}^1 & \Gamma_{12}{}^2 & b_{12}f \\ \Gamma_{22}{}^1 & \Gamma_{22}{}^2 & b_{22}f \\ b^1{}_2 & b^2{}_2 & \omega_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{2}e^{\phi}Hf \\ 0 & R^- & Q^-f \\ -2e^{-\phi}Q^- & -H & 0 \end{pmatrix}. \tag{C.4.37}$$

In component form the equations are

$$\begin{aligned}
&(i)\;\; D_-R^+ + 2e^{-\phi}Q^+Q^-f = 0, \\
&(ii)\;\; \left[D_-Q^+ + \frac{1}{2}e^{\phi}D_+H + Q^+(D_-\phi - R^-)\right]f = 0, \\
&(iii) D_+R^- + 2e^{-\phi}Q^-Q^+f = 0, \\
&(iv)\;\; \left[D_+Q^- - \frac{1}{2}e^{\phi}D_-H + Q^-(D_+\phi - R^+)\right]f = 0, \\
&(v)\;\; D_+Q^- - \frac{1}{2}e^{\phi}D_-H + Q^-(D_+\phi - R^+) = 0, \\
&(vi)\, D_-Q^+ + \frac{1}{2}e^{\phi}D_+H + Q^+(D_-\phi - R^-) = 0.
\end{aligned} \tag{C.4.38}$$

If the equations (C.4.38.vi) and (C.4.38.v) hold, then equations (C.4.38.ii) and (C.4.38.iv) are identically satisfied respectively. By adding (C.4.38.i) and (C.4.38.iii) we obtain

$$D_-R^+ + D_+R^- = 0, \tag{C.4.39}$$

which is the compatibility condition for $f$, i.e. (C.4.27). Therefore the Gauss-Codazzi equations are reduced to the four linearly independent equations

$$\begin{aligned}
&(i)\;\; D_+R^- + D_-R^+ = 0, \\
&(ii)\;\; D_-R^+ + 2e^{-\phi}Q^+Q^-f = 0, \\
&(iii) D_+Q^- - \frac{1}{2}e^{\phi}D_-H + Q^-(D_+\phi - R^+) = 0, \\
&(iv)\, D_-Q^+ + \frac{1}{2}e^{\phi}D_+H + Q^+(D_-\phi - R^-) = 0.
\end{aligned} \tag{C.4.40}$$

The Grassmann-valued PDEs (C.4.40) involve six dependent functions of the independent variables $x_+$, $x_-$, $\theta^+$ and $\theta^-$ including one bosonic function $\phi$ and the five fermionic functions $H$, $Q^\pm$ and $R^\pm$ together with one bosonic function $f$ of $x_+$ and $x_-$. Hence, if we restrict ourselves to the case where $f$ is a bosonic constant, then from (C.4.26) the Gauss-Codazzi equations become

$$\begin{aligned} &(i)\ \ D_-D_+\phi + 2e^{-\phi}Q^+Q^-f = 0,\\ &(ii)\ D_+Q^- - \frac{1}{2}e^{\phi}D_-H = 0,\\ &(iii) D_-Q^+ + \frac{1}{2}e^{\phi}D_+H = 0, \end{aligned} \tag{C.4.41}$$

which resemble the classical GC equations (C.3.1) taking into account that the $H^2$ term vanishes. The equations (C.4.41) have some terms whose signs differ from those of the classical equations. We get an underdetermined system of three PDEs for four dependent variables $H$, $Q^\pm$ and $\phi$.

Under the above assumptions we obtain the following result.

**Proposition C.4.1** (Structural SUSY equations for a moving frame on a surface)**.** *For any fermionic superfield $F(x_+,x_-,\theta^+,\theta^-)$ and bosonic superfield $N(x_+,x_-,\theta^+,\theta^-)$ satisfying the normalization conditions* (C.4.8), (C.4.9), (C.4.13) *and* (C.4.33)*, the bosonic moving frame $\Omega = (D_+F, D_-F, N)^T$ on a surface immersed in the superspace $\mathbb{R}^{(1,1|3)}$ satisfies the SUSY GW equations* (C.4.35)*. The ZCC* (C.4.36)*, which is the compatibility condition of the SUSY GW equations* (C.4.35) *expressed in terms of the matrices $A_+$ and $A_-$, is equivalent to the SUSY GC equations* (C.4.40).

## C.5. Geometric aspects of the supersymetric fermionic conformally parametrized surfaces

In this section, we discuss certain aspects of Grassmann variables in conjunction with differential geometry and supersymmetry analysis. Let us define $d_\pm = d\theta^\pm + idx_\pm\partial_{\theta^\pm}$ to be the infinitesimal displacement in the direction of $D_\pm$. The first fundamental form is given by

$$\begin{aligned} I &= \left\langle \begin{pmatrix} d_+ & d_- \end{pmatrix}, \begin{pmatrix} d_+ & d_- \end{pmatrix} \begin{pmatrix} \langle D_+F, D_+F\rangle & \langle D_+F, D_-F\rangle \\ -\langle D_+F, D_-F\rangle & \langle D_-F, D_-F\rangle \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} d_+ & d_- \end{pmatrix}, \begin{pmatrix} d_+ & d_- \end{pmatrix} \begin{pmatrix} g_{11}f & g_{12}f \\ -g_{12}f & g_{22}f \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} d_+ & d_- \end{pmatrix}, \begin{pmatrix} d_+ & d_- \end{pmatrix} Rf \right\rangle \\ &= f\left(d_+^2 g_{11} + 2d_+d_-g_{12} + d_-^2 g_{22}\right) = f d_+d_-e^{\phi}, \end{aligned} \tag{C.5.1}$$

and the discriminant $g$ is defined to be

$$g = g_{11}g_{22} + g_{12}^2 = \frac{1}{4}e^{2\phi}. \tag{C.5.2}$$

Hence the covariant metric is given by

$$g_{ij}g^{jk} = \delta_i^k, \qquad \begin{pmatrix} g_{11} & g_{21} \\ g_{12} & g_{22} \end{pmatrix} \begin{pmatrix} g^{11} & g^{21} \\ g^{12} & g^{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{C.5.3}$$

such that

$$g^{11} = g^{22} = 0, \qquad g^{12} = g^{21} = 2e^{-\phi}. \tag{C.5.4}$$

The second fundamental form is

$$\begin{aligned} I\!I &= \left\langle \begin{pmatrix} d_+ & d_- \end{pmatrix}, \begin{pmatrix} d_+ & d_- \end{pmatrix} \begin{pmatrix} \langle D_+^2 F, N\rangle & \langle D_- D_+ F, N\rangle \\ -\langle D_- D_+ F, N\rangle & \langle D_-^2 F, N\rangle \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} d_+ & d_- \end{pmatrix}, \begin{pmatrix} d_+ & d_- \end{pmatrix} \begin{pmatrix} b_{11} f & b_{12} f \\ -b_{12} f & b_{22} f \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} d_+ & d_- \end{pmatrix}, \begin{pmatrix} d_+ & d_- \end{pmatrix} S f \right\rangle \\ &= f\left(d_+^2 b_{11} + 2 d_+ d_- b_{12} + d_-^2 b_{22}\right) = f\left(d_+^2 Q^+ + d_+ d_- (e^{\phi} H) + d_-^2 Q^-\right), \end{aligned} \tag{C.5.5}$$

and the discriminant is defined to be

$$b = b_{11} b_{22} + b_{12}^2 = b_{11} b_{22} = Q^+ Q^-. \tag{C.5.6}$$

In order to compute the first and second fundamental forms, we have assumed that $(d\theta^j \lrcorner\, \partial_{\theta^i}) = 0$ for $i, j = 1, 2$. Making use of (C.5.2) and (C.5.6), the Gaussian curvature is defined as

$$\mathcal{K} = \det(SR^{-1}) = \frac{b_{11} b_{22} + (b_{12})^2}{g_{11} g_{22} + (g_{12})^2} = 4e^{-2\phi} Q^+ Q^-, \tag{C.5.7}$$

which is a bosonic bodiless function, where

$$R = \begin{pmatrix} g_{11} & g_{12} \\ -g_{12} & g_{22} \end{pmatrix} = \frac{1}{2} e^{\phi} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} b_{11} & b_{12} \\ -b_{12} & b_{22} \end{pmatrix} = \begin{pmatrix} Q^+ & \frac{1}{2} e^{\phi} H \\ -\frac{1}{2} e^{\phi} H & Q^- \end{pmatrix},$$

while the mean curvature $H$ is a fermionic function satisfying

$$H = \frac{1}{2} \operatorname{tr}(SR^{-1}). \tag{C.5.8}$$

Under the above assumptions on the SUSY version of the GC equations (C.4.40) we can provide a SUSY analogue of the Bonnet Theorem.

**Proposition C.5.1** (Fermionic supersymmetric extension of the Bonnet theorem)**.** *Given a SUSY conformal metric, $M = f d_+ d_- e^{\phi}$, of a conformally parametrized surface $\mathcal{S}$, the Hopf differentials $d_\pm^2 Q^\pm$ and a mean curvature function $H$ defined on a Riemann surface $\mathcal{R}$ satisfying the GC equation* (C.4.40)*, there exists a vector-valued fermionic immersion function, $F = (F_1, F_2, F_3) : \tilde{\mathcal{R}} \to \mathbb{R}^{(1,1|3)}$,* *with the fundamental forms*

$$I = f d_+ d_- e^{\phi}, \qquad I\!I = f(d_+^2 Q^+ + d_+ d_-(He^{\phi}) + d_-^2 Q^-), \tag{C.5.9}$$

*where* $\tilde{\mathcal{R}}$ *is the universal covering of the Riemann surface* $\mathcal{R}$ *and* $\mathbb{R}^{(1,1|3)}$ *is the superspace. The immersion function* $F$ *is unique up to affine transformations in the superspace* $\mathbb{R}^{(1,1|3)}$.

The proof of this proposition is analogous to that given in [**21**]. Note that it is straightforward to construct surfaces on the superspace $\mathbb{R}^{(1,1|3)}$ related to integrable equations using a SUSY version of the Sym-Tafel formula for immersion [**112**, **113**]. However, it is nontrivial to identify those surfaces which have an invariant geometrical characterization. A list of such surfaces is known in the classical case [**19**] but, to our knowledge, an identification of such surfaces is an open problem in the case of surfaces immersed in a superspace.

## C.6. Symmetries of the fermionic supersymmetric Gauss–Codazzi equations

By a symmetry supergroup $G$ of a SUSY system, we mean a local supergroup of transformations acting on the Cartesian product $\mathcal{X} \times \mathcal{U}$ of supermanifolds, where $\mathcal{X}$ is the space of four independent variables $(x_+, x_-, \theta^+, \theta^-)$ and $\mathcal{U}$ is the space of seven dependent superfields $(\phi, H, Q^+, Q^-, R^+, R^-, f)$. Solutions of the GC equations (C.4.40) are mapped to solutions of (C.4.40) by the action of the group $G$ on the functions $\phi, H, Q^+, Q^-, R^+, R^-$ and $f$ of $(x_+, x_-, \theta^+, \theta^-)$. When we perform the symmetry reductions, we need to take into consideration the fact that the bosonic function $f$ introduced in (C.4.8) depends only on $x_+$ and $x_-$ or is constant. If $G$ is a Lie supergroup as described in [**80**] and [**120**], it can be associated with its Lie superalgebra $\mathfrak{g}$ whose elements are infinitesimal symmetries of equations (C.4.40). We have made use of the theory described in the book by Olver [**95**] in order to determine a superalgebra of infinitesimal symmetries. The SUSY GC equations (C.4.40) are invariant under the following six bosonic symmetry generators

$$\begin{aligned}
P_+ &= \partial_{x_+}, \qquad P_- = \partial_{x_-}, \\
C_0 &= H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} - 2f\partial_f, \\
K_0 &= -H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} + 2\partial_\phi, \\
K_1 &= -2x_+\partial_{x_+} - \theta^+\partial_{\theta^+} + 2Q^+\partial_{Q^+} + R^+\partial_{R^+} + \partial_\phi, \\
K_2 &= -2x_-\partial_{x_-} - \theta^-\partial_{\theta^-} + 2Q^-\partial_{Q^-} + R^-\partial_{R^-} + \partial_\phi,
\end{aligned} \tag{C.6.1}$$

together with the three fermionic generators

$$J_+ = \partial_{\theta^+} + i\theta^+\partial_{x_+}, \qquad J_- = \partial_{\theta^-} + i\theta^-\partial_{x_-}, \qquad W = \partial_H. \tag{C.6.2}$$

The symmetry generators $W$ and $P_\pm$ represent a fermionic translation of $H$ and bosonic translations in the $x_\pm$ direction respectively, $J_\pm$ represent SUSY transformations and $C_0$, $K_0$, $K_1$ and $K_2$ represent dilations. The commutation table (anticommutation for two fermionic symmetries) for the generators of the superalgebra of equations (C.4.40) is given in table C.1. The decomposition of the superalgebra (C.4.40) is given by

TABLE C.1. Commutation table for the Lie superalgebra $\mathfrak{g}$ spanned by the vector fields (C.6.1) and (C.6.2). In the case of two fermionic generators $J_+$ and/or $J_-$ and/or $W$ we have anticommutation rather than commutation.

| | $K_1$ | $P_+$ | $J_+$ | $K_2$ | $P_-$ | $J_-$ | $K_0$ | $C_0$ | $W$ |
|---|---|---|---|---|---|---|---|---|---|
| $K_1$ | 0 | $2P_+$ | $J_+$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $P_+$ | $-2P_+$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $J_+$ | $-J_+$ | 0 | $2iP_+$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $K_2$ | 0 | 0 | 0 | 0 | $2P_-$ | $J_-$ | 0 | 0 | 0 |
| $P_-$ | 0 | 0 | 0 | $-2P_-$ | 0 | 0 | 0 | 0 | 0 |
| $J_-$ | 0 | 0 | 0 | $-J_-$ | 0 | $2iP_-$ | 0 | 0 | 0 |
| $K_0$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $W$ |
| $C_0$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $-W$ |
| $W$ | 0 | 0 | 0 | 0 | 0 | 0 | $-W$ | $W$ | 0 |

$$\mathfrak{g} = \{\{K_1\} \Supset \{P_+, J_+\}\} \oplus \{\{K_2\} \Supset \{P_-, J_-\}\} \oplus \{\{K_0, C_0\} \Supset \{W\}\}. \tag{C.6.3}$$

In equation (C.6.3) the braces $\{\cdot,\cdot\}$ denote the set of generators listed in (C.6.2) and (C.6.2).

However, if we consider the case where $D_\pm f = 0$, the equations (C.4.41) are invariant under the five bosonic generators

$$\begin{aligned} P_+ &= \partial_{x_+}, \qquad P_- = \partial_{x_-}, \\ K_0 &= -H\partial_H + Q^+\partial_{Q^+} + Q^-\partial_{Q^-} + 2\partial_\phi, \\ K_1 &= -2x_+\partial_{x_+} - \theta^+\partial_{\theta^+} + 2Q^+\partial_{Q^+} + \partial_\phi, \\ K_2 &= -2x_-\partial_{x_-} - \theta^-\partial_{\theta^-} + 2Q^-\partial_{Q^-} + \partial_\phi, \end{aligned} \tag{C.6.4}$$

and the three fermionic generators

$$J_+ = \partial_{\theta^+} + i\theta^+\partial_{x_+}, \qquad J_- = \partial_{\theta^-} + i\theta^-\partial_{x_-}, \qquad W = \partial_H. \tag{C.6.5}$$

The commutation table for the generators of the superalgebra of equations (C.4.41) is given in table C.2. This Lie superalgebra $\mathfrak{g}$ can be decomposed into the following

TABLE C.2. Commutation table for the Lie superalgebra $\mathfrak{g}$ spanned by the vector fields (C.6.4) and (C.6.5). In the case of two fermionic generators $J_+$ and/or $J_-$ and/or $W$ we have anticommutation rather than commutation.

| | $K_1$ | $P_+$ | $J_+$ | $K_2$ | $P_-$ | $J_-$ | $K_0$ | $W$ |
|---|---|---|---|---|---|---|---|---|
| $K_1$ | 0 | $2P_+$ | $J_+$ | 0 | 0 | 0 | 0 | 0 |
| $P_+$ | $-2P_+$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $J_+$ | $-J_+$ | 0 | $2iP_+$ | 0 | 0 | 0 | 0 | 0 |
| $K_2$ | 0 | 0 | 0 | 0 | $2P_-$ | $J_-$ | 0 | 0 |
| $P_-$ | 0 | 0 | 0 | $-2P_-$ | 0 | 0 | 0 | 0 |
| $J_-$ | 0 | 0 | 0 | $-J_-$ | 0 | $2iP_-$ | 0 | 0 |
| $K_0$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $W$ |
| $W$ | 0 | 0 | 0 | 0 | 0 | 0 | $-W$ | 0 |

combination of direct and semi-direct sums

$$\mathfrak{g} = \{\{K_1\} \supset\!\!\!+ \{P_+, J_+\}\} \oplus \{\{K_2\} \supset\!\!\!+ \{P_-, J_-\}\} \oplus \{\{K_0\} \supset\!\!\!+ \{W\}\}. \qquad \text{(C.6.6)}$$

In equation (C.6.6) the braces $\{\cdot, ..., \cdot\}$ denote the set of generators listed in (C.6.4) and (C.6.5). The one-dimensional subalgebras of this superalgebra can be classified into conjugacy classes.

## C.7. ONE-DIMENSIONAL SUBALGEBRAS OF THE SYMMETRY SUPERALGEBRA $\mathfrak{g}$

When constructing a list of representative one-dimensional subalgebras of the superalgebra $\mathfrak{g}$, it would be inadequate to consider the $\mathbb{R}$ or $\mathbb{C}$ span of the generators (C.6.1) and (C.6.2) because the odd generators $J_+, J_-$ and $W$ are multiplied by odd parameters in the list of subalgebras presented in appendix C.9. One is therefore led to consider a $\mathfrak{g}$ which is a supermanifold. That is, $\mathfrak{g}$ contains sums of any even combinations of $P^+, P^-, C_0, K_0, K_1$ and $K_2$ (multiplied by even parameters in $\Lambda_{\text{even}}$) and odd combinations of $J_+, J_-$ and $W$ (multiplied by odd parameters in $\Lambda_{\text{odd}}$). Therefore $\mathfrak{g}$ is an even Lie module. This leads to the following consideration. For a given $X \in \mathfrak{g}$, the subalgebras $\mathfrak{X}$ and $\mathfrak{X}'$ spanned by $X$ and $X' = aX$ with $a \in \Lambda_{\text{even}} \backslash \mathbb{C}$ are not isomorphic in general, i.e. $\mathfrak{X}' \subset \mathfrak{X}$.

It should be noted that subalgebras obtained by multiplying other subalgebras by bodiless elements of $\Lambda_{even}$ do not provide us with anything new for the purpose of symmetry reduction. It is not particularly useful to consider a subalgebra of the form e.g. $\{P_+ + \underline{\eta}_1\underline{\eta}_2 P_-\}$, since there is no limit to the number of odd parameters $\underline{\eta}_k$ that can be used to construct even coefficients. While such subalgebras may allow for more freedom in the choice of invariants, we then encounter the problem of non-standard invariants [**68**,**69**,**B7**]. Such non-standard invariants, which do not lead to standard reductions or invariant solutions, are found for several other SUSY hydrodynamic-type systems [**66**, **67**]. For the SUSY GC equations (C.4.40), the one-dimensional subalgebras $\mathfrak{g}_3, \mathfrak{g}_7, \mathfrak{g}_{19}, \mathfrak{g}_{25}, \mathfrak{g}_{46}, \mathfrak{g}_{74}$ and $\mathfrak{g}_{158}$ listed in appendix C.9 have such non-standard invariants.

Subgroups within the same conjugacy class lead to invariant solutions that are equivalent in the sense that a suitable symmetry can transform one to the other. It is therefore unnecessary to consider reductions with respect to algebras which are conjugate to each other. We make use of the methods for classifying direct and semi-direct sums of algebras as described in [**120**] in order to classify the Lie superalgebra (C.6.3) under the action of the supergroup generated by $\mathfrak{g}$. More specifically, we generalize these methods to the case of a superalgebra involving both even and odd generators. Specifically, the Goursat twist method is used for the case of direct sums of algebras. Here the superalgebra (C.6.3) contains two isomorphic copies of the 3-dimensional algebra $\mathfrak{g}_1 = \{\{K_1\} \Supset \{P_+, J_+\}\}$, the other copy being $\mathfrak{g}_2 = \{\{K_2\} \Supset \{P_-, J_-\}\}$ together with the three-dimensional algebra $\{\{K_0, C_0\} \Supset \{W\}\}$ . This allows us to adapt the classification for 3-dimensional algebras as described in [**97**] to the SUSY case. We therefore begin our classification by considering the twisted one-dimensional subalgebras of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$. Under the action of a one-parameter group generated by the vector field

$$X = \alpha K_1 + \beta P_+ + \underline{\eta} J_+ + \delta K_2 + \lambda P_- + \underline{\rho} J_-, \tag{C.7.1}$$

where $\alpha, \beta, \delta, \lambda \in \Lambda_{even}$ and $\underline{\eta}, \underline{\rho} \in \Lambda_{odd}$, the one-dimensional subalgebra $Y = P_+ + aP_-, a \in \Lambda_{even}$ transforms under the Baker–Campbell–Hausdorff formula

$$e^X Y e^{-X} = Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] + \frac{1}{3!}[X, [X, [X, Y]]] + ... \tag{C.7.2}$$

to $e^{-2\alpha} P_+ + e^{-2\delta} a P_-$. By an appropriate choice of $\alpha$ and $\delta$, the factor $e^{2\alpha - 2\delta} a$ can be rescaled to either 1 or $-1$. Hence, we get a twisted subalgebra $\mathfrak{g}_{14} = \{P_+ + \epsilon P_-, \epsilon = \pm 1\}$.

As another example, consider a twisted algebra of the form $\{K_1 + \underline{\zeta} W\}$, where $\underline{\zeta}$ is a fermionic parameter. Through the Baker–Campbell–Hausdorff formula (C.7.2), the vector field $Y = K_1 + \underline{\zeta} W$ transforms through

$$X = \alpha K_1 + \beta P_+ + \underline{\eta} J_+ + \gamma K_2 + \delta P_- + \underline{\lambda} J_- + \rho K_0 + \sigma C_0 + \underline{\tau} W, \tag{C.7.3}$$

(where $\alpha, \beta, \gamma, \delta, \rho, \sigma \in \Lambda_{\text{even}}$ and $\underline{\eta}, \underline{\lambda}, \underline{\tau} \in \Lambda_{\text{odd}}$) to

$$e^X Y e^{-X} = K_1 + e^{\rho - \sigma} \underline{\zeta} W - \frac{\beta}{\alpha}(e^{2\alpha} - 1) P_+ - \frac{1}{\alpha}(e^{\alpha} - 1) \underline{\eta} J_+. \tag{C.7.4}$$

Through a suitable choice of $\beta$ and $\underline{\eta}$, the last two terms of the expression (C.7.4) can be eliminated, so we obtain the twisted subalgebra $\mathfrak{g}_{32} = \{K_1 + \underline{\zeta} W\}$. Continuing the classification in a similar way, involving twisted and non-twisted subalgebras according to [**120**], we obtain the list of one-dimensional subalgebras given in appendix C.9 in the Appendix. These representative subalgebras allow us to determine invariant solutions of the SUSY GC equations (C.4.40) using the SRM. For the specific case where $f$ is constant (i.e. the SUSY GC equations (C.4.41)), the one-dimensional subalgebras of the resulting Lie symmetry superalgebra (C.6.6) can be found by taking the limit where the coefficients of $C_0$ tend to zero in the subalgebra listed in appendix C.9 and withdrawing repeated subalgebras, while rescaling appropiately.

## C.8. Invariant solutions of the fermionic supersymmetric GC equations

The SRM allows us to obtain invariant solutions of the GC equations (C.4.40). We proceed as follows. For each one-dimensional subalgebra listed in appendix C.9 (admitting standard invariants) we can find the orbit of the corresponding SUSY subgroup, which can be parametrized in terms of a bosonic symmetry variable $\xi$ and two fermionic symmetry variables, say $\eta$ and $\sigma$, which in turn are expressed in terms of $\theta^+$ and $\theta^-$, respectively. The superfields $\mathcal{U} = (H, Q^+, Q^-, R^+, R^-, \phi, f)$ are expanded in terms of the fermionic invariants $\eta$ and $\sigma$ with some coefficients expressed in terms of a bosonic symmetry variable $\xi$. Substituting these expanded forms of the superfields $\mathcal{U}$ into the GC equations (C.4.40) we reduce these equations to many possible differential subsystems involving even and odd functions. Solving these subsystems, we determine the invariant solutions and provide some geometrical interpretation of the associated surfaces. To illustrate this approach, we present three examples.

**1.** For the subalgebra $\mathfrak{g}_{124} = \{P_+ + \epsilon P_- + aK_0, \epsilon = \pm 1, a \neq 0\}$, the orbit of the corresponding group of the SUSY GC equations (C.4.40) can be parametrized as follows

$$\begin{aligned}
&H = e^{-ax_+}[h_0(\xi) + \theta^+\theta^- h_1(\xi)], && R^+ = r_0^+(\xi) + \theta^+\theta^- r_1^+(\xi),\\
&Q^+ = e^{ax_+}[q_0^+(\xi) + \theta^+\theta^- q_1^+(\xi)], && R^- = r_0^-(\xi) + \theta^+\theta^- r_1^-(\xi),\\
&Q^- = e^{ax_+}[q_0^-(\xi) + \theta^+\theta^- q_1^-(\xi)], && \phi = \varphi_0(\xi) + \theta^+\theta^- \varphi_1(\xi) + 2ax_+,\\
&f = \psi(\xi),
\end{aligned} \tag{C.8.1}$$

where the fermionic functions $H, Q^\pm$ and $R^\pm$ are expressed in terms of the bosonic symmetry variable $\xi = x_+ - \epsilon x_-$ and the fermionic symmetry variables $\theta^+$ and

$\theta^-$, while the bosonic functions $\varphi_0, \varphi_1$ and $\psi$ are expressed in terms of $\xi$ only. A corresponding invariant solution is given by

$$\begin{aligned} H &= -2\underline{C}_0^+\underline{C}_0^- e^{-ax_+}\left[\epsilon e^{-\varphi_0}\underline{m}_0^+ + i\theta^+\theta^-(e^{-\varphi_0}\underline{m}_0^+)_\xi\right], \\ Q^+ &= -e^{ax_+}\underline{C}_0^+\underline{C}_0^-\left[\underline{m}_0^+ + i\theta^+\theta^-((\underline{m}_0^+)_\xi + \epsilon a\underline{m}_0^+)\right], \\ Q^- &= e^{ax_+}\underline{C}_0^+\underline{C}_0^-\left[\underline{m}_0^- + i\theta^+\theta^-(\epsilon a\underline{m}_0^- + (\underline{m}_0^-)_\xi)\right], \\ \phi = \varphi_0(\xi) &+ i\theta^+\theta^-(\varphi_0)_\xi + 2ax_+, \quad R^+ = \underline{C}_0^+, \quad R^- = \underline{C}_0^-, \quad f = \psi(\xi), \end{aligned} \tag{C.8.2}$$

where the fermionic functions $\underline{m}_0^+$, $\underline{m}_0^-$ and the bosonic function $\varphi_0$ of the symmetry variable $\xi$ satisfy the differential constraint

$$[e^{-\varphi_0}(\underline{m}_0^- - \epsilon\underline{m}_0^+)]_\xi + \epsilon a\underline{m}_0^- e^{-\varphi_0} = 0. \tag{C.8.3}$$

Here $\psi$ is an arbitrary bosonic function of $\xi$, while $\underline{C}_0^+$ and $\underline{C}_0^-$ are arbitrary fermionic constants.

The first and second fundamental forms of the surface $S$ associated with the solution (C.8.2) are given by

$$\begin{aligned} I =& \psi e^{\varphi_0+2ax_+}d_+d_-\left[1 + \theta^+\theta^-\varphi_1\right], \\ II =& e^{ax_+}\underline{C}_0^+\underline{C}_0^-\psi\Big[d_+^2\left(\underline{m}_0^+ + i\theta^+\theta^-\left[(\underline{m}_0^+)_\xi + \epsilon a\underline{m}_0^+\right]\right) \\ & - 2d_+d_-\left(\epsilon\underline{m}_0^+ + i\theta^+\theta^-\left[(\underline{m}_0^+)_\xi - \epsilon i\varphi_1\underline{m}_0^+ - (\varphi_0)_\xi\underline{m}_0^+\right]\right) \\ & + d_-^2\left(\underline{m}_0^- + i\theta^+\theta^-\left[(\underline{m}_0^-)_\xi + \epsilon a\underline{m}_0^-\right]\right)\Big]. \end{aligned} \tag{C.8.4}$$

The Gaussian curvature (C.5.7) takes the form

$$\mathcal{K} = 0. \tag{C.8.5}$$

In particular when $a = 0$, which corresponds to the subalgebra $\mathfrak{g}_{14} = \{P_+ + \epsilon P_-\}$, the orbits of the group of the SUSY GC equations (C.4.40) can be parametrized in such a way that $H, Q^\pm$ and $R^\pm$ are fermionic functions of the bosonic symmetry variable $\xi = x_- - \epsilon x_+$, and the fermionic coordinates $\theta^+$ and $\theta^-$ while $\phi$ is a bosonic function of $\xi, \theta^+$ and $\theta^-$, and $\psi$ is a bosonic function of $\xi$ only. Under the assumption that the unknown functions take the form

$$\begin{aligned} &H = h_0(\xi) + \theta^+\theta^- h_1(\xi), \quad R^\pm = r_0^\pm(\xi) + \theta^+\theta^- r_1^\pm(\xi), \\ &Q^\pm = q_0^\pm(\xi) + \theta^+\theta^- q_1^\pm(\xi), \phi = \varphi_0(\xi) + \theta^+\theta^-\varphi_1(\xi), \qquad f = \psi(\xi), \end{aligned} \tag{C.8.6}$$

the corresponding invariant solution of the SUSY GC equations (C.4.40) is given by

$$\begin{aligned}
H =&2\underline{C}_0^-\underline{C}_0^+\underline{l}\left[\int e^{-\varphi_0}d\xi + i\theta^+\theta^- e^{-\varphi_0}\right] + \underline{C}, \qquad \epsilon = 1,\\
Q^+ =&\underline{C}_0^-\underline{C}_0^+\underline{l}e^{\varphi_0}\int e^{-\varphi_0}d\xi + \underline{C}_0^- B_0^+ e^{\varphi_0}\\
&+ i\theta^+\theta^-\underline{C}_0^-\left[\underline{C}_0^+\underline{l}\left(e^{\varphi_0}(\varphi_0)_\xi\int e^{-\varphi_0}d\xi + 1\right) + B_0^+ e^{\varphi_0}(\varphi_0)_\xi\right],\\
Q^- =&\underline{C}_0^+\underline{C}_0^-\underline{l}e^{\varphi_0}\int e^{-\varphi_0}d\xi + \underline{C}_0^+ B_0^- e^{\varphi_0}\\
&+ i\theta^+\theta^-\underline{C}_0^+\left[\underline{C}_0^-\underline{l}\left(e^{\varphi_0}(\varphi_0)_\xi\int e^{-\varphi_0}d\xi + 1\right) + B_0^- e^{\varphi_0}(\varphi_0)_\xi\right],\\
R^+ =&\underline{C}_0^+, \qquad R^- = \underline{C}_0^-,\\
\phi =&\varphi_0(\xi) + i\theta^+\theta^-(\varphi_0(\xi))_\xi, \qquad f = \psi(\xi),
\end{aligned} \tag{C.8.7}$$

where $\varphi_0$ and $\psi$ are bosonic functions of the symmetry variable $\xi = x_- - x_+$, while $\underline{C}_0^\pm, \underline{C}$ and $\underline{l}$ are arbitrary fermionic constants and $B_0^\pm$ are bosonic constants satisfying the algebraic constraint

$$\underline{C}_0^+ B_0^- + \underline{C}_0^- B_0^+ = 0. \tag{C.8.8}$$

For the solution (C.8.7), the tangent vectors are linearly dependent, so the immersion defines curves instead of surfaces.

**2.** For the subalgebra $\mathfrak{g}_{41} = \{C_0 + \epsilon P_+, \epsilon = \pm 1\}$, the orbit of the corresponding group of the SUSY GC equations (C.4.40) can be parametrized as follows

$$\begin{aligned}
H &= e^{\epsilon x_+}[h_0(x_-) + \theta^+\theta^- h_1(x_-)],\\
Q^\pm &= e^{\epsilon x_+}[q_0^\pm(x_-) + \theta^+\theta^- q_1^\pm(x_-)],\\
R^\pm &= r_0^\pm(x_-) + \theta^+\theta^- r_1^\pm(x_-),\\
\phi &= \varphi_0(x_-) + \theta^+\theta^-\varphi_1(x_-), \quad f = e^{-2\epsilon x_+}\psi(x_-),
\end{aligned} \tag{C.8.9}$$

where the bosonic symmetry variable is $x_-$ and the fermionic symmetry variables are $\theta^+$ and $\theta^-$. An invariant solution of the SUSY GC equations (C.4.40) takes the form

$$\begin{aligned}
&H = 2i\epsilon e^{\epsilon x_+ - \varphi_0}\left[\underline{C}_0^+ E_1 - \frac{\epsilon}{E_0}\underline{C}_0^+(A_0E_1 - A_1E_0)x_- + \theta^+\theta^- A_0\underline{C}_0^+\right],\\
&Q^+ = \underline{C}_0^+ e^{\epsilon x_+}(E_0 + \theta^+\theta^- E_1), \qquad Q^- = \underline{C}_0^+ e^{\epsilon x_+}(A_0 + \theta^+\theta^- A_1),\\
&R^+ = \underline{C}_0^+, \quad R^- = \underline{C}_0^+,\\
&\phi = \varphi_0(x_-) + \theta^+\theta^-\varphi_1(x_-), \quad f = e^{-2\epsilon x_+}\psi(x_-),
\end{aligned} \tag{C.8.10}$$

where the bosonic functions $\varphi_0$ and $\varphi_1$ satisfy the conditions

$$E_0\underline{C}_0^+\varphi_0 = \underline{C} - \epsilon A_0\underline{C}_0^+ x_-, \qquad \underline{C}_0^+\varphi_1 = \frac{\epsilon}{E_0^2}\underline{C}_0^+(A_0E_1 - A_1E_0)x_-, \tag{C.8.11}$$

respectively. Here $\psi$ is an arbitrary bosonic function of $x_-$ and $A_0, A_1, E_0$ and $E_1$ are arbitrary bosonic constants, while $\underline{C}_0^+$ and $\underline{C}$ are arbitrary fermionic constants.

The first and second fundamental forms of the surface $S$ (C.8.10) are given by

$$\begin{aligned} I &= e^{\varphi_0 - 2\epsilon x_+}\psi d_+ d_-(1 + \theta^+\theta^-\varphi_1), \\ I\!I &= \underline{C}_0^+ e^{-\epsilon x_+}\psi \Big[ d_+^2(E_0 + \theta^+\theta^- E_1) + d_-^2(A_0 + \theta^+\theta^- A_1) \\ &\quad + 2i\epsilon d_+ d_- \left( E_1 - \frac{\epsilon}{E_0}(A_0E_1 - A_1E_0)x_- \right) \\ &\quad + 2i\epsilon d_+ d_- \left( \theta^+\theta^- \left[ A_0\underline{C}_0^+ + E_1\varphi_1 - \frac{\epsilon}{E_0}(A_0E_1 - A_1E_0)x_-\varphi_1 \right] \right) \Big]. \end{aligned} \tag{C.8.12}$$

The Gaussian curvature (C.5.7) takes the form

$$\mathcal{K} = 0. \tag{C.8.13}$$

**3.** For the subalgebra $\mathfrak{g}_{35} = \{K_1 + aK_0 + bC_0, a \neq 0, b \neq 0\}$, we obtain the following parametrization of the orbit of the corresponding group of the SUSY GC equations (C.4.40)

$$\begin{aligned} H &= (x_+)^{(a-b)/2}[h_0(x_-) + \eta\theta^- h_1(x_-)], \qquad R^+ = (x_+)^{-1/2}[r_0^+(x_-) + \eta\theta^- r_1^+(x_-)], \\ Q^+ &= (x_+)^{-(a+b+2)/2}[q_0^+(x_-) + \eta\theta^- q_1^+(x_-)], \qquad R^- = r_0^-(x_-) + \eta\theta^- r_1^-(x_-), \\ Q^- &= (x_+)^{-(a+b)/2}[q_0^-(x_-) + \eta\theta^- q_1^-(x_-)], \\ \phi &= \varphi_0(x_-) + \eta\theta^-\varphi_1(x_-) - \frac{2a+1}{2}\ln x_+, \qquad f = (x_+)^b\psi(x_-), \end{aligned}$$

where the bosonic symmetry variable is $x_-$ and the fermionic symmetry variables are $\eta = (x_+)^{-1/2}\theta^+$ and $\theta^-$. A corresponding invariant solution of the SUSY GC equations (C.4.40) has the form

$$\begin{aligned} H &= (x_+)^{(a-b)/2} e^{A_0(a-b)x_-/2E_0} \left[ \underline{C} + i(x_+)^{-1/2}\theta^+\theta^-(a-b+1)A_0\underline{C}_0^+ e^{A_0x_-/2E_0} \right], \\ Q^+ &= \underline{C}_0^+(x_+)^{-(a+b+2)/2}\left[ E_0 + (x_+)^{-1/2}\theta^+\theta^- E_1 \right], \quad R^+ = \underline{C}_0^+(x_+)^{-1/2}, \\ Q^- &= A_0\underline{C}_0^+(x_+)^{-(a+b)/2}\left[ 1 + (x_+)^{-1/2}\theta^+\theta^-\frac{E_1}{E_0} \right], \qquad f = (x_+)^b\psi(x_-), \\ R^- &= \underline{C}_0^+, \quad \phi = \frac{A_0}{2E_0}(b-a-1)x_- + (x_+)^{-1/2}\theta^+\theta^-\varphi_1(x_-) - \frac{2a+1}{2}\ln x_+, \end{aligned} \tag{C.8.14}$$

where the bosonic function $\varphi_1$ of $x_-$ satisfies the constraint

$$\underline{C}_0^+\varphi_1 = \frac{E_1}{E_0}\underline{C}_0^+ + i\frac{(a-b)}{4E_0}\underline{C}e^{-A_0x_-/2E_0}, \tag{C.8.15}$$

and where $\psi$ is an arbitrary bosonic function $x_-$. Here $\underline{C}_0^+$ and $\underline{C}$ are arbitrary fermionic constants while $E_0, E_1$ and $A_0$ are arbitrary bosonic constants.

The first and second fundamental forms of the surface $S$ (C.8.14) are given by

$$\begin{aligned}
I =&(x_+)^{(2b-2a-1)/2}\exp\left(\frac{A_0}{2E_0}(b-a-1)x_-\right)\psi\\
&\times d_+d_-(1+(x_+)^{-1/2}\theta^+\theta^-\varphi_1),\\
I\!I =&(x_+)^{(b-a)/2}\psi\Big[\underline{C}_0^+(x_+)^{-1}d_+^2(E_0+(x_+)^{-1/2}\theta^+\theta^-E_1)\\
&+A_0\underline{C}_0^+d_-^2(1+(x_+)^{-1/2}\theta^+\theta^-E_1/E_0)\\
&+(x_+)^{-1/2}e^{A_0x_-/2E_0}d_+d_-\,(\underline{C}\\
&+(x_+)^{-1/2}\theta^+\theta^-\left[i\underline{C}\varphi_1+(a-b+1)A_0\underline{C}_0^+e^{A_0x_-/2E_0}\right])\Big].
\end{aligned}$$

The Gaussian curvature (C.5.7) takes the form

$$\mathcal{K} = 0. \tag{C.8.16}$$

Note that in our SUSY adaption of the classical geometric interpretation of surfaces in $\mathbb{R}^3$, the surfaces obtained in the three examples are composed of planar points or parabolic points.

## C.9. Conclusions

In this paper, we have formulated a fermionic SUSY extension of the GW and GC equation ((C.4.35) and (C.4.40) respectively) for conformally parametrized surfaces immersed in a Grassmann superspace $\mathbb{R}^{(1,1|3)}$. It is interesting and significant to note that the obtained SUSY GW equations (C.4.35) and GC equations (C.4.40) for a SUSY version of a moving frame resemble the form of the classical equations (C.3.8) and (C.3.1), respectively. The SUSY extension of the GW equations (C.4.35) is obtained in the Bianchi form. The zero curvature condition (C.4.36) for the fermionic SUSY GC equations (C.4.41) with the bosonic constant $f$ differs from its classical counterpart (C.3.2) in that it involves an anticommutator instead of a commutator. In addition, the signs of the last two terms change. The form of the zero-curvature condition (C.4.36) for the fermionic SUSY GC equations (C.4.41) differs from that for the previously established bosonic SUSY extension of the GC equations, as well as from the bosonic ZCC for the SUSY sine-Gordon equation (C.3.20). It should be noted that equation (C.4.36) does not involve the matrix $E$ given by (C.3.25). The SUSY sine-Gordon equation (C.3.20) does not determine a conformal parametrization of a surface. Hence, the surfaces associated with the SUSY sine-Gordon equation (C.3.20) and the link between the SUSY sine-Gordon model and the SUSY GW and GC equations require a separate investigation similar to the one performed in this paper.

The symmetries found for the obtained SUSY GC equation (C.4.40) include four bosonic dilations $C_0, K_0, K_1$ and $K_2$, two bosonic translations $P_+$ and $P_-$, and one fermionic shift $W$ together with the supersymmetry operators $J_+$ and $J_-$. This is in contrast with the symmetry algebra of the classical case, which is infinite-dimensional, and whose largest finite-dimensional subalgebra (C.3.15) contains three dilations $e_0, e_3$ and $e_4$, two translations $e_1$ and $e_2$, and two conformal transformations $e_5$ and $e_6$. Also, the classical Lie symmetry algebra $A$ contains a center $e_0$, while its SUSY counterpart does not. The classification list (by conjugacy classes) of the one-dimensional subalgebras of the Lie symmetry algebra $\mathfrak{g}$ for the fermionic SUSY extension of the GC equations (C.4.40) includes 199 subalgebras, which is different from both the equivalent classification for the classical model (which has 16 subalgebras) and that for the bosonic SUSY extension of this model (which has 99 subalgebras) [**B7**].

The first and second fundamental forms for SUSY conformally parametrized surfaces were established for the fermionic SUSY extension of the GC equations (C.4.40). These fundamental forms (C.5.1) and (C.5.5) differ in their signs from the classical case (equations (B.2.7) and (B.2.8) in [**B7**]). Also, we have established an analogue of the Bonnet Theorem for the fermionic SUSY GC equations (C.4.40). Three examples of solutions of the fermionic SUSY GC equations (C.4.40) are presented. For all three case, we found that the Gaussian curvature $K$ vanishes. However, the mean curvature $H$ is not zero. The relation $K \leqslant H^2$, present in the classical case, loses its meaning in the fermionic SUSY case.

This research could be extended in several directions. It could be beneficial to compute an exhaustive list of all symmetries of the fermionic SUSY GC equations (C.4.40) and to compare them to the classical case. The computation of such a list would require the development of a computer algebra Lie symmetry application capable of handling both even and odd Grassmann variables. Another open problem to be considered is whether all integrable SUSY systems possess non-standard invariants. It would be worthwhile to verify whether the conjecture proposed in [**29**,**85**] extends to all integrable SUSY models. It could also be worth attempting to establish a SUSY analogue of Noether's Theorem in order to study the conservation laws of such SUSY models. Finally, it would be interesting to investigate how the integrable characteristics, such as Hamiltonian structure and conserved quantities manifest themselves in surfaces for the SUSY case. These subjects will be investigated in our future work.

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## Annexe. Classification of the one-dimensionnal subalgebras of the Lie superalgebra (C.6.3).

Table C.3. Classification of the one-dimensional subalgebras of the symmetry superalgebra $\mathfrak{g}$ of the equations (C.4.40) into conjugacy classes. Here $\epsilon = \pm 1$, the parameters $a, b, c$ are non-zero bosonic constants, $\underline{\mu}$,$\underline{\nu}$,$\underline{\zeta}$ are non-zero fermionic constants.

| No | Subalgebra | No | Subalgebra |
|---|---|---|---|
| $\mathfrak{g}_1$ | $\{K_1^f\}$ | $\mathfrak{g}_2$ | $\{P_+\}$ |
| $\mathfrak{g}_3$ | $\{\underline{\mu}J_+\}$ | $\mathfrak{g}_4$ | $\{P_+ + \underline{\mu}J_+\}$ |
| $\mathfrak{g}_5$ | $\{K_2^f\}$ | $\mathfrak{g}_6$ | $\{P_-\}$ |
| $\mathfrak{g}_7$ | $\{\underline{\nu}J_-\}$ | $\mathfrak{g}_8$ | $\{P_- + \underline{\nu}J_-\}$ |
| $\mathfrak{g}_9$ | $\{K_1^f + aK_2^f\}$ | $\mathfrak{g}_{10}$ | $\{K_1^f + \epsilon P_-\}$ |
| $\mathfrak{g}_{11}$ | $\{K_1^f + \underline{\nu}J_-\}$ | $\mathfrak{g}_{12}$ | $\{K_1^f + \epsilon P_- + \underline{\nu}J_-\}$ |
| $\mathfrak{g}_{13}$ | $\{K_2^f + \epsilon P_+\}$ | $\mathfrak{g}_{14}$ | $\{P_+ + \epsilon P_-\}$ |
| $\mathfrak{g}_{15}$ | $\{P_+ + \underline{\nu}J_-\}$ | $\mathfrak{g}_{16}$ | $\{P_+ + \epsilon P_- + \underline{\nu}J_-\}$ |
| $\mathfrak{g}_{17}$ | $\{K_2^f + \underline{\mu}J_+\}$ | $\mathfrak{g}_{18}$ | $\{P_- + \underline{\mu}J_+\}$ |
| $\mathfrak{g}_{19}$ | $\{\underline{\mu}J_+ + \underline{\nu}J_-\}$ | $\mathfrak{g}_{20}$ | $\{P_- + \underline{\mu}J_+ + \underline{\nu}J_-\}$ |
| $\mathfrak{g}_{21}$ | $\{K_2^f + \epsilon P_+ + \underline{\mu}J_+\}$ | $\mathfrak{g}_{22}$ | $\{P_+ + \epsilon P_- + \underline{\mu}J_+\}$ |
| $\mathfrak{g}_{23}$ | $\{P_+ + \underline{\mu}J_+ + \underline{\nu}J_-\}$ | $\mathfrak{g}_{24}$ | $\{P_+ + \epsilon P_- + \underline{\mu}J_+ + \underline{\nu}J_-\}$ |
| $\mathfrak{g}_{25}$ | $\{\underline{\zeta}W\}$ | $\mathfrak{g}_{26}$ | $\{K_0\}$ |
| $\mathfrak{g}_{27}$ | $\{C_0\}$ | $\mathfrak{g}_{28}$ | $\{K_0 + aC_0\}$ |
| $\mathfrak{g}_{29}$ | $\{K_0 + \underline{\zeta}W\}$ | $\mathfrak{g}_{30}$ | $\{C_0 + \underline{\zeta}W\}$ |
| $\mathfrak{g}_{31}$ | $\{K_0 + aC_0 + \underline{\zeta}W\}$ | $\mathfrak{g}_{32}$ | $\{K_1^f + \underline{\zeta}W\}$ |
| $\mathfrak{g}_{33}$ | $\{K_1^f + aK_0\}$ | $\mathfrak{g}_{34}$ | $\{K_1^f + aC_0\}$ |
| $\mathfrak{g}_{35}$ | $\{K_1^f + aK_0 + bC_0\}$ | $\mathfrak{g}_{36}$ | $\{K_1^f + aK_0 + \underline{\zeta}W\}$ |
| $\mathfrak{g}_{37}$ | $\{K_1^f + aC_0 + \underline{\zeta}W\}$ | $\mathfrak{g}_{38}$ | $\{K_1^f + aK_0 + bC_0 + \underline{\zeta}W\}$ |
| $\mathfrak{g}_{39}$ | $\{P_+ + \underline{\zeta}W\}$ | $\mathfrak{g}_{40}$ | $\{K_0 + \epsilon P_+\}$ |
| $\mathfrak{g}_{41}$ | $\{C_0 + \epsilon P_+\}$ | $\mathfrak{g}_{42}$ | $\{K_0 + aC_0 + \epsilon P_+\}$ |
| $\mathfrak{g}_{43}$ | $\{K_0 + \epsilon P_+ + \underline{\zeta}W\}$ | $\mathfrak{g}_{44}$ | $\{C_0 + \epsilon P_+ + \underline{\zeta}W\}$ |

TABLE C.3. (Continued)

| No. | Subalgebra | No. | Subalgebra |
|---|---|---|---|
| $\mathfrak{g}_{45}$ | $\{K_0 + aC_0 + \epsilon P_+ + \underline{\zeta} W\}$ | $\mathfrak{g}_{46}$ | $\{\underline{\mu} J_+ + \underline{\zeta} W\}$ |
| $\mathfrak{g}_{47}$ | $\{K_0 + \underline{\mu} J_+\}$ | $\mathfrak{g}_{48}$ | $\{C_0 + \underline{\mu} J_+\}$ |
| $\mathfrak{g}_{49}$ | $\{K_0 + aC_0 + \underline{\mu} J_+\}$ | $\mathfrak{g}_{50}$ | $\{K_0 + \underline{\mu} J_+ + \underline{\zeta} W\}$ |
| $\mathfrak{g}_{51}$ | $\{C_0 + \underline{\mu} J_+ + \underline{\zeta} W\}$ | $\mathfrak{g}_{52}$ | $\{K_0 + aC_0 + \underline{\mu} J_+ + \underline{\zeta} W\}$ |
| $\mathfrak{g}_{53}$ | $\{P_+ + \underline{\mu} J_+ + \underline{\zeta} W\}$ | $\mathfrak{g}_{54}$ | $\{K_0 + \epsilon P_+ + \underline{\mu} J_+\}$ |
| $\mathfrak{g}_{55}$ | $\{C_0 + \epsilon P_+ + \underline{\mu} J_+\}$ | $\mathfrak{g}_{56}$ | $\{K_0 + aC_0 + \epsilon P_+ + \underline{\mu} J_+\}$ |
| $\mathfrak{g}_{57}$ | $\{K_0 + \epsilon P_+ + \underline{\mu} J_+ + \underline{\zeta} W\}$ | $\mathfrak{g}_{58}$ | $\{C_0 + \epsilon P_+ + \underline{\mu} J_+ + \underline{\zeta} W\}$ |
| $\mathfrak{g}_{59}$ | $\{K_0 + aC_0 + \epsilon P_+ + \underline{\mu} J_+ + \underline{\zeta} W\}$ | $\mathfrak{g}_{60}$ | $\{K_2^f + \underline{\zeta} W\}$ |
| $\mathfrak{g}_{61}$ | $\{K_2^f + aK_0\}$ | $\mathfrak{g}_{62}$ | $\{K_2^f + aC_0\}$ |
| $\mathfrak{g}_{63}$ | $\{K_2^f + aK_0 + bC_0\}$ | $\mathfrak{g}_{64}$ | $\{K_2^f + aK_0 + \underline{\zeta} W\}$ |
| $\mathfrak{g}_{65}$ | $\{K_2^f + aC_0 + \underline{\zeta} W\}$ | $\mathfrak{g}_{66}$ | $\{K_2^f + aK_0 + bC_0 + \underline{\zeta} W\}$ |
| $\mathfrak{g}_{67}$ | $\{P_- + \underline{\zeta} W\}$ | $\mathfrak{g}_{68}$ | $\{K_0 + \epsilon P_-\}$ |
| $\mathfrak{g}_{69}$ | $\{C_0 + \epsilon P_-\}$ | $\mathfrak{g}_{70}$ | $\{K_0 + aC_0 + \epsilon P_-\}$ |
| $\mathfrak{g}_{71}$ | $\{K_0 + \epsilon P_- + \underline{\zeta} W\}$ | $\mathfrak{g}_{72}$ | $\{C_0 + \epsilon P_- + \underline{\zeta} W\}$ |
| $\mathfrak{g}_{73}$ | $\{K_0 + aC_0 + \epsilon P_- + \underline{\zeta} W\}$ | $\mathfrak{g}_{74}$ | $\{\underline{\nu} J_- + \underline{\zeta} W\}$ |
| $\mathfrak{g}_{75}$ | $\{K_0 + \underline{\nu} J_-\}$ | $\mathfrak{g}_{76}$ | $\{C_0 + \underline{\nu} J_-\}$ |
| $\mathfrak{g}_{77}$ | $\{K_0 + aC_0 + \underline{\nu} J_-\}$ | $\mathfrak{g}_{78}$ | $\{K_0 + \underline{\nu} J_- + \underline{\zeta} W\}$ |
| $\mathfrak{g}_{79}$ | $\{C_0 + \underline{\nu} J_- + \underline{\zeta} W\}$ | $\mathfrak{g}_{80}$ | $\{K_0 + aC_0 + \underline{\nu} J_- + \underline{\zeta} W\}$ |
| $\mathfrak{g}_{81}$ | $\{P_- + \underline{\nu} J_- + \underline{\zeta} W\}$ | $\mathfrak{g}_{82}$ | $\{K_0 + \epsilon P_- + \underline{\nu} J_-\}$ |
| $\mathfrak{g}_{83}$ | $\{C_0 + \epsilon P_- + \underline{\nu} J_-\}$ | $\mathfrak{g}_{84}$ | $\{K_0 + aC_0 + \epsilon P_- + \underline{\nu} J_-\}$ |
| $\mathfrak{g}_{85}$ | $\{K_0 + \epsilon P_- + \underline{\nu} J_- + \underline{\zeta} W\}$ | $\mathfrak{g}_{86}$ | $\{C_0 + \epsilon P_- + \underline{\nu} J_- + \underline{\zeta} W\}$ |
| $\mathfrak{g}_{87}$ | $\{K_0 + aC_0 + \epsilon P_- + \underline{\nu} J_- + \underline{\zeta} W\}$ | $\mathfrak{g}_{88}$ | $\{K_1^f + aK_2^f + \underline{\zeta} W\}$ |
| $\mathfrak{g}_{89}$ | $\{K_0 + aK_1^f + bK_2^f\}$ | $\mathfrak{g}_{90}$ | $\{C_0 + aK_1^f + bK_2^f\}$ |
| $\mathfrak{g}_{91}$ | $\{K_0 + aC_0 + bK_1^f + cK_2^f\}$ | $\mathfrak{g}_{92}$ | $\{K_0 + aK_1^f + bK_2^f + \underline{\zeta} W\}$ |
| $\mathfrak{g}_{93}$ | $\{C_0 + aK_1^f + bK_2^f + \underline{\zeta} W\}$ | $\mathfrak{g}_{94}$ | $\{K_0 + aC_0 + bK_1^f + cK_2^f + \underline{\zeta} W\}$ |
| $\mathfrak{g}_{95}$ | $\{K_1^f + \epsilon P_- + \underline{\zeta} W\}$ | $\mathfrak{g}_{96}$ | $\{K_0 + aK_1^f + \epsilon P_-\}$ |
| $\mathfrak{g}_{97}$ | $\{C_0 + aK_1^f + \epsilon P_-\}$ | $\mathfrak{g}_{98}$ | $\{K_0 + aC_0 + bK_1^f + \epsilon P_-\}$ |
| $\mathfrak{g}_{99}$ | $\{K_0 + aK_1^f + \epsilon P_- + \underline{\zeta} W\}$ | $\mathfrak{g}_{100}$ | $\{C_0 + aK_1^f + \epsilon P_- + \underline{\zeta} W\}$ |
| $\mathfrak{g}_{101}$ | $\{K_0 + aC_0 + bK_1^f + \epsilon P_- + \underline{\zeta} W\}$ | $\mathfrak{g}_{102}$ | $\{K_1^f + \underline{\nu} J_- + \underline{\zeta} W\}$ |
| $\mathfrak{g}_{103}$ | $\{K_0 + aK_1^f + \underline{\nu} J_-\}$ | $\mathfrak{g}_{104}$ | $\{C_0 + aK_1^f + \underline{\nu} J_-\}$ |
| $\mathfrak{g}_{105}$ | $\{K_0 + aC_0 + bK_1^f + \underline{\nu} J_-\}$ | $\mathfrak{g}_{106}$ | $\{K_0 + aK_1^f + \underline{\nu} J_- + \underline{\zeta} W\}$ |
| $\mathfrak{g}_{107}$ | $\{C_0 + aK_1^f + \underline{\nu} J_- + \underline{\zeta} W\}$ | $\mathfrak{g}_{108}$ | $\{K_0 + aC_0 + bK_1^f + \underline{\nu} J_- + \underline{\zeta} W\}$ |
| $\mathfrak{g}_{109}$ | $\{K_1^f + \epsilon P_- + \underline{\nu} J_- + \underline{\zeta} W\}$ | $\mathfrak{g}_{110}$ | $\{K_0 + aK_1^f + \epsilon P_- + \underline{\nu} J_-\}$ |
| $\mathfrak{g}_{111}$ | $\{C_0 + aK_1^f + \epsilon P_- + \underline{\nu} J_-\}$ | $\mathfrak{g}_{112}$ | $\{K_0 + aC_0 + bK_1^f + \epsilon P_- + \underline{\nu} J_-\}$ |
| $\mathfrak{g}_{113}$ | $\{K_0 + aK_1^f + \epsilon P_- + \underline{\nu} J_- + \underline{\zeta} W\}$ | $\mathfrak{g}_{114}$ | $\{C_0 + aK_1^f + \epsilon P_- + \underline{\nu} J_- + \underline{\zeta} W\}$ |
| $\mathfrak{g}_{115}$ | $\{K_0 + aC_0 + bK_1^f + \epsilon P_- + \underline{\nu} J_- + \underline{\zeta} W\}$ | $\mathfrak{g}_{116}$ | $\{K_2^f + \epsilon P_+ + \underline{\zeta} W\}$ |

TABLE C.3. (Continued)

| No. | Subalgebra | No. | Subalgebra |
|---|---|---|---|
| $\mathfrak{g}_{117}$ | $\{K_0 + aK_2^f + \epsilon P_+\}$ | $\mathfrak{g}_{118}$ | $\{C_0 + aK_2^f + \epsilon P_+\}$ |
| $\mathfrak{g}_{119}$ | $\{K_0 + aC_0 + bK_2^f + \epsilon P_+\}$ | $\mathfrak{g}_{120}$ | $\{K_0 + aK_2^f + \epsilon P_+ + \underline{\zeta}W\}$ |
| $\mathfrak{g}_{121}$ | $\{C_0 + aK_2^f + \epsilon P_+ + \underline{\zeta}W\}$ | $\mathfrak{g}_{122}$ | $\{K_0 + aC_0 + bK_2^f + \epsilon P_+ + \underline{\zeta}W\}$ |
| $\mathfrak{g}_{123}$ | $\{P_+ + \epsilon P_- + \underline{\zeta}W\}$ | $\mathfrak{g}_{124}$ | $\{P_+ + \epsilon P_- + aK_0\}$ |
| $\mathfrak{g}_{125}$ | $\{P_+ + \epsilon P_- + aC_0\}$ | $\mathfrak{g}_{126}$ | $\{P_+ + \epsilon P_- + aK_0 + bC_0\}$ |
| $\mathfrak{g}_{127}$ | $\{P_+ + \epsilon P_- + aK_0 + \underline{\zeta}W\}$ | $\mathfrak{g}_{128}$ | $\{P_+ + \epsilon P_- + aC_0 + \underline{\zeta}W\}$ |
| $\mathfrak{g}_{129}$ | $\{P_+ + \epsilon P_- + aK_0 + bC_0\underline{\zeta}W\}$ | $\mathfrak{g}_{130}$ | $\{P_+ + \underline{\nu}J_- + \underline{\zeta}W\}$ |
| $\mathfrak{g}_{131}$ | $\{K_0 + \epsilon P_+ + \underline{\nu}J_-\}$ | $\mathfrak{g}_{132}$ | $\{C_0 + \epsilon P_+ + \underline{\nu}J_-\}$ |
| $\mathfrak{g}_{133}$ | $\{K_0 + aC_0 + \epsilon P_+ + \underline{\nu}J_-\}$ | $\mathfrak{g}_{134}$ | $\{K_0 + \epsilon P_+ + \underline{\nu}J_- + \underline{\zeta}W\}$ |
| $\mathfrak{g}_{135}$ | $\{C_0 + \epsilon P_+ + \underline{\nu}J_- + \underline{\zeta}W\}$ | $\mathfrak{g}_{136}$ | $\{K_0 + aC_0 + \epsilon P_+ + \underline{\nu}J_- + \underline{\zeta}W\}$ |
| $\mathfrak{g}_{137}$ | $\{P_+ + \epsilon P_- + \underline{\nu}J_- + \underline{\zeta}W\}$ | $\mathfrak{g}_{138}$ | $\{P_+ + \epsilon P_- + aK_0 + \underline{\nu}J_-\}$ |
| $\mathfrak{g}_{139}$ | $\{P_+ + \epsilon P_- + aC_0 + \underline{\nu}J_-\}$ | $\mathfrak{g}_{140}$ | $\{P_+ + \epsilon P_- + aK_0 + bC_0 + \underline{\nu}J_-\}$ |
| $\mathfrak{g}_{141}$ | $\{P_+ + \epsilon P_- + aK_0 + \underline{\nu}J_- + \underline{\zeta}W\}$ | $\mathfrak{g}_{142}$ | $\{P_+ + \epsilon P_- + aC_0 + \underline{\nu}J_- + \underline{\zeta}W\}$ |
| $\mathfrak{g}_{143}$ | $\{P_+ + \epsilon P_- + aK_0 + bC_0$ $+\underline{\nu}J_- + \underline{\zeta}W\}$ | $\mathfrak{g}_{144}$ | $\{K_2^f + \underline{\mu}J_+ + \underline{\zeta}W\}$ |
| $\mathfrak{g}_{145}$ | $\{K_0 + aK_2^f + \underline{\mu}J_+\}$ | $\mathfrak{g}_{146}$ | $\{C_0 + aK_2^f + \underline{\mu}J_+\}$ |
| $\mathfrak{g}_{147}$ | $\{K_0 + aC_0 + bK_2^f + \underline{\mu}J_+\}$ | $\mathfrak{g}_{148}$ | $\{K_0 + aK_2^f + \underline{\mu}J_+ + \underline{\zeta}W\}$ |
| $\mathfrak{g}_{149}$ | $\{C_0 + aK_2^f + \underline{\mu}J_+ + \underline{\zeta}W\}$ | $\mathfrak{g}_{150}$ | $\{K_0 + aC_0 + bK_2^f + \underline{\mu}J_+ + \underline{\zeta}W\}$ |
| $\mathfrak{g}_{151}$ | $\{P_- + \underline{\mu}J_+ + \underline{\zeta}W\}$ | $\mathfrak{g}_{152}$ | $\{K_0 + \epsilon P_- + \underline{\mu}J_+\}$ |
| $\mathfrak{g}_{153}$ | $\{C_0 + \epsilon P_- + \underline{\mu}J_+\}$ | $\mathfrak{g}_{154}$ | $\{K_0 + aC_0 + \epsilon P_- + \underline{\mu}J_+\}$ |
| $\mathfrak{g}_{155}$ | $\{K_0 + \epsilon P_- + \underline{\mu}J_+ + \underline{\zeta}W\}$ | $\mathfrak{g}_{156}$ | $\{C_0 + \epsilon P_- + \underline{\mu}J_+ + \underline{\zeta}W\}$ |
| $\mathfrak{g}_{157}$ | $\{K_0 + aC_0 + \epsilon P_- + \underline{\mu}J_+ + \underline{\zeta}W\}$ | $\mathfrak{g}_{158}$ | $\{\underline{\mu}J_+ + \underline{\nu}J_- + \underline{\zeta}W\}$ |
| $\mathfrak{g}_{159}$ | $\{K_0 + \underline{\mu}J_+ + \underline{\nu}J_-\}$ | $\mathfrak{g}_{160}$ | $\{C_0 + \underline{\mu}J_+ + \underline{\nu}J_-\}$ |
| $\mathfrak{g}_{161}$ | $\{K_0 + aC_0 + \underline{\mu}J_+ + \underline{\nu}J_-\}$ | $\mathfrak{g}_{162}$ | $\{K_0 + \underline{\mu}J_+ + \underline{\nu}J_- + \underline{\zeta}W\}$ |
| $\mathfrak{g}_{163}$ | $\{C_0 + \underline{\mu}J_+ + \underline{\nu}J_- + \underline{\zeta}W\}$ | $\mathfrak{g}_{164}$ | $\{K_0 + aC_0 + \underline{\mu}J_+ + \underline{\nu}J_- + \underline{\zeta}W\}$ |
| $\mathfrak{g}_{165}$ | $\{P_- + \underline{\mu}J_+ + \underline{\nu}J_- + \underline{\zeta}W\}$ | $\mathfrak{g}_{166}$ | $\{K_0 + \epsilon P_- + \underline{\mu}J_+ + \underline{\nu}J_-\}$ |
| $\mathfrak{g}_{167}$ | $\{C_0 + \epsilon P_- + \underline{\mu}J_+ + \underline{\nu}J_-\}$ | $\mathfrak{g}_{168}$ | $\{K_0 + aC_0 + \epsilon P_- + \underline{\mu}J_+ + \underline{\nu}J_-\}$ |
| $\mathfrak{g}_{169}$ | $\{K_0 + \epsilon P_- + \underline{\mu}J_+ + \underline{\nu}J_- + \underline{\zeta}W\}$ | $\mathfrak{g}_{170}$ | $\{C_0 + \epsilon P_- + \underline{\mu}J_+ + \underline{\nu}J_- + \underline{\zeta}W\}$ |
| $\mathfrak{g}_{171}$ | $\{K_0 + aC_0 + \epsilon P_- + \underline{\mu}J_+$ $+\underline{\nu}J_- + \underline{\zeta}W\}$ | $\mathfrak{g}_{172}$ | $\{K_2^f + \epsilon P_+ + \underline{\mu}J_+ + \underline{\zeta}W\}$ |
| $\mathfrak{g}_{173}$ | $\{K_0 + aK_2^f + \epsilon P_+ + \underline{\mu}J_+\}$ | $\mathfrak{g}_{174}$ | $\{C_0 + aK_2^f + \epsilon P_+ + \underline{\mu}J_+\}$ |
| $\mathfrak{g}_{175}$ | $\{K_0 + aC_0 + bK_2^f + \epsilon P_+ + \underline{\mu}J_+\}$ | $\mathfrak{g}_{176}$ | $\{K_0 + aK_2^f + \epsilon P_+ + \underline{\mu}J_+ + \underline{\zeta}W\}$ |
| $\mathfrak{g}_{177}$ | $\{C_0 + aK_2^f + \epsilon P_+ + \underline{\mu}J_+ + \underline{\zeta}W\}$ | $\mathfrak{g}_{178}$ | $\{K_0 + aC_0 + bK_2^f + \epsilon P_+$ $+\underline{\mu}J_+ + \underline{\zeta}W\}$ |
| $\mathfrak{g}_{179}$ | $\{P_+ + \epsilon P_- + \underline{\mu}J_+ + \underline{\zeta}W\}$ | $\mathfrak{g}_{180}$ | $\{P_+ + \epsilon P_- + aK_0 + \underline{\mu}J_+\}$ |
| $\mathfrak{g}_{181}$ | $\{P_+ + \epsilon P_- + aC_0 + \underline{\mu}J_+\}$ | $\mathfrak{g}_{182}$ | $\{P_+ + \epsilon P_- + aK_0 + bC_0 + \underline{\mu}J_+\}$ |
| $\mathfrak{g}_{183}$ | $\{P_+ + \epsilon P_- + aK_0 + \underline{\mu}J_+ + \underline{\zeta}W\}$ | $\mathfrak{g}_{184}$ | $\{P_+ + \epsilon P_- + aC_0 + \underline{\mu}J_+ + \underline{\zeta}W\}$ |
| $\mathfrak{g}_{185}$ | $\{P_+ + \epsilon P_- + aK_0 + bC_0$ $+\underline{\mu}J_+ + \underline{\zeta}W\}$ | $\mathfrak{g}_{186}$ | $\{P_+ + \underline{\mu}J_+ + \underline{\nu}J_- + \underline{\zeta}W\}$ |

TABLE C.3. (Continued)

| No. | Subalgebra | No. | Subalgebra |
|---|---|---|---|
| $\mathfrak{g}_{187}$ | $\{K_0 + \epsilon P_+ + \underline{\mu} J_+ + \underline{\nu} J_-\}$ | $\mathfrak{g}_{188}$ | $\{C_0 + \epsilon P_+ + \underline{\mu} J_+ + \underline{\nu} J_-\}$ |
| $\mathfrak{g}_{189}$ | $\{K_0 + aC_0 + \epsilon P_+ + \underline{\mu} J_+ + \underline{\nu} J_-\}$ | $\mathfrak{g}_{190}$ | $\{K_0 + \epsilon P_+ + \underline{\mu} J_+ + \underline{\nu} J_- + \underline{\zeta} W\}$ |
| $\mathfrak{g}_{191}$ | $\{C_0 + \epsilon P_+ + \underline{\mu} J_+ + \underline{\nu} J_- + \underline{\zeta} W\}$ | $\mathfrak{g}_{192}$ | $\{K_0 + aC_0 + \epsilon P_+ + \underline{\mu} J_+$ $+ \underline{\nu} J_- + \underline{\zeta} W\}$ |
| $\mathfrak{g}_{193}$ | $\{P_+ + \epsilon P_- + \underline{\mu} J_+ + \underline{\nu} J_- + \underline{\zeta} W\}$ | $\mathfrak{g}_{194}$ | $\{P_+ + \epsilon P_- + aK_0 + \underline{\mu} J_+ + \underline{\nu} J_-\}$ |
| $\mathfrak{g}_{195}$ | $\{P_+ + \epsilon P_- + aC_0 + \underline{\mu} J_+ + \underline{\nu} J_-\}$ | $\mathfrak{g}_{196}$ | $\{P_+ + \epsilon P_- + aK_0 + bC_0$ $+ \underline{\mu} J_+ + \underline{\nu} J_-\}$ |
| $\mathfrak{g}_{197}$ | $\{P_+ + \epsilon P_- + aK_0 + \underline{\mu} J_+$ $+ \underline{\nu} J_- + \underline{\zeta} W\}$ | $\mathfrak{g}_{198}$ | $\{P_+ + \epsilon P_- + aC_0 + \underline{\mu} J_+$ $+ \underline{\nu} J_- + \underline{\zeta} W\}$ |
| $\mathfrak{g}_{199}$ | $\{P_+ + \epsilon P_- + aK_0 + bC_0$ $+ \underline{\mu} J_+ + \underline{\nu} J_- + \underline{\zeta} W\}$ | | |