Allocation via Deferred-Acceptance under Responsive Priorities*

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Abstract

In many economic environments—such as college admissions, student placements at public schools, and university housing allocation—indivisible objects with capacity constraints are assigned to a set of agents when each agent receives at most one object and monetary compensations are not allowed. In these important applications the agent-proposing deferred-acceptance algorithm with responsive priorities (called responsive DA-rule) performs well and economists have successfully implemented responsive DA-rules or slight variants thereof. First, for house allocation problems we characterize the class of responsive DA-rules by a set of basic and intuitive properties, namely, *unavailable type invariance*, *individual rationality*, *weak non-wastefulness*, *resource-monotonicity*, *truncation invariance*, and *strategy-proofness*. We extend this characterization to the full class of allocation problems with capacity constraints by replacing *resource-monotonicity* with *two-agent consistent conflict resolution*. An alternative characterization of responsive DA-rules is obtained using *unassigned objects invariance*, *individual rationality*, *weak non-wastefulness*, *weak consistency*, and *strategy-proofness*. Various characterizations of the class of "acyclic" responsive DA-rules are obtained by using the properties *efficiency*, *group strategy-proofness*, and *consistency*.

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1 Introduction

We study the allocation of indivisible objects with capacity constraints to a set of agents when each agent receives at most one object and monetary compensations are not possible. Important applications of this model are the assignment of students to public (private) schools, college or university admissions, and college or university housing allocation. We assume that agents (e.g., students) in these situations have strict preferences over the (object) types (e.g., admission to a specific school, college, or university or dormitory rooms of a certain type or in a certain building) and that (object) types might come with a capacity constraint (the maximal number of students a school, college, or university can admit or the maximal number of dormitory rooms of the same type). An allocation rule is a systematic way of solving any allocation problem (with capacity constraints).

The search for "good" rules is the subject of many recent contributions, but most of them deal with the special case of house allocation problems, when exactly one object of each type is available (for instance, Abdulkadiroğlu and Sönmez, 1998; Ehlers, 2002; Ehlers and Klaus, 2003, 2007, 2009; Ergin, 2002; Kesten, 2009; Pápai, 2000). In most papers that study the allocation of indivisible objects with capacity constraints, externally prescribed priorities are also specified; the corresponding class of problems is usually referred to as "school choice problems" or "student placement problems" (see Sönmez and Ünver, 2009, for a recent survey). Balinski and Sönmez (1999) were the first to formulate the allocation problem based on priorities which in many real life situation naturally arise, e.g., in school choice students who live closer to a school and/or have siblings attending a school have higher priority at that school (Abdulkadiroğlu and Sönmez, 2003). The agents' priorities for a certain type are captured by an ordering of the agents. A priority structure is a profile specifying for each type a priority ordering. Given agents' priorities, it is natural to require that the allocation is "stable" with respect to the priorities. This means that there should be no agent who—conditional on higher priority—envies another agent (for receiving a better object). Given a priority structure, Gale and Shapley's (1962) famous deferred acceptance algorithm (an algorithm which has been extensively applied in practice, see Roth, 2008) can be used to find the agent-optimal stable allocation for any problem with capacity constraints and responsive priorities. The agents' priorities for one type can be interpreted as this type's preference. Of course, since sets of agents are assigned to a type, in general priorities may depend on the whole set of agents and not only on individual agents. However, if priorities over sets of agents are responsive with respect to the priority ordering over individual agents, then in determining stable allocations we only need to know the priority orderings over individual agents. It is also this feature that makes the agent-proposing deferred-acceptance rule easily applicable in practice. We call a rule which is based on the agents-proposing deferred-acceptance algorithm with responsive priorities a *responsive DA-rule*.

Note that we do not a priori assume that priorities are externally given. The only two other papers that consider this more general model of object allocation with multiple copies of each type and capacity constraints we are aware of are Ehlers and Klaus (2006) and Kojima and Manea (2009). Kojima and Manea (2009) point out that "Despite the importance of deferred acceptance rules in both theory and practice, no axiomatization has yet been obtained in an object allocation setting with unspecified priorities." Then, they proceed to provide two characterizations of deferred acceptance rules with so-called acceptant substitutable priorities (a larger class of rules than the class of responsive DA-rules which is based on priorities that are determined by a choice function that reflects substitutability in preferences over sets of agents; see also Hatfield and Milgrom, 2005). For this class of DA-rules, priority orderings over individual agents do not suffice—the priorities over sets of agents must be known. In their characterizations, Kojima and Manea (2009) use two new monotonicity properties (*individually rational monotonicity* and *weak Maskin monotonicity*) together with *non-wastefulness* and *population-monotonicity*.[1]

We consider situations where resources may change, i.e., it could be that additional objects are available. When the change of the environment is exogenous, it would be unfair if the agents who were not responsible for this change were treated unequally. We apply this idea of solidarity and require that if additional resources become available, then all agents (weakly) gain. This requirement is called *resource-monotonicity* (Chun and Thomson, 1988). Various recent studies (Ehlers and Klaus, 2003, 2009; Kesten, 2009) consider resource-monotonic rules for housing markets (the restricted model where only one object per type is available) and demonstrate that *resource-monotonicity* together with *efficiency* and some other properties characterizes a small class of responsive DA-rules, the class of mixed dictator-pairwise-exchange rules, that are based on acyclic priority structures. Here, we only impose the mild efficiency requirement of *weak non-wastefulness*[2] as well as the very basic and intuitive properties of *individual rationality*[3] and *unavailable type invariance*.[4] We also impose the invariance property *truncation invariance*[5] which is in spirit a weak form of Nash's IIA from bargaining. Our last property is the well-known strategic robustness condition of *strategy-proofness*.[6] First, we show that these elementary and intuitive properties characterize, for house allocation problems, the class of responsive DA-rules that are based on the agent-proposing deferred-acceptance algorithm with responsive priority structures (Theorem 1). Second, we extend this characterization to the class of all problems with capacity constraints, by

[1]In their first characterization of deferred acceptance rules for acceptable substitutable priorities, Kojima and Manea (2009, Theorem 1) use *non-wastefulness* and *individually rational monotonicity*. A second characterization (Kojima and Manea, 2009, Theorem 2) is in terms of *non-wastefulness*, *population monotonicity*, and *weak Maskin monotonicity*. We discuss how our and Kojima and Manea's (2009) results relate in Section 5.

[2]No agent who does not receive any object would prefer to obtain a real object that is not assigned.

[3]Each agent weakly prefers his allotment to not receiving any object.

[4]The rule only depends on the set of available objects.

[5]If an agent reduces his set of acceptable objects without changing the preference between any two real objects and his assigned object remains acceptable under the new preference, then the rule should choose the same allocation for the initial profile and the new one.

[6]No agent can manipulate the allocation to his advantage by lying about his preferences.

replacing *resource-monotonicity* with the new property of *two-agent consistent conflict resolution*[7] (Theorem 2).

Another situation of interest is the change of the set of agents and objects because agents leave with their allotments. *Consistency* requires that the allocation for the "reduced economy" allocates the remaining objects to the remaining agents in the same way as before (see Thomson 2009 for an extensive survey of this property in various economic models). Since many rules do not satisfy *consistency*, we introduce *weak consistency*, which only requires that agents who received the null object in the original economy still receive the null object in any reduced economy. We obtain a third characterization of the class of responsive DA-rules by *unassigned objects invariance*,[8] *individual rationality*, *weak non-wastefulness*, *weak consistency*, and *strategy-proofness* (Theorem 3).

Finally, various characterizations of the class of "acyclic" responsive DA-rules are obtained by using the properties *efficiency*, *group strategy-proofness*, or *consistency* (Corollaries 1, 2, and 3).

The paper is organized as follows. In Section 2 we introduce the allocation model of indivisible objects with capacity constraints, properties of rules, and the class of responsive DA-rules. In Section 3 we state and proof our first two characterizations of the class of responsive DA-rules (Theorems 1 and 2). In Section 4 we state and prove our third characterization of the class of responsive DA-rules (Theorem 3). Section 5 concludes with a brief discussion of our results compared to those of Kojima and Manea (2009). In Appendices A and B we discuss the independence of properties in Theorems 1, 2, and 3.

2 Allocation with Variable Resources

Let $N = \{1, \ldots, n\}$ denote the finite set of agents with $n \geq 2$. Let O denote the set of potential (real) object types or types for short. We assume that O contains at least two elements and that O is finite.[9] Not receiving any real type is called "receiving the null object." Let $\emptyset$ represent the *null object*.

Each agent $i \in N$ is equipped with a preference relation R_i over all types $O \cup \{\emptyset\}$. The preference relation R_i is strict, i.e., R_i is a linear order over $O \cup \{\emptyset\}$. Given $x, y \in O \cup \{\emptyset\}$, $x \, P_i \, y$ means that agent i strictly prefers x to y (and $x \neq y$) and $x \, R_i \, y$ means that agent i weakly prefers x to y (and $x \, P_i \, y$ or $x = y$). Let $\mathcal{R}$ denote the class of all linear orders over $O \cup \{\emptyset\}$, and $\mathcal{R}^N$ the set of *(preference) profiles* $R = (R_i)_{i \in N}$ such that for all $i \in N$, $R_i \in \mathcal{R}$.

Given $R \in \mathcal{R}^N$ and $M \subseteq N$, let R_M denote the profile $(R_i)_{i \in M}$. It is the restriction of R to the set of agents M. We also use the notation $R_{-M} = R_{N \setminus M}$ and $R_{-i} = R_{N \setminus \{i\}}$.

[7] In each maximal conflict situation in which two agents compete for the same object under the same profile and in which one of the agents receives the object, the conflict is resolved consistently in that the same agent always receives the object.

[8] The rule only depends on the set of assigned objects.

[9] Our results remain unchanged when O is infinite. For expositional convenience, finiteness of O is assumed.

Given $O' \subseteq O \cup \{\emptyset\}$, let $R_i|_{O'}$ denote the restriction of R_i to O' and $R|_{O'} = (R_i|_{O'})_{i \in N}$. Given $i \in N$ and $R_i \in \mathcal{R}$, object $x \in O$ is *acceptable under* R_i if $x \; P_i \; \emptyset$. Let $A(R_i) = \{x \in O : x \; P_i \; \emptyset\}$ denote the *set of acceptable objects under* R_i.

For each type $x \in O$, at most $\overline{q}_x \in \mathbb{N}$ copies are available in any economy with $1 \leq \overline{q}_x \leq |N|$. Let $\mathcal{Q} = \times_{x \in O}\{0, 1, \ldots, \overline{q}_x\}$. Let $q_x \in \{0, 1, \ldots, \overline{q}_x\}$ denote the number of available copies of object x (or the capacity of object x). The null object is always available without scarcity and therefore we set $q_\emptyset = \infty$.

An *(allocation) problem (with capacity constraints)* consists of a preference profile $R \in \mathcal{R}^N$ and a capacity vector $q = (q_x)_{x \in O}$ such that for all types $x \in O$, $0 \leq q_x \leq \overline{q}_x$. The set of all problems is $\mathcal{R}^N \times \mathcal{Q}$. Given a capacity vector q, let $O_+(q) = \{x \in O : q_x > 0\}$ denote the set of *available real types* under q. The set of available types is $O_+(q) \cup \{\emptyset\}$ and includes the null object, which is available in any problem.

An allocation problem where at most one copy of each type is available, i.e., for all $x \in O$, $\bar{q}_x = 1$, is called a *house allocation problem.*

Each agent i is to be allocated exactly one object in $O \cup \{\emptyset\}$ taking capacity constraints into account. An *allocation for capacity vector* q is a list $a = (a_i)_{i \in N}$ such that for all $i \in N$, $a_i \in O \cup \{\emptyset\}$, and any real type $x \in O$ is not assigned more than q_x times, i.e., for all $x \in O$, $|\{i \in N : a_i = x\}| \leq q_x$. Note that $\emptyset$, the null object, can be assigned to any number of agents and that not all real objects have to be assigned. Let $\mathcal{A}_q$ denote the set of all allocations for q. Let $\mathcal{A} = \bigcup_{q \in \mathcal{Q}} \mathcal{A}_q$.

An *(allocation) rule* is a function $\varphi : \mathcal{R}^N \times \mathcal{Q} \to \mathcal{A}$ such that for all problems $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, $\varphi(R, q) \in \mathcal{A}_q$, i.e., for all $i \in N$, $\varphi_i(R, q) \in O_+(q) \cup \{\emptyset\}$. By the feasibility assumption included in the definition of the set of allocations for any capacity vector q, each agent receives an available type. Given $i \in N$, we call $\varphi_i(R, q)$ the *allotment* of agent i at $\varphi(R, q)$.

A natural requirement for a rule is that the chosen allocation depends only on preferences over the set of available types. Given a capacity vector q, a type x is *unavailable* if $q_x = 0$.

Unavailable Object Type Invariance: For all $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$ and all $R' \in \mathcal{R}^N$ such that $R|_{O_+(q) \cup \{\emptyset\}} = R'|_{O_+(q) \cup \{\emptyset\}}$, $\varphi(R, q) = \varphi(R', q)$.

By *individual rationality* each agent should weakly prefer his allotment to the null object (which may represent an outside option such as off-campus housing in the context of university housing allocation, or private schools or home schooling in the context of student placement in public schools).

Individual Rationality: For all $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$ and all $i \in N$, $\varphi_i(R, q) \; R_i \; \emptyset$.

Next, we introduce two properties that require a rule to not waste any resources. First, *non-wastefulness* (Balinski and Sönmez, 1999) requires that no agent prefers an available object that is not assigned to his allotment.

Non-Wastefulness: For all $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, all $x \in O_+(q)$, and all $i \in N$, if $x \, P_i \, \varphi_i(R, q)$, then $|\{j \in N : \varphi_j(R, q) = x\}| = q_x$.

Note that Kojima and Manea's (2009) *non-wastefulness* condition is equivalent to *individual rationality* and *non-wastefulness* as stated here.[10] Next, we weaken *non-wastefulness* by requiring that no agent receives the null object while he prefers an available object that is not assigned.

Weak Non-Wastefulness: For all $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, all $x \in O_+(q)$, and all $i \in N$, if $x \, P_i \, \varphi_i(R, q)$ and $\varphi_i(R, q) = \emptyset$, then $|\{j \in N : \varphi_j(R, q) = x\}| = q_x$.

Of course, no resources are wasted if a rule is *(Pareto) efficient*.

Efficiency: For all $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, there exists no allocation $a \in \mathcal{A}_q$ such that for all $i \in N$, $a_i \, R_i \, \varphi_i(R, q)$, and for some $j \in N$, $a_j \, P_j \, \varphi_j(R, q)$.

Note that *efficiency* implies *individual rationality* and *(weak) non-wastefulness*.

When the set of objects varies, another natural requirement is *resource-monotonicity*. It describes the effect of a change in the available resource on the welfare of the agents. A rule is *resource-monotonic* if the availability of more real objects has a (weakly) positive effect on all agents.

Resource-Monotonicity: For all $R \in \mathcal{R}^N$ and all $q, q' \in \mathcal{Q}$, if for all $x \in O$, $q_x \leq q'_x$, then for all $i \in N$, $\varphi_i(R, q') \, R_i \, \varphi_i(R, q)$.

A truncation strategy is a preference relation that ranks the real objects in the same way as the corresponding original preference relation and each real object which is acceptable under the truncation strategy is also acceptable under the original preference relation. Formally, given $i \in N$ and $R_i \in \mathcal{R}$, a strategy $\bar{R}_i \in \mathcal{R}$ is a *truncation (strategy) of* R_i if (t1) $\bar{R}_i|_O = R_i|_O$ and (t2) $A(\bar{R}_i) \subseteq A(R_i)$. Loosely speaking, a truncation strategy of R_i is obtained by moving the null object "up".

If an agent truncates his preference in a way such that his allotment remains acceptable under the truncated preference, then *truncation invariance* requires that the allocation is the same under both profiles.

Truncation Invariance: For all $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, all $i \in N$, and all $\bar{R}_i \in \mathcal{R}_i$, if $\bar{R}_i$ is a truncation of R_i and $\varphi_i(R, q)$ is acceptable under $\bar{R}_i$ (i.e., $\varphi_i(R, q) \in A(\bar{R}_i)$) and $\bar{R} = (\bar{R}_i, R_{-i})$, then $\varphi(\bar{R}, q) = \varphi(R, q)$.

To the best of our knowledge, any mechanism that is used in real life satisfies this property (e.g., any mechanism based on or equivalent to the famous deferred-acceptance mechanism or the so-called priority mechanisms, Roth and Sotomayor, 1990, Sections 5.4.1 and 5.5.1).

[10] **Kojima and Manea (2009) Non-Wastefulness:** For all $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, all $x \in O_+(q) \cup \{\emptyset\}$ with $|\{j \in N : \varphi_j(R, q) = x\}| < q_x$, and all $i \in N$, $\varphi_i(R, q) \, R_i \, x$.

Remark 1. ***Truncation Invariance***
Truncation invariance could be interpreted as a variant of Nash's IIA in bargaining: namely, we may see the allocation where everybody receives the null object as the disagreement point and the set of *individually rational* allocations as the "bargaining set". Start with a problem (R, q) and suppose that the set of *individually rational* allocations shrinks under $\bar{R}$ and $\varphi(R, q)$ remains *individually rational* under $\bar{R}$ (i.e., for all $i \in N$, $\bar{R}_i$ is a truncation of R_i such that $\varphi_i(R, q) \in A(\bar{R}_i)$). Then $\varphi(R, q)$ should be still chosen for the problem $(\bar{R}, q)$. ◇

Strategy-proofness requires that no agent can ever benefit from misrepresenting his preferences.

Strategy-Proofness: For all $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, all $i \in N$, and all $\bar{R}_i \in \mathcal{R}$, $\varphi_i(R, q)\ R_i\ \varphi_i((\bar{R}_i, R_{-i}), q)$.

The following strengthening of *strategy-proofness* requires that no group of agents can ever benefit by misrepresenting their preferences.

Group Strategy-Proofness: For all $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, all $M \subseteq N$, and all $\bar{R}_M \in \mathcal{R}^M$, if for all $i \in M$, $\varphi_i((\bar{R}_M, R_{-M}), q)\ R_i\ \varphi_i(R, q)$, then for all $i \in M$, $\varphi_i((\bar{R}_M, R_{-M}), q) = \varphi_i(R, q)$.

Our last property applies to situations when two agents compete for the same object in a maximal conflict situation, i.e., they have the same preference relation with only one acceptable real object $x \in O$. *Two-agent consistent conflict resolution* then requires that if in two problems two agents compete for the same object in a maximal conflict situation under the same preference profile and in both problems one of them receives the object, the conflict is resolved consistently in that it has to be the same agent in both problems who "wins the conflict" and receives the object.

We denote a preference relation with only one acceptable object $x \in O$ by R^x, i.e., $A(R^x) = \{x\}$. We denote the set of all preference relations that have $x \in O$ as unique acceptable object by $\mathcal{R}^x$.

Two-Agent Consistent Conflict Resolution: For all $R \in \mathcal{R}^N$, all $q, q' \in \mathcal{Q}$, and all $R^x \in \mathcal{R}^x$, if $R_i = R_j = R^x$ and $\{\varphi_i(R, q), \varphi_j(R, q)\} = \{\varphi_i(R, q'), \varphi_j(R, q')\} = \{x, \emptyset\}$, then for $k \in \{i, j\}$, $\varphi_k(R, q) = \varphi_k(R, q')$.

Given type x, let $\succ_x$ denote a *priority ordering on* N, e.g., $\succ_x$: $1\ 2\ \ldots\ n$ means that agent 1 has higher priority for type x than agent 2, who has higher priority for type x than agent 3, etc. Let $\succ \equiv (\succ_x)_{x \in O}$ denote a *priority structure.* Then, given a priority structure $\succ$ and a problem (R, q), we can interpret $(R, \succ, q)$ as a college admissions problem (Gale and Shapley, 1962; Roth and Sotomayor, 1990) where the set of agents N corresponds to the set of students, the set of types O corresponds to the set of colleges, the capacity vector q describes colleges' quota, preferences R correspond to students' preferences over colleges, and the priority structure $\succ$ corresponds to colleges' preferences over students and colleges' preferences over sets of students are responsive (we define and discuss responsiveness in Remark 2). Stability is an important requirement for many real-life matching markets and it turns out to be essential in our context of allocating indivisible objects to agents.

Stability under $\succ$: Given $(R,q) \in \mathcal{R}^N \times \mathcal{Q}$, an *allocation* $a \in \mathcal{A}_q$ *is stable under* $\succ$ if there exists no agent-object pair $(i,x) \in N \times O \cup \{\emptyset\}$ such that $x \, P_i \, \varphi_i(R,q)$ and (s1) $|\{j \in N : \varphi_j(R,q) = x\}| < q_x$ or (s2) there exists $k \in N$ such that $\varphi_k(R,q) = x$ and $i \succ_x k$.[11]

For any college admissions problem with responsive preferences $(R, \succ, q)$, we denote by $DA^{\succ}(R,q)$ the student/agent-optimal allocation that is obtained by using Gale and Shapley's (1962) student/agent-proposing deferred-acceptance algorithm.

Responsive *DA*-Rules: A rule φ is a responsive DA-rule if there exists a priority structure $\succ$ such that for each $(R,q) \in \mathcal{R}^N \times \mathcal{Q}$, $\varphi(R,q) = DA^{\succ}(R,q)$.

Note that the above condition of stability under $\succ$ takes only blocking by individual agents and by agent-type pairs into account. This is sometimes referred to as "pairwise" stability. However, one may also consider group stability where blocking is allowed by arbitrary groups of agents and types. For college admissions problems with responsive preferences, pairwise stability and group stability coincide. In such environments it suffices to know the priority orderings over individual agents and the implementation of DA-rules is much easier than it would be for more general college preferences (e.g., substitutable preferences). From now on, we will use the term *responsive priorities* when referring to the responsiveness of college preferences in the associated college admissions problem.[12]

Remark 2. ***Responsive Priorities***
Let $x \in O$, $0 < q_x \leq \bar{q}_x$, and $\succ_x$ be a priority ordering. Let $2^N_{q_x}$ denote the set of all subsets of N that do not exceed the capacity q_x, i.e., $2^N_{q_x} \equiv \{S \subseteq N \mid |S| \leq q_x\}$. Let P_x denote a *priority relation* on $2^N_{q_x}$, i.e., P_x strictly orders all sets in $2^N_{q_x}$. Then, P_x is responsive to $\succ_x$ if the following two conditions hold: (r1) for all $S \in 2^N_{q_x}$ such that $|S| < q_x$ and all $i \in N \setminus S$, $S \cup \{i\} \; P_x \; S$ and (r2) for all $S \in 2^N_{q_x}$ such that $|S| < q_x$ and all $i, j \in N \setminus S$, $(S \cup \{i\}) \; P_x \; (S \cup \{j\})$ if and only if $i \succ_x j$. When formulating (r1) we implicitly assume that each object finds all agents acceptable. Let $\mathcal{P}(\succ_x)$ denote the set of all priority relations that are responsive to $\succ_x$.

Now, given $(R,q) \in \mathcal{R}^N \times \mathcal{Q}$ and a priority relation profile $P_O \in \times_{x \in O} \mathcal{P}(\succ_x)$, an *allocation* $a \in \mathcal{A}_q$ *is group stable under* P_O if there exists no coalition (consisting possibly of both agents and types) that blocks allocation a.[13]

It is known that given $(R,q) \in \mathcal{R}^N \times \mathcal{Q}$ and a responsive priority relation profile $P_O \in \times_{x \in O} \mathcal{P}(\succ_x)$, an allocation $a \in \mathcal{A}_q$ is group stable under P_O if and only if $a \in \mathcal{A}_q$ is stable under

[11] Note that if a rule is *non-wasteful*, then (s1) can never occur. However, *weak non-wastefulness* together with *individual rationality* does not imply (s1).

[12] Correspondingly, we would call priorities *substitutable* when referring to the substitutability of college preferences in the associated college admissions problem.

[13] Formally, given (R,q) and P_O, a coalition $S \subseteq N \cup O$ blocks $a \in \mathcal{A}_q$ if there exists an allocation $b \in \mathcal{A}_q$ such that (g1) for all $i \in S \cap N$, $b_i \in S \cap O$, (g2) for all $i \in S \cap N$, $b_i \; R_i \; a_i$, (g3) for all $x \in S \cap O$, $i \in \{j \in N \mid b_j = x\}$ implies $i \in S \cup \{j \in N \mid a_j = x\}$, (g4) for all $x \in S \cap O$, $\{j \in N \mid b_j = x\} \; R_x \; \{j \in N \mid a_j = x\}$, and (g5) for at least one member of S, (g2) or (g4) holds with strict preference.

$\succ$. In other words, group stability is identical with stability for responsive priorities and the set of group stable matchings is invariant with respect to the responsive preference extensions of $\succ$. This implies that for the implementation of any responsive DA-rule, we only need to know the priority orderings over individual agents and not the complete priority relations over sets of agents. This makes the application of responsive DA-rules very easy and convenient in real-life matching markets. ◇

We next introduce an acyclicity condition, due to Ergin (2002).

Cycles and Acyclicity: Given a priority structure $\succ$, a *cycle* consists of distinct $x, y \in O$ and $i, j, k \in N$ such that the following two conditions are satisfied:
cycle condition $i \succ_x j \succ_x k$ and $k \succ_y i$ and
c-scarcity condition there exist disjoint sets $N_x, N_y \subseteq N \setminus \{i, j, k\}$ (possibly $N_x = \emptyset$ or $N_y = \emptyset$) such that $N_x \subseteq \{l \in N : l \succ_x j\}$, $N_y \subseteq \{l \in N : l \succ_y i\}$, $|N_x| = q_x - 1$, and $|N_y| = q_y - 1$.

A priority structure is *acyclic* if no cycles exist. For house allocation problems, the cycle condition is sufficient to establish the existence of a cycle. For other problems, the c-scarcity condition limits the definition of a cycle to cases where there indeed might exist problems such that agents i, j, and k compete for types x and y.

A responsive DA-rule is *acyclic* if the associated priority structure is acyclic. Ergin (2002, Theorem 1) shows that the acyclicity of the priority structure $\succ$ is equivalent to *efficiency* or *group strategy-proofness* of the induced responsive DA-rule $DA^{\succ}$. Crawford (1991) studied comparative statics of DA-rules. From his results it follows that responsive DA-rules are *resource-monotonic*.[14]

3 Resource-Monotonicity and Two-Agent Consistent Conflict Resolution

3.1 Resource-Monotonicity for House Allocation Problems

We first prove a characterization of the class of all responsive DA-rules for house allocation problems.

Theorem 1. *On the class of house allocation problems, responsive DA-rules are the only rules satisfying unavailable type invariance, individual rationality, weak non-wastefulness, resource-monotonicity, truncation invariance, and strategy-proofness.*

[14]To be precise, Crawford (1991) only discusses "*type resource-monotonicity*," i.e., *resource-monotonicity* when extra types are added. Note that our *resource-monotonicity* property is different in that also changes of the capacity of types are considered. However, by using the well-known technique of transforming a college admissions problem with responsive preferences into a related marriage problem (see Roth and Sotomayor, 1990, Section 5.3.1) and then applying Crawford (1991, Theorem 1) one can easily show that responsive DA-rules are *resource-monotonic*.

We prove Theorem 1 in Section 3.3. This proof also reveals the following additional result. Let φ be a rule. Call φ *stable* if there exists a priority structure $\succ$ such that for each problem $(R,q) \in \mathcal{R}^N \times \mathcal{Q}$, $\varphi(R,q)$ is stable under $\succ$.

Proposition 1. *On the class of house allocation problems, if a rule satisfies unavailable type invariance, individual rationality, weak non-wastefulness, resource-monotonicity, and truncation invariance, then it is stable.*

Theorem 1 and Ergin's (2002, Theorem 1) result imply the following characterizations of the class of acyclic responsive DA-rules for house allocation problems.

Corollary 1. *On the class of house allocation problems,*
(a) *responsive DA-rules with acyclic priority structures are the only rules satisfying unavailable type invariance, efficiency, resource-monotonicity, truncation invariance, and strategy-proofness;*
(b) *responsive DA-rules with acyclic priority structures are the only rules satisfying unavailable type invariance, individual rationality, weak non-wastefulness, resource-monotonicity, truncation invariance, and group strategy-proofness.*

The following example demonstrates that Theorem 1 does not hold on the class of all problems.

Example 1. The following rule f, which is not a responsive DA-rule, satisfies *unavailable type invariance*, *individual rationality*, *weak non-wastefulness*, *resource-monotonicity*, *truncation invariance*, and *strategy-proofness*. Let $N = \{1,2,3\}$, $O = \{x,y\}$, and $\bar{q}_x = 2$ and $\bar{q}_y = 1$. Furthermore, $\succ_x$: 1 2 3, $\succ'_x$: 1 3 2, and $\succ_y$: 1 2 3. Let $\succ = (\succ_x, \succ_y)$ and $\succ' = (\succ'_x, \succ_y)$. Then, for each problem $(R,q) \in \mathcal{R}^N \times \mathcal{Q}$,

$$f(R,q) = \begin{cases} DA^{\succ'}(R,q) & \text{if } q_x = 2 \text{ and } x \text{ is agent 1's favorite object in } O_+(q) \text{ and} \\ DA^{\succ}(R,q) & \text{otherwise.} \end{cases}$$

It is easy to see that f satisfies *unavailable type invariance*, *individual rationality*, *weak non-wastefulness*, *truncation invariance*, and *strategy-proofness*.

For *resource-monotonicity*, let $R \in \mathcal{R}^N$ and $q, q' \in \mathcal{Q}$ be such that for all $z \in O$, $q_z \leq q'_z$. We only have to check for possible violations of *resource-monotonicity* when f uses a different priority structure for (R,q) and (R,q'). Hence, $q'_x = 2$.

If $q_x = 2$, then x is agent 1's favorite object in $O_+(q)$ and y is agent 1's favorite object in $O_+(q')$. But then $q_y = 0$, $q'_y = 1$ and $f_1(R,q') = y$. But now two copies of object x are available for agents 2 and 3 in (R,q') and *resource-monotonicity* is satisfied.

Otherwise $q_x < 2$ and x must be agent 1's favorite object in $O_+(q')$.

If $q_x = 0$, then the only violation of *resource-monotonicity* could be that an agent received y at $f(R,q)$ and $\emptyset$ at $f(R,q')$. However, since object y is allocated according to the same priority ordering, this cannot happen.

If $q_x = 1$, then $f_1(R, q) = x$ and none of the agents 2 or 3 can obtain x. Hence, in terms of allocating object x is does not matter if priority ordering $\succ_x$ or $\succ'_x$ is used. Hence, it is as if $\succ'$ is used for both problems and no violation of *resource-monotonicity* occurs. $\diamond$

3.2 Two-Agent Consistent Conflict Resolution

Next, we obtain a characterization of responsive DA-rules for problems with capacity constraints by replacing *resource-monotonicity* in Theorem 1 with *two-agent consistent conflict resolution*.

Theorem 2. *Responsive DA-rules are the only rules satisfying unavailable type invariance, individual rationality, weak non-wastefulness, two-agent consistent conflict resolution, truncation invariance, and strategy-proofness.*

We discuss the independence of properties in Theorems 1 and 2 in Appendix A.

Theorem 2 and Ergin's (2002, Theorem 1) result imply the following characterizations of the class of acyclic responsive DA-rules for house allocation problems.

Corollary 2.
(a) *Responsive DA-rules with acyclic priority structures are the only rules satisfying unavailable type invariance, efficiency, two-agent consistent conflict resolution, truncation invariance, and strategy-proofness.*
(b) *Responsive DA-rules with acyclic priority structures are the only rules satisfying unavailable type invariance, individual rationality, weak non-wastefulness, two-agent consistent conflict resolution, truncation invariance, and group strategy-proofness.*

3.3 Proof of Proposition 1 and Theorems 1 and 2

We prove Theorem 2, but we will specify how certain parts of the proof have to be adjusted for Theorem 1 and Proposition 1.

It is easy to verify that responsive DA-rules satisfy the properties of Theorems 1 and 2. Conversely, let φ be a rule satisfying the properties of Theorem 2 (Theorem 1 respectively). First, we "calibrate the priority structure using maximal conflict preference profiles".

Let $x \in O$ and let $R^x \in \mathcal{R}^x$ (i.e., $A(R^x) = x$). Let $R^\emptyset \in \mathcal{R}$ be such that $A(R^\emptyset) = \emptyset$.

For any $S \subseteq N$, let $R^x_S = (R^x_i)_{i \in S}$ such that for all $i \in S$, $R^x_i = R^x$, and similarly $R^\emptyset_S = (R^\emptyset_i)_{i \in S}$ such that for all $i \in S$, $R^\emptyset_i = R^\emptyset$.

Let 1_x denote the capacity vector q such that $q_x = 1$ and for all $z \in O \setminus \{x\}$, $q_z = 0$. Similarly, for $y \in O \setminus \{x\}$ let 1_{xy} denote the capacity vector q such that $q_x = 1$, $q_y = 1$, and for all $z \in O \setminus \{x, y\}$, $q_z = 0$.

Consider the problem $(R^x_N, 1_x)$. By *weak non-wastefulness*, for some $i \in N$, $\varphi_i(R^x_N, 1_x) = x$, say $i = 1$. Then, for all $i \in N \setminus \{1\}$, we set $1 \succ_x i$.

Next consider the problem $((R^{\emptyset}_1, R^x_{-1}), 1_x)$. By *weak non-wastefulness* and *individual rationality*, for some $i \in N \setminus \{1\}$, $\varphi_i((R^{\emptyset}_1, R^x_{-1}), 1_x) = x$, say $i = 2$. Then, for all $i \in N \setminus \{1, 2\}$, we set $2 \succ_x i$.

By induction, we obtain $\succ_x$ for any type x and thus a priority structure $\succ = (\succ_x)_{x \in O}$.

Lemma 1. *For all $R \in \mathcal{R}^N$ and all $x \in O$, if for some $j \in N$, $\varphi_j(R, 1_x) = x$, then for all $i \in N \setminus \{j\}$, $x \in A(R_i)$ implies $j \succ_x i$.*

Proof. Let $R \in \mathcal{R}^N$ and $x \in O$. Without loss of generality, suppose $1 \succ_x 2 \succ_x \cdots \succ_x n$. Let $S = \{i \in N : x \in A(R_i)\}$ and let $j = \min S$. We prove Lemma 1 by showing that $\varphi_j(R, 1_x) = x$. In the sequel, when using *two-agent consistent conflict resolution* we often also implicitly apply *weak non-wastefulness* and *individual rationality*.

Note that for all $i \in N \setminus S$, $\emptyset P_i x$. We partition the set $N \setminus S$ into the "lower" set $L = \{1, \ldots, j-1\}$ (possibly $L = \emptyset$) and the "upper" set $U = N \setminus (L \cup S)$ (possibly $U = \emptyset$). Note that by *unavailable type invariance*, $\varphi(R, 1_x) = \varphi((R^{\emptyset}_L, R^x_S, R^{\emptyset}_U), 1_x)$.

By the construction of $\succ_x$, $\varphi_j((R^{\emptyset}_L, R^x_{S\cup U}), 1_x) = x$. Hence, if $U = \emptyset$, then $\varphi_j(R, 1_x) = x$ and for all $i \in N$, $x \in A(R_i)$ implies $j \succ_x i$.

Step 1: Let $k \in U$. We prove that $\varphi_j((R^{\emptyset}_L, R^x_{S\cup(U\setminus\{k\})}, R^{\emptyset}_k), 1_x) = \varphi_j((R^{\emptyset}_L, R^x_{S\cup U}), 1_x) = x$.

By *two-agent consistent conflict resolution*, $\varphi_j((R^{\emptyset}_L, R^x_{S\cup U}), 1_{xy}) = x$. Let $y \in O \setminus \{x\}$ and $R'_k \in \mathcal{R}$ be such that $R'_k : y\ x\ \emptyset\ \ldots$. By *strategy-proofness*, $\varphi_k((R^{\emptyset}_L, R^x_{S\cup(U\setminus\{k\})}), R'_k), 1_{xy}) \neq x$. If $S \cup (U \setminus \{k\}) = \{j\}$, then by *weak non-wastefulness* and *individual rationality*, $\varphi_j((R^{\emptyset}_L, R^x_{S\cup(U\setminus\{k\})}), R'_k), 1_{xy}) = x$. Otherwise, by *two-agent consistent conflict resolution*, $\varphi_j((R^{\emptyset}_L, R^x_{S\cup(U\setminus\{k\})}), R'_k), 1_{xy}) = x$. By *weak non-wastefulness* and *individual rationality*, $\varphi_k((R^{\emptyset}_L, R^x_{S\cup(U\setminus\{k\})}), R'_k), 1_{xy}) = y$.

Let $R''_k \in \mathcal{R}$ be such that $R''_k : y\ \emptyset\ x\ \ldots$ and $R''_k|_O = R'_k|_O$. By *strategy-proofness*, $\varphi_k((R^{\emptyset}_L, R^x_{S\cup(U\setminus\{k\})}), R''_k), 1_{xy}) = y$. By *weak non-wastefulness* and *individual rationality*, for some $l \in S \cup (U \setminus \{k\})$, $\varphi_l((R^{\emptyset}_L, R^x_{S\cup(U\setminus\{k\})}, R''_k), 1_{xy}) = x$.

Now R''_k is a truncation of R'_k and both $y \in A(R''_k)$ and $\varphi_k((R^{\emptyset}_L, R^x_{S\cup(U\setminus\{k\})}, R'_k), 1_{xy}) = y$. Thus, $\varphi((R^{\emptyset}_L, R^x_{S\cup(U\setminus\{k\})}, R''_k), 1_{xy}) = \varphi((R^{\emptyset}_L, R^x_{S\cup(U\setminus\{k\})}, R'_k), 1_{xy})$. Hence, $j = l$ and $\varphi_j((R^{\emptyset}_L, R^x_{S\cup(U\setminus\{k\})}, R''_k), 1_{xy}) = x$.

By *individual rationality*, $\varphi_k((R^{\emptyset}_L, R^x_{S\cup(U\setminus\{k\})}), R''_k), 1_x) \neq x$. If $S \cup (U \setminus \{k\}) = \{j\}$, then by *weak non-wastefulness* and *individual rationality*, $\varphi_j((R^{\emptyset}_L, R^x_{S\cup(U\setminus\{k\})}), R''_k), 1_x) = x$. Otherwise, by *two-agent consistent conflict resolution*, $\varphi_j((R^{\emptyset}_L, R^x_{S\cup(U\setminus\{k\})}), R''_k), 1_x) = x$. Thus, by *unavailable type invariance*, $\varphi_j((R^{\emptyset}_L, R^x_{S\cup(U\setminus\{k\})}, R^{\emptyset}_k), 1_x) = \varphi_j((R^{\emptyset}_L, R^x_{S\cup(U\setminus\{k\})}), R''_k), 1_x) = x$.

Steps 2, ...: Let $U = \{k_1, \ldots, k_l\}$. Then using the same arguments as above, it follows that $x = \varphi_j((R^{\emptyset}_L, R^x_{S\cup U}), 1_x) = \varphi_j((R^{\emptyset}_L, R^x_{S\cup(U\setminus\{k_1\})}, R^{\emptyset}_{k_1}), 1_x) = \varphi_j((R^{\emptyset}_L, R^x_{S\cup(U\setminus\{k_1,k_2\})}), R^{\emptyset}_{\{k_1,k_2\}}), 1_x) = \ldots = \varphi_j((R^{\emptyset}_L, R^x_S \cup k_l, R^{\emptyset}_{U\setminus\{k_l\}}), 1_x) = \varphi_j((R^{\emptyset}_L, R^x_S, R^{\emptyset}_U), 1_x) = \varphi_j(R, 1_x)$. Hence, we obtain the desired result that $\varphi_j(R, 1_x) = x$. $\square$

The proof of Lemma 1 as part of the proof of Theorem 1 and Proposition 1 uses the following *Step 1* (without using *two-agent consistent conflict resolution* and *strategy-proofness*).

Step 1: Let $k \in U$. We prove that $\varphi_j((R^{\emptyset}_L, R^x_{S\cup(U\setminus\{k\})}, R^{\emptyset}_k), 1_x) = \varphi_j((R^{\emptyset}_L, R^x_{S\cup U}), 1_x) = x$.

Let $R'_k \in \mathcal{R}$ be such that $R'_k : y \; x \; \emptyset \; \ldots$. By *unavailable type invariance*, $\varphi_j((R^{\emptyset}_L, R^x_{S\cup(U\setminus\{k\})}), R'_k), 1_x) = \varphi_j((R^{\emptyset}_L, R^x_{S\cup U}), 1_x) = x$. Now by *resource-monotonicity*, $\varphi_j((R^{\emptyset}_L, R^x_{S\cup(U\setminus\{k\})}, R'_k), 1_{xy}) = x$. By *weak non-wastefulness* and *individual rationality*, $\varphi_k((R^{\emptyset}_L, R^x_{S\cup(U\setminus\{k\})}), R'_k), 1_{xy}) = y$.

On the other hand, suppose that $\varphi_j((R^{\emptyset}_L, R^x_{S\cup(U\setminus\{k\})}, R^{\emptyset}_k), 1_x) \neq x$. Then, by *individual rationality*, $\varphi_k((R^{\emptyset}_L, R^x_{S\cup(U\setminus\{k\})}, R^{\emptyset}_k), 1_x) \neq x$. Let $R''_k \in \mathcal{R}$ be such that $R''_k : y \; \emptyset \; x \; \ldots$ and $R''_k|_O = R'_k|_O$. By *unavailable type invariance*, $\varphi((R^{\emptyset}_L, R^x_{S\cup(U\setminus\{k\})}, R''_k), 1_x) = \varphi((R^{\emptyset}_L, R^x_{S\cup(U\setminus\{k\})}, R^{\emptyset}_k), 1_x)$. Thus, by *weak non-wastefulness* and *individual rationality*, for some $l \in S \cup (U \setminus \{k\})$, $\varphi_l((R^{\emptyset}_L, R^x_{S\cup(U\setminus\{k\})}, R''_k), 1_x) = x$.

Now by *resource-monotonicity*, $\varphi_l((R^{\emptyset}_L, R^x_{S\cup(U\setminus\{k\})}, R''_k), 1_{xy}) = x$. By *weak non-wastefulness* and *individual rationality*, $\varphi_k((R^{\emptyset}_L, R^x_{S\cup(U\setminus\{k\})}, R''_k), 1_{xy}) = y$. Now R''_k is a truncation of R'_k and both $y \in A(R''_k)$ and $\varphi_k((R^{\emptyset}_L, R^x_{S\cup(U\setminus\{k\})}, R'_k), 1_{xy}) = y$. Since $\varphi_j((R^{\emptyset}_L, R^x_{S\cup(U\setminus\{k\})}, R''_k), 1_{xy}) \neq \varphi_j((R^{\emptyset}_L, R^x_{S\cup(U\setminus\{k\})}, R'_k), 1_{xy})$, this is a contradiction to *truncation invariance*. Hence, $\varphi_j((R^{\emptyset}_L, R^x_{S\cup(U\setminus\{k\})}, R^{\emptyset}_k), 1_x) = x$.

With this adjusted proof and the following lemma, Proposition 1 follows.

Lemma 2. *For all house allocation problems $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, $\varphi(R, q)$ is stable under $\succ$.*

Proof. Let $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$ be a house allocation problem. Assume that $\varphi(R, q)$ is not stable under $\succ$. Then, there exists an agent-object pair $(i, x) \in N \times O \cup \{\emptyset\}$ such that $x \; P_i \; \varphi_i(R, q)$ and (s1) $\{j \in N : \varphi_j(R, q) = x\} = \emptyset$ or (s2) there exists $k \in N$ such that $\varphi_k(R, q) = x$ and $i \succ_x k$. By *individual rationality*, $x \neq \emptyset$.

Let $\bar{R} \in \mathcal{R}^N$ be such that (a) for all $j \in N$ such that $\varphi_j(R, q) \neq \emptyset$, $\bar{R}_j$ is a truncation of R_j such that there exists no $y \in O \setminus \{\varphi_j(R, q)\}$ with $\varphi_j(R, q) \; \bar{R}_j \; y \; \bar{R}_j \; \emptyset$ and (b) for all $j \in N$ such that $\varphi_j(R, q) = \emptyset$, $\bar{R}_j = R_j$. (By *individual rationality*, $\bar{R}_j$ in (a) is well-defined as truncation of R_j.) By *truncation invariance*, $\varphi(\bar{R}, q) = \varphi(R, q)$ and $(i, x) \in N \times O$ is such that $x \; \bar{P}_i \; \varphi_i(\bar{R}, q)$ and (s1') $\{j \in N : \varphi_j(\bar{R}, q) = x\} = \emptyset$ or (s2') there exists $k \in N$ such that $\varphi_k(\bar{R}, q) = x$ and $i \succ_x k$.

Let $S = \{j \in N : x \; \bar{P}_j \; \varphi_j(\bar{R}, q)\}$. Note that $i \in S$. Let $T = \{j \in N : \varphi_j(\bar{R}, q) = x\}$. Note that $|T| \in \{0, 1\}$. By *unavailable type invariance*, $\varphi(\bar{R}, 1_x) = \varphi((\bar{R}^x_{S\cup T}, \bar{R}^{\emptyset}_{-(S\cup T)}), 1_x)$.

If for $j \in S$, $\varphi_j(\bar{R}, 1_x) = x$, then by *resource-monotonicity*, $\varphi_j(\bar{R}, q) \; \bar{R}_j \; x$, contradicting $x \; \bar{P}_j \; \varphi_j(\bar{R}, q)$. Hence, for all $j \in S$, $\varphi_j(\bar{R}, 1_x) = \emptyset$. Note that by construction of $\bar{R}$, for all $j \in N \setminus (S \cup T)$, $\emptyset \; \bar{P}_j \; x$. Hence, by *individual rationality*, for all $j \in N \setminus (S \cup T)$, $\varphi_j(\bar{R}, 1_x) = \emptyset$. If $T = \emptyset$, then for all $j \in N$, $\varphi_j(\bar{R}, 1_x) = \emptyset$. Since $\varphi_i(\bar{R}, 1_x) = \emptyset$ and $x \; \bar{P}_i \; \emptyset$ this contradicts *weak non-wastefulness*. Hence, $T = \{l\}$ and by *weak non-wastefulness*, $\varphi_l(\bar{R}, 1_x) = x$. By Lemma 1, $(*)$ $l \succ_x i$.

Recall that $T = \{j \in N : \varphi_j(\bar{R}, q) = x\}$. Hence, $l = k$ and (s1') cannot be the case. Thus, by (s2') $i \succ_x l$, which contradicts $(*)$. $\square$

For any number $q_x \in \{0, 1, \ldots, \overline{q}_x\}$, let $q_x \circ 1_x = (q_x, 0_{O\setminus\{x\}})$ denote the capacity vector with exactly q_x copies of object x.

Lemma 3. *For all $R \in \mathcal{R}^N$ and all $q_x \in \{1, \ldots, \overline{q}_x\}$, if $T = \{j \in N : \varphi_j(R, q_x \circ 1_x) = x\}$, then for all $j \in T$ and $i \in N \setminus T$, $x \in A(R_i)$ implies $j \succ_x i$.*

Note that Lemma 3 is the extension of "house allocation problem Lemma 1" to the general class of problems. Recall that rule f as introduced in Example 1 satisfies all properties of Theorem 2 except *two-agent consistent conflict resolution*. The calibration step at the beginning of Section 3.3 applied to f yields the priority ordering $\succ_x$: 1 2 3 for object x. However, for preferences $R_1 = R_2 = R_3 : x\ \emptyset\ y$, we have $T = \{1, 3\}$ and in contradiction to Lemma 3, $x \in A(R_2)$ and $2 \succ_x 3$.

Proof. Let $R \in \mathcal{R}^N$, $q_x \in \{1, \ldots, \overline{q}_x\}$, $T = \{j \in N : \varphi_j(R, q_x \circ 1_x) = x\}$, and $S = \{i \in N : x \in A(R_i)\}$. By *individual rationality*, $T \subseteq S$. Let $j \in T$ and $i \in S \setminus T$. We prove Lemma 3 by showing that $j \succ_x i$. In the sequel, when using *two-agent consistent conflict resolution* we often also implicitly apply *weak non-wastefulness* and *individual rationality*.

If $q_x = 1$, then by Lemma 1, $j \succ_x i$.

Next, suppose that $q_x = 2$. Note that by *unavailable type invariance*, $\varphi(R, q_x \circ 1_x) = \varphi((R^x_S, R^{\emptyset}_{-S}), q_x \circ 1_x)$. Let $k \in S$ be such that $\varphi_k((R^x_S, R^{\emptyset}_{-S}), 1_x) = x$. By Lemma 1, $k \succ_x i$. By *two-agent consistent conflict resolution*, $\varphi_k((R^x_S, R^{\emptyset}_{-S}), q_x \circ 1_x) = x$. Hence, $k \in T$ and $k \succ_x i$.

By *weak non-wastefulness* and *individual rationality*, there exists $h \in T \setminus \{k\}$ such that $\varphi_h((R^x_S, R^{\emptyset}_{-S}), q_x \circ 1_x) = x$. Let $y \in O \setminus \{x\}$ and $R'_k \in \mathcal{R}$ be such that $R'_k : y\ x\ \emptyset\ \ldots$. By *unavailable type invariance*, $\varphi((R^x_{S\setminus\{k\}}, R^{\emptyset}_{-S}, R'_k), q_x \circ 1_x) = \varphi((R^x_S, R^{\emptyset}_{-S}), q_x \circ 1_x)$. Hence, $\varphi_h((R^x_{S\setminus\{k\}}, R^{\emptyset}_{-S}, R'_k), q_x \circ 1_x) = x$.

Next, we replace one x objects by one y object. By *weak non-wastefulness* and *individual rationality*, $\varphi_k((R^x_{S\setminus\{k\}}, R^{\emptyset}_{-S}, R'_k), 1_{xy}) = y$. By *two-agent consistent conflict resolution*, $\varphi_h((R^x_{S\setminus\{k\}}, R^{\emptyset}_{-S}, R'_k), 1_{xy}) = x$. By *truncation invariance*, $\varphi_h((R^x_{S\setminus\{k\}}, R^{\emptyset}_{-S}, R^y_k), 1_{xy}) = x$. By *two-agent consistent conflict resolution*, $\varphi_h((R^x_{S\setminus\{k\}}, R^{\emptyset}_{-S}, R^y_k), 1_x) = x$. Hence, by Lemma 1, $h \succ_x i$.

Now by induction on q_x the conclusion of Lemma 3 follows. $\square$

Lemma 4. *For all $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, $\varphi(R, q)$ is stable under $\succ$.*

Proof. Let $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$ and assume that $\varphi(R, q)$ is not stable under $\succ$. Then, there exists an agent-object pair $(i, x) \in N \times O \cup \{\emptyset\}$ such that $x\, P_i\, \varphi_i(R, q)$ and (s1) $|\{j \in N : \varphi_j(R, q) = x\}| < q_x$ or (s2) there exists $k \in N$ such that $\varphi_k(R, q) = x$ and $i \succ_x k$. By *individual rationality*, $x \neq \emptyset$.

Let $\bar{R} \in \mathcal{R}^N$ be such that (a) for all $j \in N$ such that $\varphi_j(R,q) \neq \emptyset$, $\bar{R}_j$ is a truncation of R_j such that there exists no $y \in O \setminus \{\varphi_j(R,q)\}$ with $\varphi_j(R,q)\, \bar{R}_j\, y\, \bar{R}_j\, \emptyset$ and (b) for all $j \in N$ such that $\varphi_j(R,q) = \emptyset$, $\bar{R}_j = R_j$. (By *individual rationality*, $\bar{R}_j$ in (a) is well-defined as truncation of R_j.) By *truncation invariance*, $\varphi(\bar{R},q) = \varphi(R,q)$ and $(i,x) \in N \times O$ is such that $x\, \bar{P}_i\, \varphi_i(\bar{R},q)$ and (s1') $|\{j \in N : \varphi_j(\bar{R},q) = x\}| < q_x$ or (s2') there exists $k \in N$ such that $\varphi_k(\bar{R},q) = x$ and $i \succ_x k$.

Let $S = \{j \in N : x\, \bar{P}_j\, \varphi_j(\bar{R},q)\}$. Note that $i \in S$. Let $T = \{j \in N : \varphi_j(\bar{R},q) = x\}$. By *unavailable type invariance*, $\varphi(\bar{R}, q_x \circ 1_x) = \varphi((\bar{R}^x_{S\cup T}, \bar{R}^\emptyset_{N\setminus(S\cup T)}), q_x \circ 1_x)$.

Step 1: Assume that for $j \in S$, $\varphi_j(\bar{R}, q_x \circ 1_x) = x$. Then, by *strategy-proofness* and *individual rationality*, $\varphi_j((\bar{R}_{-j}, R^x_j), q_x \circ 1_x) = x$ and $\varphi_j((\bar{R}_{-j}, R^x_j), q) = \emptyset$. By *weak non-wastefulness*, $|\{k \in N : \varphi_k((\bar{R}_{-j}, R^x_j), q) = x\}| = q_x$. By *individual rationality* there exists $l \in N$ such that $x \in A(\bar{R}_l)$, and $\varphi_l((\bar{R}_{-j}, R^x_j), q) = x$ and $\varphi_l((\bar{R}_{-j}, R^x_j), q_x \circ 1_x) = \emptyset$. By Lemma 3, $j \succ_x l$. Next, by *strategy-proofness*, $\varphi_l((\bar{R}_{-\{j,l\}}, R^x_j, R^x_l), q) = x$. By *unavailable type invariance*, $\varphi((\bar{R}_{-\{j,l\}}, R^x_j, R^x_l), q_x \circ 1_x) = \varphi((\bar{R}_{-j}, R^x_j), q_x \circ 1_x)$; in particular, $\varphi_j((\bar{R}_{-\{j,l\}}, R^x_j, R^x_l), q_x \circ 1_x) = x$ and $\varphi_l((\bar{R}_{-\{j,l\}}, R^x_j, R^x_l), q_x \circ 1_x) = \emptyset$. Thus, $\varphi_j((\bar{R}_{-\{j,l\}}, R^x_j, R^x_l), q) = \emptyset$ would contradict *two-agent consistent conflict resolution* because then $\{\varphi_j((\bar{R}_{-\{j,l\}}, R^x_j, R^x_l), q), \varphi_l((\bar{R}_{-\{j,l\}}, R^x_j, R^x_l), q)\} = \{\varphi_j((\bar{R}_{-\{j,l\}}, R^x_j, R^x_l), q_x \circ 1_x), \varphi_l((\bar{R}_{-\{j,l\}}, R^x_j, R^x_l), q_x \circ 1_x)\} = \{x, \emptyset\}$. Hence, $\varphi_j((\bar{R}_{-\{j,l\}}, R^x_j, R^x_l), q) = x$. Thus, there exists an agent $j_2 \in N \setminus \{j,l\}$ such that $\varphi_{j_2}((\bar{R}_{-\{j,l\}}, R^x_j, R^x_l), q) = \emptyset$ and $\varphi_{j_2}((\bar{R}_{-\{j,l\}}, R^x_j, R^x_l), q_x \circ 1_x) = x$. However, $\bar{R}_{j_2} = R^x_{j_2}$ would contradict *two-agent consistent conflict resolution* because then $\{\varphi_{j_2}((\bar{R}_{-\{j,l\}}, R^x_j, R^x_l), q), \varphi_l((\bar{R}_{-\{j,l\}}, R^x_j, R^x_l), q)\} = \{\varphi_{j_2}((\bar{R}_{-\{j,l\}}, R^x_j, R^x_l), q_x \circ 1_x), \varphi_l((\bar{R}_{-\{j,l\}}, R^x_j, R^x_l), q_x \circ 1_x)\} = \{x, \emptyset\}$. Hence, $\bar{R}_{j_2} \neq R^x_{j_2}$.

Steps 2, …: Step 2 replicates Step 1 with the starting preference profile $(\bar{R}_{-\{j,l\}}, R^x_j, R^x_l)$ and with agent j_2 in the role of agent j. Throughout the step, we strictly increase the number of agents with preferences R^x (at least by agent j_2). Furthermore, the step ends with the existence of another agent j_3 with $\bar{R}_{j_3} \neq R^x_{j_3}$ with whom we proceed to Step 3. Since the number of agents is finite, we obtain a contradiction in finitely many steps.

Final Step: With the previous steps we have established that for all $j \in S$, $\varphi_j(\bar{R}, q_x \circ 1_x) = \emptyset$. Note that by construction of $\bar{R}$, for all $j \in N \setminus (S \cup T)$, $\emptyset\, \bar{P}_j\, x$. Hence, by *individual rationality*, for all $j \in N \setminus (S \cup T)$, $\varphi_j(\bar{R}, q_x \circ 1_x) = \emptyset$. Note that by *individual rationality*, for all $j \in T$, $x \in A(\bar{R}_j)$. Hence, by *weak non-wastefulness*, for all $j \in T$, $\varphi_j(\bar{R}, q_x \circ 1_x) = x$ and $|T| = q_x$. By Lemma 3, we have $(*)$ that for all $j \in T$ and all $l \in S$, $j \succ_x l$.

Recall that $T = \{j \in N : \varphi_j(\bar{R},q) = x\}$. Hence, (s1') cannot be the case. Thus, by (s2') there exists $k \in N$ such that $\varphi_k(\bar{R},q) = x$ and $i \succ_x k$. Recall that $i \in S$ and that $\varphi_k(\bar{R},q) = x$ implies that $k \in T$ so that $i \succ_x k$ contradicts $(*)$. □

So far we have established that for any rule φ that satisfies the properties of Theorem 2 (Theorem 1), there exists a priority ordering $\succ$ such that for any $(R,q) \in \mathcal{R}^N \times \mathcal{Q}$, $\varphi(R,q)$ is stable under $\succ$. Hence, in the terminology of two-sided matching, the rule φ picks a stable allocation for the many-to-one two-sided market where types have responsive preferences over sets of agents who consume the objects based on the priority structure $\succ$ and agents have strict preferences over objects based on preferences R (see Roth and Sotomayor, 1990, Chapter 5). For these markets it is well-known that the responsive DA-rule is the only *strategy-proof* stable matching rule. For completeness, we provide a proof which uses some standard results from many-to-one two-sided markets with responsive preferences.

Lemma 5. *For all* $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, $\varphi(R, q) = DA^{\succ}(R, q)$.

Proof. Let $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$ and assume that $\varphi(R, q) \neq DA^{\succ}(R, q)$. By Lemma 4, $\varphi(R, q)$ is stable under $\succ$. Thus, since $DA^{\succ}(R, q)$ is the agent-optimal stable matching (Roth and Sotomayor, 1990, Corollary 5.9), for all $i \in N$, $DA_i^{\succ}(R, q)\, R_i\, \varphi_i(R, q)$. Since $\varphi(R, q) \neq DA^{\succ}(R, q)$, there exists $i \in N$ such that $DA_i^{\succ}(R, q)\, P_i\, \varphi_i(R, q)$. By *individual rationality*, $\varphi_i(R, q)\, R_i\, \emptyset$. Thus, $DA_i^{\succ}(R, q) \neq \emptyset$.

Let $\bar{R}_i$ be such that $\bar{R}_i|_O = R_i|_O$ and there is no $x \in O \setminus \{DA_i^{\succ}(R, q)\}$ such that $DA_i^{\succ}(R, q)\, \bar{R}_i\, x\, \bar{R}_i\, \emptyset$.

Then, for problem $(\bar{R}_i, R_{-i})$, $DA^{\succ}(R, q)$ is stable under $\succ$. By Lemma 4, for problem $(\bar{R}_i, R_{-i})$, $\varphi_i((\bar{R}_i, R_{-i}), q)$ is stable under $\succ$. Hence, by $DA_i^{\succ}(R, q) \neq \emptyset$ and the fact that the set of agents receiving the null object is identical for any two stable matchings (Roth and Sotomayor, 1990, Theorem 5.12), we have $\varphi_i((\bar{R}_i, R_{-i}), q) \neq \emptyset$. By definition of $\bar{R}_i$, we now have $\varphi_i((\bar{R}_i, R_{-i}), q)\, \bar{R}_i\, DA_i^{\succ}(R, q)$. Thus, by $\bar{R}_i|_O = R_i|_O$ and $DA_i^{\succ}(R, q) \bar{P}_i \emptyset$, we have $\varphi_i(\bar{R}_i, R_{-i}, q) R_i DA_i^{\succ}(R, q)$. Hence, $\varphi_i(\bar{R}_i, R_{-i}, q)\, P_i\, \varphi_i(R, q)$, which contradicts *strategy-proofness*. □

4 Consistency

Let N again denote the finite set of agents, but the set of agents who are present in a problem can vary. We define the set of all nonempty subsets of N by $\mathcal{N} \equiv \{M : M \subseteq N \text{ and } M \neq \emptyset\}$. An *(allocation) problem (with capacity constraints)* now consists of a set $N' \in \mathcal{N}$ of agents, a preference profile $R \in \mathcal{R}^{N'}$, and a capacity vector q. We denote the set of all problems by $\bigcup_{N' \in \mathcal{N}} \mathcal{R}^{N'} \times \mathcal{Q}$. We adjust our previous model, definitions, and properties by simply replacing the domain of problems $\mathcal{R}^N \times \mathcal{Q}$ by the variable population domain $\bigcup_{N' \in \mathcal{N}} \mathcal{R}^{N'} \times \mathcal{Q}$. Given $N' \in \mathcal{N}$ and $R \in \mathcal{R}^{N'}$, for any $M' \in \mathcal{N}$ such that $M' \subseteq N'$, let $R_{M'}$ denote the profile $(R_i)_{i \in M'}$.

A requirement for a rule that is very much in the spirit of *unavailable type invariance* is *unassigned objects invariance*: the chosen allocation does not depend on the unconsumed or unassigned objects. Given a problem $(R, q) \in \bigcup_{N' \in \mathcal{N}} \mathcal{R}^{N'} \times \mathcal{Q}$, we define by $q(\varphi(R, q))$ the *capacity vector of assigned objects*: for all $x \in O$, $q_x(\varphi(R, q)) = |\{j \in N' : \varphi_j(R, q) = x\}|$.

Unassigned Objects Invariance: For all $(R, q) \in \bigcup_{N' \in \mathcal{N}} \mathcal{R}^{N'} \times \mathcal{Q}$, $\varphi(R, q) = \varphi(R, q(\varphi(R, q)))$.

Consistency is one of the key properties in many frameworks with variable population scenarios. Thomson (2009) provides an extensive survey of consistency in various applications. Consistency requires that if some agents leave a problem with their allotments, then the rule should allocate the remaining objects among the agents who did not leave in the same way as in the original problem.

Consistency: For all $M', N' \in \mathcal{N}$ such that $M' \subseteq N'$, all $R \in \mathcal{R}^{N'}$, all $q \in \mathcal{Q}$, and all $i \in M'$, $\varphi_i(R, q) = \varphi_i(R_{M'}, \tilde{q})$ where $\tilde{q}_x = q_x - |\{j \in N' \setminus M' : \varphi_j(R, q) = x\}|$ for all $x \in O$.

It follows from Ergin (2002, Theorem 1) that the only responsive DA-rules satisfying *consistency* are the ones with an acyclic priority structure.

The following property is a weak consistency property that all responsive DA-rules satisfy. It requires that if some agents leave a problem with their allotments, then an agent who did not leave and who received the null object, still receives the null object. In other words, allocations only need to be consistent with respect to the agents who receive the null object.

Weak Consistency: For all $M', N' \in \mathcal{N}$ such that $M' \subseteq N'$, all $R \in \mathcal{R}^{N'}$, all $q \in \mathcal{Q}$, and all $i \in M'$, if $\varphi_i(R, q) = \emptyset$, then $\varphi_i(R_{M'}, \tilde{q}) = \emptyset$ where $\tilde{q}_x = q_x - |\{j \in N' \setminus M' : \varphi_j(R, q) = x\}|$ for all $x \in O$.

Theorem 3. *Responsive DA-rules are the only rules satisfying unassigned objects invariance, individual rationality, weak non-wastefulness, weak consistency, and strategy-proofness.*

Theorem 3 and Ergin's (2002, Theorem 1) result imply the following characterization of the class of acyclic responsive DA-rules.

Corollary 3.
Responsive DA-rules with acyclic priority structures are the only rules satisfying unassigned objects invariance, individual rationality, weak non-wastefulness, consistency, and strategy-proofness.

Proof of Theorem 3

It is easy to verify that responsive DA-rules satisfy the properties of Theorem 3. Conversely, let φ be a rule satisfying these properties. First, we "calibrate the priority structure using maximal conflict preference profiles".

Let $x \in O$ and let $R^x \in \mathcal{R}^x$ (i.e., $A(R^x) = x$). Let $R^\emptyset \in \mathcal{R}$ be such that $A(R^\emptyset) = \emptyset$.

For any $S \subseteq N$, let $R^x_S = (R^x_i)_{i \in S}$ such that for all $i \in S$, $R^x_i = R^x$, and similarly $R^\emptyset_S = (R^\emptyset_i)_{i \in S}$ such that for all $i \in S$, $R^\emptyset_i = R^\emptyset$.

Let 1_x denote the capacity vector q such that $q_x = 1$ and for all $z \in O \setminus \{x\}$, $q_z = 0$.

Consider the problem $(R^x_N, 1_x)$. By *weak non-wastefulness*, for some $i \in N$, $\varphi_i(R^x_N, 1_x) = x$, say $i = 1$. Then, for all $i \in N \setminus \{1\}$, we set $1 \succ_x i$.

Next consider the problem $(R^x_{N\setminus\{1\}}, 1_x)$. By *weak non-wastefulness* and *individual rationality*, for some $i \in N \setminus \{1\}$, $\varphi_i(R^x_{N\setminus\{1\}}, 1_x) = x$, say $i = 2$. Then, for all $i \in N \setminus \{1, 2\}$, we set $2 \succ_x i$.

By induction, we obtain $\succ_x$ for any type x and thus a priority structure $\succ = (\succ_x)_{x \in O}$.

Lemma 6. *For all $N' \in \mathcal{N}$, all $R' \in \mathcal{R}^{N'}$, and all $x \in O$, if for some $j \in N$, $\varphi_j(R', 1_x) = x$, then for all $i \in N' \setminus \{j\}$, $x \in A(R'_i)$ implies $j \succ_x i$.*

Proof. Let $N' \in \mathcal{N}$, $R' \in \mathcal{R}^{N'}$, and $x \in O$. Without loss of generality, suppose $1 \succ_x 2 \succ_x \cdots \succ_x n$. Let $S = \{i \in N' : x \in A(R'_i)\}$ and let $j = \min S$. We prove Lemma 6 by showing that $\varphi_j(R', 1_x) = x$.

By *weak non-wastefulness* and *individual rationality*, for some $l \in S$, $\varphi_l(R', 1_x) = x$. Assume, by contradiction, that $l \neq j$. Thus, $\varphi_j(R', 1_x) = \emptyset$. Hence, by *weak consistency*, $\varphi_j(R'_{\{j,l\}}, 1_x) = \emptyset$. By *weak non-wastefulness*, $\varphi_l(R'_{\{j,l\}}, 1_x) = x$. By *strategy-proofness*, $\varphi_l((R'_j, R^x_l), 1_x) = x$ and $\varphi_j((R^x_j, R^x_l), 1_x) = \emptyset$. By *weak non-wastefulness*, $\varphi_l((R^x_j, R^x_l), 1_x) = x$.

Let $L = \{1, \ldots, j-1\}$ (possibly $L = \emptyset$). By the construction of $\succ_x$, $\varphi_j(R^x_{N\setminus L}, 1_x) = x$. Thus, $\varphi_l(R^x_{N\setminus L}, 1_x) = \emptyset$. Hence, by *weak consistency*, $\varphi_l(R^x_{\{j,l\}}, 1_x) = \emptyset$. By *weak non-wastefulness*, $\varphi_j(R^x_{\{j,l\}}, 1_x) = x$. Since $R^x_{\{j,l\}} = (R^x_j, R^x_l)$ and $l \neq j$, we have established a contradiction. Thus, $l = j$ and $\varphi_j(R', 1_x) = x$. □

Let $\hat{\mathcal{R}} = \{R_i \in \mathcal{R} : |A(R_i)| \leq 1\}$.

Lemma 7.
(a) *For all $N' \in \mathcal{N}$ and all $(R, q) \in \mathcal{R}^{N'} \times \mathcal{Q}$, if $|N'| = 2$, then $\varphi(R, q)$ is stable under $\succ$.*
(b) *For all $N' \in \mathcal{N}$ and all $(R, q) \in \hat{\mathcal{R}}^{N'} \times \mathcal{Q}$, $\varphi(R, q)$ is stable under $\succ$.*

Proof. In order to show **(a)**, suppose that $\varphi(R, q)$ is not stable under $\succ$. Then, there exists an agent-object pair $(i, x) \in N' \times O \cup \{\emptyset\}$ such that $x \, P_i \, \varphi_i(R, q)$ and (s1) $|\{j \in N' : \varphi_j(R, q) = x\}| < q_x$ or (s2) there exists $k \in N'$ such that $\varphi_k(R, q) = x$ and $i \succ_x k$. Without loss of generality, let i be the agent ranked highest according to $\succ_x$ to form such an agent-object blocking pair.

Let $N' = \{i, k\}$ and $R^x_i \in \mathcal{R}^x$ as in the calibration step used to define $\succ_x$. By *strategy-proofness*, $\varphi_i((R^x_i, R_k), q) = \emptyset$. Hence, from *weak non-wastefulness* we obtain both $\varphi_k((R^x_i, R_k), q) = x$ and $q_x = 1$. By *individual rationality*, $x \, P_k \, \emptyset$. Let $R^x_k = R^x_i$. By *strategy-proofness*, $\varphi_k((R^x_i, R^x_k), q) = x$. By *individual rationality*, $\varphi_i((R^x_i, R^x_k), q) = \emptyset$. Then, using *weak consistency*, *unassigned objects invariance*, and the definition of $\succ_x$, it follows that $k \succ_x i$. Hence, for $\varphi(R, q)$ we must have (s1) $|\{j \in N' : \varphi_j(R, q) = x\}| < q_x$.

Recall that we obtained $q_x = 1$. Hence, (s1) implies that object x is not assigned at $\varphi(R, q)$. Because i is the agent ranked highest according to $\succ_x$ to form an agent-object blocking pair, $\varphi_k(R, q) \, P_k \, x$. Let $\varphi_k(R, q) = y \in O$. Recall that $\varphi_k((R^x_i, R_k), q) = x$. Then, by *strategy-proofness*, $\varphi_k(R^x_i, R^y_k, q) = \emptyset$. By *individual rationality*, $\varphi_i(R^x_i, R^y_k, q) \neq y$. But this implies a contradiction to *weak non-wastefulness* for k and y.

In order to show **(b)**, suppose that $\varphi(R,q)$ is not stable under $\succ$. Then, there exists an agent-object pair $(i,x) \in N' \times O \cup \{\emptyset\}$ such that $x \, P_i \, \varphi_i(R,q)$ and (s1) $|\{j \in N' : \varphi_j(R,q) = x\}| < q_x$ or (s2) there exists $k \in N'$ such that $\varphi_k(R,q) = x$ and $i \succ_x k$. Without loss of generality, let i be the agent ranked highest according to $\succ_x$ to form such an agent-object blocking pair.

Since $R_i \in \hat{\mathcal{R}}$, we have $A(R_i) = \{x\}$ and $\varphi_i(R,q) = \emptyset$. By *weak non-wastefulness*, (s1) cannot occur, i.e., we have (s2) there exists $k \in N'$ such that $\varphi_k(R,q) = x$ and $i \succ_x k$.

By *weak consistency*, $\varphi_i(R_{\{i,k\}}, \tilde{q}) = \emptyset$ (where $\tilde{q}_z = q_z - |\{j \in N' \setminus \{i,k\} : \varphi_j(R,q) = z\}|$). By *weak non-wastefulness*, $\varphi_k(R_{\{i,k\}}, \tilde{q}) = x$. Hence, $\varphi(R_{\{i,k\}}, \tilde{q})$ is not stable under $\succ$, which contradicts (a). □

Lemma 8. *For all $N' \in \mathcal{N}$ and all $(R,q) \in \mathcal{R}^{N'} \times \mathcal{Q}$, $\varphi(R,q)$ is stable under $\succ$.*

Proof. For any profile $R \in \mathcal{R}^{N'}$, let $\hat{N}(R) = \{i \in N' : R_i \notin \hat{\mathcal{R}}\}$. We prove that $\varphi(R,q)$ is stable under $\succ$ by induction on $|\hat{N}(R)|$.

Induction Basis: For $|\hat{N}(R)| = 0$, Lemma 7 (b) implies that $\varphi(R,q)$ is stable under $\succ$.

Induction Hypothesis: Assume that $\varphi(R,q)$ is stable under $\succ$ for any $R \in \mathcal{R}^{N'}$ such that $|\hat{N}(R)| \leq k$.

Induction Step: Let $R \in \mathcal{R}^{N'}$ be such that $|\hat{N}(R)| = k+1$. Suppose that $\varphi(R,q)$ is not stable under $\succ$. Then, there exists an agent-object pair $(i,x) \in N' \times O \cup \{\emptyset\}$ such that $x \, P_i \, \varphi_i(R,q)$ and (s1) $|\{j \in N' : \varphi_j(R,q) = x\}| < q_x$ or (s2) there exists $l \in N'$ such that $\varphi_l(R,q) = x$ and $i \succ_x l$. Without loss of generality, let i be the agent ranked highest according to $\succ_x$ to form such an agent-object blocking pair.

If $\varphi_i(R,q) = \emptyset$, then by *weak non-wastefulness*, we have (s2) there exists $l \in N'$ such that $\varphi_l(R,q) = x$ and $i \succ_x l$. By *weak consistency*, $\varphi_i(R_{\{i,l\}}, \tilde{q}) = \emptyset$ (where $\tilde{q}_z = q_z - |\{j \in N' \setminus \{i,l\} : \varphi_j(R,q) = z\}|$ for all $z \in O$). By *weak non-wastefulness*, $\varphi_l(R_{\{i,l\}}, \tilde{q}) = x$. Hence, $\varphi(R_{\{i,l\}}, \tilde{q})$ is not stable under $\succ$, which contradicts Lemma 7 (a).

Thus, $\varphi_i(R,q) \neq \emptyset$ and $R_i \notin \hat{\mathcal{R}}$. Let $R_i^x \in \mathcal{R}^x$. By *strategy-proofness*, $\varphi_i((R_i^x, R_{-i}), q) = \emptyset$. Note that $|\hat{N}(R_i^x, R_{-i})| = k$ and by the induction hypothesis, $\varphi((R_i^x, R_{-i}), q)$ is stable under $\succ$.

For (s2) there exists $l \in N'$ such that $\varphi_l(R,q) = x$ and $i \succ_x l$. Hence, by $\varphi_i((R_i^x, R_{-i}), q) = \emptyset$ and stability, $\varphi_l((R_i^x, R_{-i}), q) \neq x$. For (s1) $|\{j \in N' : \varphi_j(R,q) = x\}| < q_x$, $\varphi_i((R_i^x, R_{-i}), q) = \emptyset$ and *weak non-wastefulness* imply $|\{j \in N' : \varphi_j((R_i^x, R_{-i}), q) = x\}| = q_x$. Hence, in both cases (s1) and (s2) there exists $j \in N' \setminus \{i\}$ such that $\varphi_j((R_i^x, R_{-i}), q) = x \neq \varphi_j(R,q)$.

Thus, stability, $\varphi_j((R_i^x, R_{-i}), q) = x$, and $\varphi_i((R_i^x, R_{-i}), q) = \emptyset$ imply $j \succ_x i$. Because i is the agent ranked highest according to $\succ_x$ to form an agent-object blocking pair, $\varphi_j(R,q) \, P_j \, x$ and $R_j \notin \hat{\mathcal{R}}$.

Step 1: Let $\varphi_j(R,q) = y \in O$ and $R_j^y \in \mathcal{R}^y$. By *strategy-proofness*, $\varphi_j((R_j^y, R_{-j}), q) = y$. Recall that $\varphi_j((R_i^x, R_{-i}), q) = x$. Then, by *strategy-proofness* and *individual rationality*, $\varphi_j((R_i^x, R_j^y, R_{-i,j}), q) = \emptyset$.

Note that both $|\hat{N}(R_j^y, R_{-j})| \leq k$ and $|\hat{N}(R_i^x, R_j^y, R_{-i,j})| \leq k$ and the induction hypothesis applies to both profiles. By *weak non-wastefulness*, there exists $h \in N'$ such that $\varphi_h((R_i^x, R_j^y, R_{-i,j}), q) = y \neq \varphi_h((R_j^y, R_{-j}), q)$. By stability and $\varphi_j((R_i^x, R_j^y, R_{-i,j}), q) = \emptyset$, we have $h \succ_y j$. If $y \, P_h \, \varphi_h((R_j^y, R_{-j}), q)$, then by $h \succ_y j$ and $\varphi_j((R_j^y, R_{-j}), q) = y$, $\varphi((R_j^y, R_{-j}), q)$ is not stable under $\succ$, a contradiction.

Thus, $\varphi_h((R_j^y, R_{-j}), q) \, R_h \, y$ and $P_h \notin \hat{\mathcal{R}}$.

Step 2: Let $\varphi_h((R_j^y, R_{-j}), q) = z \in O$. As in Step 1, we use *strategy-proofness* to replace R_h by R_h^z in the problems $((R_j^y, R_{-j}), q)$ and $((R_i^x, R_j^y, R_{-i,j}), q)$. Then again we find a new agent h' with $R_{h'} \notin \hat{\mathcal{R}}$, etc.

Because the set of agents is finite, continuing along the lines of Steps 1 and 2 will ultimately lead to a contradiction. □

The proof that for all $N' \in \mathcal{N}$ and $(R, q) \in \bigcup_{N' \in \mathcal{N}} \mathcal{R}^{N'} \times \mathcal{Q}$, $\varphi(R, q) = DA^{\succ}(R, q)$ is similar to the proof of Lemma 5 (which only uses *strategy-proofness*).

5 Concluding Remarks

We present three characterizations of the class of DA-rules with responsive priorities, i.e., any of these rules determines the outcome by solely using the priority orderings over individual agents (see Remark 2). In contrast, Kojima and Manea (2009) characterize the class of DA-rules with so-called acceptant substitutable priorities: a larger class of rules than the class of responsive DA-rules that is based on priorities that are determined by a choice function that reflects substitutability in preferences over sets of agents (see also Hatfield and Milgrom, 2005).

Formally, Kojima and Manea (2009) define a *priority for a type $x \in O$ with capacity q_x* as a (choice) correspondence $C_{q_x} : 2^N \to 2^N$ satisfying for all $M \subseteq N$, $C_{q_x}(M) \subseteq M$ and $|C_{q_x}(M)| \leq q_x$. C_{q_x} is *substitutable* if for all $M' \subseteq M \subseteq N$, $C_{q_x}(M) \cap M' \subseteq C_{q_x}(M')$. C_{q_x} is *acceptant* if for all $M \subseteq N$, $|C_{q_x}(M)| = \min\{q_x, |M|\}$. Clearly, taking a linear order $\succ_x$ over agents and defining C_{q_x} by choosing the $\min\{q_x, |M|\}$ best agents in M according to $\succ_x$ defines an acceptant substitutable priority. This particular class of priorities coincides with the class of (acceptant) responsive priorities employed in this paper (see Remark 2). Note that if $q_x = 1$, then the class of acceptant substitutable priorities coincide with our class of (acceptant) responsive priorities.

Since in our model resources can change, acceptant substitutable priorities for a type $x \in O$ in our model are modeled by a profile $C_x = (C_{q_x})_{1 \leq q_x \leq \bar{q}_x}$. *Resource-monotonicity* of a DA-rule with acceptant substitutable priorities $C = (C_{q_x})_{x \in O, 1 \leq q_x \leq \bar{q}_x}$ is then guaranteed if for all $M \subseteq N$, all $x \in O$, and all $q_x \leq q'_x$, $i \in C_{q_x}(M)$ implies $i \in C_{q'_x}(M)$ (see also Crawford, Theorem 1, which strictly speaking only applies to adding a type $x \in O$ and not to increasing capacities).

We next present an example of an acceptant substitutable priority that is not responsive.

Example 2. Let $N = \{s_1, s_2, j_1, j_2\}$ be four economists looking for a new academic position. Furthermore, s_1, s_2 are seniors with specializations 1 or 2 and j_1, j_2 are juniors with specializations 1 or 2. The intuition behind the priorities that we define is the following. An economics department z has the following preferences for hiring. If only one economist can be hired the priority ranking for hiring is $s_1 \succ_z s_2 \succ_z j_1 \succ_z j_2$. However, if two positions can be filled, the department always would like to fill both positions. So, $\bar{q}_z = 2$. Furthermore, they would like to hire the two seniors; but if only one senior s_i can be hired, then they are interested in also hiring the junior s_i in the same field. To be more specific, we assume that the department's priority ranking for hiring two economists is $\{s_1, s_2\} \succ_z \{s_1, j_1\} \succ_z \{s_2, j_2\} \succ_z \{s_1, j_2\} \succ_z \{s_2, j_1\} \succ_z \{j_1, j_2\}$.
Note that the department loosely speaking has lexicographic preferences. Priorities C_{q_z} based on these preferences work as follows. For $q_z = 2$ (the case $q_z = 1$ is obvious) it follows that

$$C_{q_z}(M) = \begin{cases} M & \text{if } |M| \leq 2, \\ \{s_1, s_2\} & \text{if } \{s_1, s_2\} \subsetneq M, \\ \{s_i, j_i\} & \text{if } \{s_i, j_i\} \subsetneq M, \ i \in \{1, 2\}. \end{cases}$$

It is easily verified that $C_z = (C_{q_z})_{1 \leq q_z \leq 2}$ is an acceptant and substitutable priority. Since $C_{q_z=2}(\{s_1, j_1, j_2\}) = \{s_1, j_1\}$ and $C_{q_z=2}(\{s_2, j_1, j_2\}) = \{s_2, j_2\}$, $C_{q_z=2}$ is not responsive. ◇

Next, we show which of our properties a DA-rule based on acceptant substitutable priorities as described in Example 2 violates. We refer to Kojima and Manea (2009) for the definition of the DA-rule based on a acceptant substitutable priority structure – with our extended notion of acceptant substitutable priorities it is straightforward how this definition extends to our setup which allows for changing capacities and agents. In other words, we point out which properties that characterize responsive DA-rules might fail for some acceptant substitutable DA-rules.

Note that for housing markets the class of responsive priorities and the class of acceptant substitutable priorities coincides. Hence, for Theorem 1 we cannot give any examples of acceptant substitutable DA-rules violating any of the properties.

The following example demonstrates that some acceptant substitutable DA-rules violate *two-agent consistent conflict resolution* and *weak consistency*.

Example 3. Let $N = \{s_1, s_2, j_1, j_2\}$, $O = \{x, y\}$, $\bar{q}_x = \bar{q}_y = 2$, and $C_x = C_y$ as defined in Example 2. Let q be such that $q_x = 2$ and $q_y = 0$ and q' be such that $q'_x = 1$ and $q'_y = 1$. Furthermore, let R be such that $R_{s_1} : \emptyset \ x \ y$, $R_{s_2} : y \ x \ \emptyset$, and $R_{j_1} = R_{j_2} : x \ \emptyset \ y$. Then, $\varphi^C_{s_2}(R, q) = \varphi^C_{j_2}(R, q) = x$, $\varphi^C_{s_1}(R, q) = \varphi^C_{j_1}(R, q) = \emptyset$ and $\varphi^C_{s_2}(R, q') = y$, $\varphi^C_{j_1}(R, q') = x$, $\varphi^C_{s_1}(R, q') = \varphi^C_{j_2}(R, q') = \emptyset$. Hence, we have a violation of *two-agent consistent conflict resolution* as used in Theorem 2.
Next, let $M' = \{j_1, j_2\} \subseteq \{s_2, j_1, j_2\} = N'$. Then, $\varphi^C_{s_2}(R_{N'}, q) = \varphi^C_{j_2}(R_{N'}, q) = x$, $\varphi^C_{j_1}(R_{N'}, q) = \emptyset$ and $\varphi^C_{j_1}(R_{M'}, \tilde{q}) = x$ where $\tilde{q}_x = 1$ and $\tilde{q}_y = 0$. Hence, we have a violation of *weak consistency* as used in Theorem 3. ◇

Notice that any acceptant priority structure satisfies the *law of demand*: for all $x \in O$ and all $M' \subseteq N' \subseteq N$, $|C_x(M')| \leq |C_x(N')|$. Hence, by Hatfield and Milgrom (2005, Theorem 11), all acceptant substitutable DA-rules satisfy *strategy-proofness*.

Non-wastefulness (see definition in Footnote 10) as used by Kojima and Manea (2009) implies *weak non-wastefulness*, and *individual rationality*. Furthermore, *non-wastefulness* is equivalent to the absence of agent-object pairs $(i, x) \in N \cup \{\emptyset\}$ such that $x \; P_i \; \varphi_i(R, q)$ and (s1) $|\{j \in N : \varphi_j(R, q) = x\}| < q_x$. Thus, *non-wastefulness* already incorporates an important part of stability, which our axioms imply (in the case of Proposition 1 even without employing *strategy-proofness*).

In the characterizations obtained by Kojima and Manea (2009), next to *non-wastefulness*, two monotonicity properties are employed: *individually rational (IR) monotonicity* and *weak Maskin monotonicity*. We first define both monotonicity properties (using the equivalent "unilateral" definitions).

Given $i \in N$ and $R_i \in \mathcal{R}$, a strategy $\bar{R}_i \in \mathcal{R}$ is an *individually rational (IR) monotonic transformation of* R_i *at* $\varphi_i(R, q)$ if for all $x \in O$, $x \; \bar{P}_i \; \varphi_i(R, q)$ and $x \; \bar{P}_i \; \emptyset$ imply $x \; P_i \; \varphi_i(R, q)$.

Individually Rational (IR) Monotonicity: For all $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, all $i \in N$, and all $\bar{R}_i \in \mathcal{R}_i$, if $\bar{R}_i$ is an IR monotonic transformation of R_i at $\varphi_i(R, q)$ and $\bar{R} = (\bar{R}_i, R_{-i})$, then for all $j \in N$, $\varphi_j(\bar{R}, q) \; \bar{R}_j \; \varphi_j(R, q)$.

Given $i \in N$ and $R_i \in \mathcal{R}$, a strategy $\bar{R}_i \in \mathcal{R}$ is a *monotonic transformation of* R_i *at* $\varphi_i(R, q)$ if for all $x \in O \cup \{\emptyset\}$, $x \; \bar{P}_i \; \varphi_i(R, q)$ implies $x \; P_i \; \varphi_i(R, q)$.

Weak Maskin Monotonicity: For all $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, all $i \in N$, and all $\bar{R}_i \in \mathcal{R}_i$, if $\bar{R}_i$ is a Maskin monotonic transformation of R_i at $\varphi_i(R, q)$ and $\bar{R} = (\bar{R}_i, R_{-i})$, then for all $j \in N$, $\varphi_j(\bar{R}, q) \; \bar{R}_j \; \varphi_j(R, q)$.

Proposition 2.
(a) *If* φ *satisfies IR monotonicity, then it satisfies truncation invariance.*
(b) *If* φ *satisfies weak Maskin monotonicity, then it satisfies truncation invariance.*

Proof. Assume that (a) φ satisfies *IR monotonicity* or that (b) φ satisfies *weak Maskin monotonicity*. Let $i \in N$ and $R_i \in \mathcal{R}$ and $\bar{R}_i \in \mathcal{R}$ be a truncation (strategy) of R_i, i.e., (t1) $\bar{R}_i|_O = R_i|_O$ and (t2) $A(\bar{R}_i) \subseteq A(R_i)$, and $\varphi_i(R, q) \in A(\bar{R}_i)$. Let $\bar{R} = (\bar{R}_i, R_{-i})$.

Thus, for all $x \in O$, $x \; \bar{P}_i \; \varphi_i(R, q) \; \bar{P}_i \; \emptyset$ implies $x \; P_i \; \varphi_i(R, q)$. Then, $\bar{R}_i$ is an *individually rational (IR) monotonic transformation of* R_i *at* $\varphi_i(R, q)$ and $\bar{R}_i$ is a *Maskin monotonic transformation of* R_i *at* $\varphi_i(R, q)$. Thus, (a) by *IR monotonicity* or (b) by *weak Maskin monotonicity*, for all $j \in N$, $\varphi_j(\bar{R}, q) \; \bar{R}_j \; \varphi_j(R, q)$.

Since $\varphi_i(R, q) \in A(\bar{R}_i)$, we also have $\varphi_i(\bar{R}, q) \; \bar{P}_i \; \emptyset$. Then, by (t1), for all $x \in O$, $x \; P_i \; \varphi_i(\bar{R}, q) \; P_i \; \emptyset$ implies $x \; \bar{P}_i \; \varphi_i(\bar{R}, q)$. Then, R_i is an *individually rational (IR) monotonic transformation of* $\bar{R}_i$

at $\varphi_i(\bar{R}, q)$ and R_i is a *Maskin monotonic transformation of* $\bar{R}_i$ at $\varphi_i(\bar{R}, q)$. Thus, (a) by *IR monotonicity* or (b) by *weak Maskin monotonicity*, for all $j \in N$, $\varphi_j(R, q)\ R_j\ \varphi_j(\bar{R}, q)$.

Now, for all $j \in N$, $\varphi_j(\bar{R}, q)\ \bar{R}_j\ \varphi_j(R, q)$ and $\varphi_j(R, q)\ R_j\ \varphi_j(\bar{R}, q)$. Hence, since $\varphi_j(R, q), \varphi_j(\bar{R}, q) \in O$, by (t1), for all $j \in N$, $\varphi_j(R, q) = \varphi_j(\bar{R}, q)$. Thus, φ satisfies *truncation invariance*. □

Next, there is no relation between *strategy-proofness* and *IR monotonicity* or *weak Maskin monotonicity*. Clearly, *strategy-proofness* does not imply *IR monotonicity* or *weak Maskin monotonicity* (since *strategy-proofness* only makes a statement for the agent who unilaterally changes his preferences, but not for all agents $j \in N$). The following rule satisfies *IR monotonicity* and *weak Maskin monotonicity*, but not *strategy-proofness*.

Example 4. Let $N = \{1, 2\}$ and $O = \{a, b\}$ with $\bar{q}_x = \bar{q}_y = 1$. Then, for each problem, φ assigns to agent 1 his most preferred available object and to agent 2 his most preferred available object if agent 1 did not already receive it and otherwise $\emptyset$. It is easy to see that this rule satisfies *IR monotonicity* and *weak Maskin monotonicity*: whenever an agent i receives his most preferred choice under R_i, then any IR or Maskin monotonic transformation R'_i of R_i at this top choice must have the same top choice as R_i and hence the same allocation (and if agent 2 receives $\emptyset$, then no IR or Maskin monotonic transformation R'_i of R_i at $\emptyset$ changes the allocation). This rule φ is not *strategy-proof*: for $R_1 = R_2 :\ x\ y\ \emptyset$, we have $\varphi_2(R, O) = \emptyset$ whereas for $R'_2 :\ y\ x\ \emptyset$, $\varphi_2((R_1, R'_2), O) = y$. ◇

Our characterizations of responsive DA-rules are based on intuitive and simple axioms. These results further support the use of responsive DA-algorithm as a practical solution in real-life matching markets.

Appendix

A Independence of Properties in Theorems 1 and 2

Example 1 shows that on the domain of all problems, *two-agent consistent conflict resolution* is independent from all other properties in Theorem 2.

Not *two-agent consistent conflict resolving*: In Example 1, let $R \in \mathcal{R}$ be such that $y\, P_1\, x\, P_1\, \emptyset$ and $R_2 = R_3 = R^x$. Let $q \in \mathcal{Q}$ be such that $q_x = 2$ and $q_y = 0$. Then $\varphi_2(R, q) = \emptyset$ and $\varphi_3(R, q) = x$ whereas $\varphi_2(R, 1_{xy}) = x$ and $\varphi_3(R, 1_{xy}) = \emptyset$.

In the following we consider house allocation problems, i.e., for all $x \in O$, $\overline{q}_x = 1$. Therefore, instead of denoting capacity vectors, we simply denote the set of available real objects, e.g., for $O' \subseteq O$, $O' \neq \emptyset$, (R, O') denotes a problem where one copy of each type in O' is available.

For any strict order π of agents in N, we denote the corresponding *serial dictatorship rule* by f^π; for example, if $\pi : 1\ 2\ \ldots\ (n-1)\ n$, then f^π works as follows: for each problem (R, O'), first agent 1 chooses his preferred object in O', then agent 2 chooses his preferred object from the remaining objects $O' \setminus \{f_1^\pi(R, O')\}$, etc. Note that for each strict order π of N, $f^\pi = DA^{\succ^\pi}$ where $\succ^\pi$ equals the priority order where for all $x \in O$, $\succ_x^\pi = \pi$. Thus, each serial dictatorship rule f^π satisfies *unavailable type invariance*, *individual rationality*, *weak non-wastefulness*, *resource-monotonicity* (*two-agent consistent conflict resolution*), *truncation invariance*, and *strategy-proofness*.

The following examples establish the independence of the properties (properties not mentioned in the examples follow easily).

Not *unavailable type invariant*: Let $n \geq 3$ and $\pi : 1\ 2\ 3\ \ldots\ (n-1)\ n$ and $\pi' : 1\ n\ (n-1)\ \ldots\ 3\ 2$. Then, for each problem (R, O'),

$$\varphi(R, O') = \begin{cases} f^\pi(R, O') & \text{if } A(R_1) = \emptyset \text{ and} \\ f^{\pi'}(R, O') & \text{otherwise.} \end{cases}$$

Not *individually rational*: Let $\pi : 1\ 2\ \ldots\ (n-1)\ n$. For each $R_n \in \mathcal{R}$, let $\hat{R}_n$ be such that $A(\hat{R}_n) = O$ and $\hat{R}_n|O = R_n|O$. Then, for each problem (R, O'),

$$\varphi(R, O') = f^\pi((R_{-n}, \hat{R}_n), O').$$

Not *weakly non-wasteful*: Fix an object $y \in O$ and $\pi : 1\ 2\ \ldots\ (n-1)\ n$. Then, for each problem (R, O'),

$$\varphi(R, O') = f^\pi(R, O' \setminus \{y\}).$$

Not *resource-monotonic* (and not *two-agent consistent conflict resolving*): Let π and π' be two distinct strict orders of agents in N. Then, for each problem (R, O'),

$$\varphi(R, O') = \begin{cases} f^\pi(R, O') & \text{if } O' = O \text{ and} \\ f^{\pi'}(R, O') & \text{otherwise.} \end{cases}$$

Not *truncation invariant*: Let $N = \{1, 2, 3\}$, $O = \{x, y\}$, $\succ_x$: 1 2 3, $\succ'_x$: 2 1 3, and $\succ_y$: 3 1 2. Let $\succ = (\succ_x, \succ_y)$ and $\succ' = (\succ'_x, \succ_y)$. Then, for each problem (R, O'),

$$\varphi(R, O') = \begin{cases} DA^{\succ}(R, O') & \text{if } \emptyset\, P_3\, x \text{ and } x \in O' \text{ and} \\ DA^{\succ'}(R, O') & \text{otherwise.} \end{cases}$$

Let $R_1 : x\ \emptyset\ y$, $R_2 : x\ \emptyset\ y$, $R_3 : y\ \emptyset\ x$, and $R'_3 : y\ x\ \emptyset$. Let $R = (R_1, R_2, R_3)$ and $R' = (R_1, R_2, R'_3)$. Note that R_3 is a truncation of R'_3 and $\varphi_3(R, \{x, y\}) = y = \varphi_3(R', \{x, y\})$. However, $\varphi_1(R, \{x, y\}) = x$ and $\varphi_2(R', \{x, y\}) = x$; a contradiction of *truncation invariance*. Next, we show *two-agent consistent conflict resolution*, *strategy-proofness* and *resource-monotonicity* for this rule.

For *two-agent consistent conflict resolution*, consider (R, O') and (R, O''). Since the allocation of y always follows the same priority order, *two-agent consistent conflict resolution* could only be violated for x. But then $x \in O'$ and $x \in O''$ and $\varphi(R, O') = \varphi(R, O'')$.

For *strategy-proofness*, note that agents 1 and 2 cannot change the priority structure by reporting a false preference relation. Consider agent 3 and a problem (R, O'). Obviously, if $|O'| = 1$, then agent 3 cannot profitably manipulate by reporting a false preference relation. Let $O' = \{x, y\}$. Now if agent 3's first choice is y or $\emptyset$ (or agent 3 receives his first choice), then agent 3 receives his first choice under (R, O') and agent 3 cannot profitably manipulate. Let agent 3's first choice be x.

If $R_3 : x\ \emptyset\ y$ and agent 3 does not receive his first choice, then $\varphi_3(R, \{x, y\}) = \emptyset$ and agent 3 can only change the priority structure by reporting a preference relation R'_3 with $\emptyset\ P'_3\ x$. But then by *individual rationality*, $\varphi_3((R_{-3}, R'_3), \{x, y\}) \neq x$ and agent 3 cannot profitably manipulate by reporting a false preference relation.

If $R_3 : x\ y\ \emptyset$ and agent 3 does not receive his first choice, then $\varphi_3(R, \{x, y\}) = y$. Now the same argument as above establishes that agent 3 can never receive x by reporting a false preference relation.

For *resource-monotonicity*, let $|O'| = 1$ and $R \in \mathcal{R}^N$. If there is a violation of *resource-monotonicity*, then φ must use different priority structures for (R, O') and $(R, \{x, y\})$. But then we must have $O' = \{y\}$ and both $\varphi(R, y) = DA^{\succ'}(R, y)$ and $\varphi(R, \{x, y\}) = DA^{\succ}(R, \{x, y\})$.

If $yP_3\emptyset$, then $\varphi_3(R, y) = y$ and $\varphi_1(R, y) = \varphi_2(R, y) = \emptyset$, and all agents weakly prefer $\varphi(R, \{x, y\})$ to $\varphi(R, y)$.

If $\emptyset\ P_3\ y$, then $\varphi_3(R, y) = \emptyset$. Since $\varphi(R, \{x, y\}) = DA^{\succ}(R, \{x, y\})$, we have $\emptyset\ P_3\ x$ and by *individual rationality*, $\varphi_3(R, \{x, y\}) = \emptyset$. Note that $1 \succ_x 2$, $1 \succ_y 2$, and $1 \succ'_y 2$ (the latter because $\succ_y = \succ'_y$). Let $\pi : 1\ 2\ 3$. Then, $\varphi(R, y) = DA^{\succ'}(R, y) = f^{\pi}(R, y)$ and $\varphi(R, \{x, y\}) = DA^{\succ}(R, \{x, y\}) = f^{\pi}(R, \{x, y\})$. Hence, *resource-monotonicity* is satisfied.

Not *strategy-proof*: Let $\succ$ be a priority structure. Then, the responsive DA-rule based on the object-optimal matching that is obtained by using Gale and Shapley's (1962) object-proposing deferred-acceptance algorithm satisfies all properties except *strategy-proofness*.

B Independence of Properties in Theorem 3

In the following we consider house allocation problems, i.e., for all $x \in O$, $\overline{q}_x = 1$. Therefore, instead of denoting capacity vectors, we simply denote the set of available real objects, e.g., for $O' \subseteq O$, $O' \neq \emptyset$, (R, O') denotes a problem where one copy of each type in O' is available.

For any strict order π of agents in N, we denote the corresponding *serial dictatorship rule* by f^{π}; for example, if $\pi : 1\ 2\ \ldots\ (n-1)\ n$, then f^{π} works as follows: for each problem (R, O') such that $R \in R^{N'}$, first agent $\min N'$ chooses his preferred object in O', then agent $\min N' \setminus \{\min N'\}$ chooses his preferred object from the remaining objects $O' \setminus \{f_1^{\pi}(R, O')\}$, etc. Note that for each strict order π of N, $f^{\pi} = DA^{\succ^{\pi}}$ where $\succ^{\pi}$ equals the priority order where for all $x \in O$, $\succ_x^{\pi} = \pi$. Thus, each serial dictatorship rule f^{π} satisfies *unassigned objects invariance*, *individual rationality*, *weak non-wastefulness*, *weak consistency*, *truncation invariance*, and *strategy-proofness*.

The following examples establish the independence of the properties (properties not mentioned in the examples follow easily).

Not *unassigned objects invariant*: Ehlers and Klaus (2007, Example 1) introduce a rule that violates *unassigned objects invariance*, but satisfies *efficiency* and *consistency* (and the other properties of Theorem 3).

Not *individually rational*: Let $\pi : 1\ 2\ \ldots\ (n-1)\ n$. For each $R_n \in \mathcal{R}$, let $\hat{R}_n$ be such that $A(\hat{R}_n) = O$ and $\hat{R}_n|O = R_n|O$. Then, for each problem (R, O') such that $R \in R^{N'}$,

$$\varphi(R, O') = \begin{cases} f^{\pi}((R_{-n}, \hat{R}_n), O') & \text{if } n \in N' \text{ and} \\ f^{\pi}(R, O') & \text{otherwise.} \end{cases}$$

Not *weakly non-wasteful*: Fix an object $y \in O$ and $\pi : 1\ 2\ \ldots\ (n-1)\ n$. Then, for each problem (R, O') such that $R \in R^{N'}$,

$$\varphi(R, O') = f^{\pi}(R, O' \setminus \{y\}).$$

Not *weakly consistent*: Let $\pi : 1\ 2\ 3\ \ldots\ (n-1)\ n$ and $\pi' : 1\ n\ (n-1)\ \ldots\ 3\ 2$. Then, for each problem (R, O') such that $R \in R^{N'}$,

$$\varphi(R, O') = \begin{cases} f^{\pi}(R, O') & \text{if } 1 \in N' \text{ and} \\ f^{\pi'}(R, O') & \text{otherwise.} \end{cases}$$

Not *strategy-proof*: Let $\succ$ be a priority structure. Then, the responsive DA-rule based on the object-optimal matching that is obtained by using Gale and Shapley's (1962) object-proposing deferred-acceptance algorithm satisfies all properties except *strategy-proofness*.[15]

References

Abdulkadiroğlu, A., T. Sönmez (2003): School Choice: a Mechanism Design Approach, *American Economic Review* 93, 729–747.

Balinski, M., and T. Sönmez (1999): A Tale of Two Mechanisms: Student Placement, *Journal of Economic Theory* 84, 73–94.

Chun, Y., and W. Thomson (1988): Monotonicity Properties of Bargaining Solutions when Applied to Economics, *Mathematical Social Sciences* 15, 11–27.

[15]Using Wako (2008), it can be seen that *unassigned objects invariance*, *individual rationality*, *weak non-wastefulness*, and *weak consistency* do not imply *stability*. More precisely, let $\bar{q}_x = 1$ for all $x \in O$. Then by Wako (2008), for any profile R and any priority structure $\succ$ there exists a unique *individually rational* von Neumann-Morgenstern (vNM) stable set. Since this set is a lattice, it contains an element which is weakly preferred by all agents to any other allocation in this set. Now (given $\succ$) we can define the agent-optimal vNM-rule choosing for each profile R the agent-optimal element in the *individually rational* vNM stable set. It is easy to verify that this rule satisfies all properties in Theorem 3 except for *strategy-proofness*.

Crawford, V.P. (1991): Comparative Statics in Matching Markets, *Journal of Economic Theory* 54, 389–400.

Ehlers, L. (2002): Coalitional Strategy-Proof House Allocation, *Journal of Economic Theory* 105, 298–317.

Ehlers, L., and B. Klaus (2003): Resource-Monotonic House Allocation, *International Journal of Game Theory* 32, 545–560.

Ehlers, L., and B. Klaus (2006): Efficient Priority Rules, *Games and Economic Behavior* 55, 372–384.

Ehlers, L., and B. Klaus (2007): Consistent House Allocation, *Economic Theory* 30, 561–574.

Ehlers, L., and B. Klaus (2009): Corrigendum to "Resource-Monotonicity for House Allocation Problems", Harvard Business School Working Paper # 09-110.

Ergin, H.İ. (2002): Efficient Resource Allocation on the Basis of Priorities, *Econometrica* 70, 2489–2497.

Gale, D., and L. Shapley (1962): College Admissions and the Stability of Marriage, *American Mathematical Monthly* 69, 9–15.

Hatfield, J.W., and Milgrom, P. (2005): Matching with Contracts, *American Economic Review* 95, 913–935.

Kesten, O. (2009): Coalitional Strategy-Proofness and Resource Monotonicity for House Allocation Problems, *International Journal of Game Theory* 38, 17–21.

Kojima, F., and M. Manea (2009): Axioms for Deferred Acceptance, *Econometrica*, forthcoming.

Pápai, S. (2000): Strategyproof Assignment by Hierarchical Exchange, *Econometrica* 68, 1403–1433.

Roth, A.E. (2008): Deferred Acceptance Algorithm: History, Theory, Practice, and Open Questions, *International Journal of Game Theory* 36, 537–569.

Roth, A.E., and M.A.O. Sotomayor (1990): Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis. Econometric Society Monograph Series. New York: Cambridge University Press.

Sönmez, T., and M.U. Ünver (2009): Matching, Allocation, and Exchange of Discrete Resources, Jess Benhabib, Alberto Bisin, and Matthew Jackson (eds.) Handbook of Social Economics, Elsevier, forthcoming.

Thomson, W. (2009): Consistent Allocation Rules, book manuscript.

Wako, J. (2008): "A Note on Existence and Uniqueness of vNM Stable Sets in Marriage Games," Working Paper.